MAIN ARTICLE

The concept of energy in the analysis of system dynamics models

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Abstract

This paper applies for the first time the Newtonian concept of energy to stock–flow systems and employs it to relate system behaviour to model structure. Kinetic energy and work done are defined analytically using the concepts of loop impact and force from the Newtonian Interpretative Framework and are examined numerically within system dynamics simulations. The energy analogy is used to analyse models by understanding how loops of different orders and types act as energy sources, sinks and exchange, using the system dynamics equivalent of the work–energy theorem. It is shown that energy describes the cumulative effects of feedback on stock behaviour, in contrast to the instantaneous description given by existing methods of loop dominance. This approach gives additional insight over that of existing dominance methods, capturing the dynamical influence of loops even when not dominant. The analogy's explanatory power provides quantitative and informal analysis.

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Introduction

An important feature of dynamical modelling, whatever the methodology, is the ability to relate the model's structure to its behaviour. The model structure is an explicit representation of the hypotheses and assumptions of the model, usually expressed in variables, equations and diagrams, with rules that act as a framework for model construction and interpretation. The behaviour of the variables over time can then be examined in the light of the structure, providing tests for model validity and also insight into the problem being analysed. A good modelling methodology is one that is not only capable of numerical and analytic computation, but also allows for informal explanations of behaviour using metaphorical concepts that appeal to a wide audience.

In system dynamics, model structure is expressed in stocks, flows, auxiliary variables and the causal connections between these elements (Forrester, 1961). In a given model, stock behaviour can be rigorously computed from the model

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structure. A further level of structure is the concept of feedback, where a stock level exerts control on one of its own flows, either directly, or via other stocks. Model behaviour may then be explained in terms of the relative importance of feedback loops, such as the shifting loop dominance narrative of the S-shaped growth in the limits-to-growth model (Barlas, 2002, sec. 4.4; Senge, 2006, pp. 95–104). Such informal explanations of model behaviour are widespread throughout the field.

The value of the feedback dominance narrative has led to the development of methods that quantify the effect of loops on stock behaviour at any given point in time (Kampmann and Oliva, 2020). For example, pathway methods, such as the Pathway Participation Metric (PPM) (Mojtahedzadeh *et al.*, 2004), Loop Impact (Hayward and Boswell, 2014) and Loops That Matter (LTM) (Schoenberg *et al.*, 2020), determine how changes in stock values are propagated through the system. Although their pathway measures are different, these three methods are related and provide the same dominance analysis when applied to a single stock (Schoenberg *et al.*, 2021). LTM goes further and determines a single dominance explanation for a whole system of stocks using a loop score. However, in common with PPM, the loop measures are relative to each other and have no transparent conceptual interpretation. By contrast, loop impact is an absolute measure and directly related to the curvature in stock behaviour (Hayward and Boswell, 2014; Hayward and Roach, 2017).

The main alternative to pathway methods is Eigenvalue elasticity analysis (EEA), where loop gains are related to system behaviour using a linearised matrix representation of the system (Forrester, 1982; Kampmann, 2012; Oliva, 2020). Like the pathway methods, EEA produces similar dominance analyses of stock behaviour with a distinct explanation of oscillations, connecting it with loops through complex eigenvalues. However, EEA's reliance on eigenvalues means this explanation has no easy conceptual connection with a variable's behaviour. By contrast, PPM identifies dominant structures in oscillations using complex numbers and half-cycles to determine metrics associated with pathway stability and frequency, thus avoiding the computation of eigenvalues (Mojtahedzadeh, 2008, 2011). The Loop Impact method has taken a different approach, explaining oscillations by using the average value of loop impact over one cycle to reduce complexity (Hayward and Roach, 2019). The resulting analysis examined behaviour over a period—a cumulative view, in contrast to the instantaneous view given by the existing methods. Likewise, LTM has introduced a total loop score over a period to explain oscillations (Schoenberg, 2020, pp. 189–194).

A rich understanding of the relationship between structure and behaviour is unlikely to be enabled by analytical tools using a single measure alone, and it is proposed here that the use of loop dominance analysis is enhanced by consideration of a cumulative measure in addition to an instantaneous one. Previous work has established the value of the Newtonian Interpretative

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Framework for quantifying the effect of one stock on another while offering an intuitive narrative interpretation of this effect as analogous to a force (Hayward and Roach, 2017, 2019). System behaviour may then be described using the equivalent of Newton's three laws of motion, relating them to loop dominance analysis using the Loop Impact method. In mechanics, energy measures the *total* effect of a force, thus giving a cumulative view of the influence of a force over a time interval (Leech, 1965, p. 6). The quantification of energy in system dynamics models derives naturally from the underpinning mathematics of stock–flow systems, extending the Newtonian Interpretative Framework. Energy provides a cumulative view of feedback that augments the instantaneous interpretation of system behaviour, enabling a richer narrative explanation.

The energy perspective offers three additional benefits for model construction and analysis. First, energy conservation provides a powerful explanatory principle for change-of-state variables, both numerical and informal. Second, in systems where energy is not conserved, a consideration of possible energy sources and dissipative sinks, identified with feedback loops, can assist model development. Third, a consideration of the rate of change of energy, called *power*, will enable a system-wide comparison of loops through different stocks.

This paper will introduce the concept of energy in a system dynamics model that provides a numerical description of the cumulative effects of feedback loops on stock behaviour. This extension to the framework will enable a new analytical dominance method that measures the influence of loops by the energy they transfer to a stock–flow subsystem over an interval of time, an approach not previously used in system dynamics analysis, complementing existing dominance methods.

Structure and behaviour

As an example of the relationship between system structure and behaviour, consider a model with overshoot and collapse (Figure 1) (Breierova, 1997; Ford, 2010; Kunsch, 2006). The reinforcing loop R_1, due to births, drives the deer's growth, balanced by deaths, loop B_1, whose effectiveness depends on the availability of food, loop B_3. The vegetation follows a limits-to-growth model, where R_2 drives growth, with B_2 representing capacity resistance due to the limited area and vegetation density. The second-order loop B_3 reduces vegetation through consumption, controlling the deer population due to the limited food availability. Overconsumption by the deer leads to the collapse of vegetation and, subsequently, the deer's extinction (Figure 2).

A narrative can be constructed that explains the stock behaviour (Figure 2), using the five loops of the model (Figure 1). For example, Breierova ascribes the slowdown and decline of deer numbers initially to loop B_3 then

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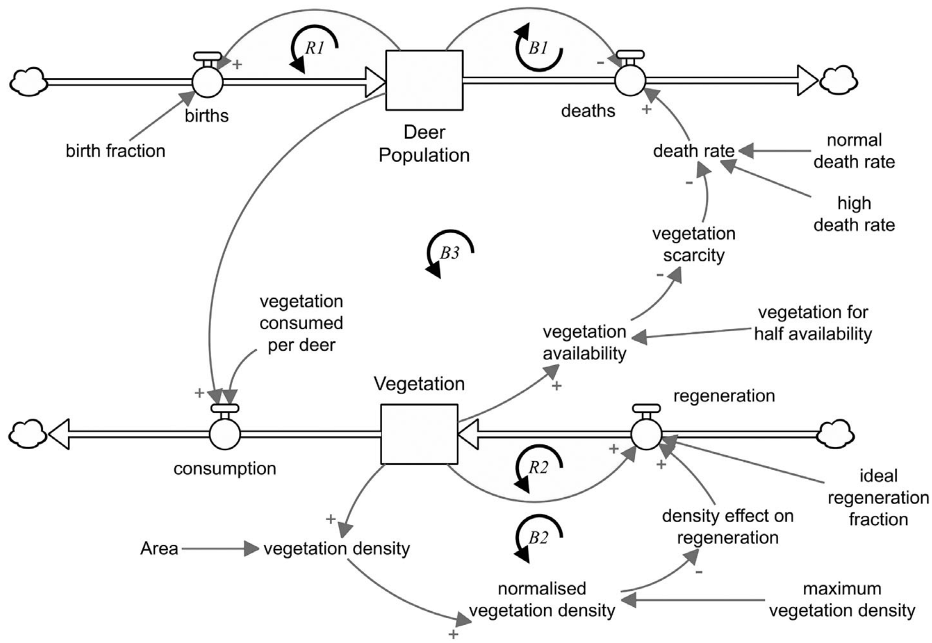

Fig. 1. Overshoot and collapse model. Equations in Appendix C

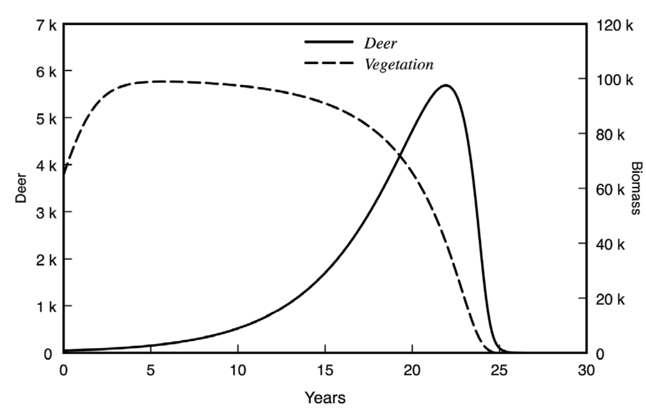

Fig. 2. Overshoot and collapse model of Figure 1. Deer numbers and vegetation biomass. Parameter values in Appendix C

to the death process B_1 (Breierova, 1997, p. 24). Sterman states that the early behaviour is driven by R_1 and that the later collapse is caused by the *gain in strength of the balancing loops* (Sterman, 2000, p. 123). In her 1977 talk, Donella Meadows describes the later model behaviour as dominated by the whole vegetation subsystem, adding that *this part of the system takes over and runs everything for a little while* (Meadows, 2017). Meadows elaborates that loop R_2 is also important at this stage, *running in a bad direction*. These

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examples connect feedback loop influence with different time intervals of the simulation, with the transitions between loop dominance occurring at specific times. These are the types of descriptions that the aforementioned current loop dominance methods attempt to quantify, providing instantaneous measures of the *strength* of the loops.

However, other questions could be asked that connect structure with behaviour. Two notable questions follow. (Q1) Which part of the system should be modified to avoid collapse and produce the stable and oscillatory modes noted by Meadows (2008, p. 69)? (Q2) Is it possible to attribute dominance to a *part of the system*, as indicated by Meadows, cited earlier? Answers to these questions require a view of a loop's influence over the whole simulation period, not just at particular moments in time. It is this cumulative quantification of loop influence that the proposed concept of energy is intended to capture. This paper will define energy within the Newtonian Interpretative Framework (Hayward and Roach, 2017), further developing the concept of loop impact. The subsequent definitions will then be applied to model analysis, before the two questions above are answered in the context of this model.

The concept of energy in a stock–flow system

Force and the Newtonian Interpretative Framework

The Newtonian Interpretive Framework uses the concept of force as a quantitative measure of the effect of one stock on another via one of its flows. Following Hayward and Roach (2017), a *force* between two stocks occurs when a source stock has a *causal pathway* to a target stock, such that changes in the source cause the target stock to accelerate—that is, to deviate from linear behaviour. A causal pathway is an individual connection from a stock to a flow, which may be composed of any number of converters, but not other stocks. This pathway represents cause and effect—the effect that one stock, the *source*, has on another, the *target*. Thus a causal pathway does not contain stocks. For example, in Figure 3 (Eqs (1) and (2)), there is a causal pathway from source stock y to target stock x via flow f_2. Thus y exerts a force on x, provided that y is changing; that is, its flow $g(t) \neq 0$, where t is time:

$$\dot{x} = by \tag{1}$$

$$\dot{y} = g(t) \tag{2}$$

Differentiating Eq. (1) by t puts the equation for x in Newtonian, second-order, form:

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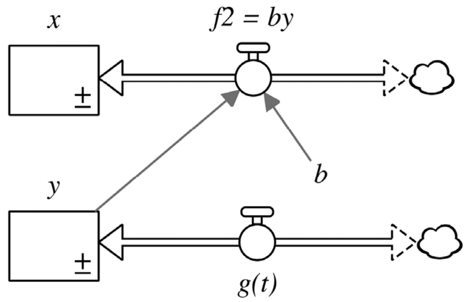

Fig. 3. Stock x subject to a single force from y

$$\ddot{x} = b\dot{y} = bg(t) \tag{3}$$

If $g(t) = 0$, then the change in x is uniform, $\ddot{x} = 0$, which is the system dynamics equivalent of Newton's first law (Hayward and Roach, 2017). If $g(t) \neq 0$, then Eq. (3) is the equivalent of Newton's second law, where b^{-1} represents the *mass* of x with respect to y. In system dynamics, mass converts the force of the source stock into the acceleration of the target stock. The value of y measures the amount of *momentum* imparted by stock y to x. The larger the value of y, the greater the rate of change of x, for a fixed mass b^{-1}.

The *impact* of a force is its contribution to a stock's acceleration divided by the net rate of change of that stock (Hayward and Boswell, 2014). Dividing Eq. (3) by $\dot{x}$ gives the impact equation:

$$\frac{\ddot{x}}{\dot{x}} = b\frac{\dot{y}}{\dot{x}} = \mathrm{I}_{\underline{yx}} \tag{4}$$

where $\mathrm{I}_{\underline{yx}} = b\dot{y}/\dot{x}$ is the impact of y on x (Hayward and Roach, 2017). The underline subscripts indicate the causal pathway, reading from left to right. The impact measures the curvature of stock behaviour over time, dimensions T^{-1}, such that a constant impact results in exponential stock behaviour (Hayward and Roach, 2019). As this model only has a single force, the impact $\mathrm{I}_{\underline{yx}}$ is responsible for the whole of the acceleration of x.

Figure 4a shows the behaviour of stock x for a constant negative force, given by $g(t) = -1$. The target stock x is initially growing as it has an initial positive momentum given by $y_0 = 15$. The force from y opposes this growth, bringing x to rest momentarily at $t = 15$. The impact of the force along the pathway from y to x is negative during this phase where change is resisted, becoming infinite at x's maximum. After x has peaked, the impact of y is positive, accelerating x downwards. This narrative is an example of the use of the force metaphor in system dynamics (Hayward and Roach, 2017, 2019).

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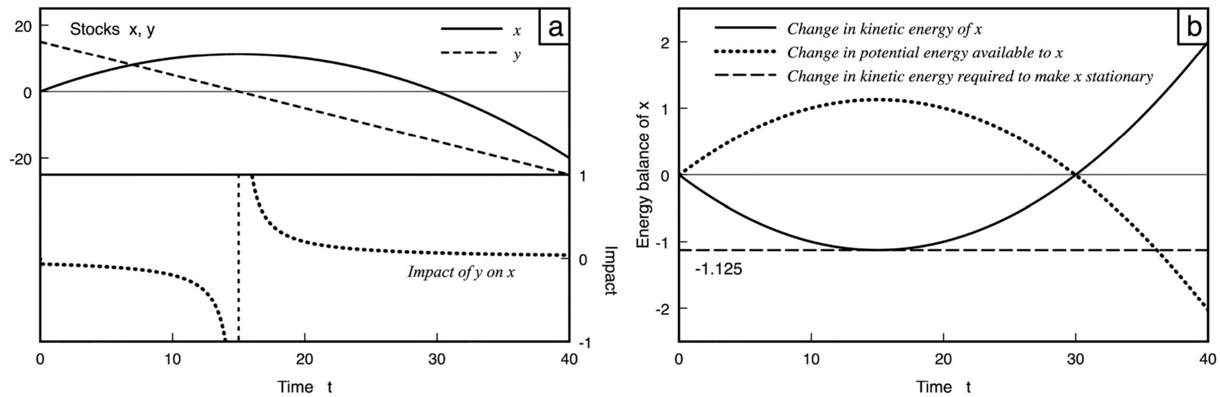

Fig. 4. Stock x subject to a constant, conservative force from stock y; Figure 3, (Eqs (1) and (2)), $b=0.1$, $g(t)=-1$, $x_0=0$, $y_0=15$. (a) Stock behaviour and force impact; (b) change in kinetic energy (ΔKE) and potential energy (ΔPE) of stock x. The dashed line indicates the change in kinetic energy required to make x stationary, $\Delta\text{KE}=-1.125$

Energy, power and work done

The Newtonian Interpretative Framework is now extended to include the concept of energy. In mechanics, energy is usually defined by integrating the equations of motion over distance and determining the energy conservation law (Leech, 1965; Tymms, 2015). The same procedure can be used in system dynamics by integrating the second-order form of the stock–flow equations over the stock value. For the model in Figure 3, integrate Eq. (3) by x from its initial value x_0, at time t_0, to its final value, x_f at time t_f. The integral of the left-hand side of Eq. (3) provides a definition of the *kinetic energy* of stock x:

$$\Delta\text{KE}_x \triangleq \int_{x_0}^{x_f} \frac{d^2x}{dt^2}\,dx = \frac{1}{2}\dot{x}_f{}^2 - \frac{1}{2}\dot{x}_0{}^2 \tag{5}$$

Unlike mechanics, the equivalent of mass, b^{-1}, is not included in the stock–flow version of kinetic energy. This definition ensures that the kinetic energy of a stock measures attributes of the stock alone, rather than its interaction with other system elements[1]. Thus energy has dimensions X^2T^{-2}.

The integral of the right-hand side of Eq. (3) represents the *work done* by the force from y on the stock x:

[1]In the Newtonian Interpretive Framework, mass is viewed as converting the effect of a source stock on a target stock; that is, force is converted into acceleration $F/m \rightarrow \ddot{x}$. Thus, the stock–flow equivalent of mass is a property of the stock's interaction with other stocks, rather than of the stock itself.

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$$\mathrm{WD}_{\underline{yx}} \triangleq \int_{x_0}^{x_f} b\frac{dy}{dt}dx \tag{6}$$

In mechanics, work done measures the total or cumulative effect of a force, as indicated by the integral Eq. (6) (Leech, 1965). Rewriting this expression in terms of impact, Eq. (4)

$$\mathrm{WD}_{\underline{yx}} \triangleq \int_{t_0}^{t_f} \mathrm{I}_{\underline{yx}}\dot{x}^2 dt \tag{7}$$

shows that in a stock–flow system, work done represents the cumulative effect of a causal connection between stocks. Equation (7)—the time integral of the impact multiplied by the net flow squared—gives a general formula for the numerical and analytical computation of the work done by one stock on another (see Appendices A and C). The integrand in Eq. (7) describes the rate at which energy is injected or removed into the stock. In mechanics this concept is called *power* (Tymms, 2015). Thus, the power of the causal link between the stocks can be defined as

$$\mathrm{P}_{\underline{yx}} = \mathrm{I}_{\underline{yx}}\dot{x}^2 \tag{8}$$

It follows from Eqs (5) and (7) that the equation of motion, Eq. (1), for the single force model of Figure 3 may be represented by the *energy balance equation* for the stock x:

$$\Delta\mathrm{KE}_x = \mathrm{WD}_{\underline{yx}} \tag{9}$$

The invariant Eq. (9) is called the work–energy theorem in mechanics, and it states that the work done by an external force is turned into changes in kinetic energy (Tymms, 2015). However, in contrast to its expression in mechanics, the definition of work done in a stock–flow system in Eq. (6) includes the mass of x with respect to y, ensuring the units of a stock's energy balance equation are independent of all units external to that stock. In summary, kinetic energy and work done in stock–flow systems measure the activity of stocks and the causal connections between stocks, respectively.

For the single force model (Figure 3), stock y performs work on x. Figure 4b plots the *change* in kinetic energy of stock x. This kinetic energy is easily computed in stock–flow simulation from the net flow of x. Initially, x has a kinetic energy of $\frac{1}{2}\dot{x}_0{}^2 = \frac{1}{2}(by_0)^2 = 1.125$, because of the initial value of y. Thus, the work done by y has reduced x's kinetic energy by 1.125 by $t = 15$, when the flow of x is momentarily zero. Thus, x has been brought to "rest"—

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that is, to a stationary state. Subsequently, y injects energy into x indefinitely, giving a positive change in kinetic energy compared with its initial value for $t > 30$. The single force model is unstable because the stock y is always performing work on x, increasing its kinetic energy.

In general, the expression for the work done by y on x is dependent on the values of x between the beginning and end time, the path of the stock $\mathrm{WD}_{\underline{yx}} = \int_{t_0}^{t_f} b\dot{y}\dot{x}dt$. This integral can be evaluated in stock–flow simulation using an accumulation (see Appendix A). However, under certain circumstances, this work done may be independent of the path of x, enabling the integral to be computed exactly without such knowledge. In such cases the stock y represents a conservative force, and a *potential energy* for y on x can be defined, turning the work–energy theorem into the conservation of energy of x. The change in potential energy is defined as the negative of the work done and represents the capacity of the force to do work.

The single force model (Figure 3), with a *constant* force, is conservative. Let $g(t) = k$, a constant. Then the change in potential energy of x due to the force from y, called $\Delta\mathrm{PE}_{\underline{yx}}$, is, from Eq. (6):

$$\Delta\mathrm{PE}_{\underline{yx}} = -\mathrm{WD}_{\underline{yx}} = -\int_{x_0}^{x_f} bkdx = -bk\left(x_f - x_0\right) \tag{10}$$

In this case, the change in potential energy depends only on the beginning and end value of x. Thus energy for x is conserved by y making energy available for x, $\Delta\mathrm{KE}_x + \Delta\mathrm{PE}_{\underline{yx}} = 0$.

Figure 4b compares the change in potential energy with that of kinetic energy for the specific value of $k = -1$. Initially, x loses kinetic energy, storing it as the potential energy of the force from y. Once x is stationary (t = 15), the kinetic energy starts increasing, as x loses its potential energy. When the potential energy returns to its initial value at $t = 30$, $\Delta\mathrm{PE}_{\underline{yx}} = 0$, stock x has returned to its initial value, because of the conservative nature of the force from y. The stock x increasingly becomes more negative, gaining kinetic energy from the effect of y. In practice, most forces will not be conservative with an associated law of energy conservation. However, if they are conservative, the exchange of energy between kinetic and potential energies provides an additional explanation of stock behaviour in terms of model structure, as will be shown for the single second-order balancing loop (Figure 12).

Generally, the system dynamics model of Figure 3 (Eqs (1) and (2)), is not conservative, even when $g(t)$ is constant. Whereas the kinetic energy of x changes, that of y is constant. Stock x does not gain kinetic energy in a *transfer* of energy from stock y, but from the *effect* of y via the causal pathway; that is, y does work on x. The action of one stock on another injects energy

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into a system, creating instability rather than equilibrium. In order to dissipate a stock's energy, an energy sink is required. Such a sink is provided by first-order feedback, as will be demonstrated by extending the single force model (Figure 3) with a draining process (Figure 7).

Sources, sinks and first-order feedback

In first-order feedback, a stock's value influences its flows directly without any other intermediate stocks in the loop. Consider a generic birth–death model, composed of two first-order loops: a compounding process R and a draining process B (Figure 5). From an energy perspective, the stock x is doing work on itself. From Eq. (6), the work done via R is $\mathrm{WD}_{\underline{xx}}(R) = \int_{x_0}^{x_f} a\dot{x}dx > 0$. Thus, the reinforcing loop is adding energy to the stock, provided that the initial kinetic energy is not zero. By contrast, the balancing loop is removing energy from the stock, $\mathrm{WD}_{\underline{xx}}(B) = -\int_{x_0}^{x_f} b\dot{x}dx < 0$. The Newtonian Interpretative Framework interprets first-order balancing loops as frictional processes (Hayward and Roach, 2017); thus it is natural to think of these as energy sinks. By analogy, first-order reinforcing loops can be interpreted as energy sources.

Consider the case where $a > b$, so that x grows exponentially (Figure 6a). Applying the work–energy theorem to this system gives the energy balance equation $\Delta\mathrm{KE}_x = \mathrm{WD}_{\underline{xx}}(R) + \mathrm{WD}_{\underline{xx}}(B)$ (compare with Eq. (9)). Thus, comparing the kinetic energy of the stock with the work done by both feedback loops shows that the kinetic energy of x is growing because more work is being done on x by R than dissipated by loop B (Figure 6b). That is, the stock's energy source R is larger than the energy sink B due to the frictional force. Because power is proportional to impact (Eq. (8)), R is a more *powerful* loop than B because it has the greater impact, delivering more energy per unit time. Thus, the concept of energy provides a new way of explaining stock behaviour in terms of model structure. Although potential energy cannot be defined in this non-conservative system, the energy balance can still help interpret system behaviour by examining how work done is transferred into a stock's kinetic energy.

As the stock–flow system is linear, the loop impacts (values a and $-b$) are constant (Figure 6a). However, the work done by each loop is not constant,

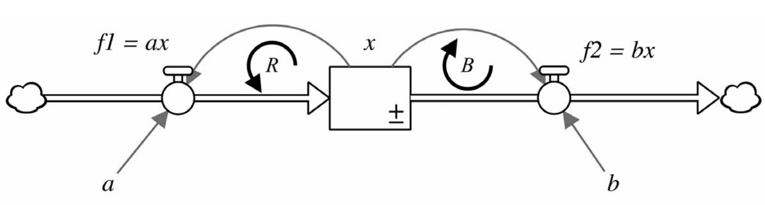

Fig. 5. Compounding R and draining B processes (energy source and sink respectively), $\dot{x} = ax - bx$

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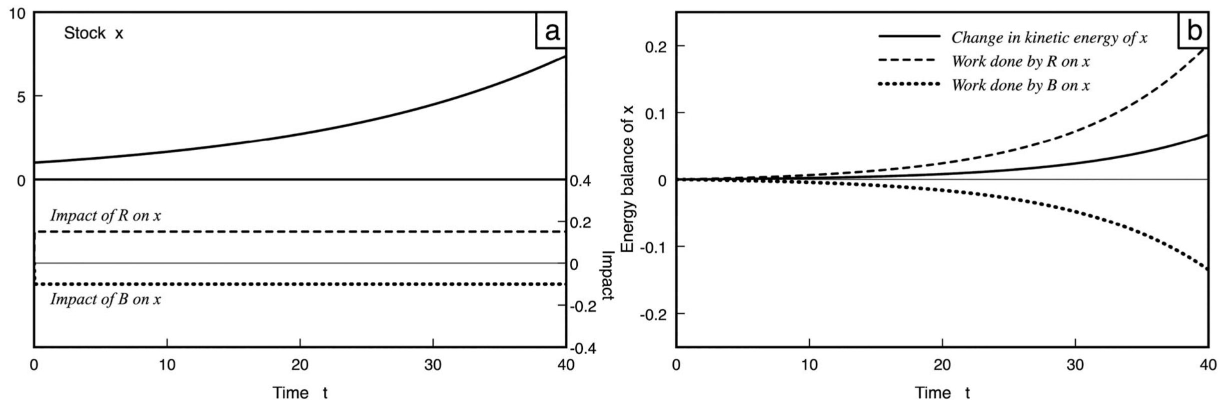
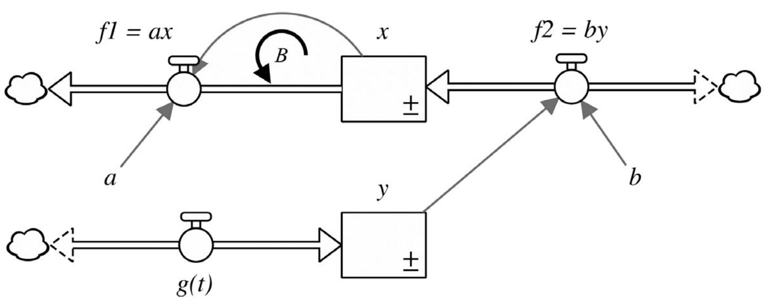


Fig. 6. Stock x subject to a compounding and draining process; Figure 5, $a = 0.15$, $b = 0.1$, $x_0 = 1$. (a) Stock behaviour and impact; (b) change in kinetic energy and work done for x

Fig. 7. External force and draining process B (energy sink). $\dot{x} = -ax + by$, $\dot{y} = g(t)$



as work done describes a cumulative effect rather than the instantaneous effect represented by loop impact. Nevertheless, because of the constant impact, the energy sink will never exceed the energy source. If instead $b > a$, then the stock would undergo exponential decline as more energy is being dissipated than being generated. Thus, the presence of a first-order balancing loop provides the possibility of stability to a system, depending on parameter values.

To further examine the role of first-order balancing feedback in energy removal, consider a stock x with a force from y (Figure 3), now also subject to a draining process, an opposing force (Figure 7). Let $g(t) > 0$. The force from y is injecting energy into x, whereas loop B is removing energy. Figure 8a shows the growth in x slowing. Comparing the impacts, it is clear that the force from y is greater than that of loop B but is tending to the same value as time progresses. Thus, the forces balance and the change in x becomes uniform, the equivalent of Newton's first law. From an energy perspective, B is removing much of the energy imparted by y, rather than it being turned into

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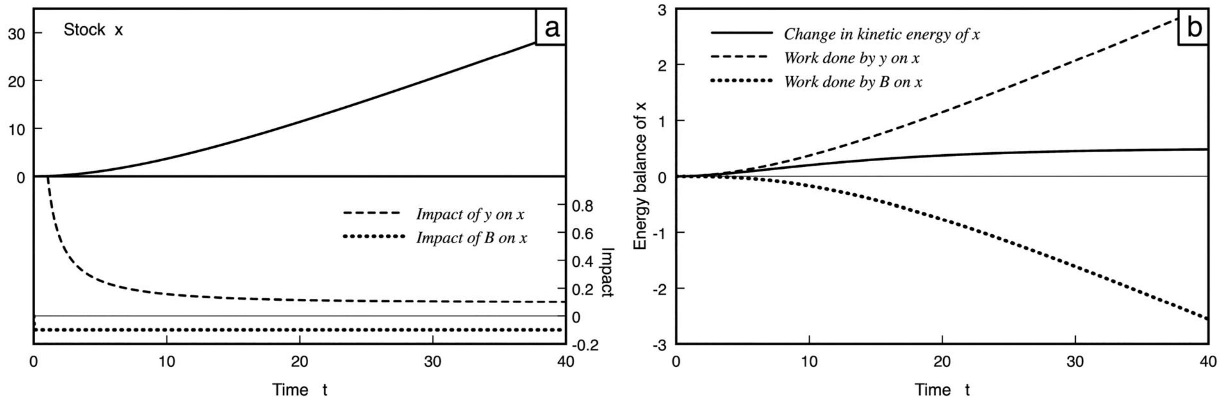
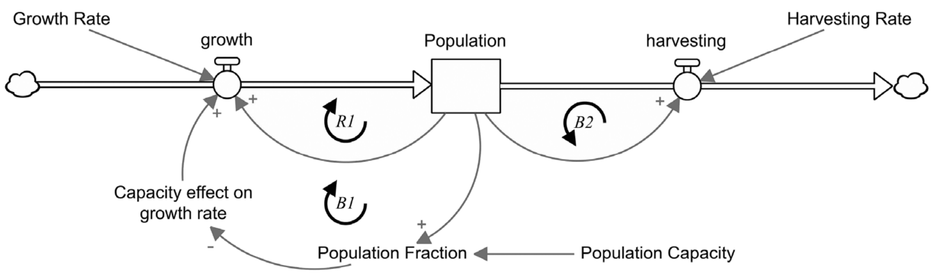


Fig. 8. Single stock x subject to a constant, conservative force from stock y; Figure 7, $a = 0.1$, $b = 0.1$, $g(t) = 1$, $x_0 = 0$, $y_0 = 0$. (a) Stock behaviour and impact; (b) change in kinetic energy and work done for x

Fig. 9. Limits-to-growth model with harvesting (polarities added). Equations in Appendix C



x's kinetic energy (Figure 8b). Nevertheless, kinetic energy becomes constant and is not reduced to zero. Thus, although the addition of a first-order balancing loop has given the possibility of stability, it has not guaranteed it.

The above definitions of kinetic energy and work are next applied to a single-stock system dynamics model. Two associated analysis techniques will be presented. First, a new approach to feedback loop dominance that compares the energy injected into the system by reinforcing loops with that removed by balancing loops—a cumulative approach to loop dominance. This approach applies the concept of dominance to work done in a similar way that the Loop Impact method applies dominance to the forces exerted by loops. Second, energy balance is examined by comparing kinetic energy with work done, treating initial kinetic energy and reinforcing loops as input, and the stock's kinetic energy and balancing loops as output.

Limits to growth with harvesting

An examination of energy balance can be used to enhance the Newtonian interpretation, and loop dominance analysis, of models. Consider a limits-to-growth

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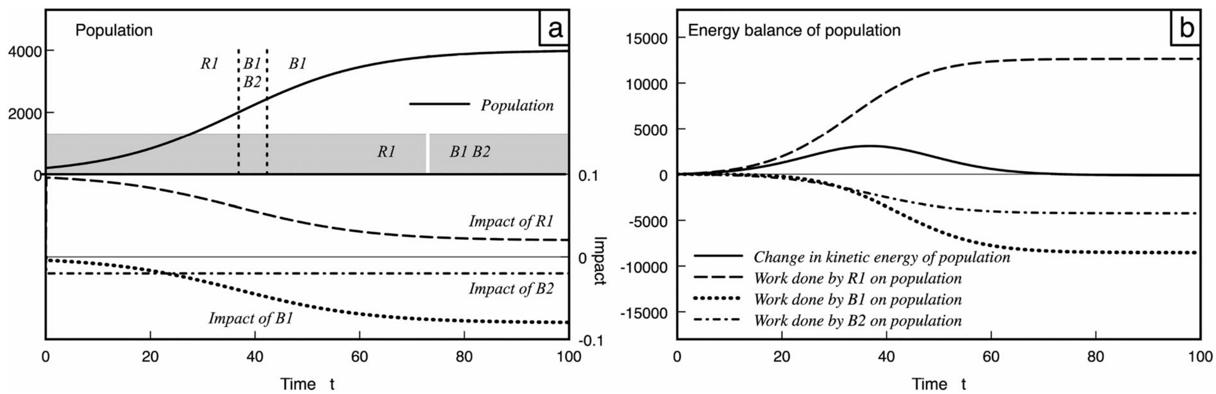

Fig. 10. Limits-to-growth model with harvesting; Figure 9, Growth Rate = 10%, Population Capacity = 5000, Harvesting Rate = 2%, Initial Population = 200. (a) Population with regions of dominance indicated by vertical lines for loop impact and shaded areas for work done; (b) change in kinetic energy and work done for x

model with harvesting (Figure 9). The growth of the animal population is driven by the reinforcing loop R_1 and opposed by limited population capacity, balancing loop B_1. The interaction of the two loops makes the model nonlinear. Without harvesting, the population reaches equilibrium at *population capacity*, falling short of that limit if harvesting is introduced, loop B_2.

In the Newtonian Interpretative Framework, the three loops are viewed as three forces acting on the stock with their regions of dominance determined by the Loop Impact method (Figure 10a) (Hayward and Boswell, 2014). Initially, the driving force R_1 is dominant, accelerating the population's growth. However, the impact of this driving force is falling as that of the opposing force B_1 is increasing in magnitude. With the help of the harvesting loop B_2, the opposing forces exceed R_1, slowing the growth in the population by applying friction. For most of the simulation, the capacity loop B_1 has more impact than the harvesting loop B_2 (Figure 10a). There is only a brief period in the loop impact analysis, $37 < t < 43$, where B_2 is involved in dominance (with B_1). It is not clear from this instantaneous approach to dominance analysis why the effect of loop B_2 prevents the population from reaching capacity. This shortfall is usually explained using stability analysis, a mathematical rather than a conceptual explanation.

From the cumulative perspective provided by energy, the population system in Figure 9 has one energy source, R_1, and two energy sinks, B_1 and B_2, representing diffusion processes. The loop B_1 removes energy from the population, limiting its growth. The harvesting loop B_2 is an additional drain of energy on the population, preventing it from reaching its carrying capacity. The loop R_1 can be viewed as input energy, along with the initial kinetic energy, with the balancing loops and kinetic energy the outputs. Figure 10b displays the energy balance on the stock, showing R_1 injecting energy

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into the system, with the change in kinetic energy of the population being the largest recipient early on—hence the accelerating growth. Both balancing loops are doing work to take energy away from the population. By $t = 31$, B_2 has removed as much input energy as B_1, reducing the change in kinetic energy and thus slowing population growth. By the time equilibrium has been reached, and the balancing loops have removed all the input energy, a third of that energy has been taken by the harvesting loop B_2.

The regions of dominance for the work done by the three loops are indicated by the shaded area in Figure 10a.[2] This new technique compares the cumulative effects of the feedback loops on the system and contrasts it with the instantaneous effects of loop impact indicated by the vertical dashed lines. The shaded area in Figure 10a shows that to overcome R_1, B_1 requires the cumulative effect of B_2, whose contribution is not evident in the dominance of loop impact (except for a brief period around $t = 40$). Figure 10b shows that by $t = 100$, B_2 has dissipated approximately one third of the energy of the system. Thus, over the entire time horizon, B_2 is having a more significant cumulative effect on the system than that indicated by the loop impact analysis. Although the effect of B_2 on the curvature of stock x at a given instant, represented by loop impact, is relatively small, its long-term effect, represented by work done—the accumulation of loop impact—is much higher. The energy drains from both balancing loops are required to exceed the energy deposited by the reinforcing loop in order to bring the population to rest.

The significance of the effect of the harvesting loop B_2 on energy depletion can be examined by comparing the kinetic energy with the energy diffused by the balancing loops. Rearranging the energy balancing equation, $\Delta \mathrm{KE}_x = \mathrm{WD}(R_1) + \mathrm{WD}(B_1) + \mathrm{WD}(B_2)$, allows inputs and outputs to be compared:

$$\mathrm{Inputs} = \mathrm{WD}(R_1) + \mathrm{KE}_{x_0} = \mathrm{KE}_x + |\mathrm{WD}(B_1)| + |\mathrm{WD}(B_2)| = \mathrm{Outputs} \tag{11}$$

where KE_{x_0} is the initial kinetic energy. At $t = 0$, the loops have done no work, and all the initial energy is kinetic (Figure 11a). Up to $t = 30$, the harvesting loop is diffusing most of the energy. The early reduction in kinetic energy slows after $t = 10$ as the balancing loops have to diffuse an increasing amount of energy from R_1; however, the early effect of harvesting has significantly slowed the population growth. This effect depends on the harvesting rate. Figure 11b compares the relative amount of energy diffused by B_2 with B_1 for different harvesting rates. The greater the harvesting rate, the greater the proportion of energy diffused by B_2, from 18% for 1% harvesting up to 57% for 4% harvesting. The corresponding shortfall of the equilibrium population from capacity ranges from 10% to 40%. The

[2]Loop dominance using work done utilises the same algorithm to describe multiple loop influence as the Loop Impact method (Hayward and Boswell, 2014).

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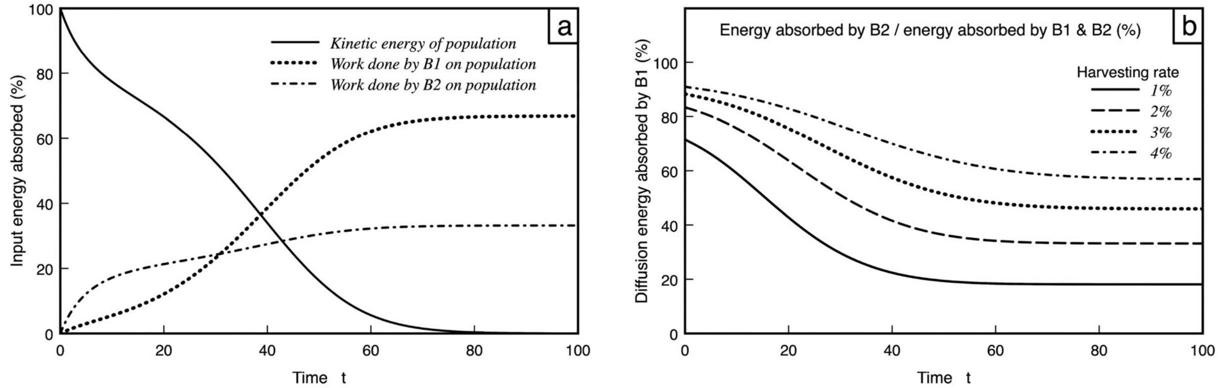

Fig. 11. Limits-to-growth model with harvesting; Figure 9, Growth Rate = 10%, Population Capacity = 5000, Initial Population = 200. (a) Energy balance (Eq. (11)), energy absorbed by kinetic energy, and diffused by B_1 and B_2 from inputs R_1 and initial kinetic energy as a percentage of total, for Harvesting Rate = 2%. (b) Energy diffused (work done) by B_2 as a percentage of the total energy diffused by B_1 and B_2 for harvesting rates 1–4%

additional energy removed by harvesting reduces the maximum possible value of the population. It can be proved analytically that the shortfall in equilibrium population from capacity is directly related to the fraction of energy removed by harvesting (Appendix C).

Energy and higher order feedback

General considerations

In the previous section, it was shown that in a system with more than one stock, such as the single force model (Figure 3), there is an energy balance equation for each stock–flow subsystem—for example, Eqs (9) and (11). However, although the change in kinetic energy of a given stock is equal to the work done on the stock by its first-order feedback loops and through the influence of other stocks, in general, there is no single energy invariant for the whole system. That is, a general dynamical system is not conservative (Strogatz, 2018). Instead, the influence of a source stock y injects energy into a target stock x. If there is feedback from the target back to the source, there is no guarantee that the work done by y on x balances with that from x to y as, in general, the energy units of the source and target stocks are different. Nevertheless, the two techniques already introduced—work done dominance and energy balance—can be applied to individual stock–flow subsystems and reveal a useful interpretative narrative. Further, the two-stock conservative system considered next, the single second-order balancing loop of Figure 12, will help link the energy narratives of connected subsystems.

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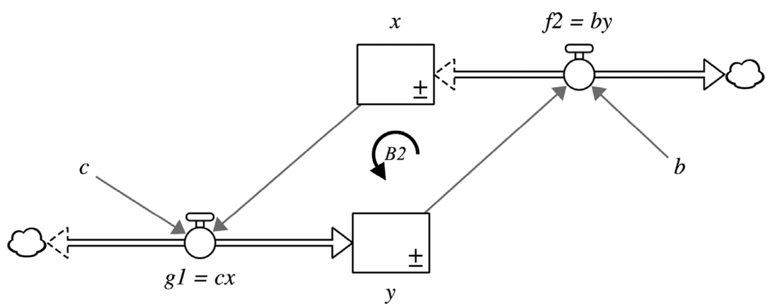
Fig. 12. Model with a single second-order balancing loop. $\dot{x} = -by$, $\dot{y} = cx$



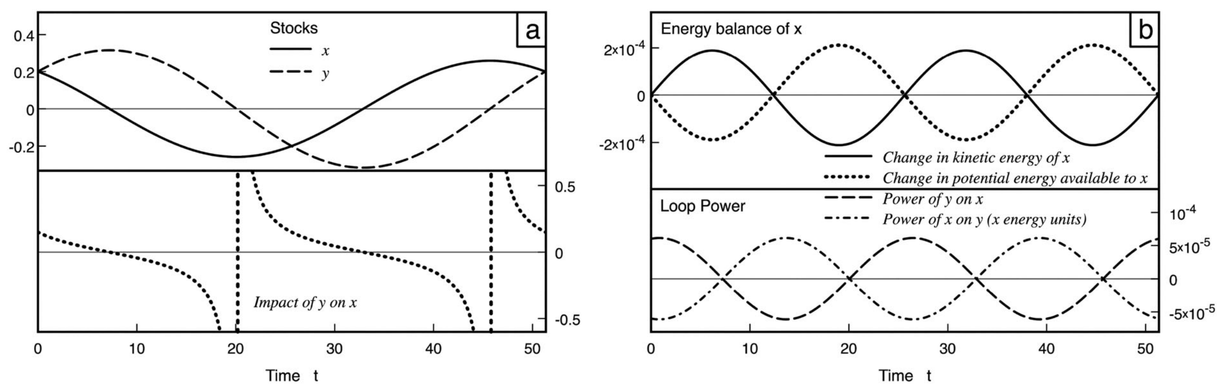


Fig. 13. Single second-order balancing loop of Figure 12; $b = 0.1$, $c = 0.15$, $x_0 = y_0 = 0.2$. (a) Stock behaviour and impact of y on x over one cycle; (b) change in kinetic energy and potential energy of x; power of stocks on each other, showing loop power is zero, $\mathrm{P}_{\underline{yx}} + \frac{b}{c}\mathrm{P}_{\underline{xy}} = 0$

Energy conservation and second-order balancing loops

A model with a single second-order balancing loop is given in Figure 12, with the two causal links between the stocks interpreted as forces. The stock values oscillate with the loop B_2 responsible for the whole of the behaviour (Figure 13a). Turning points in stock behaviour are explained by the loop impact becoming momentarily infinite and changing sign as the associated flow becomes zero (Hayward and Boswell, 2014). The frequency of oscillation is determined by the loop gain $|G| = bc$. In the Newtonian Interpretive Framework, the inverse of the loop gain describes the mass of the loop. A lower frequency oscillation is explained by the loop having a greater mass and giving more inertial resistance to the two mutual forces (Hayward and Roach, 2017).

Although a consideration of forces at each instant explains the curvature of the stock values, it does not provide a reason for the indefinite bounded oscillation behaviour. From an energy perspective, the total kinetic energy is conserved, $\mathrm{KE}_x/b + \mathrm{KE}_y/c = 0$, with appropriate unit conversion. As such, no energy is lost from the system, which is only possible with undamped

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oscillations. Both stocks have well-defined potential energies with respect to each other. Thus, for example, the kinetic energy of y makes potential energy available for x, $\Delta \mathrm{KE}_x = -\Delta \mathrm{PE}_{\underline{yx}}$. As y loses kinetic energy, potential energy is transferred to x in the form of its kinetic energy (Figure 13b). Once y is at rest, and all the kinetic energy is in x, energy then flows back from x to y. The oscillatory behaviour follows from the conservation law preventing the stocks from growing indefinitely or finding rest simultaneously.

Thus, a second-order balancing loop exchanges energies between two stock–flow subsystems in a conserved manner. Although this result has been demonstrated for a linear system, it is also true for nonlinear second-order loops (Appendix B). One corollary is that, unlike a first-order balancing loop, the second-order equivalent does not, on its own, bring stability or long-term control to a system.

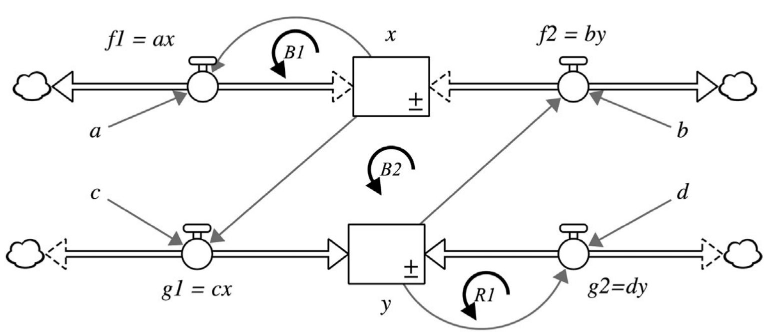
The conservative nature of this loop can also be demonstrated by comparing the *rate* at which energy is injected and removed from the stocks—that is, the power of the two causal links in loop B_2 (Figure 12). These powers are equal and opposite once they are placed in the same units $\mathrm{P}_{\underline{yx}} = -\frac{b}{c}\mathrm{P}_{\underline{xy}}$ (Figure 13b). It follows that a single power can be defined for the second-order loop by adding the two link powers together. For a balancing loop this power is zero, $\mathrm{P}(B_2) = \mathrm{P}_{\underline{yx}} + \frac{b}{c}\mathrm{P}_{\underline{xy}} = 0$. Thus, in the whole system, energy is neither added nor removed by this loop.

It is easy to show that a second-order reinforcing loop injects energy into a system, like its first-order equivalent. For this loop, energy is not conserved, the power of the loop is non-zero and potential energy cannot be defined. Thus, neither type of second-order loop can dissipate energy from a system. For energy removal at least one first-order balancing loop is required, as is shown in the next example.

Second-order linear system with first-order loops

The usefulness of a cumulative measure in loop dominance analysis and of an energy perspective for narrative explanation can be demonstrated by

Fig. 14. Linear second-order system, $\dot{x} = -ax - by$, $\dot{y} = cx + dy$



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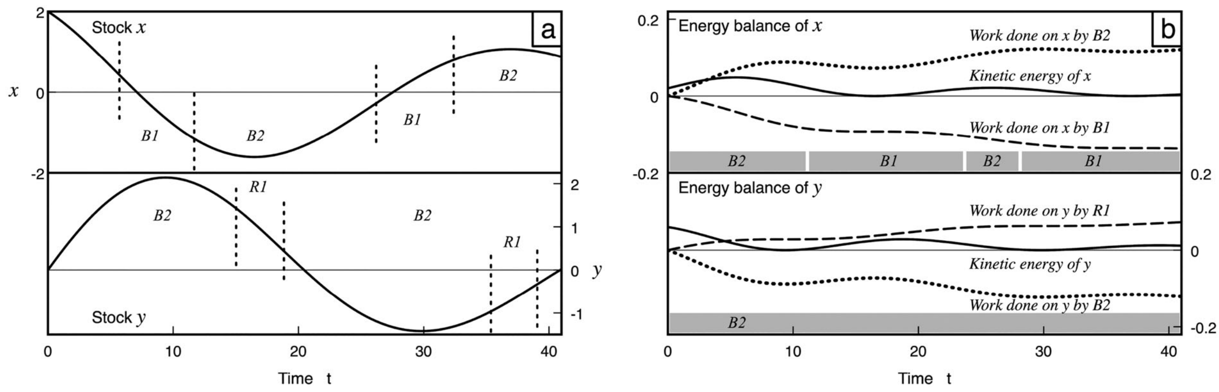

Fig. 15. Linear second-order system of Figure 14; $a = 0.1$, $b = 0.15$, $c = 0.2$, $d = 0.06$, $x_0 = 2$, $y_0 = 0$. Reinforcing loop R_1, balancing loops B_1, B_2. (a) Stock behaviour and regions of dominance in loop impact over one cycle, period 41; (b) kinetic energy and work done for stocks (in units of x's energy), with regions of dominance in work done given by the shaded area

considering a second-order linear system (Figure 14), where first-order loops have been added to the second-order balancing loop of Figure 12. This model is a special case of the general linear system (Hayward and Roach, 2017, 2019). Before proceeding further, it is clear that stability is possible for this system as it contains a first-order balancing loop—the only means by which energy can be removed from the system. Any other feedback structure either exchanges energy between the stocks or injects energy. Thus, the energy perspective can indicate the possible behaviour of a system before a simulation or analysis has taken place.

In the Newtonian Interpretive Framework, each stock exerts two forces: one associated with a first-order loop, B_1 on x and R_1 on y, and the other force a part of the second-order loop B_2. The influence of these forces can be compared using their loop impacts (Figure 15a). The damped oscillation is caused by shifts of loop impact dominance between the first- and second-order loops over one cycle, with the second-order loop B_2 responsible for the turning points. Although this is an accurate description of the rise and fall of stock behaviour from an instantaneous viewpoint, it does not succinctly explain the oscillatory or damped behaviour of the system.

The energy balance on each stock, the cumulative view of loop influence, provides further clarity to the system behaviour (Figure 15b). The oscillation is due to the exchange of energy between the two stock–flow subsystems via the balancing loop B_2 as in the case of the single second-order balancing loop (Figure 12). Although the model in Figure 14 is not a conserved system, the work done by B_2 on stock x balances that on y, $\mathrm{WD}_{\underline{yx}}(B_2) = -b/c\mathrm{WD}_{\underline{xy}}(B_2)$, allowing for unit conversion, (dotted curves in Figure 15b). Loop B_2 removes energy from y while injecting it into x and

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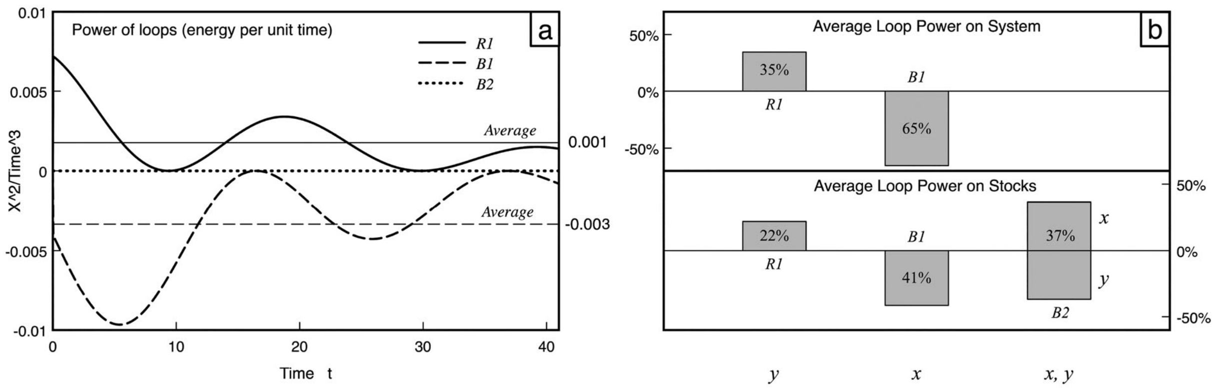

Fig. 16. Loop power over one cycle for the linear second-order system of Figure 14. (a) Power on system over time of R_1, B_1, B_2 and average powers; (b) average powers on system (energy injection and removal) and on stocks (including energy transfer)

vice versa. The opposing signs of the two expressions for work done describe a mutual flow of energy that results in oscillation.

Regions of dominance for the work done by each loop on each stock are given in Figure 15b, indicated by the shaded area. The energy transferred from y to x by B_2 always dominates over the energy source, R_1, reducing the kinetic energy of y over one cycle (period 41). Thus, y is damped. Loop B_1 is an energy sink, which is larger than the energy transferred from y by B_2 over the cycle. The brief change in dominance for the work done on x during the cycle results from transients. Thus, the system is damped because the sink, loop B_1, removes more energy than is input by the source, loop R_1.

The energy balance can be further illustrated by considering the power of the loops. The power of B_2 on the system is zero as it removes as much energy from y as it injects into x. Thus, only the two first-order loops need to be compared. Although power dominance changes between the two loops during the cycle, the *average* power of B_1 over the cycle is numerically larger than that of R_1 (Figure 16a). The system stabilises because B_1 is more *powerful* than R_1, as it has the greater influence on the energy of the system and, therefore, on the rate of change of the stocks. This result supports Hayward and Roach's (2019) assertion that oscillations are best understood by averaging over one cycle. It also demonstrates the explanatory value of cumulative measures as, although power is an instantaneous measure, average power over the cycle is an accumulation, the work done divided by the period.

It follows that the dominance of the loops on the *system* can be displayed in a cumulative form, (Figure 16b, top). The chart shows B_1 responsible for 65% of the energy change over one cycle, displaying clearly the system's stability. Loop B_2 is excluded as it is only transferring energy within the

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Table 1. Energy implications for feedback loops of different order

	First order	Second order	Third and higher order
Balancing loop	Sink	Exchange	Source
Reinforcing loop	Source	Source	Source

system. Additionally, it is possible to compare the work done by the loops on all the *stocks* using the average power of all the causal links (Figure 16b, bottom). This chart shows that B_1 is more powerful than B_2 with R_1 the weakest. Loop B_2 is transferring more energy between stocks than R_1 is injecting into y; thus, y's rate of change is reduced. Likewise, B_1 is removing more energy than B_2 is transferring; thus, x's rate of change is reduced and the system stabilises. This latter result can be compared with the *total loop score* provided by the LTM method, which is also cumulative. Over one cycle, the total loop scores are weighted $R_1 : B_1 : B_2 = 24\% : 33\% : 43\%$, indicating B_2 has the largest effect. However, it is unclear how this result can be interpreted. The energy perspective has the advantage of explaining dominance by the loop that does the most work in changing stock behaviours.

Feedback loops higher than second order

Systems composed solely of a third-order loop, or higher, are unstable. Such loops contain a dominant positive eigenvalue, leading to the stocks increasing exponentially, or with growing oscillations (Cinquin and Demongeot, 2002; Hayward and Boswell, 2014; Mojtahedzadeh and Richardson, 1995; Moxnes and Davidsen, 2016). Energy is not conserved in any of these loops, and in the limit, as time goes to infinity, they are always net energy sources. The only loop structure that removes energy from a system in the long term is the first-order balancing loop. It follows that a system dynamics model can only be stable if it contains at least one first-order balancing loop. A related result connecting balancing loops with stability was proved by Sato (2016, theorem 6.1), stating that in any system, one or more balancing loops are dominant in the region of a stable equilibrium point. The energy viewpoint suggests that care is needed in ensuring all first-order loops are correctly included in the model.

In summary, every feedback loop in a system dynamics model is associated with an energy source, sink or conserved exchange, as summarised in Table 1. These identifications form part of the Newtonian Interpretative Framework, outlined in the glossary.

Overshoot and collapse

The overshoot and collapse model of Figure 1 can now be examined using the concept of energy to quantify the accumulated influences of its

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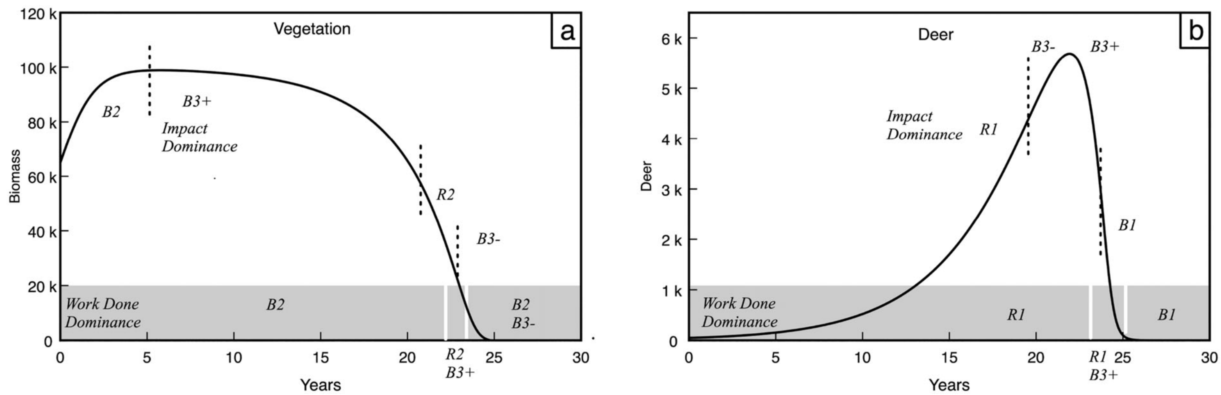

Fig. 17. Overshoot and collapse of Figure 1, model A. Regions of dominance in loop impact are indicated by vertical dashed lines, with the plus sign on B_3+ denoting a force with positive impact (minus sign is negative impact). Regions of dominance in work done are given by the shaded area, with the plus sign on B_3+ indicating the loop is an energy source (minus sign denoting a sink). Loop names are placed under the axis where needed. (a) Vegetation biomass; (b) Deer numbers. Parameter values in Appendix C

loops. The model equations follow those of Breierova (1997), with graphical relationships replaced by matching analytical functions to improve the computational accuracy of the loop analysis (see Appendix C). The model is first analysed for a constant per capita consumption rate of vegetation, referred to as *Model A*. This model will be compared with *Model B*, given later, where the per capita consumption depends on vegetation availability.

Before discussing a quantitative analysis of the model, consider an informal examination of its loop structure. The model has two sources, R_1 and R_2, and two sinks, B_1 and B_2. Thus, energy will be drained from each stock–flow subsystem making stability possible. Further, the second-order loop, B_3, will exchange energy from one stock–flow subsystem to the other. Therefore, even if diffusion through one of the sinks is insufficient to counteract the energy source in one stock–flow subsystem, excess energy could be transferred to the other stock–flow subsystem. For example, the deer could be stabilised through diffusion in the vegetation system.

First, an instantaneous view of the influence of the loops is considered. In the Newtonian Interpretative Framework, each stock in model A is subject to three forces associated with the feedback loops (Figure 1). The loop dominance, given by the Loop Impact method, is superimposed on the stock behaviour (Figure 17). The initial growth in vegetation is slowing down as the capacity loop B_2 dominates (Figure 17a). The change from growth to decline is determined by the consumption loop B_3 acting as an accelerating force, with R_2 dominating later. This reinforcing loop is accelerating the vegetation's decline, confirming Meadow's description cited earlier: *running in a bad direction*. Although there is a short period where B_3 is resistive (from

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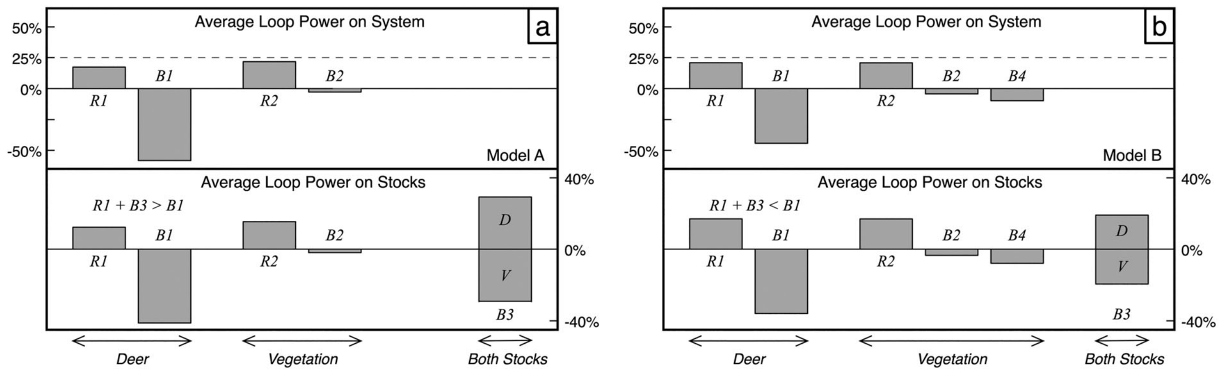

Fig. 18. Overshoot and collapse models A and B compared over one cycle. Average powers on system (energy injection and removal), and on stocks (including energy transfer). (a) Model A, Figure 1; (b) model B; Figure 1 modified by Figure 19. Parameter values in Appendix C

23 years), this control on the vegetation's decline occurs too late to prevent collapse.

The deer numbers grow through R_1, then slow down through the second-order loop B_3, changing growth to an accelerating decline (Figure 17b). The only resistive force on the deer, B_1, slows the decline until extinction is reached at 24.8 years. This pattern confirms Sterman's informal description that the balancing loops are *gaining in strength*, cited earlier (Sterman, 2000, p. 123). Although the loop impacts have successfully explained the *curvature* in stock behaviour—an instantaneous view—they have not completely explained why both stocks collapse to zero, as the same impact dominance pattern occurs with recovery and oscillation when, for example, the natural death rate of deer is increased (Ford, 2010; Hayward and Roach, 2019; Meadows, 2008). The collapse is not a consequence of an effect that happens in an instant, but an accumulation of effects over the cycle.

Comparing the work done by the energy sources and sinks on each stock provides a cumulative dominance pattern (shaded areas in Figure 17). For the vegetation, energy diffusion through B_2 dominates until year 22.2, when a net energy injection occurs through R_2 and B_3. Although the balance shifts back to energy diffusion at year 23.4, the vegetation becomes extinct by 24.8 years. For the deer, energy sources dominate until year 25.5, after the extinction of the vegetation, showing that loop B_3 has failed to bring stability to the deer subsystem, a necessary requirement for their numbers to recover.

The failure of control by B_3 is also demonstrated by comparing the powers of the loops, averaged over the entire collapse cycle (Figure 18a, model A).[3] The

[3]Although the units of energy in the two subsystems cannot be harmonised due to the nonlinearities of the system, loop powers can always be placed in the same units (see Appendix B).

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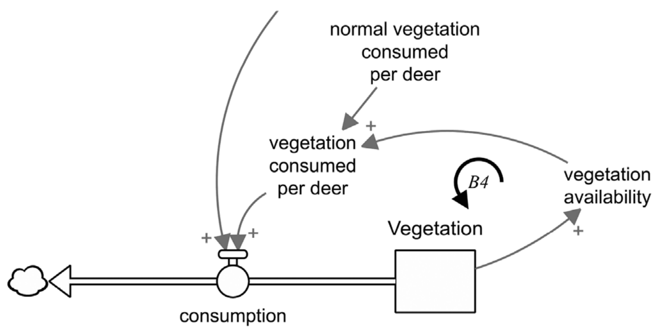

Fig. 19. Overshoot and collapse model B; changes from model A, Figure 1

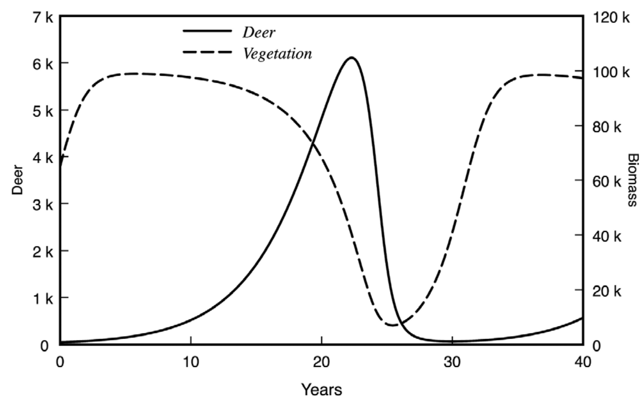

Fig. 20. Overshoot and collapse model B, Figures 1 and 19. Deer numbers and vegetation biomass. Parameter values in Appendix C

top diagram shows that the total energy injected into the system through R_1 and R_2 is less than that removed by balancing loops, $\mathrm{P}(R_1)+\mathrm{P}(R_2)=39\%<50\%$; thus, system stability is possible. The stock view (bottom diagram) shows that most of the vegetation's input energy is being removed through transfer to the deer subsystem, $\mathrm{P}(B_3)$, rather than through diffusion, $\mathrm{P}(B_2)$. The energy transferred into the deer stock, together with that injected by the deer's growth, R_1, exceeds that removed by death, B_1, $\mathrm{P}(R_1)+\mathrm{P}(B_3)=41.5\%>|\mathrm{P}(B_1)|=41.2\%$. Collapse occurs because the system has failed to diffuse all the energy injected through the reinforcing loops. In particular, too much vegetation energy has been transferred to the deer subsystem rather than being removed through diffusion. Although the falling vegetation numbers effectively increase the deer deaths, there is insufficient control on vegetation decline, leading to the energy imbalance. Thus, the behaviour of the vegetation subsystem has dominated the dynamics, providing a cumulative view of Meadows' (2017) comment: *this part of the system takes over and runs everything for a little while*. Although B_3 is responsible for deer behaviour from 20 to 23 years (Figure 17b)—an *instantaneous* view of the loop dominating for *a little while*—the energy viewpoint goes further,

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showing that it is the *cumulative* effect of the vegetation subsystem which is responsible for the ultimate instability in the deer subsystem. This result answers question (Q2) (proposed earlier) for this system.

For recovery to take place, the vegetation subsystem needs to remove energy more effectively. This effect is demonstrated by adding a further energy sink to the vegetation—model B (Figure 19)—and comparing it with the original—model A (Figure 1). The additional loop, B_4, captures the hypothesis that the ability of deer to consume is reduced when vegetation biomass falls to a low level. The behaviour mode now changes to one of cycles, avoiding collapse (Figure 20). The average loop powers over one oscillatory cycle are compared in Figure 18b. The system view (top diagram) still shows a net energy outflow. The stock view (bottom diagram) shows that a smaller proportion of energy is transferred from the vegetation subsystem to the deer by B_3 compared with model A (Figure 18a). Instead, energy is being diffused through the new loop B_4 as per capita vegetation consumption is reduced for lower deer numbers. This lower energy transfer just tips the balance in the deer subsystem, $\mathrm{P}(R_1) + \mathrm{P}(B_3) = 35.913\% < |\mathrm{P}(B_1)| = 35.914\%$, with more energy diffused than injected over one cycle. Thus, both vegetation and deer numbers have been controlled and collapse is averted. The behaviour continues in undamped cycles. If average power over a later cycle had been used, the sink B_1 would have exactly balanced R_1 and B_3 as the earlier transients have decayed.

The energy analysis over one cycle (Figure 18) has helped indicate which loop could be modified in order to avoid collapse, addressing question (Q1), posed earlier. This cumulative viewpoint has reinterpreted the model structure using energy sources, sinks and internal transfer, using them to describe model behaviour and inform structural modifications for alternative behavioural scenarios. Further, the approach has highlighted that an entire stock–flow subsystem is a suitable structure to explain behaviour in another subsystem, providing a quantitative way of describing its effect in terms of energy transfer.

Discussion and conclusion

This paper has introduced the Newtonian concept of energy into system dynamics modelling, employing it, in a narrative manner, to relate system behaviour to model structure. A form of kinetic energy was defined for a stock–flow subsystem (Eq. (5)) and shown to be in balance with the total work done on the stock obtained by summing the contribution from each stock-to-stock causal pathway. Work done was defined as an integral over an interval of time, and related to the accumulated effect of the force of one stock on another (Eqs (6) and (7)), where force is measured by stock and loop impact as defined by Hayward and Boswell (2014). The power of a feedback

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loop was defined as the rate of energy exchange between its stocks (Eq. (8)), given suitable unit conversions. This concept of energy extends the Newtonian Interpretative Framework of Hayward and Roach (2017), picturing first-order reinforcing loops as energy sources and first-order balancing loops as sinks, with the connections between stocks exchanging energy from one stock–flow subsystem to another. This energy perspective furnishes a cumulative understanding of loop influence on the system, augmenting the instantaneous interpretation of the Loop Impact method. Whereas the latter attributes stock acceleration at points in time to the balance of feedback loops, regarded as forces, the energy approach describes stock and system behaviour over a whole period, using the accumulated effects of the loops. In particular, the work done by a feedback loop measures its contribution to the final state of a stock compared with its initial state. Cumulative loop dominance was displayed alongside stock graphs, indicating the progress of the accumulation of energy over time (e.g., Figure 17) and as a bar chart of the average loop power over a period (e.g., Figure 18). This paper has illustrated five ways in which the energy perspective is useful in system dynamics modelling.

First, the cumulative measure of energy can highlight loops that have a greater influence on behaviour than would have been indicated by loop impact, or any other instant measure of a loop such as PPM or LTM. For example, the harvesting loop B_2 in the limits-to-growth model (Figure 9) has a significant effect on the final state of the system. Although this loop is never a dominant force, it is a significant drain of energy from the system (Figure 11), lowering the equilibrium value of the population. Likewise, in the model with overshoot and collapse (Figure 1), the cumulative effect of the deer birth loop, R_1, has a more significant influence on the long-term behaviour than that suggested by a loop impact analysis (Figure 17). Further, the energy balance over the whole cycle (Figure 18) highlights the weakness of the vegetation capacity loop, B_2, with too much energy being transferred to the deer subsystem, leading to collapse. This insight suggests collapse may be avoided by adding an additional loop on vegetation to increase energy diffusion, controlling its decline through reducing consumption by the deer.

Second, the cumulative measure of loop influence given by work done provides a useful way of describing oscillations, avoiding the use of complex numbers as in EEA and PPM (Kampmann, 2012; Mojtahedzadeh, 2008). Energy is conserved in a system composed of a single second-order balancing loop (Figure 12), with the rate of energy transfer, or power, along the two causal links of the loop being equal and opposite (Figure 13b). Thus, the net power of this second-order balancing loop is zero, determining the undamped oscillatory behaviour. When first-order loops are added, (Figure 14), the effect of the second-order balancing loop is to transfer energy from one stock–flow subsystem to the other over one cycle (Figure 16). Stability, or

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otherwise, is determined by the relative powers of the sources and sinks. The same approach can be applied to nonlinear systems, such as the model with overshoot and collapse (Figure 1), where energy transfer over a cycle helps determine the final state by simplifying the description of the oscillation. A similar cumulative approach was taken by Hayward and Roach (2019), where the behaviour of an oscillating predator–prey system was explained by considering the average value of loop impacts over one cycle. This method generalises into the energy definition introduced in this paper, giving the concept a natural physical analogy in the Newtonian Interpretative Framework, with the added advantage of an associated conservation law.

Third, the energy perspective enables loops through different stocks to be compared by considering their power once adjustments are made for different stock units. This system-wide approach eliminates energy transfers that balance each other and can help determine stability over a period by comparing the average powers of the sources and sinks (e.g., Figures 16b and 18). Thus, power as a measure of loop influence across a system augments loop impact, which is tied to individual stock–flow systems.

Fourth, the energy concepts are supported by formulae derived mathematically from the system dynamics equations, giving quantitative confidence to the narrative explanation of behaviour. The kinetic energy of a stock and the work done by each feedback loop have numerical values with well-defined units, giving precise descriptions of energy flows in a stock–flow system. For example, in the limits-to-growth model with harvesting, the change in energy diffusion due to the two balancing loops can be measured and compared for different harvesting rates (Figure 11b). Also, analytical results can be proved, such as the relationship between energy exchange in a second-order balancing loop (Figure 12; see Appendix B). Thus, the use of energy as an explanation of model behaviour can be supported by precise formulae and computations.

Fifth, and perhaps even more useful, is the informal explanation of behaviour using energy. For example, a system with a first-order reinforcing loop, an energy source, cannot stabilise unless sufficient energy is removed by sinks in the form of balancing loops. Thus, the linear system shown in Figure 5, with $a > b$, cannot stabilise as its sink is always less than its source, illustrated in Figure 6. By contrast, a nonlinear system, such as the limits-to-growth model with harvesting, stabilises because its sink, loop B_1, can drain more energy than that deposited through the source as capacity resistance increases with population size. Describing the drop in equilibrium due to harvesting by an additional drain of energy is another example of the informal explanatory power of the energy concept. A non-quantitative, informal examination of model loop structure, viewed as energy sources, sinks and transfers, can provide useful pointers to possible behaviour, as was indicated earlier for the overshoot and collapse model.

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Further, an informal energy consideration of a system dynamics model may also assist model construction. An examination of potential first-order balancing loops in the model would check that appropriate energy sinks have been included and, if required, are correctly formulated. For example, in the overshoot and collapse model version B, the inclusion of the loop B_4, which models the reduction in per capita vegetation consumption as food becomes scarce, helps avoid the collapse of the deer population through its additional diffusion of energy. To correctly model real-world behaviour, a modeller needs to ensure that this effect is justified. Moreover, the type of functional response is important as a linear dependence on deer numbers (e.g., Kunsch, 2006) will remove more energy than a nonlinear response (e.g., Breierova, 1997), thus making collapse less likely. These considerations will be particularly relevant if the archetype is applied to a different scenario where the population and resource may be of completely different types. In summary, an informal energy viewpoint can help challenge model assumptions and boundaries at the construction phase.

It is unlikely that any single measure of loop influence can completely explain system behaviour due to the complexities caused by nonlinearities and model size. Instead, multiple measures will be needed to interpret different behaviour patterns. Extending the Newtonian Interpretative Framework to include energy and work done provides such an additional measure, whose cumulative properties complement and augment the instantaneous measure of loop impact. In both the cumulative and instantaneous approaches, the loop structures explain behaviour. The force analogy of loop impact describes stock curvature—behaviour at points in time. In contrast, the energy analogy is used to explain the structural origins of oscillation, damping and equilibrium—behaviour over intervals of time. This cumulative viewpoint reveals new insights into the connection between system structure and behaviour, enhancing existing dominance methods. The authors will investigate the scope of applicability of this framework using additional models in a future publication.

Glossary

Force: If stock y influences a stock x then the force of y on x is the net rate of change of y (e.g., Figure 3). Force represents the ability of one stock to cause acceleration in another stock.

Impact: The impact of a stock y on a stock x is the ratio of the acceleration of x due to the force from y compared with the net rate of change of x (e.g., Figure 3; Eq. (4)). Impact measures the curvature in the behaviour of x imparted by y in units *per unit time*, and represents the ability of a force to change motion—that is, the rate of change of a stock.

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Mass: The mass of a stock x with respect to a stock y is inversely proportional to the sensitivity of x to changes in y (e.g., b^{-1} in Figure 3). In feedback loops, the inverse of the loop gain represents the mass of the loop (e.g., $(bc)^{-1}$ in Figures 12 and 14) (Hayward and Roach, 2017).

Kinetic energy: The kinetic energy of a stock represents the energy contained in its *change* of value, a half of its net flow squared (Eq. (5)).

Work done: If a stock y influences a stock x then the work done by y on x is the change in the kinetic energy of x induced by y. Work done is the cumulative effect of loop impact (e.g., Eq. (7)).

Work–energy theorem: The theorem states that the work done on a single stock–flow system by all its loops and external influences is turned into changes in the kinetic energy of the stock.

Energy balance: The energy balance equation is an invariant resulting from the work–energy theorem containing the stock's kinetic energy and the work done on the stock by loops and external influences (e.g., Eqs (9) and (11)).

Energy source: A first-order reinforcing loop is an energy source, injecting energy into a stock–flow subsystem, increasing the kinetic energy of the stock.

Energy sink: A first-order balancing loop is an energy sink, removing energy from a stock–flow subsystem, reducing the kinetic energy of the stock. See *Diffusion*.

Diffusion: The process of removing energy from a stock–flow subsystem through a first-order balancing loop—an energy sink. This loop acts like a frictional, or dissipative, force, where the stock acts on itself to resist its own changes.

Potential energy: When a system is conservative its total energy is conserved. This allows for the work done by one stock on another to be expressed as a potential energy source. The potential energy in one stock–flow subsystem can then be transferred into the kinetic energy of another stock–flow subsystem (e.g., Figures 4, 12 and 13).

Power: Power is the rate of energy injected into, or removed from, a stock–flow system by the action of another stock (Eq. (8)). In a second-order balancing loop, units can be chosen so that its net power into the system is zero. Average power over a period is equivalent to work done.

Energy transfer: Energy is transferred between stock–flow subsystems in a conserved way if the whole system is conservative. However, the concept of transfer can be used in a non-conserved system where there is a second-order balancing loop (e.g., Figure 16).

Biographies

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Paul Alun Roach is a professor of computing mathematics at the University of South Wales (USW), where he teaches courses in modelling, system dynamics, logic and artificial intelligence. He was awarded his BSc in computer science and PhD in computational geometry from Cardiff University. Before taking up a post at USW, he worked as an instructional materials designer for clients including the International Labour Organization. Paul's research interests focus on the modelling of contemporary collectible markets, and on applications of artificial intelligence and data science to business rule generation and healthcare service problems.

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Appendix A: Numerical computation of energy and work done

The kinetic energy of a stock (Eq. (5)) is computed from its net flow. For example, the calculation for stock x in the second-order model (Figure 14) is given in Figure 21 and Table 2. The stock *Initial KEx* captures the kinetic

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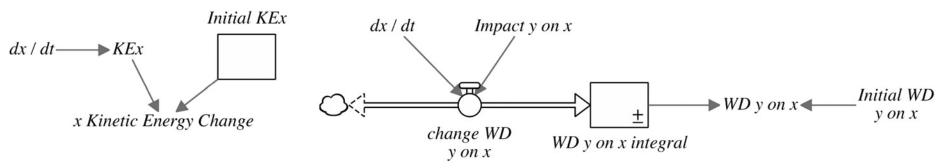

Fig. 21. System dynamics diagram for energy computation

Table 2. Model equations for change in kinetic energy

"dx_/_dt"	=	f1_copy+f2_copy
KEx	=	0.5*"dx_/_dt"^2
INIT Initial_KEx	=	KEx
x_Kinetic_Energy_Change	=	KEx-Initial_KEx

Table 3. Model equations for loop impact and work done

Impact_y_on_x	=	SAFEDIV(DERIVN(f2_copy, 1), "dx_/_dt")
change_WD_y_on_x	=	Impact_y_on_x*"dx_/_dt"^2
Initial_WD_y_on_x	=	0
WD_y_on_x_integral(t)	=	WD_y_on_x_integral(t - dt) + (change_work_done_y_on_x)*dt
INIT WD_y_on_x_integral	=	Initial_WD_y_on_x
WD_y_on_x	=	WD_y_on_x_integral-Initial_WD_y_on_x

energy at $t=0$, so that the *change* in kinetic energy over the run can be computed. The net flow dx/dt is computed from *copies* of the flows f_1 and f_2, f1_copy = a*x, f2_copy = b*y. (The flows are required in the calculations of the loop impacts where the use of copies avoids the computational errors introduced by differentiating actual flows.)

The work done by a feedback loop is calculated numerically using the integral over time of its loop impact multiplied by the net flow squared (Eq. (7)). (The loop impact is computed using the numerical method of Hayward and Boswell (2014).) The integral is computed by accumulating into a stock for work done. For example, the work done by y on x, part of loop B_2 (see Figure 14), is given in Figure 21 and Table 3. Again, the form of the net flow dx/dt must be based on copies of the flows to avoid numerical inaccuracies in the integration of the stock for work done. Also, the Euler integration method is required as Runge–Kutta methods will introduce inaccuracies due to the use of derivatives in the loop impact. The actual work done, *WD y on x*, is this integral stock minus the stock's initial value. This initial value is an arbitrary reference point and has no effect on the energy balance.

Alternatively, loop impacts may be computed analytically and coded into the simulation model (Hayward and Roach, 2017). The work done may be computed from Eq. (6). For the example above, the right-hand side of

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$\mathrm{WD}_{\underline{yx}} = -\int_{x_0}^{x_f} b\dot{y}\,dx = \int_{t_0}^{t_f} b(cx+dy)(ax+by)dt$ can be coded into the simulation. The advantage of analytical coding is numerical accuracy.

Appendix B: Balancing loops and energy

A first-order balancing loop removes energy from a single stock–flow system, even if the loop is nonlinear. Consider the system $\dot{x} = -f(x)$, where $f(x)$ is monotonically increasing, $f'(x) > 0$, to ensure the feedback is balancing. Differentiating the equation by t, and integrating by x gives the energy balance. Thus, the work done by the balancing loop is shown to be negative:

$$\mathrm{WD}_{\underline{xx}}(B) = -\int_{x_0}^{x_f} f'(x)\dot{x}\,dx = \int_{f(x_0)}^{f(x_f)} f\,df = \frac{1}{2}\left[f\left(x_f\right)^2 - f(x_0)^2\right] < 0$$

as $f\left(x_f\right) < f(x_0)$ due to $x_f < x_0$ in a declining system. If the loop is part of a more complex system where the stock may grow, $\dot{x} > 0$, then expressing the work done as a time integral $\mathrm{WD}_{\underline{xx}}(B) = -\int_{t_0}^{t_f} f'(x)\dot{x}^2 dt$ shows the expression is invariant under a sign change in $\dot{x}$, and thus remains negative.

A system with a single second-order balancing loop conserves energy regardless of the nonlinearity of the loop. Consider a system with a single second-order loop: $\dot{x} = f(y)$, $\dot{y} = g(x)$, where f, g are monotonic functions. The Jacobian of such a system has zero in its diagonals due to the absence of first-order loops. Thus, the stability of the system depends solely on the determinant of the Jacobian: $-f'(y)g'(x)$ (Strogatz, 2018). In general, this gives either neutral stability or instability, depending on the sign. Thus, the system cannot be stable, and therefore no energy diffusion is possible with a second-order loop.

For a balancing loop, the two functions' gradients will have opposing signs, for example, $f'(y) < 0$, $g'(x) > 0$. Thus, the determinant is positive, and the system is neutral with energy conserved. Kinetic energy is exchanged between the two stocks without gain or loss subject to a conserved invariant. The *powers* of the two causal links can be placed in the same units using the conversion factor $|f'(y)/g'(x)|$ on y's power. Thus, a single power for a second-order loop may always be found, whatever its nonlinearity, with a unit transformation that gives the balancing loop zero power. However, it will only be possible to harmonise the units of *work done* in a nonlinear loop in a limited number of cases due to its integral definition.

For an example of a nonlinear second-order loop with an invariant consider $\dot{x} = -by^3$, $\dot{y} = cx$. The system has a single balancing loop with invariant $\mathrm{KE}_y - \mathrm{KE}_{y_0} = -\frac{bc}{4}\left(2\frac{\mathrm{KE}_x}{b^2}\right)^{2/3} + \frac{bc}{4}\left(2\frac{\mathrm{KE}_{x_0}}{b^2}\right)^{2/3}$. There is an equivalent relationship between the work done on each stock: $\mathrm{WD}_{\underline{yx}} = 4b^{1/2}c^{-3/2}\mathrm{WD}_{\underline{xy}}^{3/2}$, demonstrating the conserved flow of energy between each stock, though in a far from

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straightforward manner. Energy transfer is easier to demonstrate using the link powers as $P_{\underline{yx}} = -\frac{3by^2}{c} P_{\underline{xy}}$.

It is important to note that energy conservation is ensured only if the two functions in the causal links remain monotonic. For example, the single loop in $\dot{x} = -by^2$, $\dot{y} = cx$ is only balancing while $y > 0$. Once y becomes negative, the loop becomes reinforcing. The system is then unstable, energy is no longer conserved, and the kinetic energies of both stocks increase indefinitely.

For systems with a single balancing loop of order three or above, there is always at least one positive eigenvalue, and such systems are unstable (Mojtahedzadeh and Richardson, 1995). Therefore, in all loops of order higher than two, energy is injected into the system in the long term. Thus, in a system dynamics model, energy is only diffused through first-order balancing loops. Their presence is essential if equilibrium or long-term control is required.

Appendix C: System equations

Limits to growth with harvesting

Let the variables in Figure 9 be represented by the symbols in Table 4. The model equations are given by

$$\frac{dx}{dt} = G - H \quad G = gcx \quad c = 1 - f$$
$$f = \frac{x}{x_m} \quad H = hx$$

Thus, the causally connected differential equation is given by

$$\frac{dx}{dt} = gx_{\underline{G}}\left(1 - \frac{x_{\underline{\underline{fcG}}}}{x_m}\right) - hx_{\underline{H}} \tag{12}$$

where the underlined subscripts identify the causal pathways (Hayward and Roach, 2017), all feedback loops in this case. The only stable equilibrium is $x_{\text{eq}} = x_m(1 - h/g)$.

Table 4. Variables of the limits to growth model

Variable	Symbol	Variable	Symbol
Population	x	Population capacity	x_m
growth	G	Growth rate	g
harvesting	H	Harvesting rate	h
Population fraction	f	Capacity effect on growth rate	c
Time	t		

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The impacts for the forces associated with the loops are derived by pathway differentiation on Eq. (12):

$$\mathrm{I}_{\underline{xGx}}(R_1) = \frac{\partial \dot{x}}{\partial x}\bigg\|_{\underline{G}} = g\left(1 - \frac{x}{x_m}\right) \tag{13}$$

$$\mathrm{I}_{\underline{xfcGx}}(B_1) = \frac{\partial \dot{x}}{\partial x}\bigg\|_{\underline{fcG}} = -\frac{gx}{x_m} \tag{14}$$

$$\mathrm{I}_{\underline{xHx}}(B_2) = \frac{\partial \dot{x}}{\partial x}\bigg\|_{\underline{H}} = -h \tag{15}$$

where, for example, the pathway derivative is the partial derivative along the specified pathway $\frac{\partial \dot{x}}{\partial x}\big\|_{\underline{G}} = \frac{\partial \dot{x}}{\partial x_{\underline{G}}}$ (Hayward and Roach, 2017, 2019). The impacts, used to determine loop dominance, can also be estimated using the numerical method of Hayward and Boswell (2014).

The work done by a first-order feedback loop is computed by integrating the loop's impact multiplied by the net flow squared (Eq. (7)). Although work done can be estimated numerically, following Appendix A, in the limits-to-growth model it can be computed analytically by changing the integration variable to x. For example:

$$\mathrm{WD}(B_1) = \int_{t_0}^{t_f} \mathrm{I}_{\underline{xfGx}}(B_1)\dot{x}^2\, dt = -\int_{x_0}^{x_f} \frac{gx}{x_m}\left(gx - g\frac{x^2}{x_m} - hx\right) dx$$

Assuming x_0 is small compared with the population capacity, the work done by each of the two balancing loops becomes

$$\mathrm{WD}(B_1) = -\frac{1}{12} g^2 x_m^2 \left(1 - \frac{h}{g}\right)^4$$

$$\mathrm{WD}(B_2) = -\frac{1}{6} g h x_m^2 \left(1 - \frac{h}{g}\right)^3$$

With some rearrangement, the shortfall from capacity of the equilibrium population, x_{eq}, is related to the ratio of the energy removed by the two balancing loops:

$$\frac{x_{\mathrm{eq}}}{x_m} = \frac{2}{\left(2 + \frac{\mathrm{WD}(B_2)}{\mathrm{WD}(B_1)}\right)}$$

showing that if more of the energy removed is taken by the harvesting loop B_2, the more the population falls short of capacity.

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Overshoot and collapse

Let the variables in Figures 1 and 19 be represented by the symbols in Table 5. The model equations for model A (Figure 1) are given by

$$
\begin{aligned}
\frac{dD}{dt} &= B - E & \frac{dV}{dt} &= R - C \\
B &= bD & E &= dD \\
R &= rV\rho_e & C &= c_a D \\
\alpha &= \frac{\frac{V}{V_h}}{1+\frac{V}{V_h}} & \sigma &= 1-\alpha = \frac{1}{1+\frac{V}{V_h}} \\
\rho_e &= 1-\nu^2 & \nu &= \frac{\rho}{\rho_m} \\
\rho &= \frac{V}{A} & d &= d_n + (d_h - d_n)\sigma
\end{aligned}
$$

The actual death rate, d, and the density effect on vegetation, ρ_e, are often modelled by graphical converters (Breierova, 1997). Here, both have been replaced by continuous functions with similar behaviour patterns (Turchin, 2003, Ch. 4), to allow for smoother values of loop impact.

Thus, the causally connected differential equations are given by

$$
\frac{dD}{dt} = bD_{\underline{R_1}} - \left(d_n + \frac{d_h - d_n}{1+\frac{V_{\underline{B_3}}}{V_h}} \right) D_{\underline{B_1}} \tag{16}
$$

Table 5. Variables of the overshoot and collapse model

Variable	Symbol	Variable	Symbol
Deer Population	D	Vegetation	V
births	B	deaths	E
regeneration	R	consumption	C
birth fraction	b	normal death rate	d_n
vegetation for half availability	V_h	high death rate	d_h
normal vegetation consumed per deer	c_n	death rate	d
vegetation consumed per deer	c_a	Area	A
vegetation density	ρ	maximum vegetation density	ρ_m
density effect on regeneration	ρ_e	normalised vegetation density	ν
vegetation availability	α	vegetation scarcity	σ
ideal regeneration fraction	r	Time	t

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$$\frac{dV}{dt} = rV_{\underline{R_2}}\left[1-\left(\frac{V_{\underline{B_2}}}{A\rho_m}\right)^2\right] - c_a D_{\underline{B_3}} \tag{17}$$

where, for brevity, pathways have been labelled by the feedback loop name associated with the sequence of variables in the pathway.

The parameter values used for model A in Figures 2, 17 and 18a are: $D_0 = 50$ deer, $V_0 = 65{,}000$ biomass, $b = 0.5\text{yr}^{-1}$, $d_n = 0.067\text{yr}^{-1}$, $d_h = 4\text{yr}^{-1}$, $c_a = 6$ biomass deer^{-1} yr^{-1}, $r = 0.5$ yr^{-1}, $A = 1000\,\text{km}^2$, $\rho_m = 100$ biomass km^{-2}, $V_h = 5000$ biomass.

For model B (Figures 18b and 20), the vegetation consumed per deer, c_a, is determined by vegetation availability α: $c_a = c_n\alpha$, where the normal vegetation consumed per deer $c_n = 6$ biomass deer^{-1}yr^{-1}. Thus, the last term of Eq. (17) is replaced by

$$-c_n\left(\frac{\frac{V_{\underline{\underline{B_4}}}}{V_h}}{1+\frac{V_{\underline{\underline{B_4}}}}{V_h}}\right)D_{\underline{B_3}}$$

which ensures per capita consumption tends to zero as $V \to 0$. It follows that vegetation cannot become zero in a finite time and the variables cycle rather than collapse.

As with the limits-to-growth model, loop impacts and work done can be computed analytically or numerically. The analytic computations are given in the model simulation file.

Supporting information

Additional supporting information may be found in the online version of this article at the publisher's website.

Appendix S1: Supporting Information

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