# MODELS THAT USE LOGARITHMIC AND POWER FUNCTIONS

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## Abstract

*Nowadays, a tremendous speed of computer technologies is being observed. This caused intensive numerous researches in the fields of data protection, information processing and image processing. This article is devoted to computer graphics. To create and manipulate realistic objects in computer graphics one needs to have powerful computers to perform a lot of complicated computations. Though, GPU in modern visual adapters is able to handle very complicated operations during short period of time, a lot of other devices (e.g. hand-held devices) not equipped with GPU require simplification of computations for image shading. Herein we suggest approximation of BRDF (Bidirectional Reflectance Distribution Function) which hard to get computed, by two quadratic polynomials.*

## The Available Achievements and Problem Definition

The intensity of pixels' color according to the Phong method is detected by using following formula:

$$I = I_a k_a + I_l (k_d \cos\psi + k_s \cos^n \gamma) \qquad (1)$$

where $I_a, I_l$ - intensities of sparse and directed light sources correspondingly, $k_a, k_d, k_s$ - sparse, diffusive and reflecting light coefficients, $\psi$ - angle between the direction of light and normal vector, $\gamma$- angle between the direction of reflected light and the observer, $n$ - surface brightness coefficient, $\cos^n \gamma$ - BRDF, represents surface optical properties.

In shading if the surfaces, the most resource-intensive procedure is computation of the $\cos^n \gamma$, that is used in Phong and Blin illumination models [1]. It is to be mentioned, that $n$ varies in diapason from 0 to 1. Among most successful achievements in BRDF approximation following are to be mentioned:

- The Schlick's Approximation [2];
- An approximation by the $\cos^k(\sqrt{n/k} \cdot \lambda)$ function [3];
- An approximation by Taylor series suggest by R.F.Lyon [4].

Though, they reached rather satisfactory results, these approaches remain to have 2 pitfalls:

- High value of the approximation error;
- Either performing operations with "angle" operand or division operation presence.

Hereby, a persuasive evidence of necessity of more efficient BRDF approximation comes. Finally, we can state a problem: To provide BRDF approximation having high accuracy and having neither division operation nor equation in $\gamma$.

## The Solution

Let's represent function $cos^n\gamma$ by the equivalent formula:

$$cos^n\gamma = 2^{n \cdot log_2(cos\gamma)}. \qquad (2)$$

Such a representation of the BRDF allows its easy hardware definition, because $log_2(cos\gamma)$ function could be represented by table and $2^x$ function – could be approximated. Let's investigate the variation interval of the product $n \cdot \log_2 \cos\gamma$ for the highlight epicenter. The distribution function has the cusp $\gamma = arctg\frac{1}{\sqrt{n-1}}$, that separates highlights epicenter from its fading zone. Let's prove, that

$$\lim(n \cdot \log_2 \cos\gamma) = \log_2\left(\frac{n-1}{n}\right)^{\frac{n}{2}}.$$

If change function argument $\gamma$ to its boundary value we receive:

$$\lim_{\gamma \to arctg\frac{1}{\sqrt{n-1}}}(n \cdot log_2 \cos\gamma) = n \cdot log_2 \cos(arctg\frac{1}{\sqrt{n-1}}).$$

According to the formula [73], we have: $\cos(arctg a) = \frac{1}{\sqrt{1+a^2}}$. So that

$$\lim_{\gamma \to arctg\frac{1}{\sqrt{n-1}}}(n \cdot \log_2 \cos\gamma) = n \cdot \log_2 \frac{1}{\sqrt{1+\frac{1}{n-1}}} = n \cdot \log_2 \sqrt{\frac{n-1}{n}} = \log_2 \left(\frac{n-1}{n}\right)^{\frac{n}{2}}.$$

With the received formula we find, that for $n = 2$

$$\lim_{\gamma \to arctg\frac{1}{\sqrt{n-1}}}(n \cdot log_2 \cos\gamma) = -1 \cdot$$

Since $\lim_{n\to\infty}\sqrt{\frac{n-1}{n}} = 1$, than for $n \to \infty$ $\lim_{\gamma \to arctg\frac{1}{\sqrt{n-1}}}(log_2 \cos\gamma) = 0 \cdot$

The maximal value of $n$ is 1000. For this value: $\lim_{\gamma \to arctg\frac{1}{\sqrt{n-1}}}(n \cdot \log_2 \cos\gamma) = -0{,}722$. Thus, for highlight epicenter values of $n \cdot \log_2 \cos\gamma$ for all $n$ are placed on the interval from -1 to -0, 722.

Let's examine the approximation of the $2^x$ function by the $f(x) = A \cdot log_2(1-x) + B$ on the interval - $[-1,0]$. This interval was chosen because it covers not only epicenter of the highlight, but a considerable part of the fading zone either (30%). Let's find unknown parameters A and B.

At $x = 0$ we have: $A \cdot log_2(1-x) + B = A \cdot log_2(1) + B = 1$. From the last equation we see, that $B = 1$.

At $x = -1$ we have: $A \cdot log_2(1-x) + B = A \cdot log_2(2) + B = \frac{1}{2}$. Inasmuch as $B = 1$, than $A = -\frac{1}{2}$. Thus, at the interval $[-1,0]$ we see:

$$2^x \approx 1 - \frac{1}{2} log_2(1-x) \cdot$$

Considering (1) we receive:

$$\cos^n\gamma = 2^{n \cdot log_2(\cos\gamma)} = 1 - \frac{1}{2} log_2(1 - n \cdot log_2 \cos\gamma) \cdot$$

Let's estimate the $\cos^n\gamma$ approximation error by the function

$$\Omega(\gamma, n) = 1 - \frac{1}{2} log_2(1 - n \cdot log_2 \cos\gamma) \cdot \quad (3)$$

An absolute approximation mistake is

$$\Delta = \cos^n\gamma - \Omega(\gamma, n) \; . \quad (4)$$

Let's find out maximum of the approximation error. If to derivate expression $\cos^n\gamma - 1 + \frac{1}{2} log_2(1 - n \cdot log_2 \cos\gamma)$ we have:

$$\frac{\partial\Delta}{\partial\gamma} = -n \cdot \cos^{n-1}\gamma \cdot \sin\gamma + \frac{n \cdot \sin\gamma}{\cos\gamma \cdot \ln^2 2 \cdot \left(1 - n \cdot \frac{\ln(\cos\gamma)}{\ln 2}\right)} \cdot$$

The roots of the expression, that determine extreme are:

$$\gamma_1 = arc\,cos(exp(\frac{W(\frac{-1}{4\cdot ln2})+ln2}{n}))\,,\ \gamma_2 = arc\,cos(exp(\frac{W(-1)+ln2}{n})),\ \gamma_3 = 0\,,$$

where $W()$ - is a Lambert function [61].

It is well known [1], that $W(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cdot n^{n-2}}{(n-1)!}x^n$ . Let's find

$$W(\frac{-1}{4\cdot ln2}) = -0,814,\ \ W(-1) = -1,23\,\cdot$$

Hence, critical values of the argument occurs in the following points

$$\gamma_1 = 0\,,\ \gamma_2 = arc\,cos(exp(\frac{-0.814+ln2}{n}))\,,\gamma_3 = arc\,cos(exp(\frac{-1.23+ln2}{n}))\,\cdot$$

The analysis showed, that critical points are extremums and they are situated inside the interval: $\left[0,\ arctg\frac{1}{\sqrt{n-1}}\right]$, at that $\gamma_1<\gamma_2<\gamma_3$ . If substitute $\gamma_1,\gamma_2,\gamma_3$ into (2.11) we get:

$$\left|\Delta(\gamma_0)\right| = 0,\quad \left|\Delta(\gamma_1)\right| = \ 2{,}098\times 10^{-3},\quad \left|\Delta(\gamma_2)\right| = 1{,}703\times 10^{-3}.$$

Comparing received data we can state that $max\left|\Delta(\gamma)\right| = \ 2.098\times 10^{-3}$ and takes place at $\gamma = arc\,cos(exp(\frac{-0.814+ln2}{n}))\,\cdot$

We also analyzed maximal relative error of BRDF approximation by $\Omega(\gamma,n)$ function for highlight epicenter. Error reaches maximum at the point $ar\,cos(exp\frac{1+ln2\cdot W(\frac{-1}{4\cdot ln2})}{n\cdot W(\frac{-1}{4\cdot ln2})})$ and doesn't exceeds 0,29%, that confirms high accuracy of highlight epicenter representation.

The approximation of $2^x$ function by the different polynomials was also analyzed. [2]. The usage of one polynomial for whole variation interval wasn't successful, 'cause at high accuracy of highlight epicenter representation we have artifacts at blooming zone. This happened because approximating curve didn't reach 0-level and changed behavior from decreasing to increasing and this could cause unnatural object illumination. So that we tried piecewise-functional approximation of BRDF.

Solution is to divide BRDF variation interval into 2 subintervals and approximate function at each one of them by the quadratic polynomial. Approximating functions have to have equal 2-order derivatives at the adherent point. In contrast to spline interpolation, where intervals are defined, this approach supposes adherent point of mentioned functions is to be found. It is necessary generating functions to touch at the connection point, or to have non-significant difference between ordinates for the full BRDF definition. One of the function is supposed to approximate highlight epicenter with high accuracy, the other one – it's blooming zone. The Tchebyshev approximating polynomial of $K-1$ order for the function $f(x)$ on the interval $[a,b]$ can be found by this formula:

$$f(x)\approx\left[\sum_{k=0}^{K-1}c_kT_k\left(\frac{x-0{,}5(b+a)}{0{,}5(b-a)}\right)\right]-\frac{c_0}{2}\,,$$

where $c_j=\frac{2}{K}\sum_{k=1}^{K}f\left[cos\left(\frac{\pi(k-0{,}5)}{K}\right)\cdot 0{,}5(b-a)+0{,}5(b+a)\right]cos\left(\frac{\pi\cdot j\cdot(k-0{,}5)}{K}\right)$; $T_0(x)=1$; $T_1(x)=x$;

$T_n(x)=cos(n\cdot arccos(x))$.

The following formulas for $2^x$ approximation have been received:

$f_1(\gamma)=1+0{,}617\cdot n\cdot log_2(cos\gamma)+0{,}124\cdot(n\cdot log_2(cos\gamma))^2$ - at the $[-2,0]$ interval ;

$f_2(\gamma)=0{,}64+0{,}227\cdot n\cdot log_2(cos\gamma)+0{,}02\cdot(n\cdot log_2(cos\gamma))^2$ - at the $[-8,-2]$ interval.

The $f_1(x)$ function was used to reproduce highlight and a part of fading zone, and $f_2(\gamma)$ function – for the blooming zone.

The approximation interval choosing has following ground. If level $\frac{1}{2^q}$ is given, then from the equation $cos^n(\gamma) = \frac{1}{2^q}$ we find border value of the angle $arccos(exp(-ln(2)\cdot\frac{q}{n}).$ As a rule $q = 8$. The maximal value of the variable is being searched is:

$$n \cdot log_2 \, cos \, arccos(exp(-ln(2) \cdot \frac{8}{n}) = 8 \cdot$$

Let's find these functions adherent point and solve an equation

$$\begin{gathered} 1+0,617 \cdot n \cdot log_2 \left(cos\,\gamma\right) + 0,124 \cdot n \cdot log_2 \left(cos\,\gamma\right)^2 = \\ 0,645 + 0,232 \cdot n \cdot log_2 \left(cos\,\gamma\right) + 0,02 \cdot n \cdot log_2 \left(cos\,\gamma\right)^2 . \end{gathered} \tag{5}$$

in $\gamma$ . Solution is $\gamma = ar\,cos(exp\frac{-1.205}{n}).$

The derivative of (5) is:

$$-0,385 \cdot n \cdot \frac{sin(x)}{cos(x) \cdot ln 2} - 0,208 \cdot n^2 \cdot \frac{ln(cos\,x)}{(ln 2)^2} \cdot \frac{sin(x)}{cos(x)} = 0 \cdot$$

The equation has solution at $\gamma = ar\,cos(exp(-1.282 / n)).$

Comparing found values we can draw a conclusion, that inflection point and the point, where two function derivatives coincide, are very close to each other, that meets prescribe requirements. The figure 1 depicts $cos^n \gamma,\ f_1(\gamma),\ f_2(\gamma)$ graphics at $n = 10$ .

Analysis has shown absolute approximation error doesn't exceeds 0,0154 (fig. 2) , and relative error - 2,3 %. For blooming zone, where BRDF value tends to zero absolute error doesn't exceeds 0,05. This confirms rather high accuracy of BRDF approximation.

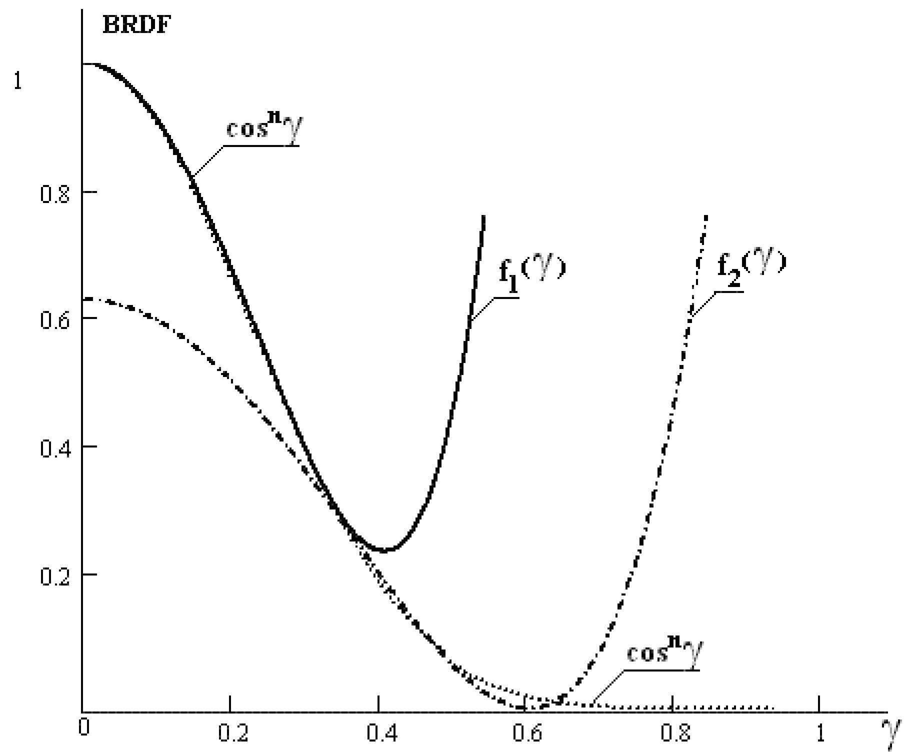


Fig.1. The Graphics of $cos^n \gamma,\ f_1(\gamma),\ f_2(\gamma)$

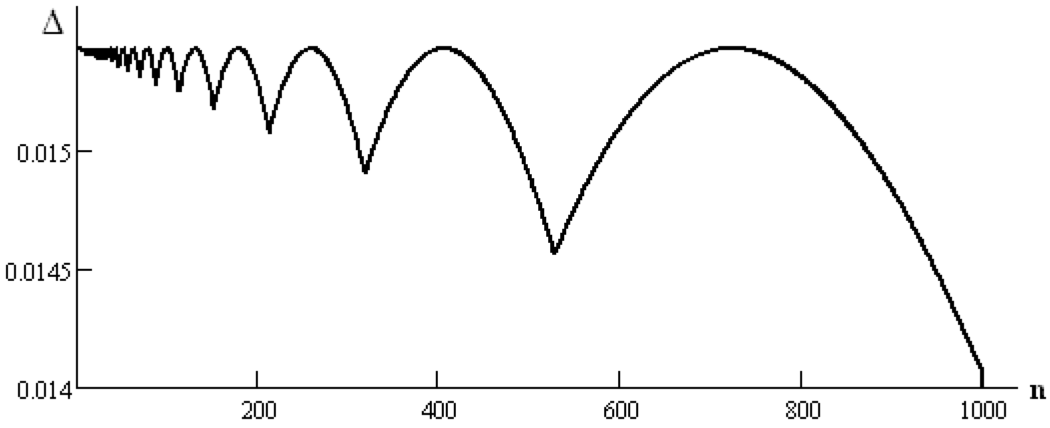


Fig.2. The Graphic of the absolute error for BRDF approximation by $f_1(x)$ function

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