

# A Predicate Transformer for Choreographies (Technical Report)

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# A Predicate Transformer for Choreographies (Technical Report)

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**Abstract.** This technical report contains detailed definitions, auxiliary lemmas, main theorems, and proofs related to the paper *A Predicate Transformer for Choreographies*, published in the proceedings of ESOP 2022. The aim of this document is to provide a comprehensive reference guide for the theory presented in the paper. In contrast to the paper, we do not split the presentation into a base calculus, an extension with blocking if/while-statements, and an extension with non-blocking if/while-statements. Instead, we present the whole calculus of global programs, including all extensions, at once.

We now present a detailed mapping of definitions, lemmas, and theorems in the paper into this document, section by section. Along the way, we also mention some cosmetic differences.

## Setting the Stage: Data and Conditions (§3)

### – Data (§3.1)

- The grammar of roles, variables, values, and expressions is given in Defn. 1.
- An excerpt of the definition of the evaluation function is given in Defn. 4.

### – Conditions (§3.2)

- The grammar of conditions is given in Defn. 5. The extra syntactic constraints are given in Rem. 1.
- The definition of the interpretation function is given in Defn. 13.

## Global Programs: Base Calculus (§4)

### – Syntax and Semantics (§4.1)

- The grammar of global actions is given in Defn. 14. The extra syntactic constraints are given in Rem. 2. For simplicity, we also allow  $\tau$  to be a global action.

We note that collections of private decisions together have an extra subscript. This makes it possible to capture the operational semantics of both global programs and local programs uniformly in one set of rules. See below for details.

- The grammar of global programs is given in Defn. 21. In this definition,  $\xi$  ranges over conjunctions; see also Defn. 5.

We note that blocking if/while-statements have an additional “ $R$ ”, and that non-blocking if/while statements have an additional “ $|_0$ ”. As a result, blocking and non-blocking if/while-statements in global programs have the same grammatical form as in local programs. This makes it possible to capture the operational semantics of both global programs and local programs uniformly in one set of rules. See below for details.

- The subjects function is given in Defn. 24. In this definition,  $A$  ranges over both global programs and local programs; see also Defn. 21.
- The abstract termination rules for global programs are given in Defn. 28. In this definition,  $A$  ranges over both global programs and local programs; see also Defn. 21.
- The abstract reduction rules for global programs are given in Defn. 31. In this definition,  $A$  ranges over both global programs and local programs; see also Defn. 21. Furthermore,  $\xi^+$  and  $\xi^-$  denote the positive and negative versions of  $\xi$ ; see also Defn. 12. Furthermore,  $\xi \upharpoonright r$  denotes the projection of  $\xi$  onto  $r$ ; see also Defn. 11. We note that rule  $[\rightarrow 1\text{-NIF1}]$  should be used only with local programs; this is ensured because, grammatically,  $n=0$  for global programs.
- The concrete termination rule for families of local programs is given in Defn. 30. In this definition,  $\mathcal{A}$  ranges over both global programs and families of local programs.

- The concrete reduction rules for global programs are given in Defn. 33. In this definition,  $\mathcal{A}$  ranges over both global programs and families of local programs, while  $\dot{\gamma}$  ranges over concrete global actions; see also Defn. 14. Furthermore,  $\text{con}_{\mathcal{S}}(\gamma)$  denotes the concretisation of  $\gamma$ ; see also Defn. 19, and  $\text{effect}(\dot{\gamma}, \mathcal{S})$  denotes the effect of  $\dot{\gamma}$ ; see also Defn. 20.
- The well-formedness relation is given in Defn. 27.
- **Predicate Transformer (§4.2)**
  - The predicate transformer function is given in Defn. 38.
- **Deadlock Freedom and Functional Correctness (§4.3)**
  - Lemma 1 in the paper corresponds to Lem. 88; it also extends the result to the concrete layer of the operational semantics.
  - Lemma 2 in the paper corresponds to Lem. 91; it also extends the result to the concrete layer of the operational semantics.
  - Lemma 3 in the paper corresponds to Lem. 89; it also extends the result to the concrete layer of the operational semantics.
  - Lemma 4 in the paper corresponds to Lem. 93; it also extends the result to the concrete layer of the operational semantics.
  - Theorem 1 in the paper corresponds to Thm. 1 and Thm. 2. In their formulation,  $\xrightarrow{F}^*$  denotes reachability using any concrete global action; see also Defn. 36.

**Global Programs: If/While-Statements (§5)** – Similar to §4.

**Global Programs: Non-Blocking If/While-Statements (§6)** – Similar to §4.

#### Local Programs and Projection (§7)

- **Syntax and Semantics (§7.1)**
  - The grammar of local actions is given in Defn. 14. The extra syntactic constraints are given in Rem. 2.
  - The grammar of local programs is given in Defn. 21. In this definition,  $\xi$  ranges over conjunctions; see also Defn. 5.  
We note that while-statements have an additional “`{true}`”. As a result, while-statements in local programs have the same grammatical form as in global programs. This makes it possible to capture the operational semantics of both global programs and local programs uniformly in one set of rules.
  - The abstract termination rules for local programs are given in Defn. 28. In this definition,  $A$  ranges over both global programs and local programs; see also Defn. 21.
  - The abstract reduction rules for local programs are given in Defn. 31. In this definition,  $A$  ranges over both global programs and local programs; see also Defn. 21. Furthermore,  $\xi^+$  and  $\xi^-$  denote the positive and negative versions of  $\xi$ ; see also Defn. 12. Furthermore,  $\xi \upharpoonright r$  denotes the projection of  $\xi$  onto  $r$ ; see also Defn. 11.
  - The abstract termination rule for families of local programs is given in Defn. 29.
  - The abstract reduction rules for families of local programs are given in Defn. 32.
  - The concrete termination rule for families of local programs is given in Defn. 30. In this definition,  $\mathcal{A}$  ranges over both global programs and families of local programs.
  - The concrete reduction rules for families of local programs are given in Defn. 33. In this definition,  $\mathcal{A}$  ranges over both global programs and families of local programs, while  $\dot{\gamma}$  ranges over concrete global actions; see also Defn. 14. Furthermore,  $\text{con}_{\mathcal{S}}(\gamma)$  denotes the concretisation of  $\gamma$ ; see also Defn. 19, and  $\text{effect}(\dot{\gamma}, \mathcal{S})$  denotes the effect of  $\dot{\gamma}$ ; see also Defn. 20.
  - The projection function is given in Defn. 26.
- **Operational Equivalence (§7.2)**
  - Lemma 7 in the paper corresponds to Lem. 59.
  - Lemma 8 in the paper corresponds to Lem. 64.
  - Theorem 6 in the paper corresponds to Thm. 3.
- **Additional Proof Steps of Thm. 6 (§C)**
  - Relation  $\Rightarrow\Leftarrow$  is given in Defn. 40.
  - The correspondence relation is given in Defn. 41.
  - Weak bisimilarity is given in Defn. 39.
  - Lemma 9 in the paper corresponds to Lem. 97
  - Lemma 10 in the paper corresponds to Lem. 96

## Part I

# Definitions, Propositions, Lemmas, Theorems



## 1 Data

**Definition 1 (Grammar).** *Excerpt:*

$$p, q, r ::= A \mid B \mid C \mid \dots \quad (\mathbb{R})$$

$$x, y, z ::= x \mid y \mid z \mid \dots \quad (\mathbb{X})$$

$$t, u, v ::= \text{error} \mid \text{true} \mid \text{false} \mid 0 \mid 1 \mid 2 \mid \dots \quad (\mathbb{V})$$

$$c, d, e ::= r.x \mid v \mid e_1 == e_2 \mid e_1 < e_2 \mid e_1 \&\& e_2 \mid !e \mid e_1 + e_2 \mid \dots \quad (\mathbb{E})$$

**Definition 2 (Substitution).** *Excerpt:*

$$r.x[d/q.y] = \begin{cases} d & \text{if } r.x = q.y \\ q.y & \text{else} \end{cases}$$

$$v[d/q.y] = v$$

$$e_1 == e_2[d/q.y] = (e_1[d/q.y]) == (e_2[d/q.y])$$

$$\mathcal{S}[u/q.y] = \{r \mapsto \mathcal{S}(r) \mid q \neq r\} \cup \{q \mapsto \{x \mapsto \mathcal{S}(q)(x) \mid x \neq y\} \cup \{y \mapsto u\}\}$$

**Definition 3 (Role-qualified variables).** *Excerpt:*

$$\text{rx}(r.x) = \{r.x\}$$

$$\text{rx}(v) = \emptyset$$

$$\text{rx}(e_1 == e_2) = \text{rx}(e_1) \cup \text{rx}(e_2)$$

**Proposition 1.**  $q.y \notin \text{rx}(e)$  implies  $e[d/q.y] = e$

**Proposition 2.**

$$\left[ \begin{array}{l} q_1.y_1 \neq q_2.y_2 \\ \text{and } q_1.y_1 \notin \text{rx}(d_2) \\ \text{and } q_2.y_2 \notin \text{rx}(d_1) \end{array} \right] \text{ implies } e[d_1/q_1.y_1][d_2/q_2.y_2] = e[d_2/q_2.y_2][d_1/q_1.y_1]$$

**Definition 4 (Evaluation).** *Excerpt:*

$$\text{eval}_{\mathcal{S}}(r.x) = \begin{cases} \mathcal{S}(r)(x) & \text{if } \mathcal{S}(r)(x) \in \mathbb{V} \\ \text{nil} & \text{else} \end{cases}$$

$$\text{eval}_{\mathcal{S}}(v) = v$$

$$\text{eval}_{\mathcal{S}}(e_1 == e_2) = \begin{cases} \text{true} & \text{if } \text{eval}_{\mathcal{S}}(e_1) = \text{eval}_{\mathcal{S}}(e_2) \\ \text{false} & \text{else} \end{cases}$$

**Proposition 3.**  $\text{eval}_{\mathcal{S}}(e[d/q.y]) = \text{eval}_{\mathcal{S}[\text{eval}_{\mathcal{S}}(d)/q.y]}(e)$

## 2 Conditions

**Definition 5 (Grammar).**

$$\begin{aligned} \phi, \chi, \psi &::= e \mid \neg\psi \mid \psi_1 \wedge \psi_2 \mid \forall\psi & (\Psi) \\ \xi &::= \mathbf{true} \mid \bigwedge\{e_r\}_{r \in R} \mid \bigwedge\{\neg e_r\}_{r \in R} & (\Xi) \end{aligned}$$

*Remark 1.*

- $\xi = \bigwedge\{e_r\}_{r \in R}$  **implies**  $[\{\hat{r} \mid \hat{r}.\hat{x} \in \mathbf{rx}(e_r)\} = \{r\} \text{ for every } r \in R \neq \emptyset]$
- $\xi = \bigwedge\{\neg e_r\}_{r \in R}$  **implies**  $[\{\hat{r} \mid \hat{r}.\hat{x} \in \mathbf{rx}(e_r)\} = \{r\} \text{ for every } r \in R \neq \emptyset]$

**Definition 6 (Macros).**

$$\psi_1 \Rightarrow \psi_2 \stackrel{\text{def}}{=} \neg(\psi_1 \wedge \neg\psi_2)$$

**Definition 7 (Substitution).**

$$\begin{aligned} e[d/q.y] &= \text{Defn. 2} \\ (\neg\psi)[d/q.y] &= \neg(\psi[d/q.y]) \\ (\psi_1 \wedge \psi_2)[d/q.y] &= \psi_1[d/q.y] \wedge \psi_2[d/q.y] \\ (\forall\psi)[d/q.y] &= \forall\psi \end{aligned}$$

**Definition 8 (Role-qualified variables).**

$$\begin{aligned} \mathbf{rx}(e) &= \text{Defn. 3} \\ \mathbf{rx}(\neg\psi) &= \mathbf{rx}(\psi) \\ \mathbf{rx}(\psi_1 \wedge \psi_2) &= \mathbf{rx}(\psi_1) \cup \mathbf{rx}(\psi_2) \\ \mathbf{rx}(\forall\psi) &= \emptyset \end{aligned}$$

**Lemma 1.**  $q.y \notin \mathbf{rx}(\psi)$  **implies**  $\psi[d/q.y] = \psi$

*Proof.* See page 32. □

**Lemma 2.**

$$\left[ \begin{array}{l} q_1.y_1 \neq q_2.y_2 \\ \mathbf{and} \ q_1.y_1 \notin \mathbf{rx}(d_2) \\ \mathbf{and} \ q_2.y_2 \notin \mathbf{rx}(d_1) \end{array} \right] \mathbf{implies} \ \psi[d_1/q_1.y_1][d_2/q_2.y_2] = \psi[d_2/q_2.y_2][d_1/q_1.y_1]$$

*Proof.* See page 32. □

**Definition 9 (Reads/writes).**

$$\mathbf{read}(\psi) = \mathbf{rx}(\psi) \qquad \mathbf{write}(\psi) = \emptyset$$

**Lemma 3.**

1.  $\mathbf{read}(\mathbf{false}) = \emptyset = \mathbf{read}(\mathbf{true})$
2.  $\mathbf{read}(\neg\psi) = \mathbf{read}(\psi)$
3.  $\mathbf{read}(\psi_1 \wedge \psi_2) = \mathbf{read}(\psi_1) \cup \mathbf{read}(\psi_2)$
4.  $\mathbf{read}(\psi_1 \Rightarrow \psi_2) = \mathbf{read}(\psi_1) \cup \mathbf{read}(\psi_2)$
5.  $\mathbf{write}(\neg\psi) = \mathbf{write}(\psi)$
6.  $\mathbf{write}(\psi_1 \wedge \psi_2) = \mathbf{write}(\psi_1) \cup \mathbf{write}(\psi_2)$
7.  $\mathbf{write}(\psi_1 \Rightarrow \psi_2) = \mathbf{write}(\psi_1) \cup \mathbf{write}(\psi_2)$

*Proof.* See page 32. □

**Definition 10 (Subjects).**

$$\text{subj}(\psi) = \{r \mid r.x \in \text{rx}(\psi)\}$$

**Lemma 4.**

1.  $\text{subj}(\text{false}) = \emptyset = \text{subj}(\text{true})$
2.  $\text{subj}(\neg\psi) = \text{subj}(\psi)$
3.  $\text{subj}(\psi_1 \wedge \psi_2) = \text{subj}(\psi_1) \cup \text{subj}(\psi_2)$
4.  $\text{subj}(\psi_1 \Rightarrow \psi_2) = \text{subj}(\psi_1) \cup \text{subj}(\psi_2)$

*Proof.* See page 33. □

**Lemma 5.**

1.  $\text{read}(\psi) \subseteq \{r.x \mid r \in \text{subj}(\psi)\}$
2.  $\text{write}(\psi) \subseteq \{r.x \mid r \in \text{subj}(\psi)\}$

*Proof.* See page 33. □

**Definition 11 (Projection).**

$$\begin{aligned} \text{true} \upharpoonright r &= \text{true} \\ \bigwedge \{e_{\hat{r}}\}_{\hat{r} \in R} \upharpoonright r &= \begin{cases} e_r & \text{if } : r \in R \\ \text{true} & \text{else} \end{cases} \\ \bigwedge \{\neg e_{\hat{r}}\}_{\hat{r} \in R} \upharpoonright r &= \begin{cases} \neg e_r & \text{if } : r \in R \\ \text{true} & \text{else} \end{cases} \end{aligned}$$

**Lemma 6.**  $(\xi \upharpoonright r) \upharpoonright r = \xi \upharpoonright r$

*Proof.* See page 33. □

**Lemma 7.**  $\text{rx}(\xi \upharpoonright r) \subseteq \text{rx}(\xi)$

*Proof.* See page 33. □

**Lemma 8.**

1.  $\text{read}(\xi \upharpoonright r) \subseteq \text{read}(\xi)$
2.  $\text{write}(\xi \upharpoonright r) \subseteq \text{write}(\xi)$

*Proof.* See page 34. □

**Lemma 9.**

1.  $r \in \text{subj}(\xi)$  **implies**  $\text{subj}(\xi \upharpoonright r) = \{r\}$
2.  $r \notin \text{subj}(\xi)$  **implies**  $\text{subj}(\xi \upharpoonright r) = \emptyset$

*Proof.* See page 34. □

**Lemma 10.**

1.  $\text{subj}(\xi \upharpoonright r) \subseteq \text{subj}(\xi)$
2.  $\text{subj}(\xi \upharpoonright r) \subseteq \{r\}$

*Proof.* See page 34. □



**Definition 12 (Derivatives).**

$$\begin{aligned} \text{true}^- &= \text{true} & \text{true}^{\equiv} &= \text{true} \\ \xi^+ = \xi & \quad \bigwedge \{e_r\}_{r \in R}^- = \bigwedge \{\neg e_r\}_{r \in R} & \quad \bigwedge \{e_r\}_{r \in R}^{\equiv} = \bigwedge \{e_{r_1} \Rightarrow e_{r_2}\}_{r_1, r_2 \in R} \\ & \quad \bigwedge \{\neg e_r\}_{r \in R}^- = \bigwedge \{e_r\}_{r \in R} & \quad \bigwedge \{\neg e_r\}_{r \in R}^{\equiv} = \bigwedge \{\neg e_{r_1} \Rightarrow \neg e_{r_2}\}_{r_1, r_2 \in R} \end{aligned}$$

**Lemma 11.**

1.  $(\xi \upharpoonright r)^+ = \xi^+ \upharpoonright r$
2.  $(\xi \upharpoonright r)^- = \xi^- \upharpoonright r$

*Proof.* See page 35. □

**Lemma 12.**

1.  $\text{read}(\xi^+) = \text{read}(\xi)$
2.  $\text{read}(\xi^-) = \text{read}(\xi)$
3.  $\text{read}(\xi^{\equiv}) = \text{read}(\xi)$
4.  $\text{write}(\xi^+) = \text{write}(\xi)$
5.  $\text{write}(\xi^-) = \text{write}(\xi)$
6.  $\text{write}(\xi^{\equiv}) = \text{write}(\xi)$

*Proof.* See page 35. □

**Lemma 13.**

1.  $\text{subj}(\xi^+) = \text{subj}(\xi)$
2.  $\text{subj}(\xi^-) = \text{subj}(\xi)$
3.  $\text{subj}(\xi^{\equiv}) = \text{subj}(\xi)$

*Proof.* See page 36. □

**Definition 13 (Satisfaction).**

$$\begin{aligned} \llbracket e \rrbracket &= \{\mathcal{S} \mid \text{eval}_{\mathcal{S}}(e) = \text{true}\} \\ \llbracket \neg \psi \rrbracket &= \llbracket \text{true} \rrbracket \setminus \llbracket \psi \rrbracket \\ \llbracket \psi_1 \wedge \psi_2 \rrbracket &= \llbracket \psi_1 \rrbracket \cap \llbracket \psi_2 \rrbracket \\ \llbracket \forall \psi \rrbracket &= \begin{cases} \llbracket \text{true} \rrbracket & \text{if } \llbracket \psi \rrbracket = \llbracket \text{true} \rrbracket \\ \llbracket \text{false} \rrbracket & \text{else} \end{cases} \end{aligned}$$

**Lemma 14.**

1.  $\mathcal{S} \in \llbracket \psi[d/q.y] \rrbracket$  implies  $\mathcal{S}[\text{eval}_{\mathcal{S}}(d)/q.y] \in \llbracket \psi \rrbracket$
2.  $\mathcal{S} \notin \llbracket \psi[d/q.y] \rrbracket$  implies  $\mathcal{S}[\text{eval}_{\mathcal{S}}(d)/q.y] \notin \llbracket \psi \rrbracket$

*Proof.* See page 36. □

**Lemma 15.**  $\emptyset = \llbracket \text{false} \rrbracket \subseteq \llbracket \psi \rrbracket \subseteq \llbracket \text{true} \rrbracket$

*Proof.* See page 37. □

**Lemma 16.**

1.  $\llbracket \forall (\psi_1 \wedge \psi_2) \rrbracket = \llbracket \forall \psi_1 \rrbracket \cap \llbracket \forall \psi_2 \rrbracket$
2.  $\llbracket \forall (\psi_1 \wedge \psi_2) \rrbracket = \llbracket \forall \psi_1 \wedge \forall \psi_2 \rrbracket$

*Proof.* See page 37. □

**Lemma 17.**  $\llbracket \psi_1 \Rightarrow \psi_2 \rrbracket = \llbracket \neg \psi_2 \Rightarrow \neg \psi_1 \rrbracket$

*Proof.* See page 38. □

**Lemma 18.**

1.  $\llbracket \psi \Rightarrow (\psi_1 \wedge \psi_2) \rrbracket = \llbracket \psi \Rightarrow \psi_1 \rrbracket \cap \llbracket \psi \Rightarrow \psi_2 \rrbracket$
2.  $\llbracket \psi \Rightarrow (\psi_1 \wedge \psi_2) \rrbracket = \llbracket (\psi \Rightarrow \psi_1) \wedge (\psi \Rightarrow \psi_2) \rrbracket$

*Proof.* See page 38. □

**Lemma 19.**

1.  $\llbracket \psi_1 \rrbracket \cap \llbracket \psi_1 \Rightarrow \psi_2 \rrbracket = \llbracket \psi_1 \rrbracket \cap \llbracket \psi_2 \rrbracket$
2.  $\llbracket \psi_1 \wedge (\psi_1 \Rightarrow \psi_2) \rrbracket = \llbracket \psi_1 \wedge \psi_2 \rrbracket$

*Proof.* See page 38. □

**Lemma 20.**  $\llbracket \psi_1 \rrbracket \subseteq \llbracket \psi_2 \rrbracket$  **implies**  $\llbracket \psi_1[d/q.y] \rrbracket \subseteq \llbracket \psi_2[d/q.y] \rrbracket$

*Proof.* See page 38. □

**Lemma 21.**

1.  $\llbracket \psi_1 \rrbracket \subseteq \llbracket \psi_2 \rrbracket$  **implies**  $\llbracket \forall \psi_1 \rrbracket \subseteq \llbracket \forall \psi_2 \rrbracket$
2.  $\llbracket \psi_1 \rrbracket = \llbracket \psi_2 \rrbracket$  **implies**  $\llbracket \forall \psi_1 \rrbracket = \llbracket \forall \psi_2 \rrbracket$

*Proof.* See page 38. □

**Lemma 22.**

1.  $\llbracket \psi_1 \rrbracket \subseteq \llbracket \psi_2 \rrbracket$  **implies**  $\llbracket \psi \Rightarrow \psi_1 \rrbracket \subseteq \llbracket \psi \Rightarrow \psi_2 \rrbracket$
2.  $\llbracket \psi_1 \rrbracket = \llbracket \psi_2 \rrbracket$  **implies**  $\llbracket \psi \Rightarrow \psi_1 \rrbracket = \llbracket \psi \Rightarrow \psi_2 \rrbracket$

*Proof.* See page 39. □

**Lemma 23.**  $\llbracket \psi_1 \Rightarrow \psi_2 \rrbracket = \llbracket \text{true} \rrbracket$  **implies**  $\llbracket \psi_1 \rrbracket \subseteq \llbracket \psi_2 \rrbracket$

*Proof.* See page 39. □

**Lemma 24.**

1.  $\llbracket \forall \psi \rrbracket \subseteq \llbracket \psi \rrbracket$
2.  $\llbracket \psi \rrbracket \cap \Sigma_1 \subseteq \Sigma_2$  **implies**  $\llbracket \forall \psi \rrbracket \cap \Sigma_1 \subseteq \Sigma_2$
3.  $\Sigma_1 \subseteq \llbracket \forall \psi \rrbracket \cap \Sigma_2$  **implies**  $\Sigma_1 \subseteq \llbracket \psi \rrbracket \cap \Sigma_2$

*Proof.* See page 39. □

**Lemma 25.**

1.  $\llbracket \psi_2 \rrbracket \subseteq \llbracket \psi_1 \Rightarrow \psi_2 \rrbracket$
2.  $\llbracket \psi_1 \Rightarrow \psi_2 \rrbracket \cap \Sigma_1 \subseteq \Sigma_2$  **implies**  $\llbracket \psi_2 \rrbracket \cap \Sigma_1 \subseteq \Sigma_2$
3.  $\Sigma_1 \subseteq \llbracket \psi_2 \rrbracket \cap \Sigma_2$  **implies**  $\Sigma_1 \subseteq \llbracket \psi_1 \Rightarrow \psi_2 \rrbracket \cap \Sigma_2$

*Proof.* See page 39. □

**Lemma 26.**

1.  $\llbracket \xi \rrbracket = \llbracket \bigwedge \{ \xi \upharpoonright r \}_{r \in \text{subj}(\xi)} \rrbracket$
2.  $\text{subj}(\xi) \subseteq R$  **implies**  $\llbracket \xi \rrbracket = \llbracket \bigwedge \{ \xi \upharpoonright r \}_{r \in R} \rrbracket$

*Proof.* See page 40. □

**Lemma 27.**

1.  $S \in \llbracket \xi^{\equiv} \rrbracket$  **implies**  $[S \in \llbracket \xi^+ \rrbracket \text{ or } S \in \llbracket \xi^- \rrbracket]$
2.  $S \in \llbracket \xi^{\equiv} \rrbracket$  **implies**  $[S \in \llbracket \xi^+ \uparrow r \rrbracket \text{ or } S \in \llbracket \xi^- \uparrow r \rrbracket]$

*Proof.* See page 40. □

**Lemma 28.**

1.  $r \in \text{subj}(\xi)$  **implies**  $\llbracket \xi^+ \uparrow r \rrbracket \cap \llbracket \xi^- \uparrow r \rrbracket = \emptyset$ .
2.  $r \in \text{subj}(\xi)$  **implies**  $\llbracket \xi^{\equiv} \rrbracket \cap \llbracket \xi^+ \uparrow r \rrbracket \subseteq \llbracket \xi^+ \rrbracket$
3.  $r \in \text{subj}(\xi)$  **implies**  $\llbracket \xi^{\equiv} \rrbracket \cap \llbracket \xi^+ \uparrow r \rrbracket \subseteq \llbracket \bigwedge \{ \xi^+ \uparrow \hat{r} \}_{\hat{r} \in \text{subj}(\xi) \setminus \{r\}} \rrbracket$
4.  $r \in \text{subj}(\xi)$  **implies**  $\llbracket \xi^{\equiv} \rrbracket \cap \llbracket \xi^- \uparrow r \rrbracket \subseteq \llbracket \xi^- \rrbracket$
5.  $r \in \text{subj}(\xi)$  **implies**  $\llbracket \xi^{\equiv} \rrbracket \cap \llbracket \xi^- \uparrow r \rrbracket \subseteq \llbracket \bigwedge \{ \xi^- \uparrow \hat{r} \}_{\hat{r} \in \text{subj}(\xi) \setminus \{r\}} \rrbracket$

*Proof.* See page 41. □

**Lemma 29.**

1.  $r \in R \subseteq \text{subj}(\xi)$  **implies**  $\llbracket \bigwedge \{ \xi^- \uparrow \hat{r} \}_{\hat{r} \in R} \rrbracket \cap \llbracket \xi^+ \uparrow r \rrbracket = \llbracket \mathbf{false} \rrbracket$
2.  $r \in R \subseteq \text{subj}(\xi)$  **implies**  $\llbracket \bigwedge \{ \xi^+ \uparrow \hat{r} \}_{\hat{r} \in R} \rrbracket \cap \llbracket \xi^- \uparrow r \rrbracket = \llbracket \mathbf{false} \rrbracket$

*Proof.* See page 43. □

### 3 Actions

**Definition 14 (Grammar).**

$$\alpha ::= \tau \mid q.y := e \mid p.e \rightarrow q.y \mid i_R^R \mid pq!e \mid pq?y \mid i_{\{r\}}^R \quad (\mathbf{A})$$

$$\gamma ::= \tau \mid q.y := e \mid p.e \rightarrow q.y \mid i_R^R \quad (\mathbf{\Gamma})$$

$$\lambda ::= \tau \mid q.y := e \mid pq!e \mid pq?y \mid i_{\{r\}}^R \quad (\mathbf{\Lambda})$$

*Remark 2.*

- $\alpha = q.y := e$  implies  $\{r \mid r.x \in \text{rx}(e)\} \subseteq \{q\}$
- $\alpha = p.e \rightarrow q.y$  implies  $[p \neq q \text{ and } \{r \mid r.x \in \text{rx}(e)\} \subseteq \{p\}]$
- $\alpha = i_R^R$  implies  $R \neq \emptyset$
- $\alpha = pq!e$  implies  $[p \neq q \text{ and } \{r \mid r.x \in \text{rx}(e)\} \subseteq \{q\}]$
- $\alpha = pq?y$  implies  $p \neq q$
- $\alpha = i_{\{r\}}^R$  implies  $r \in R$
- $\gamma = q.y := e$  implies  $\{r \mid r.x \in \text{rx}(e)\} \subseteq \{q\}$
- $\gamma = p.e \rightarrow q.y$  implies  $[p \neq q \text{ and } \{r \mid r.x \in \text{rx}(e)\} \subseteq \{p\}]$
- $\gamma = i_R^R$  implies  $R \neq \emptyset$
- $\lambda = q.y := e$  implies  $\{r \mid r.x \in \text{rx}(e)\} \subseteq \{q\}$
- $\lambda = pq!e$  implies  $[p \neq q \text{ and } \{r \mid r.x \in \text{rx}(e)\} \subseteq \{q\}]$
- $\lambda = pq?y$  implies  $p \neq q$
- $\lambda = i_{\{r\}}^R$  implies  $r \in R$

**Definition 15 (Reads/writes).**

$\text{read}(\tau)$	$= \emptyset$	$\text{write}(\tau)$	$= \emptyset$
$\text{read}(q.y := e)$	$= \text{rx}(e)$	$\text{write}(q.y := e)$	$= \{q.y\}$
$\text{read}(p.e \rightarrow q.y)$	$= \text{rx}(e)$	$\text{write}(p.e \rightarrow q.y)$	$= \{q.y\}$
$\text{read}(i_R^R)$	$= \emptyset$	$\text{write}(i_R^R)$	$= \emptyset$
$\text{read}(pq!e)$	$= \text{rx}(e)$	$\text{write}(pq!e)$	$= \emptyset$
$\text{read}(pq?y)$	$= \emptyset$	$\text{write}(pq?y)$	$= \{q.y\}$
$\text{read}(i_{\{r\}}^R)$	$= \emptyset$	$\text{write}(i_{\{r\}}^R)$	$= \emptyset$

**Definition 16 (Subjects).**

$$\begin{aligned} \text{subj}(\tau) &= \emptyset \\ \text{subj}(q.y := e) &= \{q\} \\ \text{subj}(p.e \rightarrow q.y) &= \{p, q\} \\ \text{subj}(i_R^R) &= R \\ \text{subj}(pq!e) &= \{p\} \\ \text{subj}(pq?y) &= \{q\} \\ \text{subj}(i_{\{r\}}^R) &= \{r\} \end{aligned}$$

**Lemma 30.**  $\alpha \neq \tau$  implies  $\text{subj}(\alpha) \neq \emptyset$

*Proof.* See page 44. □

**Lemma 31.**

1.  $\text{read}(\alpha) \subseteq \{r.x \mid r \in \text{subj}(\alpha)\}$
2.  $\text{write}(\alpha) \subseteq \{r.x \mid r \in \text{subj}(\alpha)\}$

*Proof.* See page 44. □

**Definition 17 (Channels).**

$$\begin{aligned}
\text{chan}(\tau) &= \emptyset \\
\text{chan}(q.y := e) &= \emptyset \\
\text{chan}(p.e \rightarrow q.y) &= \{pq\} \\
\text{chan}(i_R^R) &= \{R\} \\
\text{chan}(pq!e) &= \{pq\} \\
\text{chan}(pq?y) &= \{pq\} \\
\text{chan}(i_{\{r\}}^R) &= \{R\}
\end{aligned}$$

**Definition 18 (Projection).**

$$\begin{aligned}
\tau \upharpoonright r &= \tau \\
q.y := e \upharpoonright r &= \begin{cases} q.y := e & \text{if } r = q \\ \tau & \text{else} \end{cases} \\
p.e \rightarrow q.y \upharpoonright r &= \begin{cases} pq!e & \text{if } p = r \neq q \\ pq?y & \text{if } p \neq r = q \\ \tau & \text{else} \end{cases} \\
i_R^R \upharpoonright r &= \begin{cases} i_{\{r\}}^R & \text{if } r \in R \\ \tau & \text{else} \end{cases}
\end{aligned}$$

**Lemma 32.**

1.  $r \in \text{subj}(\gamma)$  implies  $\text{subj}(\gamma \upharpoonright r) = \{r\}$
2.  $r \notin \text{subj}(\gamma)$  implies  $\text{subj}(\gamma \upharpoonright r) = \emptyset$

*Proof.* See page 44. □

**Lemma 33.**  $\text{subj}(\gamma \upharpoonright r) \subseteq \text{subj}(\gamma)$

*Proof.* See page 45. □

**Lemma 34.**  $r \notin \text{subj}(\gamma)$  implies  $\gamma \upharpoonright r = \tau$

*Proof.* See page 45. □

**Lemma 35.**  $\text{chan}(\gamma \upharpoonright r) \subseteq \text{chan}(\gamma)$

*Proof.* See page 45. □

**Definition 19 (Concretisation).**

$$\begin{aligned}
\text{con}_{\mathcal{S}}(\tau) &= \tau \\
\text{con}_{\mathcal{S}}(q.y := e) &= q.y := \text{eval}_{\mathcal{S}}(e) \\
\text{con}_{\mathcal{S}}(p.e \rightarrow q.y) &= p.\text{eval}_{\mathcal{S}}(e) \rightarrow q.y \\
\text{con}_{\mathcal{S}}(i_R^R) &= i_R^R
\end{aligned}$$

**Definition 20 (Effects).**

$$\begin{aligned}
\text{effect}(\tau, \mathcal{S}) &= \mathcal{S} \\
\text{effect}(q.y := v, \mathcal{S}) &= \mathcal{S}[v/q.y] \\
\text{effect}(p.v \rightarrow q.y, \mathcal{S}) &= \mathcal{S}[v/q.y] \\
\text{effect}(i_R^R, \mathcal{S}) &= \mathcal{S}
\end{aligned}$$

## 4 Programs

**Definition 21 (Grammar).**

$$\begin{aligned} \circ & ::= ; \mid \| \\ A & ::= \xi \mid \alpha \mid \mathbf{skip} \mid A_1 \circ A_2 \mid R.\mathbf{if} \xi A_1 A_2 \mid R.\mathbf{while} \xi \{\psi\} A \mid \end{aligned} \quad (\mathbb{A})$$

$$\begin{aligned} & \quad \mathbf{if} \xi|_n A_1|_{R_1} A_2|_{R_2} \mid \mathbf{while} \xi|_n \{\psi\} A|_{\emptyset} \\ G & ::= \xi \mid \gamma \mid \mathbf{skip} \mid G_1 \circ G_2 \mid R.\mathbf{if} \xi G_1 G_2 \mid R.\mathbf{while} \xi \{\psi\} G \mid \end{aligned} \quad (\mathbb{G})$$

$$\begin{aligned} & \quad \mathbf{if} \xi|_0 G_1|_{R_1} G_2|_{R_2} \mid \mathbf{while} \xi|_0 \{\psi\} G|_{\emptyset} \\ L & ::= \xi \mid \lambda \mid \mathbf{skip} \mid L_1 \circ L_2 \mid R.\mathbf{if} \xi L_1 L_2 \mid R.\mathbf{while} \xi \{\mathbf{true}\} L \mid \end{aligned} \quad (\mathbb{L})$$

$$\begin{aligned} & \quad \mathbf{if} \xi|_n L_1|_{R_1} L_2|_{R_2} \mid \mathbf{while} \xi|_n \{\mathbf{true}\} L|_{\emptyset} \\ \underline{G} & ::= q.y := e \mid p.e \rightarrow q.y \mid \mathbf{skip} \mid \underline{G}_1 \circ \underline{G}_2 \mid R.\mathbf{if} \xi \underline{G}_1 \underline{G}_2 \mid \end{aligned} \quad (\mathbb{G})$$

$$\mathbf{if} \xi|_0 \underline{G}_1|_{R_1} \underline{G}_2|_{R_2} \mid \mathbf{if} \xi|_0 G_1|_{\emptyset} \underline{G}_2|_{R_2 \cup \{r\}} \mid \mathbf{if} \xi|_0 \underline{G}_1|_{R_1 \cup \{r\}} G_2|_{\emptyset}$$

**Definition 22 (Reads/writes).**

$$\begin{aligned} \text{read}(\xi) & = \text{Defn. 9} \\ \text{read}(\alpha) & = \text{Defn. 15} \\ \text{read}(\mathbf{skip}) & = \emptyset \\ \text{read}(A_1 \circ A_2) & = \text{read}(A_1) \cup \text{read}(A_2) \\ \text{read}(R.\mathbf{if} \xi A_1 A_2) & = \text{read}(\xi) \cup \text{read}(A_1) \cup \text{read}(A_2) \\ \text{read}(R.\mathbf{while} \xi \{\psi\} A) & = \text{read}(\xi) \cup \text{read}(A) \\ \text{read}(\mathbf{if} \xi|_n A_1|_{R_1} A_2|_{R_2}) & = (\text{read}(\xi) \setminus \{r.x \mid r \in R_1 \cup R_2\}) \cup \\ & \quad (\text{read}(A_1) \setminus \{r.x \mid r \in R_2\}) \cup (\text{read}(A_2) \setminus \{r.x \mid r \in R_1\}) \\ \text{read}(\mathbf{while} \xi|_n \{\psi\} A|_{\emptyset}) & = \text{read}(\xi) \cup \text{read}(A) \end{aligned}$$

$$\begin{aligned} \text{write}(\xi) & = \text{Defn. 9} \\ \text{write}(\alpha) & = \text{Defn. 15} \\ \text{write}(\mathbf{skip}) & = \emptyset \\ \text{write}(A_1 \circ A_2) & = \text{write}(A_1) \cup \text{write}(A_2) \\ \text{write}(R.\mathbf{if} \xi A_1 A_2) & = \text{write}(\xi) \cup \text{write}(A_1) \cup \text{write}(A_2) \\ \text{write}(R.\mathbf{while} \xi \{\psi\} A) & = \text{write}(\xi) \cup \text{write}(A) \\ \text{write}(\mathbf{if} \xi|_n A_1|_{R_1} A_2|_{R_2}) & = (\text{write}(\xi) \setminus \{r.x \mid r \in R_1 \cup R_2\}) \cup \\ & \quad (\text{write}(A_1) \setminus \{r.x \mid r \in R_2\}) \cup (\text{write}(A_2) \setminus \{r.x \mid r \in R_1\}) \\ \text{write}(\mathbf{while} \xi|_n \{\psi\} A|_{\emptyset}) & = \text{write}(\xi) \cup \text{write}(A) \end{aligned}$$

**Lemma 36.**

1.  $R_1 = \emptyset = R_2$  implies  $\text{read}(\mathbf{if} \xi|_n A_1|_{R_1} A_2|_{R_2}) = \text{read}(\xi) \cup \text{read}(A_1) \cup \text{read}(A_2)$
2.  $R_1 = \emptyset \neq R_2$  implies  $\left[ \begin{array}{l} \text{read}(\mathbf{if} \xi|_n A_1|_{R_1} A_2|_{R_2}) = \\ (\text{read}(\xi) \setminus \{r.x \mid r \in R_2\}) \cup (\text{read}(A_1) \setminus \{r.x \mid r \in R_2\}) \cup \text{read}(A_2) \end{array} \right]$
3.  $R_1 \neq \emptyset = R_2$  implies  $\left[ \begin{array}{l} \text{read}(\mathbf{if} \xi|_n A_1|_{R_1} A_2|_{R_2}) = \\ (\text{read}(\xi) \setminus \{r.x \mid r \in R_1\}) \cup \text{read}(A_1) \cup (\text{read}(A_2) \setminus \{r.x \mid r \in R_1\}) \end{array} \right]$
4.  $R_1 = \emptyset = R_2$  implies  $\text{write}(\mathbf{if} \xi|_n A_1|_{R_1} A_2|_{R_2}) = \text{write}(\xi) \cup \text{write}(A_1) \cup \text{write}(A_2)$
5.  $R_1 = \emptyset \neq R_2$  implies  $\left[ \begin{array}{l} \text{write}(\mathbf{if} \xi|_n A_1|_{R_1} A_2|_{R_2}) = \\ (\text{write}(\xi) \setminus \{r.x \mid r \in R_2\}) \cup (\text{write}(A_1) \setminus \{r.x \mid r \in R_2\}) \cup \text{write}(A_2) \end{array} \right]$
6.  $R_1 \neq \emptyset = R_2$  implies  $\left[ \begin{array}{l} \text{write}(\mathbf{if} \xi|_n A_1|_{R_1} A_2|_{R_2}) = \\ (\text{write}(\xi) \setminus \{r.x \mid r \in R_1\}) \cup \text{write}(A_1) \cup (\text{write}(A_2) \setminus \{r.x \mid r \in R_1\}) \end{array} \right]$

*Proof.* See page 47. □

**Definition 23 (Non-interference).**

$$\frac{\begin{array}{l} \text{read}(A_1) \cap \text{read}(A_2) = \emptyset \quad \text{read}(A_1) \cap \text{write}(A_2) = \emptyset \\ \text{write}(A_1) \cap \text{read}(A_2) = \emptyset \quad \text{write}(A_1) \cap \text{write}(A_2) = \emptyset \end{array}}{A_1 \# A_2} \text{[#]}$$

**Lemma 37.**  $A_1 \# A_2$  implies  $A_2 \# A_1$

*Proof.* See page 47. □

**Lemma 38.**

1.  $A_1 \circ A_2 \# A$  implies  $[A_1 \# A \text{ and } A_2 \# A]$
2.  $R.\text{if } \xi A_1 A_2 \# A$  implies  $[\xi \# A \text{ and } A_1 \# A \text{ and } A_2 \# A]$
3.  $R.\text{while } \xi \{\psi\} \hat{A} \# A$  implies  $[\xi \# A \text{ and } \hat{A} \# A]$
4.  $[\text{if } \xi|_n A_1|_{R_1} A_2|_{R_2} \# A \text{ and } R_1 = \emptyset = R_2]$  implies  $[\xi \# A \text{ and } A_1 \# A \text{ and } A_2 \# A]$
5.  $[\text{if } \xi|_n A_1|_{R_1} A_2|_{R_2} \# A \text{ and } R_1 = \emptyset \neq R_2]$  implies  $\left[ \begin{array}{l} \xi \upharpoonright r \# A \text{ for every} \\ r \in \text{subj}(\xi) \setminus R_2 \end{array} \right] \text{ and } A_2 \# A$
6.  $[\text{if } \xi|_n A_1|_{R_1} A_2|_{R_2} \# A \text{ and } R_1 \neq \emptyset = R_2]$  implies  $\left[ \begin{array}{l} \xi \upharpoonright r \# A \text{ for every} \\ r \in \text{subj}(\xi) \setminus R_1 \end{array} \right] \text{ and } A_1 \# A$
7.  $\text{while } \xi|_n \{\psi\} \hat{A}|_{\emptyset} \# A$  implies  $[\xi \# A \text{ and } \hat{A} \# A]$

*Proof.* See page 47. □

**Lemma 39.**

1.  $[\xi_r \# A \text{ for every } r \in R]$  implies  $\bigwedge \{\xi_r^+\}_{r \in R} \# A$
2.  $[\xi_r \# A \text{ for every } r \in R]$  implies  $\bigwedge \{\xi_r^-\}_{r \in R} \# A$

*Proof.* See page 48. □

**Definition 24 (Subjects).**

$$\begin{aligned} \text{subj}(\xi) &= \text{Defn. 10} \\ \text{subj}(\alpha) &= \text{Defn. 16} \\ \text{subj}(\text{skip}) &= \emptyset \\ \text{subj}(A_1 \circ A_2) &= \text{subj}(A_1) \cup \text{subj}(A_2) \\ \text{subj}(R.\text{if } \xi A_1 A_2) &= \text{subj}(\xi) \cup \text{subj}(A_1) \cup \text{subj}(A_2) \\ \text{subj}(R.\text{while } \xi \{\psi\} A) &= \text{subj}(\xi) \cup \text{subj}(A) \\ \text{subj}(\text{if } \xi|_n A_1|_{R_1} A_2|_{R_2}) &= (\text{subj}(\xi) \setminus (R_1 \cup R_2)) \cup (\text{subj}(A_1) \setminus R_2) \cup (\text{subj}(A_2) \setminus R_1) \\ \text{subj}(\text{while } \xi|_n \{\psi\} A|_{\emptyset}) &= \text{subj}(\xi) \cup \text{subj}(A) \end{aligned}$$

**Lemma 40.**

1.  $\text{read}(A) \subseteq \{r.x \mid r \in \text{subj}(A)\}$
2.  $\text{write}(A) \subseteq \{r.x \mid r \in \text{subj}(A)\}$

*Proof.* See page 49. □

**Lemma 41.**  $\text{subj}(A_1) \cap \text{subj}(A_2) = \emptyset$  implies  $A_1 \# A_2$

*Proof.* See page 49. □

**Definition 25 (Channels).**

$$\begin{aligned}
 \text{chan}(\xi) &= \emptyset \\
 \text{chan}(\alpha) &= \text{Defn. 17} \\
 \text{chan}(\mathbf{skip}) &= \emptyset \\
 \text{chan}(A_1 \circ A_2) &= \text{chan}(A_1) \cup \text{chan}(A_2) \\
 \text{chan}(R.\mathbf{if} \xi A_1 A_2) &= \{R\} \cup \text{chan}(A_1) \cup \text{chan}(A_2) \\
 \text{chan}(R.\mathbf{while} \xi \{\psi\} A) &= \{R\} \cup \text{chan}(A) \\
 \text{chan}(\mathbf{if} \xi|_n A_1|_{R_1} A_2|_{R_2}) &= \{\{r\} \mid r \in \text{subj}(\xi)\} \cup \text{chan}(A_1) \cup \text{chan}(A_2) \\
 \text{chan}(\mathbf{while} \xi|_n \{\psi\} A|_\emptyset) &= \{\{r\} \mid r \in \text{subj}(\xi)\} \cup \text{chan}(A)
 \end{aligned}$$

**Definition 26 (Projection).**

$$\begin{aligned}
 \xi \upharpoonright r &= \text{Defn. 11} \\
 \gamma \upharpoonright r &= \text{Defn. 18} \\
 \mathbf{skip} \upharpoonright r &= \mathbf{skip} \\
 G_1 \circ G_2 \upharpoonright r &= (G_1 \upharpoonright r) \circ (G_2 \upharpoonright r) \\
 R.\mathbf{if} \xi G_1 G_2 \upharpoonright r &= R.\mathbf{if} (\xi \upharpoonright r) (G_1 \upharpoonright r) (G_2 \upharpoonright r) \\
 R.\mathbf{while} \xi \{\psi\} G \upharpoonright r &= R.\mathbf{while} (\xi \upharpoonright r) \{\mathbf{true}\} (G \upharpoonright r) \\
 \mathbf{if} \xi|_0 G_1|_{R_1} G_2|_{R_2} \upharpoonright r &= \mathbf{if} (\xi \upharpoonright r)|_{\text{subj}(\xi) \setminus (R_1 \cup R_2 \cup \{r\})} (G_1 \upharpoonright r)|_{R_1 \cap \{r\}} (G_2 \upharpoonright r)|_{R_2 \cap \{r\}} \\
 \mathbf{while} \xi|_0 \{\psi\} G|_\emptyset \upharpoonright r &= \mathbf{while} (\xi \upharpoonright r)|_{\text{subj}(\xi) \setminus \{r\}} \{\mathbf{true}\} (G \upharpoonright r)|_\emptyset \\
 \{G\} \upharpoonright R &= \{G \upharpoonright r\}_{r \in R}
 \end{aligned}$$

**Lemma 42.**

1.  $r \in \text{subj}(G)$  implies  $\text{subj}(G \upharpoonright r) = \{r\}$
2.  $r \notin \text{subj}(G)$  implies  $\text{subj}(G \upharpoonright r) = \emptyset$

*Proof.* See page 50. □

**Lemma 43.**  $\text{subj}(G \upharpoonright r) \subseteq \text{subj}(G)$

*Proof.* See page 56. □

**Lemma 44.**  $r \in \text{subj}(G)$  implies  $r \in \text{subj}(G \upharpoonright r)$

*Proof.* See page 56. □

**Lemma 45.**  $\text{chan}(G \upharpoonright r) \subseteq \text{chan}(G)$

*Proof.* See page 56. □



**Definition 27 (Well-formedness).**

$$\begin{array}{c}
\frac{q \in R}{\checkmark_R(q.y := e)} [\checkmark\text{-ACT1}] \quad \frac{p, q \in R}{\checkmark_R(p.e \rightarrow q.y)} [\checkmark\text{-ACT2}] \quad \frac{R \neq \emptyset}{\checkmark_R(\text{skip})} [\checkmark\text{-SKIP}] \\
\\
\frac{\checkmark_R(G_1) \quad \checkmark_R(G_2)}{\checkmark_R(G_1 ; G_2)} [\checkmark\text{-SEQ}] \quad \frac{\checkmark_R(G_1) \quad \checkmark_R(G_2) \quad \text{chan}(G_1) \cap \text{chan}(G_2) = \emptyset}{\checkmark_R(G_1 \parallel G_2)} [\checkmark\text{-PAR}] \\
\\
\frac{\checkmark_R(G_1) \quad \checkmark_R(G_2) \quad R = \text{subj}(\xi)}{\checkmark_R(R.\text{if } \xi G_1 G_2)} [\checkmark\text{-IF}] \quad \frac{\checkmark_R(G) \quad R = \text{subj}(\xi)}{\checkmark_R(R.\text{while } \xi \{\psi\} G)} [\checkmark\text{-WHILE}] \\
\\
\frac{\checkmark_R(G_1) \quad \checkmark_R(G_2) \quad R = \text{subj}(\xi) \quad R_1, R_2 \subseteq \text{subj}(\xi) \quad R_1 \neq \emptyset \text{ implies } R_2 = \emptyset \quad R_2 \neq \emptyset \text{ implies } R_1 = \emptyset}{\checkmark_R(\text{if } \xi|_0 G_1|_{R_1} G_2|_{R_2})} [\checkmark\text{-NIF}] \quad \frac{\checkmark_R(G) \quad R = \text{subj}(\xi)}{\checkmark_R(\text{while } \xi|_0 \{\psi\} G|_\emptyset)} [\checkmark\text{-NWHILE}] \\
\\
\frac{\checkmark_R(G)}{\checkmark_R(\{G\})} [\checkmark\text{-2}]
\end{array}$$

**Lemma 46.**  $\checkmark_R(\text{while } \xi|_0 \{\psi\} G|_\emptyset) \text{ implies } \checkmark_R(\text{if } \xi|_0 (G ; \text{while } \xi|_0 \{\psi\} G|_\emptyset)|_\emptyset \text{ skip}|_\emptyset)$

*Proof.* See page 57. □

**Lemma 47.**  $\checkmark_R(G) \text{ implies } \text{subj}(G) \subseteq R \neq \emptyset$

*Proof.* See page 57. □

## 5 Termination

**Definition 28 (Termination, part 1).**

$$\frac{}{\text{skip} \downarrow} [\downarrow\text{-SKIP}] \quad \frac{A_1 \downarrow \quad A_2 \downarrow}{A_1 ; A_2 \downarrow} [\downarrow\text{-SEQ}] \quad \frac{A_1 \downarrow \quad A_2 \downarrow}{A_1 \parallel A_2 \downarrow} [\downarrow\text{-PAR}]$$

$$\frac{\text{subj}(\xi) = R_1 \cup R_2 \quad R_1 \neq \emptyset \text{ implies } A_1 \downarrow \quad R_2 \neq \emptyset \text{ implies } A_2 \downarrow}{\text{if } \xi|_0 A_1|_{R_1} A_2|_{R_2} \downarrow} [\downarrow\text{-NIF}]$$

**Lemma 48.**  $[G \downarrow \text{ and } \checkmark_R(G)]$  implies  $\text{subj}(G) = \emptyset$

*Proof.* See page 58. □

**Lemma 49.**  $[G \downarrow \text{ and } \checkmark_R(G)]$  implies  $(G \upharpoonright r) \downarrow$

*Proof.* See page 58. □

**Definition 29 (Termination, part 2).**

$$\frac{G \downarrow}{\{G\} \downarrow} [\downarrow\text{-GLOB}] \quad \frac{L_r \downarrow \text{ for every } r \in R}{\{L_r\}_{r \in R} \downarrow} [\downarrow\text{-Locs}]$$

**Definition 30 (Termination, part 3).**

$$\frac{\mathcal{A} \downarrow}{(\mathcal{A}, \mathcal{S}) \downarrow} [\downarrow\text{-3}]$$

## 6 Reduction

**Definition 31 (Reduction, part 1).**

$$\begin{array}{c}
\frac{}{\alpha \xrightarrow{\text{true}, \alpha} \text{skip}} \text{[}\rightarrow\text{1-ACT]} \quad \frac{A_1 \xrightarrow{\xi, \alpha} A'_1}{A_1 ; A_2 \xrightarrow{\xi, \alpha} A'_1 ; A_2} \text{[}\rightarrow\text{1-SEQ1]} \\
\frac{\text{subj}(A_1) \cap (\text{subj}(\xi) \cup \text{subj}(\alpha)) = \emptyset \quad A_2 \xrightarrow{\xi, \alpha} A'_2}{A_1 ; A_2 \xrightarrow{\xi, \alpha} A_1 ; A'_2} \text{[}\rightarrow\text{1-SEQ2]} \\
\frac{A_1 \xrightarrow{\xi, \alpha} A'_1}{A_1 \parallel A_2 \xrightarrow{\xi, \alpha} A'_1 \parallel A_2} \text{[}\rightarrow\text{1-PAR1]} \quad \frac{A_2 \xrightarrow{\xi, \alpha} A'_2}{A_1 \parallel A_2 \xrightarrow{\xi, \alpha} A_1 \parallel A'_2} \text{[}\rightarrow\text{1-PAR2]} \\
\frac{}{R.\text{if } \xi \ A_1 \ A_2 \xrightarrow{\xi^+, 1^R_{\text{subj}(\xi)}} A_1} \text{[}\rightarrow\text{1-IF1]} \quad \frac{}{R.\text{if } \xi \ A_1 \ A_2 \xrightarrow{\xi^-, 2^R_{\text{subj}(\xi)}} A_2} \text{[}\rightarrow\text{1-IF2]} \\
\frac{}{R.\text{while } \xi \ \{\psi\} \ A \xrightarrow{\xi^+, 1^R_{\text{subj}(\xi)}} A ; R.\text{while } \xi \ \{\psi\} \ A} \text{[}\rightarrow\text{1-WHILE1]} \\
\frac{}{R.\text{while } \xi \ \{\psi\} \ A \xrightarrow{\xi^-, 2^R_{\text{subj}(\xi)}} \text{skip}} \text{[}\rightarrow\text{1-WHILE2]} \\
\frac{n > 0}{\text{if } \xi|_n \ A_1|_{R_1} \ A_2|_{R_2} \xrightarrow{\text{true}, \tau} \text{if } \xi|_{n-1} \ A_1|_{R_1} \ A_2|_{R_2}} \text{[}\rightarrow\text{1-NIF1]} \\
\frac{r \in \text{subj}(\xi) \setminus (R_1 \cup R_2)}{\text{if } \xi|_n \ A_1|_{R_1} \ A_2|_{R_2} \xrightarrow{\xi^+ \uparrow r, 1^R_{\{r\}}} \text{if } \xi|_n \ A_1|_{R_1 \cup \{r\}} \ A_2|_{R_2}} \text{[}\rightarrow\text{1-NIF2]} \\
\frac{r \in \text{subj}(\xi) \setminus (R_1 \cup R_2)}{\text{if } \xi|_n \ A_1|_{R_1} \ A_2|_{R_2} \xrightarrow{\xi^- \uparrow r, 2^R_{\{r\}}} \text{if } \xi|_n \ A_1|_{R_1} \ A_2|_{R_2 \cup \{r\}}} \text{[}\rightarrow\text{1-NIF3]} \\
\frac{A_1 \xrightarrow{\hat{\xi}, \alpha} A'_1 \quad \text{subj}(\hat{\xi}) \cup \text{subj}(\alpha) \subseteq R_1 \setminus R_2}{\text{if } \xi|_n \ A_1|_{R_1} \ A_2|_{R_2} \xrightarrow{\hat{\xi}, \alpha} \text{if } \xi|_n \ A'_1|_{R_1} \ A_2|_{R_2}} \text{[}\rightarrow\text{1-NIF4]} \\
\frac{A_2 \xrightarrow{\hat{\xi}, \alpha} A'_2 \quad \text{subj}(\hat{\xi}) \cup \text{subj}(\alpha) \subseteq R_2 \setminus R_1}{\text{if } \xi|_n \ A_1|_{R_1} \ A_2|_{R_2} \xrightarrow{\hat{\xi}, \alpha} \text{if } \xi|_n \ A_1|_{R_1} \ A'_2|_{R_2}} \text{[}\rightarrow\text{1-NIF5]} \\
\frac{\text{if } \xi|_n \ (A ; \text{while } \xi|_n \ \{\psi\} \ A|_{\emptyset})|_{\emptyset} \ \text{skip}|_{\emptyset} \xrightarrow{\hat{\xi}, \alpha} A'}{\text{while } \xi|_n \ \{\psi\} \ A|_{\emptyset} \xrightarrow{\hat{\xi}, \alpha} A'} \text{[}\rightarrow\text{1-NWHILE]}
\end{array}$$

**Lemma 50.**  $[A \xrightarrow{\xi, \alpha} A' \text{ and } (\xi, \alpha) \neq (\text{true}, \tau)] \text{ implies } \alpha \neq \tau$

*Proof.* See page 60. □

**Lemma 51.**

1.  $A \xrightarrow{\xi, \alpha} A'$  implies  $\text{read}(\xi) \cup \text{read}(\alpha) \cup \text{read}(A') \subseteq \text{read}(A)$
2.  $A \xrightarrow{\xi, \alpha} A'$  implies  $\text{read}(\xi) \cup \text{write}(\alpha) \cup \text{write}(A') \subseteq \text{write}(A)$

*Proof.* See page 60. □

**Lemma 52.**

1.  $[A_1 \# A_2 \text{ and } A_2 \xrightarrow{\xi, \alpha} A'_2] \text{ implies } A_1 \# \xi$

2.  $[A_2 \# A_1 \text{ and } A_2 \xrightarrow{\xi, \alpha} A'_2]$  implies  $\xi \# A_1$
3.  $[A_1 \# A_2 \text{ and } A_2 \xrightarrow{\xi, \alpha} A'_2]$  implies  $A_1 \# \alpha$
4.  $[A_2 \# A_1 \text{ and } A_2 \xrightarrow{\xi, \alpha} A'_2]$  implies  $\alpha \# A_1$
5.  $[A_1 \# A_2 \text{ and } A_2 \xrightarrow{\xi, \alpha} A'_2]$  implies  $A_1 \# A'_2$
6.  $[A_2 \# A_1 \text{ and } A_2 \xrightarrow{\xi, \alpha} A'_2]$  implies  $A'_2 \# A_1$

*Proof.* See page 62. □

**Lemma 53.**

1.  $A \xrightarrow{\xi, \alpha} A'$  implies  $\text{subj}(\xi) \cup \text{subj}(\alpha) \cup \text{subj}(A') \subseteq \text{subj}(A)$
2.  $A \xrightarrow{\xi, \alpha} A'$  implies  $\text{subj}(\xi) \subseteq \text{subj}(\alpha)$
3.  $A \xrightarrow{\text{true}, \tau} A'$  implies  $\text{subj}(A) = \text{subj}(A')$

*Proof.* See page 63. □

**Lemma 54.**  $A \xrightarrow{\xi, \alpha} A'$  implies  $\text{chan}(\alpha) \cup \text{chan}(A') \subseteq \text{chan}(A)$

*Proof.* See page 66. □

**Lemma 55.**  $[A \downarrow \text{ and } A \xrightarrow{\xi, \alpha} A']$  implies  $[A' \downarrow \text{ and } \xi = \text{true} \text{ and } \alpha = \tau]$

*Proof.* See page 67. □

**Lemma 56.**

1.  $[A \xrightarrow{\text{true}, \tau} A'_1 \text{ and } A \xrightarrow{\xi, \alpha} A'_2]$  implies  $\left[ \begin{array}{l} [[A'_1 \xrightarrow{\xi, \alpha} A'' \text{ and } A'_2 \xrightarrow{\text{true}, \tau} A''] \text{ for some } A''] \\ \text{or } [\xi = \text{true} \text{ and } \alpha = \tau \text{ and } A'_1 = A'_2] \end{array} \right]$
2.  $A \xrightarrow{\text{true}, \tau} A'_1 \xrightarrow{\xi, \alpha} A''$  implies  $[A \xrightarrow{\xi, \alpha} A'_2 \xrightarrow{\text{true}, \tau} A'' \text{ for some } A'_2]$

*Proof.* See page 67. □

**Lemma 57.**  $\checkmark_R(G)$  implies  $[G \downarrow \text{ or } [G \xrightarrow{\xi, \gamma} G' \text{ for some } G', \xi, \gamma]]$

*Proof.* See page 72. □

**Lemma 58.**

1.  $[G \xrightarrow{\xi, \gamma} G' \text{ and } \checkmark_R(G)]$  implies  $\text{subj}(\gamma) \neq \emptyset$
2.  $[G \xrightarrow{\xi, \gamma} G' \text{ and } \checkmark_R(G)]$  implies  $\gamma \neq \tau$

*Proof.* See page 74. □

**Lemma 59.**

1.  $[G \xrightarrow{\xi, \gamma} G' \text{ and } \checkmark_R(G) \text{ and } r \in R \cap \text{subj}(\gamma)]$  implies  $(G \upharpoonright r) \xrightarrow{\xi \upharpoonright r, \gamma \upharpoonright r} (G' \upharpoonright r)$
2.  $[G \xrightarrow{\xi, \gamma} G' \text{ and } \checkmark_R(G) \text{ and } r \in R \setminus \text{subj}(\gamma)]$  implies  $(G \upharpoonright r) \xrightarrow{\text{true}, \tau} (G' \upharpoonright r)$

*Proof.* See page 74. □

**Lemma 60.**  $\left[ \begin{array}{l} (G \upharpoonright r) \xrightarrow{\xi_r, \lambda_r} L'_r \text{ and} \\ \checkmark_R(G) \text{ and } \lambda_r \neq \tau \end{array} \right]$  implies  $\left[ \begin{array}{l} [\text{subj}(\gamma) \subseteq \text{subj}(G) \text{ and } \lambda_r = \gamma \upharpoonright r] \\ \text{for some } \gamma \end{array} \right]$

*Proof.* See page 78. □

**Lemma 61.**  $\left[ \begin{array}{l} (G \upharpoonright q) \xrightarrow{\xi_q, q.y := e} L'_q \\ \text{and } \checkmark_R(G) \end{array} \right]$  implies  $\left[ \begin{array}{l} G \xrightarrow{\xi, q.y := e} G' \text{ and} \\ [L'_q = G' \upharpoonright q \text{ and } \xi_q = \xi \upharpoonright q] \\ \text{for some } G', \xi \end{array} \right]$

*Proof.* See page 80. □

**Lemma 62.**

$$\left[ \begin{array}{l} (G \upharpoonright p) \xrightarrow{\xi_p, pq!e} L'_p \text{ and} \\ (G \upharpoonright q) \xrightarrow{\xi_q, pq?y} L'_q \text{ and} \\ \sqrt{R}(G) \end{array} \right] \text{ implies } \left[ \begin{array}{l} \left[ G \xrightarrow{\xi, p.e \rightarrow q.y} G' \text{ and } L'_p = G' \upharpoonright p \text{ and } \xi_p = \xi \upharpoonright p \right] \\ \text{and } L'_q = G' \upharpoonright q \text{ and } \xi_q = \xi \upharpoonright q \\ \text{for some } G', \xi \end{array} \right]$$

*Proof.* See page 82. □

**Lemma 63.**

$$\left[ \begin{array}{l} (G \upharpoonright r) \xrightarrow{\xi_r, i_{\hat{r}}^{\hat{R}}} L'_r \\ \text{and } r \in \hat{R} \text{ and} \\ \left[ (G \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i_{\hat{r}}^{\hat{R}}} L'_{\hat{r}} \right] \\ \text{for every } \hat{r} \in \hat{R} \\ \text{and } \sqrt{R}(G) \end{array} \right] \text{ implies } \left[ \begin{array}{l} \left[ G \xrightarrow{\xi, i_{\hat{R}}^{\hat{R}}} G' \text{ and } \right. \\ \left. \left[ L'_{\hat{r}} = G' \upharpoonright \hat{r} \right] \right. \\ \left. \text{for every } \hat{r} \in \hat{R} \right] \text{ and } \left[ \xi_{\hat{r}} = \xi \upharpoonright \hat{r} \right] \\ \text{for some } G', \xi \end{array} \right]$$

*Proof.* See page 86. □

**Lemma 64.**

$$\left[ \begin{array}{l} \left[ (G \upharpoonright r) \xrightarrow{\xi_r, \gamma \upharpoonright r} L'_r \right] \\ \text{for every} \\ r \in \text{subj}(\gamma) \\ \text{and } \sqrt{R}(G) \text{ and } \gamma \neq \tau \end{array} \right] \text{ implies } \left[ \begin{array}{l} \left[ G \xrightarrow{\xi, \gamma} G' \text{ and } \right. \\ \left. \left[ L'_r = G' \upharpoonright r \right] \right. \\ \left. \text{for every } r \in \text{subj}(\gamma) \right] \\ \text{for some } G', \xi \end{array} \right] \text{ and } \left[ \begin{array}{l} \xi_r = \xi \upharpoonright r \\ \text{for every} \\ r \in \text{subj}(\gamma) \end{array} \right]$$

*Proof.* See page 91. □

**Definition 32 (Reduction, part 2).**

$$\begin{array}{c} \frac{G \xrightarrow{\xi, \gamma} G'}{\{G\} \xrightarrow{\xi, \gamma} \{G'\}} \text{ [}\rightarrow\text{2-GLOB]} \\ \frac{L_{\hat{r}} \xrightarrow{\xi, \tau} L'_{\hat{r}} \quad \hat{r} \in R}{\{L_r\}_{r \in R} \xrightarrow{\xi, \tau} \{L_r\}_{r \in R \setminus \{\hat{r}\}} \cup \{L'_r\}_{r \in \{\hat{r}\}}} \text{ [}\rightarrow\text{2-Locs1]} \\ \frac{L_q \xrightarrow{\xi, q.y := e} L'_q \quad q \in R}{\{L_r\}_{r \in R} \xrightarrow{\xi, q.y := e} \{L_r\}_{r \in R \setminus \{q\}} \cup \{L'_r\}_{r \in \{q\}}} \text{ [}\rightarrow\text{2-Locs2]} \\ \frac{L_p \xrightarrow{\xi_p, pq!e} L'_p \quad L_q \xrightarrow{\xi_q, pq?y} L'_q \quad p, q \in R}{\{L_r\}_{r \in R} \xrightarrow{\xi_p \wedge \xi_q, p.e \rightarrow q.y} \{L_r\}_{r \in R \setminus \{p, q\}} \cup \{L'_r\}_{r \in \{p, q\}}} \text{ [}\rightarrow\text{2-Locs3]} \\ \frac{[L_r \xrightarrow{\xi_r, i_{\hat{r}}^{\hat{R}}} L'_r \text{ for every } r \in \hat{R}] \quad \hat{R} \subseteq R}{\{L_r\}_{r \in R} \xrightarrow{\bigwedge \{\xi_r\}_{r \in \hat{R}}, i_{\hat{R}}^{\hat{R}}} \{L_r\}_{r \in R \setminus \hat{R}} \cup \{L'_r\}_{r \in \hat{R}}} \text{ [}\rightarrow\text{2-Locs4]} \end{array}$$

**Lemma 65.**  $[A \xrightarrow{\xi, \gamma} A' \text{ and } (\xi, \gamma) \neq (\text{true}, \tau)] \text{ implies } \gamma \neq \tau$

*Proof.* See page 91. □

**Lemma 66.**  $\{L_r\}_{r \in R} \xrightarrow{\xi, \gamma} \{L'_r\}_{r \in R} \text{ implies } \text{subj}(\gamma) \subseteq R$

*Proof.* See page 92. □

**Lemma 67.**

$$\begin{aligned}
 1. \{L_r\}_{r \in R} \xrightarrow{\xi, \tau} \{L'_r\}_{r \in R} \text{ implies } & \left[ \left[ \begin{array}{l} L_{\hat{r}} \xrightarrow{\xi, \tau} L'_{\hat{r}} \text{ and } \hat{r} \in R \text{ and} \\ [L_r = L'_r \text{ for every } r \in R \setminus \{\hat{r}\}] \end{array} \right] \right] \\
 & \text{for some } \hat{r} \\
 2. [\{L_r\}_{r \in R} \xrightarrow{\xi, \gamma} \{L'_r\}_{r \in R} \text{ and } \gamma \neq \tau] \text{ implies } & \left[ \left[ \begin{array}{l} [L_r \xrightarrow{\xi_r, \gamma \uparrow r} L'_r \text{ for every } r \in \text{subj}(\gamma)] \\ \text{and } \xi = \bigwedge \{\xi_r\}_{r \in \text{subj}(\gamma)} \text{ and} \\ [L_r = L'_r \text{ for every } r \in R \setminus \text{subj}(\gamma)] \end{array} \right] \right] \\
 & \text{for some } \{\xi_r\}_{r \in \text{subj}(\gamma)}
 \end{aligned}$$

*Proof.* See page 92. □

**Lemma 68.**

$$\left[ \begin{array}{l} [L_r \xrightarrow{\xi_r, \gamma \uparrow r} L'_r \text{ for every } r \in \text{subj}(\gamma) \subseteq R] \\ \text{and } [L_r = L'_r \text{ for every } r \in R \setminus \text{subj}(\gamma)] \\ \text{and } \gamma \neq \tau \end{array} \right] \text{ implies } \{L_r\}_{r \in R} \xrightarrow{\bigwedge \{\xi_r\}_{r \in \text{subj}(\gamma), \gamma}} \{L'_r\}_{r \in R}$$

*Proof.* See page 92. □

**Definition 33 (Reduction, part 3).**

$$\frac{\mathcal{A} \xrightarrow{\xi, \gamma} \mathcal{A}' \quad \mathcal{S} \in \llbracket \xi \rrbracket \quad \dot{\gamma} = \text{con}_{\mathcal{S}}(\gamma)}{(\mathcal{A}, \mathcal{S}) \xrightarrow{\dot{\gamma}} (\mathcal{A}', \text{effect}(\dot{\gamma}, \mathcal{S}))} \text{[}\rightarrow 3\text{]}$$

## 7 Reachability

**Definition 34 (Reachability, part 1).**

$$\frac{}{A \xrightarrow{\Pi, *}_* A} \text{ [}\rightarrow^* \text{1-BASE]} \quad \frac{A \xrightarrow{\Pi, *} A^* \xrightarrow{\pi} A^\dagger \quad \pi \in \Pi}{A \xrightarrow{\Pi, *} A^\dagger} \text{ [}\rightarrow^* \text{1-STEP]} \quad \frac{A \xrightarrow{\{\text{true}, \tau\}}_* A^\dagger}{A \Longrightarrow A^\dagger} \text{ [}\Rightarrow \text{1]}$$

**Lemma 69.**

1.  $\left[ A \xrightarrow{\{\text{true}, \tau\}}_* A^\dagger \text{ and } A \downarrow \right]$  implies  $A^\dagger \downarrow$
2.  $\left[ A \xrightarrow{\{\text{true}, \tau\}}_* A^\dagger \right]$  and  $A \xrightarrow{\xi, \alpha} A'_2$  implies  $\left[ \left[ A^\dagger \xrightarrow{\xi, \alpha} A^\ddagger \text{ and } \left[ A'_2 \xrightarrow{\{\text{true}, \tau\}}_* A^\ddagger \right] \text{ for some } A^\ddagger \right] \right]$
3.  $A \xrightarrow{\{\text{true}, \tau\}}_* A^\dagger \xrightarrow{\xi, \alpha} A^\ddagger$  implies  $\left[ A \xrightarrow{\xi, \alpha} A'_2 \xrightarrow{\{\text{true}, \tau\}}_* A^\ddagger \text{ for some } A'_2 \right]$

*Proof.* See page 94. □

**Lemma 70.**

1.  $A^\dagger \Leftarrow A \downarrow$  implies  $A^\dagger \downarrow$
2.  $A_1 \Longrightarrow A^\dagger \Leftarrow A_2 \xrightarrow{\text{true}, \tau} A'_2$  implies  $\left[ A_1 \Longrightarrow A^\dagger \Leftarrow A'_2 \text{ for some } A^\dagger \right]$
3.  $A_1 \Longrightarrow A^\dagger \Leftarrow A_2 \xrightarrow{\xi, \alpha} A'_2$  implies  $\left[ A_1 \xrightarrow{\xi, \alpha} A'_1 \Longrightarrow A^\dagger \Leftarrow A'_2 \text{ for some } A'_1, A^\dagger \right]$

*Proof.* See page 94. □

**Definition 35 (Reachability, part 2).**

$$\frac{}{A \xrightarrow{\Pi, *} A} \text{ [}\rightarrow^* \text{2-BASE]} \quad \frac{A \xrightarrow{\Pi, *} A^* \xrightarrow{\pi} A^\dagger \quad \pi \in \Pi}{A \xrightarrow{\Pi, *} A^\dagger} \text{ [}\rightarrow^* \text{2-STEP]} \quad \frac{A \xrightarrow{\{\text{true}, \tau\}}_* A^\dagger}{A \Longrightarrow A^\dagger} \text{ [}\Rightarrow \text{2]}$$

**Lemma 71.**  $A \xrightarrow{\Pi, *} A^\dagger \xrightarrow{\Pi, *} A^\S$  implies  $A \xrightarrow{\Pi, *} A^\S$

*Proof.* See page 95. □

**Lemma 72.**

1.  $\left[ L_{\hat{r}} \xrightarrow{\{\text{true}, \tau\}}_* L_{\hat{r}}^\dagger \text{ and } \hat{r} \in R \right]$  implies  $\{L_r\}_{r \in R} \xrightarrow{\{\text{true}, \tau\}}_* \{L_r\}_{r \in R \setminus \{\hat{r}\}} \cup \{L_{\hat{r}}^\dagger\}_{r \in \{\hat{r}\}}$
2.  $\left[ L_r \xrightarrow{\{\text{true}, \tau\}}_* L_r^\dagger \text{ for every } r \in \hat{R} \subseteq R \right]$  implies  $\{L_r\}_{r \in R} \xrightarrow{\{\text{true}, \tau\}}_* \{L_r\}_{r \in R \setminus \hat{R}} \cup \{L_r^\dagger\}_{r \in \hat{R}}$
3.  $\left[ L_r \xrightarrow{\{\text{true}, \tau\}}_* L_r^\dagger \text{ for every } r \in R \right]$  implies  $\{L_r\}_{r \in R} \xrightarrow{\{\text{true}, \tau\}}_* \{L_r^\dagger\}_{r \in R}$

*Proof.* See page 95. □

**Definition 36 (Reachability, part 3).**

$$\frac{}{(A, S) \xrightarrow{\Pi, *} (A, S)} \text{ [}\rightarrow^* \text{3-BASE]} \quad \frac{(A, S) \xrightarrow{\Pi, *} (A^*, S^*) \xrightarrow{\pi} (A^\dagger, S^\dagger) \quad \pi \in \Pi}{(A, S) \xrightarrow{\Pi, *} (A^\dagger, S^\dagger)} \text{ [}\rightarrow^* \text{3-STEP]} \\ \frac{(A, S) \xrightarrow{\{\tau\}}_* (A^\dagger, S^\dagger)}{(A, S) \Longrightarrow (A^\dagger, S^\dagger)} \text{ [}\Rightarrow \text{3]}$$

**Lemma 73.**  $A \xrightarrow{\{\text{true}, \tau\}}_* A^\dagger$  implies  $(A, S) \xrightarrow{\{\tau\}}_* (A^\dagger, S)$

*Proof.* See page 96. □

**Lemma 74.**  $A \Longrightarrow A^\dagger$  implies  $(A, S) \Longrightarrow (A^\dagger, S)$

*Proof.* See page 96. □

## 8 Theorems: Deadlock Freedom and Functional Correctness

**Definition 37 (Preconditions of actions).**

$$\begin{aligned}\Phi(\tau, \chi) &= \chi \\ \Phi(q.y := e, \chi) &= \chi[e/q.y] \\ \Phi(p.e \rightarrow q.y, \chi) &= \chi[e/q.y] \\ \Phi(i_R^R, \chi) &= \chi\end{aligned}$$

**Lemma 75.**

1.  $\Phi(\gamma, \neg\chi) = \neg\Phi(\gamma, \chi)$
2.  $\Phi(\gamma, \chi_1 \wedge \chi_2) = \Phi(\gamma, \chi_1) \wedge \Phi(\gamma, \chi_2)$
3.  $\Phi(\gamma, \chi_1 \Rightarrow \chi_2) = \Phi(\gamma, \chi_1) \Rightarrow \Phi(\gamma, \chi_2)$

*Proof.* See page 97. □

**Lemma 76.**  $\text{write}(\gamma) \cap \text{read}(\chi) = \emptyset$  implies  $\Phi(\gamma, \chi) = \chi$

*Proof.* See page 97. □

**Lemma 77.**

$$\left[ \begin{array}{l} \text{read}(\gamma_1) \cap \text{write}(\gamma_2) = \emptyset \text{ and} \\ \text{write}(\gamma_1) \cap \text{read}(\gamma_2) = \emptyset \text{ and} \\ \text{write}(\gamma_1) \cap \text{write}(\gamma_2) = \emptyset \end{array} \right] \text{ implies } \Phi(\gamma_1, \Phi(\gamma_2, \chi)) = \Phi(\gamma_2, \Phi(\gamma_1, \chi))$$

*Proof.* See page 97. □

**Lemma 78.**  $\llbracket \chi_1 \rrbracket \subseteq \llbracket \chi_2 \rrbracket$  implies  $\llbracket \Phi(\gamma, \chi_1) \rrbracket \subseteq \llbracket \Phi(\gamma, \chi_2) \rrbracket$

*Proof.* See page 98. □

**Lemma 79.**  $S \in \llbracket \Phi(\gamma, \chi) \rrbracket$  implies  $\text{effect}(\text{con}_S(\gamma), S) \in \llbracket \chi \rrbracket$

*Proof.* See page 98. □

**Definition 38 (Preconditions of programs).**

$$\begin{aligned}\Phi(\xi, \chi) &= \xi \wedge \chi \\ \Phi(\gamma, \chi) &= \text{Defn. 37} \\ \Phi(\text{skip}, \chi) &= \chi \\ \Phi(G_1 ; G_2, \chi) &= \Phi(G_1, \Phi(G_2, \chi)) \\ \Phi(G_1 \parallel G_2, \chi) &= \begin{cases} \Phi(G_1, \Phi(G_2, \chi)) & \text{if : } G_1, G_2 \in \underline{\mathbb{G}} \text{ and } G_1 \# G_2 \\ \text{false} & \text{else} \end{cases} \\ \Phi(R.\text{if } \xi \ G_1 \ G_2, \chi) &= (\xi^+ \Rightarrow \Phi(G_1, \chi)) \wedge (\xi^- \Rightarrow \Phi(G_2, \chi)) \wedge \xi^\equiv \\ \Phi(R.\text{while } \xi \ \{\psi\} \ G, \chi) &= \psi \wedge \forall(\psi \Rightarrow ((\xi^+ \Rightarrow \Phi(G, \psi)) \wedge (\xi^- \Rightarrow \chi) \wedge \xi^\equiv)) \\ \Phi(\text{if } \xi|_0 \ G_1|_{R_1} \ G_2|_{R_2}, \chi) &= \begin{cases} (\xi^+ \Rightarrow \Phi(G_1, \chi)) \wedge (\xi^- \Rightarrow \Phi(G_2, \chi)) \wedge \xi^\equiv & \text{if : } R_1 = \emptyset = R_2 \\ \Phi(G_2, \chi) \wedge \bigwedge \{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_2} & \text{if : } R_1 = \emptyset \neq R_2 \\ \Phi(G_1, \chi) \wedge \bigwedge \{\xi^+ \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_1} & \text{if : } R_1 \neq \emptyset = R_2 \\ \text{false} & \text{if : } R_1 \neq \emptyset \neq R_2 \end{cases} \\ \Phi(\text{while } \xi|_0 \ \{\psi\} \ G|_{\emptyset}, \chi) &= \psi \wedge \forall(\psi \Rightarrow ((\xi^+ \Rightarrow \Phi(G, \psi)) \wedge (\xi^- \Rightarrow \chi) \wedge \xi^\equiv)) \\ \Phi(\{G\}, \chi) &= \Phi(G, \chi)\end{aligned}$$



**Lemma 80.**  $\llbracket \phi(\text{while } \xi|_0 \{ \psi \} G|_{\emptyset}, \chi) \rrbracket \subseteq \llbracket \phi(\text{if } \xi|_0 (G; \text{while } \xi|_0 \{ \psi \} G|_{\emptyset})|_{\emptyset} \text{skip}|_{\emptyset}, \chi) \rrbracket$

*Proof.* See page 98. □

**Lemma 81.**

1.  $\llbracket \chi_1 \rrbracket \subseteq \llbracket \chi_2 \rrbracket$  **implies**  $\llbracket \phi(G, \chi_1) \rrbracket \subseteq \llbracket \phi(G, \chi_2) \rrbracket$
2.  $\llbracket \chi_1 \rrbracket = \llbracket \chi_2 \rrbracket$  **implies**  $\llbracket \phi(G, \chi_1) \rrbracket = \llbracket \phi(G, \chi_2) \rrbracket$

*Proof.* See page 99. □

**Lemma 82.**  $\llbracket \phi(G, \chi_1 \wedge \chi_2) \rrbracket = \llbracket \phi(G, \chi_1) \wedge \phi(G, \chi_2) \rrbracket$

*Proof.* See page 100. □

**Lemma 83.**  $\llbracket \phi(\underline{G}, \text{false}) \rrbracket = \llbracket \text{false} \rrbracket$

*Proof.* See page 103. □

**Lemma 84.**  $\llbracket \phi(\underline{G}, \chi) \rrbracket \neq \emptyset$  **implies**  $\llbracket \chi \rrbracket \neq \emptyset$

*Proof.* See page 104. □

**Lemma 85.**  $\underline{G} \# \xi$  **implies**  $\llbracket \phi(\underline{G}, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\underline{G}, \xi) \rrbracket$

*Proof.* See page 105. □

**Lemma 86.**  $\underline{G} \# \gamma$  **implies**  $\llbracket \phi(\underline{G}, \phi(\gamma, \chi)) \rrbracket = \llbracket \phi(\gamma, \phi(\underline{G}, \chi)) \rrbracket$

*Proof.* See page 108. □

**Lemma 87.**  $\llbracket \sqrt{R}(G) \text{ and } \llbracket \phi(G, \chi) \rrbracket \neq \emptyset \rrbracket$  **implies**  $\llbracket \text{subj}(G) = R \text{ or } G \in \underline{\mathbb{G}} \rrbracket$

*Proof.* See page 110. □

**Lemma 88.**

1.  $\llbracket \sqrt{R}(G) \text{ and } G \downarrow \rrbracket$  **implies**  $\llbracket \phi(G, \chi) \rrbracket \subseteq \llbracket \chi \rrbracket$
2.  $\llbracket \sqrt{R}(\mathcal{G}) \text{ and } \mathcal{G} \downarrow \rrbracket$  **implies**  $\llbracket \phi(\mathcal{G}, \chi) \rrbracket \subseteq \llbracket \chi \rrbracket$
3.  $\llbracket \sqrt{R}(\mathcal{G}) \text{ and } \mathcal{S} \in \llbracket \phi(\mathcal{G}, \chi) \rrbracket \text{ and } (\mathcal{G}, \mathcal{S}) \downarrow \rrbracket$  **implies**  $\mathcal{S} \in \llbracket \chi \rrbracket$

*Proof.* See page 112. □

**Lemma 89.**

1.  $\llbracket \sqrt{R}(G) \text{ and } \llbracket \phi(G, \chi) \wedge \xi \rrbracket \neq \emptyset \text{ and } G \xrightarrow{\xi, \gamma} G' \rrbracket$  **implies**  $\sqrt{R}(G')$
2.  $\llbracket \sqrt{R}(\mathcal{G}) \text{ and } \llbracket \phi(\mathcal{G}, \chi) \wedge \xi \rrbracket \neq \emptyset \text{ and } \mathcal{G} \xrightarrow{\xi, \gamma} \mathcal{G}' \rrbracket$  **implies**  $\sqrt{R}(\mathcal{G}')$
3.  $\llbracket \sqrt{R}(\mathcal{G}) \text{ and } \mathcal{S} \in \llbracket \phi(\mathcal{G}, \chi) \wedge \xi \rrbracket \text{ and } (\mathcal{G}, \mathcal{S}) \xrightarrow{\gamma} (\mathcal{G}', \mathcal{S}') \rrbracket$  **implies**  $\sqrt{R}(\mathcal{G}')$

*Proof.* See page 113. □

**Lemma 90.**  $\llbracket G \in \underline{\mathbb{G}} \text{ and } \sqrt{R}(G) \text{ and } G \xrightarrow{\xi, \gamma} G' \rrbracket$  **implies**  $\llbracket G' \in \underline{\mathbb{G}} \text{ or } \llbracket \phi(G, \chi) \wedge \xi \rrbracket = \emptyset \rrbracket$

*Proof.* See page 116. □

**Lemma 91.**

1.  $\llbracket \sqrt{R}(G) \text{ and } G \xrightarrow{\xi, \gamma} G' \rrbracket$  **implies**  $\llbracket \phi(G, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G', \chi)) \rrbracket$
2.  $\llbracket \sqrt{R}(\mathcal{G}) \text{ and } \mathcal{G} \xrightarrow{\xi, \gamma} \mathcal{G}' \rrbracket$  **implies**  $\llbracket \phi(\mathcal{G}, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\gamma, \phi(\mathcal{G}', \chi)) \rrbracket$
3.  $\llbracket \sqrt{R}(\mathcal{G}) \text{ and } \mathcal{S} \in \llbracket \phi(\mathcal{G}, \chi) \rrbracket \text{ and } (\mathcal{G}, \mathcal{S}) \xrightarrow{\gamma} (\mathcal{G}', \mathcal{S}') \rrbracket$  **implies**  $\mathcal{S}' \in \llbracket \phi(G', \chi) \rrbracket$

*Proof.* See page 118. □

**Lemma 92.**  $\left[ \begin{array}{l} \sqrt{R}(\mathcal{G}) \text{ and } \mathcal{S} \in \llbracket \Phi(\mathcal{G}, \chi) \rrbracket \\ \text{and } (\mathcal{G}, \mathcal{S}) \xrightarrow{\hat{r}}^* (\mathcal{G}^\dagger, \mathcal{S}^\dagger) \end{array} \right] \text{ implies } \left[ \sqrt{R}(\mathcal{G}^\dagger) \text{ and } \mathcal{S}^\dagger \in \llbracket \Phi(\mathcal{G}^\dagger, \chi) \rrbracket \right]$

*Proof.* See page 124. □

**Lemma 93.**

1.  $\left[ \sqrt{R}(G) \text{ and } \mathcal{S} \in \llbracket \Phi(G, \chi) \rrbracket \right] \text{ implies } \left[ G \downarrow \text{ or } \left[ \begin{array}{l} G \xrightarrow{\xi, \gamma} G' \text{ and } \mathcal{S} \in \llbracket \xi \rrbracket \\ \text{for some } G', \xi, \gamma \end{array} \right] \right]$
2.  $\left[ \sqrt{R}(\mathcal{G}) \text{ and } \mathcal{S} \in \llbracket \Phi(\mathcal{G}, \chi) \rrbracket \right] \text{ implies } \left[ \mathcal{G} \downarrow \text{ or } \left[ \begin{array}{l} \mathcal{G} \xrightarrow{\xi, \gamma} \mathcal{G}' \text{ and } \mathcal{S} \in \llbracket \xi \rrbracket \\ \text{for some } \mathcal{G}', \xi, \gamma \end{array} \right] \right]$
3.  $\left[ \sqrt{R}(\mathcal{G}) \text{ and } \mathcal{S} \in \llbracket \Phi(\mathcal{G}, \chi) \rrbracket \right] \text{ implies } \left[ (\mathcal{G}, \mathcal{S}) \downarrow \text{ or } \left[ \begin{array}{l} (\mathcal{G}, \mathcal{S}) \xrightarrow{\dot{\gamma}} (\mathcal{G}', \mathcal{S}') \\ \text{for some } \mathcal{G}', \mathcal{S}', \dot{\gamma} \end{array} \right] \right]$

*Proof.* See page 124. □

**Theorem 1 (Deadlock freedom).**

$$\left[ \begin{array}{l} \sqrt{R}(\mathcal{G}) \text{ and } \mathcal{S} \in \llbracket \Phi(\mathcal{G}, \chi) \rrbracket \\ \text{and } (\mathcal{G}, \mathcal{S}) \xrightarrow{\hat{r}}^* (\mathcal{G}^\dagger, \mathcal{S}^\dagger) \end{array} \right] \text{ implies } \left[ (\mathcal{G}^\dagger, \mathcal{S}^\dagger) \downarrow \text{ or } \left[ \begin{array}{l} (\mathcal{G}^\dagger, \mathcal{S}^\dagger) \xrightarrow{\dot{\gamma}} (\mathcal{G}^\ddagger, \mathcal{S}^\ddagger) \\ \text{for some } \mathcal{G}^\ddagger, \mathcal{S}^\ddagger, \dot{\gamma} \end{array} \right] \right]$$

*Proof.* See page 127. □

**Theorem 2 (Functional correctness).**

$$\left[ \sqrt{R}(\mathcal{G}) \text{ and } \mathcal{S} \in \llbracket \Phi(\mathcal{G}, \chi) \rrbracket \text{ and } (\mathcal{G}, \mathcal{S}) \xrightarrow{\hat{r}}^* (\mathcal{G}^\dagger, \mathcal{S}^\dagger) \downarrow \right] \text{ implies } \mathcal{S}^\dagger \in \llbracket \chi \rrbracket$$

*Proof.* See page 127. □

## 9 Theorem: Operational Equivalence

**Definition 39 (Weak bisimilarity).**

$$\begin{array}{c}
\left[ \left[ (\mathcal{A}_2, \mathcal{S}_2) \Longrightarrow (\mathcal{A}_2^\dagger, \mathcal{S}_2^\dagger) \downarrow \right] \right. \\
\left. \begin{array}{c} \text{for some } \mathcal{A}_2^\dagger, \mathcal{S}_2^\dagger \\ \text{for every } (\mathcal{A}_1, \mathcal{S}_1) \downarrow \end{array} \right] \quad \left[ (\mathcal{A}'_1, \mathcal{S}'_1) \approx (\mathcal{A}_2, \mathcal{S}_2) \right. \\
\left. \begin{array}{c} \text{for every} \\ (\mathcal{A}_1, \mathcal{S}_1) \xrightarrow{\tau} (\mathcal{A}'_1, \mathcal{S}'_1) \end{array} \right] \\
\left[ \left[ \begin{array}{c} (\mathcal{A}'_1, \mathcal{S}'_1) \approx (\mathcal{A}_2^\natural, \mathcal{S}_2^\natural) \text{ and} \\ [(\mathcal{A}_2, \mathcal{S}_2) \Longrightarrow (\mathcal{A}_2^\dagger, \mathcal{S}_2^\dagger) \xrightarrow{\dot{\gamma}} (\mathcal{A}_2^\ddagger, \mathcal{S}_2^\ddagger) \Longrightarrow (\mathcal{A}_2^\natural, \mathcal{S}_2^\natural)] \\ \text{for some } \mathcal{A}_2^\dagger, \mathcal{A}_2^\ddagger, \mathcal{A}_2^\natural, \mathcal{S}_2^\dagger, \mathcal{S}_2^\ddagger, \mathcal{S}_2^\natural \end{array} \right] \right] \\
\left[ \begin{array}{c} \text{for every } [(\mathcal{A}_1, \mathcal{S}_1) \xrightarrow{\dot{\gamma}} (\mathcal{A}'_1, \mathcal{S}'_1) \text{ and } \dot{\gamma} \neq \tau] \end{array} \right] \\
\left[ \left[ (\mathcal{A}_1, \mathcal{S}_1) \Longrightarrow (\mathcal{A}'_1, \mathcal{S}'_1) \downarrow \right] \right. \\
\left. \begin{array}{c} \text{for some } \mathcal{A}'_1, \mathcal{S}'_1 \\ \text{for every } (\mathcal{A}_2, \mathcal{S}_2) \downarrow \end{array} \right] \quad \left[ (\mathcal{A}'_2, \mathcal{S}'_2) \approx (\mathcal{A}_1, \mathcal{S}_1) \right. \\
\left. \begin{array}{c} \text{for every} \\ (\mathcal{A}_2, \mathcal{S}_2) \xrightarrow{\tau} (\mathcal{A}'_2, \mathcal{S}'_2) \end{array} \right] \\
\left[ \left[ \begin{array}{c} (\mathcal{A}'_1, \mathcal{S}'_1) \approx (\mathcal{A}'_2, \mathcal{S}'_2) \text{ and} \\ [(\mathcal{A}_1, \mathcal{S}_1) \Longrightarrow (\mathcal{A}'_1, \mathcal{S}'_1) \xrightarrow{\dot{\gamma}} (\mathcal{A}'_2, \mathcal{S}'_2) \Longrightarrow (\mathcal{A}'_1, \mathcal{S}'_1)] \\ \text{for some } \mathcal{A}'_1, \mathcal{A}'_2, \mathcal{S}'_1, \mathcal{S}'_2 \end{array} \right] \right] \\
\left[ \begin{array}{c} \text{for every } [(\mathcal{A}_2, \mathcal{S}_2) \xrightarrow{\dot{\gamma}} (\mathcal{A}'_2, \mathcal{S}'_2) \text{ and } \dot{\gamma} \neq \tau] \end{array} \right] \\
\hline
(\mathcal{A}_1, \mathcal{S}_1) \approx (\mathcal{A}_2, \mathcal{S}_2) \quad [\approx]
\end{array}$$

**Definition 40 (Biconvergence).**

$$\frac{L_1 \Longrightarrow L^\dagger \Leftarrow L_2 \text{ for some } L^\dagger}{L_1 \rightleftharpoons L_2} [\rightleftharpoons]$$

**Lemma 94.**  $L_1 \rightleftharpoons L_2$  implies  $L_2 \rightleftharpoons L_1$

*Proof.* See page 128. □

**Lemma 95.**

1.  $L_1 \rightleftharpoons L_2 \downarrow$  implies  $[L_1 \Longrightarrow L^\dagger \downarrow \text{ for some } L^\dagger]$
2.  $L_1 \rightleftharpoons L_2 \xrightarrow{\text{true}, \tau} L'_2$  implies  $L_1 \rightleftharpoons L'_2$
3.  $L_1 \rightleftharpoons L_2 \xrightarrow{\xi, \lambda} L'_2$  implies  $[L_1 \xrightarrow{\xi, \lambda} L'_1 \rightleftharpoons L'_2 \text{ for some } L'_1]$

*Proof.* See page 128. □

**Definition 41 (Correspondence).**

$$\frac{\sqrt{R}(G) \quad \mathcal{S} \in \llbracket \phi(G, \text{true}) \rrbracket \quad (G \upharpoonright r) \rightleftharpoons L_r \text{ for every } r \in R}{(\{G\}, \mathcal{S}) \bowtie (\{L_r\}_{r \in R}, \mathcal{S})} [\bowtie]$$

**Lemma 96.**  $[\sqrt{R}(G) \text{ and } \mathcal{S} \in \llbracket \phi(G, \text{true}) \rrbracket]$  implies  $(G, \mathcal{S}) \bowtie (G \upharpoonright R, \mathcal{S})$

*Proof.* See page 128. □

**Lemma 97.**

1.  $\left[ \begin{array}{c} (\mathcal{G}, \mathcal{S}) \bowtie (\mathcal{L}, \mathcal{S}) \text{ and} \\ (\mathcal{G}, \mathcal{S}) \downarrow \end{array} \right]$  implies  $[(\mathcal{L}, \mathcal{S}) \Longrightarrow (\mathcal{L}^\dagger, \mathcal{S}^\dagger) \downarrow \text{ for some } \mathcal{L}^\dagger, \mathcal{S}^\dagger]$
2.  $\left[ \begin{array}{c} (\mathcal{G}, \mathcal{S}) \bowtie (\mathcal{L}, \mathcal{S}) \text{ and} \\ (\mathcal{G}, \mathcal{S}) \xrightarrow{\tau} (\mathcal{G}', \mathcal{S}') \end{array} \right]$  implies  $(\mathcal{G}', \mathcal{S}') \bowtie (\mathcal{L}, \mathcal{S})$

3.  $\left[ \begin{array}{l} (\mathcal{G}, \mathcal{S}) \bowtie (\mathcal{L}, \mathcal{S}) \text{ and} \\ (\mathcal{G}, \mathcal{S}) \xrightarrow{\dot{\gamma}} (\mathcal{G}', \mathcal{S}') \text{ and } \dot{\gamma} \neq \tau \end{array} \right] \text{ implies } \left[ \left[ \begin{array}{l} (\mathcal{G}', \mathcal{S}') \bowtie (\mathcal{L}^\sharp, \mathcal{S}^\sharp) \text{ and} \\ (\mathcal{L}, \mathcal{S}) \Longrightarrow (\mathcal{L}^\dagger, \mathcal{S}^\dagger) \xrightarrow{\dot{\gamma}} (\mathcal{L}^\ddagger, \mathcal{S}^\ddagger) \Longrightarrow (\mathcal{L}^\sharp, \mathcal{S}^\sharp) \\ \text{for some } \mathcal{L}^\dagger, \mathcal{L}^\ddagger, \mathcal{L}^\sharp, \mathcal{S}^\dagger, \mathcal{S}^\ddagger, \mathcal{S}^\sharp \end{array} \right] \right]$
4.  $\left[ \begin{array}{l} (\mathcal{G}, \mathcal{S}) \bowtie (\mathcal{L}, \mathcal{S}) \text{ and} \\ (\mathcal{L}, \mathcal{S}) \downarrow \end{array} \right] \text{ implies } [(\mathcal{G}, \mathcal{S}) \Longrightarrow (\mathcal{G}^\dagger, \mathcal{S}^\dagger) \downarrow \text{ for some } \mathcal{G}^\dagger, \mathcal{S}^\dagger]$
5.  $\left[ \begin{array}{l} (\mathcal{G}, \mathcal{S}) \bowtie (\mathcal{L}, \mathcal{S}) \text{ and} \\ (\mathcal{L}, \mathcal{S}) \xrightarrow{\tau} (\mathcal{L}', \mathcal{S}') \end{array} \right] \text{ implies } (\mathcal{G}, \mathcal{S}) \bowtie (\mathcal{L}', \mathcal{S}')$
6.  $\left[ \begin{array}{l} (\mathcal{G}, \mathcal{S}) \bowtie (\mathcal{L}, \mathcal{S}) \text{ and} \\ (\mathcal{L}, \mathcal{S}) \xrightarrow{\dot{\gamma}} (\mathcal{L}', \mathcal{S}') \text{ and } \dot{\gamma} \neq \tau \end{array} \right] \text{ implies } \left[ \left[ \begin{array}{l} (\mathcal{G}^\sharp, \mathcal{S}^\sharp) \bowtie (\mathcal{L}', \mathcal{S}') \text{ and} \\ (\mathcal{G}, \mathcal{S}) \Longrightarrow (\mathcal{G}^\dagger, \mathcal{S}^\dagger) \xrightarrow{\dot{\gamma}} (\mathcal{G}^\ddagger, \mathcal{S}^\ddagger) \Longrightarrow (\mathcal{G}^\sharp, \mathcal{S}^\sharp) \\ \text{for some } \mathcal{G}^\dagger, \mathcal{G}^\ddagger, \mathcal{G}^\sharp, \mathcal{S}^\dagger, \mathcal{S}^\ddagger, \mathcal{S}^\sharp \end{array} \right] \right]$

*Proof.* See page 128. □

**Corollary 1.**  $(\mathcal{G}, \mathcal{S}) \bowtie (\mathcal{L}, \mathcal{S})$  implies  $(\mathcal{G}, \mathcal{S}) \approx (\mathcal{L}, \mathcal{S})$

**Theorem 3.**  $[\sqrt{R}(\mathcal{G}) \text{ and } \mathcal{S} \in \llbracket \Phi(\mathcal{G}, \text{true}) \rrbracket]$  implies  $(\mathcal{G}, \mathcal{S}) \approx (\mathcal{G} \upharpoonright R, \mathcal{S})$

*Proof.* See page 131. □



**Part II**

**Proofs**



## A Data



## B Conditions

*Proof (of Lem. 1).* By Defn. 5:

- **Base:**  $\psi = e$ , for some  $e$ .  
Recall  $q.y \notin \text{rx}(\psi)$ . Then,  $q.y \notin \text{rx}(e)$ . Then, by Prop. 1,  $e[d/q.y] = e$ . Then,  $\psi[d/q.y] = \psi$ .
- **Step:**  $\psi = \neg\hat{\psi}$ , for some  $\hat{\psi}$ .  
Recall  $q.y \notin \text{rx}(\psi)$ . Then,  $q.y \notin \text{rx}(\neg\hat{\psi})$ . Then, by Defn. 8,  $q.y \notin \text{rx}(\hat{\psi})$ . Then, by induction,  $\hat{\psi}[d/q.y] = \hat{\psi}$ . Then,  $\neg(\hat{\psi}[d/q.y]) = \neg\hat{\psi}$ . Then, by Defn. 7,  $(\neg\hat{\psi})[d/q.y] = \neg\hat{\psi}$ . Then,  $\psi[d/q.y] = \psi$ .
- **Step:**  $\psi = \psi_1 \wedge \psi_2$ , for some  $\psi_1, \psi_2$ .  
Recall  $q.y \notin \text{rx}(\psi)$ . Then,  $q.y \notin \text{rx}(\psi_1 \wedge \psi_2)$ . Then, by Defn. 8,  $q.y \notin \text{rx}(\psi_1) \cup \text{rx}(\psi_2)$ . Then,  $q.y \notin \text{rx}(\psi_1)$  and  $q.y \notin \text{rx}(\psi_2)$ . Then, by induction,  $\psi_1[d/q.y] = \psi_1$  and  $\psi_2[d/q.y] = \psi_2$ . Then,  $\psi_1[d/q.y] \wedge \psi_2[d/q.y] = \psi_1 \wedge \psi_2$ . Then, by Defn. 7,  $(\psi_1 \wedge \psi_2)[d/q.y] = \psi_1 \wedge \psi_2$ . Then,  $\psi[d/q.y] = \psi$ .
- **Step:**  $\psi = \forall\hat{\psi}$ , for some  $\hat{\psi}$ .  
By Defn. 7,  $(\forall\hat{\psi})[d/q.y] = \forall\hat{\psi}$ . Then,  $\psi[d/q.y] = \psi$ . □

*Proof (of Lem. 2).* By Defn. 5:

- **Base:**  $\psi = e$ , for some  $e$ .  
Recall  $q_1.y_1 \neq q_2.y_2$  and  $q_1.y_1 \notin \text{rx}(d_2)$  and  $q_2.y_2 \notin \text{rx}(d_1)$ . Then, by Prop. 2,  $e[d_1/q_1.y_1][d_2/q_2.y_2] = e[d_2/q_2.y_2][d_1/q_1.y_1]$ . Then,  $\psi[d_1/q_1.y_1][d_2/q_2.y_2] = \psi[d_2/q_2.y_2][d_1/q_1.y_1]$ .
- **Step:**  $\psi = \neg\hat{\psi}$ , for some  $\hat{\psi}$ .  
Recall  $q_1.y_1 \neq q_2.y_2$  and  $q_1.y_1 \notin \text{rx}(d_2)$  and  $q_2.y_2 \notin \text{rx}(d_1)$ . Then, by induction,  $\hat{\psi}[d_1/q_1.y_1][d_2/q_2.y_2] = \hat{\psi}[d_2/q_2.y_2][d_1/q_1.y_1]$ . Then,  $\neg(\hat{\psi}[d_1/q_1.y_1][d_2/q_2.y_2]) = \neg(\hat{\psi}[d_2/q_2.y_2][d_1/q_1.y_1])$ . Then, by Defn. 7,  $(\neg(\hat{\psi}[d_1/q_1.y_1][d_2/q_2.y_2]))[d_1/q_1.y_1] = (\neg(\hat{\psi}[d_2/q_2.y_2][d_1/q_1.y_1]))[d_1/q_1.y_1]$ . Then, by Defn. 7,  $(\neg\hat{\psi})[d_1/q_1.y_1][d_2/q_2.y_2] = (\neg\hat{\psi})[d_2/q_2.y_2][d_1/q_1.y_1]$ . Then,  $\psi[d_1/q_1.y_1][d_2/q_2.y_2] = \psi[d_2/q_2.y_2][d_1/q_1.y_1]$ .
- **Step:**  $\psi = \psi_1 \wedge \psi_2$ , for some  $\psi_1, \psi_2$ .  
Recall  $q_1.y_1 \neq q_2.y_2$  and  $q_1.y_1 \notin \text{rx}(d_2)$  and  $q_2.y_2 \notin \text{rx}(d_1)$ . Then, by induction,  $\psi_1[d_1/q_1.y_1][d_2/q_2.y_2] = \psi_1[d_2/q_2.y_2][d_1/q_1.y_1]$  and  $\psi_2[d_1/q_1.y_1][d_2/q_2.y_2] = \psi_2[d_2/q_2.y_2][d_1/q_1.y_1]$ . Then,  $\psi_1[d_1/q_1.y_1][d_2/q_2.y_2] \wedge \psi_2[d_1/q_1.y_1][d_2/q_2.y_2] = \psi_1[d_2/q_2.y_2][d_1/q_1.y_1] \wedge \psi_2[d_2/q_2.y_2][d_1/q_1.y_1]$ . Then, by Defn. 7,  $(\psi_1[d_1/q_1.y_1] \wedge \psi_2[d_1/q_1.y_1])[d_2/q_2.y_2] = (\psi_1[d_2/q_2.y_2] \wedge \psi_2[d_2/q_2.y_2])[d_1/q_1.y_1]$ . Then, by Defn. 7,  $(\psi_1 \wedge \psi_2)[d_1/q_1.y_1][d_2/q_2.y_2] = (\psi_1 \wedge \psi_2)[d_2/q_2.y_2][d_1/q_1.y_1]$ . Then,  $\psi[d_1/q_1.y_1][d_2/q_2.y_2] = \psi[d_2/q_2.y_2][d_1/q_1.y_1]$ .
- **Step:**  $\psi = \forall\hat{\psi}$ , for some  $\hat{\psi}$ .  
Recall  $\forall\hat{\psi} = \forall\hat{\psi}$ . Then, by Defn. 7,  $(\forall\hat{\psi})[d_2/q_2.y_2] = (\forall\hat{\psi})[d_1/q_1.y_1]$ . Then, by Defn. 7,  $(\forall\hat{\psi})[d_1/q_1.y_1][d_2/q_2.y_2] = (\forall\hat{\psi})[d_2/q_2.y_2][d_1/q_1.y_1]$ . Then,  $\psi[d_1/q_1.y_1][d_2/q_2.y_2] = \psi[d_2/q_2.y_2][d_1/q_1.y_1]$ . □

*Proof (of Lem. 3).*

1. By Defn. 3,  $\text{rx}(\text{false}) = \emptyset = \text{rx}(\text{true})$ . Then, by Defn. 9,  $\text{read}(\text{false}) = \emptyset = \text{read}(\text{true})$ .
2. By Defn. 9,  $\text{read}(\neg\psi) = \text{rx}(\neg\psi)$ . Then, by Defn. 8,  $\text{read}(\neg\psi) = \text{rx}(\psi)$ . Then, by Defn. 9,  $\text{read}(\neg\psi) = \text{read}(\psi)$ . □
3. Similar to 2. □
4. Recall  $\text{read}(\psi_1 \Rightarrow \psi_2) = \text{read}(\psi_1 \Rightarrow \psi_2)$ . Then, by Defn. 6,  $\text{read}(\psi_1 \Rightarrow \psi_2) = \text{read}(\neg(\psi_1 \wedge \neg\psi_2))$ . Then, by 1,  $\text{read}(\psi_1 \Rightarrow \psi_2) = \text{read}(\psi_1 \wedge \neg\psi_2)$ . Then, by 2,  $\text{read}(\psi_1 \Rightarrow \psi_2) = \text{read}(\psi_1) \cup \text{read}(\neg\psi_2)$ . Then, by 1,  $\text{read}(\psi_1 \Rightarrow \psi_2) = \text{read}(\psi_1) \cup \text{read}(\psi_2)$ . □

5. By Defn. 9,  $\text{write}(\text{false}) = \emptyset = \text{write}(\text{true})$ . □
6. Recall  $\emptyset = \emptyset$ . Then, by Defn. 9,  $\text{write}(\neg\psi) = \text{write}(\psi)$ . □
7. Similar to 6. □
8. Recall  $\emptyset = \emptyset \cup \emptyset$ . Then, by Defn. 9,  $\text{write}(\neg(\psi_1 \wedge \neg\psi_2)) = \text{write}(\psi_1) \cup \text{write}(\psi_2)$ . Then, by Defn. 6,  $\text{write}(\psi_1 \Rightarrow \psi_2) = \text{write}(\psi_1) \cup \text{write}(\psi_2)$ . □

*Proof (of Lem. 4).*

1. Recall  $\{r \mid r.x \in \emptyset\} = \emptyset = \{r \mid r.x \in \emptyset\}$ . Then, by Defn. 3,  $\{r \mid r.x \in \text{rx}(\text{false})\} = \emptyset = \{r \mid r.x \in \text{rx}(\text{true})\}$ . Then, by Defn. 10,  $\text{subj}(\text{false}) = \emptyset = \text{subj}(\text{true})$ . □
2. By Defn. 10,  $\text{subj}(\neg\psi) = \{r \mid r.x \in \text{rx}(\neg\psi)\}$ . Then, by Defn. 8,  $\text{subj}(\neg\psi) = \{r \mid r.x \in \text{rx}(\psi)\}$ . Then, by Defn. 10,  $\text{subj}(\neg\psi) = \text{subj}(\psi)$ . □
3. By Defn. 10,  $\text{subj}(\psi_1 \wedge \psi_2) = \{r \mid r.x \in \text{rx}(\psi_1 \wedge \psi_2)\}$ . Then, by Defn. 8,  $\text{subj}(\psi_1 \wedge \psi_2) = \{r \mid r.x \in \text{rx}(\psi_1) \cup \text{rx}(\psi_2)\}$ . Then,  $\text{subj}(\psi_1 \wedge \psi_2) = \{r \mid r.x \in \text{rx}(\psi_1)\} \cup \{r \mid r.x \in \text{rx}(\psi_2)\}$ . Then, by Defn. 10,  $\text{subj}(\psi_1 \wedge \psi_2) = \text{subj}(\psi_1) \cup \text{subj}(\psi_2)$ . □
4. Recall  $\text{subj}(\psi_1 \Rightarrow \psi_2) = \text{subj}(\psi_1 \Rightarrow \psi_2)$ . Then, by Defn. 6,  $\text{subj}(\psi_1 \Rightarrow \psi_2) = \text{subj}(\neg(\psi_1 \wedge \neg\psi_2))$ . Then, by 1,  $\text{subj}(\psi_1 \Rightarrow \psi_2) = \text{subj}(\psi_1 \wedge \neg\psi_2)$ . Then, by 2,  $\text{subj}(\psi_1 \Rightarrow \psi_2) = \text{subj}(\psi_1) \cup \text{subj}(\neg\psi_2)$ . Then, by 1,  $\text{subj}(\psi_1 \Rightarrow \psi_2) = \text{subj}(\psi_1) \cup \text{subj}(\psi_2)$ . □

*Proof (of Lem. 5).*

1. Recall  $\text{rx}(\psi) \subseteq \text{rx}(\psi)$ . Then,  $\text{rx}(\psi) \subseteq \{r.x \mid r \in \{\hat{r} \mid \hat{r}.\hat{x} \in \text{rx}(\psi)\}\}$ . Then, by Defn. 9,  $\text{read}(\psi) \subseteq \{r.x \mid r \in \{\hat{r} \mid \hat{r}.\hat{x} \in \text{rx}(\psi)\}\}$ . Then, by Defn. 10,  $\text{read}(\psi) \subseteq \{r.x \mid r \in \text{subj}(\psi)\}$ . □
2. Recall  $\emptyset \subseteq \{r.x \mid r \in \text{subj}(\psi)\}$ . Then, by Defn. 9,  $\text{write}(\psi) \subseteq \{r.x \mid r \in \text{subj}(\psi)\}$ . □

*Proof (of Lem. 6).* By Defn. 5:

- **Base:**  $\xi = \text{true}$ .  
Recall  $(\xi \upharpoonright r) \upharpoonright r = (\xi \upharpoonright r) \upharpoonright r$ . Then,  $(\xi \upharpoonright r) \upharpoonright r = (\text{true} \upharpoonright r) \upharpoonright r$ . Then, by Defn. 11,  $(\xi \upharpoonright r) \upharpoonright r = \text{true} \upharpoonright r$ . Then,  $(\xi \upharpoonright r) \upharpoonright r = \xi \upharpoonright r$ .
- **Base:**  $\xi = \bigwedge \{e_{\hat{r}}\}_{\hat{r} \in R}$ , for some  $\{e_{\hat{r}}\}_{\hat{r} \in R}$ .  
By Defn. 11:  
  - **Case:**  $\bigwedge \{e_{\hat{r}}\}_{\hat{r} \in R} \upharpoonright r = e_r$ .  
Recall  $r \in \{r\}$ . Then, by Defn. 11,  $e_r \upharpoonright r = e_r$ . Then,  $(\bigwedge \{e_{\hat{r}}\}_{\hat{r} \in R} \upharpoonright r) \upharpoonright r = \bigwedge \{e_{\hat{r}}\}_{\hat{r} \in R} \upharpoonright r$ . Then,  $(\xi \upharpoonright r) \upharpoonright r = \xi \upharpoonright r$ .
  - **Case:**  $\bigwedge \{e_{\hat{r}}\}_{\hat{r} \in R} \upharpoonright r = \text{true}$ .  
By Defn. 11,  $\text{true} \upharpoonright r = \text{true}$ . Then,  $(\bigwedge \{e_{\hat{r}}\}_{\hat{r} \in R} \upharpoonright r) \upharpoonright r = \bigwedge \{e_{\hat{r}}\}_{\hat{r} \in R} \upharpoonright r$ . Then,  $(\xi \upharpoonright r) \upharpoonright r = \xi \upharpoonright r$ .
- **Base:**  $\xi = \bigwedge \{-e_{\hat{r}}\}_{\hat{r} \in R}$ , for some  $\{e_{\hat{r}}\}_{\hat{r} \in R}$ . Similar to case “ $\xi = \bigwedge \{e_{\hat{r}}\}_{\hat{r} \in R}$ , for some  $\{e_{\hat{r}}\}_{\hat{r} \in R}$ ”. □

*Proof (of Lem. 7).* By Defn. 5:

- **Base:**  $\xi = \text{true}$ .  
Recall  $\emptyset \subseteq \text{rx}(\xi)$ . Then, by Defn. 3,  $\text{rx}(\text{true}) \subseteq \text{rx}(\xi)$ . Then, by Defn. 11,  $\text{rx}(\text{true} \upharpoonright r) \subseteq \text{rx}(\xi)$ . Then,  $\text{rx}(\xi \upharpoonright r) \subseteq \text{rx}(\xi)$ .
- **Base:**  $\xi = \bigwedge \{e_{\hat{r}}\}_{\hat{r} \in R}$ , for some  $\{e_{\hat{r}}\}_{\hat{r} \in R}$ .  
By Defn. 11:  
  - **Case:**  $\bigwedge \{e_{\hat{r}}\}_{\hat{r} \in R} \upharpoonright r = e_r$  and  $r \in R$ .  
    - \* Recall  $r \in R$ . Then,  $R = (R \setminus \{r\}) \cup \{r\}$ .
    - \* Recall  $\text{rx}(e_r) \subseteq \bigcup \{\text{rx}(e_{\hat{r}})\}_{\hat{r} \in R \setminus \{r\}} \cup \text{rx}(e_r)$ . Then,  $\text{rx}(e_r) \subseteq \bigcup \{\text{rx}(e_{\hat{r}})\}_{\hat{r} \in (R \setminus \{r\}) \cup \{r\}}$ . Then,  $\text{rx}(e_r) \subseteq \bigcup \{\text{rx}(e_{\hat{r}})\}_{\hat{r} \in R}$ . Then, by Defn. 8,  $\text{rx}(e_r) \subseteq \text{rx}(\bigwedge \{e_{\hat{r}}\}_{\hat{r} \in R})$ . Then,  $\text{rx}(\bigwedge \{e_{\hat{r}}\}_{\hat{r} \in R} \upharpoonright r) \subseteq \text{rx}(\bigwedge \{e_{\hat{r}}\}_{\hat{r} \in R})$ . Then,  $\text{rx}(\xi \upharpoonright r) \subseteq \text{rx}(\xi)$ .

- **Case:**  $\bigwedge\{e_{\hat{r}}\}_{\hat{r} \in R} \uparrow r = \mathbf{true}$ .  
Recall  $\emptyset \subseteq \mathbf{rx}(\xi)$ . Then, by Defn. 3,  $\mathbf{rx}(\mathbf{true}) \subseteq \mathbf{rx}(\xi)$ . Then,  $\mathbf{rx}(\bigwedge\{e_{\hat{r}}\}_{\hat{r} \in R} \uparrow r) \subseteq \mathbf{rx}(\xi)$ .  
Then,  $\mathbf{rx}(\xi \uparrow r) \subseteq \mathbf{rx}(\xi)$ .
- **Base:**  $\xi = \bigwedge\{\neg e_{\hat{r}}\}_{\hat{r} \in R}$ , for some  $\{e_{\hat{r}}\}_{\hat{r} \in R}$ . Similar to case “ $\xi = \bigwedge\{e_{\hat{r}}\}_{\hat{r} \in R}$ , for some  $\{e_{\hat{r}}\}_{\hat{r} \in R}$ ”. □

*Proof (of Lem. 8).*

1. By Lem. 7,  $\mathbf{rx}(\xi \uparrow r) \subseteq \mathbf{rx}(\xi)$ . Then, by Defn. 9,  $\mathbf{read}(\xi \uparrow r) \subseteq \mathbf{read}(\xi)$ . □
2. Recall  $\emptyset \subseteq \emptyset$ . Then, by Defn. 9,  $\mathbf{write}(\xi \uparrow r) \subseteq \mathbf{write}(\xi)$ . □

*Proof (of Lem. 9).*

1. By Defn. 5:
  - **Base:**  $\xi = \mathbf{true}$ .  
Recall  $r \in \mathbf{subj}(\xi)$ . Then,  $r \in \mathbf{subj}(\mathbf{true})$ . Then, by Lem. 4,  $r \in \emptyset$ . Then, **false**.
  - **Base:**  $\xi = \bigwedge\{e_{\hat{r}}\}_{\hat{r} \in R}$ , for some  $\{e_{\hat{r}}\}_{\hat{r} \in R}$ .
    - Recall  $\xi = \bigwedge\{e_{\hat{r}}\}_{\hat{r} \in R}$ . Then, by Rem. 1,  $\{\hat{r} \mid \hat{r}.\hat{x} \in \mathbf{rx}(e_{\hat{r}})\} = \{\hat{r}\}$  for every  $\hat{r} \in R$ .  
Then, by Defn. 10,  $\mathbf{subj}(e_{\hat{r}}) = \{\hat{r}\}$  for every  $\hat{r} \in R$ .
    - Recall  $r \in \mathbf{subj}(\xi)$ . Then,  $r \in \mathbf{subj}(\bigwedge\{e_{\hat{r}}\}_{\hat{r} \in R})$ . Then, by Lem. 4,  $r \in \bigcup\{\mathbf{subj}(e_{\hat{r}})\}_{\hat{r} \in R}$ .  
Then,  $r \in \bigcup\{\{\hat{r}\}\}_{\hat{r} \in R}$ . Then,  $r \in \{\hat{r} \mid \hat{r} \in R\}$ . Then,  $r \in R$ .
    - Recall  $r \in R$ . Then, by Defn. 11,  $\bigwedge\{e_{\hat{r}}\}_{\hat{r} \in R} \uparrow r = e_r$ .
    - Recall  $r \in R$ . Then,  $\mathbf{subj}(e_r) = \{r\}$ .
    - Recall  $\mathbf{subj}(\xi \uparrow r) = \mathbf{subj}(\xi \uparrow r)$ . Then,  $\mathbf{subj}(\xi \uparrow r) = \mathbf{subj}(\bigwedge\{e_{\hat{r}}\}_{\hat{r} \in R} \uparrow r)$ . Then,  $\mathbf{subj}(\xi \uparrow r) = \mathbf{subj}(e_r)$ . Then,  $\mathbf{subj}(\xi \uparrow r) = \{r\}$ .
  - **Base:**  $\xi = \bigwedge\{\neg e_{\hat{r}}\}_{\hat{r} \in R}$ , for some  $\{e_{\hat{r}}\}_{\hat{r} \in R}$ . Similar to case “ $\xi = \bigwedge\{e_{\hat{r}}\}_{\hat{r} \in R}$ , for some  $\{e_{\hat{r}}\}_{\hat{r} \in R}$ ”. □
2. By Defn. 5:
  - **Base:**  $\xi = \mathbf{true}$ .  
By Lem. 4,  $\mathbf{subj}(\mathbf{true}) = \emptyset$ . Then, by Defn. 11,  $\mathbf{subj}(\mathbf{true} \uparrow r) = \emptyset$ . Then,  $\mathbf{subj}(\xi \uparrow r) = \emptyset$ .
  - **Base:**  $\xi = \bigwedge\{e_{\hat{r}}\}_{\hat{r} \in R}$ , for some  $\{e_{\hat{r}}\}_{\hat{r} \in R}$ .
    - Recall  $\xi = \bigwedge\{e_{\hat{r}}\}_{\hat{r} \in R}$ . Then, by Rem. 1,  $\{\hat{r} \mid \hat{r}.\hat{x} \in \mathbf{rx}(e_{\hat{r}})\} = \{\hat{r}\}$  for every  $\hat{r} \in R$ .  
Then, by Defn. 10,  $\mathbf{subj}(e_{\hat{r}}) = \{\hat{r}\}$  for every  $\hat{r} \in R$ .
    - Recall  $r \notin \mathbf{subj}(\xi)$ . Then,  $r \notin \mathbf{subj}(\bigwedge\{e_{\hat{r}}\}_{\hat{r} \in R})$ . Then, by Lem. 4,  $r \notin \bigcup\{\mathbf{subj}(e_{\hat{r}})\}_{\hat{r} \in R}$ .  
Then,  $r \notin \bigcup\{\{\hat{r}\}\}_{\hat{r} \in R}$ . Then,  $r \notin \{\hat{r} \mid \hat{r} \in R\}$ . Then,  $r \notin R$ . Then, by Defn. 11,  $\bigwedge\{e_{\hat{r}}\}_{\hat{r} \in R} \uparrow r = \mathbf{true}$ .
    - By Lem. 4,  $\mathbf{subj}(\mathbf{true}) = \emptyset$ . Then, by Defn. 11,  $\mathbf{subj}(\bigwedge\{e_{\hat{r}}\}_{\hat{r} \in R} \uparrow r) = \emptyset$ . Then,  $\mathbf{subj}(\xi \uparrow r) = \emptyset$ .
  - **Base:**  $\xi = \bigwedge\{\neg e_{\hat{r}}\}_{\hat{r} \in R}$ , for some  $\{e_{\hat{r}}\}_{\hat{r} \in R}$ . Similar to case “ $\xi = \bigwedge\{e_{\hat{r}}\}_{\hat{r} \in R}$ , for some  $\{e_{\hat{r}}\}_{\hat{r} \in R}$ ”. □

*Proof (of Lem. 10).*

1. Recall:
  - **Case:**  $r \in \mathbf{subj}(\xi)$ .
    - Recall  $r \in \mathbf{subj}(\xi)$ . Then, by Lem. 9,  $\mathbf{subj}(\xi \uparrow r) = \{r\}$ .
    - Recall  $r \in \mathbf{subj}(\xi)$ . Then,  $\{r\} \subseteq \mathbf{subj}(\xi)$ . Then,  $\mathbf{subj}(\xi \uparrow r) \subseteq \mathbf{subj}(\xi)$ .
  - **Case:**  $r \notin \mathbf{subj}(\xi)$ .
    - Recall  $r \notin \mathbf{subj}(\xi)$ . Then, by Lem. 9,  $\mathbf{subj}(\xi \uparrow r) = \emptyset$ .
    - Recall  $\emptyset \subseteq \mathbf{subj}(\xi)$ . Then,  $\mathbf{subj}(\xi \uparrow r) \subseteq \mathbf{subj}(\xi)$ . □
2. Recall:
  - **Case:**  $r \in \mathbf{subj}(\xi)$ .  
Recall  $r \in \mathbf{subj}(\xi)$ . Then, by Lem. 9,  $\mathbf{subj}(\xi \uparrow r) = \{r\}$ . Then,  $\mathbf{subj}(\xi \uparrow r) \subseteq \{r\}$ .
  - **Case:**  $r \notin \mathbf{subj}(\xi)$ .

- Recall  $r \notin \text{subj}(\xi)$ . Then, by Lem. 9,  $\text{subj}(\xi \uparrow r) = \emptyset$ .
- Recall  $\emptyset \subseteq \{r\}$ . Then,  $\text{subj}(\xi \uparrow r) \subseteq \{r\}$ . □

*Proof (of Lem. 11).*

1. By Defn. 12,  $(\xi \uparrow r)^+ = \xi \uparrow r = \xi^+ \uparrow r$ . Then,  $(\xi \uparrow r)^+ = \xi^+ \uparrow r$ . □
2. By Defn. 5:
  - **Case:**  $\xi = \text{true}$ .  
 Recall  $(\xi \uparrow r)^- = (\xi \uparrow r)^-$ . Then,  $(\xi \uparrow r)^- = (\text{true} \uparrow r)^-$ . Then, by Defn. 11,  $(\xi \uparrow r)^- = \text{true}^-$ . Then, by Defn. 12,  $(\xi \uparrow r)^- = \text{true}$ . Then, by Defn. 11,  $(\xi \uparrow r)^- = \text{true} \uparrow r$ . Then, by Defn. 12,  $(\xi \uparrow r)^- = \text{true}^- \uparrow r$ . Then,  $(\xi \uparrow r)^- = \xi^- \uparrow r$ .
  - **Case:**  $\xi = \bigwedge \{e_{\hat{r}}\}_{\hat{r} \in R}$ , for some  $\{e_{\hat{r}}\}_{\hat{r} \in R}$ .  
 Recall:
    - **Case:**  $r \in R$ .
      - \* Recall  $r \in R$ . Then, by Defn. 11,  $\bigwedge \{e_{\hat{r}}\}_{\hat{r} \in R} \uparrow r = e_r$  and  $\bigwedge \{\neg e_{\hat{r}}\}_{\hat{r} \in R} \uparrow r = \neg e_r$ .
      - \* Recall  $(\xi \uparrow r)^- = (\xi \uparrow r)^-$ . Then,  $(\xi \uparrow r)^- = (\bigwedge \{e_{\hat{r}}\}_{\hat{r} \in R} \uparrow r)^-$ . Then,  $(\xi \uparrow r)^- = e_r^-$ . Then, by Defn. 12,  $(\xi \uparrow r)^- = \neg e_r$ . Then,  $(\xi \uparrow r)^- = \bigwedge \{\neg e_{\hat{r}}\}_{\hat{r} \in R} \uparrow r$ . Then, by Defn. 12,  $(\xi \uparrow r)^- = \bigwedge \{e_{\hat{r}}\}_{\hat{r} \in R}^- \uparrow r$ . Then,  $(\xi \uparrow r)^- = \xi^- \uparrow r$ .
    - **Case:**  $\bigwedge \{e_{\hat{r}}\}_{\hat{r} \in R} \uparrow r = \text{true}$ .
      - \* Recall  $r \notin R$ . Then, by Defn. 11,  $\bigwedge \{e_{\hat{r}}\}_{\hat{r} \in R} \uparrow r = \text{true}$  and  $\bigwedge \{\neg e_{\hat{r}}\}_{\hat{r} \in R} \uparrow r = \text{true}$ .
      - \* Recall  $(\xi \uparrow r)^- = (\xi \uparrow r)^-$ . Then,  $(\xi \uparrow r)^- = (\bigwedge \{e_{\hat{r}}\}_{\hat{r} \in R} \uparrow r)^-$ . Then,  $(\xi \uparrow r)^- = \text{true}$ . Then,  $(\xi \uparrow r)^- = \bigwedge \{\neg e_{\hat{r}}\}_{\hat{r} \in R} \uparrow r$ . Then, by Defn. 12,  $(\xi \uparrow r)^- = \bigwedge \{e_{\hat{r}}\}_{\hat{r} \in R}^- \uparrow r$ . Then,  $(\xi \uparrow r)^- = \xi^- \uparrow r$ .
  - **Case:**  $\xi = \bigwedge \{\neg e_{\hat{r}}\}_{\hat{r} \in R}$ , for some  $\{e_{\hat{r}}\}_{\hat{r} \in R}$ . Similar to case “ $\xi = \bigwedge \{e_{\hat{r}}\}_{\hat{r} \in R}$ , for some  $\{e_{\hat{r}}\}_{\hat{r} \in R}$ ”. □

*Proof (of Lem. 12).*

1. By Defn. 12,  $\xi^+ = \xi$ . Then,  $\text{read}(\xi^+) = \text{read}(\xi)$ . □
2. By Defn. 5:
  - **Base:**  $\xi = \text{true}$ .  
 Recall  $\text{read}(\xi^-) = \text{read}(\xi^-)$ . Then,  $\text{read}(\xi^-) = \text{read}(\text{true}^-)$ . Then, by Defn. 12,  $\text{read}(\xi^-) = \text{read}(\text{true})$ . Then,  $\text{read}(\xi^-) = \text{read}(\xi)$ .
  - **Base:**  $\xi = \bigwedge \{e_r\}_{r \in R}$ , for some  $\{e_r\}_{r \in R}$ .  
 Recall  $\text{read}(\xi^-) = \text{read}(\xi^-)$ . Then,  $\text{read}(\xi^-) = \text{read}(\bigwedge \{e_r\}_{r \in R}^-)$ . Then, by Defn. 12,  $\text{read}(\xi^-) = \text{read}(\bigwedge \{\neg e_r\}_{r \in R})$ . Then, by Lem. 3,  $\text{read}(\xi^-) = \bigcup \{\text{read}(\neg e_r)\}_{r \in R}$ . Then, by Lem. 3,  $\text{read}(\xi^-) = \bigcup \{\text{read}(e_r)\}_{r \in R}$ . Then, by Lem. 3,  $\text{read}(\xi^-) = \text{read}(\bigwedge \{e_r\}_{r \in R})$ . Then,  $\text{read}(\xi^-) = \text{read}(\xi)$ .
  - **Base:**  $\xi = \bigwedge \{\neg e_r\}_{r \in R}$ , for some  $\{e_r\}_{r \in R}$ . Similar to case “ $\xi = \bigwedge \{e_r\}_{r \in R}$ , for some  $\{e_r\}_{r \in R}$ ”. □
3. By Defn. 5:
  - **Base:**  $\xi = \text{true}$ .  
 Recall  $\text{read}(\xi^{\equiv}) = \text{read}(\xi^{\equiv})$ . Then,  $\text{read}(\xi^{\equiv}) = \text{read}(\text{true}^{\equiv})$ . Then, by Defn. 12,  $\text{read}(\xi^{\equiv}) = \text{read}(\text{true})$ . Then,  $\text{read}(\xi^{\equiv}) = \text{read}(\xi)$ .
  - **Base:**  $\xi = \bigwedge \{e_r\}_{r \in R}$ , for some  $\{e_r\}_{r \in R}$ .  
 Recall  $\text{read}(\xi^{\equiv}) = \text{read}(\xi^{\equiv})$ . Then,  $\text{read}(\xi^{\equiv}) = \text{read}(\bigwedge \{e_r\}_{r \in R}^{\equiv})$ . Then, by Defn. 12,  $\text{read}(\xi^{\equiv}) = \text{read}(\bigwedge \{e_{r_1} \Rightarrow e_{r_2}\}_{r_1, r_2 \in R})$ . Then, by Lem. 3,  $\text{read}(\xi^{\equiv}) = \bigcup \{\text{read}(e_{r_1} \Rightarrow e_{r_2})\}_{r_1, r_2 \in R}$ . Then, by Lem. 3,  $\text{read}(\xi^{\equiv}) = \bigcup \{\text{read}(e_{r_1}) \cup \text{read}(e_{r_2})\}_{r_1, r_2 \in R}$ . Then,  $\text{read}(\xi^{\equiv}) = \bigcup \{\text{read}(e_r)\}_{r \in R}$ . Then, by Lem. 3,  $\text{read}(\xi^{\equiv}) = \text{read}(\bigwedge \{e_r\}_{r \in R})$ . Then,  $\text{read}(\xi^{\equiv}) = \text{read}(\xi)$ .
  - **Base:**  $\xi = \bigwedge \{\neg e_r\}_{r \in R}$ , for some  $\{e_r\}_{r \in R}$ . Similar to case “ $\xi = \bigwedge \{e_r\}_{r \in R}$ , for some  $\{e_r\}_{r \in R}$ ”. □
4. By Defn. 9,  $\text{write}(\xi^+) = \emptyset = \text{write}(\xi)$ . Then,  $\text{write}(\xi^+) = \text{write}(\xi)$ . □
5. Similar to 4. □

6. Similar to 4. □

*Proof (of Lem. 13).*

1. By Defn. 12,  $\xi^+ = \xi$ . Then,  $\text{subj}(\xi^+) = \text{subj}(\xi)$ . □
2. By Defn. 5:
  - **Base:**  $\xi = \text{true}$ .  
Recall  $\text{subj}(\xi^-) = \text{subj}(\xi^-)$ . Then,  $\text{subj}(\xi^-) = \text{subj}(\text{true}^-)$ . Then, by Defn. 12,  $\text{subj}(\xi^-) = \text{subj}(\text{true})$ . Then,  $\text{subj}(\xi^-) = \text{subj}(\xi)$ .
  - **Base:**  $\xi = \bigwedge\{e_r\}_{r \in R}$ , for some  $\{e_r\}_{r \in R}$ .  
Recall  $\text{subj}(\xi^-) = \text{subj}(\xi^-)$ . Then,  $\text{subj}(\xi^-) = \text{subj}(\bigwedge\{e_r\}_{r \in R}^-)$ . Then, by Defn. 12,  $\text{subj}(\xi^-) = \text{subj}(\bigwedge\{\neg e_r\}_{r \in R})$ . Then, by Lem. 4,  $\text{subj}(\xi^-) = \bigcup\{\text{subj}(\neg e_r)\}_{r \in R}$ . Then, by Lem. 4,  $\text{subj}(\xi^-) = \bigcup\{\text{subj}(e_r)\}_{r \in R}$ . Then, by Lem. 4,  $\text{subj}(\xi^-) = \text{subj}(\bigwedge\{e_r\}_{r \in R})$ . Then,  $\text{subj}(\xi^-) = \text{subj}(\xi)$ .
  - **Base:**  $\xi = \bigwedge\{\neg e_r\}_{r \in R}$ , for some  $\{e_r\}_{r \in R}$ . Similar to case “ $\xi = \bigwedge\{e_r\}_{r \in R}$ , for some  $\{e_r\}_{r \in R}$ ”. □
3. By Defn. 5:
  - **Base:**  $\xi = \text{true}$ .  
Recall  $\text{subj}(\xi^\equiv) = \text{subj}(\xi^\equiv)$ . Then,  $\text{subj}(\xi^\equiv) = \text{subj}(\text{true}^\equiv)$ . Then, by Defn. 12,  $\text{subj}(\xi^\equiv) = \text{subj}(\text{true})$ . Then,  $\text{subj}(\xi^\equiv) = \text{subj}(\xi)$ .
  - **Base:**  $\xi = \bigwedge\{e_r\}_{r \in R}$ , for some  $\{e_r\}_{r \in R}$ .  
Recall  $\text{subj}(\xi^\equiv) = \text{subj}(\xi^\equiv)$ . Then,  $\text{subj}(\xi^\equiv) = \text{subj}(\bigwedge\{e_r\}_{r \in R}^\equiv)$ . Then, by Defn. 12,  $\text{subj}(\xi^\equiv) = \text{subj}(\bigwedge\{e_{r_1} \Rightarrow e_{r_2}\}_{r_1, r_2 \in R})$ . Then, by Lem. 4,  $\text{subj}(\xi^\equiv) = \bigcup\{\text{subj}(e_{r_1} \Rightarrow e_{r_2})\}_{r_1, r_2 \in R}$ . Then, by Lem. 4,  $\text{subj}(\xi^\equiv) = \bigcup\{\text{subj}(e_{r_1}) \cup \text{subj}(e_{r_2})\}_{r_1, r_2 \in R}$ . Then,  $\text{subj}(\xi^\equiv) = \bigcup\{\text{subj}(e_r)\}_{r \in R}$ . Then, by Lem. 4,  $\text{subj}(\xi^\equiv) = \text{subj}(\bigwedge\{e_r\}_{r \in R})$ . Then,  $\text{subj}(\xi^\equiv) = \text{subj}(\xi)$ .
  - **Base:**  $\xi = \bigwedge\{\neg e_r\}_{r \in R}$ , for some  $\{e_r\}_{r \in R}$ . Similar to case “ $\xi = \bigwedge\{e_r\}_{r \in R}$ , for some  $\{e_r\}_{r \in R}$ ”. □

*Proof (of Lem. 14).* By Defn. 5:

- **Base:**  $\psi = e$ , for some  $e$ .
  1. Recall  $\mathcal{S} \in \llbracket \psi[d/q.y] \rrbracket$ . Then,  $\mathcal{S} \in \llbracket e[d/q.y] \rrbracket$ . Then, by Defn. 13,  $\mathcal{S} \in \{\hat{\mathcal{S}} \mid \text{eval}_{\mathcal{S}}(e[d/q.y]) = \text{true}\}$ . Then,  $\text{eval}_{\mathcal{S}}(e[d/q.y]) = \text{true}$ . Then, by Prop. 3,  $\text{eval}_{\mathcal{S}[\text{eval}_{\mathcal{S}}(d)/q.y]}(e) = \text{true}$ . Then,  $\mathcal{S}[\text{eval}_{\mathcal{S}}(d)/q.y] \in \{\hat{\mathcal{S}} \mid \text{eval}_{\hat{\mathcal{S}}}(e) = \text{true}\}$ . Then, by Defn. 13,  $\mathcal{S}[\text{eval}_{\mathcal{S}}(d)/q.y] \in \llbracket e \rrbracket$ . Then,  $\mathcal{S}[\text{eval}_{\mathcal{S}}(d)/q.y] \in \llbracket \psi \rrbracket$ .
  2. Similar to case 1.
- **Step:**  $\psi = \neg\hat{\psi}$ , for some  $\hat{\psi}$ .
  1.
    - By Defn. 4,  $\text{eval}_{\mathcal{S}[\text{eval}_{\mathcal{S}}(d)/q.y]}(\text{true}) = \text{true}$ . Then,  $\mathcal{S}[\text{eval}_{\mathcal{S}}(d)/q.y] \in \{\hat{\mathcal{S}} \mid \text{eval}_{\hat{\mathcal{S}}}(\text{true}) = \text{true}\}$ . Then, by Defn. 13,  $\mathcal{S}[\text{eval}_{\mathcal{S}}(d)/q.y] \in \llbracket \text{true} \rrbracket$ .
    - Recall  $\mathcal{S} \in \llbracket \psi[d/q.y] \rrbracket$ . Then,  $\mathcal{S} \in \llbracket (\neg\hat{\psi})[d/q.y] \rrbracket$ . Then, by Defn. 7,  $\mathcal{S} \in \llbracket \neg(\hat{\psi}[d/q.y]) \rrbracket$ . Then, by Defn. 13,  $\mathcal{S} \in \llbracket \text{true} \rrbracket \setminus \llbracket \hat{\psi}[d/q.y] \rrbracket$ . Then,  $\mathcal{S} \notin \llbracket \hat{\psi}[d/q.y] \rrbracket$ . Then, by induction,  $\mathcal{S}[\text{eval}_{\mathcal{S}}(d)/q.y] \notin \llbracket \hat{\psi} \rrbracket$ .
    - Recall  $\mathcal{S}[\text{eval}_{\mathcal{S}}(d)/q.y] \in \llbracket \text{true} \rrbracket$  and  $\mathcal{S}[\text{eval}_{\mathcal{S}}(d)/q.y] \notin \llbracket \hat{\psi} \rrbracket$ . Then,  $\mathcal{S}[\text{eval}_{\mathcal{S}}(d)/q.y] \in \llbracket \text{true} \rrbracket \setminus \llbracket \hat{\psi} \rrbracket$ . Then, by Defn. 13,  $\mathcal{S}[\text{eval}_{\mathcal{S}}(d)/q.y] \in \llbracket \neg\hat{\psi} \rrbracket$ . Then,  $\mathcal{S}[\text{eval}_{\mathcal{S}}(d)/q.y] \in \llbracket \psi \rrbracket$ .
  2. Recall  $\mathcal{S} \notin \llbracket \psi[d/q.y] \rrbracket$ . Then,  $\mathcal{S} \notin \llbracket (\neg\hat{\psi})[d/q.y] \rrbracket$ . Then, by Defn. 7,  $\mathcal{S} \notin \llbracket \neg(\hat{\psi}[d/q.y]) \rrbracket$ . Then, by Defn. 13,  $\mathcal{S} \notin \llbracket \text{true} \rrbracket \setminus \llbracket \hat{\psi}[d/q.y] \rrbracket$ . Then:
    - **Case:**  $\mathcal{S} \notin \llbracket \text{true} \rrbracket$ .  
By Defn. 4,  $\text{eval}_{\mathcal{S}}(\text{true}) = \text{true}$ . Then,  $\mathcal{S} \in \{\hat{\mathcal{S}} \mid \text{eval}_{\hat{\mathcal{S}}}(\text{true}) = \text{true}\}$ . Then, by Defn. 13,  $\mathcal{S} \in \llbracket \text{true} \rrbracket$ . Then, **false**.

- **Case:**  $\mathcal{S} \in \llbracket \hat{\psi}[d/q.y] \rrbracket$ . Then, by induction,  $\mathcal{S}[d/q.y] \in \llbracket \hat{\psi} \rrbracket$ . Then,  $\mathcal{S}[d/q.y] \notin \llbracket \mathbf{true} \rrbracket \setminus \llbracket \hat{\psi} \rrbracket$ . Then, by Defn. 13,  $\mathcal{S}[\mathbf{eval}_{\mathcal{S}}(d)/q.y] \notin \llbracket \neg \hat{\psi} \rrbracket$ . Then,  $\mathcal{S}[\mathbf{eval}_{\mathcal{S}}(d)/q.y] \notin \llbracket \psi \rrbracket$ .
- **Step:**  $\psi = \psi_1 \wedge \psi_2$ , for some  $\psi_1, \psi_2$ .
  1. Recall  $\mathcal{S} \in \llbracket \psi[d/q.y] \rrbracket$ . Then,  $\mathcal{S} \in \llbracket (\psi_1 \wedge \psi_2)[d/q.y] \rrbracket$ . Then, by Defn. 7,  $\mathcal{S} \in \llbracket \psi_1[d/q.y] \wedge \psi_2[d/q.y] \rrbracket$ . Then, by Defn. 13,  $\mathcal{S} \in \llbracket \psi_1[d/q.y] \rrbracket \cap \llbracket \psi_2[d/q.y] \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \psi_1[d/q.y] \rrbracket$  and  $\mathcal{S} \in \llbracket \psi_2[d/q.y] \rrbracket$ . Then, by induction,  $\mathcal{S}[\mathbf{eval}_{\mathcal{S}}(d)/q.y] \in \llbracket \psi_1 \rrbracket$  and  $\mathcal{S}[\mathbf{eval}_{\mathcal{S}}(d)/q.y] \in \llbracket \psi_2 \rrbracket$ . Then,  $\mathcal{S}[\mathbf{eval}_{\mathcal{S}}(d)/q.y] \in \llbracket \psi_1 \rrbracket \cap \llbracket \psi_2 \rrbracket$ . Then, by Defn. 13,  $\mathcal{S}[\mathbf{eval}_{\mathcal{S}}(d)/q.y] \in \llbracket \psi_1 \wedge \psi_2 \rrbracket$ . Then,  $\mathcal{S}[\mathbf{eval}_{\mathcal{S}}(d)/q.y] \in \llbracket \psi \rrbracket$ .
  2. Similar to case 1.
- **Step:**  $\psi = \forall \hat{\psi}$ , for some  $\hat{\psi}$ .
  1. By Defn. 13:
    - **Case:**  $\llbracket \forall \hat{\psi} \rrbracket = \llbracket \mathbf{true} \rrbracket$ .  
By Lem. 15,  $\{\mathcal{S}[\mathbf{eval}_{\mathcal{S}}(d)/q.y]\} \subseteq \llbracket \mathbf{true} \rrbracket$ . Then,  $\mathcal{S}[\mathbf{eval}_{\mathcal{S}}(d)/q.y] \in \llbracket \mathbf{true} \rrbracket$ . Then,  $\mathcal{S}[\mathbf{eval}_{\mathcal{S}}(d)/q.y] \in \llbracket \forall \hat{\psi} \rrbracket$ . Then,  $\mathcal{S}[\mathbf{eval}_{\mathcal{S}}(d)/q.y] \in \llbracket \psi \rrbracket$ .
    - **Case:**  $\llbracket \forall \hat{\psi} \rrbracket = \llbracket \mathbf{false} \rrbracket$ .  
Recall  $\mathcal{S} \notin \emptyset$ . Then,  $\mathcal{S} \notin \llbracket \mathbf{false} \rrbracket$ . Then,  $\mathcal{S} \notin \llbracket \forall \hat{\psi} \rrbracket$ . Then, by Defn. 7,  $\mathcal{S} \notin \llbracket (\forall \hat{\psi})[d/q.y] \rrbracket$ . Then,  $\mathcal{S} \notin \llbracket \psi[d/q.y] \rrbracket$ . Then, **false**.
  2. By Defn. 13:
    - **Case:**  $\llbracket \forall \hat{\psi} \rrbracket = \llbracket \mathbf{true} \rrbracket$ .  
By Defn. 4,  $\mathbf{eval}_{\mathcal{S}}(\mathbf{true}) = \mathbf{true}$ . Then,  $\mathcal{S} \in \{\hat{\mathcal{S}} \mid \mathbf{eval}_{\hat{\mathcal{S}}}(\mathbf{true}) = \mathbf{true}\}$ . Then, by Defn. 13,  $\mathcal{S} \in \llbracket \mathbf{true} \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \forall \hat{\psi} \rrbracket$ . Then, by Defn. 7,  $\mathcal{S} \in \llbracket (\forall \hat{\psi})[d/q.y] \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \psi[d/q.y] \rrbracket$ . Then, **false**.
    - **Case:**  $\llbracket \forall \hat{\psi} \rrbracket = \llbracket \mathbf{false} \rrbracket$ .  
Recall  $\mathcal{S}[\mathbf{eval}_{\mathcal{S}}(d)/q.y] \notin \emptyset$ . Then,  $\mathcal{S}[\mathbf{eval}_{\mathcal{S}}(d)/q.y] \notin \llbracket \mathbf{false} \rrbracket$ . Then,  $\mathcal{S}[\mathbf{eval}_{\mathcal{S}}(d)/q.y] \notin \llbracket \forall \hat{\psi} \rrbracket$ . Then,  $\mathcal{S}[\mathbf{eval}_{\mathcal{S}}(d)/q.y] \notin \llbracket \psi \rrbracket$ . □

*Proof (of Lem. 15).*

- By Defn. 4,  $\mathbf{eval}_{\mathcal{S}}(\mathbf{false}) \neq \mathbf{true}$  for every  $\mathcal{S}$ . Then,  $\mathcal{S} \notin \{\hat{\mathcal{S}} \mid \mathbf{eval}_{\hat{\mathcal{S}}}(\mathbf{true}) = \mathbf{true}\}$  for every  $\mathcal{S}$ . Then, by Defn. 13,  $\mathcal{S} \notin \llbracket \mathbf{false} \rrbracket$  for every  $\mathcal{S}$ . Then,  $\llbracket \mathbf{false} \rrbracket = \emptyset$ .
- Recall  $\emptyset \subseteq \llbracket \psi \rrbracket$ . Then,  $\llbracket \mathbf{false} \rrbracket \subseteq \llbracket \psi \rrbracket$ .
- By Defn. 4,  $\mathbf{eval}_{\mathcal{S}}(\mathbf{true}) = \mathbf{true}$  for every  $\mathcal{S}$ . Then,  $\mathcal{S} \in \{\hat{\mathcal{S}} \mid \mathbf{eval}_{\hat{\mathcal{S}}}(\mathbf{true}) = \mathbf{true}\}$  for every  $\mathcal{S}$ . Then, by Defn. 13,  $\mathcal{S} \in \llbracket \mathbf{true} \rrbracket$  for every  $\mathcal{S}$ . Then,  $\mathcal{S} \in \llbracket \psi \rrbracket$  implies  $\mathcal{S} \in \llbracket \mathbf{true} \rrbracket$  for every  $\mathcal{S}$ . Then,  $\llbracket \psi \rrbracket \subseteq \llbracket \mathbf{true} \rrbracket$ . □

*Proof (of Lem. 16).*

1. Recall:
  - **Case:**  $\llbracket \psi_1 \wedge \psi_2 \rrbracket = \llbracket \mathbf{true} \rrbracket$ .
    - Recall  $\llbracket \psi_1 \wedge \psi_2 \rrbracket = \llbracket \mathbf{true} \rrbracket$ . Then, by Defn. 13,  $\llbracket \psi_1 \rrbracket \cap \llbracket \psi_2 \rrbracket = \llbracket \mathbf{true} \rrbracket$ . Then,  $\llbracket \psi_1 \rrbracket = \llbracket \mathbf{true} \rrbracket$  and  $\llbracket \psi_2 \rrbracket = \llbracket \mathbf{true} \rrbracket$ . Then, by Defn. 13,  $\llbracket \forall \psi_1 \rrbracket = \llbracket \mathbf{true} \rrbracket$  and  $\llbracket \forall \psi_2 \rrbracket = \llbracket \mathbf{true} \rrbracket$ . Then,  $\llbracket \forall \psi_1 \rrbracket \cap \llbracket \forall \psi_2 \rrbracket = \llbracket \mathbf{true} \rrbracket$ .
    - Recall  $\llbracket \psi_1 \wedge \psi_2 \rrbracket = \llbracket \mathbf{true} \rrbracket$ . Then, by Defn. 13,  $\llbracket \forall (\psi_1 \wedge \psi_2) \rrbracket = \llbracket \mathbf{true} \rrbracket$ . Then,  $\llbracket \forall (\psi_1 \wedge \psi_2) \rrbracket = \llbracket \forall \psi_1 \rrbracket \cap \llbracket \forall \psi_2 \rrbracket$ .
  - **Case:**  $\llbracket \psi_1 \wedge \psi_2 \rrbracket \neq \llbracket \mathbf{true} \rrbracket$ .
    - Recall  $\llbracket \psi_1 \wedge \psi_2 \rrbracket \neq \llbracket \mathbf{true} \rrbracket$ . Then, by Defn. 13,  $\llbracket \psi_1 \rrbracket \cap \llbracket \psi_2 \rrbracket \neq \llbracket \mathbf{true} \rrbracket$ . Then,  $\llbracket \psi_1 \rrbracket \neq \llbracket \mathbf{true} \rrbracket$  or  $\llbracket \psi_2 \rrbracket \neq \llbracket \mathbf{true} \rrbracket$ . Then, by Defn. 13,  $\llbracket \forall \psi_1 \rrbracket = \llbracket \mathbf{false} \rrbracket$  or  $\llbracket \forall \psi_2 \rrbracket = \llbracket \mathbf{false} \rrbracket$ . Then, by Lem. 15,  $\llbracket \forall \psi_1 \rrbracket = \emptyset$  or  $\llbracket \forall \psi_2 \rrbracket = \emptyset$ . Then,  $\llbracket \forall \psi_1 \rrbracket \cap \llbracket \forall \psi_2 \rrbracket = \emptyset$ . Then, by Lem. 15, Then,  $\llbracket \forall \psi_1 \rrbracket \cap \llbracket \forall \psi_2 \rrbracket = \llbracket \mathbf{false} \rrbracket$ .
    - Recall  $\llbracket \psi_1 \wedge \psi_2 \rrbracket \neq \llbracket \mathbf{true} \rrbracket$ . Then, by Defn. 13,  $\llbracket \forall (\psi_1 \wedge \psi_2) \rrbracket = \llbracket \mathbf{false} \rrbracket$ . Then,  $\llbracket \forall (\psi_1 \wedge \psi_2) \rrbracket = \llbracket \forall \psi_1 \rrbracket \cap \llbracket \forall \psi_2 \rrbracket$ . □
2. By 1,  $\llbracket \forall (\psi_1 \wedge \psi_2) \rrbracket = \llbracket \forall \psi_1 \rrbracket \cap \llbracket \forall \psi_2 \rrbracket$ . Then, by Defn. 13,  $\llbracket \forall (\psi_1 \wedge \psi_2) \rrbracket = \llbracket \forall \psi_1 \wedge \forall \psi_2 \rrbracket$ . □







*Proof (of Lem. 26).*

1. By Defn. 5:

– **Base:**  $\xi = \mathbf{true}$ .

Recall  $\llbracket \mathbf{true} \rrbracket = \bigcap \{ \llbracket \mathbf{true} \rrbracket \}_{r \in \text{subj}(\xi)}$ . Then, by Defn. 11,  $\llbracket \mathbf{true} \rrbracket = \bigcap \{ \llbracket \mathbf{true} \uparrow r \rrbracket \}_{r \in \text{subj}(\xi)}$ . Then, by Defn. 13,  $\llbracket \mathbf{true} \rrbracket = \llbracket \bigwedge \{ \mathbf{true} \uparrow r \}_{r \in \text{subj}(\xi)} \rrbracket$ . Then,  $\llbracket \xi \rrbracket = \llbracket \bigwedge \{ \xi \uparrow r \}_{r \in \text{subj}(\xi)} \rrbracket$ .

– **Base:**  $\xi = \bigwedge \{ e_{\hat{r}} \}_{\hat{r} \in \hat{R}}$ , for some  $\{ e_{\hat{r}} \}_{\hat{r} \in \hat{R}}$ .

- Recall  $\xi = \bigwedge \{ e_{\hat{r}} \}_{\hat{r} \in \hat{R}}$ . Then, by Rem. 1,  $\{ \hat{r} \mid \hat{r}.\hat{x} \in \text{rx}(e_{\hat{r}}) \} = \{ \hat{r} \}$  for every  $\hat{r} \in \hat{R}$ .
- By Lem. 40,  $\text{subj}(\bigwedge \{ e_{\hat{r}} \}_{\hat{r} \in \hat{R}}) = \bigcup \{ \text{subj}(e_{\hat{r}}) \}_{\hat{r} \in \hat{R}}$ . Then,  $\text{subj}(\xi) = \bigcup \{ \text{subj}(e_{\hat{r}}) \}_{\hat{r} \in \hat{R}}$ . Then, by Defn. 10,  $\text{subj}(\xi) = \bigcup \{ \{ \hat{r} \mid \hat{r}.\hat{x} \in \text{rx}(e_{\hat{r}}) \} \}_{\hat{r} \in \hat{R}}$ . Then,  $\text{subj}(\xi) = \bigcup \{ \{ \hat{r} \} \}_{\hat{r} \in \hat{R}}$ . Then,  $\text{subj}(\xi) = \{ \hat{r} \mid \hat{r} \in \hat{R} \}$ . Then,  $\text{subj}(\xi) = \hat{R}$ .
- By Defn. 11,  $\bigwedge \{ e_{\hat{r}} \}_{\hat{r} \in \hat{R}} \uparrow r = e_r$  for every  $r \in \hat{R}$ . Then,  $\xi \uparrow r = e_r$  for every  $r \in \text{subj}(\xi)$ . Then,  $\bigwedge \{ \xi \uparrow r \}_{r \in \text{subj}(\xi)} = \bigwedge \{ e_r \}_{r \in \text{subj}(\xi)}$ . Then,  $\bigwedge \{ \xi \uparrow r \}_{r \in \text{subj}(\xi)} = \xi$ . Then, by Defn. 13,  $\llbracket \bigwedge \{ \xi \uparrow r \}_{r \in \text{subj}(\xi)} \rrbracket = \llbracket \xi \rrbracket$ .

– **Base:**  $\xi = \bigwedge \{ \neg e_{\hat{r}} \}_{\hat{r} \in \hat{R}}$ , for some  $\{ e_{\hat{r}} \}_{\hat{r} \in \hat{R}}$ . Similar to case “ $\xi = \bigwedge \{ e_{\hat{r}} \}_{\hat{r} \in \hat{R}}$ , for some  $\{ e_{\hat{r}} \}_{\hat{r} \in \hat{R}}$ ”. □

2. By Defn. 5:

– **Base:**  $\xi = \mathbf{true}$ .

Recall  $\llbracket \mathbf{true} \rrbracket = \bigcap \{ \llbracket \mathbf{true} \rrbracket \}_{r \in R}$ . Then, by Defn. 11,  $\llbracket \mathbf{true} \rrbracket = \bigcap \{ \llbracket \mathbf{true} \uparrow r \rrbracket \}_{r \in R}$ . Then, by Defn. 13,  $\llbracket \mathbf{true} \rrbracket = \llbracket \bigwedge \{ \mathbf{true} \uparrow r \}_{r \in R} \rrbracket$ . Then,  $\llbracket \xi \rrbracket = \llbracket \bigwedge \{ \xi \uparrow r \}_{r \in R} \rrbracket$ .

– **Base:**  $\xi = \bigwedge \{ e_{\hat{r}} \}_{\hat{r} \in \hat{R}}$ , for some  $\{ e_{\hat{r}} \}_{\hat{r} \in \hat{R}}$ .

- Recall  $\xi = \bigwedge \{ e_{\hat{r}} \}_{\hat{r} \in \hat{R}}$ . Then, by Rem. 1,  $\{ \hat{r} \mid \hat{r}.\hat{x} \in \text{rx}(e_{\hat{r}}) \} = \{ \hat{r} \}$  for every  $\hat{r} \in \hat{R}$ .
- By Lem. 40,  $\text{subj}(\bigwedge \{ e_{\hat{r}} \}_{\hat{r} \in \hat{R}}) = \bigcup \{ \text{subj}(e_{\hat{r}}) \}_{\hat{r} \in \hat{R}}$ . Then,  $\text{subj}(\xi) = \bigcup \{ \text{subj}(e_{\hat{r}}) \}_{\hat{r} \in \hat{R}}$ . Then, by Defn. 10,  $\text{subj}(\xi) = \bigcup \{ \{ \hat{r} \mid \hat{r}.\hat{x} \in \text{rx}(e_{\hat{r}}) \} \}_{\hat{r} \in \hat{R}}$ . Then,  $\text{subj}(\xi) = \bigcup \{ \{ \hat{r} \} \}_{\hat{r} \in \hat{R}}$ . Then,  $\text{subj}(\xi) = \{ \hat{r} \mid \hat{r} \in \hat{R} \}$ . Then,  $\text{subj}(\xi) = \hat{R}$ .
- By Defn. 11,  $\bigwedge \{ e_{\hat{r}} \}_{\hat{r} \in \hat{R}} \uparrow r = \mathbf{true}$  for every  $r \notin \hat{R}$ . Then,  $\xi \uparrow r = \mathbf{true}$  for every  $r \notin \text{subj}(\xi)$ . Then,  $\xi \uparrow r = \mathbf{true}$  for every  $r \in R \setminus \text{subj}(\xi)$ .
- Recall  $\text{subj}(\xi) \subseteq R$ . Then,  $R = (R \setminus \text{subj}(\xi)) \cup \text{subj}(\xi)$ .
- By Lem. 15,  $\llbracket \bigwedge \{ \xi \uparrow r \}_{r \in \text{subj}(\xi)} \rrbracket \subseteq \llbracket \mathbf{true} \rrbracket$ . Then,  $\llbracket \bigwedge \{ \xi \uparrow r \}_{r \in \text{subj}(\xi)} \rrbracket \subseteq \bigcap \{ \llbracket \mathbf{true} \rrbracket \}_{r \in R \setminus \text{subj}(\xi)}$ . Then,  $\llbracket \bigwedge \{ \xi \uparrow r \}_{r \in \text{subj}(\xi)} \rrbracket \subseteq \bigcap \{ \llbracket \xi \uparrow r \rrbracket \}_{r \in R \setminus \text{subj}(\xi)}$ . Then, by Defn. 13,  $\bigcap \{ \llbracket \xi \uparrow r \rrbracket \}_{r \in \text{subj}(\xi)} \subseteq \bigcap \{ \llbracket \xi \uparrow r \rrbracket \}_{r \in R \setminus \text{subj}(\xi)}$ . Then,  $\bigcap \{ \llbracket \xi \uparrow r \rrbracket \}_{r \in \text{subj}(\xi)} = \bigcap \{ \llbracket \xi \uparrow r \rrbracket \}_{r \in \text{subj}(\xi)} \cap \bigcap \{ \llbracket \xi \uparrow r \rrbracket \}_{r \in R \setminus \text{subj}(\xi)}$ . Then,  $\bigcap \{ \llbracket \xi \uparrow r \rrbracket \}_{r \in \text{subj}(\xi)} = \bigcap \{ \llbracket \xi \uparrow r \rrbracket \}_{r \in (R \setminus \text{subj}(\xi)) \cup \text{subj}(\xi)}$ . Then,  $\bigcap \{ \llbracket \xi \uparrow r \rrbracket \}_{r \in \text{subj}(\xi)} = \bigcap \{ \llbracket \xi \uparrow r \rrbracket \}_{r \in R}$ . Then, by Defn. 13,  $\llbracket \bigwedge \{ \xi \uparrow r \}_{r \in \text{subj}(\xi)} \rrbracket = \llbracket \bigwedge \{ \xi \uparrow r \}_{r \in R} \rrbracket$ . Then, by 1,  $\llbracket \xi \rrbracket = \llbracket \bigwedge \{ \xi \uparrow r \}_{r \in R} \rrbracket$ .

– **Base:**  $\xi = \bigwedge \{ \neg e_{\hat{r}} \}_{\hat{r} \in \hat{R}}$ , for some  $\{ e_{\hat{r}} \}_{\hat{r} \in \hat{R}}$ . Similar to case “ $\xi = \bigwedge \{ e_{\hat{r}} \}_{\hat{r} \in \hat{R}}$ , for some  $\{ e_{\hat{r}} \}_{\hat{r} \in \hat{R}}$ ”. □

*Proof (of Lem. 27).*

1. By Defn. 5:

– **Base:**  $\xi = \mathbf{true}$ .

Recall  $\mathcal{S} \in \llbracket \xi^{\equiv} \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \mathbf{true}^{\equiv} \rrbracket$ . Then, by Defn. 12,  $\mathcal{S} \in \llbracket \mathbf{true} \rrbracket$ . Then, by Defn. 12,  $\mathcal{S} \in \llbracket \mathbf{true}^+ \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \xi^+ \rrbracket$ .

– **Base:**  $\xi = \bigwedge \{ e_r \}_{r \in R}$ , for some  $\{ e_r \}_{r \in R}$ .

- Recall  $\xi = \bigwedge \{ e_r \}_{r \in R}$ . Then, by Rem. 1,  $R \neq \emptyset$ . Then,  $\hat{r} \in R$ , for some  $\hat{r}$ .
- By Lem. 15,  $\llbracket e_{\hat{r}} \rrbracket \subseteq \llbracket \mathbf{true} \rrbracket$ . Then,  $\llbracket \mathbf{true} \rrbracket = (\llbracket \mathbf{true} \rrbracket \setminus \llbracket e_{\hat{r}} \rrbracket) \cup \llbracket e_{\hat{r}} \rrbracket$ .
- By Defn. 4,  $\text{eval}_{\mathcal{S}}(\mathbf{true}) = \mathbf{true}$ . Then,  $\mathcal{S} \in \{ \hat{\mathcal{S}} \mid \text{eval}_{\mathcal{S}}(\mathbf{true}) = \mathbf{true} \}$ . Then, by Defn. 13,  $\mathcal{S} \in \llbracket \mathbf{true} \rrbracket$ . Then,  $\mathcal{S} \in (\llbracket \mathbf{true} \rrbracket \setminus \llbracket e_{\hat{r}} \rrbracket) \cup \llbracket e_{\hat{r}} \rrbracket$ . Then, by Defn. 13,  $\mathcal{S} \in \llbracket \neg e_{\hat{r}} \rrbracket \cup \llbracket e_{\hat{r}} \rrbracket$ . Then:
  - \* **Case:**  $\mathcal{S} \in \llbracket e_{\hat{r}} \rrbracket$ .
    - Recall  $\hat{r} \in R$ . Then,  $R = (R \setminus \{ \hat{r} \}) \cup \{ \hat{r} \}$ .

- Recall  $\mathcal{S} \in \llbracket \xi^{\equiv} \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \bigwedge \{e_r\}_{r \in R}^{\equiv} \rrbracket$ . Then, by Defn. 12,  $\mathcal{S} \in \llbracket \bigwedge \{e_{r_1} \Rightarrow e_{r_2}\}_{r_1, r_2 \in R} \rrbracket$ . Then, by Defn. 13,  $\mathcal{S} \in \bigcap \{ \llbracket e_{r_1} \Rightarrow e_{r_2} \rrbracket \}_{r_1, r_2 \in R}$ . Then,  $\mathcal{S} \in \bigcap \{ \llbracket e_{r_1} \Rightarrow e_{r_2} \rrbracket \}_{r_1 \in (R \setminus \{\hat{r}\}) \cup \{\hat{r}\}, r_2 \in R}$ . Then,  $\mathcal{S} \in \bigcap \{ \llbracket e_{r_1} \Rightarrow e_{r_2} \rrbracket \}_{r_1 \in R \setminus \{\hat{r}\}, r_2 \in R} \cap \bigcap \{ \llbracket e_{\hat{r}} \Rightarrow e_{r_2} \rrbracket \}_{r_2 \in R}$ . Then,  $\mathcal{S} \in \bigcap \{ \llbracket e_{\hat{r}} \Rightarrow e_{r_2} \rrbracket \}_{r_2 \in R}$ .
- Recall  $\mathcal{S} \in \llbracket e_{\hat{r}} \rrbracket$  and  $\mathcal{S} \in \bigcap \{ \llbracket e_{\hat{r}} \Rightarrow e_{r_2} \rrbracket \}_{r_2 \in R}$ . Then,  $\mathcal{S} \in \llbracket e_{\hat{r}} \rrbracket \cap \bigcap \{ \llbracket e_{\hat{r}} \Rightarrow e_{r_2} \rrbracket \}_{r_2 \in R}$ . Then, by Lem. 19,  $\mathcal{S} \in \llbracket e_{\hat{r}} \rrbracket \cap \bigcap \{ \llbracket e_{r_2} \rrbracket \}_{r_2 \in R}$ . Then,  $\mathcal{S} \in \bigcap \{ \llbracket e_{r_2} \rrbracket \}_{r_2 \in R}$ . Then, by Defn. 13,  $\mathcal{S} \in \llbracket \bigwedge \{e_{r_2}\}_{r_2 \in R} \rrbracket$ . Then, by Defn. 12,  $\mathcal{S} \in \llbracket \bigwedge \{e_{r_2}\}_{r_2 \in R}^+ \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \xi^+ \rrbracket$ .
- \* **Case:**  $\mathcal{S} \in \llbracket \neg e_{\hat{r}} \rrbracket$ .
  - Recall  $\hat{r} \in R$ . Then,  $R = (R \setminus \{\hat{r}\}) \cup \{\hat{r}\}$ .
  - Recall  $\mathcal{S} \in \llbracket \xi^{\equiv} \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \bigwedge \{e_r\}_{r \in R}^{\equiv} \rrbracket$ . Then, by Defn. 12,  $\mathcal{S} \in \llbracket \bigwedge \{e_{r_1} \Rightarrow e_{r_2}\}_{r_1, r_2 \in R} \rrbracket$ . Then, by Defn. 13,  $\mathcal{S} \in \bigcap \{ \llbracket e_{r_1} \Rightarrow e_{r_2} \rrbracket \}_{r_1, r_2 \in R}$ . Then, by Lem. 17,  $\mathcal{S} \in \bigcap \{ \llbracket \neg e_{r_2} \Rightarrow \neg e_{r_1} \rrbracket \}_{r_1, r_2 \in R}$ . Then,  $\mathcal{S} \in \bigcap \{ \llbracket \neg e_{r_2} \Rightarrow \neg e_{r_1} \rrbracket \}_{r_1 \in R, r_2 \in (R \setminus \{\hat{r}\}) \cup \{\hat{r}\}}$ . Then,  $\mathcal{S} \in \bigcap \{ \llbracket \neg e_{r_2} \Rightarrow \neg e_{r_1} \rrbracket \}_{r_1 \in R, r_2 \in R \setminus \{\hat{r}\}} \cap \bigcap \{ \llbracket \neg e_{\hat{r}} \Rightarrow \neg e_{r_1} \rrbracket \}_{r_1 \in R}$ . Then,  $\mathcal{S} \in \bigcap \{ \llbracket \neg e_{\hat{r}} \Rightarrow \neg e_{r_1} \rrbracket \}_{r_1 \in R}$ .
  - Recall  $\mathcal{S} \in \llbracket \neg e_{\hat{r}} \rrbracket$  and  $\mathcal{S} \in \bigcap \{ \llbracket \neg e_{\hat{r}} \Rightarrow \neg e_{r_1} \rrbracket \}_{r_1 \in R}$ . Then,  $\mathcal{S} \in \llbracket \neg e_{\hat{r}} \rrbracket \cap \bigcap \{ \llbracket \neg e_{\hat{r}} \Rightarrow \neg e_{r_1} \rrbracket \}_{r_1 \in R}$ . Then, by Lem. 19,  $\mathcal{S} \in \llbracket \neg e_{\hat{r}} \rrbracket \cap \bigcap \{ \llbracket \neg e_{r_1} \rrbracket \}_{r_1 \in R}$ . Then,  $\mathcal{S} \in \bigcap \{ \llbracket \neg e_{r_1} \rrbracket \}_{r_1 \in R}$ . Then, by Defn. 13,  $\mathcal{S} \in \llbracket \bigwedge \{e_{r_1}\}_{r_1 \in R}^- \rrbracket$ . Then, by Defn. 12,  $\mathcal{S} \in \llbracket \bigwedge \{e_{r_1}\}_{r_1 \in R}^- \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \xi^- \rrbracket$ .
- **Base:**  $\xi = \bigwedge \{ \neg e_r \}_{r \in R}$ , for some  $\{e_r\}_{r \in R}$ . Similar to case “ $\xi = \bigwedge \{e_r\}_{r \in R}$ , for some  $\{e_r\}_{r \in R}$ ”. □
- 2. Recall  $\mathcal{S} \in \llbracket \xi^{\equiv} \rrbracket$ . Then, by 1:
  - **Case:**  $\mathcal{S} \in \llbracket \xi^+ \rrbracket$ .  
By Defn. 5:
    - **Base:**  $\xi = \mathbf{true}$ .  
Recall  $\mathcal{S} \in \llbracket \xi^+ \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \mathbf{true}^+ \rrbracket$ . Then, by Defn. 11,  $\mathcal{S} \in \llbracket (\mathbf{true} \upharpoonright r)^+ \rrbracket$ . Then, by Lem. 11,  $\mathcal{S} \in \llbracket \mathbf{true}^+ \upharpoonright r \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \xi^+ \upharpoonright r \rrbracket$ .
    - **Base:**  $\xi = \bigwedge \{e_{\hat{r}}\}_{\hat{r} \in R}$ , for some  $\{e_{\hat{r}}\}_{\hat{r} \in R}$ .  
By Defn. 11:
      - \* **Case:**  $\bigwedge \{e_{\hat{r}}\}_{\hat{r} \in R} \upharpoonright r = e_r$  and  $r \in R$ .
        - Recall  $r \in R$ . Then,  $R = (R \setminus \{r\}) \cup \{r\}$ .
        - Recall  $\mathcal{S} \in \llbracket \xi^+ \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \bigwedge \{e_{\hat{r}}\}_{\hat{r} \in R}^+ \rrbracket$ . Then, by Defn. 12,  $\mathcal{S} \in \llbracket \bigwedge \{e_{\hat{r}}\}_{\hat{r} \in R} \rrbracket$ . Then, by Defn. 13,  $\mathcal{S} \in \bigcap \{ \llbracket e_{\hat{r}} \rrbracket \}_{\hat{r} \in R}$ . Then,  $\mathcal{S} \in \bigcap \{ \llbracket e_{\hat{r}} \rrbracket \}_{\hat{r} \in (R \setminus \{r\}) \cup \{r\}}$ . Then,  $\mathcal{S} \in \bigcap \{ \llbracket e_{\hat{r}} \rrbracket \}_{\hat{r} \in R \setminus \{r\}} \cap \llbracket e_r \rrbracket$ . Then,  $\mathcal{S} \in \llbracket e_r \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \bigwedge \{e_{\hat{r}}\}_{\hat{r} \in R} \upharpoonright r \rrbracket$ . Then, by Defn. 12,  $\mathcal{S} \in \llbracket \bigwedge \{e_{\hat{r}}\}_{\hat{r} \in R}^+ \upharpoonright r \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \xi^+ \upharpoonright r \rrbracket$ .
      - \* **Case:**  $\bigwedge \{e_{\hat{r}}\}_{\hat{r} \in R} \upharpoonright r = \mathbf{true}$ .  
By Defn. 4,  $\mathbf{eval}_{\mathcal{S}}(\mathbf{true}) = \mathbf{true}$  for every  $\mathcal{S}$ . Then,  $\mathcal{S} \in \{\hat{\mathcal{S}} \mid \mathbf{eval}_{\hat{\mathcal{S}}}(\mathbf{true}) = \mathbf{true}\}$  for every  $\mathcal{S}$ . Then, by Defn. 13,  $\mathcal{S} \in \llbracket \mathbf{true} \rrbracket$ . Then, by Defn. 12,  $\mathcal{S} \in \llbracket \mathbf{true}^+ \rrbracket$ . Then,  $\mathcal{S} \in \llbracket (\bigwedge \{e_{\hat{r}}\}_{\hat{r} \in R} \upharpoonright r)^+ \rrbracket$ . Then, by Lem. 11,  $\mathcal{S} \in \llbracket \bigwedge \{e_{\hat{r}}\}_{\hat{r} \in R}^+ \upharpoonright r \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \xi^+ \upharpoonright r \rrbracket$ .
  - **Base:**  $\xi = \bigwedge \{ \neg e_r \}_{r \in R}$ , for some  $\{e_r\}_{r \in R}$ . Similar to case “ $\xi = \bigwedge \{e_r\}_{r \in R}$ , for some  $\{e_r\}_{r \in R}$ ”. □
  - **Case:**  $\mathcal{S} \in \llbracket \xi^- \rrbracket$ . Similar to case  $\mathcal{S} \in \llbracket \xi^+ \rrbracket$ . □

*Proof (of Lem. 28).*

1. By Defn. 5:
  - **Base:**  $\xi = \mathbf{true}$ .  
Recall  $r \notin \emptyset$ . Then, by Lem. 4,  $r \notin \mathbf{subj}(\mathbf{true})$ . Then,  $r \notin \mathbf{subj}(\xi)$ . Then, **false**.
  - **Base:**  $\xi = \bigwedge \{e_{\hat{r}}\}_{\hat{r} \in R}$ , for some  $\{e_{\hat{r}}\}_{\hat{r} \in R}$ .  
Recall:
    - **Case:**  $r \in R$ .



- $\llbracket \xi^{\equiv} \rrbracket \cap \llbracket \xi^+ \uparrow r \rrbracket \subseteq \bigcup \{ \llbracket \bigwedge \{ e_{\hat{r}} \}_{\hat{r} \in R} \uparrow \hat{r} \rrbracket \}_{\hat{r} \in R \setminus \{r\}}$ . Then, by Defn. 13,  
 $\llbracket \xi^{\equiv} \rrbracket \cap \llbracket \xi^+ \uparrow r \rrbracket \subseteq \llbracket \bigwedge \{ \bigwedge \{ e_{\hat{r}} \}_{\hat{r} \in R} \uparrow \hat{r} \}_{\hat{r} \in R \setminus \{r\}} \rrbracket$ . Then, by Defn. 12,  
 $\llbracket \xi^{\equiv} \rrbracket \cap \llbracket \xi^+ \uparrow r \rrbracket \subseteq \llbracket \bigwedge \{ \bigwedge \{ e_{\hat{r}} \}_{\hat{r} \in R} \uparrow \hat{r} \}_{\hat{r} \in R \setminus \{r\}} \rrbracket$ . Then,  
 $\llbracket \xi^{\equiv} \rrbracket \cap \llbracket \xi^+ \uparrow r \rrbracket \subseteq \llbracket \bigwedge \{ \xi^+ \uparrow \hat{r} \}_{\hat{r} \in \text{subj}(\xi) \setminus \{r\}} \rrbracket$ .
- **Base:**  $\xi = \bigwedge \{ \neg e_{\hat{r}} \}_{\hat{r} \in R}$ , for some  $\{ e_{\hat{r}} \}_{\hat{r} \in R}$ . Similar to case “ $\xi = \bigwedge \{ e_{\hat{r}} \}_{\hat{r} \in R}$ , for some  $\{ e_{\hat{r}} \}_{\hat{r} \in R}$ ”. □
4. By Defn. 5:
- **Base:**  $\xi = \text{true}$ .  
 Recall  $r \notin \emptyset$ . Then, by Lem. 4,  $r \notin \text{subj}(\text{true})$ . Then,  $r \notin \text{subj}(\xi)$ . Then, **false**.
  - **Base:**  $\xi = \bigwedge \{ e_{\hat{r}} \}_{\hat{r} \in R}$ , for some  $\{ e_{\hat{r}} \}_{\hat{r} \in R}$ .  
 By Defn. 11:
    - **Case:**  $\bigwedge \{ \neg e_{\hat{r}} \}_{\hat{r} \in R} \uparrow r = \neg e_r$  and  $r \in R$ .
      - \* Recall  $r \in R$ . Then,  $R = (R \setminus \{r\}) \cup \{r\}$ .
      - \* Recall  $\llbracket \xi^{\equiv} \rrbracket \cap \llbracket \xi^- \uparrow r \rrbracket \subseteq \llbracket \xi^{\equiv} \rrbracket \cap \llbracket \xi^- \uparrow r \rrbracket$ . Then,  
 $\llbracket \xi^{\equiv} \rrbracket \cap \llbracket \xi^- \uparrow r \rrbracket \subseteq \llbracket \bigwedge \{ e_{\hat{r}} \}_{\hat{r} \in R} \rrbracket \cap \llbracket \bigwedge \{ e_{\hat{r}} \}_{\hat{r} \in R} \rrbracket$ . Then, by Defn. 12,  
 $\llbracket \xi^{\equiv} \rrbracket \cap \llbracket \neg \xi \uparrow r \rrbracket \subseteq \llbracket \bigwedge \{ e_{r_1} \Rightarrow e_{r_2} \}_{r_1, r_2 \in R} \rrbracket \cap \llbracket \bigwedge \{ \neg e_{\hat{r}} \}_{\hat{r} \in R} \uparrow r \rrbracket$ . Then,  
 $\llbracket \xi^{\equiv} \rrbracket \cap \llbracket \neg \xi \uparrow r \rrbracket \subseteq \llbracket \bigwedge \{ e_{r_1} \Rightarrow e_{r_2} \}_{r_1, r_2 \in R} \rrbracket \cap \llbracket \neg e_r \rrbracket$ . Then, by Defn. 13,  
 $\llbracket \xi^{\equiv} \rrbracket \cap \llbracket \xi^- \uparrow r \rrbracket \subseteq \bigcap \{ \llbracket e_{r_1} \Rightarrow e_{r_2} \rrbracket \}_{r_1, r_2 \in R} \cap \llbracket \neg e_r \rrbracket$ . Then, by Lem. 17,  
 $\llbracket \xi^{\equiv} \rrbracket \cap \llbracket \xi^- \uparrow r \rrbracket \subseteq \bigcap \{ \llbracket \neg e_{r_2} \Rightarrow \neg e_{r_1} \rrbracket \}_{r_1, r_2 \in R} \cap \llbracket \neg e_r \rrbracket$ . Then,  
 $\llbracket \xi^{\equiv} \rrbracket \cap \llbracket \xi^- \uparrow r \rrbracket \subseteq \bigcap \{ \llbracket \neg e_{r_1} \Rightarrow \neg e_{r_2} \rrbracket \}_{r_1 \in R, r_2 \in (R \setminus \{r\}) \cup \{r\}} \cap \llbracket \neg e_r \rrbracket$ . Then,  
 $\llbracket \xi^{\equiv} \rrbracket \cap \llbracket \xi^- \uparrow r \rrbracket \subseteq \bigcap \{ \llbracket \neg e_{r_2} \Rightarrow \neg e_{r_1} \rrbracket \}_{r_1 \in R, r_2 \in R \setminus \{r\}} \cap \bigcap \{ \llbracket \neg e_{r_1} \rrbracket \}_{r_1 \in R} \cap \llbracket \neg e_r \rrbracket$ .  
 Then, by Lem. 19,  
 $\llbracket \xi^{\equiv} \rrbracket \cap \llbracket \xi^- \uparrow r \rrbracket \subseteq \bigcap \{ \llbracket \neg e_{r_2} \Rightarrow \neg e_{r_1} \rrbracket \}_{r_1 \in R, r_2 \in R \setminus \{r\}} \cap \bigcap \{ \llbracket \neg e_{r_1} \rrbracket \}_{r_1 \in R} \cap \llbracket \neg e_r \rrbracket$ . Then,  
 $\llbracket \xi^{\equiv} \rrbracket \cap \llbracket \xi^- \uparrow r \rrbracket \subseteq \bigcap \{ \llbracket \neg e_{r_1} \rrbracket \}_{r_1 \in R}$ . Then, by Defn. 13,  
 $\llbracket \xi^{\equiv} \rrbracket \cap \llbracket \xi^- \uparrow r \rrbracket \subseteq \llbracket \bigwedge \{ \neg e_{r_1} \}_{r_1 \in R} \rrbracket$ . Then, by Defn. 12,  
 $\llbracket \xi^{\equiv} \rrbracket \cap \llbracket \xi^- \uparrow r \rrbracket \subseteq \llbracket \bigwedge \{ e_{r_1} \}_{r_1 \in R} \rrbracket$ . Then,  $\llbracket \xi^{\equiv} \rrbracket \cap \llbracket \xi^- \uparrow r \rrbracket \subseteq \llbracket \xi^- \rrbracket$ .
    - **Case:**  $r \notin R$ .
      - \* Recall  $\xi = \bigwedge \{ e_{\hat{r}} \}_{\hat{r} \in R}$ . Then, by Rem. 1,  $\{ \hat{r} \mid \hat{r} \cdot \hat{x} \in \text{rx}(e_{\hat{r}}) \} = \{ \hat{r} \}$  for every  $\hat{r} \in R$ .
      - \* By Lem. 40,  $\text{subj}(\bigwedge \{ e_{\hat{r}} \}_{\hat{r} \in R}) = \bigcup \{ \text{subj}(e_{\hat{r}}) \}_{\hat{r} \in R}$ . Then,  $\text{subj}(\xi) = \bigcup \{ \text{subj}(e_{\hat{r}}) \}_{\hat{r} \in R}$ .  
 Then, by Defn. 10,  $\text{subj}(\xi) = \bigcup \{ \{ \hat{r} \mid \hat{r} \cdot \hat{x} \in \text{rx}(e_{\hat{r}}) \} \}_{\hat{r} \in R}$ . Then,  
 $\text{subj}(\xi) = \bigcup \{ \{ \hat{r} \} \}_{\hat{r} \in R}$ . Then,  $\text{subj}(\xi) = \{ \hat{r} \mid \hat{r} \in R \}$ . Then,  $\text{subj}(\xi) = R$ .
      - \* Recall  $r \notin R$ . Then,  $r \notin \text{subj}(\xi)$ . Then, **false**.
  - **Base:**  $\xi = \bigwedge \{ \neg e_{\hat{r}} \}_{\hat{r} \in R}$ , for some  $\{ e_{\hat{r}} \}_{\hat{r} \in R}$ . Similar to case “ $\xi = \bigwedge \{ e_{\hat{r}} \}_{\hat{r} \in R}$ , for some  $\{ e_{\hat{r}} \}_{\hat{r} \in R}$ ”. □
5. Similar to 4. □

*Proof (of Lem. 29).*

1. - Recall  $r \in R$ . Then,  $R = (R \setminus \{r\}) \cup \{r\}$ .  
 - Recall  $r \in R \subseteq \text{subj}(\xi)$ . Then,  $r \in \text{subj}(\xi)$ . Then, by Lem. 28,  $\llbracket \xi^+ \uparrow r \rrbracket \cap \llbracket \xi^- \uparrow r \rrbracket = \llbracket \text{false} \rrbracket$ .  
 - Recall  $\llbracket \bigwedge \{ \xi^- \uparrow \hat{r} \}_{\hat{r} \in R} \rrbracket \cap \llbracket \xi^+ \uparrow r \rrbracket \subseteq \llbracket \bigwedge \{ \xi^- \uparrow \hat{r} \}_{\hat{r} \in R} \rrbracket \cap \llbracket \xi^+ \uparrow r \rrbracket$ . Then, by Defn. 13,  
 $\llbracket \bigwedge \{ \xi^- \uparrow \hat{r} \}_{\hat{r} \in R} \rrbracket \cap \llbracket \xi^+ \uparrow r \rrbracket \subseteq \bigcap \{ \llbracket \xi^- \uparrow \hat{r} \rrbracket \}_{\hat{r} \in R} \cap \llbracket \xi^+ \uparrow r \rrbracket$ . Then,  
 $\llbracket \bigwedge \{ \xi^- \uparrow \hat{r} \}_{\hat{r} \in R} \rrbracket \cap \llbracket \xi^+ \uparrow r \rrbracket \subseteq \bigcap \{ \llbracket \xi^- \uparrow \hat{r} \rrbracket \}_{\hat{r} \in (R \setminus \{r\}) \cup \{r\}} \cap \llbracket \xi^+ \uparrow r \rrbracket$ . Then,  
 $\llbracket \bigwedge \{ \xi^- \uparrow \hat{r} \}_{\hat{r} \in R} \rrbracket \cap \llbracket \xi^+ \uparrow r \rrbracket \subseteq \bigcap \{ \llbracket \xi^- \uparrow \hat{r} \rrbracket \}_{\hat{r} \in R \setminus \{r\}} \cap \llbracket \xi^- \uparrow r \rrbracket \cap \llbracket \xi^+ \uparrow r \rrbracket$ . Then,  
 $\llbracket \bigwedge \{ \xi^- \uparrow \hat{r} \}_{\hat{r} \in R} \rrbracket \cap \llbracket \xi^+ \uparrow r \rrbracket \subseteq \llbracket \xi^- \uparrow r \rrbracket \cap \llbracket \xi^+ \uparrow r \rrbracket$ . Then,  
 $\llbracket \bigwedge \{ \xi^- \uparrow \hat{r} \}_{\hat{r} \in R} \rrbracket \cap \llbracket \xi^+ \uparrow r \rrbracket \subseteq \llbracket \text{false} \rrbracket$ .
2. Similar to 2. □

## C Actions

*Proof (of Lem. 30).* By Defn. 14:

- **Base:**  $\alpha = \tau$ . Then, **false**.
- **Base:**  $\alpha = q.y := e$ , for some  $q, y, e$ .  
Recall  $\{q\} \neq \emptyset$ . Then, by Defn. 16,  $\text{subj}(q.y := e) \neq \emptyset$ . Then,  $\text{subj}(\alpha) \neq \emptyset$ .
- **Base:**  $\alpha = p.e \rightarrow q.y$ , for some  $p, q, y, e$ . Similar to case “ $\alpha = q.y := e$ , for some  $q, y, e$ ”.
- **Base:**  $\alpha = i_R^R$ , for some  $R$ .  
Recall  $\alpha = i_R^R$ . Then, by Rem. 2,  $R \neq \emptyset$ . Then, by Defn. 16,  $\text{subj}(i_R^R) \neq \emptyset$ . Then,  $\text{subj}(\alpha) \neq \emptyset$ .
- **Base:**  $\alpha = pq!e$ , for some  $p, q, e$ . Similar to case “ $\alpha = q.y := e$ , for some  $q, y, e$ ”.
- **Base:**  $\alpha = pq?y$ , for some  $p, q, y$ . Similar to case “ $\alpha = q.y := e$ , for some  $q, y, e$ ”.
- **Base:**  $\alpha = i_{\{r\}}^R$ , for some  $p, q, y$ . Similar to case “ $\alpha = q.y := e$ , for some  $q, y, e$ ”. □

*Proof (of Lem. 31).*

1. By Defn. 14:

- **Base:**  $\alpha = \tau$ .  
Recall  $\emptyset \subseteq \{r.x \mid r \in \text{subj}(\alpha)\}$ . Then, by Defn. 15,  $\text{read}(\tau) \subseteq \{r.x \mid r \in \text{subj}(\alpha)\}$ . Then,  $\text{read}(\alpha) \subseteq \{r.x \mid r \in \text{subj}(\alpha)\}$ .
- **Base:**  $\alpha = q.y := e$ , for some  $q, y, e$ .  
Recall  $\alpha = q.y := e$ . Then, by Rem. 2,  $\{r \mid r.x \in \text{rx}(e)\} \subseteq \{q\}$ . Then,  $\hat{r} \in \{q\}$  for every  $\hat{r} \in \{r \mid r.x \in \text{rx}(e)\}$ . Then,  $\hat{r}.x \in \{q\}$  for every  $\hat{r}.x \in \text{rx}(e)$ . Then,  $\text{rx}(e) \subseteq \{r.x \mid r \in \{q\}\}$ . Then, by Defn. 15,  $\text{read}(q.y := e) \subseteq \{r.x \mid r \in \{q\}\}$ . Then, by Defn. 16,  $\text{read}(q.y := e) \subseteq \{r.x \mid r \in \text{subj}(q.y := e)\}$ . Then,  $\text{read}(\alpha) \subseteq \{r.x \mid r \in \text{subj}(\alpha)\}$ .
- **Base:**  $\alpha = p.e \rightarrow q.y$ , for some  $p, q, y, e$ .  
Recall  $\alpha = p.e \rightarrow q.y$ . Then, by Rem. 2,  $\{r \mid r.x \in \text{rx}(e)\} \subseteq \{p\}$ . Then,  $\{r \mid r.x \in \text{rx}(e)\} \subseteq \{p, q\}$ . Then,  $\hat{r} \in \{p, q\}$  for every  $\hat{r} \in \{r \mid r.x \in \text{rx}(e)\}$ . Then,  $\hat{r} \in \{p, q\}$  for every  $\hat{r}.x \in \text{rx}(e)$ . Then,  $\text{rx}(e) \subseteq \{r.x \mid r \in \{p, q\}\}$ . Then, by Defn. 15,  $\text{read}(q.y := e) \subseteq \{r.x \mid r \in \{p, q\}\}$ . Then, by Defn. 16,  $\text{read}(q.y := e) \subseteq \{r.x \mid r \in \text{subj}(q.y := e)\}$ . Then,  $\text{read}(\alpha) \subseteq \{r.x \mid r \in \text{subj}(\alpha)\}$ .
- **Base:**  $\alpha = i_R^R$ , for some  $R$ . Similar to case  $\alpha = \tau$ .
- **Base:**  $\alpha = pq!e$ , for some  $p, q, e$ . Similar to case “ $\alpha = q.y := e$ , for some  $q, y, e$ ”.
- **Base:**  $\alpha = pq?y$ , for some  $p, q, y$ . Similar to case  $\alpha = \tau$ .
- **Base:**  $\alpha = i_R^R r$ , for some  $R, r$ . Similar to case  $\alpha = \tau$ . □

2. By Defn. 14:

- **Base:**  $\alpha = \tau$ .  
Recall  $\emptyset \subseteq \{r.x \mid r \in \text{subj}(\alpha)\}$ . Then, by Defn. 15,  $\text{write}(\tau) \subseteq \{r.x \mid r \in \text{subj}(\alpha)\}$ . Then,  $\text{write}(\alpha) \subseteq \{r.x \mid r \in \text{subj}(\alpha)\}$ .
- **Base:**  $\alpha = q.y := e$ , for some  $q, y, e$ .  
Recall  $\{q.y\} \subseteq \{q.y\}$ . Then,  $\{q.y\} \subseteq \{r.x \mid r \in \{q\}\}$ . Then, by Defn. 15,  $\text{write}(q.y := e) \subseteq \{r.x \mid r \in \{q\}\}$ . Then, by Defn. 16,  $\text{write}(q.y := e) \subseteq \{r.x \mid r \in \text{subj}(q.y := e)\}$ . Then,  $\text{write}(\alpha) \subseteq \{r.x \mid r \in \text{subj}(\alpha)\}$ .
- **Base:**  $\alpha = i_R^R$ , for some  $R$ . Similar to case  $\alpha = \tau$ .
- **Base:**  $\alpha = p.e \rightarrow q.y$ , for some  $p, q, y, e$ . Similar to case “ $\alpha = q.y := e$ , for some  $q, y, e$ ”.
- **Base:**  $\alpha = pq!e$ , for some  $p, q, e$ . Similar to case  $\alpha = \tau$ .
- **Base:**  $\alpha = pq?y$ , for some  $p, q, y$ . Similar to case “ $\alpha = q.y := e$ , for some  $q, y, e$ ”.
- **Base:**  $\alpha = i_{\{r\}}^R$ , for some  $R, r$ . Similar to case  $\alpha = \tau$ . □

*Proof (of Lem. 32).*

1. By Defn. 14:

- **Base:**  $\gamma = \tau$ .  
Recall  $r \notin \emptyset$ . Then, by Defn. 16,  $r \notin \text{subj}(\tau)$ . Then,  $r \notin \text{subj}(\gamma)$ . Then, **false**.

- **Base:**  $\gamma = q.y := e$ , for some  $q, y, e$ .  
 By Defn. 18:
    - **Case:**  $q.y := e \upharpoonright r = q.y := e$  and  $r = q$ .  
 Recall  $q.y := e \upharpoonright r = q.y := e$ . Then,  $\text{subj}(q.y := e \upharpoonright r) = \text{subj}(q.y := e)$ . Then, by Defn. 16,  $\text{subj}(q.y := e \upharpoonright r) = \{q\}$ . Then,  $\text{subj}(\gamma \upharpoonright r) = \{r\}$ .
    - **Case:**  $r \neq q$ . Then,  $r \notin \{q\}$ . Then, by Defn. 16,  $r \notin \text{subj}(q.y := e)$ . Then,  $r \notin \text{subj}(\gamma)$ . Then, **false**.
  - **Base:**  $\gamma = p.e \rightarrow q.y$ , for some  $p, q, y, e$ .  
 By Defn. 18:
    - **Case:**  $p.e \rightarrow q.y \upharpoonright r = pq!e$  and  $p = r \neq q$ .  
 Recall  $p.e \rightarrow q.y \upharpoonright r = pq!e$ . Then,  $\text{subj}(p.e \rightarrow q.y \upharpoonright r) = \text{subj}(pq!e)$ . Then, by Defn. 16,  $\text{subj}(p.e \rightarrow q.y \upharpoonright r) = \{p\}$ . Then,  $\text{subj}(\gamma \upharpoonright r) = \{r\}$ .
    - **Case:**  $p.e \rightarrow q.y \upharpoonright r = pq?y$  and  $p \neq r = q$ . Similar to case “ $p.e \rightarrow q.y \upharpoonright r = pq!e$  and  $p = r \neq q$ ”.
    - **Case:**  $p.e \rightarrow q.y \upharpoonright r = \tau$  and  $p \neq r \neq q$ .  
 Recall  $p \neq r \neq q$ . Then,  $r \notin \{p, q\}$ . Then, by Defn. 16,  $r \notin \text{subj}(p.e \rightarrow q.y)$ . Then,  $r \notin \text{subj}(\gamma)$ . Then, **false**.
  - **Base:**  $\gamma = i_R^R$ , for some  $R$ .  
 By Defn. 18:
    - **Case:**  $i_R^R \upharpoonright r = i_{\{r\}}^R$ . Then,  $\text{subj}(i_R^R \upharpoonright r) = \text{subj}(i_{\{r\}}^R)$ . Then, by Defn. 16,  $\text{subj}(i_R^R \upharpoonright r) = \{r\}$ . Then,  $\text{subj}(\gamma \upharpoonright r) = \{r\}$ .
    - **Case:**  $r \notin R$ . Then, by Defn. 16,  $r \notin \text{subj}(i_R^R)$ . Then,  $r \notin \text{subj}(\gamma)$ . Then, **false**. □
2. By Defn. 14:
- **Base:**  $\gamma = \tau$ .  
 By Defn. 16,  $\text{subj}(\tau) = \emptyset$ . Then, by Defn. 18,  $\text{subj}(\tau \upharpoonright r) = \emptyset$ . Then,  $\text{subj}(\gamma \upharpoonright r) = \emptyset$ .
  - **Base:**  $\gamma = q.y := e$ , for some  $q, y, e$ .  
 Recall  $r \notin \text{subj}(\gamma)$ . Then,  $r \notin \text{subj}(q.y := e)$ . Then, by Defn. 16,  $r \notin \{q\}$ . Then,  $r \neq q$ . Then, by Defn. 18,  $q.y := e \upharpoonright r = \tau$ . Then,  $\text{subj}(q.y := e \upharpoonright r) = \text{subj}(\tau)$ . Then, by Defn. 16,  $\text{subj}(q.y := e \upharpoonright r) = \emptyset$ . Then,  $\text{subj}(\gamma \upharpoonright r) = \emptyset$ .
  - **Base:**  $\gamma = p.e \rightarrow q.y$ , for some  $p, q, y, e$ . Similar to case “ $\gamma = q.y := e$ , for some  $q, y, e$ ”.
  - **Base:**  $\gamma = i_R^R$ , for some  $R$ . Similar to case “ $\gamma = q.y := e$ , for some  $q, y, e$ ”. □

*Proof (of Lem. 33).* Recall:

- **Case:**  $r \in \text{subj}(\gamma)$ .
  - Recall  $r \in \text{subj}(\gamma)$ . Then, by Lem. 32,  $\text{subj}(\gamma \upharpoonright r) = \{r\}$ .
  - Recall  $r \in \text{subj}(\gamma)$ . Then,  $\{r\} \subseteq \text{subj}(\gamma)$ . Then,  $\text{subj}(\gamma \upharpoonright r) \subseteq \text{subj}(\gamma)$ .
- **Case:**  $r \notin \text{subj}(\gamma)$ .
  - Recall  $r \notin \text{subj}(\gamma)$ . Then, by Lem. 32,  $\text{subj}(\gamma \upharpoonright r) = \emptyset$ .
  - Recall  $\emptyset \subseteq \text{subj}(\gamma)$ . Then,  $\text{subj}(\gamma \upharpoonright r) \subseteq \text{subj}(\gamma)$ . □

*Proof (of Lem. 34).* By Defn. 14:

- **Base:**  $\gamma = \tau$ .  
 By Defn. 16,  $\tau \upharpoonright r = \tau$ . Then,  $\gamma \upharpoonright r = \tau$ .
- **Base:**  $\gamma = q.y := e$ , for some  $q, y, e$ .  
 Recall  $r \notin \text{subj}(\gamma)$ . Then,  $r \notin \text{subj}(q.y := e)$ . Then, by Defn. 16,  $r \notin \{q\}$ . Then,  $r \neq q$ . Then, by Defn. 18,  $q.y := e \upharpoonright r = \tau$ . Then,  $\alpha \upharpoonright r = \tau$ .
- **Base:**  $\gamma = p.e \rightarrow q.y$ , for some  $p, q, y, e$ . Similar to case “ $\gamma = q.y := e$ , for some  $q, y, e$ ”.
- **Base:**  $\gamma = i_R^R$ , for some  $R$ . Similar to case “ $\gamma = q.y := e$ , for some  $q, y, e$ ”. □

*Proof (of Lem. 35).* By Defn. 14:

– **Base:**  $\gamma = \tau$ .

Recall  $\emptyset \subseteq \text{chan}(\gamma)$ . Then, by Defn. 17,  $\text{chan}(\tau) \subseteq \text{chan}(\gamma)$ . Then, by Defn. 18,  $\text{chan}(\tau \upharpoonright r) \subseteq \text{chan}(\gamma)$ . Then,  $\text{chan}(\gamma \upharpoonright r) \subseteq \text{chan}(\gamma)$ .

– **Base:**  $\gamma = q.y := e$ , for some  $q, y, e$ .

By Defn. 18:

• **Case:**  $q.y := e \upharpoonright r = q.y := e$ .

Recall  $\emptyset \subseteq \text{chan}(\gamma)$ . Then, by Defn. 17,  $\text{chan}(q.y := e) \subseteq \text{chan}(\gamma)$ . Then,  $\text{chan}(q.y := e \upharpoonright r) \subseteq \text{chan}(\gamma)$ . Then,  $\text{chan}(\gamma \upharpoonright r) \subseteq \text{chan}(\gamma)$ .

• **Case:**  $q.y := e \upharpoonright r = \tau$ . Similar to case  $q.y := e \upharpoonright r = q.y := e$ .

– **Base:**  $\gamma = p.e \rightarrow q.y$ , for some  $p, q, y, e$ .

By Defn. 18:

• **Case:**  $p.e \rightarrow q.y \upharpoonright r = pq!e$ .

Recall  $\{pq\} \subseteq \{pq\}$ . Then, by Defn. 17,  $\text{chan}(pq!e) \subseteq \text{chan}(p.e \rightarrow q.y)$ . Then,  $\text{chan}(p.e \rightarrow q.y \upharpoonright r) \subseteq \text{chan}(p.e \rightarrow q.y)$ . Then,  $\text{chan}(\gamma \upharpoonright r) \subseteq \text{chan}(\gamma)$ .

• **Case:**  $p.e \rightarrow q.y \upharpoonright r = pq?y$ . Similar to case  $p.e \rightarrow q.y \upharpoonright r = pq!e$ .

• **Case:**  $p.e \rightarrow q.y \upharpoonright r = \tau$ .

Recall  $\emptyset \subseteq \text{chan}(\gamma)$ . Then, by Defn. 17,  $\text{chan}(\tau) \subseteq \text{chan}(\gamma)$ . Then,  $\text{chan}(p.e \rightarrow q.y \upharpoonright r) \subseteq \text{chan}(\gamma)$ . Then,  $\text{chan}(\gamma \upharpoonright r) \subseteq \text{chan}(\gamma)$ .

– **Base:**  $\gamma = i_R^R$ , for some  $R$ .

By Defn. 18:

• **Case:**  $i_R^R \upharpoonright r = i_{\{r\}}^R$ .

Recall  $\{R\} \subseteq \{R\}$ . Then, by Defn. 17,  $\text{chan}(i_{\{r\}}^R) \subseteq \text{chan}(i_R^R)$ . Then,  $\text{chan}(i_R^R \upharpoonright r) \subseteq \text{chan}(i_R^R)$ . Then,  $\text{chan}(\gamma \upharpoonright r) \subseteq \text{chan}(\gamma)$ .

• **Case:**  $i_R^R \upharpoonright r = \tau$ .

Recall  $\emptyset \subseteq \text{chan}(\gamma)$ . Then, by Defn. 17,  $\text{chan}(\tau) \subseteq \text{chan}(\gamma)$ . Then,  $\text{chan}(i_R^R \upharpoonright r) \subseteq \text{chan}(\gamma)$ . Then,  $\text{chan}(\gamma \upharpoonright r) \subseteq \text{chan}(\gamma)$ .  $\square$

## D Programs

*Proof (of Lem. 36).*

1. By Defn. 22,  $\text{read}(\text{if } \xi|_n A_1|_{R_1} A_2|_{R_2}) = (\text{read}(\xi) \setminus \{r.x \mid r \in R_1 \cup R_2\}) \cup (\text{read}(A_1) \setminus \{r.x \mid r \in R_2\}) \cup (\text{read}(A_2) \setminus \{r.x \mid r \in R_1\})$ . Then,  $\text{read}(\text{if } \xi|_n A_1|_{R_1} A_2|_{R_2}) = (\text{read}(\xi) \setminus \{r.x \mid r \in \emptyset \cup \emptyset\}) \cup (\text{read}(A_1) \setminus \{r.x \mid r \in \emptyset\}) \cup (\text{read}(A_2) \setminus \{r.x \mid r \in \emptyset\})$ . Then,  $\text{read}(\text{if } \xi|_n A_1|_{R_1} A_2|_{R_2}) = (\text{read}(\xi) \setminus \{r.x \mid r \in \emptyset\}) \cup (\text{read}(A_1) \setminus \{r.x \mid r \in \emptyset\}) \cup (\text{read}(A_2) \setminus \{r.x \mid r \in \emptyset\})$ . Then,  $\text{read}(\text{if } \xi|_n A_1|_{R_1} A_2|_{R_2}) = (\text{read}(\xi) \setminus \emptyset) \cup (\text{read}(A_1) \setminus \emptyset) \cup (\text{read}(A_2) \setminus \emptyset)$ . Then,  $\text{read}(\text{if } \xi|_n A_1|_{R_1} A_2|_{R_2}) = \text{read}(\xi) \cup \text{read}(A_1) \cup \text{read}(A_2)$ . □
2. By Defn. 22,  $\text{read}(\text{if } \xi|_n A_1|_{R_1} A_2|_{R_2}) = (\text{read}(\xi) \setminus \{r.x \mid r \in R_1 \cup R_2\}) \cup (\text{read}(A_1) \setminus \{r.x \mid r \in R_2\}) \cup (\text{read}(A_2) \setminus \{r.x \mid r \in R_1\})$ . Then,  $\text{read}(\text{if } \xi|_n A_1|_{R_1} A_2|_{R_2}) = (\text{read}(\xi) \setminus \{r.x \mid r \in \emptyset \cup R_2\}) \cup (\text{read}(A_1) \setminus \{r.x \mid r \in R_2\}) \cup (\text{read}(A_2) \setminus \{r.x \mid r \in \emptyset\})$ . Then,  $\text{read}(\text{if } \xi|_n A_1|_{R_1} A_2|_{R_2}) = (\text{read}(\xi) \setminus \{r.x \mid r \in R_2\}) \cup (\text{read}(A_1) \setminus \{r.x \mid r \in R_2\}) \cup (\text{read}(A_2) \setminus \{r.x \mid r \in \emptyset\})$ . Then,  $\text{read}(\text{if } \xi|_n A_1|_{R_1} A_2|_{R_2}) = (\text{read}(\xi) \setminus \{r.x \mid r \in R_2\}) \cup (\text{read}(A_1) \setminus \{r.x \mid r \in R_2\}) \cup (\text{read}(A_2) \setminus \emptyset)$ . Then,  $\text{read}(\text{if } \xi|_n A_1|_{R_1} A_2|_{R_2}) = (\text{read}(\xi) \setminus \{r.x \mid r \in R_2\}) \cup (\text{read}(A_1) \setminus \{r.x \mid r \in R_2\}) \cup \text{read}(A_2)$ . □
3. Similar to 2. □
4. Similar to 1. □
5. Similar to 2. □
6. Similar to 2. □

*Proof (of Lem. 37).* Recall  $A_1 \# A_2$ . Then, by Defn. 23,  $\text{read}(A_1) \cap \text{read}(A_2) = \emptyset$  and  $\text{read}(A_1) \cap \text{write}(A_2) = \emptyset$  and  $\text{write}(A_1) \cap \text{read}(A_2) = \emptyset$  and  $\text{write}(A_1) \cap \text{write}(A_2) = \emptyset$ . Then,  $\text{read}(A_2) \cap \text{read}(A_1) = \emptyset$  and  $\text{write}(A_2) \cap \text{read}(A_1) = \emptyset$  and  $\text{read}(A_2) \cap \text{write}(A_1) = \emptyset$  and  $\text{write}(A_2) \cap \text{write}(A_1) = \emptyset$ . Then, by [#],  $A_2 \# A_1$ . □

*Proof (of Lem. 38).*

1. Recall  $A_1 \circ A_2 \# A$ . Then, by Defn. 23,  $\text{read}(A_1 \circ A_2) \cap \text{read}(A) = \emptyset$  and  $\text{read}(A_1 \circ A_2) \cap \text{write}(A) = \emptyset$  and  $\text{write}(A_1 \circ A_2) \cap \text{read}(A) = \emptyset$  and  $\text{write}(A_1 \circ A_2) \cap \text{write}(A) = \emptyset$ . Then, by Defn. 22,  $(\text{read}(A_1) \cup \text{read}(A_2)) \cap \text{read}(A) = \emptyset$  and  $(\text{read}(A_1) \cup \text{read}(A_2)) \cap \text{write}(A) = \emptyset$  and  $(\text{write}(A_1) \cup \text{write}(A_2)) \cap \text{read}(A) = \emptyset$  and  $(\text{write}(A_1) \cup \text{write}(A_2)) \cap \text{write}(A) = \emptyset$ . Then,  $\text{read}(A_1) \cap \text{read}(A) = \emptyset$  and  $\text{read}(A_1) \cap \text{write}(A) = \emptyset$  and  $\text{write}(A_1) \cap \text{read}(A) = \emptyset$  and  $\text{write}(A_1) \cap \text{write}(A) = \emptyset$  and  $\text{read}(A_2) \cap \text{read}(A) = \emptyset$  and  $\text{read}(A_2) \cap \text{write}(A) = \emptyset$  and  $\text{write}(A_2) \cap \text{read}(A) = \emptyset$  and  $\text{write}(A_2) \cap \text{write}(A) = \emptyset$ . Then, by [#],  $A_1 \# A$  and  $A_2 \# A$ . □
2. Similar to 1. □
3. Similar to 1. □
4. – Recall  $R_1 = \emptyset = R_2$ . Then, by Lem. 36  $\text{read}(\text{if } \xi|_n A_1|_{R_1} A_2|_{R_2}) = \text{read}(\xi) \cup \text{read}(A_1) \cup \text{read}(A_2)$  and  $\text{write}(\text{if } \xi|_n A_1|_{R_1} A_2|_{R_2}) = \text{write}(\xi) \cup \text{write}(A_1) \cup \text{write}(A_2)$ .  
 – Recall  $\text{if } \xi|_n A_1|_{R_1} A_2|_{R_2} \# A$ . Then, by Defn. 23,  $\text{read}(\text{if } \xi|_n A_1|_{R_1} A_2|_{R_2}) \cap \text{read}(A) = \emptyset$  and  $\text{read}(\text{if } \xi|_n A_1|_{R_1} A_2|_{R_2}) \cap \text{write}(A) = \emptyset$  and  $\text{write}(\text{if } \xi|_n A_1|_{R_1} A_2|_{R_2}) \cap \text{read}(A) = \emptyset$  and  $\text{write}(\text{if } \xi|_n A_1|_{R_1} A_2|_{R_2}) \cap \text{write}(A) = \emptyset$ . Then,  $(\text{read}(\xi) \cup \text{read}(A_1) \cup \text{read}(A_2)) \cap \text{read}(A) = \emptyset$  and  $(\text{read}(\xi) \cup \text{read}(A_1) \cup \text{read}(A_2)) \cap \text{write}(A) = \emptyset$  and  $(\text{write}(\xi) \cup \text{write}(A_1) \cup \text{write}(A_2)) \cap \text{read}(A) = \emptyset$  and  $(\text{write}(\xi) \cup \text{write}(A_1) \cup \text{write}(A_2)) \cap \text{write}(A) = \emptyset$ . Then,  $\text{read}(\xi) \cap \text{read}(A) = \emptyset$  and  $\text{read}(\xi) \cap \text{write}(A) = \emptyset$  and  $\text{write}(\xi) \cap \text{read}(A) = \emptyset$  and  $\text{write}(\xi) \cap \text{write}(A) = \emptyset$  and  $\text{read}(A_1) \cap \text{read}(A) = \emptyset$  and  $\text{read}(A_1) \cap \text{write}(A) = \emptyset$  and  $\text{write}(A_1) \cap \text{read}(A) = \emptyset$  and  $\text{write}(A_1) \cap \text{write}(A) = \emptyset$  and  $\text{read}(A_2) \cap \text{read}(A) = \emptyset$  and  $\text{read}(A_2) \cap \text{write}(A) = \emptyset$  and  $\text{write}(A_2) \cap \text{read}(A) = \emptyset$  and  $\text{write}(A_2) \cap \text{write}(A) = \emptyset$  and  $\text{read}(A) \cap \text{write}(A) = \emptyset$ . □



- $\text{write}(A_2) \cap \text{read}(A) = \emptyset$  and  $\text{write}(A_2) \cap \text{write}(A) = \emptyset$ . Then, by [#],  $\xi \# A$  and  $A_1 \# A$  and  $A_2 \# A$ .  $\square$
5. – Recall  $R_1 = \emptyset \neq R_2$ . Then, by Lem. 36,  
 $\text{read}(\text{if } \xi|_n A_1|_{R_1} A_2|_{R_2}) = (\text{read}(\xi) \setminus \{r.x \mid r \in R_2\}) \cup (\text{read}(A_1) \setminus \{r.x \mid r \in R_2\}) \cup \text{read}(A_2)$   
and  $\text{write}(\text{if } \xi|_n A_1|_{R_1} A_2|_{R_2}) = (\text{write}(\xi) \setminus \{r.x \mid r \in R_2\}) \cup (\text{write}(A_1) \setminus \{r.x \mid r \in R_2\}) \cup \text{write}(A_2)$ .
- Recall  $\text{if } \xi|_n A_1|_{R_1} A_2|_{R_2} \# A$ . Then, by Defn. 23,  $\text{read}(\text{if } \xi|_n A_1|_{R_1} A_2|_{R_2}) \cap \text{read}(A) = \emptyset$  and  $\text{read}(\text{if } \xi|_n A_1|_{R_1} A_2|_{R_2}) \cap \text{write}(A) = \emptyset$  and  $\text{write}(\text{if } \xi|_n A_1|_{R_1} A_2|_{R_2}) \cap \text{read}(A) = \emptyset$  and  $\text{write}(\text{if } \xi|_n A_1|_{R_1} A_2|_{R_2}) \cap \text{write}(A) = \emptyset$ . Then,  
 $((\text{read}(\xi) \setminus \{r.x \mid r \in R_2\}) \cup (\text{read}(A_1) \setminus \{r.x \mid r \in R_2\})) \cup \text{read}(A_2) \cap \text{read}(A) = \emptyset$  and  
 $((\text{read}(\xi) \setminus \{r.x \mid r \in R_2\}) \cup (\text{read}(A_1) \setminus \{r.x \mid r \in R_2\})) \cup \text{read}(A_2) \cap \text{write}(A) = \emptyset$  and  
 $((\text{write}(\xi) \setminus \{r.x \mid r \in R_2\}) \cup (\text{write}(A_1) \setminus \{r.x \mid r \in R_2\})) \cup \text{write}(A_2) \cap \text{read}(A) = \emptyset$  and  
 $((\text{write}(\xi) \setminus \{r.x \mid r \in R_2\}) \cup (\text{write}(A_1) \setminus \{r.x \mid r \in R_2\})) \cup \text{write}(A_2) \cap \text{write}(A) = \emptyset$ .  
Then,  $(\text{read}(\xi) \setminus \{r.x \mid r \in R_2\}) \cap \text{read}(A) = \emptyset$  and  
 $(\text{read}(\xi) \setminus \{r.x \mid r \in R_2\}) \cap \text{write}(A) = \emptyset$  and  $(\text{write}(\xi) \setminus \{r.x \mid r \in R_2\}) \cap \text{read}(A) = \emptyset$  and  
 $(\text{write}(\xi) \setminus \{r.x \mid r \in R_2\}) \cap \text{write}(A) = \emptyset$  and  $\text{read}(A_2) \cap \text{read}(A) = \emptyset$  and  
 $\text{read}(A_2) \cap \text{write}(A) = \emptyset$  and  $\text{write}(A_2) \cap \text{read}(A) = \emptyset$  and  $\text{write}(A_2) \cap \text{write}(A) = \emptyset$ .
- Recall  $\hat{r} \notin R_2$  for every  $\hat{r} \in \text{subj}(\xi) \setminus R_2$ . Then,  $\{\hat{r}\} \cap R_2 = \emptyset$  for every  $\hat{r} \in \text{subj}(\xi) \setminus R_2$ .  
Then,  $\{r.x \mid r \in \{\hat{r}\}\} \cap \{r.x \mid r \in R_2\} = \emptyset$  for every  $\hat{r} \in \text{subj}(\xi) \setminus R_2$ .
- By Lem. 5,  $\text{read}(\xi \upharpoonright \hat{r}) \subseteq \{r.x \mid r \in \text{subj}(\xi \upharpoonright \hat{r})\}$  and  $\text{write}(\xi \upharpoonright \hat{r}) \subseteq \{r.x \mid r \in \text{subj}(\xi \upharpoonright \hat{r})\}$ , for every  $\hat{r} \in \text{subj}(\xi)$ . Then, by Lem. 9,  $\text{read}(\xi \upharpoonright \hat{r}) \subseteq \{r.x \mid r \in \{\hat{r}\}\}$  and  $\text{write}(\xi \upharpoonright \hat{r}) \subseteq \{r.x \mid r \in \{\hat{r}\}\}$ , for every  $\hat{r} \in \text{subj}(\xi) \setminus R_2$ .
- Recall  $\{r.x \mid r \in \{\hat{r}\}\} \cap \{r.x \mid r \in R_2\} = \emptyset$  and  $\text{read}(\xi \upharpoonright \hat{r}) \subseteq \{r.x \mid r \in \{\hat{r}\}\}$  and  $\text{write}(\xi \upharpoonright \hat{r}) \subseteq \{r.x \mid r \in \{\hat{r}\}\}$ , for every  $\hat{r} \in \text{subj}(\xi) \setminus R_2$ . Then,  
 $\text{read}(\xi \upharpoonright \hat{r}) \cap \{r.x \mid r \in R_2\} = \emptyset$  and  $\text{write}(\xi \upharpoonright \hat{r}) \cap \{r.x \mid r \in R_2\} = \emptyset$ , for every  $\hat{r} \in \text{subj}(\xi) \setminus R_2$ . Then,  $\text{read}(\xi \upharpoonright \hat{r}) \setminus \{r.x \mid r \in R_2\} = \text{read}(\xi \upharpoonright \hat{r})$  and  $\text{write}(\xi \upharpoonright \hat{r}) \setminus \{r.x \mid r \in R_2\} = \text{write}(\xi \upharpoonright \hat{r})$ , for every  $\hat{r} \in \text{subj}(\xi) \setminus R_2$ .
- By Lem. 8,  $\text{read}(\xi \upharpoonright \hat{r}) \subseteq \text{read}(\xi)$  and  $\text{write}(\xi \upharpoonright \hat{r}) \subseteq \text{write}(\xi)$ , for every  $\hat{r} \in \text{subj}(\xi)$ . Then,  
 $\text{read}(\xi \upharpoonright \hat{r}) \setminus \{r.x \mid r \in R_2\} \subseteq \text{read}(\xi) \setminus \{r.x \mid r \in R_2\}$ , and  
 $\text{write}(\xi \upharpoonright \hat{r}) \setminus \{r.x \mid r \in R_2\} \subseteq \text{write}(\xi) \setminus \{r.x \mid r \in R_2\}$ , for every  $\hat{r} \in \text{subj}(\xi)$ . Then,  
 $\text{read}(\xi \upharpoonright \hat{r}) \setminus \{r.x \mid r \in R_2\} \subseteq \text{read}(\xi) \setminus \{r.x \mid r \in R_2\}$  and  
 $\text{write}(\xi \upharpoonright \hat{r}) \setminus \{r.x \mid r \in R_2\} \subseteq \text{write}(\xi) \setminus \{r.x \mid r \in R_2\}$ , for every  $\hat{r} \in \text{subj}(\xi) \setminus R_2$ . Then,  
 $\text{read}(\xi \upharpoonright \hat{r}) \subseteq \text{read}(\xi) \setminus \{r.x \mid r \in R_2\}$  and  $\text{write}(\xi \upharpoonright \hat{r}) \subseteq \text{write}(\xi) \setminus \{r.x \mid r \in R_2\}$ , for every  $\hat{r} \in \text{subj}(\xi) \setminus R_2$ .
- Recall  $(\text{read}(\xi) \setminus \{r.x \mid r \in R_2\}) \cap \text{read}(A) = \emptyset$  and  $(\text{read}(\xi) \setminus \{r.x \mid r \in R_2\}) \cap \text{write}(A) = \emptyset$  and  $(\text{write}(\xi) \setminus \{r.x \mid r \in R_2\}) \cap \text{read}(A) = \emptyset$  and  $(\text{write}(\xi) \setminus \{r.x \mid r \in R_2\}) \cap \text{write}(A) = \emptyset$  and  $\text{read}(\xi \upharpoonright \hat{r}) \subseteq \text{read}(\xi) \setminus \{r.x \mid r \in R_2\}$  and  $\text{write}(\xi \upharpoonright \hat{r}) \subseteq \text{write}(\xi) \setminus \{r.x \mid r \in R_2\}$ , for every  $\hat{r} \in \text{subj}(\xi) \setminus R_2$ . Then,  $\text{read}(\xi \upharpoonright \hat{r}) \cap \text{read}(A) = \emptyset$  and  $\text{read}(\xi \upharpoonright \hat{r}) \cap \text{write}(A) = \emptyset$  and  $\text{write}(\xi \upharpoonright \hat{r}) \cap \text{read}(A) = \emptyset$  and  $\text{write}(\xi \upharpoonright \hat{r}) \cap \text{write}(A) = \emptyset$ , for every  $\hat{r} \in \text{subj}(\xi) \setminus R_2$ . Then, by [#],  $\xi \upharpoonright \hat{r} \# A$  for every  $\hat{r} \in \text{subj}(\xi) \setminus R_2$ .
- Recall  $\text{read}(A_2) \cap \text{read}(A) = \emptyset$  and  $\text{read}(A_2) \cap \text{write}(A) = \emptyset$  and  $\text{write}(A_2) \cap \text{read}(A) = \emptyset$  and  $\text{write}(A_2) \cap \text{write}(A) = \emptyset$ . Then, by [#], and  $A_2 \# A$ .  $\square$
6. Similar to 5.  $\square$
7. Similar to 1.  $\square$

*Proof (of Lem. 39).*

1. Recall  $\xi_r \# A$  for every  $r \in R$ . Then, by Defn. 23,  $\text{read}(\xi_r) \cap \text{read}(A) = \emptyset$  and  $\text{read}(\xi_r) \cap \text{write}(A) = \emptyset$  and  $\text{write}(\xi_r) \cap \text{read}(A) = \emptyset$  and  $\text{write}(\xi_r) \cap \text{write}(A) = \emptyset$  and  $\text{read}(\xi_2) \cap \text{read}(A) = \emptyset$ , for every  $r \in R$ . Then,  $\bigcup \{\text{read}(\xi_r)\}_{r \in R} \cap \text{read}(A) = \emptyset$  and  $\bigcup \{\text{read}(\xi_r)\}_{r \in R} \cap \text{write}(A) = \emptyset$  and  $\bigcup \{\text{write}(\xi_r)\}_{r \in R} \cap \text{read}(A) = \emptyset$  and  $\bigcup \{\text{write}(\xi_r)\}_{r \in R} \cap \text{write}(A) = \emptyset$ . Then, by Lem. 12,  $\bigcup \{\text{read}(\xi_r^+)\}_{r \in R} \cap \text{read}(A) = \emptyset$  and  $\bigcup \{\text{read}(\xi_r^+)\}_{r \in R} \cap \text{write}(A) = \emptyset$  and  $\bigcup \{\text{write}(\xi_r^+)\}_{r \in R} \cap \text{read}(A) = \emptyset$  and

- $\bigcup \{\text{write}(\xi_r^+)\}_{r \in R} \cap \text{write}(A) = \emptyset$ . Then, by Lem. 3,  $\text{read}(\bigwedge \{\xi_r^+\}_{r \in R}) \cap \text{read}(A) = \emptyset$  and  $\text{read}(\bigwedge \{\xi_r^+\}_{r \in R}) \cap \text{write}(A) = \emptyset$  and  $\text{write}(\bigwedge \{\xi_r^+\}_{r \in R}) \cap \text{read}(A) = \emptyset$  and  $\text{write}(\bigwedge \{\xi_r^+\}_{r \in R}) \cap \text{write}(A) = \emptyset$ . Then, by [#],  $\bigwedge \{\xi_r^+\}_{r \in R} \# A$ . □
2. Similar to 1. □

*Proof (of Lem. 40).*

1. By Defn. 21:
  - **Base:**  $A = \xi$ , for some  $\xi$ .  
By Lem. 5,  $\text{read}(\psi) \subseteq \{r.x \mid r \in \text{subj}(\psi)\}$ . Then,  $\text{read}(A) \subseteq \{r.x \mid r \in \text{subj}(A)\}$ .
  - **Base:**  $A = \alpha$ , for some  $\alpha$ .  
By Lem. 31,  $\text{read}(\alpha) \subseteq \{r.x \mid r \in \text{subj}(\alpha)\}$ . Then,  $\text{read}(A) \subseteq \{r.x \mid r \in \text{subj}(A)\}$ .
  - **Base:**  $A = \text{skip}$ .  
Recall  $\emptyset \subseteq \{r.x \mid r \in \text{subj}(A)\}$ . Then, by Defn. 22,  $\text{read}(\text{skip}) \subseteq \{r.x \mid r \in \text{subj}(A)\}$ . Then,  $\text{read}(A) \subseteq \{r.x \mid r \in \text{subj}(A)\}$ .
  - **Step:**  $A = A_1 \circ A_2$ , for some  $A_1, A_2$ .  
By induction,  $\text{read}(A_1) \subseteq \{r.x \mid r \in \text{subj}(A_1)\}$  and  $\text{read}(A_2) \subseteq \{r.x \mid r \in \text{subj}(A_2)\}$ . Then,  $\text{read}(A_1) \cup \text{read}(A_2) \subseteq \{r.x \mid r \in \text{subj}(A_1)\} \cup \{r.x \mid r \in \text{subj}(A_2)\}$ . Then,  $\text{read}(A_1) \cup \text{read}(A_2) \subseteq \{r.x \mid r \in \text{subj}(A_1) \cup \text{subj}(A_2)\}$ . Then, by Defn. 22,  $\text{read}(A_1 \circ A_2) \subseteq \{r.x \mid r \in \text{subj}(A_1) \cup \text{subj}(A_2)\}$ . Then, by Defn. 24,  $\text{read}(A_1 \circ A_2) \subseteq \{r.x \mid r \in \text{subj}(A_1 \circ A_2)\}$ . Then,  $\text{read}(A) \subseteq \{r.x \mid r \in \text{subj}(A)\}$ .
  - **Step:**  $A = R.\text{if } \xi A_1 A_2$ , for some  $R, A_1, A_2, \xi$ . Similar to case “ $A = A_1 \circ A_2$ , for some  $A_1, A_2$ ”.
  - **Step:**  $A = R.\text{while } \xi \{\psi\} \hat{A}$ , for some  $R, \hat{A}, \psi, \xi$ . Similar to case “ $A = A_1 \circ A_2$ , for some  $A_1, A_2$ ”.
  - **Step:**  $A = \text{if } \xi|_n A_1|_{R_1} A_2|_{R_2}$ , for some  $R_1, R_2, A_1, A_2, \xi, n$ .
    - By induction,  $\text{read}(\xi) \subseteq \{r.x \mid r \in \text{subj}(\xi)\}$ . Then,  $\text{read}(\xi) \setminus \{r.x \mid r \in R_1 \cup R_2\} \subseteq \{r.x \mid r \in \text{subj}(\xi)\} \setminus \{r.x \mid r \in R_1 \cup R_2\}$ . Then,  $\text{read}(\xi) \setminus \{r.x \mid r \in R_1 \cup R_2\} \subseteq \{r.x \mid r \in \text{subj}(\xi) \setminus (R_1 \cup R_2)\}$ .
    - By induction,  $\text{read}(A_1) \subseteq \{r.x \mid r \in \text{subj}(A_1)\}$ . Then,  $\text{read}(A_1) \setminus \{r.x \mid r \in R_2\} \subseteq \{r.x \mid r \in \text{subj}(A_1)\} \setminus \{r.x \mid r \in R_2\}$ . Then,  $\text{read}(A_1) \setminus \{r.x \mid r \in R_2\} \subseteq \{r.x \mid r \in \text{subj}(A_1) \setminus R_2\}$ .
    - By induction,  $\text{read}(A_2) \subseteq \{r.x \mid r \in \text{subj}(A_2)\}$ . Then,  $\text{read}(A_2) \setminus \{r.x \mid r \in R_1\} \subseteq \{r.x \mid r \in \text{subj}(A_2)\} \setminus \{r.x \mid r \in R_1\}$ . Then,  $\text{read}(A_2) \setminus \{r.x \mid r \in R_1\} \subseteq \{r.x \mid r \in \text{subj}(A_2) \setminus R_1\}$ .
    - Recall  $\text{read}(\xi) \setminus \{r.x \mid r \in R_1 \cup R_2\} \subseteq \{r.x \mid r \in \text{subj}(\xi) \setminus (R_1 \cup R_2)\}$  and  $\text{read}(A_1) \setminus \{r.x \mid r \in R_2\} \subseteq \{r.x \mid r \in \text{subj}(A_1) \setminus R_2\}$  and  $\text{read}(A_2) \setminus \{r.x \mid r \in R_1\} \subseteq \{r.x \mid r \in \text{subj}(A_2) \setminus R_1\}$ . Then,  $(\text{read}(\xi) \setminus \{r.x \mid r \in R_1 \cup R_2\}) \cup (\text{read}(A_1) \setminus \{r.x \mid r \in R_2\}) \cup (\text{read}(A_2) \setminus \{r.x \mid r \in R_1\}) \subseteq \{r.x \mid r \in \text{subj}(\xi) \setminus (R_1 \cup R_2)\} \cup \{r.x \mid r \in \text{subj}(A_1) \setminus R_2\} \cup \{r.x \mid r \in \text{subj}(A_2) \setminus R_1\}$ . Then,  $(\text{read}(\xi) \setminus \{r.x \mid r \in R_1 \cup R_2\}) \cup (\text{read}(A_1) \setminus \{r.x \mid r \in R_2\}) \cup (\text{read}(A_2) \setminus \{r.x \mid r \in R_1\}) \subseteq \{r.x \mid r \in (\text{subj}(\xi) \setminus (R_1 \cup R_2)) \cup (\text{subj}(A_1) \setminus R_2) \cup (\text{subj}(A_2) \setminus R_1)\}$ . Then, by Defn. 22,  $\text{read}(\text{if } \xi|_n A_1|_{R_1} A_2|_{R_2}) \subseteq \{r.x \mid r \in (\text{subj}(\xi) \setminus (R_1 \cup R_2)) \cup (\text{subj}(A_1) \setminus R_2) \cup (\text{subj}(A_2) \setminus R_1)\}$ . Then, by Defn. 24,  $\text{read}(\text{if } \xi|_n A_1|_{R_1} A_2|_{R_2}) \subseteq \{r.x \mid r \in \text{subj}(\text{if } \xi|_n A_1|_{R_1} A_2|_{R_2})\}$ . Then,  $\text{read}(A) \subseteq \{r.x \mid r \in \text{subj}(A)\}$ .
  - **Step:**  $A = \text{while } \xi|_n \{\psi\} \hat{A}|_\emptyset$ , for some  $\hat{A}, \psi, \xi$ . Similar to case “ $A = A_1 \circ A_2$ , for some  $A_1, A_2$ ”.
2. Similar to 1. □

*Proof (of Lem. 41).*

- Recall  $\text{subj}(A_1) \cap \text{subj}(A_2) = \emptyset$ . Then,  $\{r.x \mid r \in \text{subj}(A_1)\} \cap \{r.x \mid r \in \text{subj}(A_2)\} = \emptyset$ .

- By Lem. 40,  $\text{read}(A_1) \subseteq \{r.x \mid r \in \text{subj}(A_1)\}$  and  $\text{write}(A_1) \subseteq \{r.x \mid r \in \text{subj}(A_1)\}$  and  $\text{read}(A_2) \subseteq \{r.x \mid r \in \text{subj}(A_2)\}$  and  $\text{write}(A_2) \subseteq \{r.x \mid r \in \text{subj}(A_2)\}$ .
- Recall  $\{r.x \mid r \in \text{subj}(A_1)\} \cap \{r.x \mid r \in \text{subj}(A_2)\} = \emptyset$  and  $\text{read}(A_1) \subseteq \{r.x \mid r \in \text{subj}(A_1)\}$  and  $\text{read}(A_2) \subseteq \{r.x \mid r \in \text{subj}(A_2)\}$ . Then,  $\text{read}(A_1) \cap \text{read}(A_2) = \emptyset$ .
- Recall  $\{r.x \mid r \in \text{subj}(A_1)\} \cap \{r.x \mid r \in \text{subj}(A_2)\} = \emptyset$  and  $\text{read}(A_1) \subseteq \{r.x \mid r \in \text{subj}(A_1)\}$  and  $\text{write}(A_2) \subseteq \{r.x \mid r \in \text{subj}(A_2)\}$ . Then,  $\text{read}(A_1) \cap \text{write}(A_2) = \emptyset$ .
- Recall  $\{r.x \mid r \in \text{subj}(A_1)\} \cap \{r.x \mid r \in \text{subj}(A_2)\} = \emptyset$  and  $\text{write}(A_1) \subseteq \{r.x \mid r \in \text{subj}(A_1)\}$  and  $\text{read}(A_2) \subseteq \{r.x \mid r \in \text{subj}(A_2)\}$ . Then,  $\text{write}(A_1) \cap \text{read}(A_2) = \emptyset$ .
- Recall  $\{r.x \mid r \in \text{subj}(A_1)\} \cap \{r.x \mid r \in \text{subj}(A_2)\} = \emptyset$  and  $\text{write}(A_1) \subseteq \{r.x \mid r \in \text{subj}(A_1)\}$  and  $\text{write}(A_2) \subseteq \{r.x \mid r \in \text{subj}(A_2)\}$ . Then,  $\text{write}(A_1) \cap \text{write}(A_2) = \emptyset$ .
- Recall  $\text{read}(A_1) \cap \text{read}(A_2) = \emptyset$  and  $\text{read}(A_1) \cap \text{write}(A_2) = \emptyset$  and  $\text{write}(A_1) \cap \text{read}(A_2) = \emptyset$  and  $\text{write}(A_1) \cap \text{write}(A_2) = \emptyset$ . Then, by [#],  $A_1 \# A_2$ .  $\square$

*Proof (of Lem. 42).* By Defn. 21:

- **Base:**  $G = \xi$ , for some  $\xi$ .
  1. Recall  $r \in \text{subj}(G)$ . Then,  $r \in \text{subj}(\xi)$ . Then, by Lem. 9,  $\text{subj}(\xi \upharpoonright r) = \{r\}$ . Then,  $\text{subj}(G \upharpoonright r) = \{r\}$ .
  2. Recall  $r \notin \text{subj}(G)$ . Then,  $r \notin \text{subj}(\xi)$ . Then, by Lem. 9,  $\text{subj}(\xi \upharpoonright r) = \emptyset$ . Then,  $\text{subj}(G \upharpoonright r) = \emptyset$ .
- **Base:**  $G = \gamma$ , for some  $\gamma$ .
  1. Recall  $r \in \text{subj}(G)$ . Then,  $r \in \text{subj}(\gamma)$ . Then, by Lem. 32,  $\text{subj}(\gamma \upharpoonright r) = \{r\}$ . Then,  $\text{subj}(G \upharpoonright r) = \{r\}$ .
  2. Recall  $r \notin \text{subj}(G)$ . Then,  $r \notin \text{subj}(\gamma)$ . Then, by Lem. 32,  $\text{subj}(\gamma \upharpoonright r) = \emptyset$ . Then,  $\text{subj}(G \upharpoonright r) = \emptyset$ .
- **Base:**  $G = \text{skip}$ .
  1. Recall  $r \in \text{subj}(G)$ . Then,  $r \in \text{subj}(\text{skip})$ . Then, by Defn. 24,  $r \in \emptyset$ . Then, **false**.
  2. By Defn. 24,  $\text{subj}(\text{skip}) = \emptyset$ . Then, by Defn. 26,  $\text{subj}(\text{skip} \upharpoonright r) = \emptyset$ . Then,  $\text{subj}(G \upharpoonright r) = \emptyset$ .
- **Step:**  $G = G_1 \circ G_2$ , for some  $G_1, G_2$ .
  1. Recall:
    - **Case:**  $r \in \text{subj}(G_1)$  and  $r \in \text{subj}(G_2)$ . Then, by induction,  $\text{subj}(G_1 \upharpoonright r) = \{r\}$  and  $\text{subj}(G_2 \upharpoonright r) = \{r\}$ . Then,  $\text{subj}(G_1 \upharpoonright r) \cup \text{subj}(G_2 \upharpoonright r) = \{r\} \cup \{r\}$ . Then,  $\text{subj}(G_1 \upharpoonright r) \cup \text{subj}(G_2 \upharpoonright r) = \{r\}$ . Then, by Defn. 24,  $\text{subj}((G_1 \upharpoonright r) \circ (G_2 \upharpoonright r)) = \{r\}$ . Then, by Defn. 26,  $\text{subj}(G_1 \circ G_2 \upharpoonright r) = \{r\}$ . Then,  $\text{subj}(G \upharpoonright r) = \{r\}$ .
    - **Case:**  $r \in \text{subj}(G_1)$  and  $r \notin \text{subj}(G_2)$ . Similar to case “ $r \in \text{subj}(G_1)$  and  $r \in \text{subj}(G_2)$ ”.
    - **Case:**  $r \notin \text{subj}(G_1)$  and  $r \in \text{subj}(G_2)$ . Similar to case “ $r \in \text{subj}(G_1)$  and  $r \in \text{subj}(G_2)$ ”.
    - **Case:**  $r \notin \text{subj}(G_1)$  and  $r \notin \text{subj}(G_2)$ .  
Recall  $r \in \text{subj}(G)$ . Then,  $r \in \text{subj}(G_1 \circ G_2)$ . Then, by Defn. 24,  $r \in \text{subj}(G_1) \cup \text{subj}(G_2)$ . Then,  $r \in \text{subj}(G_1)$  or  $r \in \text{subj}(G_2)$ . Then, **false**.
  2. Recall  $r \notin \text{subj}(G)$ . Then,  $r \notin \text{subj}(G_1 \circ G_2)$ . Then, by Defn. 24,  $r \notin \text{subj}(G_1) \cup \text{subj}(G_2)$ . Then,  $r \notin \text{subj}(G_1)$  and  $r \notin \text{subj}(G_2)$ . Then, by induction,  $\text{subj}(G_1 \upharpoonright r) = \emptyset$  and  $\text{subj}(G_2 \upharpoonright r) = \emptyset$ . Then,  $\text{subj}(G_1 \upharpoonright r) \cup \text{subj}(G_2 \upharpoonright r) = \emptyset \cup \emptyset$ . Then,  $\text{subj}(G_1 \upharpoonright r) \cup \text{subj}(G_2 \upharpoonright r) = \emptyset$ . Then, by Defn. 24,  $\text{subj}((G_1 \upharpoonright r) \circ (G_2 \upharpoonright r)) = \emptyset$ . Then, by Defn. 26,  $\text{subj}(G_1 \circ G_2 \upharpoonright r) = \emptyset$ . Then,  $\text{subj}(G \upharpoonright r) = \emptyset$ .
- **Step:**  $G = R.\text{if } \xi G_1 G_2$ , for some  $R, G_1, G_2, \xi$ .
  1. Recall:
    - **Case:**  $r \in \text{subj}(\xi)$  and  $r \in \text{subj}(G_1)$  and  $r \in \text{subj}(G_2)$ . Then, by induction,  $\text{subj}(\xi \upharpoonright r) = \{r\}$  and  $\text{subj}(G_1 \upharpoonright r) = \{r\}$  and  $\text{subj}(G_2 \upharpoonright r) = \{r\}$ . Then,  $\text{subj}(\xi \upharpoonright r) \cup \text{subj}(G_1 \upharpoonright r) \cup \text{subj}(G_2 \upharpoonright r) = \{r\} \cup \{r\} \cup \{r\}$ . Then,  $\text{subj}(\xi \upharpoonright r) \cup \text{subj}(G_1 \upharpoonright r) \cup \text{subj}(G_2 \upharpoonright r) = \{r\}$ . Then, by Defn. 24,  $\text{subj}(R.\text{if } (\xi \upharpoonright r) (G_1 \upharpoonright r) (G_2 \upharpoonright r)) = \{r\}$ . Then, by Defn. 26,  $\text{subj}(R.\text{if } \xi G_1 G_2 \upharpoonright r) = \{r\}$ . Then,  $\text{subj}(G \upharpoonright r) = \{r\}$ .

- **Case:**  $r \in \text{subj}(\xi)$  and  $r \in \text{subj}(G_1)$  and  $r \notin \text{subj}(G_2)$ . Similar to case “ $r \in \text{subj}(\xi)$  and  $r \in \text{subj}(G_1)$  and  $r \in \text{subj}(G_2)$ ”.
  - **Case:**  $r \in \text{subj}(\xi)$  and  $r \notin \text{subj}(G_1)$  and  $r \in \text{subj}(G_2)$ . Similar to case “ $r \in \text{subj}(\xi)$  and  $r \in \text{subj}(G_1)$  and  $r \in \text{subj}(G_2)$ ”.
  - **Case:**  $r \in \text{subj}(\xi)$  and  $r \notin \text{subj}(G_1)$  and  $r \notin \text{subj}(G_2)$ . Similar to case “ $r \in \text{subj}(\xi)$  and  $r \in \text{subj}(G_1)$  and  $r \in \text{subj}(G_2)$ ”.
  - **Case:**  $r \notin \text{subj}(\xi)$  and  $r \in \text{subj}(G_1)$  and  $r \in \text{subj}(G_2)$ . Similar to case “ $r \in \text{subj}(\xi)$  and  $r \in \text{subj}(G_1)$  and  $r \in \text{subj}(G_2)$ ”.
  - **Case:**  $r \notin \text{subj}(\xi)$  and  $r \in \text{subj}(G_1)$  and  $r \notin \text{subj}(G_2)$ . Similar to case “ $r \in \text{subj}(\xi)$  and  $r \in \text{subj}(G_1)$  and  $r \in \text{subj}(G_2)$ ”.
  - **Case:**  $r \notin \text{subj}(\xi)$  and  $r \notin \text{subj}(G_1)$  and  $r \in \text{subj}(G_2)$ . Similar to case “ $r \in \text{subj}(\xi)$  and  $r \in \text{subj}(G_1)$  and  $r \in \text{subj}(G_2)$ ”.
  - **Case:**  $r \notin \text{subj}(\xi)$  and  $r \notin \text{subj}(G_1)$  and  $r \notin \text{subj}(G_2)$ .  
 Recall  $r \in \text{subj}(G)$ . Then,  $r \in \text{subj}(R.\text{if } \xi G_1 G_2)$ . Then, by Defn. 24,  
 $r \in \text{subj}(\xi) \cup \text{subj}(G_1) \cup \text{subj}(G_2)$ . Then,  $r \in \text{subj}(\xi)$  or  $r \in \text{subj}(G_1)$  or  $r \in \text{subj}(G_2)$ .  
 Then, **false**.
2. Recall  $r \notin \text{subj}(G)$ . Then,  $r \notin \text{subj}(R.\text{if } \xi G_1 G_2)$ . Then, by Defn. 24,  
 $r \notin \text{subj}(\xi) \cup \text{subj}(G_1) \cup \text{subj}(G_2)$ . Then,  $r \notin \text{subj}(\xi)$  and  $r \notin \text{subj}(G_1)$  and  $r \notin \text{subj}(G_2)$ .  
 Then, by induction,  $\text{subj}(\xi \uparrow r) = \emptyset$  and  $\text{subj}(G_1 \uparrow r) = \emptyset$  and  $\text{subj}(G_2 \uparrow r) = \emptyset$ . Then,  
 $\text{subj}(\xi \uparrow r) \cup \text{subj}(G_1 \uparrow r) \cup \text{subj}(G_2 \uparrow r) = \emptyset \cup \emptyset \cup \emptyset$ . Then,  
 $\text{subj}(\xi \uparrow r) \cup \text{subj}(G_1 \uparrow r) \cup \text{subj}(G_2 \uparrow r) = \emptyset$ . Then, by Defn. 24,  
 $\text{subj}(R.\text{if } (\xi \uparrow r) (G_1 \uparrow r) (G_2 \uparrow r)) = \emptyset$ . Then, by Defn. 26,  $\text{subj}(R.\text{if } \xi G_1 G_2 \uparrow r) = \emptyset$ .  
 Then,  $\text{subj}(G \uparrow r) = \emptyset$ .
- **Step:**  $G = R.\text{while } \xi \{\psi\} \hat{G}$ , for some  $R, \hat{G}, \psi, \xi$ . Similar to case “ $G = G_1 \circ G_2$ , for some  $G_1, G_2$ ”.
- **Step:**  $G = \text{if } \xi|_0 G_1|_{R_1} G_2|_{R_2}$ , for some  $R_1, R_2, G_1, G_2, \xi$ .
1. Recall:
- **Case:**  $r \in \text{subj}(\xi) \setminus (R_1 \cup R_2)$  and  $r \in \text{subj}(G_1) \setminus R_2$  and  $r \in \text{subj}(G_2) \setminus R_1$ .
    - \* Recall  $r \in \text{subj}(\xi) \setminus (R_1 \cup R_2)$ . Then,  $r \in \text{subj}(\xi)$  and  $r \notin R_1 \cup R_2$ .
    - \* Recall  $r \notin R_1 \cup R_2$ . Then,  $\{r\} \cap (R_1 \cup R_2) = \emptyset$ .
    - \* Recall  $r \in \text{subj}(\xi)$ . Then, by induction,  $\text{subj}(\xi \uparrow r) = \{r\}$ . Then,  
 $\text{subj}(\xi \uparrow r) \setminus \emptyset = \{r\}$ . Then,  $\text{subj}(\xi \uparrow r) \setminus ((R_1 \cup R_2) \cap \{r\}) = \{r\}$ . Then,  
 $\text{subj}(\xi \uparrow r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\})) = \{r\}$ .
    - \* Recall  $r \in \text{subj}(G_1) \setminus R_2$ . Then,  $r \in \text{subj}(G_1)$  and  $r \notin R_2$ .
    - \* Recall  $r \notin R_2$ . Then,  $\{r\} \cap R_2 = \emptyset$ .
    - \* Recall  $r \in \text{subj}(G_1)$ . Then, by induction,  $\text{subj}(G_1 \uparrow r) = \{r\}$ . Then,  
 $\text{subj}(G_1 \uparrow r) \setminus \emptyset = \{r\}$ . Then,  $\text{subj}(G_1 \uparrow r) \setminus (R_2 \cap \{r\}) = \{r\}$ .
    - \* Recall  $r \in \text{subj}(G_2) \setminus R_1$ . Then,  $r \in \text{subj}(G_2)$  and  $r \notin R_1$ .
    - \* Recall  $r \notin R_1$ . Then,  $\{r\} \cap R_1 = \emptyset$ .
    - \* Recall  $r \in \text{subj}(G_2)$ . Then, by induction,  $\text{subj}(G_2 \uparrow r) = \{r\}$ . Then,  
 $\text{subj}(G_2 \uparrow r) \setminus \emptyset = \{r\}$ . Then,  $\text{subj}(G_2 \uparrow r) \setminus (R_1 \cap \{r\}) = \{r\}$ .
    - \* Recall  $\{r\} \cup \{r\} \cup \{r\} = \{r\}$ . Then,  $(\text{subj}(\xi \uparrow r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\}))) \cup$   
 $(\text{subj}(G_1 \uparrow r) \setminus (R_2 \cap \{r\})) \cup (\text{subj}(G_2 \uparrow r) \setminus (R_1 \cap \{r\})) = \{r\}$ . Then, by Defn. 24,  
 $\text{subj}(\text{if } (\xi \uparrow r)|_{\text{subj}(\xi) \setminus (R_1 \cup R_2 \cup \{r\})} (G_1 \uparrow r)|_{R_1 \cap \{r\}} (G_2 \uparrow r)|_{R_2 \cap \{r\}}) = \{r\}$ . Then, by  
 Defn. 26,  $\text{subj}(\text{if } \xi|_0 G_1|_{R_1} G_2|_{R_2} \uparrow r) = \{r\}$ . Then,  $\text{subj}(G \uparrow r) = \{r\}$ .
  - **Case:**  $r \in \text{subj}(\xi) \setminus (R_1 \cup R_2)$  and  $r \in \text{subj}(G_1) \setminus R_2$  and  $r \notin \text{subj}(G_2) \setminus R_1$ .
    - \* Recall  $r \in \text{subj}(\xi) \setminus (R_1 \cup R_2)$ . Then,  $r \in \text{subj}(\xi)$  and  $r \notin R_1 \cup R_2$ .
    - \* Recall  $r \notin R_1 \cup R_2$ . Then,  $\{r\} \cap (R_1 \cup R_2) = \emptyset$ .
    - \* Recall  $r \in \text{subj}(\xi)$ . Then, by induction,  $\text{subj}(\xi \uparrow r) = \{r\}$ . Then,  
 $\text{subj}(\xi \uparrow r) \setminus \emptyset = \{r\}$ . Then,  $\text{subj}(\xi \uparrow r) \setminus ((R_1 \cup R_2) \cap \{r\}) = \{r\}$ . Then,  
 $\text{subj}(\xi \uparrow r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\})) = \{r\}$ .
    - \* Recall  $r \in \text{subj}(G_1) \setminus R_2$ . Then,  $r \in \text{subj}(G_1)$  and  $r \notin R_2$ .
    - \* Recall  $r \notin R_2$ . Then,  $\{r\} \cap R_2 = \emptyset$ .











- Then,  $\text{subj}(\xi \uparrow r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\})) = \{r\} \setminus \{r\}$ . Then,  
 $\text{subj}(\xi \uparrow r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\})) = \emptyset$ .
- \* Recall  $r \in R_2$ . Then,  $\{r\} \cap R_2 = \{r\}$ .
  - \* Recall  $r \in \text{subj}(G_1)$ . Then, by induction,  $\text{subj}(G_1 \uparrow r) = \{r\}$ . Then,  
 $\text{subj}(G_1 \uparrow r) \setminus \{r\} = \{r\} \setminus \{r\}$ . Then,  $\text{subj}(G_1 \uparrow r) \setminus (R_2 \cap \{r\}) = \{r\} \setminus \{r\}$ . Then,  
 $\text{subj}(G_1 \uparrow r) \setminus (R_2 \cap \{r\}) = \emptyset$ .
  - \* Recall  $r \in R_1$ . Then,  $\{r\} \cap R_1 = \{r\}$ .
  - \* Recall  $r \in \text{subj}(G_2)$ . Then, by induction,  $\text{subj}(G_2 \uparrow r) = \{r\}$ . Then,  
 $\text{subj}(G_2 \uparrow r) \setminus \{r\} = \{r\} \setminus \{r\}$ . Then,  $\text{subj}(G_2 \uparrow r) \setminus (R_1 \cap \{r\}) = \{r\} \setminus \{r\}$ . Then,  
 $\text{subj}(G_2 \uparrow r) \setminus (R_1 \cap \{r\}) = \emptyset$ .
  - \* Recall  $\emptyset \cup \emptyset \cup \emptyset = \emptyset$ . Then,  $(\text{subj}(\xi \uparrow r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\}))) \cup (\text{subj}(G_1 \uparrow r) \setminus (R_2 \cap \{r\})) \cup (\text{subj}(G_2 \uparrow r) \setminus (R_1 \cap \{r\})) = \emptyset$ . Then, by Defn. 24,  
 $\text{subj}(\text{if } (\xi \uparrow r)|_{\text{subj}(\xi) \setminus (R_1 \cup R_2 \cup \{r\})} (G_1 \uparrow r)|_{R_1 \cap \{r\}} (G_2 \uparrow r)|_{R_2 \cap \{r\}}) = \emptyset$ . Then, by  
Defn. 26,  $\text{subj}(\text{if } \xi|_0 G_1|_{R_1} G_2|_{R_2} \uparrow r) = \emptyset$ . Then,  $\text{subj}(G \uparrow r) = \emptyset$ .
- **Step:**  $G = \text{while } \xi|_0 \{\psi\} \hat{G}|_\emptyset$ , for some  $\hat{G}, \psi, \xi$ . Similar to case “ $G = G_1 \circ G_2$ , for some  $G_1, G_2$ ”. □

*Proof (of Lem. 43).* Recall:

- **Case:**  $r \in \text{subj}(G)$ .
  - Recall  $r \in \text{subj}(G)$ . Then, by Lem. 42,  $\text{subj}(G \uparrow r) = \{r\}$ .
  - Recall  $r \in \text{subj}(G)$ . Then,  $\{r\} \subseteq \text{subj}(G)$ . Then,  $\text{subj}(G \uparrow r) \subseteq \text{subj}(G)$ .
- **Case:**  $r \notin \text{subj}(G)$ .
  - Recall  $r \notin \text{subj}(G)$ . Then, by Lem. 42,  $\text{subj}(G \uparrow r) = \emptyset$ .
  - Recall  $\emptyset \subseteq \text{subj}(G)$ . Then,  $\text{subj}(G \uparrow r) \subseteq \text{subj}(G)$ . □

*Proof (of Lem. 44).*

- Recall  $r \in \text{subj}(G)$ . Then, by Lem. 42,  $\text{subj}(G \uparrow r) = \{r\}$ .
- Recall  $r \in \{r\}$ . Then,  $r \in \text{subj}(G \uparrow r)$ . □

*Proof (of Lem. 45).* By Defn. 21:

- **Base:**  $G = \xi$ , for some  $\xi$ .  
Recall  $\emptyset \subseteq \text{chan}(G)$ . Then, by Defn. 25,  $\text{chan}(\xi \uparrow r) \subseteq \text{chan}(G)$ . Then,  $\text{chan}(G \uparrow r) \subseteq \text{chan}(G)$ .
- **Base:**  $G = \gamma$ , for some  $\gamma$ .  
By Lem. 35,  $\text{chan}(\gamma \uparrow r) \subseteq \text{chan}(\gamma)$ . Then,  $\text{chan}(G \uparrow r) \subseteq \text{chan}(G)$ .
- **Base:**  $G = \text{skip}$ .  
Recall  $\text{chan}(\text{skip}) \subseteq \text{chan}(\text{skip})$ . Then, by Defn. 26,  $\text{chan}(\text{skip} \uparrow r) \subseteq \text{chan}(\text{skip})$ . Then,  
 $\text{chan}(G \uparrow r) \subseteq \text{chan}(G)$ .
- **Step:**  $G = G_1 \circ G_2$ , for some  $G_1, G_2$ .  
By induction,  $\text{chan}(G_1 \uparrow r) \subseteq \text{chan}(G_1)$  and  $\text{chan}(G_2 \uparrow r) \subseteq \text{chan}(G_2)$ . Then,  
 $\text{chan}(G_1 \uparrow r) \cup \text{chan}(G_2 \uparrow r) \subseteq \text{chan}(G_1) \cup \text{chan}(G_2)$ . Then, by Defn. 25,  
 $\text{chan}((G_1 \uparrow r) \circ (G_2 \uparrow r)) \subseteq \text{chan}(G_1 \circ G_2)$ . Then, by Defn. 26,  
 $\text{chan}(G_1 \circ G_2 \uparrow r) \subseteq \text{chan}(G_1 \circ G_2)$ . Then,  $\text{chan}(G \uparrow r) \subseteq \text{chan}(G)$ .
- **Step:**  $G = R.\text{if } \xi G_1 G_2$ , for some  $R, G_1, G_2, \xi$ .  
By induction,  $\text{chan}(G_1 \uparrow r) \subseteq \text{chan}(G_1)$  and  $\text{chan}(G_2 \uparrow r) \subseteq \text{chan}(G_2)$ . Then,  
 $\text{chan}(G_1 \uparrow r) \cup \text{chan}(G_2 \uparrow r) \subseteq \text{chan}(G_1) \cup \text{chan}(G_2)$ . Then,  
 $\{R\} \cup \text{chan}(G_1 \uparrow r) \cup \text{chan}(G_2 \uparrow r) \subseteq \{R\} \cup \text{chan}(G_1) \cup \text{chan}(G_2)$ . Then, by Defn. 25,  
 $\text{chan}(R.\text{if } (\xi \uparrow r) (G_1 \uparrow r) (G_2 \uparrow r)) \subseteq \text{chan}(R.\text{if } \xi G_1 G_2)$ . Then, by Defn. 26,  
 $\text{chan}(R.\text{if } \xi G_1 G_2 \uparrow r) \subseteq \text{chan}(R.\text{if } \xi G_1 G_2)$ . Then,  $\text{chan}(G \uparrow r) \subseteq \text{chan}(G)$ .
- **Step:**  $G = R.\text{while } \xi \{\psi\} \hat{G}$ , for some  $R, \hat{G}, \psi, \xi$ . Similar to case “ $G = R.\text{if } \xi G_1 G_2$ , for some  $R, G_1, G_2, \xi$ ”.
- **Step:**  $G = \text{if } \xi|_0 G_1|_{R_1} G_2|_{R_2}$ , for some  $R_1, R_2, G_1, G_2, \xi$ .

- By Lem. 10,  $\text{subj}(\xi \uparrow r) \subseteq \text{subj}(\xi)$ . Then,  $\{\{\hat{r}\} \mid \hat{r} \in \text{subj}(\xi \uparrow r)\} \subseteq \{\{\hat{r}\} \mid \hat{r} \in \text{subj}(\xi)\}$ .
  - By induction,  $\text{chan}(G_1 \uparrow r) \subseteq \text{chan}(G_1)$  and  $\text{chan}(G_2 \uparrow r) \subseteq \text{chan}(G_2)$ .
  - Recall  $\{\{\hat{r}\} \mid \hat{r} \in \text{subj}(\xi \uparrow r)\} \subseteq \{\{\hat{r}\} \mid \hat{r} \in \text{subj}(\xi)\}$  and  $\text{chan}(G_1 \uparrow r) \subseteq \text{chan}(G_1)$  and  $\text{chan}(G_2 \uparrow r) \subseteq \text{chan}(G_2)$ . Then,  $\{\{\hat{r}\} \mid \hat{r} \in \text{subj}(\xi \uparrow r)\} \cup \text{chan}(G_1 \uparrow r) \cup \text{chan}(G_2 \uparrow r) \subseteq \{\{\hat{r}\} \mid \hat{r} \in \text{subj}(\xi)\} \cup \text{chan}(G_1) \cup \text{chan}(G_2)$ . Then, by Defn. 25,  $\text{chan}(\text{if } (\xi \uparrow r)|_{\text{subj}(\xi) \setminus (R_1 \cup R_2 \cup \{r\})} (G_1 \uparrow r)|_{R_1 \cap \{r\}} (G_2 \uparrow r)|_{R_2 \cap \{r\}}) \subseteq \text{chan}(\text{if } \xi|_0 G_1|_{R_1} G_2|_{R_2})$ . Then, by Defn. 26,  $\text{chan}(\text{if } \xi|_0 G_1|_{R_1} G_2|_{R_2} \uparrow r) \subseteq \text{chan}(\text{if } \xi|_0 G_1|_{R_1} G_2|_{R_2})$ . Then,  $\text{chan}(G \uparrow r) \subseteq \text{chan}(G)$ .
- **Step:**  $G = \text{while } \xi|_0 \{\psi\} \hat{G}|_\emptyset$ , for some  $\hat{G}, \psi, \xi$ . Similar to case “ $G = \text{if } \xi|_0 G_1|_{R_1} G_2|_{R_2}$ , for some  $R_1, R_2, G_1, G_2, \xi$ ”.  $\square$

*Proof (of Lem. 46).*

- Recall  $\checkmark_R(\text{while } \xi|_0 \{\psi\} G|_\emptyset)$ . Then, by Defn. 27,  $\checkmark_R(G)$  and  $R = \text{subj}(\xi)$ .
- Recall  $\checkmark_R(G)$  and  $\checkmark_R(\text{while } \xi|_0 \{\psi\} G|_\emptyset)$ . Then, by  $[\checkmark 1\text{-SEQ}]$ ,  $\checkmark_R(G; \text{while } \xi|_0 \{\psi\} G|_\emptyset)$ .
- Recall  $\checkmark_R(G)$ . Then, by Lem. 47,  $R \neq \emptyset$ . Then, by  $[\checkmark 1\text{-SKIP}]$ ,  $\checkmark_R(\text{skip})$ .
- Recall  $\checkmark_R(G; \text{while } \xi|_0 \{\psi\} G|_\emptyset)$  and  $\checkmark_R(\text{skip})$  and  $R = \text{subj}(\xi)$  and  $\emptyset, \emptyset \subseteq \text{subj}(\xi)$ , and  $\emptyset \neq \emptyset$  implies  $\emptyset = \emptyset$ , and  $\emptyset \neq \emptyset$  implies  $\emptyset = \emptyset$ . Then, by  $[\checkmark 1\text{-NIF}]$ ,  $\text{if } \xi|_0 (G; \text{while } \xi|_0 \{\psi\} G|_\emptyset)|_\emptyset \text{skip}|_\emptyset$ .  $\square$

*Proof (of Lem. 47).* Recall  $\checkmark_R(G)$ . Then, by Defn. 27:

- **Base:**  $[\checkmark 1\text{-ACT1}]$ , such that  $G = q.y := e$  and  $q \in R$ , for some  $q, y, e$ .
  - Recall  $q \in R$ . Then,  $\{q\} \subseteq R$ . Then, by Defn. 24,  $\text{subj}(q.y := e) \subseteq R$ . Then,  $\text{subj}(G) \subseteq R$ .
  - Recall  $q \in R$ . Then,  $R \neq \emptyset$ .
- **Base:**  $[\checkmark 1\text{-ACT2}]$ . Similar to case  $[\checkmark 1\text{-ACT1}]$ .
- **Base:**  $[\checkmark 1\text{-SKIP}]$ , such that  $G = \text{skip}$  and  $R \neq \emptyset$ . Recall  $\emptyset \subseteq R$ . Then, by Defn. 24,  $\text{subj}(\text{skip}) \subseteq R$ . Then,  $\text{subj}(G) \subseteq R$ .
- **Step:**  $[\checkmark 1\text{-SEQ}]$ , such that  $G = G_1; G_2$  and  $\checkmark_R(G_1)$  and  $\checkmark_R(G_2)$ , for some  $G_1, G_2$ .
  - Recall  $\checkmark_R(G_1)$  and  $\checkmark_R(G_2)$ . Then, by induction,  $\text{subj}(G_1) \subseteq R$  and  $\text{subj}(G_2) \subseteq R$  and  $R \neq \emptyset$ .
  - Recall  $\text{subj}(G_1) \subseteq R$  and  $\text{subj}(G_2) \subseteq R$ . Then,  $\text{subj}(G_1) \cup \text{subj}(G_2) \subseteq R$ . Then, by Defn. 24,  $\text{subj}(G_1; G_2) \subseteq R$ . Then,  $\text{subj}(G) \subseteq R$ .
- **Step:**  $[\checkmark 1\text{-PAR}]$ . Similar to case  $[\checkmark 1\text{-SEQ}]$ .
- **Step:**  $[\checkmark 1\text{-IF}]$ , such that  $G = R.\text{if } \xi G_1 G_2$  and  $\checkmark_R(G_1)$  and  $\checkmark_R(G_2)$  and  $R = \text{subj}(\xi)$ , for some  $G_1, G_2, \xi$ .
  - Recall  $\checkmark_R(G_1)$  and  $\checkmark_R(G_2)$ . Then, by induction,  $\text{subj}(G_1) \subseteq R$  and  $\text{subj}(G_2) \subseteq R$  and  $R \neq \emptyset$ .
  - Recall  $R \subseteq R$  and  $\text{subj}(G_1) \subseteq R$  and  $\text{subj}(G_2) \subseteq R$ . Then,  $R \cup \text{subj}(G_1) \cup \text{subj}(G_2) \subseteq R$ . Then,  $\text{subj}(\xi) \cup \text{subj}(G_1) \cup \text{subj}(G_2) \subseteq R$ . Then, by Defn. 24,  $\text{subj}(R.\text{if } \xi G_1 G_2) \subseteq R$ . Then,  $\text{subj}(G) \subseteq R$ .
- **Step:**  $[\checkmark 1\text{-WHILE}]$ . Similar to case  $[\checkmark 1\text{-IF}]$ .
- **Step:**  $[\checkmark 1\text{-NIF}]$ , such that  $G = \text{if } \xi|_0 G_1|_{R_1} G_2|_{R_2}$  and  $\checkmark_R(G_1)$  and  $\checkmark_R(G_2)$  and  $R = \text{subj}(\xi)$ , for some  $R_1, R_2, G_1, G_2, \xi$ .
  - Recall  $R \subseteq R$ . Then,  $\text{subj}(\xi) \subseteq R$ . Then,  $\text{subj}(\xi) \setminus (R_1 \cup R_2) \subseteq R$ .
  - Recall  $\checkmark_R(G_1)$  and  $\checkmark_R(G_2)$ . Then, by induction,  $\text{subj}(G_1) \subseteq R$  and  $\text{subj}(G_2) \subseteq R$  and  $R \neq \emptyset$ .
  - Recall  $\text{subj}(G_1) \subseteq R$  and  $\text{subj}(G_2) \subseteq R$ . Then,  $\text{subj}(G_1) \setminus R_2 \subseteq R$  and  $\text{subj}(G_2) \setminus R_1 \subseteq R$ .
  - Recall  $\text{subj}(\xi) \setminus (R_1 \cup R_2) \subseteq R$  and  $\text{subj}(G_1) \setminus R_2 \subseteq R$  and  $\text{subj}(G_2) \setminus R_1 \subseteq R$ . Then,  $(\text{subj}(\xi) \setminus (R_1 \cup R_2)) \cup (\text{subj}(G_1) \setminus R_2) \cup (\text{subj}(G_2) \setminus R_1) \subseteq R$ . Then, by Defn. 24,  $\text{subj}(\text{if } \xi|_0 G_1|_{R_1} G_2|_{R_2}) \subseteq R$ . Then,  $\text{subj}(G) \subseteq R$ .
- **Step:**  $[\checkmark 1\text{-NWHILE}]$ , such that  $G = \text{while } \xi|_0 \{\psi\} \hat{G}|_\emptyset$  and  $\checkmark_R(\hat{G})$  and  $R = \text{subj}(\xi)$ , for some  $\hat{G}, \psi, \xi$ .
  - Recall  $\checkmark_R(\hat{G})$ . Then, by induction,  $\text{subj}(\hat{G}) \subseteq R$  and  $R \neq \emptyset$ .
  - Recall  $R \subseteq R$  and  $\text{subj}(\hat{G}) \subseteq R$ . Then,  $R \cup \text{subj}(\hat{G}) \subseteq R$ . Then,  $\text{subj}(\xi) \cup \text{subj}(\hat{G}) \subseteq R$ . Then, by Defn. 24,  $\text{subj}(\text{while } \xi|_0 \{\psi\} \hat{G}|_\emptyset) \subseteq R$ . Then,  $\text{subj}(G) \subseteq R$ .  $\square$

## E Termination

*Proof (of Lem. 48).* Recall  $G \downarrow$ . Then, by Defn. 28:

- **Base:**  $[\downarrow 1\text{-SKIP}]$ , such that  $G = \text{skip}$ .  
By Defn. 24,  $\text{subj}(\text{skip}) = \emptyset$ . Then,  $\text{subj}(G) = \emptyset$ .
- **Step:**  $[\downarrow 1\text{-SEQ}]$ , such that  $G = G_1 ; G_2$  and  $G_1 \downarrow$  and  $G_2 \downarrow$ , for some  $G_1, G_2$ .
  - Recall  $\checkmark_R(G)$ . Then,  $\checkmark_R(G_1 ; G_2)$ . Then, by Defn. 27,  $\checkmark_R(G_1)$  and  $\checkmark_R(G_2)$ .
  - Recall  $\checkmark_R(G_1)$  and  $G_1 \downarrow$ . Then,  $\text{subj}(G_1) = \emptyset$ .
  - Recall  $\checkmark_R(G_2)$  and  $G_2 \downarrow$ . Then,  $\text{subj}(G_2) = \emptyset$ .
  - Recall  $\emptyset \cup \emptyset = \emptyset$ . Then,  $\text{subj}(G_1) \cup \text{subj}(G_2) = \emptyset$ . Then, by Defn. 24,  $\text{subj}(G_1 ; G_2) = \emptyset$ .  
Then,  $\text{subj}(G) = \emptyset$ .
- **Step:**  $[\downarrow 1\text{-PAR}]$ . Similar to case  $[\downarrow 1\text{-SEQ}]$ .
- **Step:**  $[\downarrow 1\text{-NIF}]$ , such that  $G = \text{if } \xi |_0 G_1 |_{R_1} G_2 |_{R_2}$  and  $\text{subj}(\xi) = R_1 \cup R_2$ , and  $R_1 \neq \emptyset$  implies  $G_1 \downarrow$ , and  $R_2 \neq \emptyset$  implies  $G_2 \downarrow$ , for some  $R_1, R_2, G_1, G_2, \xi$ .

Recall:

- **Case:**  $R_1 = \emptyset = R_2$ .
  - \* Recall  $\checkmark_R(G)$ . Then,  $\checkmark_R(\text{if } \xi |_0 G_1 |_{R_1} G_2 |_{R_2})$ . Then, by Defn. 27,  $R = \text{subj}(\xi)$ .
  - \* Recall  $\checkmark_R(G)$ . Then, by Lem. 47,  $R \neq \emptyset$ . Then,  $\text{subj}(\xi) \neq \emptyset$ . Then,  $R_1 \cup R_2 \neq \emptyset$ .  
Then,  $\emptyset \cup \emptyset \neq \emptyset$ . Then,  $\emptyset \neq \emptyset$ . Then, **false**.
- **Case:**  $R_1 = \emptyset \neq R_2$ .
  - \* Recall  $\text{subj}(\xi) \setminus \text{subj}(\xi) = \emptyset$ . Then,  $\text{subj}(\xi) \setminus (R_1 \cup R_2) = \emptyset$ .
  - \* Recall  $\checkmark_R(G)$ . Then,  $\checkmark_R(\text{if } \xi |_0 G_1 |_{R_1} G_2 |_{R_2})$ . Then, by Defn. 27,  $\checkmark_R(G_1)$ .
  - \* Recall  $\checkmark_R(G)$ . Then,  $\checkmark_R(\text{if } \xi |_0 G_1 |_{R_1} G_2 |_{R_2})$ . Then, by Defn. 27,  $R = \text{subj}(\xi)$ .
  - \* Recall  $\text{subj}(\xi) = R_1 \cup R_2$ . Then,  $\text{subj}(\xi) = \emptyset \cup R_2$ . Then,  $\text{subj}(\xi) = R_2$ .
  - \* Recall  $\checkmark_R(G_1)$ . Then, by Lem. 47,  $\text{subj}(G_1) \subseteq R$ . Then,  $\text{subj}(G_1) \setminus R \subseteq R \setminus R$ . Then,  $\text{subj}(G_1) \setminus \text{subj}(\xi) \subseteq R \setminus R$ . Then,  $\text{subj}(G_1) \setminus R_2 \subseteq R \setminus R$ . Then,  $\text{subj}(G_1) \setminus R_2 \subseteq \emptyset$ .  
Then,  $\text{subj}(G_1) \setminus R_2 = \emptyset$ .
  - \* Recall  $R_2 \neq \emptyset$ . Then,  $G_2 \downarrow$ .
  - \* Recall  $\checkmark_R(G)$ . Then,  $\checkmark_R(\text{if } \xi |_0 G_1 |_{R_1} G_2 |_{R_2})$ . Then, by Defn. 27,  $\checkmark_R(G_2)$ .
  - \* Recall  $G_2 \downarrow$  and  $\checkmark_R(G_2)$ . Then, by induction,  $\text{subj}(G_2) = \emptyset$ . Then,  $\text{subj}(G_2) \setminus R_1 = \emptyset \setminus R_1$ . Then,  $\text{subj}(G_2) \setminus R_1 = \emptyset$ .
  - \* Recall  $\emptyset \cup \emptyset \cup \emptyset = \emptyset$ . Then,  $(\text{subj}(\xi) \setminus (R_1 \cup R_2)) \cup (\text{subj}(G_1) \setminus R_2) \cup (\text{subj}(G_2) \setminus R_1) = \emptyset$ .  
Then, by Defn. 24,  $\text{subj}(\text{if } \xi |_0 G_1 |_{R_1} G_2 |_{R_2}) = \emptyset$ . Then,  $\text{subj}(G) = \emptyset$ .
- **Case:**  $R_1 \neq \emptyset = R_2$ . Similar to case  $R_1 = \emptyset \neq R_2$ .
- **Case:**  $R_1 \neq \emptyset \neq R_2$ .
  - \* Recall  $\text{subj}(\xi) \setminus \text{subj}(\xi) = \emptyset$ . Then,  $\text{subj}(\xi) \setminus (R_1 \cup R_2) = \emptyset$ .
  - \* Recall  $R_1 \neq \emptyset$ . Then,  $G_1 \downarrow$ .
  - \* Recall  $\checkmark_R(G)$ . Then,  $\checkmark_R(\text{if } \xi |_0 G_1 |_{R_1} G_2 |_{R_2})$ . Then, by Defn. 27,  $\checkmark_R(G_1)$ .
  - \* Recall  $G_1 \downarrow$  and  $\checkmark_R(G_1)$ . Then, by induction,  $\text{subj}(G_1) = \emptyset$ . Then,  $\text{subj}(G_1) \setminus R_2 = \emptyset \setminus R_2$ . Then,  $\text{subj}(G_1) \setminus R_2 = \emptyset$ .
  - \* Recall  $R_2 \neq \emptyset$ . Then,  $G_2 \downarrow$ .
  - \* Recall  $\checkmark_R(G)$ . Then,  $\checkmark_R(\text{if } \xi |_0 G_1 |_{R_1} G_2 |_{R_2})$ . Then, by Defn. 27,  $\checkmark_R(G_2)$ .
  - \* Recall  $G_2 \downarrow$  and  $\checkmark_R(G_2)$ . Then, by induction,  $\text{subj}(G_2) = \emptyset$ . Then,  $\text{subj}(G_2) \setminus R_1 = \emptyset \setminus R_1$ . Then,  $\text{subj}(G_2) \setminus R_1 = \emptyset$ .
  - \* Recall  $\emptyset \cup \emptyset \cup \emptyset = \emptyset$ . Then,  $(\text{subj}(\xi) \setminus (R_1 \cup R_2)) \cup (\text{subj}(G_1) \setminus R_2) \cup (\text{subj}(G_2) \setminus R_1) = \emptyset$ .  
Then, by Defn. 24,  $\text{subj}(\text{if } \xi |_0 G_1 |_{R_1} G_2 |_{R_2}) = \emptyset$ . Then,  $\text{subj}(G) = \emptyset$ .  $\square$

*Proof (of Lem. 49).* Recall  $G \downarrow$ . Then, by Defn. 28:

- **Base:**  $[\downarrow 1\text{-SKIP}]$ , such that  $G = \text{skip}$ .  
By  $[\downarrow 1\text{-SKIP}]$ ,  $\text{skip} \downarrow$ . Then, by Defn. 26,  $(\text{skip} \uparrow r) \downarrow$ . Then,  $(G \uparrow r) \downarrow$ .
- **Step:**  $[\downarrow 1\text{-SEQ}]$ , such that  $G = G_1 ; G_2$  and  $G_1 \downarrow$  and  $G_2 \downarrow$ , for some  $G_1, G_2$ .
  - Recall  $\checkmark_R(G)$ . Then,  $\checkmark_R(G_1 ; G_2)$ . Then, by Defn. 27,  $\checkmark_R(G_1)$  and  $\checkmark_R(G_2)$ .

- Recall  $\checkmark_R(G_1)$  and  $G_1 \downarrow$ . Then, by induction,  $(G_1 \uparrow r) \downarrow$ .
  - Recall  $\checkmark_R(G_2)$  and  $G_2 \downarrow$ . Then, by induction,  $(G_2 \uparrow r) \downarrow$ .
  - Recall  $(G_1 \uparrow r) \downarrow$  and  $(G_2 \uparrow r) \downarrow$ . Then, by  $[\downarrow 1\text{-SEQ}]$ ,  $(G_1 \uparrow r); (G_2 \uparrow r) \downarrow$ . Then, by Defn. 26,  $(G_1; G_2 \uparrow r) \downarrow$ . Then,  $(G \uparrow r) \downarrow$ .
- **Step:**  $[\downarrow 1\text{-PAR}]$ . Similar to case  $[\downarrow 1\text{-SEQ}]$ .
- **Step:**  $[\downarrow 1\text{-NIF}]$ , such that  $G = \mathbf{if} \xi|_0 G_1|_{R_1} G_2|_{R_2}$  and  $\text{subj}(\xi) = R_1 \cup R_2$ , and  $R_1 \neq \emptyset$  implies  $G_1 \downarrow$ , and  $R_2 \neq \emptyset$  implies  $G_2 \downarrow$ , for some  $R_1, R_2, G_1, G_2, \xi$ .

Recall:

- **Case:**  $R_1 = \emptyset = R_2$ .
  - \* Recall  $\checkmark_R(G)$ . Then,  $\checkmark_R(\mathbf{if} \xi|_0 G_1|_{R_1} G_2|_{R_2})$ . Then, by Defn. 27,  $R = \text{subj}(\xi)$ .
  - \* Recall  $\checkmark_R(G)$ . Then, by Lem. 47,  $R \neq \emptyset$ . Then,  $\text{subj}(\xi) \neq \emptyset$ . Then,  $R_1 \cup R_2 \neq \emptyset$ . Then,  $\emptyset \cup \emptyset \neq \emptyset$ . Then,  $\emptyset \neq \emptyset$ . Then, **false**.
- **Case:**  $R_1 = \emptyset \neq R_2$ .
  - \* Recall  $R_1 = \emptyset$ . Then,  $R_1 \cap \{r\} = \emptyset$ . Then,  $R_1 \cap \{r\} \neq \emptyset$  implies  $(G_1 \uparrow r) \downarrow$ .
  - \* Recall  $R_2 \neq \emptyset$ . Then,  $G_2 \downarrow$ .
  - \* Recall  $\checkmark_R(G)$ . Then,  $\checkmark_R(\mathbf{if} \xi|_0 G_1|_{R_1} G_2|_{R_2})$ . Then, by Defn. 27,  $\checkmark_R(G_2)$ .
  - \* Recall  $G_2 \downarrow$  and  $\checkmark_R(G_2)$ . Then, by induction,  $(G_2 \uparrow r) \downarrow$ . Then,  $R_2 \cap \{r\} \neq \emptyset$  implies  $(G_2 \uparrow r) \downarrow$ .
  - \* Recall  $\text{subj}(\xi) \subseteq \text{subj}(\xi)$ . Then,  $\text{subj}(\xi) \subseteq R_1 \cup R_2$ . Then,  $\text{subj}(\xi) \subseteq R_1 \cup R_2 \cup \{r\}$ . Then,  $\text{subj}(\xi) \setminus (R_1 \cup R_2 \cup \{r\}) = \emptyset$ . Then,  $|\text{subj}(\xi) \setminus (R_1 \cup R_2 \cup \{r\})| = |\emptyset|$ . Then,  $|\text{subj}(\xi) \setminus (R_1 \cup R_2 \cup \{r\})| = 0$ .
  - \* Recall:
    - **Case:**  $r \in R_1 \cup R_2$ .
      - Recall  $r \in R_1 \cup R_2$ . Then,  $r \in \text{subj}(\xi)$ . Then, by Lem. 9,  $\text{subj}(\xi \uparrow r) = \{r\}$ .
      - Recall  $r \in R_1 \cup R_2$ . Then,  $\{r\} \subseteq R_1 \cup R_2$ . Then,  $\{r\} = (R_1 \cup R_2) \cap \{r\}$ . Then,  $\text{subj}(\xi \uparrow r) = (R_1 \cup R_2) \cap \{r\}$ . Then,  $\text{subj}(\xi \uparrow r) = (R_1 \cap \{r\}) \cup (R_2 \cap \{r\})$ .
      - Recall  $\text{subj}(\xi \uparrow r) = (R_1 \cap \{r\}) \cup (R_2 \cap \{r\})$ , and  $R_1 \cap \{r\} \neq \emptyset$  implies  $(G_1 \uparrow r) \downarrow$ , and  $R_2 \cap \{r\} \neq \emptyset$  implies  $(G_2 \uparrow r) \downarrow$ . Then, by  $[\downarrow 1\text{-NIF}]$ ,  $\mathbf{if} (\xi \uparrow r)|_0 (G_1 \uparrow r)|_{R_1 \cap \{r\}} (G_2 \uparrow r)|_{R_2 \cap \{r\}} \downarrow$ . Then,  $\mathbf{if} (\xi \uparrow r)|_{|\text{subj}(\xi) \setminus (R_1 \cup R_2 \cup \{r\})|} (G_1 \uparrow r)|_{R_1 \cap \{r\}} (G_2 \uparrow r)|_{R_2 \cap \{r\}} \downarrow$ . Then, by Defn. 26,  $(\mathbf{if} \xi|_0 G_1|_{R_1} G_2|_{R_2} \uparrow r) \downarrow$ . Then,  $(G \uparrow r) \downarrow$ .
    - **Case:**  $r \notin R_1 \cup R_2$ .
      - Recall  $r \notin R_1 \cup R_2$ . Then,  $r \notin \text{subj}(\xi)$ . Then, by Lem. 9,  $\text{subj}(\xi \uparrow r) = \emptyset$ .
      - Recall  $r \notin R_1 \cup R_2$ . Then,  $\emptyset = (R_1 \cup R_2) \cap \{r\}$ . Then,  $\text{subj}(\xi \uparrow r) = (R_1 \cup R_2) \cap \{r\}$ . Then,  $\text{subj}(\xi \uparrow r) = (R_1 \cap \{r\}) \cup (R_2 \cap \{r\})$ .
      - Recall  $\text{subj}(\xi \uparrow r) = (R_1 \cap \{r\}) \cup (R_2 \cap \{r\})$ , and  $R_1 \cap \{r\} \neq \emptyset$  implies  $(G_1 \uparrow r) \downarrow$ , and  $R_2 \cap \{r\} \neq \emptyset$  implies  $(G_2 \uparrow r) \downarrow$ . Then, by  $[\downarrow 1\text{-NIF}]$ ,  $\mathbf{if} (\xi \uparrow r)|_0 (G_1 \uparrow r)|_{R_1 \cap \{r\}} (G_2 \uparrow r)|_{R_2 \cap \{r\}} \downarrow$ . Then,  $\mathbf{if} (\xi \uparrow r)|_{|\text{subj}(\xi) \setminus (R_1 \cup R_2 \cup \{r\})|} (G_1 \uparrow r)|_{R_1 \cap \{r\}} (G_2 \uparrow r)|_{R_2 \cap \{r\}} \downarrow$ . Then, by Defn. 26,  $(\mathbf{if} \xi|_0 G_1|_{R_1} G_2|_{R_2} \uparrow r) \downarrow$ . Then,  $(G \uparrow r) \downarrow$ .
- **Case:**  $R_1 \neq \emptyset = R_2$ . Similar to case  $R_1 = \emptyset \neq R_2$ .
- **Case:**  $R_1 \neq \emptyset \neq R_2$ . Similar to case  $R_1 = \emptyset \neq R_2$ . □

## F Reduction

*Proof (of Lem. 50).* Recall  $A \xrightarrow{\xi, \alpha} A'$ . Then, by Defn. 31:

- **Base:**  $[\rightarrow 1\text{-ACT}]$ , such that  $\xi = \mathbf{true}$ .  
Recall:
  - **Case:**  $\alpha = \tau$ .  
Recall  $(\xi, \alpha) \neq (\mathbf{true}, \tau)$ . Then,  $(\mathbf{true}, \tau) \neq (\mathbf{true}, \tau)$ . Then, **false**.
  - **Case:**  $\alpha \neq \tau$ .
- **Base:**  $[\rightarrow 1\text{-IF1}]$ , such that  $\alpha \neq \tau$ .
- **Base:**  $[\rightarrow 1\text{-IF2}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
- **Base:**  $[\rightarrow 1\text{-WHILE1}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
- **Base:**  $[\rightarrow 1\text{-WHILE2}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
- **Base:**  $[\rightarrow 1\text{-NIF1}]$ , such that  $\xi = \mathbf{true}$  and  $\alpha = \tau$ .  
Recall  $(\xi, \alpha) \neq (\mathbf{true}, \tau)$ . Then,  $(\mathbf{true}, \tau) \neq (\mathbf{true}, \tau)$ . Then, **false**.
- **Base:**  $[\rightarrow 1\text{-NIF2}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
- **Base:**  $[\rightarrow 1\text{-NIF3}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
- **Step:**  $[\rightarrow 1\text{-SEQ1}]$ , such that  $A_1 \xrightarrow{\xi, \alpha} A'_1$ , for some  $A_1, A'_1$ .  
Recall  $A_1 \xrightarrow{\xi, \alpha} A'_1$  and  $(\xi, \alpha) \neq (\mathbf{true}, \tau)$ . Then, by induction,  $\alpha \neq \tau$ .
- **Step:**  $[\rightarrow 1\text{-SEQ2}]$ . Similar to case  $[\rightarrow 1\text{-SEQ1}]$ .
- **Step:**  $[\rightarrow 1\text{-PAR1}]$ . Similar to case  $[\rightarrow 1\text{-SEQ1}]$ .
- **Step:**  $[\rightarrow 1\text{-PAR2}]$ . Similar to case  $[\rightarrow 1\text{-SEQ1}]$ .
- **Step:**  $[\rightarrow 1\text{-NIF4}]$ . Similar to case  $[\rightarrow 1\text{-SEQ1}]$ .
- **Step:**  $[\rightarrow 1\text{-NIF5}]$ . Similar to case  $[\rightarrow 1\text{-SEQ1}]$ .
- **Step:**  $[\rightarrow 1\text{-NWHILE}]$ . Similar to case  $[\rightarrow 1\text{-SEQ1}]$ . □

*Proof (of Lem. 51).*

1. Recall  $A \xrightarrow{\xi, \alpha} A'$ . Then, by Defn. 31:
  - **Base:**  $[\rightarrow 1\text{-ACT}]$ , such that  $A = \alpha$  and  $A' = \mathbf{skip}$  and  $\xi = \mathbf{true}$ .  
Recall  $\emptyset \cup \mathbf{read}(A) \cup \emptyset \subseteq \mathbf{read}(A)$ . Then, by Lem. 3,  $\mathbf{read}(\mathbf{true}) \cup \mathbf{read}(A) \cup \emptyset \subseteq \mathbf{read}(A)$ .  
Then, by Defn. 22,  $\mathbf{read}(\mathbf{true}) \cup \mathbf{read}(A) \cup \mathbf{read}(\mathbf{skip}) \subseteq \mathbf{read}(A)$ . Then,  
 $\mathbf{read}(\xi) \cup \mathbf{read}(\alpha) \cup \mathbf{read}(A') \subseteq \mathbf{read}(A)$ .
  - **Base:**  $[\rightarrow 1\text{-IF1}]$ , such that  $A = R.\mathbf{if} \hat{\xi} A_1 A_2$  and  $A' = A_1$  and  $\xi = \hat{\xi}^+$  and  $\alpha = 1_{\text{subj}(\hat{\xi})}^R$ ,  
for some  $R, A_1, A_2, \hat{\xi}$ .  
Recall  $\mathbf{read}(\hat{\xi}) \cup \emptyset \cup \mathbf{read}(A_1) \subseteq \mathbf{read}(\hat{\xi}) \cup \mathbf{read}(A_1) \cup \mathbf{read}(A_2)$ . Then, by Lem. 12,  
 $\mathbf{read}(\hat{\xi}^+) \cup \emptyset \cup \mathbf{read}(A_1) \subseteq \mathbf{read}(\hat{\xi}) \cup \mathbf{read}(A_1) \cup \mathbf{read}(A_2)$ . Then, by Defn. 15,  
 $\mathbf{read}(\hat{\xi}^+) \cup \mathbf{read}(1_{\text{subj}(\hat{\xi})}^R) \cup \mathbf{read}(A_1) \subseteq \mathbf{read}(\hat{\xi}) \cup \mathbf{read}(A_1) \cup \mathbf{read}(A_2)$ . Then, by Defn. 22,  
 $\mathbf{read}(\hat{\xi}^+) \cup \mathbf{read}(1_{\text{subi}(\hat{\xi})}^R) \cup \mathbf{read}(A_1) \subseteq \mathbf{read}(R.\mathbf{if} \hat{\xi} A_1 A_2)$ . Then,  
 $\mathbf{read}(\xi) \cup \mathbf{read}(\alpha) \cup \mathbf{read}(A') \subseteq \mathbf{read}(A)$ .
  - **Base:**  $[\rightarrow 1\text{-IF2}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
  - **Base:**  $[\rightarrow 1\text{-WHILE1}]$ , such that  $A = R.\mathbf{while} \hat{\xi} \{\psi\} \hat{A}$  and  $A' = \hat{A}$ ;  $R.\mathbf{while} \hat{\xi} \{\psi\} \hat{A}$  and  
 $\xi = \hat{\xi}^+$  and  $\alpha = 1_{\text{subj}(\hat{\xi})}^R$ , for some  $R, \hat{A}, \psi, \hat{\xi}$ .  
Recall  $\mathbf{read}(\hat{\xi}) \cup \emptyset \cup \mathbf{read}(\hat{A}) \cup \mathbf{read}(\hat{\xi}) \cup \mathbf{read}(\hat{A}) \subseteq \mathbf{read}(\hat{\xi}) \cup \mathbf{read}(\hat{A})$ . Then, by Lem. 12,  
 $\mathbf{read}(\hat{\xi}^+) \cup \emptyset \cup \mathbf{read}(\hat{A}) \cup \mathbf{read}(\hat{\xi}) \cup \mathbf{read}(\hat{A}) \subseteq \mathbf{read}(\hat{\xi}) \cup \mathbf{read}(\hat{A})$ . Then, by Defn. 15,  
 $\mathbf{read}(\hat{\xi}^+) \cup \mathbf{read}(1_{\text{subj}(\hat{\xi})}^R) \cup \mathbf{read}(\hat{A}) \cup \mathbf{read}(\hat{\xi}) \cup \mathbf{read}(\hat{A}) \subseteq \mathbf{read}(\hat{\xi}) \cup \mathbf{read}(\hat{A})$ . Then, by  
Defn. 22,  
 $\mathbf{read}(\hat{\xi}^+) \cup \mathbf{read}(1_{\text{subi}(\hat{\xi})}^R) \cup \mathbf{read}(\hat{A}) \cup \mathbf{read}(R.\mathbf{while} \hat{\xi} \{\psi\} \hat{A}) \subseteq \mathbf{read}(R.\mathbf{while} \hat{\xi} \{\psi\} \hat{A})$ .  
Then, by Defn. 22,  
 $\mathbf{read}(\hat{\xi}^+) \cup \mathbf{read}(1_{\text{subi}(\hat{\xi})}^R) \cup \mathbf{read}(\hat{A}; R.\mathbf{while} \hat{\xi} \{\psi\} \hat{A}) \subseteq \mathbf{read}(R.\mathbf{while} \hat{\xi} \{\psi\} \hat{A})$ . Then,  
 $\mathbf{read}(\xi) \cup \mathbf{read}(\alpha) \cup \mathbf{read}(A') \subseteq \mathbf{read}(A)$ .
  - **Base:**  $[\rightarrow 1\text{-WHILE2}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .

- **Base:** [ $\rightarrow$ 1-NIF1], such that  $A = \mathbf{if} \hat{\xi}|_n A_1|_{R_1} A_2|_{R_2}$  and  $A' = \mathbf{if} \hat{\xi}|_{n-1} A_1|_{R_1} A_2|_{R_2}$  and  $\xi = \mathbf{true}$  and  $\alpha = \tau$ , for some  $R_1, R_2, A_1, A_2, \hat{\xi}, n$ .  
Recall  $\emptyset \cup \emptyset \cup \mathbf{read}(\mathbf{if} \hat{\xi}|_{n-1} A_1|_{R_1} A_2|_{R_2}) \subseteq \mathbf{read}(\mathbf{if} \hat{\xi}|_{n-1} A_1|_{R_1} A_2|_{R_2})$ . Then, by Lem. 3,  $\mathbf{read}(\mathbf{true}) \cup \emptyset \cup \mathbf{read}(\mathbf{if} \hat{\xi}|_{n-1} A_1|_{R_1} A_2|_{R_2}) \subseteq \mathbf{read}(\mathbf{if} \hat{\xi}|_{n-1} A_1|_{R_1} A_2|_{R_2})$ . Then, by Defn. 15,  $\mathbf{read}(\mathbf{true}) \cup \mathbf{read}(\tau) \cup \mathbf{read}(\mathbf{if} \hat{\xi}|_{n-1} A_1|_{R_1} A_2|_{R_2}) \subseteq \mathbf{read}(\mathbf{if} \hat{\xi}|_{n-1} A_1|_{R_1} A_2|_{R_2})$ . Then, by Defn. 22,  $\mathbf{read}(\mathbf{true}) \cup \mathbf{read}(\tau) \cup \mathbf{read}(\mathbf{if} \hat{\xi}|_{n-1} A_1|_{R_1} A_2|_{R_2}) \subseteq (\mathbf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\}) \cup (\mathbf{read}(G_1) \setminus \{\hat{r}.x \mid \hat{r} \in R_2\}) \cup (\mathbf{read}(G_2) \setminus \{\hat{r}.x \mid \hat{r} \in R_1\})$ . Then, by Defn. 22,  $\mathbf{read}(\mathbf{true}) \cup \mathbf{read}(\tau) \cup \mathbf{read}(\mathbf{if} \hat{\xi}|_{n-1} A_1|_{R_1} A_2|_{R_2}) \subseteq \mathbf{read}(\mathbf{if} \hat{\xi}|_n A_1|_{R_1} A_2|_{R_2})$ . Then,  $\mathbf{read}(\xi) \cup \mathbf{read}(\alpha) \cup \mathbf{read}(A') \subseteq \mathbf{read}(A)$ .
- **Base:** [ $\rightarrow$ 1-NIF2], such that  $A = \mathbf{if} \hat{\xi}|_n A_1|_{R_1} A_2|_{R_2}$  and  $A' = \mathbf{if} \hat{\xi}|_n A_1|_{R_1 \cup \{r\}} A_2|_{R_2}$  and  $\xi = \hat{\xi}^+ \uparrow r$  and  $\alpha = 1\{r\}$  and  $r \in \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$ , for some  $R_1, R_2, A_1, A_2, r, \hat{\xi}, n$ .
  - Recall  $r \in \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$ . Then,  $r \notin R_1 \cup R_2$ . Then,  $\{r\} \cap (R_1 \cup R_2) = \emptyset$ .
  - By Lem. 5,  $\mathbf{read}(\hat{\xi} \uparrow r) \subseteq \{\hat{r}.x \mid \hat{r} \in \text{subj}(\hat{\xi} \uparrow r) \subseteq \{r\}\}$ . Then, by Lem. 10,  $\mathbf{read}(\hat{\xi} \uparrow r) \subseteq \{\hat{r}.x \mid \hat{r} \in \text{subj}(\hat{\xi} \uparrow r) \subseteq \{r\}\}$ . Then,  $\mathbf{read}(\hat{\xi} \uparrow r) \subseteq \{\hat{r}.x \mid \hat{r} \in \{r\}\}$ . Then,  $\mathbf{read}(\hat{\xi} \uparrow r) \subseteq \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\} \subseteq \{\hat{r}.x \mid \hat{r} \in \{r\}\} \cap \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\}$ . Then,  $\mathbf{read}(\hat{\xi} \uparrow r) \cap \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\} \subseteq \{\hat{r}.x \mid \hat{r} \in \{r\}\} \cap \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\}$ . Then,  $\mathbf{read}(\hat{\xi} \uparrow r) \cap \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\} \subseteq \{\hat{r}.x \mid \hat{r} \in \emptyset\}$ . Then,  $\mathbf{read}(\hat{\xi} \uparrow r) \cap \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\} = \emptyset$ . Then,  $\mathbf{read}(\hat{\xi} \uparrow r) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\} = \mathbf{read}(\hat{\xi} \uparrow r)$ .
  - By Lem. 8,  $\mathbf{read}(\hat{\xi} \uparrow r) \subseteq \mathbf{read}(\hat{\xi})$ . Then,  $\mathbf{read}(\hat{\xi} \uparrow r) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\} \subseteq \mathbf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\}$ . Then,  $\mathbf{read}(\hat{\xi} \uparrow r) \subseteq \mathbf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\}$ . Then, by Lem. 12,  $\mathbf{read}((\hat{\xi} \uparrow r)^+) \subseteq \mathbf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\}$ . Then, by Lem. 11,  $\mathbf{read}(\hat{\xi}^+ \uparrow r) \subseteq \mathbf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\}$ .
  - Recall  $\mathbf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\} \subseteq \mathbf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\}$ . Then,  $(\mathbf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\}) \setminus \{\hat{r}.x \mid \hat{r} \in \{r\}\} \subseteq \mathbf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\}$ . Then,  $\mathbf{read}(\hat{\xi}) \setminus (\{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\} \cup \{\hat{r}.x \mid \hat{r} \in \{r\}\}) \subseteq \mathbf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\}$ . Then,  $\mathbf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2 \cup \{r\}\} \subseteq \mathbf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\}$ .
  - Recall  $\mathbf{read}(G_2) \setminus \{\hat{r}.x \mid \hat{r} \in R_1\} \subseteq \mathbf{read}(G_2) \setminus \{\hat{r}.x \mid \hat{r} \in R_1\}$ . Then,  $(\mathbf{read}(G_2) \setminus \{\hat{r}.x \mid \hat{r} \in R_1\}) \setminus \{\hat{r}.x \mid \hat{r} \in \{r\}\} \subseteq \mathbf{read}(G_2) \setminus \{\hat{r}.x \mid \hat{r} \in R_1\}$ . Then,  $\mathbf{read}(G_2) \setminus (\{\hat{r}.x \mid \hat{r} \in R_1\} \cup \{\hat{r}.x \mid \hat{r} \in \{r\}\}) \subseteq \mathbf{read}(G_2) \setminus \{\hat{r}.x \mid \hat{r} \in R_1\}$ . Then,  $\mathbf{read}(G_2) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup \{r\}\} \subseteq \mathbf{read}(G_2) \setminus \{\hat{r}.x \mid \hat{r} \in R_1\}$ .
  - Recall  $\mathbf{read}(\hat{\xi}^+ \uparrow r) \subseteq \mathbf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\}$  and  $\mathbf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2 \cup \{r\}\} \subseteq \mathbf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\}$  and  $\mathbf{read}(G_2) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup \{r\}\} \subseteq \mathbf{read}(G_2) \setminus \{\hat{r}.x \mid \hat{r} \in R_1\}$ . Then,  $\mathbf{read}(\hat{\xi}^+ \uparrow r) \cup (\mathbf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2 \cup \{r\}\}) \cup (\mathbf{read}(G_2) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup \{r\}\}) \subseteq (\mathbf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\}) \cup (\mathbf{read}(G_2) \setminus \{\hat{r}.x \mid \hat{r} \in R_1\})$ . Then,  $\mathbf{read}(\hat{\xi}^+ \uparrow r) \cup \emptyset \cup (\mathbf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2 \cup \{r\}\}) \cup (\mathbf{read}(G_1) \setminus \{\hat{r}.x \mid \hat{r} \in R_2\}) \cup (\mathbf{read}(G_2) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup \{r\}\}) \subseteq (\mathbf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\}) \cup (\mathbf{read}(G_1) \setminus \{\hat{r}.x \mid \hat{r} \in R_2\}) \cup (\mathbf{read}(G_2) \setminus \{\hat{r}.x \mid \hat{r} \in R_1\})$ . Then, by Defn. 15,  $\mathbf{read}(\hat{\xi}^+ \uparrow r) \cup \mathbf{read}(1\{r\}) \cup (\mathbf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2 \cup \{r\}\}) \cup (\mathbf{read}(G_1) \setminus \{\hat{r}.x \mid \hat{r} \in R_2\}) \cup (\mathbf{read}(G_2) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup \{r\}\}) \subseteq (\mathbf{read}(\hat{\xi}) \setminus \{\hat{r}.x \mid \hat{r} \in R_1 \cup R_2\}) \cup (\mathbf{read}(G_1) \setminus \{\hat{r}.x \mid \hat{r} \in R_2\}) \cup (\mathbf{read}(G_2) \setminus \{\hat{r}.x \mid \hat{r} \in R_1\})$ . Then, by Defn. 22,  $\mathbf{read}(\hat{\xi}^+ \uparrow r) \cup \mathbf{read}(1\{r\}) \cup \mathbf{read}(\mathbf{if} \hat{\xi}|_n A_1|_{R_1 \cup \{r\}} A_2|_{R_2}) \subseteq \mathbf{read}(\mathbf{if} \hat{\xi}|_n A_1|_{R_1} A_2|_{R_2})$ . Then,  $\mathbf{read}(\xi) \cup \mathbf{read}(\alpha) \cup \mathbf{read}(A') \subseteq \mathbf{read}(A)$ .
- **Base:** [ $\rightarrow$ 1-NIF3]. Similar to case [ $\rightarrow$ 1-NIF2].
- **Step:** [ $\rightarrow$ 1-SEQ1], such that  $A = A_1 ; A_2$  and  $A' = A'_1 ; A_2$  and  $A_1 \xrightarrow{\xi, \alpha} A'_1$ , for some  $A_1, A'_1, A_2$ .  
Recall  $A_1 \xrightarrow{\xi, \alpha} A'_1$ . Then, by induction,  $\mathbf{read}(\xi) \cup \mathbf{read}(\alpha) \cup \mathbf{read}(A'_1) \subseteq \mathbf{read}(A_1)$ . Then,  $\mathbf{read}(\xi) \cup \mathbf{read}(\alpha) \cup \mathbf{read}(A'_1) \cup \mathbf{read}(A_2) \subseteq \mathbf{read}(A_1) \cup \mathbf{read}(A_2)$ . Then, by Defn. 22,  $\mathbf{read}(\xi) \cup \mathbf{read}(\alpha) \cup \mathbf{read}(A'_1 ; A_2) \subseteq \mathbf{read}(A_1 ; A_2)$ . Then,  $\mathbf{read}(\xi) \cup \mathbf{read}(\alpha) \cup \mathbf{read}(A') \subseteq \mathbf{read}(A)$ .

- **Step:** [ $\rightarrow$ 1-SEQ2]. Similar to case [ $\rightarrow$ 1-SEQ1].
  - **Step:** [ $\rightarrow$ 1-PAR1]. Similar to case [ $\rightarrow$ 1-SEQ1].
  - **Step:** [ $\rightarrow$ 1-PAR2]. Similar to case [ $\rightarrow$ 1-SEQ1].
  - **Step:** [ $\rightarrow$ 1-NIF4], such that  $A = \mathbf{if} \hat{\xi}|_n A_1|_{R_1} A_2|_{R_2}$  and  $A' = \mathbf{if} \hat{\xi}|_n A'_1|_{R_1} A_2|_{R_2}$  and  $A_1 \xrightarrow{\xi, \alpha} A'_1$  and  $\text{subj}(\xi) \cup \text{subj}(\alpha) \subseteq R_1 \setminus R_2$ , for some  $R_1, R_2, A_1, A'_1, A_2, \xi, n$ .
    - Recall  $\text{subj}(\xi) \cup \text{subj}(\alpha) \subseteq R_1 \setminus R_2$ . Then,  $\text{subj}(\xi) \subseteq R_1 \setminus R_2$ . Then,  $\text{subj}(\xi) \cap R_2 = \emptyset$ .
    - By Lem. 5,  $\text{read}(\xi) \subseteq \{r.x \mid r \in \text{subj}(\xi)\}$ . Then,  $\text{read}(\xi) \cap \{r.x \mid r \in R_2\} \subseteq \{r.x \mid r \in \text{subj}(\xi)\} \cap \{r.x \mid r \in R_2\}$ . Then,  $\text{read}(\xi) \cap \{r.x \mid r \in R_2\} \subseteq \{r.x \mid r \in \text{subj}(\xi) \cap R_2\}$ . Then,  $\text{read}(\xi) \cap \{r.x \mid r \in R_2\} \subseteq \{r.x \mid r \in \emptyset\}$ . Then,  $\text{read}(\xi) \cap \{r.x \mid r \in R_2\} \subseteq \emptyset$ . Then,  $\text{read}(\xi) \cap \{r.x \mid r \in R_2\} = \emptyset$ . Then,  $\text{read}(\xi) \setminus \{r.x \mid r \in R_2\} = \text{read}(\xi)$ .
    - Recall  $\text{subj}(\xi) \cup \text{subj}(\alpha) \subseteq R_1 \setminus R_2$ . Then,  $\text{subj}(\alpha) \subseteq R_1 \setminus R_2$ . Then,  $\text{subj}(\alpha) \cap R_2 = \emptyset$ .
    - By Lem. 31,  $\text{read}(\alpha) \subseteq \{r.x \mid r \in \text{subj}(\alpha)\}$ . Then,  $\text{read}(\alpha) \cap \{r.x \mid r \in R_2\} \subseteq \{r.x \mid r \in \text{subj}(\alpha)\} \cap \{r.x \mid r \in R_2\}$ . Then,  $\text{read}(\alpha) \cap \{r.x \mid r \in R_2\} \subseteq \{r.x \mid r \in \text{subj}(\alpha) \cap R_2\}$ . Then,  $\text{read}(\alpha) \cap \{r.x \mid r \in R_2\} \subseteq \{r.x \mid r \in \emptyset\}$ . Then,  $\text{read}(\alpha) \cap \{r.x \mid r \in R_2\} \subseteq \emptyset$ . Then,  $\text{read}(\alpha) \cap \{r.x \mid r \in R_2\} = \emptyset$ . Then,  $\text{read}(\alpha) \setminus \{r.x \mid r \in R_2\} = \text{read}(\alpha)$ .
    - Recall  $A_1 \xrightarrow{\xi, \alpha} A'_1$ . Then, by induction,  $\text{read}(\xi) \cup \text{read}(\alpha) \cup \text{read}(A'_1) \subseteq \text{read}(A_1)$ . Then,  $(\text{read}(\xi) \setminus \{r.x \mid r \in R_2\}) \cup (\text{read}(\alpha) \setminus \{r.x \mid r \in R_2\}) \cup (\text{read}(A'_1) \setminus \{r.x \mid r \in R_2\}) \subseteq \text{read}(A_1) \setminus \{r.x \mid r \in R_2\}$ . Then,  $\text{read}(\xi) \cup \text{read}(\alpha) \cup (\text{read}(A'_1) \setminus \{r.x \mid r \in R_2\}) \subseteq \text{read}(A_1) \setminus \{r.x \mid r \in R_2\}$ . Then,  $\text{read}(\xi) \cup \text{read}(\alpha) \cup (\text{read}(\hat{\xi}) \setminus \{r.x \mid r \in R_1 \cup R_2\}) \cup (\text{read}(A'_1) \setminus \{r.x \mid r \in R_2\}) \cup (\text{read}(A_2) \setminus \{r.x \mid r \in R_1\}) \subseteq (\text{read}(\hat{\xi}) \setminus \{r.x \mid r \in R_1 \cup R_2\}) \cup (\text{read}(A_1) \setminus \{r.x \mid r \in R_2\}) \cup (\text{read}(A_2) \setminus \{r.x \mid r \in R_1\})$ . Then, by Defn. 22,  $\text{read}(\xi) \cup \text{read}(\alpha) \cup \text{read}(\mathbf{if} \hat{\xi}|_n A'_1|_{R_1} A_2|_{R_2}) \subseteq \text{read}(\mathbf{if} \hat{\xi}|_n A_1|_{R_1} A_2|_{R_2})$ . Then,  $\text{read}(\xi) \cup \text{read}(\alpha) \cup \text{read}(A') \subseteq \text{read}(A)$ .
  - **Step:** [ $\rightarrow$ 1-NIF4]. Similar to case [ $\rightarrow$ 1-NIF5].
  - **Step:** [ $\rightarrow$ 1-NWHILE], such that  $A = \mathbf{while} \hat{\xi}|_n \{\psi\} \hat{A}|_{\emptyset}$  and  $\mathbf{if} \hat{\xi}|_n (\hat{A} ; \mathbf{while} \hat{\xi}|_n \{\psi\} \hat{A}|_{\emptyset})|_{\emptyset} \mathbf{skip}|_{\emptyset} \xrightarrow{\xi, \alpha} A'$ , for some  $\hat{A}, \psi, \hat{\xi}, n$ . Recall  $\mathbf{if} \hat{\xi}|_n (\hat{A} ; \mathbf{while} \hat{\xi}|_n \{\psi\} \hat{A}|_{\emptyset})|_{\emptyset} \mathbf{skip}|_{\emptyset} \xrightarrow{\xi, \alpha} A'$ . Then, by induction,  $\text{read}(\xi) \cup \text{read}(\alpha) \cup \text{read}(A') \subseteq \text{read}(\mathbf{if} \hat{\xi}|_n (\hat{A} ; \mathbf{while} \hat{\xi}|_n \{\psi\} \hat{A}|_{\emptyset})|_{\emptyset} \mathbf{skip}|_{\emptyset})$ . Then, by Lem. 36,  $\text{read}(\xi) \cup \text{read}(\alpha) \cup \text{read}(A') \subseteq \text{read}(\hat{\xi}) \cup \text{read}(\hat{A} ; \mathbf{while} \hat{\xi}|_n \{\psi\} \hat{A}|_{\emptyset}) \cup \text{read}(\mathbf{skip})$ . Then, by Defn. 22,  $\text{read}(\xi) \cup \text{read}(\alpha) \cup \text{read}(A') \subseteq \text{read}(\hat{\xi}) \cup \text{read}(\hat{A}) \cup \text{read}(\mathbf{while} \hat{\xi}|_n \{\psi\} \hat{A}|_{\emptyset}) \cup \emptyset$ . Then, by Defn. 22,  $\text{read}(\xi) \cup \text{read}(\alpha) \cup \text{read}(A') \subseteq \text{read}(\mathbf{while} \hat{\xi}|_n \{\psi\} \hat{A}|_{\emptyset}) \cup \text{read}(\mathbf{while} \hat{\xi}|_n \{\psi\} \hat{A}|_{\emptyset}) \cup \emptyset$ . Then,  $\text{read}(\xi) \cup \text{read}(\alpha) \cup \text{read}(A') \subseteq \text{read}(\mathbf{while} \hat{\xi}|_n \{\psi\} \hat{A}|_{\emptyset})$ . Then,  $\text{read}(\xi) \cup \text{read}(\alpha) \cup \text{read}(A') \subseteq \text{read}(A)$ . □
2. Similar to 1. □

*Proof (of Lem. 52).*

1. – Recall  $A_1 \# A_2$ . Then, by Defn. 23,  $\text{read}(A_1) \cap \text{read}(A_2) = \emptyset$  and  $\text{read}(A_1) \cap \text{write}(A_2) = \emptyset$  and  $\text{write}(A_1) \cap \text{read}(A_2) = \emptyset$  and  $\text{write}(A_1) \cap \text{write}(A_2) = \emptyset$ .
  - Recall  $A_2 \xrightarrow{\xi, \alpha} A'_2$ . Then, by Lem. 51,  $\text{read}(\xi) \cup \text{read}(\alpha) \cup \text{read}(A'_2) \subseteq \text{read}(A_2)$  and  $\text{write}(\xi) \cup \text{write}(\alpha) \cup \text{write}(A'_2) \subseteq \text{write}(A_2)$ . Then,  $\text{read}(\xi) \subseteq \text{read}(A_2)$  and  $\text{write}(\xi) \subseteq \text{write}(A_2)$ .
  - Recall  $\text{read}(A_1) \cap \text{read}(A_2) = \emptyset$  and  $\text{write}(A_1) \cap \text{read}(A_2) = \emptyset$  and  $\text{read}(\xi) \subseteq \text{read}(A_2)$ . Then,  $\text{read}(A_1) \cap \text{read}(\xi) = \emptyset$  and  $\text{write}(A_1) \cap \text{read}(\xi) = \emptyset$ .
  - Recall  $\text{read}(A_1) \cap \text{write}(A_2) = \emptyset$  and  $\text{write}(A_1) \cap \text{write}(A_2) = \emptyset$  and  $\text{write}(\xi) \subseteq \text{write}(A_2)$ . Then,  $\text{read}(A_1) \cap \text{write}(\xi) = \emptyset$  and  $\text{write}(A_1) \cap \text{write}(\xi) = \emptyset$ .
  - Recall  $\text{read}(A_1) \cap \text{read}(\xi) = \emptyset$  and  $\text{write}(A_1) \cap \text{read}(\xi) = \emptyset$  and  $\text{read}(A_1) \cap \text{write}(\xi) = \emptyset$  and  $\text{write}(A_1) \cap \text{write}(\xi) = \emptyset$ . Then, by [#],  $A_1 \# \xi$ . □
2. Similar to 1. □

3. Similar to 1. □  
 4. Similar to 1. □  
 5. Similar to 1. □  
 6. Similar to 1. □

*Proof (of Lem. 53).*

1. Recall  $A \xrightarrow{\xi, \alpha} A'$ . Then, by Defn. 31:
- **Base:**  $[\rightarrow 1\text{-ACT}]$ , such that  $A = \alpha$  and  $A' = \text{skip}$  and  $\xi = \text{true}$ .  
 Recall  $\emptyset \cup \text{subj}(A) \cup \emptyset \subseteq \text{subj}(A)$ . Then, by Lem. 4,  $\text{subj}(\text{true}) \cup \text{subj}(A) \cup \emptyset \subseteq \text{subj}(A)$ .  
 Then, by Defn. 24,  $\text{subj}(\text{true}) \cup \text{subj}(A) \cup \text{subj}(\text{skip}) \subseteq \text{subj}(A)$ . Then,  
 $\text{subj}(\xi) \cup \text{subj}(\alpha) \cup \text{subj}(A') \subseteq \text{subj}(A)$ .
  - **Base:**  $[\rightarrow 1\text{-IF1}]$ , such that  $A = R.\text{if } \hat{\xi} A_1 A_2$  and  $A' = A_1$  and  $\xi = \hat{\xi}^+$  and  $\alpha = 1_{\text{subj}(\hat{\xi})}^R$ ,  
 for some  $R, A_1, A_2, \hat{\xi}$ .
    - Recall  $\alpha = 1_{\text{subj}(\hat{\xi})}^R$ . Then,  $\text{subj}(\alpha) = \text{subj}(1_{\text{subj}(\hat{\xi})}^R)$ . Then, by Defn. 16,  
 $\text{subj}(\alpha) = \text{subj}(\hat{\xi})$ .
    - Recall  $\text{subj}(\hat{\xi}) \cup \text{subj}(\hat{\xi}) \cup \text{subj}(A_1) \subseteq \text{subj}(\hat{\xi}) \cup \text{subj}(A_1) \cup \text{subj}(A_2)$ . Then, by Lem. 13,  
 $\text{subj}(\hat{\xi}^+) \cup \text{subj}(\hat{\xi}) \cup \text{subj}(A_1) \subseteq \text{subj}(\hat{\xi}) \cup \text{subj}(A_1) \cup \text{subj}(A_2)$ . Then,  
 $\text{subj}(\hat{\xi}^+) \cup \text{subj}(\alpha) \cup \text{subj}(A_1) \subseteq \text{subj}(\hat{\xi}) \cup \text{subj}(A_1) \cup \text{subj}(A_2)$ . Then, by Defn. 24,  
 $\text{subj}(\hat{\xi}^+) \cup \text{subj}(\alpha) \cup \text{subj}(A_1) \subseteq \text{subj}(R.\text{if } \hat{\xi} A_1 A_2)$ . Then,  
 $\text{subj}(\xi) \cup \text{subj}(\alpha) \cup \text{subj}(A') \subseteq \text{subj}(A)$ .
  - **Base:**  $[\rightarrow 1\text{-IF2}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
  - **Base:**  $[\rightarrow 1\text{-WHILE1}]$ , such that  $A = R.\text{while } \hat{\xi} \{\psi\} \hat{A}$  and  $A' = \hat{A}$ ;  $R.\text{while } \hat{\xi} \{\psi\} \hat{A}$  and  
 $\xi = \hat{\xi}^+$  and  $\alpha = 1_{\text{subj}(\hat{\xi})}^R$ , for some  $R, \hat{A}, \psi, \hat{\xi}$ .
    - Recall  $\alpha = 1_{\text{subj}(\hat{\xi})}^R$ . Then,  $\text{subj}(\alpha) = \text{subj}(1_{\text{subj}(\hat{\xi})}^R)$ . Then, by Defn. 16,  
 $\text{subj}(\alpha) = \text{subj}(\hat{\xi})$ .
    - Recall  $\text{subj}(\hat{\xi}) \cup \text{subj}(\hat{\xi}) \cup \text{subj}(\hat{A}) \cup \text{subj}(\hat{\xi}) \cup \text{subj}(\hat{A}) \subseteq \text{subj}(\hat{\xi}) \cup \text{subj}(\hat{A})$ . Then, by  
 Lem. 13,  $\text{subj}(\hat{\xi}^+) \cup \text{subj}(\hat{\xi}) \cup \text{subj}(\hat{A}) \cup \text{subj}(\hat{\xi}) \cup \text{subj}(\hat{A}) \subseteq \text{subj}(\hat{\xi}) \cup \text{subj}(\hat{A})$ . Then,  
 $\text{subj}(\hat{\xi}^+) \cup \text{subj}(\alpha) \cup \text{subj}(\hat{A}) \cup \text{subj}(\hat{\xi}) \cup \text{subj}(\hat{A}) \subseteq \text{subj}(\hat{\xi}) \cup \text{subj}(\hat{A})$ . Then, by Defn. 24,  
 $\text{subj}(\hat{\xi}^+) \cup \text{subj}(\alpha) \cup \text{subj}(\hat{A}) \cup \text{subj}(R.\text{while } \hat{\xi} \{\psi\} \hat{A}) \subseteq \text{subj}(R.\text{while } \hat{\xi} \{\psi\} \hat{A})$ . Then,  
 by Defn. 24,  $\text{subj}(\hat{\xi}^+) \cup \text{subj}(\alpha) \cup \text{subj}(\hat{A}; R.\text{while } \hat{\xi} \{\psi\} \hat{A}) \subseteq \text{subj}(R.\text{while } \hat{\xi} \{\psi\} \hat{A})$ .  
 Then,  $\text{subj}(\xi) \cup \text{subj}(\alpha) \cup \text{subj}(A') \subseteq \text{subj}(A)$ .
  - **Base:**  $[\rightarrow 1\text{-WHILE2}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
  - **Base:**  $[\rightarrow 1\text{-NIF1}]$ , such that  $A = \text{if } \hat{\xi}|_n A_1|_{R_1} A_2|_{R_2}$  and  $A' = \text{if } \hat{\xi}|_{n-1} A_1|_{R_1} A_2|_{R_2}$  and  
 $\xi = \hat{\xi}^+ \uparrow r$  and  $\alpha = \tau$ , for some  $R_1, R_2, A_1, A_2, \hat{\xi}, n$ .  
 Recall  $\emptyset \cup \emptyset \cup \text{subj}(\text{if } \hat{\xi}|_{n-1} A_1|_{R_1} A_2|_{R_2}) \subseteq \text{subj}(\text{if } \hat{\xi}|_{n-1} A_1|_{R_1} A_2|_{R_2})$ . Then, by Lem. 4,  
 $\text{subj}(\text{true}) \cup \emptyset \cup \text{subj}(\text{if } \hat{\xi}|_{n-1} A_1|_{R_1} A_2|_{R_2}) \subseteq \text{subj}(\text{if } \hat{\xi}|_{n-1} A_1|_{R_1} A_2|_{R_2})$ . Then, by  
 Defn. 16,  $\text{subj}(\text{true}) \cup \text{subj}(\tau) \cup \text{subj}(\text{if } \hat{\xi}|_{n-1} A_1|_{R_1} A_2|_{R_2}) \subseteq \text{subj}(\text{if } \hat{\xi}|_{n-1} A_1|_{R_1} A_2|_{R_2})$ .  
 Then, by Defn. 24,  $\text{subj}(\text{true}) \cup \text{subj}(\tau) \cup \text{subj}(\text{if } \hat{\xi}|_{n-1} A_1|_{R_1} A_2|_{R_2}) \subseteq \text{subj}(\text{if } \hat{\xi}|_{n-1} A_1|_{R_1} A_2|_{R_2})$ .  
 $(\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)) \cup (\text{subj}(A_1) \setminus R_2) \cup (\text{subj}(A_2) \setminus R_1)$ . Then, by Defn. 24,  
 $\text{subj}(\text{true}) \cup \text{subj}(\tau) \cup \text{subj}(\text{if } \hat{\xi}|_{n-1} A_1|_{R_1} A_2|_{R_2}) \subseteq \text{subj}(\text{if } \hat{\xi}|_n A_1|_{R_1} A_2|_{R_2})$ . Then,  
 $\text{subj}(\xi) \cup \text{subj}(\alpha) \cup \text{subj}(A') \subseteq \text{subj}(A)$ .
  - **Base:**  $[\rightarrow 1\text{-NIF2}]$ , such that  $A = \text{if } \hat{\xi}|_n A_1|_{R_1} A_2|_{R_2}$  and  $A' = \text{if } \hat{\xi}|_n A_1|_{R_1 \cup \{r\}} A_2|_{R_2}$  and  
 $\xi = \hat{\xi}^+ \uparrow r$  and  $\alpha = 1_{\{r\}}^r$  and  $r \in \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$ , for some  $R_1, R_2, A_1, A_2, r, \hat{\xi}, n$ .
    - Recall  $r \in \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$ . Then,  $r \notin R_1 \cup R_2$ . Then,  $\{r\} \cap (R_1 \cup R_2) = \emptyset$ .
    - By Lem. 10,  $\text{subj}(\hat{\xi} \uparrow r) \subseteq \{r\}$ . Then,  $\text{subj}(\hat{\xi} \uparrow r) \cap (R_1 \cup R_2) \subseteq \{r\} \cap (R_1 \cup R_2)$ . Then,  
 $\text{subj}(\hat{\xi} \uparrow r) \cap (R_1 \cup R_2) \subseteq \emptyset$ . Then,  $\text{subj}(\hat{\xi} \uparrow r) \cap (R_1 \cup R_2) = \emptyset$ . Then,  
 $\text{subj}(\hat{\xi} \uparrow r) \setminus (R_1 \cup R_2) = \text{subj}(\hat{\xi} \uparrow r)$ .
    - By Lem. 10,  $\text{subj}(\hat{\xi} \uparrow r) \subseteq \text{subj}(\hat{\xi})$ . Then,  $\text{subj}(\hat{\xi} \uparrow r) \setminus (R_1 \cup R_2) \subseteq \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$ .  
 Then,  $\text{subj}(\hat{\xi} \uparrow r) \subseteq \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$ . Then, by Lem. 13,  
 $\text{subj}((\hat{\xi} \uparrow r)^+) \subseteq \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$ . Then, by Lem. 11,  
 $\text{subj}(\hat{\xi}^+ \uparrow r) \subseteq \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$ .
    - Recall  $\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2) \subseteq \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$ . Then,  
 $(\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)) \setminus \{r\} \subseteq \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$ . Then,  
 $\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\}) \subseteq \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$ .



- Recall  $\text{subj}(G_2) \setminus R_1 \subseteq \text{subj}(G_2) \setminus R_1$ . Then,  $(\text{subj}(G_2) \setminus R_1) \setminus \{r\} \subseteq \text{subj}(G_2) \setminus R_1$ . Then,  $\text{subj}(G_2) \setminus (R_1 \cup \{r\}) \subseteq \text{subj}(G_2) \setminus R_1$ .
  - Recall  $\text{subj}(\hat{\xi}^+ \uparrow r) \subseteq \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$  and  $\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\}) \subseteq \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$  and  $\text{subj}(G_2) \setminus (R_1 \cup \{r\}) \subseteq \text{subj}(G_2) \setminus R_1$ . Then,  $\text{subj}(\hat{\xi}^+ \uparrow r) \cup (\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})) \cup (\text{subj}(G_2) \setminus (R_1 \cup \{r\})) \subseteq (\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)) \cup (\text{subj}(G_2) \setminus R_1)$ . Then,  $\text{subj}(\hat{\xi}^+ \uparrow r) \cup \emptyset \cup (\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})) \cup (\text{subj}(G_1) \setminus R_2) \cup (\text{subj}(G_2) \setminus (R_1 \cup \{r\})) \subseteq (\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)) \cup (\text{subj}(G_1) \setminus R_2) \cup (\text{subj}(G_2) \setminus R_1)$ . Then, by Defn. 16,  $\text{subj}(\hat{\xi}^+ \uparrow r) \cup \text{subj}(1 \uparrow_r) \cup (\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})) \cup (\text{subj}(G_1) \setminus R_2) \cup (\text{subj}(G_2) \setminus (R_1 \cup \{r\})) \subseteq (\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)) \cup (\text{subj}(G_1) \setminus R_2) \cup (\text{subj}(G_2) \setminus R_1)$ . Then, by Defn. 24,  $\text{subj}(\hat{\xi}^+ \uparrow r) \cup \text{subj}(1 \uparrow_r) \cup \text{subj}(\text{if } \hat{\xi}|_n A_1|_{R_1 \cup \{r\}} A_2|_{R_2}) \subseteq \text{subj}(\text{if } \hat{\xi}|_n A_1|_{R_1} A_2|_{R_2})$ . Then,  $\text{subj}(\hat{\xi}) \cup \text{subj}(\alpha) \cup \text{subj}(A') \subseteq \text{subj}(A)$ .
- **Base:**  $[\rightarrow 1\text{-NIF3}]$ . Similar to case  $[\rightarrow 1\text{-NIF2}]$ .
- **Step:**  $[\rightarrow 1\text{-SEQ1}]$ , such that  $A = A_1 ; A_2$  and  $A_1 \xrightarrow{\xi, \alpha} A'_1$ , for some  $A_1, A'_1, A_2$ . Recall  $A_1 \xrightarrow{\xi, \alpha} A'_1$ . Then, by induction,  $\text{subj}(\xi) \cup \text{subj}(\alpha) \cup \text{subj}(A'_1) \subseteq \text{subj}(A_1)$ . Then,  $\text{subj}(\xi) \cup \text{subj}(\alpha) \cup \text{subj}(A'_1) \cup \text{subj}(A_2) \subseteq \text{subj}(A_1) \cup \text{subj}(A_2)$ . Then, by Defn. 24,  $\text{subj}(\xi) \cup \text{subj}(\alpha) \cup \text{subj}(A'_1 ; A_2) \subseteq \text{subj}(A_1 ; A_2)$ . Then,  $\text{subj}(\xi) \cup \text{subj}(\alpha) \cup \text{subj}(A') \subseteq \text{subj}(A)$ .
- **Step:**  $[\rightarrow 1\text{-SEQ2}]$ . Similar to case  $[\rightarrow 1\text{-SEQ1}]$ .
- **Step:**  $[\rightarrow 1\text{-PAR1}]$ . Similar to case  $[\rightarrow 1\text{-SEQ1}]$ .
- **Step:**  $[\rightarrow 1\text{-PAR2}]$ . Similar to case  $[\rightarrow 1\text{-SEQ1}]$ .
- **Step:**  $[\rightarrow 1\text{-NIF4}]$ , such that  $A = \text{if } \hat{\xi}|_n A_1|_{R_1} A_2|_{R_2}$  and  $A' = \text{if } \hat{\xi}|_n A'_1|_{R_1} A_2|_{R_2}$  and  $A_1 \xrightarrow{\xi, \alpha} A'_1$  and  $\text{subj}(\xi) \cup \text{subj}(\alpha) \subseteq R_1 \setminus R_2$ , for some  $R_1, R_2, A_1, A'_1, A_2, \hat{\xi}$ .
- Recall  $\text{subj}(\xi) \cup \text{subj}(\alpha) \subseteq R_1 \setminus R_2$ . Then,  $\text{subj}(\xi) \subseteq R_1 \setminus R_2$ . Then,  $\text{subj}(\xi) \cap R_2 = \emptyset$ . Then,  $\text{subj}(\xi) \setminus R_2 = \text{subj}(\xi)$ .
  - Recall  $\text{subj}(\xi) \cup \text{subj}(\alpha) \subseteq R_1 \setminus R_2$ . Then,  $\text{subj}(\alpha) \subseteq R_1 \setminus R_2$ . Then,  $\text{subj}(\alpha) \cap R_2 = \emptyset$ . Then,  $\text{subj}(\alpha) \setminus R_2 = \text{subj}(\alpha)$ .
  - Recall  $A_1 \xrightarrow{\xi, \alpha} A'_1$ . Then, by induction,  $\text{subj}(\xi) \cup \text{subj}(\alpha) \cup \text{subj}(A'_1) \subseteq \text{subj}(A_1)$ . Then,  $(\text{subj}(\xi) \setminus R_2) \cup (\text{subj}(\alpha) \setminus R_2) \cup (\text{subj}(A'_1) \setminus R_2) \subseteq \text{subj}(A_1) \setminus R_2$ . Then,  $\text{subj}(\xi) \cup \text{subj}(\alpha) \cup (\text{subj}(A'_1) \setminus R_2) \subseteq \text{subj}(A_1) \setminus R_2$ . Then,  $\text{subj}(\xi) \cup \text{subj}(\alpha) \cup (\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)) \cup (\text{subj}(A'_1) \setminus R_2) \cup (\text{subj}(A_2) \setminus R_1) \subseteq (\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)) \cup (\text{subj}(A_1) \setminus R_2) \cup (\text{subj}(A_2) \setminus R_1)$ . Then, by Defn. 24,  $\text{subj}(\xi) \cup \text{subj}(\alpha) \cup \text{subj}(\text{if } \hat{\xi}|_n A'_1|_{R_1} A_2|_{R_2}) \subseteq \text{subj}(\text{if } \hat{\xi}|_n A_1|_{R_1} A_2|_{R_2})$ . Then,  $\text{subj}(\xi) \cup \text{subj}(\alpha) \cup \text{subj}(A') \subseteq \text{subj}(A)$ .
- **Step:**  $[\rightarrow 1\text{-NIF5}]$ . Similar to case  $[\rightarrow 1\text{-NIF4}]$ .
- **Step:**  $[\rightarrow 1\text{-NWHILE}]$ , such that  $A = \text{while } \hat{\xi}|_n \{\psi\} \hat{A}|_\emptyset$  and  $\text{if } \hat{\xi}|_n (\hat{A} ; \text{while } \hat{\xi}|_n \{\psi\} \hat{A}|_\emptyset) \text{ skip}|_\emptyset \xrightarrow{\xi, \alpha} A'$ , for some  $\hat{A}, \psi, \hat{\xi}, n$ . Recall  $\text{if } \hat{\xi}|_n (\hat{A} ; \text{while } \hat{\xi}|_n \{\psi\} \hat{A}|_\emptyset) \text{ skip}|_\emptyset \xrightarrow{\xi, \alpha} A'$ . Then, by induction,  $\text{subj}(\xi) \cup \text{subj}(\alpha) \cup \text{subj}(A') \subseteq \text{subj}(\text{if } \hat{\xi}|_n (\hat{A} ; \text{while } \hat{\xi}|_n \{\psi\} \hat{A}|_\emptyset) \text{ skip}|_\emptyset)$ . Then, by Defn. 24,  $\text{subj}(\xi) \cup \text{subj}(\alpha) \cup \text{subj}(A') \subseteq (\text{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset)) \cup (\text{subj}(\hat{A} ; \text{while } \hat{\xi}|_n \{\psi\} \hat{A}|_\emptyset) \setminus \emptyset) \cup (\text{subj}(\text{skip}) \setminus \emptyset)$ . Then,  $\text{subj}(\xi) \cup \text{subj}(\alpha) \cup \text{subj}(A') \subseteq \text{subj}(\hat{\xi}) \cup \text{subj}(\hat{A} ; \text{while } \hat{\xi}|_n \{\psi\} \hat{A}|_\emptyset) \cup \text{subj}(\text{skip})$ . Then, by Defn. 24,  $\text{subj}(\xi) \cup \text{subj}(\alpha) \cup \text{subj}(A') \subseteq \text{subj}(\hat{\xi}) \cup \text{subj}(\hat{A}) \cup \text{subj}(\text{while } \hat{\xi}|_n \{\psi\} \hat{A}|_\emptyset) \cup \emptyset$ . Then, by Defn. 24,  $\text{subj}(\xi) \cup \text{subj}(\alpha) \cup \text{subj}(A') \subseteq \text{subj}(\text{while } \hat{\xi}|_n \{\psi\} \hat{A}|_\emptyset) \cup \text{subj}(\text{while } \hat{\xi}|_n \{\psi\} \hat{A}|_\emptyset) \cup \emptyset$ . Then,  $\text{subj}(\xi) \cup \text{subj}(\alpha) \cup \text{subj}(A') \subseteq \text{subj}(\text{while } \hat{\xi}|_n \{\psi\} \hat{A}|_\emptyset)$ . Then,  $\text{subj}(\xi) \cup \text{subj}(\alpha) \cup \text{subj}(A') \subseteq \text{subj}(A)$ . □
2. Recall  $A \xrightarrow{\xi, \alpha} A'$ . Then, by Defn. 31:
- **Base:**  $[\rightarrow 1\text{-ACT}]$ , such that  $\xi = \text{true}$ . Recall  $\emptyset \subseteq \text{subj}(\alpha)$ . Then, by Lem. 4,  $\text{subj}(\text{true}) \subseteq \text{subj}(\alpha)$ . Then,  $\text{subj}(\xi) \subseteq \text{subj}(\alpha)$ .
  - **Base:**  $[\rightarrow 1\text{-IF1}]$ , such that  $\xi = \hat{\xi}^+$  and  $\alpha = 1_{\text{subj}(\hat{\xi})}^R$ , for some  $R, \hat{\xi}$ . Recall  $\text{subj}(\xi) \subseteq \text{subj}(\hat{\xi})$ . Then,  $\text{subj}(\xi) \subseteq \text{subj}(\hat{\xi}^+)$ . Then, by Lem. 13,  $\text{subj}(\xi) \subseteq \text{subj}(\hat{\xi})$ . Then, by Defn. 16,  $\text{subj}(\xi) \subseteq \text{subj}(1_{\text{subj}(\hat{\xi})}^R)$ . Then,  $\text{subj}(\xi) \subseteq \text{subj}(\alpha)$ .

- **Base:**  $[\rightarrow 1\text{-IF2}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
  - **Base:**  $[\rightarrow 1\text{-WHILE1}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
  - **Base:**  $[\rightarrow 1\text{-WHILE2}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
  - **Base:**  $[\rightarrow 1\text{-NIF1}]$ . Similar to case  $[\rightarrow 1\text{-ACT}]$ .
  - **Base:**  $[\rightarrow 1\text{-NIF2}]$ , such that  $\xi = \hat{\xi}^+ \upharpoonright r$  and  $\alpha = 1\{\frac{r}{r}\}$  and  $r \in \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$ , for some  $R, R_1, R_2, r, \hat{\xi}$ .
    - Recall  $r \in \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$ . Then,  $r \in \text{subj}(\hat{\xi})$ . Then, by Lem. 9,  $\text{subj}(\hat{\xi} \upharpoonright r) = \{r\}$ .
    - Recall  $\text{subj}(\xi) \subseteq \text{subj}(\hat{\xi})$ . Then,  $\text{subj}(\xi) \subseteq \text{subj}(\hat{\xi}^+ \upharpoonright r)$ . Then, by Lem. 11,  $\text{subj}(\xi) \subseteq \text{subj}((\hat{\xi} \upharpoonright r)^+)$ . Then, by Lem. 13,  $\text{subj}(\xi) \subseteq \text{subj}(\hat{\xi} \upharpoonright r)$ . Then,  $\text{subj}(\xi) \subseteq \{r\}$ . Then, by Defn. 16,  $\text{subj}(\xi) \subseteq \text{subj}(1\{\frac{r}{r}\})$ . Then,  $\text{subj}(\xi) \subseteq \text{subj}(\alpha)$ .
  - **Base:**  $[\rightarrow 1\text{-NIF3}]$ . Similar to case  $[\rightarrow 1\text{-NIF2}]$ .
  - **Step:**  $[\rightarrow 1\text{-SEQ1}]$ , such that  $A_1 \xrightarrow{\xi, \alpha} A'_1$ , for some  $A_1, A'_1$ . Recall  $A_1 \xrightarrow{\xi, \alpha} A'_1$ . Then, by induction,  $\text{subj}(\xi) \subseteq \text{subj}(\alpha)$ .
  - **Step:**  $[\rightarrow 1\text{-SEQ2}]$ . Similar to case  $[\rightarrow 1\text{-SEQ1}]$ .
  - **Step:**  $[\rightarrow 1\text{-PAR1}]$ . Similar to case  $[\rightarrow 1\text{-SEQ1}]$ .
  - **Step:**  $[\rightarrow 1\text{-PAR2}]$ . Similar to case  $[\rightarrow 1\text{-SEQ1}]$ .
  - **Step:**  $[\rightarrow 1\text{-NIF4}]$ . Similar to case  $[\rightarrow 1\text{-SEQ1}]$ .
  - **Step:**  $[\rightarrow 1\text{-NIF5}]$ . Similar to case  $[\rightarrow 1\text{-SEQ1}]$ .
  - **Step:**  $[\rightarrow 1\text{-NWHILE}]$ . Similar to case  $[\rightarrow 1\text{-SEQ1}]$ . □
3. Recall  $A \xrightarrow{\text{true}, \tau} A'$ . Then, by Defn. 31:
- **Base:**  $[\rightarrow 1\text{-ACT}]$ , such that  $A = \tau$  and  $A' = \text{skip}$ . Recall  $\emptyset = \emptyset$ . Then, by Defn. 24,  $\text{subj}(\tau) = \text{subj}(\text{skip})$ . Then,  $\text{subj}(A) = \text{subj}(A')$ .
  - **Base:**  $[\rightarrow 1\text{-IF1}]$ , such that  $\tau = i_{\text{subj}(\hat{\xi})}^R$ , for some  $R, \hat{\xi}$ . Then, **false**.
  - **Base:**  $[\rightarrow 1\text{-IF2}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
  - **Base:**  $[\rightarrow 1\text{-WHILE1}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
  - **Base:**  $[\rightarrow 1\text{-WHILE2}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
  - **Base:**  $[\rightarrow 1\text{-NIF1}]$ , such that  $A = \text{if } \hat{\xi}|_n A_1|_{R_1} A_2|_{R_2}$  and  $A' = \text{if } \hat{\xi}|_{n-1} A_1|_{R_1} A_2|_{R_2}$ , for some  $R_1, R_2, A_1, A_2, \hat{\xi}, n$ . By Defn. 24,  $\text{subj}(A = \text{if } \hat{\xi}|_n A_1|_{R_1} A_2|_{R_2}) = (\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)) \cup (\text{subj}(A_1) \setminus R_2) \cup (\text{subj}(A_2) \setminus R_1) = \text{subj}(\text{if } \hat{\xi}|_{n-1} A_1|_{R_1} A_2|_{R_2})$ . Then,  $\text{subj}(\text{if } \hat{\xi}|_n A_1|_{R_1} A_2|_{R_2}) = \text{subj}(\text{if } \hat{\xi}|_{n-1} A_1|_{R_1} A_2|_{R_2})$ . Then,  $\text{subj}(A) = \text{subj}(A')$ .
  - **Base:**  $[\rightarrow 1\text{-NIF2}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
  - **Base:**  $[\rightarrow 1\text{-NIF3}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
  - **Step:**  $[\rightarrow 1\text{-SEQ1}]$ , such that  $A = A_1 ; A_2$  and  $A' = A'_1 ; A_2$  and  $A_1 \xrightarrow{\text{true}, \tau} A'_1$ , for some  $A_1, A'_1, A_2$ .
    - Recall  $A_1 \xrightarrow{\text{true}, \tau} A'_1$ . Then, by induction,  $\text{subj}(A_1) = \text{subj}(A'_1)$ .
    - By Defn. 24,  $\text{subj}(A_1 ; A_2) = \text{subj}(A_1) \cup \text{subj}(A_2)$ . Then,  $\text{subj}(A_1 ; A_2) = \text{subj}(A'_1) \cup \text{subj}(A_2)$ . Then, by Defn. 24,  $\text{subj}(A_1 ; A_2) = \text{subj}(A'_1 ; A_2)$ . Then,  $\text{subj}(A) = \text{subj}(A')$ .
  - **Step:**  $[\rightarrow 1\text{-SEQ2}]$ . Similar to case  $[\rightarrow 1\text{-SEQ1}]$ .
  - **Step:**  $[\rightarrow 1\text{-PAR1}]$ . Similar to case  $[\rightarrow 1\text{-SEQ1}]$ .
  - **Step:**  $[\rightarrow 1\text{-PAR2}]$ . Similar to case  $[\rightarrow 1\text{-SEQ1}]$ .
  - **Step:**  $[\rightarrow 1\text{-NIF4}]$ . Similar to case  $[\rightarrow 1\text{-SEQ1}]$ .
  - **Step:**  $[\rightarrow 1\text{-NIF5}]$ . Similar to case  $[\rightarrow 1\text{-SEQ1}]$ .
  - **Step:**  $[\rightarrow 1\text{-NWHILE}]$ , such that  $A = \text{while } \hat{\xi}|_n \{\psi\} \hat{A}|_\emptyset$  and  $\text{if } \hat{\xi}|_n (\hat{A} ; \text{while } \hat{\xi}|_n \{\psi\} \hat{A}|_\emptyset) |_\emptyset \text{skip} |_\emptyset \xrightarrow{\xi, \alpha} A'$ , for some  $\hat{A}, \psi, \hat{\xi}, n$ . Recall  $\text{if } \hat{\xi}|_n (\hat{A} ; \text{while } \hat{\xi}|_n \{\psi\} \hat{A}|_\emptyset) |_\emptyset \text{skip} |_\emptyset \xrightarrow{\xi, \alpha} A'$ . Then, by induction,  $\text{subj}(\text{if } \hat{\xi}|_n (\hat{A} ; \text{while } \hat{\xi}|_n \{\psi\} \hat{A}|_\emptyset) |_\emptyset \text{skip} |_\emptyset) = \text{subj}(A')$ . Then, by Defn. 24,  $(\text{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset)) \cup (\text{subj}(\hat{A} ; \text{while } \hat{\xi}|_n \{\psi\} \hat{A}|_\emptyset) \setminus \emptyset) \cup (\text{subj}(\text{skip}) \setminus \emptyset) = \text{subj}(A')$ . Then,  $\text{subj}(\hat{\xi}) \cup \text{subj}(\hat{A} ; \text{while } \hat{\xi}|_n \{\psi\} \hat{A}|_\emptyset) \cup \text{subj}(\text{skip}) = \text{subj}(A')$ . Then, by Defn. 24,  $\text{subj}(\hat{\xi}) \cup \text{subj}(\hat{A}) \cup \text{subj}(\text{while } \hat{\xi}|_n \{\psi\} \hat{A}|_\emptyset) \cup \emptyset = \text{subj}(A')$ . Then, by Defn. 24,  $\text{subj}(\text{while } \hat{\xi}|_n \{\psi\} \hat{A}|_\emptyset) \cup \text{subj}(\text{while } \hat{\xi}|_n \{\psi\} \hat{A}|_\emptyset) \cup \emptyset = \text{subj}(A')$ . Then,  $\text{subj}(\text{while } \hat{\xi}|_n \{\psi\} \hat{A}|_\emptyset) = \text{subj}(A')$ . Then,  $\text{subj}(A) = \text{subj}(A')$ . □

*Proof (of Lem. 54).* Recall  $A \xrightarrow{\xi, \alpha} A'$ . Then, by Defn. 31:

- **Base:** [ $\rightarrow$ 1-ACT], such that  $A = \alpha$  and  $A' = \mathbf{skip}$ .  
Recall  $\text{chan}(A) \cup \emptyset \subseteq \text{chan}(A)$ . Then, by Defn. 25,  $\text{chan}(A) \cup \text{chan}(\mathbf{skip}) \subseteq \text{chan}(A)$ . Then,  $\text{chan}(\alpha) \cup \text{chan}(A') \subseteq \text{chan}(A)$ .
- **Base:** [ $\rightarrow$ 1-IF1], such that  $A = R.\mathbf{if} \hat{\xi} A_1 A_2$  and  $A' = A_1$  and  $\alpha = 1_{\text{subj}(\hat{\xi})}^R$ , for some  $R, A_1, A_2, \hat{\xi}$ .  
Recall  $\{R\} \cup \text{chan}(A_1) \subseteq \{R\} \cup \text{chan}(A_1) \cup \text{chan}(A_2)$ . Then, by Defn. 17,  $\text{chan}(1_{\text{subj}(\hat{\xi})}^R) \cup \text{chan}(A_1) \subseteq \{R\} \cup \text{chan}(A_1) \cup \text{chan}(A_2)$ . Then, by Defn. 25,  $\text{chan}(1_{\text{subj}(\hat{\xi})}^R) \cup \text{chan}(A_1) \subseteq \text{chan}(R.\mathbf{if} \hat{\xi} A_1 A_2)$ . Then,  $\text{chan}(\alpha) \cup \text{chan}(A') \subseteq \text{chan}(A)$ .
- **Base:** [ $\rightarrow$ 1-IF2]. Similar to case [ $\rightarrow$ 1-IF1].
- **Base:** [ $\rightarrow$ 1-WHILE1], such that  $A = R.\mathbf{while} \hat{\xi} \{\psi\} \hat{A}$  and  $A' = \hat{A}$ ;  $R.\mathbf{while} \hat{\xi} \{\psi\} \hat{A}$  and  $\alpha = 1_{\text{subj}(\hat{\xi})}^R$ , for some  $R, \hat{A}, \psi, \hat{\xi}$ .  
Recall  $\{R\} \cup \text{chan}(\hat{A}) \cup \{R\} \cup \text{chan}(\hat{A}) \subseteq \{R\} \cup \text{chan}(\hat{A})$ . Then, by Defn. 17,  $\text{chan}(1_{\text{subj}(\hat{\xi})}^R) \cup \text{chan}(\hat{A}) \cup \{R\} \cup \text{chan}(\hat{A}) \subseteq \{R\} \cup \text{chan}(\hat{A})$ . Then, by Defn. 25,  $\text{chan}(1_{\text{subj}(\hat{\xi})}^R) \cup \text{chan}(\hat{A}) \cup \text{chan}(R.\mathbf{while} \hat{\xi} \{\psi\} \hat{A}) \subseteq \text{chan}(R.\mathbf{while} \hat{\xi} \{\psi\} \hat{A})$ . Then, by Defn. 25,  $\text{chan}(1_{\text{subj}(\hat{\xi})}^R) \cup \text{chan}(\hat{A}; R.\mathbf{while} \hat{\xi} \{\psi\} \hat{A}) \subseteq \text{chan}(R.\mathbf{while} \hat{\xi} \{\psi\} \hat{A})$ . Then,  $\text{chan}(\alpha) \cup \text{chan}(A') \subseteq \text{chan}(A)$ .
- **Base:** [ $\rightarrow$ 1-WHILE2]. Similar to case [ $\rightarrow$ 1-IF1].
- **Base:** [ $\rightarrow$ 1-NIF1], such that  $A = \mathbf{if} \hat{\xi}|_{n-1} A_1|_{R_1} A_2|_{R_2}$  and  $A' = \mathbf{if} \hat{\xi}|_{n-1} A_1|_{R_1} A_2|_{R_2}$ , for some  $R_1, R_2, A_1, A_2, \hat{\xi}, n$ .  
Recall  $\emptyset \cup \text{chan}(A') \subseteq \text{chan}(\mathbf{if} \hat{\xi}|_{n-1} A_1|_{R_1} A_2|_{R_2})$ . Then, by Defn. 17,  $\text{chan}(\tau) \cup \text{chan}(\mathbf{if} \hat{\xi}|_{n-1} A_1|_{R_1} A_2|_{R_2}) \subseteq \text{chan}(\mathbf{if} \hat{\xi}|_{n-1} A_1|_{R_1} A_2|_{R_2})$ . Then, by Defn. 25,  $\text{chan}(\tau) \cup \text{chan}(\mathbf{if} \hat{\xi}|_{n-1} A_1|_{R_1} A_2|_{R_2}) \subseteq \{\{r\} \mid r \in \text{subj}(\hat{\xi})\} \cup \text{chan}(A)$ . Then, by Defn. 25,  $\text{chan}(\tau) \cup \text{chan}(\mathbf{if} \hat{\xi}|_{n-1} A_1|_{R_1} A_2|_{R_2}) \subseteq \text{chan}(\mathbf{if} \hat{\xi}|_n A_1|_{R_1} A_2|_{R_2})$ . Then,  $\text{chan}(\alpha) \cup \text{chan}(A') \subseteq \text{chan}(A)$ .
- **Base:** [ $\rightarrow$ 1-NIF2], such that  $A = \mathbf{if} \hat{\xi}|_n A_1|_{R_1} A_2|_{R_2}$  and  $A' = \mathbf{if} \hat{\xi}|_n A_1|_{R_1 \cup \{r\}} A_2|_{R_2}$  and  $\alpha = 1_{\{r\}}^r$  and  $r \in \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$ , for some  $R_1, R_2, A_1, A_2, \hat{\xi}, r, n$ .  
Recall  $r \in \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$ . Then,  $r \in \text{subj}(\hat{\xi})$ . Then,  $\{r\} \in \{\{\hat{r}\} \mid \hat{r} \in \text{subj}(\hat{\xi})\}$ . Then,  $\{\{r\}\} \subseteq \{\{\hat{r}\} \mid \hat{r} \in \text{subj}(\hat{\xi})\}$ . Then, by Defn. 17,  $\text{chan}(1_{\{r\}}^r) \subseteq \{\{\hat{r}\} \mid \hat{r} \in \text{subj}(\hat{\xi})\}$ . Then,  $\text{chan}(1_{\{r\}}^r) \cup \{\{\hat{r}\} \mid \hat{r} \in \text{subj}(\hat{\xi})\} \cup \text{chan}(A_1) \cup \text{chan}(A_2) \subseteq \{\{\hat{r}\} \mid \hat{r} \in \text{subj}(\hat{\xi})\} \cup \text{chan}(A_1) \cup \text{chan}(A_2)$ . Then, by Defn. 25,  $\text{chan}(1_{\{r\}}^r) \cup \text{chan}(\mathbf{if} \hat{\xi}|_n A_1|_{R_1 \cup \{r\}} A_2|_{R_2}) \subseteq \text{chan}(\mathbf{if} \hat{\xi}|_n A_1|_{R_1} A_2|_{R_2})$ . Then,  $\text{chan}(\alpha) \cup \text{chan}(A') \subseteq \text{chan}(A)$ .
- **Base:** [ $\rightarrow$ 1-NIF3]. Similar to case [ $\rightarrow$ 1-NIF2].
- **Step:** [ $\rightarrow$ 1-SEQ1], such that  $A = A_1; A_2$  and  $A' = A'_1; A_2$  and  $A_1 \xrightarrow{\xi, \alpha} A'_1$ , for some  $A_1, A'_1, A_2$ .  
Recall  $A_1 \xrightarrow{\xi, \alpha} A'_1$ . Then, by induction,  $\text{chan}(\alpha) \cup \text{chan}(A'_1) \subseteq \text{chan}(A_1)$ . Then,  $\text{chan}(\alpha) \cup \text{chan}(A'_1) \cup \text{chan}(A_2) \subseteq \text{chan}(A_1) \cup \text{chan}(A_2)$ . Then, by Defn. 25,  $\text{chan}(\alpha) \cup \text{chan}(A'_1; A_2) \subseteq \text{chan}(A_1; A_2)$ . Then,  $\text{chan}(\alpha) \cup \text{chan}(A') \subseteq \text{chan}(A)$ .
- **Step:** [ $\rightarrow$ 1-SEQ2]. Similar to case [ $\rightarrow$ 1-SEQ1].
- **Step:** [ $\rightarrow$ 1-PAR1]. Similar to case [ $\rightarrow$ 1-SEQ1].
- **Step:** [ $\rightarrow$ 1-PAR2]. Similar to case [ $\rightarrow$ 1-SEQ1].
- **Step:** [ $\rightarrow$ 1-NIF4]. Similar to case [ $\rightarrow$ 1-SEQ1].
- **Step:** [ $\rightarrow$ 1-NIF5]. Similar to case [ $\rightarrow$ 1-SEQ1].
- **Step:** [ $\rightarrow$ 1-NWHILE], such that  $A = \mathbf{while} \hat{\xi}|_n \{\psi\} \hat{A}|_{\emptyset}$  and  $\mathbf{if} \hat{\xi}|_n (\hat{A}; \mathbf{while} \hat{\xi}|_n \{\psi\} \hat{A}|_{\emptyset})|_{\emptyset} \mathbf{skip}|_{\emptyset} \xrightarrow{\xi, \alpha} A'$ , for some  $\hat{A}, \psi, \hat{\xi}, n$ .  
Recall  $\mathbf{if} \hat{\xi}|_n (\hat{A}; \mathbf{while} \hat{\xi}|_n \{\psi\} \hat{A}|_{\emptyset})|_{\emptyset} \mathbf{skip}|_{\emptyset} \xrightarrow{\xi, \alpha} A'$ . Then, by induction,  $\text{chan}(\alpha) \cup \text{chan}(A') \subseteq \text{chan}(\mathbf{if} \hat{\xi}|_n (\hat{A}; \mathbf{while} \hat{\xi}|_n \{\psi\} \hat{A}|_{\emptyset})|_{\emptyset} \mathbf{skip}|_{\emptyset})$ . Then, by Defn. 25,  $\text{chan}(\alpha) \cup \text{chan}(A') \subseteq \{\{r\} \mid r \in \text{subj}(\hat{\xi})\} \cup \text{chan}(\hat{A}; \mathbf{while} \hat{\xi}|_n \{\psi\} \hat{A}|_{\emptyset}) \cup \text{chan}(\mathbf{skip})$ . Then, by Defn. 25,  $\text{chan}(\alpha) \cup \text{chan}(A') \subseteq \{\{r\} \mid r \in \text{subj}(\hat{\xi})\} \cup \text{chan}(\hat{A}) \cup \text{chan}(\mathbf{while} \hat{\xi}|_n \{\psi\} \hat{A}|_{\emptyset}) \cup \emptyset$ . Then, by Defn. 25,  $\text{chan}(\alpha) \cup \text{chan}(A') \subseteq \text{chan}(\mathbf{while} \hat{\xi}|_n \{\psi\} \hat{A}|_{\emptyset}) \cup \text{chan}(\mathbf{while} \hat{\xi}|_n \{\psi\} \hat{A}|_{\emptyset}) \cup \emptyset$ . Then,  $\text{chan}(\alpha) \cup \text{chan}(A') \subseteq \text{chan}(\mathbf{while} \hat{\xi}|_n \{\psi\} \hat{A}|_{\emptyset})$ . Then,  $\text{chan}(\alpha) \cup \text{chan}(A') \subseteq \text{chan}(A)$ .  $\square$

*Proof (of Lem. 55).* Recall  $A \downarrow$ . Then, by Defn. 28:

- **Base:** [ $\downarrow$ 1-SKIP], such that  $A = \mathbf{skip}$ .  
By Defn. 31, not  $\mathbf{skip} \xrightarrow{\xi, \alpha} A'$ . Then, not  $A \xrightarrow{\xi, \alpha} A'$ . Then, **false**.
- **Step:** [ $\downarrow$ 1-SEQ], such that  $A = A_1 ; A_2$  and  $A_1 \downarrow$  and  $A_2 \downarrow$ , for some  $A_1, A_2$ .  
Recall  $A \xrightarrow{\xi, \alpha} A'$ . Then,  $A_1 ; A_2 \xrightarrow{\xi, \alpha} A'$ . Then, by Defn. 31:
  - **Case:** [ $\rightarrow$ 1-SEQ1], such that  $A' = A_1 ; A_2$  and  $A_1 \xrightarrow{\xi, \alpha} A'_1$ , for some  $A'_1$ .
    - \* Recall  $A_1 \downarrow$  and  $A_1 \xrightarrow{\xi, \alpha} A'_1$ . Then, by induction,  $A'_1 \downarrow$  and  $\xi = \mathbf{true}$  and  $\alpha = \tau$ .
    - \* Recall  $A'_1 \downarrow$  and  $A_2 \downarrow$ . Then, by [ $\downarrow$ 1-SEQ],  $A'_1 ; A_2 \downarrow$ . Then,  $A' \downarrow$ .
  - **Case:** [ $\rightarrow$ 1-SEQ2]. Similar to case [ $\rightarrow$ 1-SEQ1].
- **Step:** [ $\downarrow$ 1-PAR]. Similar to case [ $\downarrow$ 1-SEQ].
- **Step:** [ $\downarrow$ 1-NIF], such that  $A = \mathbf{if} \hat{\xi}|_0 A_1|_{R_1} A_2|_{R_2}$  and  $\mathbf{subj}(\hat{\xi}) = R_1 \cup R_2$ , and  $R_1 \neq \emptyset$  implies  $A_1 \downarrow$ , and  $R_2 \neq \emptyset$  implies  $A_2 \downarrow$ , for some  $R_1, R_2, A_1, A_2, \hat{\xi}$ .  
Recall  $A \xrightarrow{\xi, \alpha} A'$ . Then,  $\mathbf{if} \hat{\xi}|_0 A_1|_{R_1} A_2|_{R_2} \xrightarrow{\xi, \alpha} A'$ . Then, by Defn. 31:
  - **Case:** [ $\rightarrow$ 1-NIF1], such that  $0 > 0$ . Then, **false**.
  - **Case:** [ $\rightarrow$ 1-NIF2], such that  $r \in \mathbf{subj}(\xi) \setminus (R_1 \cup R_2)$ , for some  $r$ . Then,  $r \in \mathbf{subj}(\xi) \setminus \mathbf{subj}(\alpha)$ . Then,  $r \in \emptyset$ . Then, **false**.
  - **Case:** [ $\rightarrow$ 1-NIF3]. Similar to case [ $\rightarrow$ 1-NIF2].
  - **Case:** [ $\rightarrow$ 1-NIF4], such that  $A' = \mathbf{if} \hat{\xi}|_0 A'_1|_{R_1} A_2|_{R_2}$  and  $A_1 \xrightarrow{\xi, \alpha} A'_1$  and  $\mathbf{subj}(\xi) \cup \mathbf{subj}(\alpha) \subseteq R_1 \setminus R_2$ , for some  $A'_1$ .  
Recall:
    - \* **Case:**  $R_1 = \emptyset$ .
      - Recall  $R_1 = \emptyset$ . Then,  $R_1 \neq \emptyset$  implies  $A'_1 \downarrow$ .
      - Recall  $\mathbf{subj}(\hat{\xi}) = R_1 \cup R_2$ , and  $R'_1 \neq \emptyset$  implies  $A_1 \downarrow$ , and  $R_2 \neq \emptyset$  implies  $A_2 \downarrow$ .  
Then, by [ $\downarrow$ 1-NIF],  $\mathbf{if} \hat{\xi}|_0 A'_1|_{R_1} A_2|_{R_2} \downarrow$ . Then,  $A' \downarrow$ .
      - Recall  $\mathbf{subj}(\xi) \cup \mathbf{subj}(\alpha) \subseteq R_1 \setminus R_2$ . Then,  $\mathbf{subj}(\alpha) \subseteq R_1 \setminus R_2$ . Then,  $\mathbf{subj}(\alpha) \subseteq \emptyset \setminus R_2$ . Then,  $\mathbf{subj}(\alpha) \subseteq \emptyset$ . Then,  $\mathbf{subj}(\alpha) = \emptyset$ . Then, by Lem. 30,  $\alpha = \tau$ .
      - Recall  $A_1 \xrightarrow{\xi, \alpha} A'_1$  and  $\alpha = \tau$ . Then, by Lem. 50,  $(\xi, \alpha) = (\mathbf{true}, \tau)$ . Then,  $\xi = \mathbf{true}$ .
    - \* **Case:**  $R_1 \neq \emptyset$ .
      - Recall  $R_1 \neq \emptyset$ . Then,  $A_1 \downarrow$ .
      - Recall  $A_1 \downarrow$  and  $A_1 \xrightarrow{\xi, \alpha} A'_1$ . Then, by induction,  $A'_1 \downarrow$  and  $\xi = \mathbf{true}$  and  $\alpha = \tau$ .
      - Recall  $A'_1 \downarrow$ . Then,  $R_1 \neq \emptyset$  implies  $A'_1 \downarrow$ .
      - Recall  $\mathbf{subj}(\hat{\xi}) = R_1 \cup R_2$ , and  $R'_1 \neq \emptyset$  implies  $A_1 \downarrow$ , and  $R_2 \neq \emptyset$  implies  $A_2 \downarrow$ .  
Then, by [ $\downarrow$ 1-NIF],  $\mathbf{if} \hat{\xi}|_0 A'_1|_{R_1} A_2|_{R_2} \downarrow$ . Then,  $A' \downarrow$ .
  - **Case:** [ $\rightarrow$ 1-NIF5]. Similar to case [ $\rightarrow$ 1-NIF4]. □

*Proof (of Lem. 56).*

1. Recall  $A \xrightarrow{\mathbf{true}, \tau} A'_1$ . Then, by Defn. 31:
  - **Base:** [ $\rightarrow$ 1-ACT], such that  $A = \tau$  and  $A'_1 = \mathbf{skip}$ .
    - Recall  $A \xrightarrow{\xi, \alpha} A'_2$ . Then,  $\tau \xrightarrow{\xi, \alpha} A'_2$ . Then, by Defn. 31,  $A'_2 = \mathbf{skip}$  and  $\xi = \mathbf{true}$  and  $\alpha = \tau$ .
    - Recall  $\mathbf{skip} = \mathbf{skip}$ . Then,  $A'_1 = A'_2$ .
  - **Base:** [ $\rightarrow$ 1-IF1], such that  $\tau = i_{\mathbf{subj}(\hat{\xi})}^R$ , for some  $R, \hat{\xi}$ . Then, **false**.
  - **Base:** [ $\rightarrow$ 1-IF2]. Similar to case [ $\rightarrow$ 1-IF1].
  - **Base:** [ $\rightarrow$ 1-WHILE1]. Similar to case [ $\rightarrow$ 1-IF1].
  - **Base:** [ $\rightarrow$ 1-WHILE2]. Similar to case [ $\rightarrow$ 1-IF1].
  - **Base:** [ $\rightarrow$ 1-NIF1], such that  $A = \mathbf{if} \hat{\xi}|_n A_{11}|_{R_1} A_{12}|_{R_2}$  and  $A'_1 = \mathbf{if} \hat{\xi}|_{n-1} A_{11}|_{R_1} A_{12}|_{R_2}$  and  $n > 0$ , for some  $R_1, R_2, A_{11}, A_{12}, \hat{\xi}, n$ .  
Recall  $A \xrightarrow{\xi, \alpha} A'_2$ . Then,  $\mathbf{if} \hat{\xi}|_n A_{11}|_{R_1} A_{12}|_{R_2} \xrightarrow{\xi, \alpha} A'_2$ . Then, by Defn. 31:
    - **Case:** [ $\rightarrow$ 1-NIF1], such that  $A'_2 = \mathbf{if} \hat{\xi}|_{n-1} A_{11}|_{R_1} A_{12}|_{R_2}$  and  $\xi = \mathbf{true}$  and  $\alpha = \tau$ .  
Recall  $A'_1 = \mathbf{if} \hat{\xi}|_{n-1} A_{11}|_{R_1} A_{12}|_{R_2} = A'_2$ . Then,  $A'_1 = A'_2$ .

- **Case:**  $[\rightarrow 1\text{-NIF2}]$ , such that  $A'_2 = \mathbf{if} \hat{\xi}|_n A_{11}|_{R_1 \cup \{r\}} A_{12}|_{R_2}$  and  $\xi = \hat{\xi}^+ \upharpoonright r$  and  $\alpha = 1|_{\{r\}}$  and  $r \in \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$ , for some  $r$ .
    - \* Recall  $r \in \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$ . Then, by  $[\rightarrow 1\text{-NIF2}]$ ,  
 $\mathbf{if} \hat{\xi}|_{p-1} A_{11}|_{R_1} A_{12}|_{R_2} \xrightarrow{\xi^+ \upharpoonright r, 1|_{\{r\}}} \mathbf{if} \hat{\xi}|_{n-1} A_{11}|_{R_1 \cup \{r\}} A_{12}|_{R_2}$ . Then,  
 $A'_1 \xrightarrow{\xi, \alpha} \mathbf{if} \hat{\xi}|_{n-1} A_{11}|_{R_1 \cup \{r\}} A_{12}|_{R_2}$ . Then,  $A'' = \mathbf{if} \hat{\xi}|_{n-1} A_{11}|_{R_1 \cup \{r\}} A_{12}|_{R_2}$  and  
 $A'_1 \xrightarrow{\xi, \alpha} A''$ , for some  $A''$ .
    - \* Recall  $n > 0$ . Then, by  $[\rightarrow 1\text{-NIF1}]$ ,  
 $\mathbf{if} \hat{\xi}|_n A_{11}|_{R_1 \cup \{r\}} A_{12}|_{R_2} \xrightarrow{\text{true}, \tau} \mathbf{if} \hat{\xi}|_{n-1} A_{11}|_{R_1 \cup \{r\}} A_{12}|_{R_2}$ . Then,  $A'_2 \xrightarrow{\text{true}, \tau} A''$ .
  - **Case:**  $[\rightarrow 1\text{-NIF3}]$ . Similar to case  $[\rightarrow 1\text{-NIF2}]$ .
  - **Case:**  $[\rightarrow 1\text{-NIF4}]$ . Similar to case  $[\rightarrow 1\text{-NIF2}]$ .
  - **Case:**  $[\rightarrow 1\text{-NIF5}]$ . Similar to case  $[\rightarrow 1\text{-NIF2}]$ .
- **Base:**  $[\rightarrow 1\text{-NIF2}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
- **Base:**  $[\rightarrow 1\text{-NIF3}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
- **Step:**  $[\rightarrow 1\text{-SEQ1}]$ , such that  $A = A_{11}; A_{12}$  and  $A'_1 = A'_{11}; A_{12}$  and  $A_{11} \xrightarrow{\text{true}, \tau} A'_{11}$ , for some  $A_{11}, A'_{11}, A_{12}$ .
- Recall  $A \xrightarrow{\xi, \alpha} A'_2$ . Then,  $A_{11}; A_{12} \xrightarrow{\xi, \alpha} A'_2$ . Then, by Defn. 31:
- **Case:**  $[\rightarrow 1\text{-SEQ1}]$ , such that  $A'_2 = A'_{21}; A_{12}$  and  $A_{11} \xrightarrow{\xi, \alpha} A'_{21}$ , for some  $A'_{21}$ .  
 Recall  $A_{11} \xrightarrow{\text{true}, \tau} A'_{11}$  and  $A_{11} \xrightarrow{\xi, \alpha} A'_{21}$ . Then, by induction:
    - \* **Case:**  $A'_{11} \xrightarrow{\xi, \alpha} A''_{11}$  and  $A'_{21} \xrightarrow{\text{true}, \tau} A''_{11}$ , for some  $A''_{11}$ .
      - Recall  $A'_{11} \xrightarrow{\xi, \alpha} A''_{11}$ . Then, by  $[\rightarrow 1\text{-SEQ1}]$ ,  $A'_{11}; A_{12} \xrightarrow{\xi, \alpha} A''_{11}; A_{12}$ . Then,  
 $A'_1 \xrightarrow{\xi, \alpha} A''_{11}; A_{12}$ . Then,  $A'' = A''_{11}; A_{12}$  and  $A'_1 \xrightarrow{\xi, \alpha} A''$ , for some  $A''$ .
      - Recall  $A'_{21} \xrightarrow{\text{true}, \tau} A''_{11}$ . Then, by  $[\rightarrow 1\text{-SEQ1}]$ ,  $A'_{21}; A_{12} \xrightarrow{\text{true}, \tau} A''_{11}; A_{12}$ . Then,  
 $A'_2 \xrightarrow{\text{true}, \tau} A''$ .
    - \* **Case:**  $\xi = \text{true}$  and  $\alpha = \tau$  and  $A'_{11} = A'_{21}$ .  
 Recall  $A'_{11} = A'_{21}$ . Then,  $A'_{11}; A_{12} = A'_{21}; A_{12}$ . Then,  $A'_1 = A'_2$ .
  - **Case:**  $[\rightarrow 1\text{-SEQ2}]$ , such that  $A'_2 = A_{11}; A'_{22}$  and  $\text{subj}(A_{11}) \cap (\text{subj}(\xi) \cup \text{subj}(\alpha)) = \emptyset$  and  $A_{12} \xrightarrow{\xi, \alpha} A'_{22}$ , for some  $A'_{22}$ .
    - \* Recall  $A_{11} \xrightarrow{\text{true}, \tau} A'_{11}$ . Then, by Lem. 53,  $\text{subj}(A_{11}) = \text{subj}(A'_{11})$ .
    - \* Recall  $\text{subj}(A_{11}) \cap (\text{subj}(\xi) \cup \text{subj}(\alpha)) = \emptyset$ . Then,  
 $\text{subj}(A'_{11}) \cap (\text{subj}(\xi) \cup \text{subj}(\alpha)) = \emptyset$ .
    - \* Recall  $\text{subj}(A'_{11}) \cap (\text{subj}(\xi) \cup \text{subj}(\alpha)) = \emptyset$  and  $A_{12} \xrightarrow{\xi, \alpha} A'_{22}$ . Then, by  $[\rightarrow 1\text{-SEQ2}]$ ,  
 $A'_{11}; A_{12} \xrightarrow{\xi, \alpha} A'_{11}; A'_{22}$ . Then,  $A'_1 \xrightarrow{\xi, \alpha} A'_{11}; A'_{22}$ . Then,  $A'' = A'_{11}; A'_{22}$  and  
 $A'_1 \xrightarrow{\xi, \alpha} A''$ , for some  $A''$ .
    - \* Recall  $A_{11} \xrightarrow{\text{true}, \tau} A'_{11}$ . Then, by  $[\rightarrow 1\text{-SEQ1}]$ ,  $A_{11}; A'_{22} \xrightarrow{\text{true}, \tau} A'_{11}; A'_{22}$ . Then,  
 $A'_2 \xrightarrow{\text{true}, \tau} A''$ .
- **Step:**  $[\rightarrow 1\text{-SEQ2}]$ , such that  $A = A_{11}; A_{12}$  and  $A'_1 = A_{11}; A'_{12}$  and  
 $\text{subj}(A_{11}) \cap (\text{subj}(\text{true}) \cup \text{subj}(\tau)) = \emptyset$  and  $A_{12} \xrightarrow{\text{true}, \tau} A'_{12}$ , for some  $A_{11}, A_{12}, A'_{12}$ .
- Recall  $A \xrightarrow{\xi, \alpha} A'_2$ . Then,  $A_{11}; A_{12} \xrightarrow{\xi, \alpha} A'_2$ . Then, by Defn. 31:
- **Case:**  $[\rightarrow 1\text{-SEQ1}]$ , such that  $A'_2 = A'_{21}; A_{12}$  and  $A_{11} \xrightarrow{\xi, \alpha} A'_{21}$ , for some  $A'_{21}$ .
    - \* Recall  $A_{11} \xrightarrow{\xi, \alpha} A'_{21}$ . Then, by  $[\rightarrow 1\text{-SEQ1}]$ ,  $A_{11}; A'_{12} \xrightarrow{\xi, \alpha} A'_{21}; A'_{12}$ . Then,  
 $A'_1 \xrightarrow{\xi, \alpha} A'_{21}; A'_{12}$ . Then,  $A'' = A'_{21}; A'_{12}$  and  $A'_1 \xrightarrow{\xi, \alpha} A''$ , for some  $A''$ .
    - \* Recall  $\text{subj}(A'_{21}) \cap (\emptyset \cup \emptyset) = \emptyset$ . Then, by Lem. 4,  $\text{subj}(A'_{21}) \cap (\text{subj}(\text{true}) \cup \emptyset) = \emptyset$ .  
 Then, by Defn. 16,  $\text{subj}(A'_{21}) \cap (\text{subj}(\text{true}) \cup \text{subj}(\tau)) = \emptyset$ .
    - \* Recall  $\text{subj}(A'_{21}) \cap (\text{subj}(\text{true}) \cup \text{subj}(\tau)) = \emptyset$  and  $A_{12} \xrightarrow{\text{true}, \tau} A'_{12}$ . Then,  
 $A'_{21}; A_{12} \xrightarrow{\text{true}, \tau} A'_{21}; A'_{12}$ . Then,  $A'_2 \xrightarrow{\text{true}, \tau} A''$ .
  - **Case:**  $[\rightarrow 1\text{-SEQ2}]$ , such that  $A'_2 = A_{11}; A'_{22}$  and  $\text{subj}(A_{11}) \cap (\text{subj}(\xi) \cup \text{subj}(\alpha)) = \emptyset$  and  $A_{12} \xrightarrow{\xi, \alpha} A'_{22}$ , for some  $A'_{22}$ .  
 Recall  $A_{12} \xrightarrow{\text{true}, \tau} A'_{12}$  and  $A_{12} \xrightarrow{\xi, \alpha} A'_{22}$ . Then, by induction:
    - \* **Case:**  $A'_{12} \xrightarrow{\xi, \alpha} A''_{12}$  and  $A'_{22} \xrightarrow{\text{true}, \tau} A''_{12}$ , for some  $A''_{12}$ .
      - Recall  $\text{subj}(A_{11}) \cap (\text{subj}(\xi) \cup \text{subj}(\alpha)) = \emptyset$  and  $A'_{12} \xrightarrow{\xi, \alpha} A''_{12}$ . Then, by  
 $[\rightarrow 1\text{-SEQ2}]$ ,  $A_{11}; A'_{12} \xrightarrow{\xi, \alpha} A_{11}; A''_{12}$ . Then,  $A'_1 \xrightarrow{\xi, \alpha} A_{11}; A''_{12}$ . Then,  
 $A'' = A_{11}; A''_{12}$  and  $A'_1 \xrightarrow{\xi, \alpha} A''$ , for some  $A''$ .
      - Recall  $\text{subj}(A_{11}) \cap (\text{subj}(\text{true}) \cup \text{subj}(\tau)) = \emptyset$  and  $A'_{22} \xrightarrow{\text{true}, \tau} A''_{12}$ . Then, by  
 $[\rightarrow 1\text{-SEQ2}]$ ,  $A_{11}; A'_{22} \xrightarrow{\text{true}, \tau} A_{11}; A''_{12}$ . Then,  $A'_2 \xrightarrow{\text{true}, \tau} A''$ .

- \* **Case:**  $\xi = \text{true}$  and  $\alpha = \tau$  and  $A'_{12} = A'_{22}$ .  
Recall  $A'_{12} = A'_{22}$ . Then,  $A_{11}; A'_{12} = A_{11}; A'_{22}$ . Then,  $A'_1 = A'_2$ .
- **Step:**  $[\rightarrow 1\text{-PAR1}]$ , such that  $A = A_{11} \parallel A_{12}$  and  $A'_1 = A'_{11} \parallel A_{12}$  and  $A_{11} \xrightarrow{\text{true}, \tau} A'_{11}$ , for some  $A_{11}, A'_{11}, A_{12}$ .  
Recall  $A \xrightarrow{\xi, \alpha} A'_2$ . Then,  $A_{11} \parallel A_{12} \xrightarrow{\xi, \alpha} A'_2$ . Then, by Defn. 31:
  - **Case:**  $[\rightarrow 1\text{-PAR1}]$ , such that  $A'_2 = A'_{21} \parallel A_{12}$  and  $A_{11} \xrightarrow{\xi, \alpha} A'_{21}$ , for some  $A'_{21}$ .  
Recall  $A_{11} \xrightarrow{\text{true}, \tau} A'_{11}$  and  $A_{11} \xrightarrow{\xi, \alpha} A'_{21}$ . Then, by induction:
    - \* **Case:**  $A'_{11} \xrightarrow{\xi, \alpha} A''_{11}$  and  $A'_{21} \xrightarrow{\text{true}, \tau} A''_{11}$ , for some  $A''_{11}$ .
      - Recall  $A'_{11} \xrightarrow{\xi, \alpha} A''_{11}$ . Then, by  $[\rightarrow 1\text{-PAR1}]$ ,  $A'_{11} \parallel A_{12} \xrightarrow{\xi, \alpha} A''_{11} \parallel A_{12}$ . Then,  $A'_1 \xrightarrow{\xi, \alpha} A''_{11} \parallel A_{12}$ . Then,  $A'' = A''_{11} \parallel A_{12}$  and  $A'_1 \xrightarrow{\xi, \alpha} A''$ , for some  $A''$ .
      - Recall  $A'_{21} \xrightarrow{\text{true}, \tau} A''_{11}$ . Then, by  $[\rightarrow 1\text{-PAR1}]$ ,  $A'_{21} \parallel A_{12} \xrightarrow{\text{true}, \tau} A''_{11} \parallel A_{12}$ . Then,  $A'_2 \xrightarrow{\text{true}, \tau} A''$ .
    - \* **Case:**  $\xi = \text{true}$  and  $\alpha = \tau$  and  $A'_{11} = A'_{21}$ .  
Recall  $A'_{11} = A'_{21}$ . Then,  $A'_{11} \parallel A_{12} = A'_{21} \parallel A_{12}$ . Then,  $A'_1 = A'_2$ .
  - **Case:**  $[\rightarrow 1\text{-PAR2}]$ , such that  $A'_2 = A_{11} \parallel A'_{22}$  and  $A_{12} \xrightarrow{\xi, \alpha} A'_{22}$ , for some  $A'_{22}$ .
    - \* Recall  $A_{12} \xrightarrow{\xi, \alpha} A'_{22}$ . Then, by  $[\rightarrow 1\text{-PAR2}]$ ,  $A'_{11} \parallel A_{12} \xrightarrow{\xi, \alpha} A'_{11} \parallel A'_{22}$ . Then,  $A'_1 \xrightarrow{\xi, \alpha} A'_{11} \parallel A'_{22}$ . Then,  $A'' = A'_{11} \parallel A'_{22}$  and  $A'_1 \xrightarrow{\xi, \alpha} A''$ , for some  $A''$ .
    - \* Recall  $A_{11} \xrightarrow{\text{true}, \tau} A'_{11}$ . Then, by  $[\rightarrow 1\text{-PAR1}]$ ,  $A_{11} \parallel A'_{22} \xrightarrow{\text{true}, \tau} A'_{11} \parallel A'_{22}$ . Then,  $A'_2 \xrightarrow{\text{true}, \tau} A''$ .
- **Step:**  $[\rightarrow 1\text{-PAR2}]$ . Similar to case  $[\rightarrow 1\text{-PAR1}]$ .
- **Step:**  $[\rightarrow 1\text{-NIF4}]$ , such that  $A = \text{if } \hat{\xi} |_n A_{11}|_{R_1} A_{12}|_{R_2}$  and  $A'_1 = \text{if } \hat{\xi} |_n A'_{11}|_{R_1} A_{12}|_{R_2}$  and  $A_{11} \xrightarrow{\text{true}, \tau} A'_{11}$  and  $\text{subj}(\text{true}) \cup \text{subj}(\tau) \subseteq R_1 \setminus R_2$ , for some  $R_1, R_2, A_{11}, A'_{11}, A_{12}, \hat{\xi}, n$ .  
Recall  $A \xrightarrow{\xi, \alpha} A'_2$ . Then,  $\text{if } \hat{\xi} |_n A_{11}|_{R_1} A_{12}|_{R_2} \xrightarrow{\xi, \alpha} A'_2$ . Then, by Defn. 31:
  - **Case:**  $[\rightarrow 1\text{-NIF1}]$ , such that  $A'_2 = \text{if } \hat{\xi} |_{n-1} A_{11}|_{R_1} A_{12}|_{R_2}$  and  $\xi = \text{true}$  and  $\alpha = \tau$  and  $n > 0$ .
    - \* Recall  $n > 0$ . Then, by  $[\rightarrow 1\text{-NIF1}]$ ,  
 $\text{if } \hat{\xi} |_n A_{11}|_{R_1} A_{12}|_{R_2} \xrightarrow{\text{true}, \tau} \text{if } \hat{\xi} |_{n-1} A'_{11}|_{R_1} A_{12}|_{R_2}$ . Then,  
 $A'_1 \xrightarrow{\xi, \alpha} \text{if } \hat{\xi} |_{n-1} A'_{11}|_{R_1} A_{12}|_{R_2}$ . Then,  $A'' = \text{if } \hat{\xi} |_{n-1} A'_{11}|_{R_1} A_{12}|_{R_2}$  and  $A'_1 \xrightarrow{\xi, \alpha} A''$ , for some  $A''$ .
    - \* Recall  $A_{11} \xrightarrow{\text{true}, \tau} A'_{11}$  and  $\text{subj}(\text{true}) \cup \text{subj}(\tau) \subseteq R_1 \setminus R_2$ . Then, by  $[\rightarrow 1\text{-NIF4}]$ ,  
 $\text{if } \hat{\xi} |_{n-1} A_{11}|_{R_1} A_{12}|_{R_2} \xrightarrow{\text{true}, \tau} \text{if } \hat{\xi} |_{n-1} A'_{11}|_{R_1} A_{12}|_{R_2}$ . Then,  $A'_2 \xrightarrow{\text{true}, \tau} A''$ .
  - **Case:**  $[\rightarrow 1\text{-NIF2}]$ , such that  $A'_2 = \text{if } \hat{\xi} |_n A_{11}|_{R_1 \cup \{r\}} A_{12}|_{R_2}$  and  $\xi = \hat{\xi}^+ \upharpoonright r$  and  $\alpha = 1 \upharpoonright \{r\}$  and  $r \in \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$ , for some  $r$ .
    - \* Recall  $r \in \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$ . Then, by  $[\rightarrow 1\text{-NIF2}]$ ,  
 $\text{if } \hat{\xi} |_n A_{11}|_{R_1} A_{12}|_{R_2} \xrightarrow{\hat{\xi}^+ \upharpoonright r, 1 \upharpoonright \{r\}} \text{if } \hat{\xi} |_n A'_{11}|_{R_1 \cup \{r\}} A_{12}|_{R_2}$ . Then,  
 $A'_1 \xrightarrow{\xi, \alpha} \text{if } \hat{\xi} |_n A'_{11}|_{R_1 \cup \{r\}} A_{12}|_{R_2}$ . Then,  $A'' = \text{if } \hat{\xi} |_n A'_{11}|_{R_1 \cup \{r\}} A_{12}|_{R_2}$  and  $A'_1 \xrightarrow{\xi, \alpha} A''$ , for some  $A''$ .
    - \* Recall  $\emptyset \cup \emptyset \subseteq (R_1 \cup \{r\}) \setminus R_2$ . Then, by Lem. 4,  $\text{subj}(\text{true}) \cup \emptyset \subseteq (R_1 \cup \{r\}) \setminus R_2$ . Then, by Defn. 16,  $\text{subj}(\text{true}) \cup \text{subj}(\tau) \subseteq (R_1 \cup \{r\}) \setminus R_2$ .
    - \* Recall  $A_{11} \xrightarrow{\text{true}, \tau} A'_{11}$  and  $\text{subj}(\text{true}) \cup \text{subj}(\tau) \subseteq (R_1 \cup \{r\}) \setminus R_2$ . Then, by  $[\rightarrow 1\text{-NIF4}]$ ,  $\text{if } \hat{\xi} |_{A_{11}} n |_{R_1 \cup \{r\}} A_{12}|_{R_2} \xrightarrow{\text{true}, \tau} \text{if } \hat{\xi} |_{A'_{11}} n |_{R_1 \cup \{r\}} A_{12}|_{R_2}$ . Then,  $A'_2 \xrightarrow{\text{true}, \tau} A''$ .
  - **Case:**  $[\rightarrow 1\text{-NIF3}]$ . Similar to case  $[\rightarrow 1\text{-NIF2}]$ .
  - **Case:**  $[\rightarrow 1\text{-NIF4}]$ , such that  $A'_2 = \text{if } \hat{\xi} |_n A'_{21}|_{R_1} A_{12}|_{R_2}$  and  $A_{11} \xrightarrow{\xi, \alpha} A'_{21}$  and  $\text{subj}(\xi) \cup \text{subj}(\alpha) \subseteq R_1 \setminus R_2$ , for some  $A'_{21}$ .  
Recall  $A_{11} \xrightarrow{\text{true}, \tau} A'_{11}$  and  $A_{11} \xrightarrow{\xi, \alpha} A'_{21}$ . Then, by induction:
    - \* **Case:**  $A'_{11} \xrightarrow{\xi, \alpha} A''_{11}$  and  $A'_{21} \xrightarrow{\text{true}, \tau} A''_{11}$ , for some  $A''_{11}$ .
      - Recall  $A'_{11} \xrightarrow{\xi, \alpha} A''_{11}$  and  $\text{subj}(\xi) \cup \text{subj}(\alpha) \subseteq R_1 \setminus R_2$ . Then, by  $[\rightarrow 1\text{-NIF4}]$ ,  
 $\text{if } \hat{\xi} |_n A'_{11}|_{R_1} A_{12}|_{R_2} \xrightarrow{\xi, \alpha} \text{if } \hat{\xi} |_n A''_{11}|_{R_1} A_{12}|_{R_2}$ . Then,  
 $A'_1 \xrightarrow{\xi, \alpha} \text{if } \hat{\xi} |_n A''_{11}|_{R_1} A_{12}|_{R_2}$ . Then,  $A'' = \text{if } \hat{\xi} |_n A''_{11}|_{R_1} A_{12}|_{R_2}$  and  $A'_1 \xrightarrow{\xi, \alpha} A''$ , for some  $A''$ .
      - Recall  $A'_{21} \xrightarrow{\text{true}, \tau} A''_{11}$  and  $\text{subj}(\text{true}) \cup \text{subj}(\tau) \subseteq R_1 \setminus R_2$ . Then, by  $[\rightarrow 1\text{-NIF4}]$ ,  
 $\text{if } \hat{\xi} |_n A'_{21}|_{R_1} A_{12}|_{R_2} \xrightarrow{\text{true}, \tau} \text{if } \hat{\xi} |_n A''_{11}|_{R_1} A_{12}|_{R_2}$ . Then,  $A'_2 \xrightarrow{\text{true}, \tau} A''$ .

- \* **Case:**  $\xi = \text{true}$  and  $\alpha = \tau$  and  $A'_{11} = A'_{21}$ .  
Recall  $A'_{11} = A'_{21}$ . Then,  $\text{if } \hat{\xi}|_n A'_{11}|_{R_1} A_{12}|_{R_2} = \text{if } \hat{\xi}|_n A'_{21}|_{R_1} A_{12}|_{R_2}$ . Then,  $A'_1 = A'_2$ .
  - **Case:**  $[\rightarrow 1\text{-NIF5}]$ , such that  $A'_2 = \text{if } \hat{\xi}|_n A_{11}|_{R_1} A'_{22}|_{R_2}$  and  $A_{12} \xrightarrow{\xi, \alpha} A'_{22}$  and  $\text{subj}(\xi) \cup \text{subj}(\alpha) \subseteq R_2 \setminus R_1$ , for some  $A'_{22}$ .
    - \* Recall  $A_{12} \xrightarrow{\xi, \alpha} A'_{22}$  and  $\text{subj}(\xi) \cup \text{subj}(\alpha) \subseteq R_2 \setminus R_1$ . Then, by  $[\rightarrow 1\text{-NIF5}]$ ,  $\text{if } \hat{\xi}|_n A'_{11}|_{R_1} A_{12}|_{R_2} \xrightarrow{\xi, \alpha} \text{if } \hat{\xi}|_n A'_{11}|_{R_1} A'_{22}|_{R_2}$ . Then,  $A'_1 \xrightarrow{\xi, \alpha} \text{if } \hat{\xi}|_n A'_{11}|_{R_1} A'_{22}|_{R_2}$ . Then,  $A'' = \text{if } \hat{\xi}|_n A'_{11}|_{R_1} A'_{22}|_{R_2}$  and  $A'_1 \xrightarrow{\xi, \alpha} A''$ , for some  $A''$ .
    - \* Recall  $A_{11} \xrightarrow{\text{true}, \tau} A'_{11}$  and  $\text{subj}(\text{true}) \cup \text{subj}(\tau) \subseteq R_1 \setminus R_2$ . Then, by  $[\rightarrow 1\text{-NIF4}]$ ,  $\text{if } \hat{\xi}|_n A_{11}|_{R_1} A_{22}|_{R_2} \xrightarrow{\text{true}, \tau} \text{if } \hat{\xi}|_n A'_{11}|_{R_1} A'_{22}|_{R_2}$ . Then,  $A'_2 \xrightarrow{\text{true}, \tau} A''$ .
  - **Step:**  $[\rightarrow 1\text{-NIF5}]$ . Similar to case  $[\rightarrow 1\text{-NIF4}]$ .
  - **Step:**  $[\rightarrow 1\text{-NWHILE}]$ , such that  $A = \text{while } \hat{\xi}|_n \{\psi\} \hat{A}|_{\emptyset}$  and  $\text{if } \hat{\xi}|_n (\hat{A}; \text{while } \hat{\xi}|_n \{\psi\} \hat{A}|_{\emptyset})|_{\emptyset} \text{skip}|_{\emptyset} \xrightarrow{\text{true}, \tau} A'_1$ , for some  $\hat{A}, \psi, \hat{\xi}, n$ .
    - Recall  $A \xrightarrow{\xi, \alpha} A'_2$ . Then,  $\text{while } \hat{\xi}|_n \{\psi\} \hat{A}|_{\emptyset} \xrightarrow{\xi, \alpha} A'_2$ . Then, by Defn. 31,  $\text{if } \hat{\xi}|_n (\hat{A}; \text{while } \hat{\xi}|_n \{\psi\} \hat{A}|_{\emptyset})|_{\emptyset} \text{skip}|_{\emptyset} \xrightarrow{\xi, \alpha} A'_2$ .
    - Recall  $\text{if } \hat{\xi}|_n (\hat{A}; \text{while } \hat{\xi}|_n \{\psi\} \hat{A}|_{\emptyset})|_{\emptyset} \text{skip}|_{\emptyset} \xrightarrow{\text{true}, \tau} A'_1$  and  $\text{if } \hat{\xi}|_n (\hat{A}; \text{while } \hat{\xi}|_n \{\psi\} \hat{A}|_{\emptyset})|_{\emptyset} \text{skip}|_{\emptyset} \xrightarrow{\xi, \alpha} A'_2$ . Then, by induction:
      - \* **Case:**  $\xi = \text{true}$  and  $\alpha = \tau$  and  $A'_1 = A'_2$ .
      - \* **Case:**  $A'_1 \xrightarrow{\xi, \alpha} A''$  and  $A'_2 \xrightarrow{\xi, \alpha} A''$ , for some  $A''$ . □
2. Recall  $A \xrightarrow{\text{true}, \tau} A'_1$ . Then, by Defn. 31:
- **Base:**  $[\rightarrow 1\text{-ACT}]$ , such that  $A'_1 = \text{skip}$ .  
By Defn. 31, not  $\text{skip} \xrightarrow{\xi, \alpha} A'_1$ . Then, not  $A'_1 \xrightarrow{\xi, \alpha} A'_1$ . Then, **false**.
  - **Base:**  $[\rightarrow 1\text{-IF1}]$ , such that  $\tau = i_{\text{subj}(\hat{\xi})}^R$ , for some  $R, \hat{\xi}$ . Then, **false**.
  - **Base:**  $[\rightarrow 1\text{-IF2}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
  - **Base:**  $[\rightarrow 1\text{-WHILE1}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
  - **Base:**  $[\rightarrow 1\text{-WHILE2}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
  - **Base:**  $[\rightarrow 1\text{-NIF1}]$ , such that  $A = \text{if } \hat{\xi}|_n A_{11}|_{R_1} A_{12}|_{R_2}$  and  $A'_1 = \text{if } \hat{\xi}|_{n-1} A_{11}|_{R_1} A_{12}|_{R_2}$  and  $n > 0$ , for some  $R_1, R_2, A_{11}, A_{12}, \xi, n$ .  
Recall  $A'_1 \xrightarrow{\xi, \alpha} A''$ . Then,  $\text{if } \hat{\xi}|_{n-1} A_{11}|_{R_1} A_{12}|_{R_2} \xrightarrow{\xi, \alpha} A''$ . Then, by Defn. 31:
    - **Case:**  $[\rightarrow 1\text{-NIF1}]$ , such that  $A'' = \text{if } \hat{\xi}|_{n-1-1} A_{11}|_{R_1} A_{12}|_{R_2}$  and  $\xi = \text{true}$  and  $\alpha = \tau$ .
      - \* Recall  $A \xrightarrow{\text{true}, \tau} A'_1$ . Then,  $A \xrightarrow{\xi, \alpha} A'_1$ . Then,  $A'_2 = A'_1$  and  $A \xrightarrow{\xi, \alpha} A'_2$ , for some  $A'_2$ .
      - \* Recall  $A'_1 \xrightarrow{\xi, \alpha} A''$ . Then,  $A'_2 \xrightarrow{\text{true}, \tau} A''$ .
    - **Case:**  $[\rightarrow 1\text{-NIF2}]$ ,  $A'' = \text{if } \hat{\xi}|_{n-1} A_{11}|_{R_1 \cup \{r\}} A_{12}|_{R_2}$  and  $\xi = \hat{\xi}^+ \upharpoonright r$  and  $\alpha = 1\{r\}$  and  $r \in \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$ , for some  $r$ .
      - \* Recall  $r \in \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$ . Then, by  $[\rightarrow 1\text{-NIF2}]$ ,  $\text{if } \hat{\xi}|_n A_{11}|_{R_1} A_{12}|_{R_2} \xrightarrow{\xi \upharpoonright r, 1\{r\}} \text{if } \hat{\xi}|_n A_{11}|_{R_1 \cup \{r\}} A_{12}|_{R_2}$ . Then,  $A \xrightarrow{\xi, \alpha} \text{if } \hat{\xi}|_n A_{11}|_{R_1 \cup \{r\}} A_{12}|_{R_2}$ . Then  $A'_2 = \text{if } \hat{\xi}|_n A_{11}|_{R_1 \cup \{r\}} A_{12}|_{R_2}$  and  $A \xrightarrow{\xi, \alpha} A'_2$ , for some  $A'_2$ .
      - \* Recall  $n > 0$ . Then, by  $[\rightarrow 1\text{-NIF1}]$ ,  $\text{if } \hat{\xi}|_n A_{11}|_{R_1 \cup \{r\}} A_{12}|_{R_2} \xrightarrow{\text{true}, \tau} \text{if } \hat{\xi}|_{n-1} A_{11}|_{R_1 \cup \{r\}} A_{12}|_{R_2}$ . Then,  $A'_2 \xrightarrow{\text{true}, \tau} A''$ .
    - **Case:**  $[\rightarrow 1\text{-NIF3}]$ . Similar to case  $[\rightarrow 1\text{-NIF2}]$ .
    - **Case:**  $[\rightarrow 1\text{-NIF4}]$ . Similar to case  $[\rightarrow 1\text{-NIF2}]$ .
    - **Case:**  $[\rightarrow 1\text{-NIF5}]$ . Similar to case  $[\rightarrow 1\text{-NIF2}]$ .
  - **Base:**  $[\rightarrow 1\text{-NIF2}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
  - **Base:**  $[\rightarrow 1\text{-NIF3}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
  - **Step:**  $[\rightarrow 1\text{-SEQ1}]$ , such that  $A = A_{11}; A_{12}$  and  $A'_1 = A'_{11}; A_{12}$  and  $A_{11} \xrightarrow{\text{true}, \tau} A'_{11}$ , for some  $A_{11}, A'_{11}, A_{12}$ .  
Recall  $A'_1 \xrightarrow{\xi, \alpha} A''$ . Then,  $A'_{11}; A_{12} \xrightarrow{\xi, \alpha} A''$ . Then, by Defn. 31:
    - **Case:**  $[\rightarrow 1\text{-SEQ1}]$ , such that  $A'' = A''_{11}; A_{12}$  and  $A'_{11} \xrightarrow{\xi, \alpha} A''_{11}$ , for some  $A''_{11}$ .
      - \* Recall  $A_{11} \xrightarrow{\text{true}, \tau} A'_{11} \xrightarrow{\xi, \alpha} A''_{11}$ . Then, by induction,  $A_{11} \xrightarrow{\xi, \alpha} A'_{21} \xrightarrow{\text{true}, \tau} A''_{11}$ , for some  $A'_{21}$ .
      - \* Recall  $A_{11} \xrightarrow{\xi, \alpha} A'_{21}$ . Then, by  $[\rightarrow 1\text{-SEQ1}]$ ,  $A_{11}; A_{12} \xrightarrow{\xi, \alpha} A'_{21}; A_{12}$ . Then,  $A \xrightarrow{\xi, \alpha} A'_{21}; A_{12}$ . Then,  $A'_2 = A'_{21}; A_{12}$  and  $A \xrightarrow{\xi, \alpha} A'_2$ , for some  $A'_2$ .

- \* Recall  $A'_{21} \xrightarrow{\text{true}, \tau} A''_{11}$ . Then, by  $[\rightarrow 1\text{-SEQ1}]$ ,  $A'_{21}; A_{12} \xrightarrow{\text{true}, \tau} A''_{11}; A_{12}$ . Then,  $A'_2 \xrightarrow{\text{true}, \tau} A''$ .
- **Case:**  $[\rightarrow 1\text{-SEQ2}]$ , such that  $A'' = A'_{11}; A'_{12}$  and  $\text{subj}(A'_{11}) \cap (\text{subj}(\xi) \cup \text{subj}(\alpha)) = \emptyset$  and  $A_{12} \xrightarrow{\xi, \alpha} A'_{12}$ , for some  $A'_{12}$ .
  - \* Recall  $A_{11} \xrightarrow{\text{true}, \tau} A'_{11}$ . Then, by Lem. 53,  $\text{subj}(A_{11}) = \text{subj}(A'_{11})$ .
  - \* Recall  $\text{subj}(A'_{11}) \cap (\text{subj}(\xi) \cup \text{subj}(\alpha)) = \emptyset$ . Then,  $\text{subj}(A_{11}) \cap (\text{subj}(\xi) \cup \text{subj}(\alpha)) = \emptyset$ .
  - \* Recall  $\text{subj}(A_{11}) \cap (\text{subj}(\xi) \cup \text{subj}(\alpha)) = \emptyset$  and  $A_{12} \xrightarrow{\xi, \alpha} A'_{12}$ . Then, by  $[\rightarrow 1\text{-SEQ2}]$ ,  $A_{11}; A_{12} \xrightarrow{\xi, \alpha} A_{11}; A'_{12}$ . Then,  $A \xrightarrow{\xi, \alpha} A_{11}; A'_{12}$ . Then,  $A'_2 = A_{11}; A'_{12}$  and  $A \xrightarrow{\xi, \alpha} A'_2$ , for some  $A'_2$ .
  - \* Recall  $A_{11} \xrightarrow{\text{true}, \tau} A'_{11}$ . Then, by  $[\rightarrow 1\text{-SEQ1}]$ ,  $A_{11}; A'_{12} \xrightarrow{\text{true}, \tau} A'_{11}; A'_{12}$ . Then,  $A'_2 \xrightarrow{\text{true}, \tau} A''$ .
- **Step:**  $[\rightarrow 1\text{-SEQ2}]$ , such that  $A = A_{11}; A_{12}$  and  $A'_1 = A_{11}; A'_{12}$  and  $\text{subj}(A_{11}) \cap (\text{subj}(\text{true}) \cup \text{subj}(\tau)) = \emptyset$  and  $A_{12} \xrightarrow{\text{true}, \tau} A'_{12}$ , for some  $A_{11}, A_{12}, A'_{12}$ . Recall  $A'_1 \xrightarrow{\xi, \alpha} A''$ . Then,  $A_{11}; A'_{12} \xrightarrow{\xi, \alpha} A''$ . Then, by Defn. 31:
  - **Case:**  $[\rightarrow 1\text{-SEQ1}]$ , such that  $A'' = A'_{11}; A'_{12}$  and  $A_{11} \xrightarrow{\xi, \alpha} A'_{11}$ , for some  $A'_{11}$ .
    - \* Recall  $A_{11} \xrightarrow{\xi, \alpha} A'_{11}$ . Then, by  $[\rightarrow 1\text{-SEQ1}]$ ,  $A_{11}; A_{12} \xrightarrow{\xi, \alpha} A'_{11}; A_{12}$ . Then,  $A \xrightarrow{\xi, \alpha} A'_{11}; A_{12}$ . Then,  $A'_2 = A'_{11}; A_{12}$  and  $A \xrightarrow{\xi, \alpha} A'_2$ , for some  $A'_2$ .
    - \* Recall  $\text{subj}(A'_{11}) \cap (\emptyset \cup \emptyset) = \emptyset$ . Then, by Lem. 4,  $\text{subj}(A'_{11}) \cap (\text{subj}(\text{true}) \cup \emptyset) = \emptyset$ . Then, by Defn. 16,  $\text{subj}(A'_{11}) \cap (\text{subj}(\text{true}) \cup \text{subj}(\tau)) = \emptyset$ .
    - \* Recall  $\text{subj}(A'_{11}) \cap (\text{subj}(\text{true}) \cup \text{subj}(\tau)) = \emptyset$  and  $A_{12} \xrightarrow{\text{true}, \tau} A'_{12}$ . Then, by  $[\rightarrow 1\text{-SEQ2}]$ ,  $A_{11}; A_{12} \xrightarrow{\text{true}, \tau} A'_{11}; A'_{12}$ . Then,  $A'_2 \xrightarrow{\text{true}, \tau} A''$ .
  - **Case:**  $[\rightarrow 1\text{-SEQ2}]$ , such that  $A'' = A_{11}; A'_{12}$  and  $\text{subj}(A_{11}) \cap (\text{subj}(\xi) \cup \text{subj}(\alpha)) = \emptyset$  and  $A'_{12} \xrightarrow{\xi, \alpha} A'_{12}$ , for some  $A'_{12}$ .
    - \* Recall  $A_{12} \xrightarrow{\text{true}, \tau} A'_{12} \xrightarrow{\xi, \alpha} A'_{12}$ . Then, by induction,  $A_{12} \xrightarrow{\xi, \alpha} A'_{22} \xrightarrow{\text{true}, \tau} A'_{12}$ , for some  $A'_{22}$ .
    - \* Recall  $\text{subj}(A_{11}) \cap (\text{subj}(\xi) \cup \text{subj}(\alpha)) = \emptyset$  and  $A_{12} \xrightarrow{\xi, \alpha} A'_{22}$ . Then, by  $[\rightarrow 1\text{-SEQ2}]$ ,  $A_{11}; A_{12} \xrightarrow{\xi, \alpha} A_{11}; A'_{22}$ . Then,  $A \xrightarrow{\xi, \alpha} A_{11}; A'_{22}$ . Then,  $A'_2 = A_{11}; A'_{22}$  and  $A \xrightarrow{\xi, \alpha} A'_2$ , for some  $A'_2$ .
    - \* Recall  $\text{subj}(A_{11}) \cap (\text{subj}(\text{true}) \cup \text{subj}(\tau)) = \emptyset$  and  $A'_{22} \xrightarrow{\text{true}, \tau} A'_{12}$ . Then, by  $[\rightarrow 1\text{-SEQ2}]$ ,  $A_{11}; A'_{22} \xrightarrow{\text{true}, \tau} A_{11}; A'_{12}$ . Then,  $A'_2 \xrightarrow{\text{true}, \tau} A''$ .
- **Step:**  $[\rightarrow 1\text{-PAR1}]$ , such that  $A = A_{11} \parallel A_{12}$  and  $A'_1 = A'_{11} \parallel A_{12}$  and  $A_{11} \xrightarrow{\text{true}, \tau} A'_{11}$ , for some  $A_{11}, A'_{11}, A_{12}$ . Recall  $A'_1 \xrightarrow{\xi, \alpha} A''$ . Then,  $A'_{11} \parallel A_{12} \xrightarrow{\xi, \alpha} A''$ . Then, by Defn. 31:
  - **Case:**  $[\rightarrow 1\text{-PAR1}]$ , such that  $A'' = A'_{11} \parallel A_{12}$  and  $A'_{11} \xrightarrow{\xi, \alpha} A'_{11}$ , for some  $A'_{11}$ .
    - \* Recall  $A_{11} \xrightarrow{\text{true}, \tau} A'_{11} \xrightarrow{\xi, \alpha} A'_{11}$ . Then, by induction,  $A_{11} \xrightarrow{\xi, \alpha} A'_{21} \xrightarrow{\text{true}, \tau} A'_{11}$ , for some  $A'_{21}$ .
    - \* Recall  $A_{11} \xrightarrow{\xi, \alpha} A'_{21}$ . Then, by  $[\rightarrow 1\text{-PAR1}]$ ,  $A_{11} \parallel A_{12} \xrightarrow{\xi, \alpha} A'_{21} \parallel A_{12}$ . Then,  $A \xrightarrow{\xi, \alpha} A'_{21} \parallel A_{12}$ . Then,  $A'_2 = A'_{21} \parallel A_{12}$  and  $A \xrightarrow{\xi, \alpha} A'_2$ , for some  $A'_2$ .
    - \* Recall  $A'_{21} \xrightarrow{\text{true}, \tau} A'_{11}$ . Then, by  $[\rightarrow 1\text{-PAR1}]$ ,  $A'_{21} \parallel A_{12} \xrightarrow{\text{true}, \tau} A'_{11} \parallel A_{12}$ . Then,  $A'_2 \xrightarrow{\text{true}, \tau} A''$ .
  - **Case:**  $[\rightarrow 1\text{-PAR2}]$ , such that  $A'' = A'_{11} \parallel A'_{12}$  and  $A_{12} \xrightarrow{\xi, \alpha} A'_{12}$ , for some  $A'_{12}$ .
    - \* Recall  $A_{12} \xrightarrow{\xi, \alpha} A'_{12}$ . Then, by  $[\rightarrow 1\text{-PAR2}]$ ,  $A_{11} \parallel A_{12} \xrightarrow{\xi, \alpha} A_{11} \parallel A'_{12}$ . Then,  $A \xrightarrow{\xi, \alpha} A_{11} \parallel A'_{12}$ . Then,  $A'_2 = A_{11} \parallel A'_{12}$  and  $A \xrightarrow{\xi, \alpha} A'_2$ , for some  $A'_2$ .
    - \* Recall  $A_{11} \xrightarrow{\text{true}, \tau} A'_{11}$ . Then, by  $[\rightarrow 1\text{-PAR1}]$ ,  $A_{11} \parallel A'_{12} \xrightarrow{\text{true}, \tau} A'_{11} \parallel A'_{12}$ . Then,  $A'_2 \xrightarrow{\text{true}, \tau} A''$ .
- **Step:**  $[\rightarrow 1\text{-PAR2}]$ . Similar to case  $[\rightarrow 1\text{-PAR1}]$ .
- **Step:**  $[\rightarrow 1\text{-NIF4}]$ , such that  $A = \mathbf{if} \hat{\xi}|_n A_{11}|_{R_1} A_{12}|_{R_2}$  and  $A'_1 = \mathbf{if} \hat{\xi}|_n A'_{11}|_{R_1} A_{12}|_{R_2}$  and  $A_{11} \xrightarrow{\text{true}, \tau} A'_{11}$  and  $\text{subj}(\text{true}) \cup \text{subj}(\tau) \subseteq R_1 \setminus R_2$ , for some  $R_1, R_2, A_{11}, A'_{11}, A_{12}, \hat{\xi}, n$ . Recall  $A'_1 \xrightarrow{\xi, \alpha} A''$ . Then,  $\mathbf{if} \hat{\xi}|_n A'_{11}|_{R_1} A_{12}|_{R_2} \xrightarrow{\xi, \alpha} A''$ . Then, by Defn. 31:
  - **Case:**  $[\rightarrow 1\text{-NIF1}]$ , such that  $A'' = \mathbf{if} \hat{\xi}|_{n-1} A'_{11}|_{R_1} A_{12}|_{R_2}$  and  $\xi = \text{true}$  and  $\alpha = \tau$  and  $n > 0$ .



- \* Recall  $n > 0$ . Then, by  $[\rightarrow 1\text{-NIF1}]$ ,  
 $\text{if } \hat{\xi}|_n A_{11}|_{R_1} A_{12}|_{R_2} \xrightarrow{\text{true}, \tau} \text{if } \hat{\xi}|_{n-1} A_{11}|_{R_1} A_{12}|_{R_2}$ . Then,  
 $A \xrightarrow{\xi, \alpha} \text{if } \hat{\xi}|_{n-1} A_{11}|_{R_1} A_{12}|_{R_2}$ . Then,  $A'_2 = \text{if } \hat{\xi}|_{n-1} A_{11}|_{R_1} A_{12}|_{R_2}$  and  $A \xrightarrow{\xi, \alpha} A'_2$ ,  
for some  $A'_2$ .
- \* Recall  $A_{11} \xrightarrow{\text{true}, \tau} A'_{11}$  and  $\text{subj}(\text{true}) \cup \text{subj}(\tau) \subseteq R_1 \setminus R_2$ . Then, by  $[\rightarrow 1\text{-NIF4}]$ ,  
 $\text{if } \hat{\xi}|_{n-1} A_{11}|_{R_1} A_{12}|_{R_2} \xrightarrow{\text{true}, \tau} \text{if } \hat{\xi}|_{n-1} A'_{11}|_{R_1} A_{12}|_{R_2}$ . Then,  $A'_2 \xrightarrow{\text{true}, \tau} A''$ .
- **Case:**  $[\rightarrow 1\text{-NIF2}]$ ,  $A'' = \text{if } \hat{\xi}|_n A'_{11}|_{R_1 \cup \{r\}} A_{12}|_{R_2}$  and  $\xi = \hat{\xi}^+ \upharpoonright r$  and  $\alpha = 1\{r\}$  and  
 $r \in \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$ , for some  $r$ .
  - \* Recall  $r \in \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$ . Then, by  $[\rightarrow 1\text{-NIF2}]$ ,  
 $\text{if } \hat{\xi}|_n A_{11}|_{R_1} A_{12}|_{R_2} \xrightarrow{\hat{\xi}^+ \upharpoonright r, 1\{r\}} \text{if } \hat{\xi}|_n A_{11}|_{R_1 \cup \{r\}} A_{12}|_{R_2}$ . Then,  
 $A \xrightarrow{\xi, \alpha} \text{if } \hat{\xi}|_n A_{11}|_{R_1 \cup \{r\}} A_{12}|_{R_2}$ . Then  $A'_2 = \text{if } \hat{\xi}|_n A_{11}|_{R_1 \cup \{r\}} A_{12}|_{R_2}$  and  
 $A \xrightarrow{\xi, \alpha} A'_2$ , for some  $A'_2$ .
  - \* Recall  $\emptyset \cup \emptyset \subseteq (R_1 \cup \{r\}) \setminus R_2$ . Then, by Lem. 4,  $\text{subj}(\text{true}) \cup \emptyset \subseteq (R_1 \cup \{r\}) \setminus R_2$ .  
Then, by Defn. 16,  $\text{subj}(\text{true}) \cup \text{subj}(\tau) \subseteq (R_1 \cup \{r\}) \setminus R_2$ .
  - \* Recall  $A_{11} \xrightarrow{\text{true}, \tau} A'_{11}$  and  $\text{subj}(\text{true}) \cup \text{subj}(\tau) \subseteq (R_1 \cup \{r\}) \setminus R_2$ . Then, by  
 $[\rightarrow 1\text{-NIF4}]$ ,  $\text{if } \hat{\xi}|_n A_{11}|_{R_1 \cup \{r\}} A_{12}|_{R_2} \xrightarrow{\text{true}, \tau} \text{if } \hat{\xi}|_n A'_{11}|_{R_1 \cup \{r\}} A_{12}|_{R_2}$ . Then,  
 $A'_2 \xrightarrow{\text{true}, \tau} A''$ .
- **Case:**  $[\rightarrow 1\text{-NIF3}]$ . Similar to case  $[\rightarrow 1\text{-NIF2}]$ .
- **Case:**  $[\rightarrow 1\text{-NIF4}]$ , such that  $A'' = \text{if } \hat{\xi}|_n A'_{11}|_{R_1} A_{12}|_{R_2}$  and  $A'_{11} \xrightarrow{\xi, \alpha} A''_{11}$  and  
 $\text{subj}(\xi) \cup \text{subj}(\alpha) \subseteq R_1 \setminus R_2$ , for some  $A''_{11}$ .
  - \* Recall  $A_{11} \xrightarrow{\text{true}, \tau} A'_{11} \xrightarrow{\xi, \alpha} A''_{11}$ . Then, by induction,  $A_{11} \xrightarrow{\xi, \alpha} A'_{21} \xrightarrow{\text{true}, \tau} A''_{11}$ , for  
some  $A'_{21}$ .
  - \* Recall  $A_{11} \xrightarrow{\xi, \alpha} A'_{21}$  and  $\text{subj}(\xi) \cup \text{subj}(\alpha) \subseteq R_1 \setminus R_2$ . Then, by  $[\rightarrow 1\text{-NIF4}]$ ,  
 $\text{if } \hat{\xi}|_n A_{11}|_{R_1} A_{12}|_{R_2} \xrightarrow{\xi, \alpha} \text{if } \hat{\xi}|_n A'_{21}|_{R_1} A_{12}|_{R_2}$ . Then,  $A \xrightarrow{\xi, \alpha} \text{if } \hat{\xi}|_n A'_{21}|_{R_1} A_{12}|_{R_2}$ .  
Then,  $A'_2 = \text{if } \hat{\xi}|_n A'_{21}|_{R_1} A_{12}|_{R_2}$  and  $A \xrightarrow{\xi, \alpha} A'_2$ , for some  $A'_2$ .
  - \* Recall  $A'_{21} \xrightarrow{\text{true}, \tau} A''_{11}$  and  $\text{subj}(\text{true}) \cup \text{subj}(\tau) \subseteq R_1 \setminus R_2$ . Then, by  $[\rightarrow 1\text{-NIF4}]$ ,  
 $\text{if } \hat{\xi}|_n A'_{21}|_{R_1} A_{12}|_{R_2} \xrightarrow{\text{true}, \tau} \text{if } \hat{\xi}|_n A''_{11}|_{R_1} A_{12}|_{R_2}$ . Then,  $A'_2 \xrightarrow{\text{true}, \tau} A''$ .
- **Case:**  $[\rightarrow 1\text{-NIF5}]$ , such that  $A'' = \text{if } \hat{\xi}|_n A'_{11}|_{R_1} A'_{12}|_{R_2}$  and  $A_{12} \xrightarrow{\xi, \alpha} A'_{12}$  and  
 $\text{subj}(\xi) \cup \text{subj}(\alpha) \subseteq R_2 \setminus R_1$ , for some  $A'_{12}$ .
  - \* Recall  $A_{12} \xrightarrow{\xi, \alpha} A'_{12}$  and  $\text{subj}(\xi) \cup \text{subj}(\alpha) \subseteq R_2 \setminus R_1$ . Then, by  $[\rightarrow 1\text{-NIF5}]$ ,  
 $\text{if } \hat{\xi}|_n A_{11}|_{R_1} A_{12}|_{R_2} \xrightarrow{\xi, \alpha} \text{if } \hat{\xi}|_n A_{11}|_{R_1} A'_{12}|_{R_2}$ . Then,  $A \xrightarrow{\xi, \alpha} \text{if } \hat{\xi}|_n A_{11}|_{R_1} A'_{12}|_{R_2}$ .  
Then,  $A'_2 = \text{if } \hat{\xi}|_n A_{11}|_{R_1} A'_{12}|_{R_2}$  and  $A \xrightarrow{\xi, \alpha} A'_2$ , for some  $A'_2$ .
  - \* Recall  $A_{11} \xrightarrow{\text{true}, \tau} A'_{11}$  and  $\text{subj}(\text{true}) \cup \text{subj}(\tau) \subseteq R_1 \setminus R_2$ . Then, by  $[\rightarrow 1\text{-NIF4}]$ ,  
 $\text{if } \hat{\xi}|_n A_{11}|_{R_1} A'_{12}|_{R_2} \xrightarrow{\text{true}, \tau} \text{if } \hat{\xi}|_n A'_{11}|_{R_1} A'_{12}|_{R_2}$ . Then,  $A'_2 \xrightarrow{\text{true}, \tau} A''$ .
- **Step:**  $[\rightarrow 1\text{-NIF5}]$ . Similar to case  $[\rightarrow 1\text{-NIF4}]$ .
- **Step:**  $[\rightarrow 1\text{-NWHILE}]$ , such that  $A = \text{while } \hat{\xi}|_n \{\psi\} \hat{A}|_{\emptyset}$  and  
 $\text{if } \hat{\xi}|_n (\hat{A}; \text{while } \hat{\xi}|_n \{\psi\} \hat{A}|_{\emptyset})|_{\emptyset} \text{skip}|_{\emptyset} \xrightarrow{\text{true}, \tau} A'_1$ , for some  $\hat{A}, \psi, \hat{\xi}, n$ .
  - Recall  $\text{if } \hat{\xi}|_n (\hat{A}; \text{while } \hat{\xi}|_n \{\psi\} \hat{A}|_{\emptyset})|_{\emptyset} \text{skip}|_{\emptyset} \xrightarrow{\text{true}, \tau} A'_1 \xrightarrow{\xi, \alpha} A''$ . Then, by induction,  
 $\text{if } \hat{\xi}|_n (\hat{A}; \text{while } \hat{\xi}|_n \{\psi\} \hat{A}|_{\emptyset})|_{\emptyset} \text{skip}|_{\emptyset} \xrightarrow{\xi, \alpha} A'_2$  and  $A'_2 \xrightarrow{\text{true}, \tau} A''$ , for some  $A'_2$ .
  - Recall  $\text{if } \hat{\xi}|_n (\hat{A}; \text{while } \hat{\xi}|_n \{\psi\} \hat{A}|_{\emptyset})|_{\emptyset} \text{skip}|_{\emptyset} \xrightarrow{\xi, \alpha} A'_2$ . Then, by  $[\rightarrow 1\text{-NWHILE}]$ ,  
 $\text{while } \hat{\xi}|_n \{\psi\} \hat{A}|_{\emptyset} \xrightarrow{\xi, \alpha} A'_2$ . Then,  $A \xrightarrow{\xi, \alpha} A'_2$ . □

*Proof (of Lem. 57).* Recall  $\checkmark_R(G)$ . Then, by Defn. 27:

- **Base:**  $[\checkmark 1\text{-ACT1}]$ , such that  $G = q.y := e$ , for some  $q, y, e$ .  
By  $[\rightarrow 1\text{-ACT}]$ ,  $q.y := e \xrightarrow{\text{true}, q.y := e} \text{skip}$ . Then,  $G' = \text{skip}$  and  $\xi = \text{true}$  and  $\gamma = q.y := e$  and  
 $G \xrightarrow{\xi, \gamma} G'$ , for some  $G', \xi, \gamma$ .
- **Base:**  $[\checkmark 1\text{-ACT2}]$ . Similar to case  $[\checkmark 1\text{-ACT1}]$ .
- **Base:**  $[\checkmark 1\text{-SKIP}]$ , such that  $G = \text{skip}$ .  
By  $[\downarrow 1\text{-SKIP}]$ ,  $\text{skip} \downarrow$ . Then,  $G \downarrow$ .
- **Step:**  $[\checkmark 1\text{-SEQ}]$ , such that  $G = G_1 ; G_2$  and  $\checkmark_R(G_1)$  and  $\checkmark_R(G_2)$ , for some  $G_1, G_2$ .  
Recall  $\checkmark_R(G_1)$  and  $\checkmark_R(G_2)$ . Then, by induction:
  - **Case:**  $G_1 \downarrow$  and  $G_2 \downarrow$ . Then, by  $[\downarrow 1\text{-SEQ}]$ ,  $G_1 ; G_2 \downarrow$ . Then,  $G \downarrow$ .

- **Case:**  $G_1 \downarrow$  and  $G_2 \xrightarrow{\xi, \gamma} G'_2$ , for some  $\xi, \gamma$ , for some  $G'_2$ .
  - \* Recall  $G_1 \downarrow$  and  $\check{\nu}_R(G_1)$ . Then, by Lem. 48,  $\text{subj}(G_1) = \emptyset$ .
  - \* Recall  $\emptyset \cap (\text{subj}(\xi) \cup \text{subj}(\gamma)) = \emptyset$ . Then,  $\text{subj}(G_1) \cap (\text{subj}(\xi) \cup \text{subj}(\gamma)) = \emptyset$ .
  - \* Recall  $\text{subj}(G_1) \cap (\text{subj}(\xi) \cup \text{subj}(\gamma)) = \emptyset$  and  $G_2 \xrightarrow{\xi, \gamma} G'_2$ . Then, by  $[\rightarrow 1\text{-SEQ2}]$ ,  $G_1; G_2 \xrightarrow{\xi, \gamma} G_1; G'_2$ . Then,  $G \xrightarrow{\xi, \gamma} G_1; G'_2$ . Then,  $G' = G_1; G'_2$  and  $G \xrightarrow{\xi, \gamma} G'$ , for some  $G'$ .
- **Case:**  $G_1 \xrightarrow{\xi, \gamma} G'_1$ .
  - Recall  $G_1 \xrightarrow{\xi, \gamma} G'_1$ . Then, by  $[\rightarrow 1\text{-SEQ1}]$ ,  $G_1; G_2 \xrightarrow{\xi, \gamma} G'_1; G_2$ . Then,  $G \xrightarrow{\xi, \gamma} G'_1; G_2$ . Then,  $G' = G'_1; G_2$  and  $G \xrightarrow{\xi, \gamma} G'$ , for some  $G'$ .
- **Step:**  $[\check{\nu} 1\text{-PAR}]$ , such that  $G = G_1 \parallel G_2$  and  $\check{\nu}_R(G_1)$  and  $\check{\nu}_R(G_2)$ , for some  $G_1, G_2$ .
  - Recall  $\check{\nu}_R(G_1)$  and  $\check{\nu}_R(G_2)$ . Then, by induction:
    - **Case:**  $G_1 \downarrow$  and  $G_2 \downarrow$ . Then, by  $[\downarrow 1\text{-PAR}]$ ,  $G_1 \parallel G_2 \downarrow$ . Then,  $G \downarrow$ .
    - **Case:**  $G_1 \downarrow$  and  $G_2 \xrightarrow{\xi, \gamma} G'_2$ , for some  $\xi, \gamma$ , for some  $G'_2$ .
      - Recall  $G_2 \xrightarrow{\xi, \gamma} G'_2$ . Then, by  $[\rightarrow 1\text{-PAR2}]$ ,  $G_1 \parallel G_2 \xrightarrow{\xi, \gamma} G_1 \parallel G'_2$ . Then,  $G \xrightarrow{\xi, \gamma} G_1 \parallel G'_2$ . Then,  $G' = G_1 \parallel G'_2$  and  $G \xrightarrow{\xi, \gamma} G'$ , for some  $G'$ .
    - **Case:**  $G_1 \xrightarrow{\xi, \gamma} G'_1$ .
      - Recall  $G_1 \xrightarrow{\xi, \gamma} G'_1$ . Then, by  $[\rightarrow 1\text{-PAR1}]$ ,  $G_1 \parallel G_2 \xrightarrow{\xi, \gamma} G'_1 \parallel G_2$ . Then,  $G \xrightarrow{\xi, \gamma} G'_1 \parallel G_2$ . Then,  $G' = G'_1 \parallel G_2$  and  $G \xrightarrow{\xi, \gamma} G'$ , for some  $G'$ .
  - **Step:**  $[\check{\nu} 1\text{-IF}]$ , such that  $G \equiv R_{\text{if}} \hat{\xi} G_1 G_2$ , for some  $G_{\text{if}}, G_2, \hat{\xi}$ .
    - By  $[\rightarrow 1\text{-IF1}]$ ,  $R_{\text{if}} \hat{\xi} G_1 G_2 \xrightarrow{\xi^+, \uparrow_{\text{subj}(\hat{\xi})}} G_1$ . Then,  $G \xrightarrow{\xi^+, \uparrow_{\text{subj}(\hat{\xi})}} G_1$ . Then,  $G' = G_1$  and  $\xi = \hat{\xi}^+$  and  $\gamma = 1_{\text{subj}(\hat{\xi})}^R$  and  $G \xrightarrow{\xi, \gamma} G'$ , for some  $G', \xi, \gamma$ .
  - **Step:**  $[\check{\nu} 1\text{-WHILE}]$ , such that  $G = R_{\text{while}} \hat{\xi} \{\psi\} \hat{G}$ , for some  $\hat{G}, \psi, \hat{\xi}$ .
    - By  $[\rightarrow 1\text{-WHILE1}]$ ,  $R_{\text{while}} \hat{\xi} \{\psi\} \hat{G} \xrightarrow{\xi^+, \uparrow_{\text{subj}(\hat{\xi})}} \hat{G}; R_{\text{while}} \hat{\xi} \{\psi\} \hat{G}$ . Then,  $G \xrightarrow{\xi^+, \uparrow_{\text{subj}(\hat{\xi})}} \hat{G}; R_{\text{while}} \hat{\xi} \{\psi\} \hat{G}$ . Then,  $G' = \hat{G}; R_{\text{while}} \hat{\xi} \{\psi\} \hat{G}$  and  $\xi = \hat{\xi}^+$  and  $\gamma = 1_{\text{subj}(\hat{\xi})}^R$  and  $G \xrightarrow{\xi, \gamma} G'$ , for some  $G', \xi, \gamma$ .
  - **Step:**  $[\check{\nu} 1\text{-NIF}]$ , such that  $G = \text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2}$  and  $\check{\nu}_R(G_1)$  and  $\check{\nu}_R(G_2)$  and  $R = \text{subj}(\hat{\xi})$  and  $R_1, R_2 \subseteq \text{subj}(\hat{\xi})$ , and  $R_1 \neq \emptyset$  implies  $R_2 = \emptyset$ , and  $R_2 \neq \emptyset$  implies  $R_1 = \emptyset$ , for some  $R_1, R_2, G_1, G_2, \hat{\xi}$ .
    - Recall:
      - **Case:**  $R_1 \cup R_2 = \text{subj}(\hat{\xi})$ .
        - Recall  $\check{\nu}_R(G_1)$  and  $\check{\nu}_R(G_2)$ . Then, by induction:
          - \* **Case:**  $G_1 \downarrow$  and  $G_2 \downarrow$ .
            - Recall  $G_1 \downarrow$ . Then,  $R_1 \neq \emptyset$  implies  $G_1 \downarrow$ .
            - Recall  $G_2 \downarrow$ . Then,  $R_2 \neq \emptyset$  implies  $G_2 \downarrow$ .
            - Recall  $R_1 \cup R_2 = \text{subj}(\hat{\xi})$ , and  $R_1 \neq \emptyset$  implies  $G_1 \downarrow$ , and  $R_2 \neq \emptyset$  implies  $G_2 \downarrow$ . Then, by  $[\downarrow 1\text{-NIF}]$ ,  $\text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2} \downarrow$ . Then,  $G \downarrow$ .
          - \* **Case:**  $G_1 \xrightarrow{\xi, \gamma} G'_1$ , for some  $G'_1$ , for some  $\xi, \gamma$ .
            - Recall  $G_1 \xrightarrow{\xi, \gamma} G'_1$ . Then, by Lem. 53,  $\text{subj}(\xi) \cup \text{subj}(\gamma) \cup \text{subj}(G'_1) \subseteq \text{subj}(G_1)$ . Then,  $\text{subj}(\xi) \cup \text{subj}(\gamma) \subseteq \text{subj}(G_1)$ .
            - Recall  $\check{\nu}_R(G_1)$ . Then, by Lem. 47,  $\text{subj}(G_1) \subseteq R$ .
            - Recall  $R_1 \neq \emptyset$  implies  $R_2 = \emptyset$ . Then,  $R_1 \setminus R_2 = R_1$ .
            - Recall  $\text{subj}(\xi) \cup \text{subj}(\gamma) \subseteq \text{subj}(G_1) \subseteq R$ . Then,  $\text{subj}(\xi) \cup \text{subj}(\gamma) \subseteq R$ . Then,  $\text{subj}(\xi) \cup \text{subj}(\gamma) \subseteq \text{subj}(\hat{\xi})$ . Then,  $\text{subj}(\xi) \cup \text{subj}(\gamma) \subseteq R_1$ . Then,  $\text{subj}(\xi) \cup \text{subj}(\gamma) \subseteq R_1 \setminus R_2$ .
            - Recall  $G_1 \xrightarrow{\xi, \gamma} G'_1$  and  $\text{subj}(\xi) \cup \text{subj}(\gamma) \subseteq R_1 \setminus R_2$ . Then, by  $[\rightarrow 1\text{-NIF4}]$ ,  $\text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2} \xrightarrow{\xi, \gamma} \text{if } \hat{\xi}|_0 G'_1|_{R_1} G_2|_{R_2}$ . Then,  $G \xrightarrow{\xi, \gamma} \text{if } \hat{\xi}|_0 G'_1|_{R_1} G_2|_{R_2}$ . Then,  $G' = \text{if } \hat{\xi}|_0 G'_1|_{R_1} G_2|_{R_2}$  and  $G \xrightarrow{\xi, \gamma} G'$ , for some  $G'$ .
        - \* **Case:**  $G_2 \xrightarrow{\xi, \gamma} G'_2$ , for some  $G'_2, \xi, \gamma$ . Similar to case  $G_1 \xrightarrow{\xi, \gamma} G'_1$ , for some  $G'_1, \xi, \gamma$ .
      - **Case:**  $R_1 \cup R_2 \neq \text{subj}(\hat{\xi})$ .
        - \* Recall  $R_1, R_2 \subseteq \text{subj}(\hat{\xi})$ . Then,  $R_1 \cup R_2 \subseteq \text{subj}(\hat{\xi})$ .
        - \* Recall  $R_1 \cup R_2 \subseteq \text{subj}(\hat{\xi})$  and  $R_1 \cup R_2 \neq \text{subj}(\hat{\xi})$ . Then,  $\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2) \neq \emptyset$ . Then,  $r \in \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$ , for some  $r$ . Then, by  $[\rightarrow 1\text{-NIF3}]$ ,  $\text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2} \xrightarrow{\xi^+, \uparrow_{\{r\}}} \text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2 \cup \{r\}}$ . Then,  $G \xrightarrow{\xi^+, \uparrow_{\{r\}}} \text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2 \cup \{r\}}$ . Then,  $G' = \text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2 \cup \{r\}}$  and  $\xi = \hat{\xi}^+ \uparrow r$  and  $\gamma = 2_{\{r\}}^{\uparrow}$  and  $G \xrightarrow{\xi, \gamma} G'$ , for some  $G', \xi, \gamma$ .

- **Step:** [ $\checkmark$ 1-NWHILE], such that  $G = \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset$  and  $R = \mathbf{subj}(\hat{\xi})$ , for some  $\hat{G}, \psi, \hat{\xi}$ . Recall  $\checkmark_R(G)$ . Then, by Lem. 47,  $R \neq \emptyset$ . Then,  $\mathbf{subj}(\hat{\xi}) \neq \emptyset$ . Then,  $r \in \mathbf{subj}(\hat{\xi})$ , for some  $r$ . Then,  $r \in \mathbf{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset)$ . Then, by [ $\rightarrow$ 1-NIF2],  
 $\mathbf{if} \hat{\xi}|_0 (\hat{G}; \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \mathbf{skip}|_\emptyset \xrightarrow{\xi^+ \uparrow r, 1 \uparrow r} \mathbf{if} \hat{\xi}|_0 (\hat{G}; \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) |_{\emptyset \cup \{r\}} \mathbf{skip}|_\emptyset$ .  
Then, by [ $\rightarrow$ 1-NWHILE],  
 $\mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset \xrightarrow{\xi^+ \uparrow r, 1 \uparrow r} \mathbf{if} \hat{\xi}|_0 (\hat{G}; \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) |_{\emptyset \cup \{r\}} \mathbf{skip}|_\emptyset$ . Then,  
 $G \xrightarrow{\xi^+ \uparrow r, 1 \uparrow r} \mathbf{if} \hat{\xi}|_0 (\hat{G}; \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) |_{\emptyset \cup \{r\}} \mathbf{skip}|_\emptyset$ . Then,  
 $G' = \mathbf{if} \hat{\xi}|_0 (\hat{G}; \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) |_{\emptyset \cup \{r\}} \mathbf{skip}|_\emptyset$  and  $\xi = \hat{\xi}^+ \uparrow r$  and  $\gamma = 1 \uparrow r$  and  $G \xrightarrow{\xi, \gamma} G'$ ,  
for some  $G', \xi, \gamma$ .  $\square$

*Proof (of Lem. 58).*

1. Recall  $G \xrightarrow{\xi, \gamma} G'$ . Then, by Defn. 31:
  - **Base:** [ $\rightarrow$ 1-ACT], such that  $G = \gamma$ . Recall  $\checkmark_R(G)$ . Then,  $\checkmark_R(\gamma)$ . Then, by Defn. 27:
    - **Case:**  $\gamma = q.y := e$ , for some  $q, y, e$ . Recall  $\{q\} \neq \emptyset$ . Then, by Defn. 16,  $\mathbf{subj}(q.y := e) \neq \emptyset$ . Then,  $\mathbf{subj}(\gamma) \neq \emptyset$ .
    - **Case:**  $\gamma = p.e \rightarrow q.y$ , for some  $p, q, y, e$ . Similar to case “ $\gamma = q.y := e$ , for some  $q, y, e$ ”.
  - **Base:** [ $\rightarrow$ 1-IF1], such that  $G = \hat{R}.\mathbf{if} \hat{\xi} G_1 G_2$  and  $\gamma = 1_{\mathbf{subj}(\hat{\xi})}^{\hat{R}}$ , for some  $\hat{R}, G_1, G_2, \hat{\xi}$ .
    - Recall  $\checkmark_R(G)$ . Then,  $\checkmark_R(\hat{R}.\mathbf{if} \hat{\xi} G_1 G_2)$ . Then, by Defn. 27,  $R = \mathbf{subj}(\hat{\xi})$ .
    - Recall  $\checkmark_R(G)$ . Then, by Lem. 47,  $R \neq \emptyset$ . Then,  $\mathbf{subj}(\hat{\xi}) \neq \emptyset$ . Then, by Defn. 16,  $\mathbf{subj}(1_{\mathbf{subj}(\hat{\xi})}^{\hat{R}}) \neq \emptyset$ . Then,  $\mathbf{subj}(\gamma) \neq \emptyset$ .
  - **Base:** [ $\rightarrow$ 1-IF2]. Similar to case [ $\rightarrow$ 1-IF1].
  - **Base:** [ $\rightarrow$ 1-WHILE1]. Similar to case [ $\rightarrow$ 1-IF1].
  - **Base:** [ $\rightarrow$ 1-WHILE2]. Similar to case [ $\rightarrow$ 1-IF1].
  - **Base:** [ $\rightarrow$ 1-NIF1], such that  $0 > 0$ . Then, **false**.
  - **Base:** [ $\rightarrow$ 1-NIF2], such that  $\gamma = 1 \uparrow r$ , for some  $r$ . Recall  $\{r\} \neq \emptyset$ . Then, by Defn. 16,  $\mathbf{subj}(1 \uparrow r) \neq \emptyset$ . Then,  $\mathbf{subj}(\gamma) \neq \emptyset$ .
  - **Base:** [ $\rightarrow$ 1-NIF3]. Similar to case [ $\rightarrow$ 1-NIF2].
  - **Step:** [ $\rightarrow$ 1-SEQ1], such that  $G = G_1; G_2$  and  $G_1 \xrightarrow{\xi, \gamma} G'_1$ , for some  $G_1, G'_1, G_2$ .
    - Recall  $\checkmark_R(G)$ . Then,  $\checkmark_R(G_1; G_2)$ . Then, by Defn. 27,  $\checkmark_R(G_1)$ .
    - Recall  $G_1 \xrightarrow{\xi, \gamma} G'_1$  and  $\checkmark_R(G_1)$ . Then, by induction,  $\mathbf{subj}(\gamma) \neq \emptyset$ .
  - **Step:** [ $\rightarrow$ 1-SEQ2]. Similar to case [ $\rightarrow$ 1-SEQ1].
  - **Step:** [ $\rightarrow$ 1-PAR1]. Similar to case [ $\rightarrow$ 1-SEQ1].
  - **Step:** [ $\rightarrow$ 1-PAR2]. Similar to case [ $\rightarrow$ 1-SEQ1].
  - **Step:** [ $\rightarrow$ 1-NIF4]. Similar to case [ $\rightarrow$ 1-SEQ1].
  - **Step:** [ $\rightarrow$ 1-NIF5]. Similar to case [ $\rightarrow$ 1-SEQ1].
  - **Step:** [ $\rightarrow$ 1-NWHILE], such that  $G = \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset$  and  $\mathbf{if} \hat{\xi}|_0 (\hat{G}; \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \mathbf{skip}|_\emptyset \xrightarrow{\xi, \gamma} G'$ , for some  $\hat{G}, \psi, \hat{\xi}$ .
    - Recall  $\checkmark_R(G)$ . Then,  $\checkmark_R(\mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset)$ . Then, by Lem. 46,  $\checkmark_R(\mathbf{if} \hat{\xi}|_0 (\hat{G}; \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \mathbf{skip}|_\emptyset)$ .
    - Recall  $\mathbf{if} \hat{\xi}|_0 (\hat{G}; \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \mathbf{skip}|_\emptyset \xrightarrow{\xi, \gamma} G'$  and  $\checkmark_R(\mathbf{if} \hat{\xi}|_0 (\hat{G}; \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \mathbf{skip}|_\emptyset)$ . Then, by induction,  $\mathbf{subj}(\gamma) \neq \emptyset$ .  $\square$
2. Recall  $G \xrightarrow{\xi, \gamma} G'$  and  $\checkmark_R(G)$ . Then, by 1,  $\mathbf{subj}(\gamma) \neq \emptyset$ . Then, by Defn. 16,  $\gamma \neq \tau$ .

*Proof (of Lem. 59).*

1. Recall  $G \xrightarrow{\xi, \gamma} G'$ . Then, by Defn. 31:
  - **Base:** [ $\rightarrow$ 1-ACT], such that  $G = \gamma$  and  $G' = \mathbf{skip}$  and  $\xi = \mathbf{true}$ . By [ $\rightarrow$ 1-ACT],  $(\gamma \uparrow r) \xrightarrow{\mathbf{true}, \gamma \uparrow r} \mathbf{skip}$ . Then, by Defn. 11,  $(\gamma \uparrow r) \xrightarrow{\mathbf{true} \uparrow r, \gamma \uparrow r} \mathbf{skip}$ . Then by Defn. 26,  $(\gamma \uparrow r) \xrightarrow{\mathbf{true} \uparrow r, \gamma \uparrow r} (\mathbf{skip} \uparrow r)$ . Then,  $(G \uparrow r) \xrightarrow{\xi \uparrow r, \gamma \uparrow r} (G' \uparrow r)$ .
  - **Base:** [ $\rightarrow$ 1-IF1], such that  $G = \hat{R}.\mathbf{if} \hat{\xi} G_1 G_2$  and  $G' = G_1$  and  $\xi = \hat{\xi}^+$  and  $\gamma = 1_{\mathbf{subj}(\hat{\xi})}^{\hat{R}}$ , for some  $\hat{R}, G_1, G_2, \hat{\xi}$ .
    - Recall  $\checkmark_R(G)$ . Then,  $\checkmark_R(\hat{R}.\mathbf{if} \hat{\xi} G_1 G_2)$ . Then, by Defn. 27,  $R = \mathbf{subj}(\hat{\xi})$ .

- Recall  $r \in R \cap \text{subj}(\gamma)$ . Then,  $r \in R$ . Then,  $r \in \text{subj}(\hat{\xi})$ .
  - Recall  $r \in \text{subj}(\hat{\xi})$ . Then, by Lem. 9,  $\text{subj}(\hat{\xi} \uparrow r) = \{r\}$ .
  - Recall  $r \in \text{subj}(\hat{\xi})$ . Then, by Defn. 18,  $1_{\text{subj}(\hat{\xi})}^{\hat{R}} \uparrow r = 1_{\{r\}}^{\hat{R}}$ .
  - By  $[\rightarrow 1\text{-IF1}]$ ,  $\hat{R}.\text{if } (\hat{\xi} \uparrow r) (G_1 \uparrow r) (G_2 \uparrow r) \xrightarrow{1_{\text{subj}(\hat{\xi})}^{\hat{R}} \uparrow r, 1_{\text{subj}(\hat{\xi} \uparrow r)}^{\hat{R}}} (G_1 \uparrow r)$ . Then, by Defn. 26,  $(\hat{R}.\text{if } \hat{\xi} G_1 G_2 \uparrow r) \xrightarrow{(\hat{\xi} \uparrow r)^+, 1_{\text{subj}(\hat{\xi} \uparrow r)}^{\hat{R}}} (G_1 \uparrow r)$ . Then, by Lem. 11,  $(\hat{R}.\text{if } \hat{\xi} G_1 G_2 \uparrow r) \xrightarrow{\hat{\xi}^+ \uparrow r, 1_{\text{subj}(\hat{\xi} \uparrow r)}^{\hat{R}}} (G_1 \uparrow r)$ . Then,  $(\hat{R}.\text{if } \hat{\xi} G_1 G_2 \uparrow r) \xrightarrow{\hat{\xi}^+ \uparrow r, 1_{\text{subj}(\hat{\xi} \uparrow r)}^{\hat{R}}} (G_1 \uparrow r)$ . Then,  $(\hat{R}.\text{if } \hat{\xi} G_1 G_2 \uparrow r) \xrightarrow{\hat{\xi}^+ \uparrow r, 1_{\text{subj}(\hat{\xi} \uparrow r)}^{\hat{R}}} (G_1 \uparrow r)$ . Then,  $(G \uparrow r) \xrightarrow{\xi \uparrow r, \gamma \uparrow r} (G' \uparrow r)$ .
- **Base:**  $[\rightarrow 1\text{-IF2}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
- **Base:**  $[\rightarrow 1\text{-WHILE1}]$ , such that  $G = \hat{R}.\text{while } \hat{\xi} \{\psi\} \hat{G}$  and  $G' = \hat{G}; \hat{R}.\text{while } \hat{\xi} \{\psi\} \hat{G}$  and  $\xi = \hat{\xi}^+$  and  $\gamma = 1_{\text{subj}(\hat{\xi})}^{\hat{R}}$ , for some  $\hat{R}, \hat{G}, \psi, \hat{\xi}$ .
- Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(\hat{R}.\text{while } \hat{\xi} \{\psi\} \hat{G})$ . Then, by Defn. 27,  $R = \text{subj}(\hat{\xi})$ .
  - Recall  $r \in R \cap \text{subj}(\gamma)$ . Then,  $r \in R$ . Then,  $r \in \text{subj}(\hat{\xi})$ .
  - Recall  $r \in \text{subj}(\hat{\xi})$ . Then, by Lem. 9,  $\text{subj}(\hat{\xi} \uparrow r) = \{r\}$ .
  - Recall  $r \in \text{subj}(\hat{\xi})$ . Then, by Defn. 18,  $1_{\text{subj}(\hat{\xi})}^{\hat{R}} \uparrow r = 1_{\{r\}}^{\hat{R}}$ .
  - By  $[\rightarrow 1\text{-WHILE1}]$ ,  $\hat{R}.\text{while } (\hat{\xi} \uparrow r) \{\text{true}\} (\hat{G} \uparrow r) \xrightarrow{(\hat{\xi} \uparrow r)^+, 1_{\text{subj}(\hat{\xi} \uparrow r)}^{\hat{R}}} (\hat{G} \uparrow r); \hat{R}.\text{while } (\hat{\xi} \uparrow r) \{\text{true}\} (\hat{G} \uparrow r)$ . Then, by Defn. 26,  $(\hat{R}.\text{while } \hat{\xi} \{\psi\} \hat{G} \uparrow r) \xrightarrow{(\hat{\xi} \uparrow r)^+, 1_{\text{subj}(\hat{\xi} \uparrow r)}^{\hat{R}}} (\hat{G} \uparrow r); (\hat{R}.\text{while } \hat{\xi} \{\psi\} \hat{G} \uparrow r)$ . Then, by Lem. 11,  $(\hat{R}.\text{while } \hat{\xi} \{\psi\} \hat{G} \uparrow r) \xrightarrow{\hat{\xi}^+ \uparrow r, 1_{\text{subj}(\hat{\xi} \uparrow r)}^{\hat{R}}} (\hat{G} \uparrow r); (\hat{R}.\text{while } \hat{\xi} \{\psi\} \hat{G} \uparrow r)$ . Then,  $(\hat{R}.\text{while } \hat{\xi} \{\psi\} \hat{G} \uparrow r) \xrightarrow{\hat{\xi}^+ \uparrow r, 1_{\text{subj}(\hat{\xi} \uparrow r)}^{\hat{R}}} (\hat{G} \uparrow r); (\hat{R}.\text{while } \hat{\xi} \{\psi\} \hat{G} \uparrow r)$ . Then,  $(\hat{R}.\text{while } \hat{\xi} \{\psi\} \hat{G} \uparrow r) \xrightarrow{\hat{\xi}^+ \uparrow r, 1_{\text{subj}(\hat{\xi} \uparrow r)}^{\hat{R}}} (\hat{G} \uparrow r); (\hat{R}.\text{while } \hat{\xi} \{\psi\} \hat{G} \uparrow r)$ . Then, by Defn. 26,  $(\hat{R}.\text{while } \hat{\xi} \{\psi\} \hat{G} \uparrow r) \xrightarrow{\hat{\xi}^+ \uparrow r, 1_{\text{subj}(\hat{\xi} \uparrow r)}^{\hat{R}}} (\hat{G}; \hat{R}.\text{while } \hat{\xi} \{\psi\} \hat{G} \uparrow r)$ . Then,  $(G \uparrow r) \xrightarrow{\xi \uparrow r, \gamma \uparrow r} (G' \uparrow r)$ .
- **Base:**  $[\rightarrow 1\text{-WHILE2}]$ . Similar to case  $[\rightarrow 1\text{-WHILE1}]$ .
- **Base:**  $[\rightarrow 1\text{-NIF1}]$ , such that  $0 > 0$ . Then, **false**.
- **Base:**  $[\rightarrow 1\text{-NIF2}]$ , such that  $G = \text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2}$  and  $G' = \text{if } \hat{\xi}|_0 G_1|_{R_1 \cup \{\hat{r}\}} G_2|_{R_2}$  and  $\xi = \hat{\xi}^+ \uparrow \hat{r}$  and  $\gamma = 1_{\{\hat{r}\}}^{\hat{R}}$  and  $\hat{r} \in \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$ , for some  $R_1, R_2, G_1, G_2, \hat{r}, \hat{\xi}$ .
- Recall  $\hat{r} \in \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$ . Then,  $\hat{r} \in \text{subj}(\hat{\xi})$  and  $\hat{r} \notin R_1 \cup R_2$ .
  - Recall  $\hat{r} \in \text{subj}(\hat{\xi})$ . Then, by Lem. 9,  $\text{subj}(\hat{\xi} \uparrow \hat{r}) = \{\hat{r}\}$ .
  - Recall  $\hat{r} \in \{\hat{r}\}$ . Then,  $\hat{r} \in \text{subj}(\hat{\xi} \uparrow \hat{r})$ .
  - Recall  $\hat{r} \notin R_1 \cup R_2$ . Then,  $\hat{r} \notin (R_1 \cup R_2) \cap \{\hat{r}\}$ . Then,  $\hat{r} \notin ((R_1 \cap \{\hat{r}\}) \cup (R_2 \cap \{\hat{r}\}))$ .
  - Recall  $\hat{r} \in \{\hat{r}\}$ . Then, by Defn. 18,  $1_{\{\hat{r}\}}^{\hat{R}} \uparrow \hat{r} = 1_{\{\hat{r}\}}^{\hat{R}}$ .
  - Recall  $r \in R \cap \text{subj}(\gamma)$ . Then,  $r \in R \cap \text{subj}(1_{\{\hat{r}\}}^{\hat{R}})$ . Then,  $r \in \text{subj}(1_{\{\hat{r}\}}^{\hat{R}})$ . Then, by Defn. 16,  $r \in \{\hat{r}\}$ . Then,  $r = \hat{r}$ .
  - Recall  $\hat{r} \in \text{subj}(\hat{\xi} \uparrow \hat{r})$  and  $\hat{r} \notin ((R_1 \cap \{\hat{r}\}) \cup (R_2 \cap \{\hat{r}\}))$ . Then,  $\hat{r} \in \text{subj}(\hat{\xi} \uparrow \hat{r}) \setminus ((R_1 \cap \{\hat{r}\}) \cup (R_2 \cap \{\hat{r}\}))$ . Then, by  $[\rightarrow 1\text{-NIF2}]$ ,  $(\hat{\xi} \uparrow \hat{r})^+ \uparrow \hat{r}, 1_{\{\hat{r}\}}^{\hat{R}}$   
 $\text{if } (\hat{\xi} \uparrow \hat{r})|_{\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{\hat{r}\})} (G_1 \uparrow \hat{r})|_{R_1 \cap \{\hat{r}\}} (G_2 \uparrow \hat{r})|_{R_2 \cap \{\hat{r}\}} \xrightarrow{(\hat{\xi} \uparrow \hat{r})^+ \uparrow \hat{r}, 1_{\{\hat{r}\}}^{\hat{R}}}$   
 $\text{if } (\hat{\xi} \uparrow \hat{r})|_{\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{\hat{r}\})} (G_1 \uparrow \hat{r})|_{(R_1 \cap \{\hat{r}\}) \cup \{\hat{r}\}} (G_2 \uparrow \hat{r})|_{R_2 \cap \{\hat{r}\}} \xrightarrow{\text{Then,}} \text{Then,}$   
 $\text{if } (\hat{\xi} \uparrow \hat{r})|_{\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{\hat{r}\})} (G_1 \uparrow \hat{r})|_{R_1 \cap \{\hat{r}\}} (G_2 \uparrow \hat{r})|_{R_2 \cap \{\hat{r}\}} \xrightarrow{(\hat{\xi} \uparrow \hat{r})^+ \uparrow \hat{r}, 1_{\{\hat{r}\}}^{\hat{R}}}$   
 $\text{if } (\hat{\xi} \uparrow \hat{r})|_{\text{subj}(\hat{\xi}) \setminus (R_1 \cup \{\hat{r}\} \cup R_2 \cup \{\hat{r}\})} (G_1 \uparrow \hat{r})|_{(R_1 \cap \{\hat{r}\}) \cup \{\hat{r}\}} (G_2 \uparrow \hat{r})|_{R_2 \cap \{\hat{r}\}} \xrightarrow{\text{Then,}} \text{Then,}$   
 $\text{if } (\hat{\xi} \uparrow \hat{r})|_{\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{\hat{r}\})} (G_1 \uparrow \hat{r})|_{R_1 \cap \{\hat{r}\}} (G_2 \uparrow \hat{r})|_{R_2 \cap \{\hat{r}\}} \xrightarrow{(\hat{\xi} \uparrow \hat{r})^+ \uparrow \hat{r}, 1_{\{\hat{r}\}}^{\hat{R}}}$   
 $\text{if } (\hat{\xi} \uparrow \hat{r})|_{\text{subj}(\hat{\xi}) \setminus (R_1 \cup \{\hat{r}\} \cup R_2 \cup \{\hat{r}\})} (G_1 \uparrow \hat{r})|_{(R_1 \cup \{\hat{r}\}) \cap (R_2 \cup \{\hat{r}\})} (G_2 \uparrow \hat{r})|_{R_2 \cap \{\hat{r}\}} \xrightarrow{\text{Then,}} \text{Then,}$   
 $\text{if } (\hat{\xi} \uparrow \hat{r})|_{\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{\hat{r}\})} (G_1 \uparrow \hat{r})|_{R_1 \cap \{\hat{r}\}} (G_2 \uparrow \hat{r})|_{R_2 \cap \{\hat{r}\}} \xrightarrow{(\hat{\xi} \uparrow \hat{r})^+ \uparrow \hat{r}, 1_{\{\hat{r}\}}^{\hat{R}}}$   
 $\text{if } (\hat{\xi} \uparrow \hat{r})|_{\text{subj}(\hat{\xi}) \setminus (R_1 \cup \{\hat{r}\} \cup R_2 \cup \{\hat{r}\})} (G_1 \uparrow \hat{r})|_{(R_1 \cap \{\hat{r}\}) \cap (R_2 \cap \{\hat{r}\})} (G_2 \uparrow \hat{r})|_{R_2 \cap \{\hat{r}\}} \xrightarrow{\text{Then,}} \text{Then, by}$   
 $\text{Defn. 26, } (\text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2} \uparrow \hat{r}) \xrightarrow{(\hat{\xi} \uparrow \hat{r})^+ \uparrow \hat{r}, 1_{\{\hat{r}\}}^{\hat{R}}} (\text{if } \hat{\xi}|_0 G_1|_{R_1 \cup \{\hat{r}\}} G_2|_{R_2} \uparrow \hat{r})$ . Then, by  
 $\text{Lem. 11, } (\text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2} \uparrow \hat{r}) \xrightarrow{(\hat{\xi} \uparrow \hat{r})^+ \uparrow \hat{r}, 1_{\{\hat{r}\}}^{\hat{R}}} (\text{if } \hat{\xi}|_0 G_1|_{R_1 \cup \{\hat{r}\}} G_2|_{R_2} \uparrow \hat{r})$ . Then,  
 $(\text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2} \uparrow \hat{r}) \xrightarrow{(\hat{\xi} \uparrow \hat{r})^+ \uparrow \hat{r}, 1_{\{\hat{r}\}}^{\hat{R}}} (\text{if } \hat{\xi}|_0 G_1|_{R_1 \cup \{\hat{r}\}} G_2|_{R_2} \uparrow \hat{r})$ . Then,  
 $(G \uparrow r) \xrightarrow{\xi \uparrow r, \gamma \uparrow r} (G' \uparrow r)$ .
- **Base:**  $[\rightarrow 1\text{-NIF3}]$ . Similar to case  $[\rightarrow 1\text{-NIF2}]$ .
- **Step:**  $[\rightarrow 1\text{-SEQ1}]$ , such that  $G = G_1; G_2$  and  $G' = G'_1; G_2$  and  $G_1 \xrightarrow{\xi, \gamma} G'_1$ , for some  $G_1, G'_1, G_2$ .
- Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(G_1; G_2)$ . Then, by Defn. 27,  $\sqrt{R}(G_1)$ .
  - Recall  $G_1 \xrightarrow{\xi, \gamma} G'_1$  and  $\sqrt{R}(G_1)$  and  $r \in R \cap \text{subj}(\gamma)$ . Then, by induction,  $(G_1 \uparrow r) \xrightarrow{\xi \uparrow r, \gamma \uparrow r} (G'_1 \uparrow r)$ . Then, by  $[\rightarrow 1\text{-SEQ1}]$ ,

- $(G_1 \upharpoonright r); (G_2 \upharpoonright r) \xrightarrow{\xi \upharpoonright r, \gamma \upharpoonright r} (G'_1 \upharpoonright r); (G_2 \upharpoonright r)$ . Then, by Defn. 26,  
 $(G_1; G_2 \upharpoonright r) \xrightarrow{\xi \upharpoonright r, \gamma \upharpoonright r} (G'_1; G_2 \upharpoonright r)$ . Then,  $(G \upharpoonright r) \xrightarrow{\xi \upharpoonright r, \gamma \upharpoonright r} (G' \upharpoonright r)$ .
- **Step:** [ $\rightarrow$ 1-SEQ2], such that  $G = G_1; G_2$  and  $G' = G_1; G'_2$  and  $\text{subj}(G_1) \cap (\text{subj}(\xi) \cup \text{subj}(\gamma)) = \emptyset$  and  $G_2 \xrightarrow{\xi, \gamma} G'_2$ , for some  $G_1, G_2, G'_2$ .
    - By Lem. 43,  $\text{subj}(G_1 \upharpoonright r) \subseteq \text{subj}(G_1)$  and  $\text{subj}(\xi \upharpoonright r) \subseteq \text{subj}(\xi)$  and  $\text{subj}(\gamma \upharpoonright r) \subseteq \text{subj}(\gamma)$ .
    - Recall  $\text{subj}(\xi \upharpoonright r) \subseteq \text{subj}(\xi)$  and  $\text{subj}(\gamma \upharpoonright r) \subseteq \text{subj}(\gamma)$ . Then,  $\text{subj}(\xi \upharpoonright r) \cup \text{subj}(\gamma \upharpoonright r) \subseteq \text{subj}(\xi) \cup \text{subj}(\gamma)$ .
    - Recall  $\text{subj}(G_1) \cap (\text{subj}(\xi) \cup \text{subj}(\gamma)) = \emptyset$  and  $\text{subj}(G_1 \upharpoonright r) \subseteq \text{subj}(G_1)$  and  $\text{subj}(\xi \upharpoonright r) \cup \text{subj}(\gamma \upharpoonright r) \subseteq \text{subj}(\xi) \cup \text{subj}(\gamma)$ . Then,  $\text{subj}(G_1 \upharpoonright r) \cap (\text{subj}(\xi \upharpoonright r) \cup \text{subj}(\gamma \upharpoonright r)) = \emptyset$ .
    - Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(G_1; G_2)$ . Then, by Defn. 27,  $\sqrt{R}(G_2)$ .
    - Recall  $G_2 \xrightarrow{\xi, \gamma} G'_2$  and  $\sqrt{R}(G_2)$  and  $r \in R \cap \text{subj}(\gamma)$ . Then, by induction,  $(G_2 \upharpoonright r) \xrightarrow{\xi \upharpoonright r, \gamma \upharpoonright r} (G'_2 \upharpoonright r)$ .
    - Recall  $\text{subj}(G_1 \upharpoonright r) \cap (\text{subj}(\xi \upharpoonright r) \cup \text{subj}(\gamma \upharpoonright r)) = \emptyset$  and  $(G_2 \upharpoonright r) \xrightarrow{\xi \upharpoonright r, \gamma \upharpoonright r} (G'_2 \upharpoonright r)$ . Then, by [ $\rightarrow$ 1-SEQ2],  $(G_1 \upharpoonright r); (G_2 \upharpoonright r) \xrightarrow{\xi \upharpoonright r, \gamma \upharpoonright r} (G_1 \upharpoonright r); (G'_2 \upharpoonright r)$ . Then, by Defn. 26,  $(G_1; G_2 \upharpoonright r) \xrightarrow{\xi \upharpoonright r, \gamma \upharpoonright r} (G_1; G'_2 \upharpoonright r)$ . Then,  $(G \upharpoonright r) \xrightarrow{\xi \upharpoonright r, \gamma \upharpoonright r} (G' \upharpoonright r)$ .
  - **Step:** [ $\rightarrow$ 1-PAR1]. Similar to case [ $\rightarrow$ 1-SEQ1].
  - **Step:** [ $\rightarrow$ 1-PAR2]. Similar to case [ $\rightarrow$ 1-SEQ1].
  - **Step:** [ $\rightarrow$ 1-NIF4], such that  $G = \text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2}$  and  $G' = \text{if } \hat{\xi}|_0 G'_1|_{R_1} G_2|_{R_2}$  and  $G_1 \xrightarrow{\xi, \gamma} G'_1$  and  $\text{subj}(\xi) \cup \text{subj}(\gamma) \subseteq R_1 \setminus R_2$ , for some  $R_1, R_2, G_1, G'_1, G_2, \hat{\xi}$ .
    - Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(\text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2})$ . Then, by Defn. 27,  $\sqrt{R}(G_1)$ .
    - Recall  $G_1 \xrightarrow{\xi, \gamma} G'_1$  and  $\sqrt{R}(G_1)$  and  $r \in R \cap \text{subj}(\gamma)$ . Then, by induction,  $(G_1 \upharpoonright r) \xrightarrow{\xi \upharpoonright r, \gamma \upharpoonright r} (G'_1 \upharpoonright r)$ .
    - By Lem. 10,  $\text{subj}(\xi \upharpoonright r) \subseteq \text{subj}(\xi)$ .
    - By Lem. 33,  $\text{subj}(\gamma \upharpoonright r) \subseteq \text{subj}(\gamma)$ .
    - Recall  $\text{subj}(\xi \upharpoonright r) \subseteq \text{subj}(\xi)$  and  $\text{subj}(\gamma \upharpoonright r) \subseteq \text{subj}(\gamma)$ . Then,  $\text{subj}(\xi \upharpoonright r) \cup \text{subj}(\gamma \upharpoonright r) \subseteq \text{subj}(\xi) \cup \text{subj}(\gamma)$ .
    - Recall  $r \in R \cap \text{subj}(\gamma)$ . Then,  $r \in \text{subj}(\gamma)$ . Then, by Lem. 32,  $\text{subj}(\gamma \upharpoonright r) = \{r\}$ .
    - Recall  $(G_1 \upharpoonright r) \xrightarrow{\xi \upharpoonright r, \gamma \upharpoonright r} (G'_1 \upharpoonright r)$ . Then, by Lem. 53,  $\text{subj}(\xi \upharpoonright r) \subseteq \text{subj}(\gamma \upharpoonright r)$ . Then,  $\text{subj}(\xi \upharpoonright r) \subseteq \{r\}$ . Then,  $\text{subj}(\xi \upharpoonright r) \cap \{r\} = \text{subj}(\xi \upharpoonright r)$ .
    - Recall  $\text{subj}(\xi \upharpoonright r) \cup \text{subj}(\gamma \upharpoonright r) \subseteq \text{subj}(\xi) \cup \text{subj}(\gamma) \subseteq R_1 \setminus R_2$ . Then,  $\text{subj}(\xi \upharpoonright r) \cup \text{subj}(\gamma \upharpoonright r) \subseteq R_1 \setminus R_2$ . Then,  $(\text{subj}(\xi \upharpoonright r) \cup \text{subj}(\gamma \upharpoonright r)) \cap \{r\} \subseteq (R_1 \setminus R_2) \cap \{r\}$ . Then,  $(\text{subj}(\xi \upharpoonright r) \cap \{r\}) \cup (\text{subj}(\gamma \upharpoonright r) \cap \{r\}) \subseteq (R_1 \cap \{r\}) \setminus (R_2 \cap \{r\})$ . Then,  $\text{subj}(\xi \upharpoonright r) \cup (\text{subj}(\gamma \upharpoonright r) \cap \{r\}) \subseteq (R_1 \cap \{r\}) \setminus (R_2 \cap \{r\})$ . Then,  $\text{subj}(\xi \upharpoonright r) \cup (\text{subj}(\gamma \upharpoonright r) \cap \text{subj}(\gamma \upharpoonright r)) \subseteq (R_1 \cap \{r\}) \setminus (R_2 \cap \{r\})$ . Then,  $\text{subj}(\xi \upharpoonright r) \cup \text{subj}(\gamma \upharpoonright r) \subseteq (R_1 \cap \{r\}) \setminus (R_2 \cap \{r\})$ .
    - Recall  $(G_1 \upharpoonright r) \xrightarrow{\xi \upharpoonright r, \gamma \upharpoonright r} (G'_1 \upharpoonright r)$  and  $\text{subj}(\xi \upharpoonright r) \cup \text{subj}(\gamma \upharpoonright r) \subseteq (R_1 \cap \{r\}) \setminus (R_2 \cap \{r\})$ . Then, by [ $\rightarrow$ 1-NIF4],  $\text{if } (\hat{\xi} \upharpoonright r)|_{\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})} (G_1 \upharpoonright r)|_{R_1 \cap \{r\}} (G_2 \upharpoonright r)|_{R_2 \cap \{r\}} \xrightarrow{\xi \upharpoonright r, \gamma \upharpoonright r} \text{if } (\hat{\xi} \upharpoonright r)|_{\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})} (G'_1 \upharpoonright r)|_{R_1 \cap \{r\}} (G_2 \upharpoonright r)|_{R_2 \cap \{r\}}$ . Then, by Defn. 26,  $(\text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2} \upharpoonright r) \xrightarrow{\xi \upharpoonright r, \gamma \upharpoonright r} (\text{if } \hat{\xi}|_0 G'_1|_{R_1} G_2|_{R_2} \upharpoonright r)$ . Then,  $(G \upharpoonright r) \xrightarrow{\xi \upharpoonright r, \gamma \upharpoonright r} (G' \upharpoonright r)$ .
  - **Step:** [ $\rightarrow$ 1-NIF5]. Similar to case [ $\rightarrow$ 1-NIF4].
  - **Step:** [ $\rightarrow$ 1-NWHILE], such that  $G = \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset$  and  $\text{if } \hat{\xi}|_0 (\hat{G}; \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \text{ skip}|_\emptyset \xrightarrow{\xi, \gamma} G'$ , for some  $\hat{G}, \psi, \hat{\xi}$ .
    - Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(\text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset)$ . Then, by Lem. 46,  $\sqrt{R}(\text{if } \hat{\xi}|_0 (\hat{G}; \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \text{ skip}|_\emptyset)$ .
    - Recall  $\text{if } \hat{\xi}|_0 (\hat{G}; \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \text{ skip}|_\emptyset \xrightarrow{\xi, \gamma} G'$  and  $\sqrt{R}(\text{if } \hat{\xi}|_0 (\hat{G}; \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \text{ skip}|_\emptyset)$  and  $r \in R \cap \text{subj}(\gamma)$ . Then, by induction,  $(\text{if } \hat{\xi}|_0 (\hat{G}; \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \text{ skip}|_\emptyset \upharpoonright r) \xrightarrow{\xi \upharpoonright r, \gamma \upharpoonright r} (G' \upharpoonright r)$ . Then, by Defn. 26,  $\text{if } (\hat{\xi} \upharpoonright r)|_{\text{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset \cup \{r\})} (\hat{G}; \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset \upharpoonright r)|_{\emptyset \cap \{r\}} (\text{skip} \upharpoonright r)|_{\emptyset \cap \{r\}} \xrightarrow{\xi \upharpoonright r, \gamma \upharpoonright r} (\text{if } \hat{\xi} \upharpoonright r)|_{\text{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset \cup \{r\})} ((\hat{G} \upharpoonright r); (\text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset \upharpoonright r))|_{\emptyset \cap \{r\}} \text{skip}|_{\emptyset \cap \{r\}} \xrightarrow{\xi \upharpoonright r, \gamma \upharpoonright r}$

$(G' \uparrow r)$ . Then, by Defn. 26,  $\text{if } (\hat{\xi} \uparrow r)|_{|\text{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset \cup \{r\})|} ((\hat{G} \uparrow r); \text{while } (\hat{\xi} \uparrow r)|_{|\text{subj}(\hat{\xi}) \setminus \{r\}|} \{\text{true}\} (G' \uparrow r)|_{\emptyset})|_{\emptyset \cap \{r\}} \text{skip}|_{\emptyset \cap \{r\}} \xrightarrow{\xi \uparrow r, \gamma \uparrow r} (G' \uparrow r)$ . Then,  $\text{if } (\hat{\xi} \uparrow r)|_{|\text{subj}(\hat{\xi}) \setminus \{r\}|} ((\hat{G} \uparrow r); \text{while } (\hat{\xi} \uparrow r)|_{|\text{subj}(\hat{\xi}) \setminus \{r\}|} \{\text{true}\} (G' \uparrow r)|_{\emptyset})|_{\emptyset} \text{skip}|_{\emptyset} \xrightarrow{\xi \uparrow r, \gamma \uparrow r} (G' \uparrow r)$ . Then, by  $[\rightarrow 1\text{-NWHILE}]$ ,  $\text{while } (\hat{\xi} \uparrow r)|_{|\text{subj}(\hat{\xi}) \setminus \{r\}|} \{\text{true}\} (G' \uparrow r)|_{\emptyset} \xrightarrow{\xi \uparrow r, \gamma \uparrow r} (G' \uparrow r)$ . Then, by Defn. 26,  $(\text{while } \hat{\xi}|_{\emptyset} \{\psi\} G|_{\emptyset} \uparrow r) \xrightarrow{\xi \uparrow r, \gamma \uparrow r} (G' \uparrow r)$ . Then,  $(G \uparrow r) \xrightarrow{\xi \uparrow r, \gamma \uparrow r} (G' \uparrow r)$ .  $\square$

2. Recall  $G \xrightarrow{\xi, \gamma} G'$ . Then, by Defn. 31:

- **Base:**  $[\rightarrow 1\text{-ACT}]$ , such that  $G = \gamma$  and  $G' = \text{skip}$  and  $\xi = \text{true}$ .
  - Recall  $r \in R \setminus \text{subj}(\gamma)$ . Then,  $r \notin \text{subj}(\gamma)$ . Then, by Lem. 34,  $\gamma \uparrow r = \tau$ .
  - By  $[\rightarrow 1\text{-ACT}]$ ,  $(\gamma \uparrow r) \xrightarrow{\text{true}, \gamma \uparrow r} \text{skip}$ . Then,  $(\gamma \uparrow r) \xrightarrow{\text{true}, \tau} \text{skip}$ . Then by Defn. 26,  $(\gamma \uparrow r) \xrightarrow{\text{true}, \tau} (\text{skip} \uparrow r)$ . Then,  $(G \uparrow r) \xrightarrow{\text{true}, \tau} (G' \uparrow r)$ .
- **Base:**  $[\rightarrow 1\text{-IF1}]$ , such that  $G = \hat{R}.\text{if } \hat{\xi} G_1 G_2$ , for some  $\hat{R}, G_1, G_2, \hat{\xi}$ .
  - Recall  $\sqrt{R}(G)$ . Then,  $\hat{R}.\text{if } \hat{\xi} G_1 G_2$ . Then, by Defn. 27,  $R = \text{subj}(\hat{\xi})$ .
  - Recall  $r \in R \setminus \text{subj}(\gamma)$ . Then,  $r \in R \setminus \text{subj}(1_{\text{subj}(\hat{\xi})}^{\hat{R}})$ . Then, by Defn. 16,  $r \in R \setminus \text{subj}(\hat{\xi})$ . Then,  $r \in R \setminus R$ . Then,  $r \in \emptyset$ . Then, **false**.
- **Base:**  $[\rightarrow 1\text{-IF2}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
- **Base:**  $[\rightarrow 1\text{-WHILE1}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
- **Base:**  $[\rightarrow 1\text{-WHILE2}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
- **Base:**  $[\rightarrow 1\text{-NIF1}]$ , such that  $0 > 0$ . Then, **false**.
- **Base:**  $[\rightarrow 1\text{-NIF2}]$ , such that  $G = \text{if } \hat{\xi}|_{\emptyset} G_1|_{R_1} G_2|_{R_2}$  and  $G' = \text{if } \hat{\xi}|_{\emptyset} G_1|_{R_1 \cup \{\hat{r}\}} G_2|_{R_2}$  and  $\xi = \hat{\xi}^+ \uparrow \hat{r}$  and  $\gamma = 1_{\{\hat{r}\}}$  and  $\hat{r} \in \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$ , for some  $R_1, R_2, G_1, G_2, \hat{r}, \hat{\xi}$ .
  - Recall  $\hat{r} \in \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$ . Then,  $\hat{r} \in \text{subj}(\hat{\xi})$  and  $\hat{r} \notin R_1 \cup R_2$ .
  - Recall  $r \in R \setminus \text{subj}(\gamma)$ . Then,  $r \in R \setminus \text{subj}(1_{\{\hat{r}\}})$ . Then,  $r \notin \text{subj}(1_{\{\hat{r}\}})$ . Then, by Defn. 16,  $r \notin \{\hat{r}\}$ . Then,  $r \neq \hat{r}$ . Then,  $\hat{r} \notin \{r\}$  and  $\{r\} \cap \{\hat{r}\} = \emptyset$ .
  - Recall  $\hat{r} \notin R_1 \cup R_2$  and  $\hat{r} \notin \{r\}$ . Then,  $\hat{r} \notin R_1 \cup R_2 \cup \{r\}$ .
  - Recall  $\hat{r} \in \text{subj}(\hat{\xi})$  and  $\hat{r} \notin R_1 \cup R_2 \cup \{r\}$ . Then,  $\hat{r} \in \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})$ . Then,  $|\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})| = |\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})| - 1$ . Then,  $|\text{subj}(\hat{\xi}) \setminus (R_1 \cup \{\hat{r}\} \cup R_2 \cup \{r\})| = |\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})| - 1$ .
  - Recall  $\hat{r} \in \text{subj}(\hat{\xi})$  and  $\hat{r} \notin R_1 \cup R_2 \cup \{\hat{r}\}$ . Then,  $\hat{r} \in \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})$ . Then,  $\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\}) \neq \emptyset$ . Then,  $|\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})| > 0$ . Then, by  $[\rightarrow 1\text{-NIF1}]$ ,  $\text{if } (\hat{\xi} \uparrow r)|_{|\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})|} (G_1 \uparrow r)|_{R_1 \cap \{r\}} (G_2 \uparrow r)|_{R_2 \cap \{r\}} \xrightarrow{\text{true}, \tau} \text{if } (\hat{\xi} \uparrow r)|_{|\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})|} (G_1 \uparrow r)|_{R_1 \cap \{r\}} (G_2 \uparrow r)|_{R_2 \cap \{r\}}$ . Then,  $\text{if } (\hat{\xi} \uparrow r)|_{|\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})|} (G_1 \uparrow r)|_{R_1 \cap \{r\}} (G_2 \uparrow r)|_{R_2 \cap \{r\}} \xrightarrow{\text{true}, \tau} \text{if } (\hat{\xi} \uparrow r)|_{|\text{subj}(\hat{\xi}) \setminus (R_1 \cup \{\hat{r}\} \cup R_2 \cup \{r\})|} (G_1 \uparrow r)|_{R_1 \cap \{r\}} (G_2 \uparrow r)|_{R_2 \cap \{r\}}$ . Then,  $\text{if } (\hat{\xi} \uparrow r)|_{|\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})|} (G_1 \uparrow r)|_{R_1 \cap \{r\}} (G_2 \uparrow r)|_{R_2 \cap \{r\}} \xrightarrow{\text{true}, \tau} \text{if } (\hat{\xi} \uparrow r)|_{|\text{subj}(\hat{\xi}) \setminus (R_1 \cup \{\hat{r}\} \cup R_2 \cup \{r\})|} (G_1 \uparrow r)|_{R_1 \cap \{r\}} (G_2 \uparrow r)|_{R_2 \cap \{r\}}$ . Then,  $\text{if } (\hat{\xi} \uparrow r)|_{|\text{subj}(\hat{\xi}) \setminus (R_1 \cup \{\hat{r}\} \cup R_2 \cup \{r\})|} (G_1 \uparrow r)|_{R_1 \cap \{r\}} (G_2 \uparrow r)|_{R_2 \cap \{r\}} \xrightarrow{\text{true}, \tau} \text{if } (\hat{\xi} \uparrow r)|_{|\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})|} (G_1 \uparrow r)|_{R_1 \cap \{r\}} (G_2 \uparrow r)|_{R_2 \cap \{r\}}$ . Then,  $\text{if } (\hat{\xi} \uparrow r)|_{|\text{subj}(\hat{\xi}) \setminus (R_1 \cup \{\hat{r}\} \cup R_2 \cup \{r\})|} (G_1 \uparrow r)|_{R_1 \cap \{r\}} (G_2 \uparrow r)|_{R_2 \cap \{r\}} \xrightarrow{\text{true}, \tau} \text{if } (\hat{\xi} \uparrow r)|_{|\text{subj}(\hat{\xi}) \setminus (R_1 \cup \{\hat{r}\} \cup R_2 \cup \{r\})|} (G_1 \uparrow r)|_{R_1 \cap \{r\}} (G_2 \uparrow r)|_{R_2 \cap \{r\}}$ . Then, by Defn. 26,  $(\text{if } \hat{\xi}|_{\emptyset} G_1|_{R_1} G_2|_{R_2} \uparrow r) \xrightarrow{\text{true}, \tau} (\text{if } \hat{\xi}|_{\emptyset} G_1|_{R_1 \cup \{\hat{r}\}} G_2|_{R_2} \uparrow r)$ . Then,  $(G \uparrow r) \xrightarrow{\text{true}, \tau} (G' \uparrow r)$ .
- **Base:**  $[\rightarrow 1\text{-NIF3}]$ . Similar to case  $[\rightarrow 1\text{-NIF2}]$ .
- **Step:**  $[\rightarrow 1\text{-SEQ1}]$ , such that  $G = G_1; G_2$  and  $G' = G'_1; G'_2$  and  $G_1 \xrightarrow{\xi, \gamma} G'_1$ , for some  $G_1, G'_1, G_2$ .
  - Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(G_1; G_2)$ . Then, by Defn. 27,  $\sqrt{R}(G_1)$ .
  - Recall  $G_1 \xrightarrow{\xi, \gamma} G'_1$  and  $\sqrt{R}(G_1)$  and  $r \in R \setminus \text{subj}(\gamma)$ . Then, by induction,  $(G_1 \uparrow r) \xrightarrow{\text{true}, \tau} (G'_1 \uparrow r)$ . Then, by  $[\rightarrow 1\text{-SEQ1}]$ ,  $(G_1 \uparrow r); (G_2 \uparrow r) \xrightarrow{\text{true}, \tau} (G'_1 \uparrow r); (G_2 \uparrow r)$ . Then, by Defn. 26,  $(G_1; G_2 \uparrow r) \xrightarrow{\text{true}, \tau} (G'_1; G_2 \uparrow r)$ . Then,  $(G \uparrow r) \xrightarrow{\text{true}, \tau} (G' \uparrow r)$ .
- **Step:**  $[\rightarrow 1\text{-SEQ2}]$ , such that  $G = G_1; G_2$  and  $G' = G_1; G'_2$  and  $G_2 \xrightarrow{\xi, \gamma} G'_2$ , for some  $G_1, G_2, G'_2$ .
  - Recall  $\text{subj}(G_1 \uparrow r) \cap (\emptyset \cup \emptyset) = \emptyset$ . Then, by Lem. 4,  $\text{subj}(G_1 \uparrow r) \cap (\text{subj}(\text{true}) \cup \emptyset) = \emptyset$ . Then, by Defn. 16,  $\text{subj}(G_1 \uparrow r) \cap (\text{subj}(\text{true}) \cup \text{subj}(\tau)) = \emptyset$ .

- Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(G_1; G_2)$ . Then, by Defn. 27,  $\sqrt{R}(G_2)$ .
  - Recall  $G_2 \xrightarrow{\xi, \gamma} G'_2$  and  $\sqrt{R}(G_2)$  and  $r \in R \setminus \text{subj}(\gamma)$ . Then, by induction,  $(G_2 \uparrow r) \xrightarrow{\text{true}, \tau} (G'_2 \uparrow r)$ .
  - Recall  $\text{subj}(G_1 \uparrow r) \cap (\text{subj}(\text{true}) \cup \text{subj}(\tau)) = \emptyset$  and  $(G_2 \uparrow r) \xrightarrow{\text{true}, \tau} (G'_2 \uparrow r)$ . Then, by  $[\rightarrow 1\text{-SEQ2}]$ ,  $(G_1 \uparrow r); (G_2 \uparrow r) \xrightarrow{\text{true}, \tau} (G_1 \uparrow r); (G'_2 \uparrow r)$ . Then, by Defn. 26,  $(G_1; G_2 \uparrow r) \xrightarrow{\text{true}, \tau} (G_1; G'_2 \uparrow r)$ . Then,  $(G \uparrow r) \xrightarrow{\text{true}, \tau} (G' \uparrow r)$ .
- **Step:**  $[\rightarrow 1\text{-PAR1}]$ . Similar to case  $[\rightarrow 1\text{-SEQ1}]$ .
- **Step:**  $[\rightarrow 1\text{-PAR2}]$ . Similar to case  $[\rightarrow 1\text{-SEQ1}]$ .
- **Step:**  $[\rightarrow 1\text{-NIF4}]$ , such that  $G = \mathbf{if} \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2}$  and  $G' = \mathbf{if} \hat{\xi}'|_0 G'_1|_{R_1} G'_2|_{R_2}$  and  $G_1 \xrightarrow{\xi, \gamma} G'_1$  and  $\text{subj}(\hat{\xi}) \cup \text{subj}(\gamma) \subseteq R_1 \setminus R_2$ , for some  $R_1, R_2, G_1, G'_1, G_2, \hat{\xi}$ .
- Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(\mathbf{if} \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2})$ . Then, by Defn. 27,  $\sqrt{R}(G_1)$ .
  - Recall  $G_1 \xrightarrow{\xi, \gamma} G'_1$  and  $\sqrt{R}(G_1)$  and  $r \in R \setminus \text{subj}(\gamma)$ . Then, by induction,  $(G_1 \uparrow r) \xrightarrow{\text{true}, \tau} (G'_1 \uparrow r)$ .
  - Recall  $\emptyset \cup \emptyset \subseteq (R_1 \cap \{r\}) \setminus (R_2 \cap \{r\})$ . Then, by Lem. 4,  $\text{subj}(\text{true}) \cup \emptyset \subseteq (R_1 \cap \{r\}) \setminus (R_2 \cap \{r\})$ . Then, by Defn. 16,  $\text{subj}(\text{true}) \cup \text{subj}(\tau) \subseteq (R_1 \cap \{r\}) \setminus (R_2 \cap \{r\})$ .
  - Recall  $(G_1 \uparrow r) \xrightarrow{\text{true}, \tau} (G'_1 \uparrow r)$  and  $\text{subj}(\text{true}) \cup \text{subj}(\tau) \subseteq (R_1 \cap \{r\}) \setminus (R_2 \cap \{r\})$ . Then, by  $[\rightarrow 1\text{-NIF4}]$ ,  $\mathbf{if} (\hat{\xi} \uparrow r)|_{\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})} (G_1 \uparrow r)|_{R_1 \cap \{r\}} (G_2 \uparrow r)|_{R_2 \cap \{r\}} \xrightarrow{\text{true}, \tau} \mathbf{if} (\hat{\xi} \uparrow r)|_{\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})} (G'_1 \uparrow r)|_{R_1 \cap \{r\}} (G_2 \uparrow r)|_{R_2 \cap \{r\}}$ . Then, by Defn. 26,  $(\mathbf{if} \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2} \uparrow r) \xrightarrow{\text{true}, \tau} (\mathbf{if} \hat{\xi}'|_0 G'_1|_{R_1} G'_2|_{R_2} \uparrow r)$ . Then,  $(G \uparrow r) \xrightarrow{\text{true}, \tau} (G' \uparrow r)$ .
- **Step:**  $[\rightarrow 1\text{-NIF5}]$ . Similar to case  $[\rightarrow 1\text{-NIF4}]$ .
- **Step:**  $[\rightarrow 1\text{-NWHILE}]$ , such that  $G = \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset$  and  $\mathbf{if} \hat{\xi}|_0 (\hat{G}; \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \mathbf{skip}|_\emptyset \xrightarrow{\xi, \gamma} G'$ , for some  $\hat{G}, \psi, \hat{\xi}$ .
- Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(\mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset)$ . Then, by Lem. 46,  $\sqrt{R}(\mathbf{if} \hat{\xi}|_0 (\hat{G}; \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \mathbf{skip}|_\emptyset)$ .
  - Recall  $\mathbf{if} \hat{\xi}|_0 (\hat{G}; \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \mathbf{skip}|_\emptyset \xrightarrow{\xi, \gamma} G'$  and  $\sqrt{R}(\mathbf{if} \hat{\xi}|_0 (\hat{G}; \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \mathbf{skip}|_\emptyset)$  and  $r \in R \setminus \text{subj}(\gamma)$ . Then, by induction,  $(\mathbf{if} \hat{\xi}|_0 (\hat{G}; \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \mathbf{skip}|_\emptyset \uparrow r) \xrightarrow{\text{true}, \tau} (G' \uparrow r)$ . Then, by Defn. 26,  $\mathbf{if} (\hat{\xi} \uparrow r)|_{\text{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset \cup \{r\})} (\hat{G}; \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset \uparrow r)|_{\emptyset \cap \{r\}} (\mathbf{skip} \uparrow r)|_{\emptyset \cap \{r\}} \xrightarrow{\text{true}, \tau} (G' \uparrow r)$ . Then, by Defn. 26,  $\mathbf{if} (\hat{\xi} \uparrow r)|_{\text{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset \cup \{r\})} ((\hat{G} \uparrow r); (\mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset \uparrow r))|_{\emptyset \cap \{r\}} \mathbf{skip}|_{\emptyset \cap \{r\}} \xrightarrow{\text{true}, \tau} (G' \uparrow r)$ . Then, by Defn. 26,  $\mathbf{if} (\hat{\xi} \uparrow r)|_{\text{subj}(\hat{\xi}) \setminus \{r\}} \{\text{true}\} (\hat{G} \uparrow r)|_\emptyset \mathbf{skip}|_{\emptyset \cap \{r\}} \xrightarrow{\text{true}, \tau} (G' \uparrow r)$ . Then,  $\mathbf{if} (\hat{\xi} \uparrow r)|_{\text{subj}(\hat{\xi}) \setminus \{r\}} ((\hat{G} \uparrow r); \mathbf{while} (\hat{\xi} \uparrow r)|_{\text{subj}(\hat{\xi}) \setminus \{r\}} \{\text{true}\} (\hat{G} \uparrow r)|_\emptyset) \mathbf{skip}|_\emptyset \xrightarrow{\text{true}, \tau} (G' \uparrow r)$ . Then, by  $[\rightarrow 1\text{-NWHILE}]$ ,  $\mathbf{while} (\hat{\xi} \uparrow r)|_{\text{subj}(\hat{\xi}) \setminus \{r\}} \{\text{true}\} (\hat{G} \uparrow r)|_\emptyset \xrightarrow{\text{true}, \tau} (G' \uparrow r)$ . Then, by Defn. 26,  $(\mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset \uparrow r) \xrightarrow{\text{true}, \tau} (G' \uparrow r)$ . Then,  $(G \uparrow r) \xrightarrow{\text{true}, \tau} (G' \uparrow r)$ .  $\square$

*Proof (of Lem. 60).* Recall  $(G \uparrow r) \xrightarrow{\xi_r, \lambda_r} L'_r$ . Then, by Defn. 31:

- **Base:**  $[\rightarrow 1\text{-ACT}]$ , such that  $G \uparrow r = \lambda_r$ .
  - Recall  $G \uparrow r = \lambda_r$ . Then, by Defn. 26,  $G = \gamma$ , for some  $\gamma$ .
  - Recall  $G \uparrow r = \lambda_r$ . Then,  $\gamma \uparrow r = \lambda_r$ .
  - Recall  $\text{subj}(G) \subseteq \text{subj}(G)$ . Then,  $\text{subj}(\gamma) \subseteq \text{subj}(G)$ .
- **Base:**  $[\rightarrow 1\text{-IF1}]$ , such that  $G \uparrow r = \hat{R}. \mathbf{if} \hat{\xi}_r L_{1r} L_{2r}$  and  $\lambda_r = 1_{\text{subj}(\hat{\xi}_r)}^{\hat{R}}$ , for some  $\hat{R}, \hat{\xi}_r, L_{1r}, L_{2r}$ .
  - Recall  $G \uparrow r = \hat{R}. \mathbf{if} \hat{\xi}_r L_{1r} L_{2r}$ . Then, by Defn. 26,  $G = \hat{R}. \mathbf{if} \hat{\xi} G_1 G_2$  and  $\hat{\xi}_r = \hat{\xi} \uparrow r$ , for some  $G_1, G_2, \hat{\xi}$ .
  - Recall  $\text{subj}(\hat{\xi}) \subseteq \text{subj}(\hat{\xi}) \cup \text{subj}(G_1) \cup \text{subj}(G_2)$ . Then, by Defn. 16,  $\text{subj}(1_{\text{subj}(\hat{\xi})}^{\hat{R}}) \subseteq \text{subj}(\hat{\xi}) \cup \text{subj}(G_1) \cup \text{subj}(G_2)$ . Then, by Defn. 24,  $\text{subj}(1_{\text{subj}(\hat{\xi})}^{\hat{R}}) \subseteq \text{subj}(\hat{R}. \mathbf{if} \hat{\xi} G_1 G_2)$ . Then,  $\text{subj}(1_{\text{subj}(\hat{\xi})}^{\hat{R}}) \subseteq \text{subj}(G)$ . Then,  $\gamma = 1_{\text{subj}(\hat{\xi})}^{\hat{R}}$  and  $\text{subj}(\gamma) \subseteq \text{subj}(G)$ , for some  $\gamma$ .
  - By Defn. 18:

- \* **Case:**  $1_{\text{subj}(\hat{\xi})}^{\text{subj}(\hat{\xi})} \upharpoonright r = 1_{\{r\}}^{\text{subj}(\hat{\xi})}$  and  $r \in \text{subj}(\hat{\xi})$ .
  - Recall  $r \in \text{subj}(\hat{\xi})$ . Then, by Lem. 9,  $\text{subj}(\hat{\xi} \upharpoonright r) = \{r\}$ .
  - Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(\hat{R}.\text{if } \hat{\xi} G_1 G_2)$ . Then, by Defn. 27,  $\hat{R} = \text{subj}(\hat{\xi})$ .
  - Recall  $1_{\text{subj}(\hat{\xi})}^{\text{subj}(\hat{\xi})} \upharpoonright r = 1_{\{r\}}^{\text{subj}(\hat{\xi})}$ . Then,  $1_{\text{subj}(\hat{\xi})}^{\text{subj}(\hat{\xi})} \upharpoonright r = 1_{\text{subj}(\hat{\xi} \upharpoonright r)}^{\text{subj}(\hat{\xi})}$ . Then,  $1_{\text{subj}(\hat{\xi})}^{\text{subj}(\hat{\xi})} \upharpoonright r = 1_{\text{subj}(\hat{\xi}_r)}^{\hat{R}}$ . Then,  $\gamma \upharpoonright r = \lambda_r$ .
- \* **Case:**  $r \notin \text{subj}(\hat{\xi})$ .
  - Recall  $r \notin \text{subj}(\hat{\xi})$ . Then, by Lem. 9,  $\text{subj}(\hat{\xi} \upharpoonright r) = \emptyset$ .
  - By Defn. 14,  $\lambda_r \neq 1_{\emptyset}^{\hat{R}}$ . Then,  $\lambda_r \neq 1_{\text{subj}(\hat{\xi} \upharpoonright r)}^{\hat{R}}$ . Then,  $\lambda_r \neq 1_{\text{subj}(\hat{\xi}_r)}^{\hat{R}}$ . Then, **false**.
- **Base:** [ $\rightarrow$ 1-IF2]. Similar to case [ $\rightarrow$ 1-IF1].
- **Base:** [ $\rightarrow$ 1-WHILE1]. Similar to case [ $\rightarrow$ 1-IF1].
- **Base:** [ $\rightarrow$ 1-WHILE2]. Similar to case [ $\rightarrow$ 1-IF1].
- **Base:** [ $\rightarrow$ 1-NIF1], such that  $\lambda_r = \tau$ .  
Recall  $\lambda_r \neq \tau$ . Then,  $\tau \neq \tau$ . Then, **false**.
- **Base:** [ $\rightarrow$ 1-NIF2], such that  $G \upharpoonright r = \text{if } \hat{\xi}_r \upharpoonright_n L_{1r} \upharpoonright_{R_{1r}} L_{2r} \upharpoonright_{R_{2r}}$  and  $L'_r = \text{if } \hat{\xi}_r \upharpoonright_n L_{1r} \upharpoonright_{R_{1r} \cup \{\hat{r}\}} L_{2r} \upharpoonright_{R_{2r}}$  and  $\lambda_r = 1_{\{\hat{r}\}}^{\hat{r}}$  and  $\hat{r} \in \text{subj}(\hat{\xi}_r) \setminus (R_{1r} \cup R_{2r})$ , for some  $R_{1r}, R_{2r}, L_{1r}, L_{2r}, \hat{r}, \xi_r, n$ .
  - Recall  $G \upharpoonright r = \text{if } \hat{\xi}_r \upharpoonright_n L_{1r} \upharpoonright_{R_{1r}} L_{2r} \upharpoonright_{R_{2r}}$ . Then, by Defn. 26,  $G = \text{if } \hat{\xi}|_0 G_1 \upharpoonright_{R_1} G_2 \upharpoonright_{R_2}$  and  $R_{1r} = R_1 \cap \{r\}$  and  $R_{2r} = R_2 \cap \{r\}$  and  $\hat{\xi}_r = \hat{\xi} \upharpoonright r$ , for some  $R_1, R_2, G_1, G_2, \hat{\xi}$ .
  - Recall  $\hat{r} \in \text{subj}(\hat{\xi}_r) \setminus (R_{1r} \cup R_{2r})$ . Then,  $\hat{r} \in \text{subj}(\hat{\xi} \upharpoonright r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\}))$ . Then,  $\hat{r} \in \text{subj}(\hat{\xi} \upharpoonright r)$  and  $\hat{r} \notin (R_1 \cap \{r\}) \cup (R_2 \cap \{r\})$ .
  - By Lem. 10,  $\text{subj}(\hat{\xi} \upharpoonright r) \subseteq \text{subj}(\hat{\xi})$  and  $\text{subj}(\hat{\xi} \upharpoonright r) \subseteq \{r\}$ .
  - Recall  $\hat{r} \in \text{subj}(\hat{\xi} \upharpoonright r) \subseteq \text{subj}(\hat{\xi})$ . Then,  $\hat{r} \in \text{subj}(\hat{\xi})$ .
  - Recall  $\hat{r} \in \text{subj}(\hat{\xi} \upharpoonright r) \subseteq \{r\}$ . Then,  $\hat{r} \in \{r\}$ . Then,  $\hat{r} = r$ .
  - Recall  $\hat{r} \notin (R_1 \cap \{r\}) \cup (R_2 \cap \{r\})$ . Then,  $\hat{r} \notin (R_1 \cap \{\hat{r}\}) \cup (R_2 \cap \{\hat{r}\})$ . Then,  $\hat{r} \notin R_1 \cap \{\hat{r}\}$  and  $\hat{r} \notin R_2 \cap \{\hat{r}\}$ . Then,  $\hat{r} \notin R_1$  and  $\hat{r} \notin R_2$ . Then,  $\hat{r} \notin R_1 \cup R_2$ .
  - Recall  $\hat{r} \in \text{subj}(\hat{\xi})$  and  $\hat{r} \notin R_1 \cup R_2$ . then,  $\hat{r} \in \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$ . Then,  $\{\hat{r}\} \subseteq \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$ . Then, by Defn. 16,  $\text{subj}(1_{\{\hat{r}\}}^{\hat{r}}) \subseteq \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$ . Then,  $\text{subj}(1_{\{\hat{r}\}}^{\hat{r}}) \subseteq (\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)) \cup (\text{subj}(G_1) \setminus R_2) \cup (\text{subj}(G_2) \setminus R_1)$ . Then, by Defn. 24,  $\text{subj}(1_{\{\hat{r}\}}^{\hat{r}}) \subseteq \text{subj}(\text{if } \hat{\xi}|_0 G_1 \upharpoonright_{R_1} G_2 \upharpoonright_{R_2})$ . Then,  $\text{subj}(1_{\{\hat{r}\}}^{\hat{r}}) \subseteq \text{subj}(G)$ . Then,  $\gamma = 1_{\{\hat{r}\}}^{\hat{r}}$  and  $\text{subj}(\gamma) \subseteq \text{subj}(G)$ , for some  $G$ .
  - Recall  $r \in \{r\}$ . Then,  $r \in \{\hat{r}\}$ . Then, by Defn. 18,  $1_{\{\hat{r}\}}^{\hat{r}} \upharpoonright r = 1_{\{\hat{r}\}}^{\hat{r}}$ . Then,  $\gamma \upharpoonright r = \lambda_r$ .
- **Base:** [ $\rightarrow$ 1-NIF3]. Similar to case [ $\rightarrow$ 1-NIF2].
- **Step:** [ $\rightarrow$ 1-SEQ1], such that  $G \upharpoonright r = L_{1r}; L_{2r}$  and  $L_{1r} \xrightarrow{\xi_r, \lambda_r} L'_{1r}$ , for some  $L_{1r}, L'_{1r}, L_{2r}$ .
  - Recall  $G \upharpoonright r = L_{1r}; L_{2r}$ . Then, by Defn. 26,  $G = G_1; G_2$  and  $L_{1r} = G_1 \upharpoonright r$ , for some  $G_1, G_2$ .
  - Recall  $L_{1r} \xrightarrow{\xi_r, \lambda_r} L'_{1r}$ . Then,  $(G_1 \upharpoonright r) \xrightarrow{\xi_r, \lambda_r} L'_{1r}$ .
  - Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(G_1; G_2)$ . Then, by Defn. 27,  $\sqrt{R}(G_1)$ .
  - Recall  $(G_1 \upharpoonright r) \xrightarrow{\xi_r, \lambda_r} L'_{1r}$  and  $\sqrt{R}(G_1)$  and  $\lambda_r \neq \tau$ . Then, by induction,  $\text{subj}(\gamma) \subseteq \text{subj}(G_1)$  and  $\lambda_r = \gamma \upharpoonright r$ , for some  $\gamma$ .
  - Recall  $\text{subj}(\gamma) \subseteq \text{subj}(G_1)$ . Then,  $\text{subj}(\gamma) \subseteq \text{subj}(G_1) \cup \text{subj}(G_2)$ . Then, by Defn. 24,  $\text{subj}(\gamma) \subseteq \text{subj}(G_1; G_2)$ . Then,  $\text{subj}(\gamma) \subseteq \text{subj}(G)$ .
- **Step:** [ $\rightarrow$ 1-SEQ2]. Similar to case [ $\rightarrow$ 1-SEQ1].
- **Step:** [ $\rightarrow$ 1-PAR1]. Similar to case [ $\rightarrow$ 1-SEQ1].
- **Step:** [ $\rightarrow$ 1-PAR2]. Similar to case [ $\rightarrow$ 1-SEQ1].
- **Step:** [ $\rightarrow$ 1-NIF4], such that  $G \upharpoonright r = \text{if } \hat{\xi}_r \upharpoonright_n L_{1r} \upharpoonright_{R_{1r}} L_{2r} \upharpoonright_{R_{2r}}$  and  $L'_r = \text{if } \hat{\xi}_r \upharpoonright_n L'_{1r} \upharpoonright_{R_{1r}} L_{2r} \upharpoonright_{R_{2r}}$  and  $L_{1r} \xrightarrow{\xi_r, \lambda_r} L'_{1r}$  and  $\text{subj}(\xi_r) \cup \text{subj}(\lambda_r) \subseteq R_{1r} \setminus R_{2r}$ , for some  $R_{1r}, R_{2r}, L_{1r}, L'_{1r}, L_{2r}, \xi_r, n$ .
  - Recall  $G \upharpoonright r = \text{if } \hat{\xi}_r \upharpoonright_n L_{1r} \upharpoonright_{R_{1r}} L_{2r} \upharpoonright_{R_{2r}}$ . Then, by Defn. 26,  $G = \text{if } \hat{\xi}|_0 G_1 \upharpoonright_{R_1} G_2 \upharpoonright_{R_2}$  and  $R_{1r} = R_1 \cap \{r\}$  and  $L_{1r} = G_1 \upharpoonright r$  and  $\hat{\xi}_r = \hat{\xi} \upharpoonright r$ , for some  $R_1, R_2, G_1, G_2, \hat{\xi}$ .
  - Recall  $L_{1r} \xrightarrow{\xi_r, \lambda_r} L'_{1r}$ . Then,  $(G_1 \upharpoonright r) \xrightarrow{\xi_r, \lambda_r} L'_{1r}$ .
  - Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(\text{if } \hat{\xi}|_0 G_1 \upharpoonright_{R_1} G_2 \upharpoonright_{R_2})$ . Then, by Defn. 27,  $\sqrt{R}(G_1)$ , and  $R_1 \neq \emptyset$  implies  $R_2 = \emptyset$ .
  - Recall  $(G_1 \upharpoonright r) \xrightarrow{\xi_r, \lambda_r} L'_{1r}$  and  $\sqrt{R}(G_1)$  and  $\lambda_r \neq \tau$ . Then, by induction,  $\text{subj}(\gamma) \subseteq \text{subj}(G_1)$  and  $\lambda_r = \gamma \upharpoonright r$ , for some  $\gamma$ .



- Recall  $\lambda_r \neq \tau$ . Then, by Lem. 30,  $\text{subj}(\lambda_r) \neq \emptyset$ . Then,  $\text{subj}(\xi_r) \cup \text{subj}(\lambda_r) \neq \emptyset$ .
  - Recall  $\emptyset \neq \text{subj}(\xi_r) \cup \text{subj}(\lambda_r) \subseteq R_{1r} \setminus R_{2r}$ . Then,  $\emptyset \neq R_{1r} \setminus R_{2r}$ . Then,  $\emptyset \neq R_{1r}$ . Then,  $\emptyset \neq R_1 \cap \{r\}$ . Then,  $\emptyset \neq R_1$ . Then,  $R_2 = \emptyset$ .
  - Recall  $\text{subj}(\gamma) \subseteq \text{subj}(G_1)$ . Then,  $\text{subj}(\gamma) \subseteq \text{subj}(G_1) \setminus \emptyset$ . Then,  $\text{subj}(\gamma) \subseteq \text{subj}(G_1) \setminus R_2$ . Then,  $\text{subj}(\gamma) \subseteq (\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)) \cup (\text{subj}(G_1) \setminus R_2) \cup (\text{subj}(G_2) \setminus R_1)$ . Then, by Defn. 24,  $\text{subj}(\gamma) \subseteq \text{subj}(\text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2})$ . Then,  $\text{subj}(\gamma) \subseteq \text{subj}(G)$ .
- **Step:**  $[\rightarrow 1\text{-NIF5}]$ . Similar to case  $[\rightarrow 1\text{-NIF4}]$ .
- **Step:**  $[\rightarrow 1\text{-NWHILE}]$ , such that  $G \uparrow r = \text{while } \hat{\xi}_r|_n \{\text{true}\} \hat{L}_r|_\emptyset$  and  $\text{if } \hat{\xi}_r|_n (\hat{L}_r; \text{while } \hat{\xi}_r|_n \{\text{true}\} \hat{L}_r|_\emptyset) |_\emptyset \text{skip}|_\emptyset \xrightarrow{\xi_r, \lambda_r} L'_r$ , for some  $\hat{L}_r, \hat{\xi}_r, n$ .
- Recall  $G \uparrow r = \text{while } \hat{\xi}_r|_n \{\text{true}\} \hat{L}_r|_\emptyset$ . Then, by Defn. 26,  $G = \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset$  and  $\hat{L}_r = \hat{G} \uparrow r$  and  $\hat{\xi}_r = \hat{\xi} \uparrow r$  and  $n = |\text{subj}(\hat{\xi}) \setminus \{r\}|$ , for some  $\hat{G}, \psi, \hat{\xi}$ .
  - Recall  $\text{if } \hat{\xi}_r|_n (\hat{L}_r; \text{while } \hat{\xi}_r|_n \{\text{true}\} \hat{L}_r|_\emptyset) |_\emptyset \text{skip}|_\emptyset \xrightarrow{\xi_r, \lambda_r} L'_r$ . Then,  $\text{if } (\hat{\xi} \uparrow r) |_{|\text{subj}(\hat{\xi}) \setminus \{r\}|} ((\hat{G} \uparrow r); \text{while } (\hat{\xi} \uparrow r) |_{|\text{subj}(\hat{\xi}) \setminus \{r\}|} \{\text{true}\} (\hat{G} \uparrow r) |_\emptyset) |_\emptyset \text{skip}|_\emptyset \xrightarrow{\xi_r, \lambda_r} L'_r$ . Then,  $\text{if } (\hat{\xi} \uparrow r) |_{|\text{subj}(\hat{\xi}) \setminus \{\emptyset \cup \emptyset \cup \{r\}\}|} ((\hat{G} \uparrow r); \text{while } (\hat{\xi} \uparrow r) |_{|\text{subj}(\hat{\xi}) \setminus \{r\}|} \{\text{true}\} (\hat{G} \uparrow r) |_\emptyset) |_{\emptyset \cap \{r\}} \text{skip}|_{\emptyset \cap \{r\}} \xrightarrow{\xi_r, \lambda_r} L'_r$ . Then, by Defn. 26,  $\text{if } (\hat{\xi} \uparrow r) |_{|\text{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset \cup \{r\})|} ((\hat{G} \uparrow r); (\text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset \uparrow r)) |_{\emptyset \cap \{r\}} \text{skip}|_{\emptyset \cap \{r\}} \xrightarrow{\xi_r, \lambda_r} L'_r$ . Then, by Defn. 26,  $\text{if } (\hat{\xi} \uparrow r) |_{|\text{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset \cup \{r\})|} (\hat{G}; \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset \uparrow r) |_{\emptyset \cap \{r\}} (\text{skip} \uparrow r) |_{\emptyset \cap \{r\}} \xrightarrow{\xi_r, \lambda_r} L'_r$ . Then, by Defn. 26,  $(\text{if } \hat{\xi}|_0 (\hat{G}; \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) |_\emptyset \text{skip}|_\emptyset \uparrow r) \xrightarrow{\xi_r, \lambda_r} L'_r$ .
  - Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(\text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset)$ . Then, by Lem. 46,  $\sqrt{R}(\text{if } \hat{\xi}|_0 (\hat{G}; \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) |_\emptyset \text{skip}|_\emptyset)$ .
  - Recall  $(\text{if } \hat{\xi}|_0 (\hat{G}; \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) |_\emptyset \text{skip}|_\emptyset \uparrow r) \xrightarrow{\xi_r, \lambda_r} L'_r$  and  $\sqrt{R}(\text{if } \hat{\xi}|_0 (\hat{G}; \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) |_\emptyset \text{skip}|_\emptyset)$  and  $\lambda_r \neq \tau$ . Then, by induction,  $\text{subj}(\gamma) \subseteq \text{subj}(\text{if } \hat{\xi}|_0 (\hat{G}; \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) |_\emptyset \text{skip}|_\emptyset)$  and  $\lambda_r = \gamma \uparrow r$ , for some  $\gamma$ .
  - Recall  $\text{subj}(\gamma) \subseteq \text{subj}(\text{if } \hat{\xi}|_0 (\hat{G}; \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) |_\emptyset \text{skip}|_\emptyset)$ . Then, by Defn. 24,  $\text{subj}(\gamma) \subseteq (\text{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset)) \cup (\text{subj}(\hat{G}; \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \setminus \emptyset) \cup (\text{subj}(\text{skip}) \setminus \emptyset)$ . Then,  $\text{subj}(\gamma) \subseteq \text{subj}(\hat{\xi}) \cup \text{subj}(\hat{G}; \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \cup \text{subj}(\text{skip})$ . Then, by Defn. 24,  $\text{subj}(\gamma) \subseteq \text{subj}(\hat{\xi}) \cup \text{subj}(\hat{G}) \cup \text{subj}(\text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \cup \emptyset$ . Then, by Defn. 24,  $\text{subj}(\gamma) \subseteq \text{subj}(\text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \cup \text{subj}(\text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \cup \emptyset$ . Then,  $\text{subj}(\gamma) \subseteq \text{subj}(\text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset)$ . Then,  $\text{subj}(\gamma) \subseteq \text{subj}(G)$ .  $\square$

*Proof (of Lem. 61).* Recall  $(G \uparrow q) \xrightarrow{\xi_q, q.y := e} L'_q$ . Then, by Defn. 31:

- **Base:**  $[\rightarrow 1\text{-ACT}]$ , such that  $G \uparrow q = q.y := e$  and  $L'_q = \text{skip}$  and  $\xi_q = \text{true}$ .
  - Recall  $G \uparrow q = q.y := e$ . Then, by Defn. 26,  $G = q.y := e$ .
  - By  $[\rightarrow 1\text{-ACT}]$ ,  $q.y := e \xrightarrow{\text{true}, q.y := e} \text{skip}$ . Then,  $G \xrightarrow{\text{true}, q.y := e} \text{skip}$ . Then,  $G' = \text{skip}$  and  $\xi = \text{true}$  and  $G \xrightarrow{\xi, q.y := e} G'$ , for some  $G', \xi$ .
  - Recall  $L'_q = \text{skip}$ . Then, by Defn. 26,  $L'_q = \text{skip} \uparrow q$ . Then,  $L'_q = G' \uparrow q$ .
  - Recall  $\xi_q = \text{true}$ . Then, by Defn. 11,  $\xi_q = \text{true} \uparrow r$ . Then,  $\xi_q = \xi \uparrow q$ .
- **Base:**  $[\rightarrow 1\text{-IF1}]$ , such that  $q.y := e = 1_{\text{subj}(\hat{\xi}_q)}$ , for some  $\hat{R}, \hat{\xi}_q$ . Then, **false**.
- **Base:**  $[\rightarrow 1\text{-IF2}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
- **Base:**  $[\rightarrow 1\text{-WHILE1}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
- **Base:**  $[\rightarrow 1\text{-WHILE2}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
- **Base:**  $[\rightarrow 1\text{-NIF1}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
- **Base:**  $[\rightarrow 1\text{-NIF2}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
- **Base:**  $[\rightarrow 1\text{-NIF3}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
- **Step:**  $[\rightarrow 1\text{-SEQ1}]$ , such that  $G \uparrow q = L_{1q}; L_{2q}$  and  $L'_q = L'_{1q}; L_{2q}$  and  $L_{1q} \xrightarrow{\xi_q, q.y := e} L'_{1q}$ , for some  $L_{1q}, L'_{1q}, L_{2q}$ .
  - Recall  $G \uparrow q = L_{1q}; L_{2q}$ . Then, by Defn. 26,  $G = G_1; G_2$  and  $L_{1q} = G_1 \uparrow q$  and  $L_{2q} = G_2 \uparrow q$ , for some  $G_1, G_2$ .
  - Recall  $L_{1q} \xrightarrow{\xi_q, q.y := e} L'_{1q}$ . Then,  $(G_1 \uparrow q) \xrightarrow{\xi_q, q.y := e} L'_{1q}$ .
  - Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(G_1; G_2)$ . Then, by Defn. 27,  $\sqrt{R}(G_1)$ .
  - Recall  $(G_1 \uparrow q) \xrightarrow{\xi_q, q.y := e} L'_{1q}$  and  $\sqrt{R}(G_1)$ . Then, by induction,  $G_1 \xrightarrow{\xi_q, q.y := e} G'_1$  and  $L'_{1q} = G'_1 \uparrow q$  and  $\xi_q = \xi \uparrow q$ , for some  $\xi$ .

- Recall  $G_1 \xrightarrow{\xi, q, y := e} G'_1$ . Then, by  $[\rightarrow 1\text{-SEQ1}]$ ,  $G_1 ; G_2 \xrightarrow{\xi, q, y := e} G'_1 ; G_2$ . Then,  $G \xrightarrow{\xi, q, y := e} G'_1 ; G_2$ . Then,  $G' = G'_1 ; G_2$  and  $G \xrightarrow{\xi, q, y := e} G'$ , for some  $G'$ .
  - Recall  $L'_{1q} = G'_1 \upharpoonright q$  and  $L_{2q} = G_2 \upharpoonright q$ . Then,  $L'_{1q} ; L_{2q} = (G'_1 \upharpoonright q) ; (G_2 \upharpoonright q)$ . Then, by Defn. 26,  $L'_{1q} ; L_{2q} = (G'_1 ; G_2) \upharpoonright q$ . Then,  $L'_q = G' \upharpoonright q$ .
- **Step:**  $[\rightarrow 1\text{-SEQ2}]$ , such that  $G \upharpoonright q = L_{1q} ; L_{2q}$  and  $L'_q = L'_{1q} ; L'_{2q}$  and  $\text{subj}(L_{1q}) \cap (\text{subj}(\xi_q) \cup \text{subj}(q, y := e)) = \emptyset$  and  $L_{2q} \xrightarrow{\xi_q, q, y := e} L'_{2q}$ , for some  $L_{1q}, L_{2q}, L'_{2q}$ .
- Recall  $G \upharpoonright q = L_{1q} ; L_{2q}$ . Then, by Defn. 26,  $G = G_1 ; G_2$  and  $L_{1q} = G_1 \upharpoonright q$  and  $L_{2q} = G_2 \upharpoonright q$ , for some  $G_1, G_2$ .
  - Recall  $L_{2q} \xrightarrow{\xi_q, q, y := e} L'_{2q}$ . Then,  $(G_2 \upharpoonright q) \xrightarrow{\xi_q, q, y := e} L'_{2q}$ .
  - Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(G_1 ; G_2)$ . Then, by Defn. 27,  $\sqrt{R}(G_2)$ .
  - Recall  $(G_2 \upharpoonright q) \xrightarrow{\xi_q, q, y := e} L'_{2q}$  and  $\sqrt{R}(G_2)$ . Then, by induction,  $G_2 \xrightarrow{\xi, q, y := e} G'_2$  and  $L'_{2q} = G'_2 \upharpoonright q$  and  $\xi_q = \xi \upharpoonright q$ , for some  $\xi$ .
  - Recall  $L_{2q} \xrightarrow{\xi_q, q, y := e} L'_{2q}$ . Then, by Lem. 53,  $\text{subj}(\xi_q) \subseteq \text{subj}(q, y := e)$ . Then,  $\text{subj}(\xi_q) \cup \text{subj}(q, y := e) = \text{subj}(q, y := e)$ .
  - Recall  $G_2 \xrightarrow{\xi, q, y := e} G'_2$ . Then, by Lem. 53,  $\text{subj}(\xi) \subseteq \text{subj}(q, y := e)$ . Then,  $\text{subj}(\xi) \cup \text{subj}(q, y := e) = \text{subj}(q, y := e)$ .
  - Recall  $\text{subj}(L_{1q}) \cap (\text{subj}(\xi_q) \cup \text{subj}(q, y := e)) = \emptyset$ . Then,  $\text{subj}(G_1 \upharpoonright q) \cap \text{subj}(q, y := e) = \emptyset$ . Then, by Defn. 16,  $\text{subj}(G_1 \upharpoonright q) \cap \{q\} = \emptyset$ . Then,  $q \notin \text{subj}(G_1 \upharpoonright q)$ . Then, by Lem. 44,  $q \notin \text{subj}(G_1)$ . Then,  $\text{subj}(G_1) \cap \{q\} = \emptyset$ . Then, by Defn. 16,  $\text{subj}(G_1) \cap \text{subj}(q, y := e) = \emptyset$ . Then,  $\text{subj}(G_1) \cap (\text{subj}(\xi) \cup \text{subj}(q, y := e)) = \emptyset$ .
  - Recall  $\text{subj}(G_1) \cap (\text{subj}(\xi) \cup \text{subj}(q, y := e)) = \emptyset$  and  $G_2 \xrightarrow{\xi, q, y := e} G'_2$ . Then, by  $[\rightarrow 1\text{-SEQ2}]$ ,  $G_1 ; G_2 \xrightarrow{\xi, q, y := e} G_1 ; G'_2$ . Then,  $G \xrightarrow{\xi, q, y := e} G_1 ; G'_2$ . Then,  $G' = G_1 ; G'_2$  and  $G \xrightarrow{\xi, q, y := e} G'$ , for some  $G'$ .
  - Recall  $L'_{1q} = G'_1 \upharpoonright q$  and  $L_{2q} = G'_2 \upharpoonright q$ . Then,  $L_{1q} ; L'_{2q} = (G'_1 \upharpoonright q) ; (G'_2 \upharpoonright q)$ . Then, by Defn. 26,  $L_{1q} ; L'_{2q} = (G'_1 ; G'_2) \upharpoonright q$ . Then,  $L'_q = G' \upharpoonright q$ .
- **Step:**  $[\rightarrow 1\text{-PAR1}]$ . Similar to case  $[\rightarrow 1\text{-SEQ1}]$ .
- **Step:**  $[\rightarrow 1\text{-PAR2}]$ . Similar to case  $[\rightarrow 1\text{-SEQ1}]$ .
- **Step:**  $[\rightarrow 1\text{-NIF4}]$ , such that  $G \upharpoonright q = \mathbf{if} \hat{\xi}_q \upharpoonright_n L_{1q} \upharpoonright_{R_{1q}} L_{2q} \upharpoonright_{R_{2q}}$  and  $L'_q = \mathbf{if} \hat{\xi}_q \upharpoonright_n L'_{1q} \upharpoonright_{R_{1q}} L'_{2q} \upharpoonright_{R_{2q}}$  and  $L_{1q} \xrightarrow{\xi_q, q, y := e} L'_{1q}$  and  $\text{subj}(\xi_q) \cup \text{subj}(q, y := e) \subseteq R_{1q} \setminus R_{2q}$ , for some  $R_{1q}, R_{2q}, L_{1q}, L'_{1q}, L_{2q}, \hat{\xi}_q, n$ .
- Recall  $G \upharpoonright q = \mathbf{if} \hat{\xi}_q \upharpoonright_n L_{1q} \upharpoonright_{R_{1q}} L_{2q} \upharpoonright_{R_{2q}}$ . Then, by Defn. 26,  $G = \mathbf{if} \hat{\xi} \upharpoonright_0 G_1 \upharpoonright_{R_1} G_2 \upharpoonright_{R_2}$  and  $R_{1q} = R_1 \cap \{q\}$  and  $R_{2q} = R_2 \cap \{q\}$  and  $L_{1q} = G_1 \upharpoonright q$  and  $L_{2q} = G_2 \upharpoonright q$  and  $\xi_q = \hat{\xi} \upharpoonright q$  and  $n = |\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{q\})|$ , for some  $R_1, R_2, G_1, G_2, \hat{\xi}$ .
  - Recall  $L_{1q} \xrightarrow{\xi_q, q, y := e} L'_{1q}$ . Then,  $(G_1 \upharpoonright q) \xrightarrow{\xi_q, q, y := e} L'_{1q}$ .
  - Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(\mathbf{if} \hat{\xi} \upharpoonright_0 G_1 \upharpoonright_{R_1} G_2 \upharpoonright_{R_2})$ . Then, by Defn. 27,  $\sqrt{R}(G_1)$ .
  - Recall  $(G_1 \upharpoonright q) \xrightarrow{\xi_q, q, y := e} L'_{1q}$  and  $\sqrt{R}(G_1)$ . Then, by induction,  $G_1 \xrightarrow{\xi, q, y := e} G'_1$  and  $L'_{1q} = G'_1 \upharpoonright q$  and  $\xi_q = \xi \upharpoonright q$ , for some  $G'_1$ , for some  $\xi$ .
  - Recall  $G_1 \xrightarrow{\xi, q, y := e} G'_1$ . Then, by Lem. 53,  $\text{subj}(\xi) \subseteq \text{subj}(q, y := e)$ . Then,  $\text{subj}(\xi) \cup \text{subj}(q, y := e) = \text{subj}(q, y := e)$ .
  - Recall  $\text{subj}(\xi_q) \cup \text{subj}(q, y := e) \subseteq R_{1q} \setminus R_{2q}$ . Then,  $\text{subj}(\xi_q) \cup \text{subj}(q, y := e) \subseteq (R_1 \cap \{q\}) \setminus (R_2 \cap \{q\})$ . Then,  $\text{subj}(q, y := e) \subseteq (R_1 \cap \{q\}) \setminus (R_2 \cap \{q\})$ . Then, by Defn. 16,  $\{q\} \subseteq (R_1 \cap \{q\}) \setminus (R_2 \cap \{q\})$ . Then,  $q \in (R_1 \cap \{q\}) \setminus (R_2 \cap \{q\})$ . Then,  $q \in R_1 \cap \{q\}$  and  $q \notin R_2 \cap \{q\}$ . Then,  $q \in R_1$  and  $q \notin R_2$ . Then,  $q \in R_1 \setminus R_2$ . Then,  $\{q\} \subseteq R_1 \setminus R_2$ . Then, by Defn. 16,  $\text{subj}(q, y := e) \subseteq R_1 \setminus R_2$ . Then,  $\text{subj}(\xi) \cup \text{subj}(q, y := e) \subseteq R_1 \setminus R_2$ .
  - Recall  $G_1 \xrightarrow{\xi, q, y := e} G'_1$  and  $\text{subj}(\xi) \cup \text{subj}(q, y := e) \subseteq R_1 \setminus R_2$ . Then, by  $[\rightarrow 1\text{-NIF4}]$ ,  $\mathbf{if} \hat{\xi} \upharpoonright_0 G_1 \upharpoonright_{R_1} G_2 \upharpoonright_{R_2} \xrightarrow{\xi, q, y := e} \mathbf{if} \hat{\xi} \upharpoonright_0 G'_1 \upharpoonright_{R_1} G_2 \upharpoonright_{R_2}$ . Then,  $G \xrightarrow{\xi, q, y := e} \mathbf{if} \hat{\xi} \upharpoonright_0 G'_1 \upharpoonright_{R_1} G_2 \upharpoonright_{R_2}$ . Then,  $G' = \mathbf{if} \hat{\xi} \upharpoonright_0 G'_1 \upharpoonright_{R_1} G_2 \upharpoonright_{R_2}$  and  $G \xrightarrow{\xi, q, y := e} G'$ , for some  $G'$ .
  - Recall  $L'_q = \mathbf{if} \hat{\xi}_q \upharpoonright_n L'_{1q} \upharpoonright_{R_{1q}} L'_{2q} \upharpoonright_{R_{2q}}$ . Then,  $L'_q = \mathbf{if} (\hat{\xi} \upharpoonright q) \upharpoonright_{|\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{q\})|} (G'_1 \upharpoonright q) \upharpoonright_{R_1 \cap \{q\}} (G_2 \upharpoonright q) \upharpoonright_{R_2 \cap \{q\}}$ . Then, by Defn. 26,  $L'_q = \mathbf{if} \hat{\xi} \upharpoonright_0 G'_1 \upharpoonright_{R_1} G_2 \upharpoonright_{R_2} \upharpoonright q$ . Then,  $L'_q = G' \upharpoonright q$ .
- **Step:**  $[\rightarrow 1\text{-NIF5}]$ . Similar to case  $[\rightarrow 1\text{-NIF4}]$ .
- **Step:**  $[\rightarrow 1\text{-NWHILE}]$ , such that  $G \upharpoonright q = \mathbf{while} \hat{\xi}_q \upharpoonright_n \{\mathbf{true}\} \hat{L}_q \upharpoonright_\emptyset$  and  $\mathbf{if} \hat{\xi}_q \upharpoonright_n (\hat{L}_q ; \mathbf{while} \hat{\xi}_q \upharpoonright_n \{\mathbf{true}\} \hat{L}_q \upharpoonright_\emptyset) \upharpoonright_\emptyset \mathbf{skip} \upharpoonright_\emptyset \xrightarrow{\xi_q, q, y := e} L'_q$ , for some  $\hat{L}_q, \hat{\xi}_q, n$ .

- Recall  $G \uparrow q = \mathbf{while} \hat{\xi}_q | n \{ \mathbf{true} \} \hat{L}_q | \emptyset$ . Then, by Defn. 26,  $G = \mathbf{while} \hat{\xi} |_0 \{ \psi \} \hat{G} | \emptyset$  and  $\hat{L}_q = \hat{G} \uparrow q$  and  $\hat{\xi}_q = \hat{\xi} \uparrow q$  and  $n = |\text{subj}(\hat{\xi}) \setminus \{q\}|$ , for some  $\hat{G}, \psi, \hat{\xi}$ .
- Recall  $\mathbf{if} \hat{\xi}_q | n (\hat{L}_q ; \mathbf{while} \hat{\xi}_q | n \{ \mathbf{true} \} \hat{L}_q | \emptyset) \mathbf{skip} | \emptyset \xrightarrow{\xi_q, q, y := e} L'_q$ . Then,  $\mathbf{if} (\hat{\xi} \uparrow q) |_{|\text{subj}(\hat{\xi}) \setminus \{q\}|} ((\hat{G} \uparrow q) ; \mathbf{while} (\hat{\xi} \uparrow q) |_{|\text{subj}(\hat{\xi}) \setminus \{q\}|} \{ \mathbf{true} \} (\hat{G} \uparrow q) | \emptyset) \mathbf{skip} | \emptyset \xrightarrow{\xi_q, q, y := e} L'_q$ . Then,  $\mathbf{if} (\hat{\xi} \uparrow q) |_{|\text{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset \cup \{q\})|} ((\hat{G} \uparrow q) ; \mathbf{while} (\hat{\xi} \uparrow q) |_{|\text{subj}(\hat{\xi}) \setminus \{q\}|} \{ \mathbf{true} \} (\hat{G} \uparrow q) | \emptyset) \mathbf{skip} |_{\emptyset \cap \{q\}} \xrightarrow{\xi_q, q, y := e} L'_q$ . Then, by Defn. 26,  $\mathbf{if} (\hat{\xi} \uparrow q) |_{|\text{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset \cup \{q\})|} ((\hat{G} \uparrow q) ; (\mathbf{while} \hat{\xi} |_0 \{ \psi \} \hat{G} | \emptyset \uparrow q) |_{\emptyset \cap \{q\}}) \mathbf{skip} |_{\emptyset \cap \{q\}} \xrightarrow{\xi_q, q, y := e} L'_q$ . Then, by Defn. 26,  $\mathbf{if} (\hat{\xi} \uparrow q) |_{|\text{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset \cup \{q\})|} (\hat{G} ; \mathbf{while} \hat{\xi} |_0 \{ \psi \} \hat{G} | \emptyset \uparrow q) |_{\emptyset \cap \{q\}} (\mathbf{skip} \uparrow q) |_{\emptyset \cap \{q\}} \xrightarrow{\xi_q, q, y := e} L'_q$ . Then, by Defn. 26,  $(\mathbf{if} \hat{\xi} |_0 (\hat{G} ; \mathbf{while} \hat{\xi} |_0 \{ \psi \} \hat{G} | \emptyset) | \emptyset) \mathbf{skip} | \emptyset \uparrow q \xrightarrow{\xi_q, q, y := e} L'_q$ .
- Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(\mathbf{while} \hat{\xi} |_0 \{ \psi \} \hat{G} | \emptyset)$ . Then, by Lem. 46,  $\sqrt{R}(\mathbf{if} \hat{\xi} |_0 (\hat{G} ; \mathbf{while} \hat{\xi} |_0 \{ \psi \} \hat{G} | \emptyset) | \emptyset) \mathbf{skip} | \emptyset$ .
- Recall  $(\mathbf{if} \hat{\xi} |_0 (\hat{G} ; \mathbf{while} \hat{\xi} |_0 \{ \psi \} \hat{G} | \emptyset) | \emptyset) \mathbf{skip} | \emptyset \uparrow q \xrightarrow{\xi_q, q, y := e} L'_q$  and  $\sqrt{R}(\mathbf{if} \hat{\xi} |_0 (\hat{G} ; \mathbf{while} \hat{\xi} |_0 \{ \psi \} \hat{G} | \emptyset) | \emptyset) \mathbf{skip} | \emptyset$ . Then, by induction,  $\mathbf{if} \hat{\xi} |_0 (\hat{G} ; \mathbf{while} \hat{\xi} |_0 \{ \psi \} \hat{G} | \emptyset) | \emptyset) \mathbf{skip} | \emptyset \xrightarrow{\xi, q, y := e} G'$  and  $L'_q = G' \uparrow q$  and  $\xi_q = \xi \uparrow q$ , for some  $G', \xi$ .
- Recall  $\mathbf{if} \hat{\xi} |_0 (\hat{G} ; \mathbf{while} \hat{\xi} |_0 \{ \psi \} \hat{G} | \emptyset) | \emptyset) \mathbf{skip} | \emptyset \xrightarrow{\xi, q, y := e} G'$ . Then, by  $[\rightarrow 1\text{-NWHILE}]$ ,  $\mathbf{while} \hat{\xi} |_0 \{ \psi \} \hat{G} | \emptyset \xrightarrow{\xi, q, y := e} G'$ . Then,  $G \xrightarrow{\xi, q, y := e} G'$ .  $\square$

*Proof (of Lem. 62).* Recall  $(G \uparrow p) \xrightarrow{\xi_p, pq!e} L'_p$ . Then, by Defn. 31:

- **Base:**  $[\rightarrow 1\text{-ACT}]$ , such that  $G \uparrow p = pq!e$  and  $L'_p = \mathbf{skip}$  and  $\xi_p = \mathbf{true}$ .
  - Recall  $G \uparrow p = pq!e$ . Then, by Defn. 26,  $G = p.e \rightarrow q.\hat{y}$ , for some  $\hat{y}$ .
  - Recall  $(G \uparrow q) \xrightarrow{\xi_q, pq!y} L'_q$ . Then,  $(p.e \rightarrow q.\hat{y} \uparrow q) \xrightarrow{\xi_q, pq!y} L'_q$ . Then, by Defn. 26,  $pq!\hat{y} \xrightarrow{\xi_q, pq!y} L'_q$ . Then, by Defn. 31,  $L'_q = \mathbf{skip}$  and  $\xi_q = \mathbf{true}$  and  $\hat{y} = y$ .
  - By  $[\rightarrow 1\text{-ACT}]$ ,  $p.e \rightarrow q.\hat{y} \xrightarrow{\mathbf{true}, p.e \rightarrow q.\hat{y}} \mathbf{skip}$ . Then,  $G \xrightarrow{\mathbf{true}, p.e \rightarrow q.\hat{y}} \mathbf{skip}$ . Then,  $G' = \mathbf{skip}$  and  $\xi = \mathbf{true}$  and  $G \xrightarrow{\xi, p.e \rightarrow q.y} G'$ , for some  $G', \xi$ .
  - Recall  $L'_p = \mathbf{skip}$  and  $L'_q = \mathbf{skip}$ . Then, by Defn. 26,  $L'_p = \mathbf{skip} \uparrow p$  and  $L'_q = \mathbf{skip} \uparrow q$ . Then,  $L'_p = G' \uparrow p$  and  $L'_q = G' \uparrow q$ .
  - Recall  $\xi_p = \mathbf{true}$  and  $\xi_q = \mathbf{true}$ . Then, by Defn. 11,  $\xi_p = \mathbf{true} \uparrow p$  and  $\xi_q = \mathbf{true} \uparrow q$ . Then,  $\xi_p = \xi \uparrow p$  and  $\xi_q = \xi \uparrow q$ .
- **Base:**  $[\rightarrow 1\text{-IF1}]$ , such that  $pq!e = 1_{\text{subj}(\hat{\xi}_p)}^{\hat{R}}$ , for some  $\hat{R}, \hat{\xi}_p$ . Then, **false**.
- **Base:**  $[\rightarrow 1\text{-IF2}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
- **Base:**  $[\rightarrow 1\text{-WHILE1}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
- **Base:**  $[\rightarrow 1\text{-WHILE2}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
- **Base:**  $[\rightarrow 1\text{-NIF1}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
- **Base:**  $[\rightarrow 1\text{-NIF2}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
- **Base:**  $[\rightarrow 1\text{-NIF3}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
- **Step:**  $[\rightarrow 1\text{-SEQ1}]$ , such that  $G \uparrow p = L_{1p} ; L_{2p}$  and  $L'_p = L'_{1p} ; L'_{2p}$  and  $L_{1p} \xrightarrow{\xi_p, pq!e} L'_{1p}$ , for some  $L_{1p}, L'_{1p}, L_{2p}$ .
  - Recall  $G \uparrow p = L_{1p} ; L_{2p}$ . Then, by Defn. 26,  $G = G_1 ; G_2$  and  $L_{1p} = G_1 \uparrow p$  and  $L_{2p} = G_2 \uparrow p$ , for some  $G_1, G_2$ .
  - Recall  $L_{1p} \xrightarrow{\xi_p, pq!e} L'_{1p}$ . Then,  $(G_1 \uparrow p) \xrightarrow{\xi_p, pq!e} L'_{1p}$ .
  - Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(G_1 ; G_2)$ . Then, by Defn. 27,  $\sqrt{R}(G_1)$ .
  - Recall  $(G \uparrow q) \xrightarrow{\xi_q, pq!y} L'_q$ . Then,  $(G_1 ; G_2 \uparrow q) \xrightarrow{\xi_q, pq!y} L'_q$ . Then, by Defn. 26,  $(G_1 \uparrow q) ; (G_2 \uparrow q) \xrightarrow{\xi_q, pq!y} L'_q$ . Then, by Defn. 31:
    - \* **Case:**  $[\rightarrow 1\text{-SEQ1}]$ , such that  $L'_q = L'_{1q} ; (G_2 \uparrow q)$  and  $(G_1 \uparrow q) \xrightarrow{\xi_q, pq!y} L'_{1q}$ , for some  $L'_{1q}$ .
      - Recall  $(G_1 \uparrow p) \xrightarrow{\xi_p, pq!e} L'_{1p}$  and  $(G_1 \uparrow q) \xrightarrow{\xi_q, pq!y} L'_{1q}$  and  $\sqrt{R}(G_1)$ . Then, by induction,  $G_1 \xrightarrow{\xi, p.e \rightarrow q.y} G'_1$  and  $L'_{1p} = G'_1 \uparrow p$  and  $\xi_p = \xi \uparrow p$  and  $L'_{1q} = G'_1 \uparrow q$  and  $\xi_q = \xi \uparrow q$ , for some  $G'_1$ , for some  $\xi$ .
      - Recall  $G_1 \xrightarrow{\xi, p.e \rightarrow q.y} G'_1$ . Then, by  $[\rightarrow 1\text{-SEQ1}]$ ,  $G_1 ; G_2 \xrightarrow{\xi, p.e \rightarrow q.y} G'_1 ; G_2$ . Then,  $G \xrightarrow{\xi, p.e \rightarrow q.y} G'_1 ; G_2$ . Then,  $G' = G'_1 ; G_2$  and  $G_1 ; G_2 \xrightarrow{\xi, p.e \rightarrow q.y} G'$ , for some  $G'$ .

- Recall  $L'_p = L'_{1p}; L_{2p}$  and  $L'_q = L'_{1q}; (G_2 \upharpoonright q)$ . Then,  $L'_p = (G'_1 \upharpoonright p); (G_2 \upharpoonright p)$  and  $L'_q = (G'_1 \upharpoonright q); (G_2 \upharpoonright q)$ . Then, by Defn. 26,  $L'_n = (G'_1; G_2) \upharpoonright p$  and  $L'_q = (G'_1; G_2) \upharpoonright q$ . Then,  $L'_p = G' \upharpoonright p$  and  $L'_q = G' \upharpoonright q$ .
- \* **Case:**  $[\rightarrow 1\text{-SEQ2}]$ , such that  $\text{subj}(G_1 \upharpoonright q) \cap (\text{subj}(\xi_q) \cup \text{subj}(pq?y)) = \emptyset$ .
  - Recall  $(G_1 \upharpoonright p) \xrightarrow{\xi_p, pq!e} L'_{1p}$  and  $\sqrt{R}(G_1)$  and  $pq!e \neq \tau$ . Then, by Lem. 60,  $\text{subj}(\gamma) \subseteq \text{subj}(G_1)$  and  $pq!e = \gamma \upharpoonright p$ , for some  $\gamma$ .
  - Recall  $pq!e = \gamma \upharpoonright p$ . Then, by Defn. 18,  $\gamma = p.e \rightarrow q.\hat{y}$ , for some  $\hat{y}$ .
  - Recall  $\text{subj}(\gamma) \subseteq \text{subj}(G_1)$ . Then,  $\text{subj}(p.e \rightarrow q.\hat{y}) \subseteq \text{subj}(G_1)$ . Then, by Defn. 16,  $\{p, q\} \subseteq \text{subj}(G_1)$ . Then,  $q \in \text{subj}(G_1)$ . Then, by Lem. 44,  $q \in \text{subj}(G_1 \upharpoonright q)$ .
  - Recall  $q \in \{q\}$ . Then, by Defn. 16,  $q \in \text{subj}(pq?y)$ . Then,  $q \in \text{subj}(\xi_q) \cup \text{subj}(pq?y)$ .
  - Recall  $q \in \text{subj}(G_1 \upharpoonright q)$  and  $q \in \text{subj}(\xi_q) \cup \text{subj}(pq?y)$ . Then,  $q \in \text{subj}(G_1 \upharpoonright q) \cap (\text{subj}(\xi_q) \cup \text{subj}(pq?y))$ . Then,  $q \in \emptyset$ . Then, **false**.
- **Step:**  $[\rightarrow 1\text{-SEQ2}]$ , such that  $G \upharpoonright p = L_{1p}; L_{2p}$  and  $L'_p = L_{1p}; L'_{2p}$  and  $\text{subj}(L_{1p}) \cap (\text{subj}(\xi_p) \cup \text{subj}(pq!e)) = \emptyset$  and  $L_{2p} \xrightarrow{\xi_p, pq!e} L'_{2p}$ , for some  $L_{1p}, L_{2p}, L'_{2p}$ .
  - Recall  $G \upharpoonright p = L_{1p}; L_{2p}$ . Then, by Defn. 26,  $G = G_1; G_2$  and  $L_{1p} = G_1 \upharpoonright p$  and  $L_{2p} = G_2 \upharpoonright p$ , for some  $G_1, G_2$ .
  - Recall  $(G \upharpoonright q) \xrightarrow{\xi_q, pq?y} L'_q$ . Then,  $(G_1; G_2 \upharpoonright q) \xrightarrow{\xi_q, pq?y} L'_q$ . Then, by Defn. 26,  $(G_1 \upharpoonright q); (G_2 \upharpoonright q) \xrightarrow{\xi_q, pq?y} L'_q$ . Then, by Defn. 31:
    - \* **Case:**  $[\rightarrow 1\text{-SEQ1}]$ , such that  $L'_q = L'_{1q}; (G_2 \upharpoonright q)$  and  $(G_1 \upharpoonright q) \xrightarrow{\xi_q, pq?y} L'_{1q}$ , for some  $L'_{1q}$ .
      - Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(G_1; G_2)$ . Then, by Defn. 27,  $\sqrt{R}(G_1)$ .
      - Recall  $(G_1 \upharpoonright q) \xrightarrow{\xi_q, pq?y} L'_{1q}$  and  $\sqrt{R}(G_1)$  and  $pq?y \neq \tau$ . Then, by Lem. 60,  $\text{subj}(\gamma) \subseteq \text{subj}(G_1)$  and  $pq?y = \gamma \upharpoonright q$ , for some  $\gamma$ .
      - Recall  $pq?y = \gamma \upharpoonright q$ . Then, by Defn. 18,  $\gamma = p.\hat{e} \rightarrow q.y$ , for some  $\hat{e}$ .
      - Recall  $\text{subj}(\gamma) \subseteq \text{subj}(G_1)$ . Then,  $\text{subj}(p.\hat{e} \rightarrow q.y) \subseteq \text{subj}(G_1)$ . Then, by Defn. 16,  $\{p, q\} \subseteq \text{subj}(G_1)$ . Then,  $p \in \text{subj}(G_1)$ . Then, by Lem. 44,  $p \in \text{subj}(G_1 \upharpoonright p)$ .
      - Recall  $p \in \{p\}$ . Then, by Defn. 16,  $p \in \text{subj}(pq!e)$ . Then,  $p \in \text{subj}(\xi_p) \cup \text{subj}(pq!e)$ .
      - Recall  $p \in \text{subj}(G_1 \upharpoonright p)$  and  $p \in \text{subj}(\xi_p) \cup \text{subj}(pq!e)$ . Then,  $p \in \text{subj}(G_1 \upharpoonright p) \cap (\text{subj}(\xi_p) \cup \text{subj}(pq!e))$ . Then,  $p \in \text{subj}(L_{1p}) \cap (\text{subj}(\xi_p) \cup \text{subj}(pq!e))$ . Then,  $p \in \emptyset$ . Then, **false**.
    - \* **Case:**  $[\rightarrow 1\text{-SEQ2}]$ , such that  $L'_q = (G_1 \upharpoonright q); L'_{2q}$  and  $\text{subj}(G_1 \upharpoonright q) \cap (\text{subj}(\xi_q) \cup \text{subj}(pq?y)) = \emptyset$  and  $(G_2 \upharpoonright q) \xrightarrow{\xi_q, pq?y} L'_{2q}$ , for some  $L'_{1q}, L'_{2q}$ .
      - Recall  $L_{2p} \xrightarrow{\xi_p, pq!e} L'_{2p}$ . Then,  $(G_2 \upharpoonright p) \xrightarrow{\xi_p, pq!e} L'_{2p}$ .
      - Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(G_1; G_2)$ . Then, by Defn. 27,  $\sqrt{R}(G_2)$ .
      - Recall  $(G_2 \upharpoonright p) \xrightarrow{\xi_p, pq!e} L'_{2p}$  and  $(G_2 \upharpoonright q) \xrightarrow{\xi_q, pq?y} L'_{2q}$  and  $\sqrt{R}(G_2)$ . Then, by induction,  $G_2 \xrightarrow{\xi_p, e \rightarrow q.y} G'_2$  and  $L'_{2n} = G'_2 \upharpoonright p$  and  $\xi_p = \xi \upharpoonright p$  and  $L'_{2q} = G'_2 \upharpoonright q$  and  $\xi_q = \xi \upharpoonright q$ , for some  $G'_2$ , for some  $\xi$ .
      - Recall  $\text{subj}(L_{1p}) \cap (\text{subj}(\xi_p) \cup \text{subj}(pq!e)) = \emptyset$ . Then,  $\text{subj}(G_1 \upharpoonright p) \cap (\text{subj}(\xi_p) \cup \text{subj}(pq!e)) = \emptyset$ .
      - Recall  $G_2 \xrightarrow{\xi_p, e \rightarrow q.y} G'_2$ . Then, by Lem. 53,  $\text{subj}(\xi) \subseteq \text{subj}(p.e \rightarrow q.y)$ . Then,  $\text{subj}(\xi) \cup \text{subj}(p.e \rightarrow q.y) = \text{subj}(p.e \rightarrow q.y)$ .
      - Recall  $\text{subj}(G_1 \upharpoonright p) \cap (\text{subj}(\xi_p) \cup \text{subj}(pq!e)) = \emptyset$  and  $\text{subj}(G_1 \upharpoonright q) \cap (\text{subj}(\xi_q) \cup \text{subj}(pq?y)) = \emptyset$ . Then,  $\text{subj}(G_1 \upharpoonright p) \cap \text{subj}(pq!e) = \emptyset$  and  $\text{subj}(G_1 \upharpoonright q) \cap \text{subj}(pq?y) = \emptyset$ . Then, by Defn. 16,  $\text{subj}(G_1 \upharpoonright p) \cap \{p\} = \emptyset$  and  $\text{subj}(G_1 \upharpoonright q) \cap \{q\} = \emptyset$ . Then,  $p \notin \text{subj}(G_1 \upharpoonright p)$  and  $q \notin \text{subj}(G_1 \upharpoonright q)$ . Then, by Lem. 44,  $p, q \notin \text{subj}(G_1)$ . Then,  $\text{subj}(G_1) \cap \{p, q\} = \emptyset$ . Then, by Defn. 16,  $\text{subj}(G_1) \cap \text{subj}(p.e \rightarrow q.y) = \emptyset$ . Then,  $\text{subj}(G_1) \cap (\text{subj}(\xi) \cup \text{subj}(p.e \rightarrow q.y)) = \emptyset$ .
      - Recall  $\text{subj}(G_1) \cap (\text{subj}(\xi) \cup \text{subj}(p.e \rightarrow q.y)) = \emptyset$  and  $G_2 \xrightarrow{\xi_p, e \rightarrow q.y} G'_2$ . Then, by  $[\rightarrow 1\text{-SEQ2}]$ ,  $G_1; G_2 \xrightarrow{\xi_p, e \rightarrow q.y} G_1; G'_2$ . Then,  $G \xrightarrow{\xi_p, e \rightarrow q.y} G_1; G'_2$ . Then,  $G' = G_1; G'_2$  and  $G_1; G_2 \xrightarrow{\xi_p, e \rightarrow q.y} G'$ , for some  $G'$ .
      - Recall  $L'_p = L_{1p}; L'_{2p}$  and  $L'_q = (G_1 \upharpoonright q); L'_{2q}$ . Then,  $L'_p = (G_1 \upharpoonright p); (G'_2 \upharpoonright p)$  and  $L'_q = (G_1 \upharpoonright q); (G'_2 \upharpoonright q)$ . Then, by Defn. 26,  $L'_n = (G_1; G'_2) \upharpoonright p$  and  $L'_q = (G_1; G'_2) \upharpoonright q$ . Then,  $L'_p = G' \upharpoonright p$  and  $L'_q = G' \upharpoonright q$ .

- **Step:** [ $\rightarrow 1$ -PAR1], such that  $G \upharpoonright p = L_{1p} \parallel L_{2p}$  and  $L'_p = L'_{1p} \parallel L_{2p}$  and  $L_{1p} \xrightarrow{\xi_p, pq!e} L'_{1p}$ , for some  $L_{1p}, L'_{1p}, L_{2p}$ .
  - Recall  $G \upharpoonright p = L_{1p} \parallel L_{2p}$ . Then, by Defn. 26,  $G = G_1 \parallel G_2$  and  $L_{1p} = G_1 \upharpoonright p$  and  $L_{2p} = G_2 \upharpoonright p$ , for some  $G_1, G_2$ .
  - Recall  $L_{1p} \xrightarrow{\xi_p, pq!e} L'_{1p}$ . Then,  $(G_1 \upharpoonright p) \xrightarrow{\xi_p, pq!e} L'_{1p}$ .
  - Recall  $(G \upharpoonright q) \xrightarrow{\xi_q, pq?y} L'_q$ . Then,  $(G_1 \parallel G_2 \upharpoonright q) \xrightarrow{\xi_q, pq?y} L'_q$ . Then, by Defn. 26,  $(G_1 \upharpoonright q) \parallel (G_2 \upharpoonright q) \xrightarrow{\xi_q, pq?y} L'_q$ . Then, by Defn. 31:
    - \* **Case:** [ $\rightarrow 1$ -PAR1], such that  $L'_q = L'_{1q} \parallel (G_2 \upharpoonright q)$  and  $(G_1 \upharpoonright q) \xrightarrow{\xi_q, pq?y} L'_{1q}$ , for some  $L'_{1q}$ .
      - Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(G_1 \parallel G_2)$ . Then, by Defn. 27,  $\sqrt{R}(G_1)$ .
      - Recall  $(G_1 \upharpoonright p) \xrightarrow{\xi_p, pq!e} L'_{1p}$  and  $(G_1 \upharpoonright q) \xrightarrow{\xi_q, pq?y} L'_{1q}$  and  $\sqrt{R}(G_1)$ . Then, by induction,  $G_1 \xrightarrow{\xi_p, e \rightarrow q, y} G'_1$  and  $L'_{1n} = G'_1 \upharpoonright p$  and  $\xi_p = \xi \upharpoonright p$  and  $L'_{1q} = G'_1 \upharpoonright q$  and  $\xi_q = \xi \upharpoonright q$ , for some  $G'_1$ , for some  $\xi$ .
      - Recall  $G_1 \xrightarrow{\xi_p, e \rightarrow q, y} G'_1$ . Then, by [ $\rightarrow 1$ -PAR1],  $G_1 \parallel G_2 \xrightarrow{\xi_p, e \rightarrow q, y} G'_1 \parallel G_2$ . Then,  $G \xrightarrow{\xi_p, e \rightarrow q, y} G'_1 \parallel G_2$ . Then,  $G' = G'_1 \parallel G_2$  and  $G_1 \parallel G_2 \xrightarrow{\xi_p, e \rightarrow q, y} G'$ , for some  $G'$ .
      - Recall  $L'_p = L'_{1p} \parallel L_{2p}$  and  $L'_q = L'_{1q} \parallel (G_2 \upharpoonright q)$ . Then,  $L'_p = (G'_1 \upharpoonright p) \parallel (G_2 \upharpoonright p)$  and  $L'_q = (G'_1 \upharpoonright q) \parallel (G_2 \upharpoonright q)$ . Then, by Defn. 26,  $L'_n = (G'_1 \parallel G_2) \upharpoonright p$  and  $L'_q = (G'_1 \parallel G_2) \upharpoonright q$ . Then,  $L'_p = G' \upharpoonright p$  and  $L'_q = G' \upharpoonright q$ .
    - \* **Case:** [ $\rightarrow 1$ -PAR2], such that  $(G_2 \upharpoonright q) \xrightarrow{\xi_q, pq?y} L'_{2q}$ , for some  $L'_{2q}$ .
      - Recall  $(G_1 \upharpoonright p) \xrightarrow{\xi_p, pq!e} L'_{1p}$  and  $(G_2 \upharpoonright q) \xrightarrow{\xi_q, pq?y} L'_{2q}$ . Then, by Lem. 54,  $\text{chan}(pq!e) \subseteq \text{chan}(G_1 \upharpoonright p)$  and  $\text{chan}(pq?y) \subseteq \text{chan}(G_2 \upharpoonright q)$ . Then, by Defn. 17,  $\{pq\} \subseteq \text{chan}(G_1 \upharpoonright p)$  and  $\{pq\} \subseteq \text{chan}(G_2 \upharpoonright q)$ . Then,  $\{pq\} \subseteq \text{chan}(G_1 \upharpoonright p) \cap \text{chan}(G_2 \upharpoonright q)$ .
      - By Lem. 45,  $\text{chan}(G_1 \upharpoonright p) \subseteq \text{chan}(G_1)$  and  $\text{chan}(G_2 \upharpoonright q) \subseteq \text{chan}(G_2)$ . Then,  $\text{chan}(G_1 \upharpoonright p) \cap \text{chan}(G_2 \upharpoonright q) \subseteq \text{chan}(G_1) \cap \text{chan}(G_2)$ .
      - Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(G_1 \parallel G_2)$ . Then, by Defn. 27,  $\text{chan}(G_1) \cap \text{chan}(G_2) = \emptyset$ .
      - Recall  $\{pq\} \subseteq \text{chan}(G_1 \upharpoonright p) \cap \text{chan}(G_2 \upharpoonright q) \subseteq \text{chan}(G_1) \cap \text{chan}(G_2) = \emptyset$ . Then,  $\{pq\} \subseteq \emptyset$ . Then, **false**.
- **Step:** [ $\rightarrow 1$ -PAR2]. Similar to [ $\rightarrow 1$ -PAR1].
- **Step:** [ $\rightarrow 1$ -NIF4], such that  $G \upharpoonright p = \mathbf{if} \xi_p | n L_{1p} |_{R_{1p}} L_{2p} |_{R_{2p}}$  and  $L'_p = \mathbf{if} \hat{\xi}_p | n L'_{1p} |_{R_{1p}} L_{2p} |_{R_{2p}}$  and  $L_{1p} \xrightarrow{\xi_p, pq!e} L'_{1p}$  and  $\text{subj}(\xi_p) \cup \text{subj}(pq!e) \subseteq R_{1p} \setminus R_{2p}$ , for some  $R_{1p}, R_{2p}, L_{1p}, L'_{1p}, L_{2p}, \hat{\xi}_p, n$ .
  - Recall  $G \upharpoonright p = \mathbf{if} \hat{\xi}_p | n L_{1p} |_{R_{1p}} L_{2p} |_{R_{2p}}$ . Then, by Defn. 26,  $G = \mathbf{if} \hat{\xi} |_0 G_1 |_{R_1} G_2 |_{R_2}$  and  $R_{1p} = R_1 \cap \{p\}$  and  $R_{2p} = R_2 \cap \{p\}$  and  $L_{1p} = G_1 \upharpoonright p$  and  $L_{2p} = G_2 \upharpoonright p$  and  $\hat{\xi}_p = \hat{\xi} \upharpoonright p$  and  $n = |\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{p\})|$ , for some  $R_1, R_2, G_1, G_2, \hat{\xi}$ .
  - Recall  $(G \upharpoonright q) \xrightarrow{\xi_q, pq?y} L'_q$ . Then,  $(\mathbf{if} \hat{\xi} |_0 G_1 |_{R_1} G_2 |_{R_2} \upharpoonright q) \xrightarrow{\xi_q, pq?y} L'_q$ . Then, by Defn. 26,  $\mathbf{if} (\hat{\xi} \upharpoonright q) |_{\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{q\})} (G_1 \upharpoonright q) |_{R_1 \cap \{q\}} (G_2 \upharpoonright q) |_{R_2 \cap \{q\}} \xrightarrow{\xi_q, pq?y} L'_q$ . Then, by Defn. 31:
    - \* **Case:** [ $\rightarrow 1$ -NIF1], such that  $pq?q = \tau$ . Then, **false**.
    - \* **Case:** [ $\rightarrow 1$ -NIF2]. Similar to case [ $\rightarrow 1$ -NIF1].
    - \* **Case:** [ $\rightarrow 1$ -NIF3]. Similar to case [ $\rightarrow 1$ -NIF1].
    - \* **Case:** [ $\rightarrow 1$ -NIF4], such that  $L'_q = \mathbf{if} (\hat{\xi} \upharpoonright q) |_{\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{q\})} L'_{1q} |_{R_1 \cap \{q\}} (G_2 \upharpoonright q) |_{R_2 \cap \{q\}}$  and  $(G_1 \upharpoonright q) \xrightarrow{\xi_q, pq?y} L'_{1q}$  and  $\text{subj}(\xi_q) \cup \text{subj}(pq?y) \subseteq (R_1 \cap \{q\}) \setminus (R_2 \cap \{q\})$ , for some  $L'_{1q}$ .
      - Recall  $L_{1p} \xrightarrow{\xi_p, pq!e} L'_{1p}$ . Then,  $(G_1 \upharpoonright p) \xrightarrow{\xi_p, pq!e} L'_{1p}$ .
      - Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(\mathbf{if} \hat{\xi} |_0 G_1 |_{R_1} G_2 |_{R_2})$ . Then, by Defn. 27,  $\sqrt{R}(G_1)$ .
      - Recall  $(G_1 \upharpoonright p) \xrightarrow{\xi_p, pq!e} L'_{1p}$  and  $(G_1 \upharpoonright q) \xrightarrow{\xi_q, pq?y} L'_{1q}$  and  $\sqrt{R}(G_1)$ . Then, by induction,  $G_1 \xrightarrow{\xi_p, e \rightarrow q, y} G'_1$  and  $L'_{1n} = G'_1 \upharpoonright p$  and  $\xi_p = \xi \upharpoonright p$  and  $L'_{1q} = G'_1 \upharpoonright q$  and  $\xi_q = \xi \upharpoonright q$ , for some  $G'_1$ , for some  $\xi$ .
      - Recall  $G_1 \xrightarrow{\xi_p, e \rightarrow q, y} G'_1$ . Then, by Lem. 53,  $\text{subj}(\xi) \subseteq \text{subj}(p.e \rightarrow q.y)$ . Then,  $\text{subj}(\xi) \cup \text{subj}(p.e \rightarrow q.y) = \text{subj}(p.e \rightarrow q.y)$ .
      - Recall  $\text{subj}(\xi_p) \cup \text{subj}(pq!e) \subseteq R_{1p} \setminus R_{2p}$ . Then,  $\text{subj}(\xi_p) \cup \text{subj}(pq!e) \subseteq (R_1 \cap \{p\}) \setminus (R_2 \cap \{p\})$ . Then,

- $\text{subj}(pq!e) \subseteq (R_1 \cap \{p\}) \setminus (R_2 \cap \{p\})$ . Then, by Defn. 16,  
 $\{p\} \subseteq (R_1 \cap \{p\}) \setminus (R_2 \cap \{p\})$ . Then,  $p \in (R_1 \cap \{p\}) \setminus (R_2 \cap \{p\})$ . Then,  
 $p \in R_1 \cap \{p\}$  and  $p \notin R_2 \cap \{p\}$ . Then,  $p \in R_1$  and  $p \notin R_2$ . Then,  $p \in R_1 \setminus R_2$ .
- Recall  $\text{subj}(\xi_q) \cup \text{subj}(pq?y) \subseteq (R_1 \cap \{q\}) \setminus (R_2 \cap \{q\})$ . Then,  
 $\text{subj}(pq?y) \subseteq (R_1 \cap \{q\}) \setminus (R_2 \cap \{q\})$ . Then, by Defn. 16,  
 $\{q\} \subseteq (R_1 \cap \{q\}) \setminus (R_2 \cap \{q\})$ . Then,  $q \in (R_1 \cap \{q\}) \setminus (R_2 \cap \{q\})$ . Then,  
 $q \in R_1 \cap \{q\}$  and  $q \notin R_2 \cap \{q\}$ . Then,  $q \in R_1$  and  $q \notin R_2$ . Then,  $q \in R_1 \setminus R_2$ .
  - Recall  $p, q \in R_1 \setminus R_2$ . Then,  $\{p, q\} \subseteq R_1 \setminus R_2$ . Then, by Defn. 16,  
 $\text{subj}(p.e \rightarrow q.y) \subseteq R_1 \setminus R_2$ . Then,  $\text{subj}(\xi) \cup \text{subj}(p.e \rightarrow q.y) \subseteq R_1 \setminus R_2$ .
  - Recall  $G_1 \xrightarrow{\xi.p.e \rightarrow q.y} G'_1$  and  $\text{subj}(\xi) \cup \text{subj}(p.e \rightarrow q.y) \subseteq R_1 \setminus R_2$ . Then, by  
 $[\rightarrow 1\text{-NIF4}]$ ,  $\text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2} \xrightarrow{\xi.p.e \rightarrow q.y} \text{if } \hat{\xi}|_0 G'_1|_{R_1} G_2|_{R_2}$ . Then,  
 $G \xrightarrow{\xi.p.e \rightarrow q.y} \text{if } \hat{\xi}|_0 G'_1|_{R_1} G_2|_{R_2}$ . Then,  $G' = \text{if } \hat{\xi}|_0 G'_1|_{R_1} G_2|_{R_2}$  and  
 $G \xrightarrow{\xi.p.e \rightarrow q.y} G'$ , for some  $G'$ .
  - Recall  $L'_p = \text{if } \hat{\xi}_p|_n L'_{1p}|_{R_{1p}} L_{2p}|_{R_{2p}}$ . Then,  
 $L'_p = \text{if } (\hat{\xi} \uparrow p)|_{\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{p\})} (G'_1 \uparrow p)|_{R_1 \cap \{p\}} (G_2 \uparrow p)|_{R_2 \cap \{p\}}$ . Then, by  
Defn. 26,  $L'_p = \text{if } \hat{\xi}|_0 G'_1|_{R_1} G_2|_{R_2} \uparrow p$ . Then,  $L'_p = G' \uparrow p$ .
  - Recall  $L'_q = \text{if } \hat{\xi}_q|_{\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{q\})} L'_{1q}|_{R_{1q}} L_{2q}|_{R_{2q}}$ . Then,  
 $L'_q = \text{if } (\hat{\xi} \uparrow q)|_{\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{q\})} (G'_1 \uparrow q)|_{R_1 \cap \{q\}} (G_2 \uparrow q)|_{R_2 \cap \{q\}}$ . Then, by  
Defn. 26,  $L'_q = \text{if } \hat{\xi}|_0 G'_1|_{R_1} G_2|_{R_2} \uparrow q$ . Then,  $L'_q = G' \uparrow q$ .
- \* **Case:**  $[\rightarrow 1\text{-NIF5}]$ , such that  $\text{subj}(\xi_q) \cup \text{subj}(pq?y) \subseteq (R_2 \cap \{q\}) \setminus (R_1 \cap \{q\})$ .
- Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(\text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2})$ . Then, by Defn. 27,  $R_1 \neq \emptyset$  implies  
 $R_2 = \emptyset$ .
  - Recall  $\text{subj}(\xi_p) \cup \text{subj}(pq!e) \subseteq R_{1p} \setminus R_{2p}$ . Then,  
 $\text{subj}(\xi_p) \cup \text{subj}(pq!e) \subseteq (R_1 \cap \{p\}) \setminus (R_2 \cap \{p\})$ . Then,  
 $\text{subj}(pq!e) \subseteq (R_1 \cap \{p\}) \setminus (R_2 \cap \{p\})$ . Then, by Defn. 16,  
 $\{p\} \subseteq (R_1 \cap \{p\}) \setminus (R_2 \cap \{p\})$ . Then,  $p \in (R_1 \cap \{p\}) \setminus (R_2 \cap \{p\})$ . Then,  
 $p \in R_1 \cap \{p\}$ . Then,  $p \in R_1$ . Then,  $R_1 \neq \emptyset$ . Then,  $R_2 = \emptyset$ .
  - Recall  $\text{subj}(\xi_q) \cup \text{subj}(pq?y) \subseteq (R_2 \cap \{q\}) \setminus (R_1 \cap \{q\})$ . Then,  
 $\text{subj}(\xi_q) \cup \text{subj}(pq?y) \subseteq (\emptyset \cap \{q\}) \setminus (R_1 \cap \{q\})$ . Then,  
 $\text{subj}(pq?y) \subseteq (\emptyset \cap \{q\}) \setminus (R_1 \cap \{q\})$ . Then, by Defn. 16,  
 $\{q\} \subseteq (\emptyset \cap \{q\}) \setminus (R_1 \cap \{q\})$ . Then,  $q \in (\emptyset \cap \{q\}) \setminus (R_1 \cap \{q\})$ . Then,  $q \in \emptyset \cap \{q\}$ .  
Then,  $q \in \emptyset$ . Then, **false**.
- **Step:**  $[\rightarrow 1\text{-NIF5}]$ . Similar to case  $[\rightarrow 1\text{-NIF4}]$ .
- **Step:**  $[\rightarrow 1\text{-NWHILE}]$ , such that  $G \uparrow p = \text{while } \hat{\xi}_p|_n \{\text{true}\} \hat{L}_p|_\emptyset$  and  
 $\text{if } \hat{\xi}_p|_n (\hat{L}_p; \text{while } \hat{\xi}_p|_n \{\text{true}\} \hat{L}_p|_\emptyset) |_\emptyset \text{skip} |_\emptyset \xrightarrow{\xi_p.pq!e} L'_p$ , for some  $\hat{L}_p, \hat{\xi}_p, n$ .
- Recall  $G \uparrow p = \text{while } \hat{\xi}_p|_n \{\text{true}\} \hat{L}_p|_\emptyset$ . Then, by Defn. 26,  $G = \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset$  and  
 $\hat{L}_p = \hat{G} \uparrow p$  and  $\hat{\xi}_p = \hat{\xi} \uparrow p$  and  $n = |\text{subj}(\hat{\xi}) \setminus \{p\}|$ , for some  $\hat{G}, \psi, \hat{\xi}$ .
  - Recall  $\text{if } \hat{\xi}_p|_n (\hat{L}_p; \text{while } \hat{\xi}_p|_n \{\text{true}\} \hat{L}_p|_\emptyset) |_\emptyset \text{skip} |_\emptyset \xrightarrow{\xi_p.pq!e} L'_p$ . Then,  
 $\text{if } (\hat{\xi} \uparrow p)|_{\text{subj}(\hat{\xi}) \setminus \{p\}} ((\hat{G} \uparrow p); \text{while } (\hat{\xi} \uparrow p)|_{\text{subj}(\hat{\xi}) \setminus \{p\}} \{\text{true}\} (\hat{G} \uparrow p)|_\emptyset) |_\emptyset \text{skip} |_\emptyset \xrightarrow{\xi_p.pq!e} L'_p$ .  
Then,  $\text{if } (\hat{\xi} \uparrow p)|_{\text{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset \cup \{p\})} ((\hat{G} \uparrow p); \text{while } (\hat{\xi} \uparrow p)|_{\text{subj}(\hat{\xi}) \setminus \{p\}} \{\text{true}\} (\hat{G} \uparrow p)|_\emptyset) |_{\emptyset \cap \{p\}} \text{skip} |_{\emptyset \cap \{p\}} \xrightarrow{\xi_p.pq!e} L'_p$ .  
Then, by Defn. 26,  
 $\text{if } (\hat{\xi} \uparrow p)|_{\text{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset \cup \{p\})} ((\hat{G} \uparrow p); (\text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset \uparrow p)) |_{\emptyset \cap \{p\}} \text{skip} |_{\emptyset \cap \{p\}} \xrightarrow{\xi_p.pq!e} L'_p$ .  
Then, by Defn. 26,  
 $\text{if } (\hat{\xi} \uparrow p)|_{\text{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset \cup \{p\})} ((\hat{G} \uparrow p); \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) |_\emptyset \text{skip} |_\emptyset \xrightarrow{\xi_p.pq!e} L'_p$ .
  - Recall  $(G \uparrow q) \xrightarrow{\xi_q.pq?y} L'_q$ . Then,  $(\text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset \uparrow q) \xrightarrow{\xi_q.pq?y} L'_q$ . Then, by Defn. 26,  
 $\text{while } (\hat{\xi} \uparrow q)|_{\text{subj}(\hat{\xi}) \setminus \{q\}} \{\text{true}\} (\hat{G} \uparrow q)|_\emptyset \xrightarrow{\xi_q.pq?y} L'_q$ . Then, by Defn. 31,  
 $\text{if } (\hat{\xi} \uparrow q)|_{\text{subj}(\hat{\xi}) \setminus \{q\}} ((\hat{G} \uparrow q); \text{while } (\hat{\xi} \uparrow q)|_{\text{subj}(\hat{\xi}) \setminus \{q\}} \{\text{true}\} (\hat{G} \uparrow q)|_\emptyset) |_\emptyset \text{skip} |_\emptyset \xrightarrow{\xi_q.pq?y} L'_q$ .  
Then,  $\text{if } (\hat{\xi} \uparrow q)|_{\text{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset \cup \{q\})} ((\hat{G} \uparrow q); \text{while } (\hat{\xi} \uparrow q)|_{\text{subj}(\hat{\xi}) \setminus \{q\}} \{\text{true}\} (\hat{G} \uparrow q)|_\emptyset) |_{\emptyset \cap \{q\}} \text{skip} |_{\emptyset \cap \{q\}} \xrightarrow{\xi_q.pq?y} L'_q$ .  
Then, by Defn. 26,  
 $\text{if } (\hat{\xi} \uparrow q)|_{\text{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset \cup \{q\})} ((\hat{G} \uparrow q); (\text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset \uparrow q)) |_{\emptyset \cap \{q\}} \text{skip} |_{\emptyset \cap \{q\}} \xrightarrow{\xi_q.pq?y} L'_q$ .  
Then, by Defn. 26,  
 $\text{if } (\hat{\xi} \uparrow q)|_{\text{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset \cup \{q\})} ((\hat{G} \uparrow q); \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) |_\emptyset \text{skip} |_\emptyset \xrightarrow{\xi_q.pq?y} L'_q$ .  
Then, by Defn. 26,  $(\text{if } \hat{\xi}|_0 (\hat{G}; \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) |_\emptyset \text{skip} |_\emptyset \uparrow q) \xrightarrow{\xi_q.pq?y} L'_q$ .

- Recall  $\checkmark_R(G)$ . Then,  $\checkmark_R(\mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset)$ . Then, by Lem. 46,  $\checkmark_R(\mathbf{if} \hat{\xi}|_0 (\hat{G}; \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \mathbf{skip}|_\emptyset)$ .
- Recall  $(\mathbf{if} \hat{\xi}|_0 (\hat{G}; \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \mathbf{skip}|_\emptyset) \xrightarrow{\xi_p, pq!e} L'_p$  and  $(\mathbf{if} \hat{\xi}|_0 (\hat{G}; \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \mathbf{skip}|_\emptyset) \xrightarrow{\xi_q, pq!y} L'_q$  and  $\checkmark_R(\mathbf{if} \hat{\xi}|_0 (\hat{G}; \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \mathbf{skip}|_\emptyset)$ . Then, by induction,  $\mathbf{if} \hat{\xi}|_0 (\hat{G}; \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \mathbf{skip}|_\emptyset \xrightarrow{\xi, p.e \rightarrow q.y} G'$  and  $L'_p = G' \upharpoonright p$  and  $\xi_p = \xi \upharpoonright p$  and  $L'_q = G' \upharpoonright q$  and  $\xi_q = \xi \upharpoonright q$ , for some  $G', \xi$ .
- Recall  $\mathbf{if} \hat{\xi}|_0 (\hat{G}; \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \mathbf{skip}|_\emptyset \xrightarrow{\xi, p.e \rightarrow q.y} G'$ . Then, by  $[\rightarrow 1\text{-NWHILE}]$ ,  $\mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset \xrightarrow{\xi, p.e \rightarrow q.y} G'$ . Then,  $G \xrightarrow{\xi, p.e \rightarrow q.y} G'$ .  $\square$

*Proof (of Lem. 63).* Recall  $(G \upharpoonright r) \xrightarrow{\xi_r, i_{\hat{R}}} L'_r$ . Then, by Defn. 31:

- **Base:**  $[\rightarrow 1\text{-ACT}]$ , such that  $G \upharpoonright r = i_{\hat{R}}$  and  $L'_r = \mathbf{skip}$  and  $\xi_r = \mathbf{true}$ .
  - Recall  $G \upharpoonright r = i_{\hat{R}}$ . Then, by Defn. 26,  $G = i_{\hat{R}}$ .
  - By Defn. 27, not  $\checkmark_R(i_{\hat{R}})$ . Then, not  $\checkmark_R(G)$ . Then, **false**.
- **Base:**  $[\rightarrow 1\text{-IF1}]$ , such that  $G \upharpoonright r = \hat{R}.\mathbf{if} \hat{\xi}_r L_{1r} L_{2r}$  and  $\hat{R} = \hat{R}$  and  $i = 1$ , for some  $\hat{R}, \hat{\xi}_r, L_{1r}, L_{2r}$ .
  - Recall  $G \upharpoonright r = \hat{R}.\mathbf{if} \hat{\xi}_r L_{1r} L_{2r}$ . Then, by Defn. 26,  $G = \hat{R}.\mathbf{if} \hat{\xi} G_1 G_2$ , for some  $G_1, G_2, \hat{\xi}$ .
  - By  $[\rightarrow 1\text{-IF1}]$ ,  $\hat{R}.\mathbf{if} \hat{\xi} G_1 G_2 \xrightarrow{\xi_r, 1_{\hat{R}}} G_1$ . Then,  $G \xrightarrow{\xi_r, i_{\hat{R}}} G_1$ . Then,  $G' = G_1$  and  $\xi = \hat{\xi}^+$  and  $G \xrightarrow{\xi, i_{\hat{R}}} G'$ , for some  $G', \xi$ .
  - Recall  $(G \upharpoonright \hat{r}) \xrightarrow{\xi_r, i_{\hat{r}}} L'_r$  for every  $\hat{r} \in \hat{R}$ . Then,  $(\hat{R}.\mathbf{if} \hat{\xi} G_1 G_2 \upharpoonright \hat{r}) \xrightarrow{\xi_r, 1_{\hat{r}}} L'_r$  for every  $\hat{r} \in \hat{R}$ . Then, by Defn. 26,  $\hat{R}.\mathbf{if} (\hat{\xi} \upharpoonright \hat{r}) (G_1 \upharpoonright \hat{r}) (G_2 \upharpoonright \hat{r}) \xrightarrow{\xi_r, 1_{\hat{r}}} L'_r$  for every  $\hat{r} \in \hat{R}$ . Then, by Defn. 31,  $L'_r = G_1 \upharpoonright \hat{r}$  for every  $\hat{r} \in \hat{R}$ . Then,  $L'_r = G' \upharpoonright \hat{r}$  for every  $\hat{r} \in \hat{R}$ .
  - Recall  $(G \upharpoonright \hat{r}) \xrightarrow{\xi_r, i_{\hat{r}}} L'_r$  for every  $\hat{r} \in \hat{R}$ . Then,  $(\hat{R}.\mathbf{if} \hat{\xi} G_1 G_2 \upharpoonright \hat{r}) \xrightarrow{\xi_r, 1_{\hat{r}}} L'_r$  for every  $\hat{r} \in \hat{R}$ . Then, by Defn. 26,  $\hat{R}.\mathbf{if} (\hat{\xi} \upharpoonright \hat{r}) (G_1 \upharpoonright \hat{r}) (G_2 \upharpoonright \hat{r}) \xrightarrow{\xi_r, 1_{\hat{r}}} L'_r$  for every  $\hat{r} \in \hat{R}$ . Then, by Defn. 31,  $\xi_r = (\hat{\xi} \upharpoonright \hat{r})^+$  for every  $\hat{r} \in \hat{R}$ . Then, by Lem. 11,  $\xi_r = \hat{\xi}^+ \upharpoonright \hat{r}$  for every  $\hat{r} \in \hat{R}$ . Then,  $\xi_r = \xi \upharpoonright \hat{r}$  for every  $r \in \hat{R}$ .
- **Base:**  $[\rightarrow 1\text{-IF2}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
- **Base:**  $[\rightarrow 1\text{-WHILE1}]$ , such that  $G \upharpoonright r = \hat{R}.\mathbf{while} \hat{\xi}_r \{\psi_r\} L_r$  and  $\hat{R} = \hat{R}$  and  $i = 1$ , for some  $\hat{R}, \psi_r, \hat{\xi}_r, L_r$ .
  - Recall  $G \upharpoonright r = \hat{R}.\mathbf{while} \hat{\xi}_r \{\psi_r\} L_r$ . Then, by Defn. 26,  $G = \hat{R}.\mathbf{while} \hat{\xi} \{\psi\} \hat{G}$ , for some  $\hat{G}, \psi, \hat{\xi}$ .
  - By  $[\rightarrow 1\text{-WHILE1}]$ ,  $\hat{R}.\mathbf{while} \hat{\xi} \{\psi\} \hat{G} \xrightarrow{\xi_r, 1_{\hat{R}}} \hat{G}$ ;  $\hat{R}.\mathbf{while} \hat{\xi} \{\psi\} \hat{G}$ . Then,  $G \xrightarrow{\xi_r, i_{\hat{R}}} \hat{G}$ ;  $\hat{R}.\mathbf{while} \hat{\xi} \{\psi\} \hat{G}$ . Then,  $G' = \hat{G}$ ;  $\hat{R}.\mathbf{while} \hat{\xi} \{\psi\} \hat{G}$  and  $\xi = \hat{\xi}^+$  and  $G \xrightarrow{\xi, i_{\hat{R}}} G'$ , for some  $G', \xi$ .
  - Recall  $(G \upharpoonright \hat{r}) \xrightarrow{\xi_r, i_{\hat{r}}} L'_r$  for every  $\hat{r} \in \hat{R}$ . Then,  $(\hat{R}.\mathbf{while} \hat{\xi} \{\psi\} \hat{G} \upharpoonright \hat{r}) \xrightarrow{\xi_r, 1_{\hat{r}}} L'_r$  for every  $\hat{r} \in \hat{R}$ . Then, by Defn. 26,  $\hat{R}.\mathbf{while} (\hat{\xi} \upharpoonright \hat{r}) \{\mathbf{true}\} (\hat{G} \upharpoonright \hat{r}) \xrightarrow{\xi_r, 1_{\hat{r}}} L'_r$  for every  $\hat{r} \in \hat{R}$ . Then, by Defn. 31,  $L'_r = (\hat{G} \upharpoonright \hat{r})$ ;  $\hat{R}.\mathbf{while} (\hat{\xi} \upharpoonright \hat{r}) \{\mathbf{true}\} (\hat{G} \upharpoonright \hat{r})$  for every  $\hat{r} \in \hat{R}$ . Then, by Defn. 26,  $L'_r = (\hat{G} \upharpoonright \hat{r})$ ;  $(\hat{R}.\mathbf{while} \hat{\xi} \{\psi\} \hat{G} \upharpoonright \hat{r})$  for every  $\hat{r} \in \hat{R}$ . Then, by Defn. 26,  $L'_r = \hat{G} \upharpoonright \hat{r}$ ;  $\hat{R}.\mathbf{while} \hat{\xi} \{\psi\} \hat{G} \upharpoonright \hat{r}$  for every  $\hat{r} \in \hat{R}$ . Then,  $L'_r = G' \upharpoonright \hat{r}$  for every  $\hat{r} \in \hat{R}$ .
  - Recall  $(G \upharpoonright \hat{r}) \xrightarrow{\xi_r, i_{\hat{r}}} L'_r$  for every  $\hat{r} \in \hat{R}$ . Then,  $(\hat{R}.\mathbf{while} \hat{\xi} \{\psi\} \hat{G} \upharpoonright \hat{r}) \xrightarrow{\xi_r, 1_{\hat{r}}} L'_r$  for every  $\hat{r} \in \hat{R}$ . Then, by Defn. 26,  $\hat{R}.\mathbf{while} (\hat{\xi} \upharpoonright \hat{r}) \{\mathbf{true}\} (\hat{G} \upharpoonright \hat{r}) \xrightarrow{\xi_r, 1_{\hat{r}}} L'_r$  for every  $\hat{r} \in \hat{R}$ . Then, by Defn. 31,  $\xi_r = (\hat{\xi} \upharpoonright \hat{r})^+$  for every  $\hat{r} \in \hat{R}$ . Then, by Lem. 11,  $\xi_r = \hat{\xi}^+ \upharpoonright \hat{r}$  for every  $\hat{r} \in \hat{R}$ . Then,  $\xi_r = \xi \upharpoonright \hat{r}$  for every  $r \in \hat{R}$ .
- **Base:**  $[\rightarrow 1\text{-WHILE2}]$ , such that  $G \upharpoonright r = \hat{R}.\mathbf{while} \hat{\xi}_r \{\psi_r\} L_r$  and  $\hat{R} = \hat{R}$  and  $i = 2$ , for some  $\hat{R}, \psi_r, \hat{\xi}_r, L_r$ .
  - Recall  $G \upharpoonright r = \hat{R}.\mathbf{while} \hat{\xi}_r \{\psi_r\} L_r$ . Then, by Defn. 26,  $G = \hat{R}.\mathbf{while} \hat{\xi} \{\psi\} \hat{G}$ , for some  $\hat{G}, \psi, \hat{\xi}$ .
  - By  $[\rightarrow 1\text{-WHILE2}]$ ,  $\hat{R}.\mathbf{while} \hat{\xi} \{\psi\} \hat{G} \xrightarrow{\xi_r, 2_{\hat{R}}} \mathbf{skip}$ . Then,  $G \xrightarrow{\xi_r, i_{\hat{R}}} \mathbf{skip}$ . Then,  $G' = \mathbf{skip}$  and  $\xi = \hat{\xi}^-$  and  $G \xrightarrow{\xi, i_{\hat{R}}} G'$ , for some  $G', \xi$ .
  - Recall  $(G \upharpoonright \hat{r}) \xrightarrow{\xi_r, i_{\hat{r}}} L'_r$  for every  $\hat{r} \in \hat{R}$ . Then,  $(\hat{R}.\mathbf{while} \hat{\xi} \{\psi\} \hat{G} \upharpoonright \hat{r}) \xrightarrow{\xi_r, 2_{\hat{r}}} L'_r$  for every  $\hat{r} \in \hat{R}$ . Then, by Defn. 26,  $\hat{R}.\mathbf{while} (\hat{\xi} \upharpoonright \hat{r}) \{\mathbf{true}\} (\hat{G} \upharpoonright \hat{r}) \xrightarrow{\xi_r, 2_{\hat{r}}} L'_r$  for every  $\hat{r} \in \hat{R}$ . Then, by Defn. 31,  $L'_r = \mathbf{skip}$  for every  $\hat{r} \in \hat{R}$ . Then, by Defn. 26,  $L'_r = \mathbf{skip} \upharpoonright \hat{r}$  for every  $\hat{r} \in \hat{R}$ . Then,  $L'_r = G' \upharpoonright \hat{r}$  for every  $r \in \hat{R}$ .

- Recall  $(G \uparrow \hat{r}) \xrightarrow{\xi_{\hat{r}, i_{\hat{r}}^{\hat{R}}}} L'_{\hat{r}}$  for every  $\hat{r} \in \hat{R}$ . Then,  $(\hat{R}.\mathbf{while} \hat{\xi} \{ \psi \} \hat{G} \uparrow \hat{r}) \xrightarrow{\xi_{\hat{r}, 2_{\hat{r}}^{\hat{R}}}} L'_{\hat{r}}$  for every  $\hat{r} \in \hat{R}$ . Then, by Defn. 26,  $\hat{R}.\mathbf{while} (\hat{\xi} \uparrow \hat{r}) \{ \mathbf{true} \} (\hat{G} \uparrow \hat{r}) \xrightarrow{\xi_{\hat{r}, 2_{\hat{r}}^{\hat{R}}}} L'_{\hat{r}}$  for every  $\hat{r} \in \hat{R}$ . Then, by Defn. 31,  $\xi_{\hat{r}} = (\hat{\xi} \uparrow \hat{r})^-$  for every  $\hat{r} \in \hat{R}$ . Then, by Lem. 11,  $\xi_{\hat{r}} = \hat{\xi}^- \uparrow \hat{r}$  for every  $\hat{r} \in \hat{R}$ . Then,  $\xi_{\hat{r}} = \hat{\xi} \uparrow \hat{r}$  for every  $r \in \hat{R}$ .
- **Base:**  $[\rightarrow 1\text{-NIF1}]$ , such that  $i_{\{r\}}^{\hat{R}} = \tau$ . Then, **false**.
- **Base:**  $[\rightarrow 1\text{-NIF2}]$ , such that  $G \uparrow r = \mathbf{if} \hat{\xi}_r \uparrow n L_{1r} \uparrow R_{1r} L_{2r} \uparrow R_{2r}$  and  $L'_r = \mathbf{if} \hat{\xi}_r \uparrow n L_{1r} \uparrow R_{1r} \cup \{r\} L_{2r} \uparrow R_{2r}$  and  $\xi_r = (\hat{\xi}_r \uparrow r)^+$  and  $\hat{R} = \{r\}$  and  $i = 1$  and  $r \in \text{subj}(\hat{\xi}_r) \setminus (R_{1r} \cup R_{2r})$ , for some  $R_{1r}, R_{2r}, L_{1r}, L_{2r}, \hat{\xi}_r, n$ .
  - Recall  $G \uparrow r = \mathbf{if} \hat{\xi}_r \uparrow n L_{1r} \uparrow R_{1r} L_{2r} \uparrow R_{2r}$ . Then, by Defn. 26,  $G = \mathbf{if} \hat{\xi} \uparrow 0 G_1 \uparrow R_1 G_2 \uparrow R_2$  and  $R_{1r} = R_1 \cap \{r\}$  and  $R_{2r} = R_2 \cap \{r\}$  and  $L_{1r} = G_1 \uparrow r$  and  $L_{2r} = G_2 \uparrow r$  and  $\hat{\xi}_r = \hat{\xi} \uparrow r$  and  $n = |\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})|$ , for some  $R_1, R_2, G_1, G_2, \hat{\xi}$ .
  - Recall  $r \in \text{subj}(\hat{\xi}_r) \setminus (R_{1r} \cup R_{2r})$ . Then,  $r \in \text{subj}(\hat{\xi} \uparrow r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\}))$ . Then,  $r \in \text{subj}(\hat{\xi} \uparrow r)$  and  $r \notin (R_1 \cap \{r\}) \cup (R_2 \cap \{r\})$ .
  - By Lem. 10,  $\text{subj}(\hat{\xi} \uparrow r) \subseteq \text{subj}(\hat{\xi})$ .
  - Recall  $r \in \text{subj}(\hat{\xi} \uparrow r) \subseteq \text{subj}(\hat{\xi})$ . Then,  $r \in \text{subj}(\hat{\xi})$ .
  - Recall  $r \notin (R_1 \cap \{r\}) \cup (R_2 \cap \{r\})$ . Then,  $r \notin R_1 \cap \{r\}$  and  $r \notin R_2 \cap \{r\}$ . Then,  $r \notin R_1$  and  $r \notin R_2$ . Then,  $r \notin R_1 \cup R_2$ .
  - Recall  $r \in \text{subj}(\hat{\xi})$  and  $r \notin R_1 \cup R_2$ . Then,  $r \in \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$ . Then, by  $[\rightarrow 1\text{-NIF2}]$ ,  $\mathbf{if} \hat{\xi} \uparrow 0 G_1 \uparrow R_1 G_2 \uparrow R_2 \xrightarrow{\xi^+ \uparrow r, 1_{\{r\}}} \mathbf{if} \hat{\xi} \uparrow 0 G_1 \uparrow R_1 \cup \{r\} G_2 \uparrow R_2$ . Then,  $G \xrightarrow{\xi^+ \uparrow r, 1_{\{r\}}} \mathbf{if} \hat{\xi} \uparrow 0 G_1 \uparrow R_1 \cup \{r\} G_2 \uparrow R_2$ . Then,  $G' = \mathbf{if} \hat{\xi} \uparrow 0 G_1 \uparrow R_1 \cup \{r\} G_2 \uparrow R_2$  and  $\xi = \hat{\xi}^+ \uparrow r$  and  $G \xrightarrow{\xi, i_{\hat{R}}^{\hat{R}}} G'$ , for some  $G', \xi$ .
  - Recall  $L'_r = \mathbf{if} \hat{\xi}_r \uparrow n L_{1r} \uparrow R_{1r} \cup \{r\} L_{2r} \uparrow R_{2r}$ . Then,  $L'_r = \mathbf{if} (\hat{\xi} \uparrow r) \uparrow |\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})| (G_1 \uparrow r) \uparrow (R_1 \cap \{r\}) \cup \{r\} (G_2 \uparrow r) \uparrow R_2 \cap \{r\}$ . Then,  $L'_r = \mathbf{if} (\hat{\xi} \uparrow r) \uparrow |\text{subj}(\hat{\xi}) \setminus (R_1 \cup \{r\}) \cup R_2 \cup \{r\})| (G_1 \uparrow r) \uparrow (R_1 \cap \{r\}) \cup \{r\} (G_2 \uparrow r) \uparrow R_2 \cap \{r\}$ . Then,  $L'_r = \mathbf{if} (\hat{\xi} \uparrow r) \uparrow |\text{subj}(\hat{\xi}) \setminus (R_1 \cup \{r\}) \cup R_2 \cup \{r\})| (G_1 \uparrow r) \uparrow (R_1 \cup \{r\}) \cap (\{r\} \cup \{r\}) (G_2 \uparrow r) \uparrow R_2 \cap \{r\}$ . Then, by Defn. 26,  $L'_r = \mathbf{if} \xi \uparrow 0 G_1 \uparrow R_1 \cup \{r\} G_2 \uparrow R_2 \uparrow r$ . Then,  $L'_r = G' \uparrow r$ . Then,  $L'_r = G' \uparrow \hat{r}$  for every  $\hat{r} \in \hat{R}$ .
  - Recall  $\xi_r = (\hat{\xi}_r \uparrow r)^+$ . Then,  $\xi_r = (\hat{\xi} \uparrow r)^+ \uparrow r$ . Then, by Lem. 11,  $\xi_r = (\hat{\xi}^+ \uparrow r) \uparrow r$ . Then,  $\xi_r = \hat{\xi} \uparrow r$ . Then,  $\xi_{\hat{r}} = \hat{\xi} \uparrow \hat{r}$  for every  $\hat{r} \in \{r\}$ . Then,  $\xi_{\hat{r}} = \hat{\xi} \uparrow \hat{r}$  for every  $\hat{r} \in \hat{R}$ .
- **Base:**  $[\rightarrow 1\text{-NIF3}]$ . Similar to case  $[\rightarrow 1\text{-NIF2}]$ .
- **Step:**  $[\rightarrow 1\text{-SEQ1}]$ , such that  $G \uparrow r = L_{1r}; L_{2r}$  and  $L_{1r} \xrightarrow{\xi_{r, i_{\{r\}}^{\hat{R}}}} L'_{1r}$ , for some  $L_{1r}, L'_{1r}, L_{2r}$ .
  - Recall  $G \uparrow r = L_{1r}; L_{2r}$ . Then, by Defn. 26,  $G = G_1; G_2$  and  $L_{1r} = G_1 \uparrow r$ , for some  $G_1, G_2$ .
  - Recall  $L_{1r} \xrightarrow{\xi_{r, i_{\{r\}}^{\hat{R}}}} L'_{1r}$ . Then,  $(G_1 \uparrow r) \xrightarrow{\xi_{r, i_{\{r\}}^{\hat{R}}}} L'_{1r}$ .
  - Recall  $\checkmark_R(G)$ . Then,  $\checkmark_R(G_1; G_2)$ . Then, by Defn. 27,  $\checkmark_R(G_1)$ .
  - Recall  $(G \uparrow \hat{r}) \xrightarrow{\xi_{\hat{r}, i_{\hat{r}}^{\hat{R}}}} L'_{\hat{r}}$  for every  $\hat{r} \in \hat{R}$ . Then,  $(G_1; G_2 \uparrow \hat{r}) \xrightarrow{\xi_{\hat{r}, i_{\hat{r}}^{\hat{R}}}} L'_{\hat{r}}$  for every  $\hat{r} \in \hat{R}$ . Then, by Defn. 26,  $(G_1 \uparrow \hat{r}); (G_2 \uparrow \hat{r}) \xrightarrow{\xi_{\hat{r}, i_{\hat{r}}^{\hat{R}}}} L'_{\hat{r}}$  for every  $\hat{r} \in \hat{R}$ . Then, by Defn. 31:
    - \* **Case:**  $L'_{\hat{r}} = L'_{1\hat{r}}; (G_2 \uparrow \hat{r})$  and  $(G_1 \uparrow \hat{r}) \xrightarrow{\xi_{\hat{r}, i_{\hat{r}}^{\hat{R}}}} L'_{1\hat{r}}$ , for every  $\hat{r} \in \hat{R}$ , for some  $\{L'_{1\hat{r}}\}_{\hat{r} \in \hat{R}}$ .
      - Recall  $(G_1 \uparrow r) \xrightarrow{\xi_{r, i_{\{r\}}^{\hat{R}}}} L'_{1r}$  and  $r \in \hat{R}$ , and  $(G_1 \uparrow \hat{r}) \xrightarrow{\xi_{\hat{r}, i_{\hat{r}}^{\hat{R}}}} L'_{1\hat{r}}$  for every  $\hat{r} \in \hat{R}$ , and  $\checkmark_R(G_1)$ . Then, by induction,  $G_1 \xrightarrow{\xi_{\hat{r}, i_{\hat{r}}^{\hat{R}}}} G'_1$ , and  $L'_{1\hat{r}} = G'_1 \uparrow \hat{r}$  for every  $\hat{r} \in \hat{R}$ , and  $\xi_{\hat{r}} = \hat{\xi} \uparrow \hat{r}$  for every  $\hat{r} \in \hat{R}$ , for some  $G'_1$ , for some  $\hat{\xi}$ .
      - Recall  $G_1 \xrightarrow{\xi_{\hat{r}, i_{\hat{r}}^{\hat{R}}}} G'_1$ . Then, by  $[\rightarrow 1\text{-SEQ1}]$ ,  $G_1; G_2 \xrightarrow{\xi_{\hat{r}, i_{\hat{r}}^{\hat{R}}}} G'_1; G_2$ . Then,  $G \xrightarrow{\xi_{\hat{r}, i_{\hat{r}}^{\hat{R}}}} G'_1; G_2$ . Then,  $G' = G'_1; G_2$  and  $G \xrightarrow{\xi_{\hat{r}, i_{\hat{r}}^{\hat{R}}}} G'$ , for some  $G'$ .
      - Recall  $L'_{\hat{r}} = L'_{1\hat{r}}; (G_2 \uparrow \hat{r})$  for every  $\hat{r} \in \hat{R}$ . Then,  $L'_{\hat{r}} = (G'_1 \uparrow \hat{r}); (G_2 \uparrow \hat{r})$  for every  $\hat{r} \in \hat{R}$ . Then, by Defn. 26,  $L'_{\hat{r}} = (G'_1; G_2) \uparrow \hat{r}$  for every  $\hat{r} \in \hat{R}$ . Then,  $L'_{\hat{r}} = G' \uparrow \hat{r}$  for every  $\hat{r} \in \hat{R}$ .
    - \* **Case:**  $\text{subj}(G_1 \uparrow \hat{r}) \cap (\text{subj}(\hat{\xi}_{\hat{r}}) \cup \text{subj}(i_{\hat{r}}^{\hat{R}})) = \emptyset$  and  $\hat{r} \in \hat{R}$ , for some  $\hat{r}$ .
      - Recall  $(G_1 \uparrow r) \xrightarrow{\xi_{r, i_{\{r\}}^{\hat{R}}}} L'_{1r}$  and  $\checkmark_R(G_1)$  and  $i_{\{r\}}^{\hat{R}} \neq \tau$ . Then, by Lem. 60,  $\text{subj}(\gamma) \subseteq \text{subj}(G_1)$  and  $i_{\{r\}}^{\hat{R}} = \gamma \uparrow r$ , for some  $\gamma$ .
      - Recall  $i_{\{r\}}^{\hat{R}} = \gamma \uparrow r$ . Then, by Defn. 18,  $\gamma = i_{\hat{R}}^{\hat{R}}$ .
      - Recall  $\text{subj}(\gamma) \subseteq \text{subj}(G_1)$ . Then,  $\text{subj}(i_{\hat{R}}^{\hat{R}}) \subseteq \text{subj}(G_1)$ . Then, by Defn. 16,  $\hat{R} \subseteq \text{subj}(G_1)$ .



- Recall  $\hat{r} \in \hat{R} \subseteq \text{subj}(G_1)$ . Then,  $\hat{r} \in \text{subj}(G_1)$ . Then, by Lem. 44,  $\hat{r} \in \text{subj}(G_1 \upharpoonright \hat{r})$ .
  - Recall  $\hat{r} \in \{\hat{r}\}$ . Then, by Defn. 16,  $\hat{r} \in \text{subj}(i_{\{\hat{r}\}}^{\hat{R}})$ . Then,  $\hat{r} \in \text{subj}(\xi_{\hat{r}}) \cup \text{subj}(i_{\{\hat{r}\}}^{\hat{R}})$ .
  - Recall  $\hat{r} \in \text{subj}(G_1 \upharpoonright \hat{r})$  and  $\hat{r} \in \text{subj}(\xi_{\hat{r}}) \cup \text{subj}(i_{\{\hat{r}\}}^{\hat{R}})$ . Then,  $\hat{r} \in \text{subj}(G_1 \upharpoonright \hat{r}) \cap (\text{subj}(\xi_{\hat{r}}) \cup \text{subj}(i_{\{\hat{r}\}}^{\hat{R}}))$ . Then,  $\hat{r} \in \emptyset$ . Then, **false**.
- **Step:**  $[\rightarrow 1\text{-SEQ2}]$ , such that  $G \upharpoonright r = L_{1r}; L_{2r}$  and  $\text{subj}(G_1 \upharpoonright r) \cap (\text{subj}(\xi_r) \cup \text{subj}(i_{\{r\}}^{\hat{R}})) = \emptyset$  and  $L_{2r} \xrightarrow{\xi_r, i_{\{r\}}^{\hat{R}}} L'_{2r}$ , for some  $L_{1r}, L_{2r}, L'_{2r}$ .
- Recall  $G \upharpoonright r = L_{1r}; L_{2r}$ . Then, by Defn. 26,  $G = G_1; G_2$  and  $L_{2r} = G_2 \upharpoonright r$ , for some  $G_1, G_2$ .
  - Recall  $(G \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i_{\{\hat{r}\}}^{\hat{R}}} L'_{\hat{r}}$  for every  $\hat{r} \in \hat{R}$ . Then,  $(G_1; G_2 \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i_{\{\hat{r}\}}^{\hat{R}}} L'_{\hat{r}}$  for every  $\hat{r} \in \hat{R}$ . Then, by Defn. 26,  $(G_1 \upharpoonright \hat{r}); (G_2 \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i_{\{\hat{r}\}}^{\hat{R}}} L'_{\hat{r}}$  for every  $\hat{r} \in \hat{R}$ . Then, by Defn. 31:
    - \* **Case:**  $L'_{\hat{r}} = L'_{1\hat{r}}; (G_2 \upharpoonright \hat{r})$  and  $(G_1 \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i_{\{\hat{r}\}}^{\hat{R}}} L'_{1\hat{r}}$  and  $\hat{r} \in \hat{R}$ , for some  $L'_{1\hat{r}}, \hat{r}$ .
      - Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(G_1; G_2)$ . Then, by Defn. 27,  $\sqrt{R}(G_1)$ .
      - Recall  $(G_1 \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i_{\{\hat{r}\}}^{\hat{R}}} L'_{1\hat{r}}$  and  $\sqrt{R}(G_1)$  and  $i_{\{\hat{r}\}}^{\hat{R}} \neq \tau$ . Then, by Lem. 60,  $\text{subj}(\gamma) \subseteq \text{subj}(G_1)$  and  $i_{\{\hat{r}\}}^{\hat{R}} = \gamma \upharpoonright \hat{r}$ , for some  $\gamma$ .
      - Recall  $i_{\{\hat{r}\}}^{\hat{R}} = \gamma \upharpoonright \hat{r}$ . Then, by Defn. 18,  $\gamma = i_{\hat{R}}$ .
      - Recall  $\text{subj}(\gamma) \subseteq \text{subj}(G_1)$ . Then,  $\text{subj}(i_{\hat{R}}) \subseteq \text{subj}(G_1)$ . Then, by Defn. 16,  $\hat{R} \subseteq \text{subj}(G_1)$ .
      - Recall  $r \in \hat{R} \subseteq \text{subj}(G_1)$ . Then,  $r \in \text{subj}(G_1)$ . Then, by Lem. 44,  $r \in \text{subj}(G_1 \upharpoonright r)$ .
      - Recall  $r \in \{r\}$ . Then, by Defn. 16,  $r \in \text{subj}(i_{\{r\}}^{\hat{R}})$ . Then,  $r \in \text{subj}(\xi_r) \cup \text{subj}(i_{\{r\}}^{\hat{R}})$ .
      - Recall  $r \in \text{subj}(G_1 \upharpoonright r)$  and  $r \in \text{subj}(\xi_r) \cup \text{subj}(i_{\{r\}}^{\hat{R}})$ . Then,  $r \in \text{subj}(G_1 \upharpoonright r) \cap (\text{subj}(\xi_r) \cup \text{subj}(i_{\{r\}}^{\hat{R}}))$ . Then,  $r \in \emptyset$ . Then, **false**.
    - \* **Case:**  $L'_{\hat{r}} = (G_1 \upharpoonright \hat{r}); L'_{2\hat{r}}$  and  $\text{subj}(G_1 \upharpoonright \hat{r}) \cap (\text{subj}(\xi_{\hat{r}}) \cup \text{subj}(i_{\{\hat{r}\}}^{\hat{R}})) = \emptyset$  and  $(G_2 \upharpoonright \hat{r}) \xrightarrow{\xi_r, i_{\{\hat{r}\}}^{\hat{R}}} L'_{2\hat{r}}$ , for every  $\hat{r} \in \hat{R}$ , for some  $\{L'_{2\hat{r}}\}_{\hat{r} \in \hat{R}}$ .
      - Recall  $L_{2r} \xrightarrow{\xi_r, i_{\{r\}}^{\hat{R}}} L'_{2r}$ . Then,  $(G_2 \upharpoonright r) \xrightarrow{\xi_r, i_{\{r\}}^{\hat{R}}} L'_{2r}$ .
      - Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(G_1; G_2)$ . Then, by Defn. 27,  $\sqrt{R}(G_2)$ .
      - Recall  $(G_2 \upharpoonright r) \xrightarrow{\xi_r, i_{\{r\}}^{\hat{R}}} L'_{2r}$  and  $r \in \hat{R}$ , and  $(G_2 \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i_{\{\hat{r}\}}^{\hat{R}}} L'_{2\hat{r}}$  for every  $\hat{r} \in \hat{R}$ , and  $\sqrt{R}(G_2)$ . Then, by induction,  $G_2 \xrightarrow{\xi, i_{\hat{R}}} G'_2$ , and  $L'_{2\hat{r}} = G'_2 \upharpoonright \hat{r}$  for every  $\hat{r} \in \hat{R}$ , and  $\xi_{\hat{r}} = \xi \upharpoonright \hat{r}$  for every  $\hat{r} \in \hat{R}$ , for some  $G'_2$ , for some  $\xi$ .
      - Recall  $\sqrt{R}(G_2)$  and  $G_2 \xrightarrow{\xi, i_{\hat{R}}} G'_2$ . Then, by Lem. 53,  $\text{subj}(\xi) \subseteq \text{subj}(i_{\hat{R}})$ . Then,  $\text{subj}(\xi) \cup \text{subj}(i_{\hat{R}}) = \text{subj}(i_{\hat{R}})$ .
      - Recall  $\text{subj}(G_1 \upharpoonright \hat{r}) \cap (\text{subj}(\xi_{\hat{r}}) \cup \text{subj}(i_{\{\hat{r}\}}^{\hat{R}})) = \emptyset$  for every  $\hat{r} \in \hat{R}$ . Then,  $\text{subj}(G_1 \upharpoonright \hat{r}) \cap \text{subj}(i_{\{\hat{r}\}}^{\hat{R}}) = \emptyset$  for every  $\hat{r} \in \hat{R}$ . Then, by Defn. 16,  $\text{subj}(G_1 \upharpoonright \hat{r}) \cap \{\hat{r}\} = \emptyset$  for every  $\hat{r} \in \hat{R}$ . Then,  $\hat{r} \notin \text{subj}(G_1 \upharpoonright \hat{r})$  for every  $\hat{r} \in \hat{R}$ . Then, by Lem. 44,  $\hat{r} \notin \text{subj}(G_1)$  for every  $\hat{r} \in \hat{R}$ . Then,  $\text{subj}(G_1) \cap \hat{R} = \emptyset$ . Then, by Defn. 16,  $\text{subj}(G_1) \cap \text{subj}(i_{\hat{R}}) = \emptyset$ . Then,  $\text{subj}(G_1) \cap (\text{subj}(\xi) \cup \text{subj}(i_{\hat{R}})) = \emptyset$ .
      - Recall  $\text{subj}(G_1) \cap (\text{subj}(\xi) \cup \text{subj}(i_{\hat{R}})) = \emptyset$  and  $G_2 \xrightarrow{\xi, i_{\hat{R}}} G'_2$ . Then, by  $[\rightarrow 1\text{-SEQ2}]$ ,  $G_1; G_2 \xrightarrow{\xi, i_{\hat{R}}} G_1; G'_2$ . Then,  $G \xrightarrow{\xi, i_{\hat{R}}} G_1; G'_2$ . Then,  $G' = G_1; G'_2$  and  $G \xrightarrow{\xi, i_{\hat{R}}} G'$ , for some  $G'$ .
      - Recall  $L'_{\hat{r}} = (G_1 \upharpoonright \hat{r}); L'_{2\hat{r}}$  for every  $\hat{r} \in \hat{R}$ . Then,  $L'_{\hat{r}} = (G_1 \upharpoonright \hat{r}); (G'_2 \upharpoonright \hat{r})$  for every  $\hat{r} \in \hat{R}$ . Then, by Defn. 26,  $L'_{\hat{r}} = (G_1; G'_2) \upharpoonright \hat{r}$  for every  $\hat{r} \in \hat{R}$ . Then,  $L'_{\hat{r}} = G' \upharpoonright \hat{r}$  for every  $\hat{r} \in \hat{R}$ .
- **Step:**  $[\rightarrow 1\text{-PAR1}]$ , such that  $G \upharpoonright r = L_{1r} \parallel L_{2r}$  and  $L_{1r} \xrightarrow{\xi_r, i_{\{r\}}^{\hat{R}}} L'_{1r}$ , for some  $L_{1r}, L'_{1r}, L_{2r}$ .
- Recall  $G \upharpoonright r = L_{1r} \parallel L_{2r}$ . Then, by Defn. 26,  $G = G_1 \parallel G_2$  and  $L_{1r} = G_1 \upharpoonright r$ , for some  $G_1, G_2$ .
  - Recall  $L_{1r} \xrightarrow{\xi_r, i_{\{r\}}^{\hat{R}}} L'_{1r}$ . Then,  $(G_1 \upharpoonright r) \xrightarrow{\xi_r, i_{\{r\}}^{\hat{R}}} L'_{1r}$ .
  - Recall  $(G \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i_{\{\hat{r}\}}^{\hat{R}}} L'_{\hat{r}}$  for every  $\hat{r} \in \hat{R}$ . Then,  $(G_1 \parallel G_2 \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i_{\{\hat{r}\}}^{\hat{R}}} L'_{\hat{r}}$  for every  $\hat{r} \in \hat{R}$ . Then, by Defn. 26,  $(G_1 \upharpoonright \hat{r}) \parallel (G_2 \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i_{\{\hat{r}\}}^{\hat{R}}} L'_{\hat{r}}$  for every  $\hat{r} \in \hat{R}$ . Then, by Defn. 31:
    - \* **Case:**  $L'_{\hat{r}} = L'_{1\hat{r}} \parallel (G_2 \upharpoonright \hat{r})$  and  $(G_1 \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i_{\{\hat{r}\}}^{\hat{R}}} L'_{1\hat{r}}$ , for every  $\hat{r} \in \hat{R}$ , for some  $\{L'_{1\hat{r}}\}_{\hat{r} \in \hat{R}}$ .
      - Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(G_1 \parallel G_2)$ . Then, by Defn. 27,  $\sqrt{R}(G_1)$ .
      - Recall  $(G_1 \upharpoonright r) \xrightarrow{\xi_r, i_{\{r\}}^{\hat{R}}} L'_{1r}$  and  $r \in \hat{R}$ , and  $(G_1 \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i_{\{\hat{r}\}}^{\hat{R}}} L'_{1\hat{r}}$  for every  $\hat{r} \in \hat{R}$ , and  $\sqrt{R}(G_1)$ . Then, by induction,  $G_1 \xrightarrow{\xi, i_{\hat{R}}} G'_1$ , and  $L'_{1\hat{r}} = G'_1 \upharpoonright \hat{r}$  for every  $\hat{r} \in \hat{R}$ , and  $\xi_{\hat{r}} = \xi \upharpoonright \hat{r}$  for every  $\hat{r} \in \hat{R}$ , for some  $G'_1$ , for some  $\xi$ .

- Recall  $G_1 \xrightarrow{\xi, i_{\hat{R}}} G'_1$ . Then, by  $[\rightarrow 1\text{-PAR1}]$ ,  $G_1 \parallel G_2 \xrightarrow{\xi, i_{\hat{R}}} G'_1 \parallel G_2$ . Then,  $G \xrightarrow{\xi, i_{\hat{R}}} G'_1 \parallel G_2$ . Then,  $G' = G'_1 \parallel G_2$  and  $G \xrightarrow{\xi, i_{\hat{R}}} G'$ , for some  $G'$ .
- Recall  $L'_{\hat{r}} = L'_{1\hat{r}} \parallel (G_2 \upharpoonright \hat{r})$  for every  $\hat{r} \in \hat{R}$ . Then,  $L'_{\hat{r}} = (G'_1 \upharpoonright \hat{r}) \parallel (G_2 \upharpoonright \hat{r})$  for every  $\hat{r} \in \hat{R}$ . Then, by Defn. 26,  $L'_{\hat{r}} = (G'_1 \parallel G_2) \upharpoonright \hat{r}$  for every  $\hat{r} \in \hat{R}$ . Then,  $L'_{\hat{r}} = G' \upharpoonright \hat{r}$  for every  $\hat{r} \in \hat{R}$ .
- \* **Case:**  $(G_2 \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i_{\{\hat{r}\}}} L'_{2\hat{r}}$  and  $\hat{r} \in \hat{R}$ , for some  $L_{2\hat{r}}, \hat{r}$ .
  - Recall  $(G_1 \upharpoonright r) \xrightarrow{\xi_r, i_{\{r\}}} L'_{1r}$  and  $(G_2 \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i_{\{\hat{r}\}}} L'_{2\hat{r}}$ . Then, by Lem. 54,  $\text{chan}(i_{\{\hat{r}\}}^{\hat{R}}) \subseteq \text{chan}(G_1 \upharpoonright r)$  and  $\text{chan}(i_{\{\hat{r}\}}^{\hat{R}}) \subseteq \text{chan}(G_2 \upharpoonright \hat{r})$ . Then, by Defn. 17,  $\{\hat{R}\} \subseteq \text{chan}(G_1 \upharpoonright r)$  and  $\{\hat{R}\} \subseteq \text{chan}(G_2 \upharpoonright \hat{r})$ . Then,  $\{\hat{R}\} \subseteq \text{chan}(G_1 \upharpoonright r) \cap \text{chan}(G_2 \upharpoonright \hat{r})$ .
  - By Lem. 45,  $\text{chan}(G_1 \upharpoonright r) \subseteq \text{chan}(G_1)$  and  $\text{chan}(G_2 \upharpoonright \hat{r}) \subseteq \text{chan}(G_2)$ . Then,  $\text{chan}(G_1 \upharpoonright r) \cap \text{chan}(G_2 \upharpoonright \hat{r}) \subseteq \text{chan}(G_1) \cap \text{chan}(G_2)$ .
  - Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(G_1 \parallel G_2)$ . Then, by Defn. 27,  $\text{chan}(G_1) \cap \text{chan}(G_2) = \emptyset$ .
  - Recall  $\{\hat{R}\} \subseteq \text{chan}(G_1 \upharpoonright r) \cap \text{chan}(G_2 \upharpoonright \hat{r}) \subseteq \text{chan}(G_1) \cap \text{chan}(G_2) = \emptyset$ . Then,  $\{\hat{R}\} \subseteq \emptyset$ . Then, **false**.
- **Step:**  $[\rightarrow 1\text{-PAR2}]$ . Similar to case  $[\rightarrow 1\text{-PAR1}]$ .
- **Step:**  $[\rightarrow 1\text{-NIF4}]$ , such that  $G \upharpoonright r = \mathbf{if} \hat{\xi}_r |_n L_{1r} |_{R_{1r}} L_{2r} |_{R_{2r}}$  and  $L'_r = \mathbf{if} \hat{\xi}_r |_n L'_{1r} |_{R_{1r}} L'_{2r} |_{R_{2r}}$  and  $L_{1r} \xrightarrow{\xi_r, i_{\{r\}}} L'_{1r}$  and  $\text{subj}(\xi_r) \cup \text{subj}(i_{\{r\}}^{\hat{R}}) \subseteq R_{1r} \setminus R_{2r}$ , for some  $R_{1r}, R_{2r}, L_{1r}, L'_{1r}, L_{2r}, \hat{\xi}_r, n$ .
  - Recall  $G \upharpoonright r = \mathbf{if} \hat{\xi}_r |_n L_{1r} |_{R_{1r}} L_{2r} |_{R_{2r}}$ . Then, by Defn. 26,  $G = \mathbf{if} \hat{\xi} |_0 G_1 |_{R_1} G_2 |_{R_2}$  and  $R_{1r} = R_1 \cap \{r\}$  and  $R_{2r} = R_2 \cap \{r\}$  and  $L_{1r} = G_1 \upharpoonright r$  and  $L_{2r} = G_2 \upharpoonright r$  and  $\xi_r = \hat{\xi} \upharpoonright r$  and  $n = |\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{r\})|$ , for some  $R_1, R_2, G_1, G_2, \hat{\xi}$ .
  - Recall  $(G \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i_{\{\hat{r}\}}} L'_{\hat{r}}$  for every  $\hat{r} \in \hat{R}$ . Then,  $(\mathbf{if} \hat{\xi} |_0 G_1 |_{R_1} G_2 |_{R_2} \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i_{\{\hat{r}\}}} L'_{\hat{r}}$  for every  $\hat{r} \in \hat{R}$ . Then, by Defn. 26,  $\mathbf{if} (\hat{\xi} \upharpoonright \hat{r}) |_{|\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{\hat{r}\})|} (G_1 \upharpoonright \hat{r}) |_{R_1 \cap \{\hat{r}\}} (G_2 \upharpoonright \hat{r}) |_{R_2 \cap \{\hat{r}\}} \xrightarrow{\xi_{\hat{r}}, i_{\{\hat{r}\}}} L'_{\hat{r}}$  for every  $\hat{r} \in \hat{R}$ . Then, by Defn. 31:
    - \* **Case:**  $i_{\{r\}}^{\hat{R}} = \tau$ . Then, **false**.
    - \* **Case:**  $\hat{R} = \{\hat{r}\}$  and  $\hat{r} \in \text{subj}(\hat{\xi} \upharpoonright \hat{r}) \setminus ((R_1 \cap \{\hat{r}\}) \cup (R_2 \cap \{\hat{r}\}))$ , for some  $\hat{r}$ .
      - Recall  $\text{subj}(\xi_r) \cup \text{subj}(i_{\{r\}}^{\hat{R}}) \subseteq R_{1r} \setminus R_{2r}$ . Then,  $\text{subj}(\xi_r) \cup \text{subj}(i_{\{r\}}^{\hat{R}}) \subseteq (R_1 \cap \{r\}) \setminus (R_2 \cap \{r\})$ . Then,  $\text{subj}(i_{\{r\}}^{\hat{R}}) \subseteq (R_1 \cap \{r\}) \setminus (R_2 \cap \{r\})$ . Then, by Defn. 16,  $\{r\} \subseteq (R_1 \cap \{r\}) \setminus (R_2 \cap \{r\})$ . Then,  $r \in (R_1 \cap \{r\}) \setminus (R_2 \cap \{r\})$ . Then,  $r \in R_1 \cap \{r\}$ . Then,  $r \in R_1$ .
      - Recall  $r \in \hat{R}$ . Then,  $r \in \{\hat{r}\}$ . Then,  $r = \hat{r}$ .
      - Recall  $\hat{r} \in \text{subj}(\hat{\xi} \upharpoonright \hat{r}) \setminus ((R_1 \cap \{\hat{r}\}) \cup (R_2 \cap \{\hat{r}\}))$ . Then,  $r \in \text{subj}(\hat{\xi} \upharpoonright r) \setminus ((R_1 \cap \{r\}) \cup (R_2 \cap \{r\}))$ . Then,  $r \notin (R_1 \cap \{r\}) \cup (R_2 \cap \{r\})$ . Then,  $r \notin R_1 \cap \{r\}$ . Then,  $r \notin R_1$ . Then, **false**.
    - \* **Case:**  $\text{subj}(\xi_{\hat{r}}) \cup \text{subj}(i_{\{\hat{r}\}}^{\hat{R}}) \subseteq (R_2 \cap \{\hat{r}\}) \setminus (R_1 \cap \{\hat{r}\})$ , for some  $\hat{r}$ .
      - Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(\mathbf{if} \hat{\xi} |_0 G_1 |_{R_1} G_2 |_{R_2})$ . Then, by Defn. 27,  $R_1 \neq \emptyset$  implies  $R_2 = \emptyset$ .
      - Recall  $\text{subj}(\xi_r) \cup \text{subj}(i_{\{r\}}^{\hat{R}}) \subseteq R_{1r} \setminus R_{2r}$ . Then,  $\text{subj}(\xi_r) \cup \text{subj}(i_{\{r\}}^{\hat{R}}) \subseteq (R_1 \cap \{r\}) \setminus (R_2 \cap \{r\})$ . Then,  $\text{subj}(i_{\{r\}}^{\hat{R}}) \subseteq (R_1 \cap \{r\}) \setminus (R_2 \cap \{r\})$ . Then, by Defn. 16,  $\{r\} \subseteq (R_1 \cap \{r\}) \setminus (R_2 \cap \{r\})$ . Then,  $r \in (R_1 \cap \{r\}) \setminus (R_2 \cap \{r\})$ . Then,  $r \in R_1 \cap \{r\}$ . Then,  $r \in R_1$ . Then,  $R_1 \neq \emptyset$ . Then,  $R_2 = \emptyset$ .
      - Recall  $\text{subj}(\xi_{\hat{r}}) \cup \text{subj}(i_{\{\hat{r}\}}^{\hat{R}}) \subseteq (R_2 \cap \{\hat{r}\}) \setminus (R_1 \cap \{\hat{r}\})$ . Then,  $\text{subj}(\xi_{\hat{r}}) \cup \text{subj}(i_{\{\hat{r}\}}^{\hat{R}}) \subseteq (\emptyset \cap \{\hat{r}\}) \setminus (R_1 \cap \{\hat{r}\})$ . Then,  $\text{subj}(i_{\{\hat{r}\}}^{\hat{R}}) \subseteq (\emptyset \cap \{\hat{r}\}) \setminus (R_1 \cap \{\hat{r}\})$ . Then, by Defn. 16,  $\{\hat{r}\} \subseteq (\emptyset \cap \{\hat{r}\}) \setminus (R_1 \cap \{\hat{r}\})$ . Then,  $\hat{r} \in (\emptyset \cap \{\hat{r}\}) \setminus (R_1 \cap \{\hat{r}\})$ . Then,  $\hat{r} \in \emptyset \cap \{\hat{r}\}$ . Then,  $\hat{r} \in \emptyset$ . Then, **false**.
    - \* **Case:**  $L'_{\hat{r}} = \mathbf{if} (\hat{\xi} \upharpoonright \hat{r}) |_{|\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{\hat{r}\})|} L'_{1\hat{r}} |_{R_1 \cap \{\hat{r}\}} (G_2 \upharpoonright \hat{r}) |_{R_2 \cap \{\hat{r}\}}$  and  $(G_1 \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i_{\{\hat{r}\}}} L'_{1\hat{r}}$  and  $\text{subj}(\xi_{\hat{r}}) \cup \text{subj}(i_{\{\hat{r}\}}^{\hat{R}}) \subseteq (R_1 \cap \{\hat{r}\}) \setminus (R_2 \cap \{\hat{r}\})$ , for every  $\hat{r} \in \hat{R}$ , for some  $\{L'_{\hat{r}}\}_{\hat{r} \in \hat{R}}$ .
      - Recall  $L_{1r} \xrightarrow{\xi_r, i_{\{r\}}} L'_{1r}$ . Then,  $(G_1 \upharpoonright r) \xrightarrow{\xi_r, i_{\{r\}}} L'_{1r}$ .

- Recall  $\sqrt{R}(G)$ .  $\sqrt{R}(\mathbf{if} \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2})$ . Then, by Defn. 27,  $\sqrt{R}(G_1)$ .
  - Recall  $(G_1 \uparrow r) \xrightarrow{\xi_r, i_{\hat{R}}^r} L'_{1r}$  and  $r \in R$ , and  $(G_1 \uparrow \hat{r}) \xrightarrow{\xi_{\hat{r}}, i_{\hat{R}}^{\hat{r}}} L'_{1\hat{r}}$  for every  $\hat{r} \in \hat{R}$ , and  $\sqrt{R}(G_1)$ . Then, by induction,  $G_1 \xrightarrow{\xi, i_{\hat{R}}} G'_1$ , and  $L'_{1\hat{r}} = G'_1 \uparrow \hat{r}$  for every  $\hat{r} \in \hat{R}$ , and  $\xi_{\hat{r}} = \xi \uparrow \hat{r}$  for every  $\hat{r} \in \hat{R}$ , for some  $G'_1$ , for some  $\xi$ .
  - Recall  $G_1 \xrightarrow{\xi, i_{\hat{R}}} G'_1$ . Then, by Lem. 53,  $\text{subj}(\xi) \subseteq \text{subj}(i_{\hat{R}}^{\hat{R}})$ . Then,  $\text{subj}(\xi) \cup \text{subj}(i_{\hat{R}}^{\hat{R}}) = \text{subj}(i_{\hat{R}}^{\hat{R}})$ .
  - Recall  $\text{subj}(\xi_{\hat{r}}) \cup \text{subj}(i_{\hat{r}}^{\hat{R}}) \subseteq (R_1 \cap \{\hat{r}\}) \setminus (R_2 \cap \{\hat{r}\})$  for every  $\hat{r} \in \hat{R}$ . Then,  $\text{subj}(i_{\hat{r}}^{\hat{R}}) \subseteq (R_1 \cap \{\hat{r}\}) \setminus (R_2 \cap \{\hat{r}\})$  for every  $\hat{r} \in \hat{R}$ . Then, by Defn. 16,  $\{\hat{r}\} \subseteq (R_1 \cap \{\hat{r}\}) \setminus (R_2 \cap \{\hat{r}\})$  for every  $\hat{r} \in \hat{R}$ . Then,  $\hat{r} \in (R_1 \cap \{\hat{r}\}) \setminus (R_2 \cap \{\hat{r}\})$  for every  $\hat{r} \in \hat{R}$ . Then,  $\hat{r} \in R_1 \cap \{\hat{r}\}$  and  $\hat{r} \notin R_2 \cap \{\hat{r}\}$ , for every  $\hat{r} \in \hat{R}$ . Then,  $\hat{r} \in R_1$  and  $\hat{r} \notin R_2$ , for every  $\hat{r} \in \hat{R}$ . Then,  $\hat{r} \in R_1 \setminus R_2$  for every  $\hat{r} \in \hat{R}$ . Then,  $\hat{R} \subseteq R_1 \setminus R_2$ . Then, by Defn. 16,  $\text{subj}(i_{\hat{R}}^{\hat{R}}) \subseteq R_1 \setminus R_2$ . Then,  $\text{subj}(\xi) \cup \text{subj}(i_{\hat{R}}^{\hat{R}}) \subseteq R_1 \setminus R_2$ .
  - Recall  $G_1 \xrightarrow{\xi, i_{\hat{R}}} G'_1$  and  $\text{subj}(\xi) \cup \text{subj}(i_{\hat{R}}^{\hat{R}}) \subseteq R_1 \setminus R_2$ . Then, by  $[\rightarrow 1\text{-NIF4}]$ ,  $\mathbf{if} \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2} \xrightarrow{\xi, i_{\hat{R}}} \mathbf{if} \hat{\xi}|_0 G'_1|_{R_1} G_2|_{R_2}$ . Then,  $G \xrightarrow{\xi, i_{\hat{R}}} \mathbf{if} \hat{\xi}|_0 G'_1|_{R_1} G_2|_{R_2}$ . Then,  $G' = \mathbf{if} \hat{\xi}|_0 G'_1|_{R_1} G_2|_{R_2}$  and  $G \xrightarrow{\xi, i_{\hat{R}}} G'$ , for some  $G'$ .
  - Recall  $L'_r = \mathbf{if} (\hat{\xi} \uparrow \hat{r})|_{\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{\hat{r}\})} L'_{1r}|_{R_1 \cap \{\hat{r}\}} (G_2 \uparrow \hat{r})|_{R_2 \cap \{\hat{r}\}}$  for every  $\hat{r} \in \hat{R}$ . Then,  $L'_r = \mathbf{if} (\hat{\xi} \uparrow \hat{r})|_{\text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2 \cup \{\hat{r}\})} (G'_1 \uparrow \hat{r})|_{R_1 \cap \{\hat{r}\}} (G_2 \uparrow \hat{r})|_{R_2 \cap \{\hat{r}\}}$  for every  $\hat{r} \in \hat{R}$ . Then, by Defn. 26,  $L'_r = \mathbf{if} \hat{\xi}|_0 G'_1|_{R_1} G_2|_{R_2} \uparrow \hat{r}$  for every  $\hat{r} \in \hat{R}$ . Then,  $L'_r = G' \uparrow \hat{r}$  for every  $\hat{r} \in \hat{R}$ .
- **Step:**  $[\rightarrow 1\text{-NIF5}]$ . Similar to case  $[\rightarrow 1\text{-NIF4}]$ .
- **Step:**  $[\rightarrow 1\text{-NWHILE}]$ , such that  $G \uparrow r = \mathbf{while} \hat{\xi}_{\hat{r}}|_n \{\mathbf{true}\} \hat{L}_r|_{\emptyset}$  and  $\mathbf{if} \hat{\xi}_r|_n (\hat{L}_r; \mathbf{while} \hat{\xi}_r|_n \{\mathbf{true}\} \hat{L}_r|_{\emptyset})|_{\emptyset} \mathbf{skip}|_{\emptyset} \xrightarrow{\xi_r, i_{\hat{R}}^r} L'_r$ , for some  $\hat{L}_r, \hat{\xi}_r, n$ .
- Recall  $G \uparrow r = \mathbf{while} \hat{\xi}_r|_n \{\mathbf{true}\} \hat{L}_r|_{\emptyset}$ . Then, by Defn. 26,  $G = \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_{\emptyset}$  and  $\hat{L}_r = \hat{G} \uparrow r$  and  $\hat{\xi}_r = \hat{\xi} \uparrow r$  and  $n = |\text{subj}(\hat{\xi}) \setminus \{r\}|$ , for some  $\hat{G}, \psi, \hat{\xi}$ .
  - Recall  $\mathbf{if} \hat{\xi}_r|_n (\hat{L}_r; \mathbf{while} \hat{\xi}_r|_n \{\mathbf{true}\} \hat{L}_r|_{\emptyset})|_{\emptyset} \mathbf{skip}|_{\emptyset} \xrightarrow{\xi_r, i_{\hat{R}}^r} L'_r$ . Then,  $\mathbf{if} (\hat{\xi} \uparrow r)|_{\text{subj}(\hat{\xi}) \setminus \{r\}} ((\hat{G} \uparrow r); \mathbf{while} (\hat{\xi} \uparrow r)|_{\text{subj}(\hat{\xi}) \setminus \{r\}} \{\mathbf{true}\} (\hat{G} \uparrow r)|_{\emptyset})|_{\emptyset} \mathbf{skip}|_{\emptyset} \xrightarrow{\xi_r, i_{\hat{R}}^r} L'_r$ . Then,  $\mathbf{if} (\hat{\xi} \uparrow r)|_{\text{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset \cup \{r\})} ((\hat{G} \uparrow r); \mathbf{while} (\hat{\xi} \uparrow r)|_{\text{subj}(\hat{\xi}) \setminus \{r\}} \{\mathbf{true}\} (\hat{G} \uparrow r)|_{\emptyset})|_{\emptyset \cap \{r\}} \mathbf{skip}|_{\emptyset \cap \{r\}} \xrightarrow{\xi_r, i_{\hat{R}}^r} L'_r$ . Then, by Defn. 26,  $\mathbf{if} (\hat{\xi} \uparrow r)|_{\text{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset \cup \{r\})} ((\hat{G} \uparrow r); (\mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_{\emptyset} \uparrow r))|_{\emptyset \cap \{r\}} \mathbf{skip}|_{\emptyset \cap \{r\}} \xrightarrow{\xi_r, i_{\hat{R}}^r} L'_r$ . Then, by Defn. 26,  $\mathbf{if} (\hat{\xi} \uparrow r)|_{\text{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset \cup \{r\})} (\hat{G}; \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_{\emptyset} \uparrow r)|_{\emptyset \cap \{r\}} (\mathbf{skip} \uparrow r)|_{\emptyset \cap \{r\}} \xrightarrow{\xi_r, i_{\hat{R}}^r} L'_r$ . Then, by Defn. 26,  $\mathbf{if} \hat{\xi}|_0 (\hat{G}; \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_{\emptyset})|_{\emptyset} \mathbf{skip}|_{\emptyset} \uparrow r \xrightarrow{\xi_r, i_{\hat{R}}^r} L'_r$ .
  - Recall  $(G \uparrow \hat{r}) \xrightarrow{\xi_{\hat{r}}, i_{\hat{R}}^{\hat{r}}} L'_r$  for every  $\hat{r} \in \hat{R}$ . Then,  $(\mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_{\emptyset} \uparrow \hat{r}) \xrightarrow{\xi_{\hat{r}}, i_{\hat{R}}^{\hat{r}}} L'_r$  for every  $\hat{r} \in \hat{R}$ . Then, by Defn. 26,  $\mathbf{while} (\hat{\xi} \uparrow \hat{r})|_{\text{subj}(\hat{\xi}) \setminus \{\hat{r}\}} \{\mathbf{true}\} (\hat{G} \uparrow \hat{r})|_{\emptyset} \xrightarrow{\xi_{\hat{r}}, i_{\hat{R}}^{\hat{r}}} L'_r$  for every  $\hat{r} \in \hat{R}$ . Then, by Defn. 31,  $\mathbf{if} (\hat{\xi} \uparrow \hat{r})|_{\text{subj}(\hat{\xi}) \setminus \{\hat{r}\}} ((\hat{G} \uparrow \hat{r}); \mathbf{while} (\hat{\xi} \uparrow \hat{r})|_{\text{subj}(\hat{\xi}) \setminus \{\hat{r}\}} \{\mathbf{true}\} (\hat{G} \uparrow \hat{r})|_{\emptyset})|_{\emptyset} \mathbf{skip}|_{\emptyset} \xrightarrow{\xi_{\hat{r}}, i_{\hat{R}}^{\hat{r}}} L'_r$  for every  $\hat{r} \in \hat{R}$ . Then,  $\mathbf{if} (\hat{\xi} \uparrow \hat{r})|_{\text{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset \cup \{\hat{r}\})} ((\hat{G} \uparrow \hat{r}); \mathbf{while} (\hat{\xi} \uparrow \hat{r})|_{\text{subj}(\hat{\xi}) \setminus \{\hat{r}\}} \{\mathbf{true}\} (\hat{G} \uparrow \hat{r})|_{\emptyset})|_{\emptyset \cap \{\hat{r}\}} \mathbf{skip}|_{\emptyset \cap \{\hat{r}\}} \xrightarrow{\xi_{\hat{r}}, i_{\hat{R}}^{\hat{r}}} L'_r$  for every  $\hat{r} \in \hat{R}$ . Then, by Defn. 26,  $\mathbf{if} (\hat{\xi} \uparrow \hat{r})|_{\text{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset \cup \{\hat{r}\})} ((\hat{G} \uparrow \hat{r}); (\mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_{\emptyset} \uparrow \hat{r}))|_{\emptyset \cap \{\hat{r}\}} \mathbf{skip}|_{\emptyset \cap \{\hat{r}\}} \xrightarrow{\xi_{\hat{r}}, i_{\hat{R}}^{\hat{r}}} L'_r$  for every  $\hat{r} \in \hat{R}$ . Then, by Defn. 26,  $\mathbf{if} (\hat{\xi} \uparrow \hat{r})|_{\text{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset \cup \{\hat{r}\})} (\hat{G}; \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_{\emptyset} \uparrow \hat{r})|_{\emptyset \cap \{\hat{r}\}} (\mathbf{skip} \uparrow \hat{r})|_{\emptyset \cap \{\hat{r}\}} \xrightarrow{\xi_{\hat{r}}, i_{\hat{R}}^{\hat{r}}} L'_r$  for every  $\hat{r} \in \hat{R}$ . Then, by Defn. 26,  $(\mathbf{if} \hat{\xi}|_0 (\hat{G}; \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_{\emptyset})|_{\emptyset} \mathbf{skip}|_{\emptyset} \uparrow \hat{r}) \xrightarrow{\xi_{\hat{r}}, i_{\hat{R}}^{\hat{r}}} L'_r$  for every  $\hat{r} \in \hat{R}$ .
  - Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(\mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_{\emptyset})$ . Then, by Lem. 46,  $\sqrt{R}(\mathbf{if} \hat{\xi}|_0 (\hat{G}; \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_{\emptyset})|_{\emptyset} \mathbf{skip}|_{\emptyset})$ .
  - Recall  $(\mathbf{if} \hat{\xi}|_0 (\hat{G}; \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_{\emptyset})|_{\emptyset} \mathbf{skip}|_{\emptyset}) \uparrow \hat{r} \xrightarrow{\xi_{\hat{r}}, i_{\hat{R}}^{\hat{r}}} L'_r$  and  $r \in \hat{R}$ , and  $(\mathbf{if} \hat{\xi}|_0 (\hat{G}; \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_{\emptyset})|_{\emptyset} \mathbf{skip}|_{\emptyset}) \uparrow \hat{r} \xrightarrow{\xi_{\hat{r}}, i_{\hat{R}}^{\hat{r}}} L'_r$  for every  $\hat{r} \in \hat{R}$ , and  $\sqrt{R}(\mathbf{if} \hat{\xi}|_0 (\hat{G}; \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_{\emptyset})|_{\emptyset} \mathbf{skip}|_{\emptyset})$ . Then, by induction,  $\mathbf{if} \hat{\xi}|_0 (\hat{G}; \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_{\emptyset})|_{\emptyset} \mathbf{skip}|_{\emptyset} \xrightarrow{\xi, i_{\hat{R}}} G'$  and  $L'_r = G' \uparrow \hat{r}$  for every  $\hat{r} \in \hat{R}$  and  $\xi_{\hat{r}} = \xi \uparrow \hat{r}$  for every  $\hat{r} \in \hat{R}$ , for some  $G', \xi$ .
  - Recall  $\mathbf{if} \hat{\xi}|_0 (\hat{G}; \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_{\emptyset})|_{\emptyset} \mathbf{skip}|_{\emptyset} \xrightarrow{\xi, i_{\hat{R}}} G'$ . Then, by  $[\rightarrow 1\text{-NWHILE}]$ ,  $\mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_{\emptyset} \xrightarrow{\xi, i_{\hat{R}}} G'$ . Then,  $G \xrightarrow{\xi, i_{\hat{R}}} G'$ . □

*Proof (of Lem. 64).* By Defn. 14:

- **Case:**  $\gamma = \tau$ . Then, **false**.
- **Case:**  $\gamma = q.y := e$ , for some  $q, y, e$ .
  - Recall  $(G \upharpoonright r) \xrightarrow{\xi_r, \gamma \upharpoonright r} L'_r$  for every  $r \in \text{subj}(\gamma)$ . Then,  $(G \upharpoonright r) \xrightarrow{\xi_r, q.y := e \upharpoonright r} L'_r$  for every  $r \in \text{subj}(q.y := e)$ . Then, by Defn. 16,  $(G \upharpoonright r) \xrightarrow{\xi_r, q.y := e \upharpoonright r} L'_r$  for every  $r \in \{q\}$ . Then,  $(G \upharpoonright q) \xrightarrow{\xi_q, q.y := e \upharpoonright q} L'_q$ . Then, by Defn. 18,  $(G \upharpoonright q) \xrightarrow{\xi_q, q.y := e} L'_q$ . Then, by Lem. 61,  $G \xrightarrow{\xi, q.y := e} G'$  and  $L'_q = G' \upharpoonright q$  and  $\xi_q = \xi \upharpoonright q$ , for some  $G', \xi$ .
  - Recall  $G \xrightarrow{\xi, q.y := e} G'$ . Then,  $G \xrightarrow{\xi, \gamma} G'$ .
  - Recall  $L'_q = G' \upharpoonright q$ . Then,  $L'_r = G' \upharpoonright r$  for every  $r \in \{q\}$ . Then, by Defn. 16,  $L'_r = G' \upharpoonright r$  for every  $r \in \text{subj}(q.y := e)$ . Then,  $L'_r = G' \upharpoonright r$  for every  $r \in \text{subj}(\gamma)$ .
  - Recall  $\xi_q = \xi \upharpoonright q$ . Then,  $\xi_r = \xi \upharpoonright r$  for every  $r \in \{q\}$ . Then, by Defn. 16,  $\xi_r = \xi \upharpoonright r$  for every  $r \in \text{subj}(q.y := e)$ . Then,  $\xi_r = \xi \upharpoonright r$  for every  $r \in \text{subj}(\gamma)$ .
- **Case:**  $\gamma = p.e \rightarrow q.y$ , for some  $p, q, y, e$ .
  - Recall  $(G \upharpoonright r) \xrightarrow{\xi_r, \gamma \upharpoonright r} L'_r$  for every  $r \in \text{subj}(\gamma)$ . Then,  $(G \upharpoonright r) \xrightarrow{\xi_r, p.e \rightarrow q.y \upharpoonright r} L'_r$  for every  $r \in \text{subj}(p.e \rightarrow q.y)$ . Then, by Defn. 16,  $(G \upharpoonright r) \xrightarrow{\xi_r, p.e \rightarrow q.y \upharpoonright r} L'_r$  for every  $r \in \{p, q\}$ . Then,  $(G \upharpoonright p) \xrightarrow{\xi_p, p.e \rightarrow q.y \upharpoonright p} L'_p$  and  $(G \upharpoonright q) \xrightarrow{\xi_q, p.e \rightarrow q.y \upharpoonright q} L'_q$ . Then, by Defn. 18,  $(G \upharpoonright p) \xrightarrow{\xi_p, pq!e} L'_p$  and  $(G \upharpoonright q) \xrightarrow{\xi_q, pq!e} L'_q$ . Then, by Defn. 18,  $(G \upharpoonright p) \xrightarrow{\xi_p, pq!e} L'_p$  and  $(G \upharpoonright q) \xrightarrow{\xi_q, pq!e} L'_q$  and  $\sqrt{R}(G)$ . Then, by Lem. 62,  $G \xrightarrow{\xi, p.e \rightarrow q.y} G'$  and  $L'_p = G' \upharpoonright p$  and  $\xi_p = \xi \upharpoonright p$  and  $L'_q = G' \upharpoonright q$  and  $\xi_q = \xi \upharpoonright q$ , for some  $G', \xi$ .
  - Recall  $G \xrightarrow{\xi, p.e \rightarrow q.y} G'$ . Then,  $G \xrightarrow{\xi, \gamma} G'$ .
  - Recall  $L'_p = G' \upharpoonright p$  and  $L'_q = G' \upharpoonright q$ . Then,  $L'_r = G' \upharpoonright r$  for every  $r \in \{p, q\}$ . Then, by Defn. 16,  $L'_r = G' \upharpoonright r$  for every  $r \in \text{subj}(p.e \rightarrow q.y)$ . Then,  $L'_r = G' \upharpoonright r$  for every  $r \in \text{subj}(\gamma)$ .
  - Recall  $\xi_p = \xi \upharpoonright p$  and  $\xi_q = \xi \upharpoonright q$ . Then,  $\xi_r = \xi \upharpoonright r$  for every  $r \in \{p, q\}$ . Then, by Defn. 16,  $\xi_r = \xi \upharpoonright r$  for every  $r \in \text{subj}(p.e \rightarrow q.y)$ . Then,  $\xi_r = \xi \upharpoonright r$  for every  $r \in \text{subj}(\gamma)$ .
- **Case:**  $\gamma = i_{\hat{R}}^{\hat{R}}$ , for some  $\hat{R}$ .
  - Recall  $(G \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, \gamma \upharpoonright \hat{r}} L'_{\hat{r}}$  for every  $\hat{r} \in \text{subj}(\gamma)$ . Then,  $(G \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i_{\hat{R}}^{\hat{R}} \upharpoonright \hat{r}} L'_{\hat{r}}$  for every  $\hat{r} \in \text{subj}(i_{\hat{R}}^{\hat{R}})$ . Then, by Defn. 16,  $(G \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i_{\hat{R}}^{\hat{R}} \upharpoonright \hat{r}} L'_{\hat{r}}$  for every  $\hat{r} \in \hat{R}$ . Then, by Defn. 18,  $(G \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i_{\hat{R}}^{\hat{R}} \upharpoonright \hat{r}} L'_{\hat{r}}$  for every  $\hat{r} \in \hat{R}$ .
  - Recall  $\gamma = i_{\hat{R}}^{\hat{R}}$ . Then, by Rem. 2,  $\hat{R} \neq \emptyset$ . Then,  $r \in \hat{R}$ , for some  $r$ .
  - Recall  $r \in \hat{R}$ . Then,  $(G \upharpoonright r) \xrightarrow{\xi_r, i_{\hat{R}}^{\hat{R}} \upharpoonright r} L'_r$ .
  - Recall  $(G \upharpoonright r) \xrightarrow{\xi_r, i_{\hat{R}}^{\hat{R}} \upharpoonright r} L'_r$  and  $r \in \hat{R}$ , and  $(G \upharpoonright \hat{r}) \xrightarrow{\xi_{\hat{r}}, i_{\hat{R}}^{\hat{R}} \upharpoonright \hat{r}} L'_{\hat{r}}$  for every  $\hat{r} \in \hat{R}$ , and  $\sqrt{R}(G)$ . Then, by Lem. 63,  $G \xrightarrow{\xi, i_{\hat{R}}^{\hat{R}}} G'$ , and  $L'_{\hat{r}} = G' \upharpoonright \hat{r}$  for every  $\hat{r} \in \hat{R}$ , and  $\xi_{\hat{r}} = \xi \upharpoonright \hat{r}$  for every  $\hat{r} \in \hat{R}$ , for some  $G', \xi$ .
  - Recall  $G \xrightarrow{\xi, i_{\hat{R}}^{\hat{R}}} G'$ . Then,  $G \xrightarrow{\xi, \gamma} G'$ .
  - Recall  $L'_{\hat{r}} = G' \upharpoonright \hat{r}$  for every  $\hat{r} \in \hat{R}$ . Then, by Defn. 16,  $L'_{\hat{r}} = G' \upharpoonright \hat{r}$  for every  $\hat{r} \in \text{subj}(i_{\hat{R}}^{\hat{R}})$ . Then,  $L'_{\hat{r}} = G' \upharpoonright \hat{r}$  for every  $\hat{r} \in \text{subj}(\gamma)$ .
  - Recall  $\xi_{\hat{r}} = \xi \upharpoonright \hat{r}$  for every  $\hat{r} \in \hat{R}$ . Then, by Defn. 16,  $\xi_{\hat{r}} = \xi \upharpoonright \hat{r}$  for every  $\hat{r} \in \text{subj}(i_{\hat{R}}^{\hat{R}})$ . Then,  $\xi_{\hat{r}} = \xi \upharpoonright \hat{r}$  for every  $\hat{r} \in \text{subj}(\gamma)$ .  $\square$

*Proof (of Lem. 65).* Recall  $A \xrightarrow{\xi, \gamma} A'$ . Then, by Defn. 32:

- **Case:**  $[\rightarrow 2\text{-GLOB}]$ , such that  $G \xrightarrow{\xi, \gamma} G'$ , for some  $G, G'$ .  
Recall  $G \xrightarrow{\xi, \gamma} G'$  and  $(\xi, \gamma) \neq (\text{true}, \tau)$ . Then, by Lem. 50,  $\gamma \neq \tau$ .
- **Case:**  $[\rightarrow 2\text{-LOCS1}]$ , such that  $\gamma = \tau$  and  $L_{\hat{r}} \xrightarrow{\xi, \tau} L'_{\hat{r}}$ , for some  $L_{\hat{r}}, L'_{\hat{r}}$ .
  - Recall  $(\xi, \gamma) \neq (\text{true}, \tau)$ . Then,  $(\xi, \tau) \neq (\text{true}, \tau)$ .
  - Recall  $L_{\hat{r}} \xrightarrow{\xi, \tau} L'_{\hat{r}}$  and  $(\xi, \tau) \neq (\text{true}, \tau)$ . Then, by Lem. 50,  $\tau \neq \tau$ . Then, **false**.
- **Case:**  $[\rightarrow 2\text{-LOCS2}]$ , such that  $\gamma \neq \tau$ .
- **Case:**  $[\rightarrow 2\text{-LOCS3}]$ . Similar to case  $[\rightarrow 2\text{-LOCS3}]$ .
- **Case:**  $[\rightarrow 2\text{-LOCS4}]$ . Similar to case  $[\rightarrow 2\text{-LOCS3}]$ .  $\square$

*Proof (of Lem. 66).* Recall  $\{L_r\}_{r \in R} \xrightarrow{\xi, \gamma} \{L'_r\}_{r \in R}$ . Then, by Defn. 32:

- **Case:**  $[\rightarrow 2\text{-LOCS1}]$ , such that  $\gamma = \tau$ .  
Recall  $\emptyset \subseteq R$ . Then, by Defn. 16,  $\text{subj}(\tau) \subseteq R$ . Then,  $\text{subj}(\gamma) \subseteq R$ .
- **Case:**  $[\rightarrow 2\text{-LOCS2}]$ , such that  $\gamma = q.y := e$  and  $q \in R$ , for some  $q, y, e$ .  
Recall  $q \in R$ . Then,  $\{q\} \subseteq R$ . Then, by Defn. 16,  $\text{subj}(q.y := e) \subseteq R$ . Then,  $\text{subj}(\gamma) \subseteq R$ .
- **Case:**  $[\rightarrow 2\text{-LOCS3}]$ . Similar to case  $[\rightarrow 2\text{-LOCS2}]$ .
- **Case:**  $[\rightarrow 2\text{-LOCS4}]$ . Similar to case  $[\rightarrow 2\text{-LOCS2}]$ . □

*Proof (of Lem. 67).*

1. – Recall  $\{L_r\}_{r \in R} \xrightarrow{\xi, \tau} \{L'_r\}_{r \in R}$ . Then, by Defn. 32,  $\{L_r\}_{r \in R} = \{L_r\}_{r \in R \setminus \{\hat{r}\}} \cup \{L_r\}_{r \in \{\hat{r}\}}$  and  $L_{\hat{r}} \xrightarrow{\xi, \tau} L'_{\hat{r}}$  and  $\hat{r} \in R$ , for some  $\hat{r}$ .  
– Recall  $\hat{r} \in R$ . Then,  $(R \setminus \{\hat{r}\}) \cup \{\hat{r}\} = R$ .  
– Recall  $\{L'_r\}_{r \in R} = \{L_r\}_{r \in R \setminus \{\hat{r}\}} \cup \{L'_r\}_{r \in \{\hat{r}\}}$ . Then,  $\{L'_r\}_{r \in (R \setminus \{\hat{r}\}) \cup \{\hat{r}\}} = \{L_r\}_{r \in R \setminus \{\hat{r}\}} \cup \{L'_r\}_{r \in \{\hat{r}\}}$ . Then,  $\{L'_r\}_{r \in R \setminus \{\hat{r}\}} \cup \{L'_r\}_{r \in \{\hat{r}\}} = \{L_r\}_{r \in R \setminus \{\hat{r}\}} \cup \{L'_r\}_{r \in \{\hat{r}\}}$ . Then,  $\{L'_r\}_{r \in R \setminus \{\hat{r}\}} = \{L_r\}_{r \in R \setminus \{\hat{r}\}}$ . Then,  $L_r = L'_r$  for every  $r \in R \setminus \{\hat{r}\}$ . □
2. By Defn. 14:
  - **Case:**  $\gamma = \tau$ . Then, **false**.
  - **Case:**  $\gamma = q.y := e$ , for some  $q, y, e$ .
    - Recall  $\{L_r\}_{r \in R} \xrightarrow{\xi, \gamma} \{L'_r\}_{r \in R}$ . Then,  $\{L_r\}_{r \in R} \xrightarrow{\xi, q.y := e} \{L'_r\}_{r \in R}$ . Then, by Defn. 32,  $\{L'_r\}_{r \in R} = \{L_r\}_{r \in R \setminus \{q\}} \cup \{L'_r\}_{r \in \{q\}}$  and  $L_q \xrightarrow{\xi, q.y := e} L'_q$  and  $q \in R$ .
    - Recall  $L_q \xrightarrow{\xi, q.y := e} L'_q$ . Then, by Defn. 18,  $L_q \xrightarrow{\xi, q.y := e \upharpoonright q^q} L'_q$ . Then,  $L_r \xrightarrow{\xi, q.y := e \upharpoonright r} L'_r$  for every  $r \in \{q\}$ . Then, by Defn. 16,  $L_r \xrightarrow{\xi, q.y := e \upharpoonright r} L'_r$  for every  $r \in \text{subj}(q.y := e)$ . Then,  $L_r \xrightarrow{\xi, \gamma \upharpoonright r} L'_r$  for every  $r \in \text{subj}(\gamma)$ .
    - Recall  $\xi = \bigwedge \{\xi\}_{r \in \{q\}}$ . Then, by Defn. 16,  $\xi = \bigwedge \{\xi\}_{r \in \text{subj}(q.y := e)}$ . Then,  $\xi = \bigwedge \{\xi\}_{r \in \text{subj}(\gamma)}$ . Then,  $\bigwedge \{\xi_r\}_{r \in \text{subj}(\gamma)} = \bigwedge \{\xi\}_{r \in \text{subj}(\gamma)}$  and  $\xi = \bigwedge \{\xi_r\}_{r \in \text{subj}(\gamma)}$ , for some  $\{\xi_r\}_{r \in \text{subj}(\gamma)}$ .
    - Recall  $q \in R$ . Then,  $(R \setminus \{q\}) \cup \{q\} = R$ .
    - Recall  $\{L'_r\}_{r \in R} = \{L_r\}_{r \in R \setminus \{q\}} \cup \{L'_r\}_{r \in \{q\}}$ . Then,  $\{L'_r\}_{r \in (R \setminus \{q\}) \cup \{q\}} = \{L_r\}_{r \in R \setminus \{q\}} \cup \{L'_r\}_{r \in \{q\}}$ . Then,  $\{L'_r\}_{r \in R \setminus \{q\}} \cup \{L'_r\}_{r \in \{q\}} = \{L_r\}_{r \in R \setminus \{q\}} \cup \{L'_r\}_{r \in \{q\}}$ . Then,  $\{L'_r\}_{r \in R \setminus \{q\}} = \{L_r\}_{r \in R \setminus \{q\}}$ . Then, by Defn. 16,  $\{L'_r\}_{r \in R \setminus \text{subj}(q.y := e)} = \{L_r\}_{r \in R \setminus \text{subj}(q.y := e)}$ . Then,  $\{L'_r\}_{r \in R \setminus \text{subj}(\gamma)} = \{L_r\}_{r \in R \setminus \text{subj}(\gamma)}$ . Then,  $L_r = L'_r$  for every  $r \in R \setminus \text{subj}(\gamma)$ .
  - **Case:**  $\gamma = p.e \rightarrow q.y$ , for some  $p, q, y, e$ . Similar to case “ $\gamma = q.y := e$ , for some  $q, y, e$ ”.
  - **Case:**  $\gamma = i_{\hat{R}}$ , for some  $\hat{R}$ . Similar to case “ $\gamma = q.y := e$ , for some  $q, y, e$ ”. □

*Proof (of Lem. 68).* By Defn. 14:

- **Case:**  $\gamma = \tau$ . Then, **false**.
- **Case:**  $\gamma = q.y := e$ , for some  $q, y, e$ .
  - By Defn. 16,  $\text{subj}(q.y := e) = \{q\}$ . Then,  $\text{subj}(\gamma) = \{q\}$ .
  - Recall  $L_r \xrightarrow{\xi_r, \gamma \upharpoonright r} L'_r$  for every  $r \in \text{subj}(\gamma)$ . Then,  $L_r \xrightarrow{\xi_r, q.y := e \upharpoonright r} L'_r$  for every  $r \in \text{subj}(\gamma)$ . Then,  $L_r \xrightarrow{\xi_r, q.y := e \upharpoonright r} L'_r$  for every  $r \in \{q\}$ . Then,  $L_q \xrightarrow{\xi_q, q.y := e \upharpoonright q} L'_q$ . Then, by Defn. 18,  $L_q \xrightarrow{\xi_q, q.y := e} L'_q$ .
  - Recall  $\text{subj}(\gamma) \subseteq R$ . Then,  $\{q\} \subseteq R$ . Then,  $q \in R$ .
  - Recall  $\text{subj}(\gamma) \subseteq R$ . Then,  $R = (R \setminus \text{subj}(\gamma)) \cup \text{subj}(\gamma)$ .

- Recall  $L_q \xrightarrow{\xi_q, q.y:=e} L'_q$  and  $q \in R$ . Then, by [ $\rightarrow$ 2-LoCS2],
  - $\{L_r\}_{r \in R} \xrightarrow{\xi_q, q.y:=e} \{L'_r\}_{r \in R \setminus \{q\}} \cup \{L'_r\}_{r \in \{q\}}$ . Then,
  - $\{L_r\}_{r \in R} \xrightarrow{\wedge \{\xi_r\}_{r \in \{q\}}, q.y:=e} \{L_r\}_{r \in R \setminus \{q\}} \cup \{L'_r\}_{r \in \{q\}}$ . Then,
  - $\{L_r\}_{r \in R} \xrightarrow{\wedge \{\xi_r\}_{r \in \text{subj}(\gamma)}, q.y:=e} \{L_r\}_{r \in R \setminus \text{subj}(\gamma)} \cup \{L'_r\}_{r \in \text{subj}(\gamma)}$ . Then,
  - $\{L_r\}_{r \in R} \xrightarrow{\wedge \{\xi_r\}_{r \in \text{subj}(\gamma)}, q.y:=e} \{L'_r\}_{r \in R \setminus \text{subj}(\gamma)} \cup \{L'_r\}_{r \in \text{subj}(\gamma)}$ . Then,
  - $\{L_r\}_{r \in R} \xrightarrow{\wedge \{\xi_r\}_{r \in \text{subj}(\gamma)}, q.y:=e} \{L'_r\}_{r \in (R \setminus \text{subj}(\gamma)) \cup \text{subj}(\gamma)}$ . Then,
  - $\{L_r\}_{r \in R} \xrightarrow{\wedge \{\xi_r\}_{r \in \text{subj}(\gamma)}, \gamma} \{L'_r\}_{r \in R}$ .

- **Case:**  $\gamma = p.e \rightarrow q.y$ , for some  $p, q, y, e$ . Similar to case “ $\gamma = q.y := e$ , for some  $q, y, e$ ”.
- **Case:**  $\gamma = i_{\hat{R}}$ , for some  $\hat{R}$ . Similar to case “ $\gamma = q.y := e$ , for some  $q, y, e$ ”. □

## G Reachability

*Proof (of Lem. 69).*

1. Recall  $A \xrightarrow{\{(true, \tau)\}}^* A^\dagger$ . Then, by Defn. 34:
  - **Base:**  $[\rightarrow^* 1\text{-BASE}]$ , such that  $A = A^\dagger$ .  
Recall  $A \downarrow$ . then,  $A^\dagger \downarrow$ .
  - **Step:**  $[\rightarrow^* 1\text{-STEP}]$ , such that  $A \xrightarrow{\{(true, \tau)\}}^* A^* \xrightarrow{\pi} A^\dagger$ , for some  $A^*, \pi$ .
    - Recall  $A \xrightarrow{\{(true, \tau)\}}^* A^*$  and  $A \downarrow$ . Then, by induction,  $A^* \downarrow$ .
    - Recall  $A^* \downarrow$  and  $A^* \xrightarrow{\pi} A^\dagger$ . Then, by Lem. 55,  $A^\dagger \downarrow$ . □
2. Recall  $A \xrightarrow{\{(true, \tau)\}}^* A_1^\dagger$ . Then, by Defn. 34:
  - **Base:**  $[\rightarrow^* 1\text{-BASE}]$ , such that  $A = A_1^\dagger$ .
    - Recall  $A \xrightarrow{\xi, \alpha} A_2'$ . Then,  $A_1^\dagger \xrightarrow{\xi, \alpha} A_2'$ . Then,  $A^\ddagger = A_2'$  and  $A_1^\dagger \xrightarrow{\xi, \alpha} A^\ddagger$ , for some  $A^\ddagger$ .
    - By  $[\rightarrow^* 1\text{-BASE}]$ ,  $A_2' \xrightarrow{\{(true, \tau)\}}^* A_2'$ . Then,  $A_2' \xrightarrow{\{(true, \tau)\}}^* A^\ddagger$ .
  - **Step:**  $[\rightarrow^* 1\text{-STEP}]$ , such that  $A \xrightarrow{\{(true, \tau)\}}^* A_1^* \xrightarrow{\pi} A_1^\dagger$  and  $\pi \in \{(true, \tau)\}$ , for some  $A_1^*, \pi$ .
    - Recall  $\pi \in \{(true, \tau)\}$ . Then,  $\pi = (true, \tau)$ .
    - Recall  $A_1^* \xrightarrow{\pi} A_1^\dagger$ . Then,  $A_1^* \xrightarrow{true, \tau} A_1^\dagger$ .
    - Recall  $A \xrightarrow{\{(true, \tau)\}}^* A_1^*$  and  $A \xrightarrow{\xi, \alpha} A_2'$ . Then, by induction,  $A_1^* \xrightarrow{\xi, \alpha} A^\dagger$  and  $A_2' \xrightarrow{\{(true, \tau)\}}^* A^\dagger$ , for some  $A^\dagger$ .
    - Recall  $A_1^* \xrightarrow{true, \tau} A_1^\dagger$  and  $A_1^* \xrightarrow{\xi, \alpha} A^\dagger$ . Then, by Lem. 56:
      - \* **Case:**  $A_1^\dagger \xrightarrow{\xi, \alpha} A^\dagger$  and  $A^\dagger \xrightarrow{true, \tau} A^\dagger$ , for some  $A^\dagger$ .  
Recall  $A_2' \xrightarrow{\{(true, \tau)\}}^* A^\dagger \xrightarrow{true, \tau} A^\dagger$  and  $(true, \tau) \in \{(true, \tau)\}$ . Then, by  $[\rightarrow^* 1\text{-STEP}]$ ,  $A_2' \xrightarrow{\{(true, \tau)\}}^* A^\dagger$ .
      - \* **Case:**  $\xi = true$  and  $\alpha = \tau$  and  $A_1^\dagger = A^\dagger$ .
        - By  $[\rightarrow^* 1\text{-BASE}]$ ,  $A_1^\dagger \xrightarrow{\{(true, \tau)\}}^* A^\dagger$ . Then,  $A_1^\dagger \xrightarrow{\{(true, \tau)\}}^* A^\dagger$ . Then,  $A^\dagger = A^\dagger$  and  $A_1^\dagger \xrightarrow{\{(true, \tau)\}}^* A^\dagger$ , for some  $A^\dagger$ .
        - Recall  $A_2' \xrightarrow{\{(true, \tau)\}}^* A^\dagger$ . Then,  $A_2' \xrightarrow{\{(true, \tau)\}}^* A^\dagger$ . □
  - 3. Recall  $A \xrightarrow{\{(true, \tau)\}}^* A_1^\dagger$ . Then, by Defn. 34:
    - **Base:**  $[\rightarrow^* 1\text{-BASE}]$ , such that  $A = A_1^\dagger$ .
      - Recall  $A_1^\dagger \xrightarrow{\xi, \alpha} A^\dagger$ . Then,  $A \xrightarrow{\xi, \alpha} A^\dagger$ . Then,  $A_2^\dagger = A^\dagger$  and  $A \xrightarrow{\xi, \alpha} A_2^\dagger$ , for some  $A_2^\dagger$ .
      - By  $[\rightarrow^* 1\text{-BASE}]$ ,  $A^\dagger \xrightarrow{\{(true, \tau)\}}^* A^\dagger$ . Then,  $A_2^\dagger \xrightarrow{\{(true, \tau)\}}^* A^\dagger$ .
    - **Step:**  $[\rightarrow^* 1\text{-STEP}]$ , such that  $A \xrightarrow{\{(true, \tau)\}}^* A_1^* \xrightarrow{\pi} A_1^\dagger$  and  $\pi \in \{(true, \tau)\}$ , for some  $A_1^*, \pi$ .
      - Recall  $\pi \in \{(true, \tau)\}$ . Then,  $\pi = (true, \tau)$ .
      - Recall  $A_1^* \xrightarrow{\pi} A_1^\dagger$ . Then,  $A_1^* \xrightarrow{true, \tau} A_1^\dagger$ .
      - Recall  $A_1^* \xrightarrow{true, \tau} A_1^\dagger \xrightarrow{\xi, \alpha} A^\dagger$ . Then, by Lem. 56,  $A_1^* \xrightarrow{\xi, \alpha} A_2^\dagger \xrightarrow{true, \tau} A^\dagger$ , for some  $A_2^\dagger$ .
      - Recall  $A \xrightarrow{\{(true, \tau)\}}^* A_1^* \xrightarrow{\xi, \alpha} A_2^\dagger$ . Then, by induction,  $A \xrightarrow{\xi, \alpha} A_2^\dagger$  and  $A_2^\dagger \xrightarrow{\{(true, \tau)\}}^* A^\dagger$ , for some  $A_2^\dagger$ .
      - Recall  $A_2^\dagger \xrightarrow{\{(true, \tau)\}}^* A^\dagger \xrightarrow{true, \tau} A^\dagger$  and  $(true, \tau) \in \{(true, \tau)\}$ . Then, by  $[\rightarrow^* 1\text{-STEP}]$ ,  $A_2^\dagger \xrightarrow{\{(true, \tau)\}}^* A^\dagger$ . □

*Proof (of Lem. 70).*

1. – Recall  $A \Longrightarrow A^\dagger$ . Then, by Defn. 34,  $A \xrightarrow{\{(true, \tau)\}}^* A^\dagger$ .  
– Recall  $A \xrightarrow{\{(true, \tau)\}}^* A^\dagger$  and  $A \downarrow$ . Then, by Lem. 69,  $A^\dagger \downarrow$ . □
2. – Recall  $A_2 \Longrightarrow A^\dagger$ . Then, by Defn. 34,  $A_2 \xrightarrow{\{(true, \tau)\}}^* A^\dagger$ .  
– Recall  $A_2 \xrightarrow{\{(true, \tau)\}}^* A^\dagger$  and  $A_2 \xrightarrow{true, \tau} A_2'$ . Then, by Lem. 69,  $A^\dagger \xrightarrow{true, \tau} A^\dagger$  and  $A_2' \xrightarrow{\{(true, \tau)\}}^* A^\dagger$ , for some  $A^\dagger$ .  
– Recall  $A_1 \Longrightarrow A^\dagger$ . Then, by Defn. 34,  $A_1 \xrightarrow{\{(true, \tau)\}}^* A^\dagger$ .  
– Recall  $A_1 \xrightarrow{\{(true, \tau)\}}^* A^\dagger \xrightarrow{true, \tau} A^\dagger$  and  $(true, \tau) \in \{(true, \tau)\}$ . Then, by  $[\rightarrow^* 1\text{-STEP}]$ ,  $A_1 \xrightarrow{\{(true, \tau)\}}^* A^\dagger$ . Then, by  $[\Rightarrow 1]$ ,  $A_1 \Longrightarrow A^\dagger$ .  
– Recall  $A_2' \xrightarrow{\{(true, \tau)\}}^* A^\dagger$ . Then, by  $[\Rightarrow 1]$ ,  $A_2' \Longrightarrow A^\dagger$ . □
3. – Recall  $A_2 \Longrightarrow A^\dagger$ . Then, by Defn. 34,  $A_2 \xrightarrow{\{(true, \tau)\}}^* A^\dagger$ .  
– Recall  $A_2 \xrightarrow{\{(true, \tau)\}}^* A^\dagger$  and  $A_2 \xrightarrow{\xi, \alpha} A_2'$ . Then, by Lem. 69,  $A^\dagger \xrightarrow{\xi, \alpha} A^\dagger$  and  $A_2' \xrightarrow{\{(true, \tau)\}}^* A^\dagger$ , for some  $A^\dagger$ .

- Recall  $A_1 \xRightarrow{\text{true}, \tau} A^\dagger$ . Then, by Defn. 34,  $A_1 \xrightarrow{\{(\text{true}, \tau)\}_*} A^\dagger$ .
- Recall  $A_1 \xrightarrow{\{(\text{true}, \tau)\}_*} A^\dagger \xrightarrow{\xi, \alpha} A^\ddagger$ . Then, by Lem. 69,  $A_1 \xrightarrow{\xi, \alpha} A'_1$  and  $A'_1 \xrightarrow{\{(\text{true}, \tau)\}_*} A^\ddagger$ , for some  $A'_1$ .
- Recall  $A'_1 \xrightarrow{\{(\text{true}, \tau)\}_*} A^\ddagger$  and  $A'_2 \xrightarrow{\{(\text{true}, \tau)\}_*} A^\ddagger$ . Then, by  $[\Rightarrow 1]$ ,  $A'_1 \xRightarrow{\text{true}, \tau} A^\ddagger$  and  $A'_2 \xRightarrow{\text{true}, \tau} A^\ddagger$ . □

*Proof (of Lem. 71).* Recall  $\mathcal{A}^\dagger \xrightarrow{\Pi} \mathcal{A}^\S$ . Then, by Defn. 35:

- **Base:**  $[\rightarrow^* 2\text{-BASE}]$ , such that  $\mathcal{A}^\dagger = \mathcal{A}^\S$ .  
Recall  $\mathcal{A} \xrightarrow{\Pi} \mathcal{A}^\dagger$ . Then,  $\mathcal{A} \xrightarrow{\Pi} \mathcal{A}^\S$ .
- **Step:**  $[\rightarrow^* 2\text{-STEP}]$ , such that  $\mathcal{A}^\dagger \xrightarrow{\Pi} \mathcal{A}^\ddagger \xrightarrow{\pi} \mathcal{A}^\S$  and  $\pi \in \Pi$ , for some  $\mathcal{A}^\ddagger, \pi$ .
  - Recall  $\mathcal{A} \xrightarrow{\Pi} \mathcal{A}^\dagger \xrightarrow{\Pi} \mathcal{A}^\ddagger$ . Then, by induction,  $\mathcal{A} \xrightarrow{\Pi} \mathcal{A}^\ddagger$ .
  - Recall  $\mathcal{A} \xrightarrow{\Pi} \mathcal{A}^\ddagger \xrightarrow{\pi} \mathcal{A}^\S$  and  $\pi \in \Pi$ . Then, by  $[\rightarrow^* 2\text{-STEP}]$ ,  $\mathcal{A} \xrightarrow{\Pi} \mathcal{A}^\S$ . □

*Proof (of Lem. 72).*

1. Recall  $L_{\hat{r}} \xrightarrow{\{(\text{true}, \tau)\}_*} L_{\hat{r}}^\dagger$ . Then, by Defn. 34:
  - **Base:**  $[\rightarrow^* 1\text{-BASE}]$ , such that  $L_{\hat{r}} = L_{\hat{r}}^\dagger$ .
    - Recall  $\hat{r} \in R$ . Then,  $R = (R \setminus \{\hat{r}\}) \cup \{\hat{r}\}$
    - Recall  $L_{\hat{r}} = L_{\hat{r}}^\dagger$ . Then,  $\{L_r\}_{r \in \{\hat{r}\}} = \{L_r^\dagger\}_{r \in \{\hat{r}\}}$ .
    - By  $[\rightarrow^* 2\text{-BASE}]$ ,  $\{L_r\}_{r \in R} \xrightarrow{\{(\text{true}, \tau)\}_*} \{L_r\}_{r \in R}$ . Then,  $\{L_r\}_{r \in R} \xrightarrow{\{(\text{true}, \tau)\}_*} \{L_r\}_{r \in (R \setminus \{\hat{r}\}) \cup \{\hat{r}\}}$ . Then,  $\{L_r\}_{r \in R} \xrightarrow{\{(\text{true}, \tau)\}_*} \{L_r\}_{r \in R \setminus \{\hat{r}\}} \cup \{L_r\}_{r \in \{\hat{r}\}}$ . Then,  $\{L_r\}_{r \in R} \xrightarrow{\{(\text{true}, \tau)\}_*} \{L_r\}_{r \in R \setminus \{\hat{r}\}} \cup \{L_r^\dagger\}_{r \in \{\hat{r}\}}$ .
  - **Step:**  $[\rightarrow^* 1\text{-STEP}]$ , such that  $L_{\hat{r}} \xrightarrow{\{(\text{true}, \tau)\}_*} L_{\hat{r}}^* \xrightarrow{\pi} L_{\hat{r}}^\dagger$  and  $\pi \in \{(\text{true}, \tau)\}$ , for some  $L_{\hat{r}}^*, \pi$ .
    - Recall  $L_{\hat{r}} \xrightarrow{\{(\text{true}, \tau)\}_*} L_{\hat{r}}^*$  and  $\hat{r} \in R$ . Then, by induction,  $\{L_r\}_{r \in R} \xrightarrow{\{(\text{true}, \tau)\}_*} \{L_r\}_{r \in R \setminus \{\hat{r}\}} \cup \{L_r^*\}_{r \in \{\hat{r}\}}$ .
    - Recall  $\pi \in \{(\text{true}, \tau)\}$ . Then,  $\pi = (\text{true}, \tau)$ .
    - Recall  $L_{\hat{r}}^* \xrightarrow{\text{true}, \tau} L_{\hat{r}}^\dagger$ . Then,  $L_{\hat{r}}^* \xrightarrow{\text{true}, \tau} L_{\hat{r}}^\dagger$ .
    - Recall  $L_{\hat{r}}^* \xrightarrow{\text{true}, \tau} L_{\hat{r}}^\dagger$  and  $\hat{r} \in (R \setminus \{\hat{r}\}) \cup \{\hat{r}\}$ . Then, by  $[\rightarrow 2\text{-LOCS1}]$ ,  $\{L_r\}_{r \in R \setminus \{\hat{r}\}} \cup \{L_r^*\}_{r \in \{\hat{r}\}} \xrightarrow{\text{true}, \tau} \{L_r\}_{r \in R \setminus \{\hat{r}\}} \cup \{L_r^*\}_{r \in \{\hat{r}\} \setminus \{\hat{r}\}} \cup \{L_r^\dagger\}_{r \in \{\hat{r}\}}$ . Then,  $\{L_r\}_{r \in R \setminus \{\hat{r}\}} \cup \{L_r^*\}_{r \in \{\hat{r}\}} \xrightarrow{\text{true}, \tau} \{L_r\}_{r \in R \setminus \{\hat{r}\}} \cup \{L_r^*\}_{r \in \setminus \{\hat{r}\}} \cup \{L_r^\dagger\}_{r \in \{\hat{r}\}}$ . Then,  $\{L_r\}_{r \in R \setminus \{\hat{r}\}} \cup \{L_r^*\}_{r \in \{\hat{r}\}} \xrightarrow{\text{true}, \tau} \{L_r\}_{r \in R \setminus \{\hat{r}\}} \cup \{L_r^\dagger\}_{r \in \{\hat{r}\}}$ .
    - Recall  $\{L_r\}_{r \in R} \xrightarrow{\{(\text{true}, \tau)\}_*} \{L_r\}_{r \in R \setminus \{\hat{r}\}} \cup \{L_r^*\}_{r \in \{\hat{r}\}} \xrightarrow{\text{true}, \tau} \{L_r\}_{r \in R \setminus \{\hat{r}\}} \cup \{L_r^\dagger\}_{r \in \{\hat{r}\}}$  and  $(\text{true}, \tau) \in \{(\text{true}, \tau)\}$ . Then, by  $[\rightarrow^* 2\text{-STEP}]$ ,  $\{L_r\}_{r \in R} \xrightarrow{\{(\text{true}, \tau)\}_*} \{L_r\}_{r \in R \setminus \{\hat{r}\}} \cup \{L_r^\dagger\}_{r \in \{\hat{r}\}}$ . □
2. Recall:
  - **Base:**  $\hat{R} = \emptyset$ .  
By  $[\rightarrow^* 2\text{-BASE}]$ ,  $\{L_r\}_{r \in R} \xrightarrow{\{(\text{true}, \tau)\}_*} \{L_r\}_{r \in R}$ . Then,  $\{L_r\}_{r \in R} \xrightarrow{\{(\text{true}, \tau)\}_*} \{L_r\}_{r \in R \setminus \emptyset}$ . Then,  $\{L_r\}_{r \in R} \xrightarrow{\{(\text{true}, \tau)\}_*} \{L_r\}_{r \in R \setminus \emptyset} \cup \{L_r^\dagger\}_{r \in \emptyset}$ . Then,  $\{L_r\}_{r \in R} \xrightarrow{\{(\text{true}, \tau)\}_*} \{L_r\}_{r \in R \setminus \hat{R}} \cup \{L_r^\dagger\}_{r \in \hat{R}}$ .
  - **Step:**  $\hat{R} = \hat{R}' \cup \{\hat{r}\}$  and  $\hat{R}' = \hat{R} \setminus \{\hat{r}\}$ , for some  $\hat{R}', \hat{r}$ .
    - Recall  $L_r \xrightarrow{\{(\text{true}, \tau)\}_*} L_r^\dagger$  for every  $r \in \hat{R}$ . Then,  $L_r \xrightarrow{\{(\text{true}, \tau)\}_*} L_r^\dagger$  for every  $r \in \hat{R}' \cup \{\hat{r}\}$ . Then,  $L_r \xrightarrow{\{(\text{true}, \tau)\}_*} L_r^\dagger$  for every  $r \in \hat{R}'$ .
    - Recall  $\hat{R} \subseteq R$ . Then,  $\hat{R}' \cup \{\hat{r}\} \subseteq R$ . Then,  $\hat{R}' \subseteq R$ .
    - Recall  $L_r \xrightarrow{\{(\text{true}, \tau)\}_*} L_r^\dagger$  for every  $r \in \hat{R}' \subseteq R$ . Then, by induction,  $\{L_r\}_{r \in R} \xrightarrow{\{(\text{true}, \tau)\}_*} \{L_r\}_{r \in R \setminus \hat{R}'} \cup \{L_r^\dagger\}_{r \in \hat{R}'}$ .
    - Recall  $\hat{r} \in \hat{R}' \cup \{\hat{r}\}$ . Then,  $\hat{r} \in \hat{R}$ .
    - Recall  $\hat{r} \in \hat{R}$ . Then,  $L_{\hat{r}} \xrightarrow{\{(\text{true}, \tau)\}_*} L_{\hat{r}}^\dagger$ .
    - Recall  $\hat{R}' \subseteq R$ . Then,  $R = (R \setminus \hat{R}') \cup \hat{R}'$ .
    - Recall  $\hat{r} \in \hat{R} \subseteq R$ . Then,  $\hat{r} \in R$ . Then,  $\hat{r} \in (R \setminus \hat{R}') \cup \hat{R}'$ .



- Recall  $L_{\hat{r}} \xrightarrow{\{\text{true}, \tau\}^*} L_{\hat{r}}^\dagger$  and  $\hat{r} \in (R \setminus \hat{R}') \cup \hat{R}'$ . Then, by 1,
 
$$\{L_r\}_{r \in R \setminus \hat{R}'} \cup \{L_r^\dagger\}_{r \in \hat{R}'} \xrightarrow{\{\text{true}, \tau\}^*} \{L_r\}_{r \in (R \setminus \hat{R}') \setminus \{\hat{r}\}} \cup \{L_r^\dagger\}_{r \in \hat{R}' \setminus \{\hat{r}\}} \cup \{L_r^\dagger\}_{r \in \{\hat{r}\}}.$$
 Then,  $\{L_r\}_{r \in R \setminus \hat{R}'} \cup \{L_r^\dagger\}_{r \in \hat{R}'}$   $\xrightarrow{\{\text{true}, \tau\}^*}$   $\{L_r\}_{r \in (R \setminus \hat{R}') \setminus \{\hat{r}\}} \cup \{L_r^\dagger\}_{r \in (\hat{R}' \setminus \{\hat{r}\}) \cup \{\hat{r}\}}.$   
 Then,  $\{L_r\}_{r \in R \setminus \hat{R}'} \cup \{L_r^\dagger\}_{r \in \hat{R}'}$   $\xrightarrow{\{\text{true}, \tau\}^*}$   $\{L_r\}_{r \in R \setminus (\hat{R}' \cup \{\hat{r}\})} \cup \{L_r^\dagger\}_{r \in \hat{R}' \cup \{\hat{r}\}}.$  Then,
 
$$\{L_r\}_{r \in R \setminus \hat{R}'} \cup \{L_r^\dagger\}_{r \in \hat{R}'}$$
  $\xrightarrow{\{\text{true}, \tau\}^*}$   $\{L_r\}_{r \in R \setminus \hat{R}} \cup \{L_r^\dagger\}_{r \in \hat{R}}.$
  - Recall  $\{L_r\}_{r \in R} \xrightarrow{\{\text{true}, \tau\}^*} \{L_r\}_{r \in R \setminus \hat{R}'} \cup \{L_r^\dagger\}_{r \in \hat{R}'}$   $\xrightarrow{\{\text{true}, \tau\}^*}$   $\{L_r\}_{r \in R \setminus \hat{R}} \cup \{L_r^\dagger\}_{r \in \hat{R}}.$  Then, by Lem. 71,  $\{L_r\}_{r \in R} \xrightarrow{\{\text{true}, \tau\}^*} \{L_r\}_{r \in R \setminus \hat{R}} \cup \{L_r^\dagger\}_{r \in \hat{R}}.$   $\square$
3. Recall  $L_r \xrightarrow{\{\text{true}, \tau\}^*} L_r^\dagger$  for every  $r \in R \subseteq R$ . Then, by 2,  $\{L_r\}_{r \in R} \xrightarrow{\{\text{true}, \tau\}^*} \{L_r\}_{r \in R \setminus R} \cup \{L_r^\dagger\}_{r \in R}.$  Then,  $\{L_r\}_{r \in R} \xrightarrow{\{\text{true}, \tau\}^*} \{L_r\}_{r \in \emptyset} \cup \{L_r^\dagger\}_{r \in R}.$  Then,  $\{L_r\}_{r \in R} \xrightarrow{\{\text{true}, \tau\}^*} \{L_r^\dagger\}_{r \in R}.$   $\square$

*Proof (of Lem. 73).* Recall  $\mathcal{A} \xrightarrow{\{\text{true}, \tau\}^*} \mathcal{A}^\dagger$ . Then, by Defn. 35:

- **Base:**  $[\rightarrow^*2\text{-BASE}]$ , such that  $\mathcal{A} = \mathcal{A}^\dagger$ .  
 By  $[\rightarrow^*3\text{-BASE}]$ ,  $(\mathcal{A}, \mathcal{S}) \xrightarrow{\{\tau\}^*} (\mathcal{A}, \mathcal{S})$ . Then,  $(\mathcal{A}, \mathcal{S}) \xrightarrow{\{\tau\}^*} (\mathcal{A}^\dagger, \mathcal{S})$ .
- **Step:**  $[\rightarrow^*2\text{-STEP}]$ , such that  $\mathcal{A} \xrightarrow{\pi \in \{\text{true}, \tau\}^*} \mathcal{A}^* \xrightarrow{\pi} \mathcal{A}^\dagger$  and  $\pi \in \{\{\text{true}, \tau\}\}$ , for some  $\mathcal{A}^*$ .
  - Recall  $\mathcal{A} \xrightarrow{\pi \in \{\text{true}, \tau\}^*} \mathcal{A}^*$ . Then, by induction,  $(\mathcal{A}, \mathcal{S}) \xrightarrow{\{\tau\}^*} (\mathcal{A}^*, \mathcal{S})$ .
  - Recall  $\pi \in \{\{\text{true}, \tau\}\}$ . Then,  $\pi = (\text{true}, \tau)$ .
  - Recall  $\mathcal{A}^* \xrightarrow{\pi} \mathcal{A}^\dagger$ . Then,  $\mathcal{A}^* \xrightarrow{\text{true}, \tau} \mathcal{A}^\dagger$ .
  - By Defn. 4,  $\text{eval}_{\mathcal{S}}(\text{true}) = \text{true}$ . Then,  $\mathcal{S} \in \{\hat{\mathcal{S}} \mid \text{eval}_{\hat{\mathcal{S}}}(\text{true}) = \text{true}\}$ . Then, by Defn. 13,  $\mathcal{S} \in \llbracket \text{true} \rrbracket$ .
  - By Defn. 19,  $\tau = \text{cons}_{\mathcal{S}}(\tau)$ .
  - Recall  $\mathcal{A}^* \xrightarrow{\text{true}, \tau} \mathcal{A}^\dagger$  and  $\mathcal{S} \in \llbracket \text{true} \rrbracket$  and  $\tau = \text{cons}_{\mathcal{S}}(\tau)$ . Then, by  $[\rightarrow 3]$ ,  $(\mathcal{A}^*, \mathcal{S}) \xrightarrow{\tau} (\mathcal{A}^\dagger, \text{effect}(\tau, \mathcal{S}))$ . Then, by Defn. 20,  $(\mathcal{A}^*, \mathcal{S}) \xrightarrow{\tau} (\mathcal{A}^\dagger, \mathcal{S})$ .
  - Recall  $(\mathcal{A}, \mathcal{S}) \xrightarrow{\{\tau\}^*} (\mathcal{A}^*, \mathcal{S}) \xrightarrow{\tau} (\mathcal{A}^\dagger, \mathcal{S})$  and  $\tau \in \{\tau\}$ . Then, by  $[\rightarrow^*3\text{-STEP}]$ ,  $(\mathcal{A}, \mathcal{S}) \xrightarrow{\{\tau\}^*} (\mathcal{A}^\dagger, \mathcal{S})$ .  $\square$

*Proof (of Lem. 74).* Recall  $\mathcal{A} \implies \mathcal{A}^\dagger$ . Then, by Defn. 36,  $\mathcal{A} \xrightarrow{\{\text{true}, \tau\}^*} \mathcal{A}^\dagger$ . Then, by Lem. 73,  $(\mathcal{A}, \mathcal{S}) \xrightarrow{\{\tau\}^*} (\mathcal{A}^\dagger, \mathcal{S})$ . Then, by  $[\implies 3]$ ,  $(\mathcal{A}, \mathcal{S}) \implies (\mathcal{A}^\dagger, \mathcal{S})$ .  $\square$

## H Theorems: Deadlock Freedom and Functional Correctness

*Proof (of Lem. 75).*

1. By Defn. 14:
  - **Base:**  $\gamma = \tau$ .  
Recall  $\neg\chi = \neg\chi$ . Then, by Defn. 37,  $\phi(\tau, \neg\chi) = \neg\phi(\tau, \chi)$ . Then,  $\phi(\gamma, \neg\chi) = \neg\phi(\gamma, \chi)$ .
  - **Base:**  $\gamma = q.y := e$ , for some  $q, y, e$ .  
By Defn. 7,  $\neg\chi[e/q.y] = \neg(\chi[e/q.y])$ . Then, by Defn. 37,  $\phi(q.y := e, \neg\chi) = \neg\phi(q.y := e, \chi)$ .  
Then,  $\phi(\gamma, \neg\chi) = \neg\phi(\gamma, \chi)$ .
  - **Base:**  $\gamma = p.e \rightarrow q.y$ , for some  $p, q, y, e$ . Similar to case “ $\gamma = q.y := e$ , for some  $q, y, e$ ”.
  - **Base:**  $\gamma = i_R^R$ , for some  $R$ . Similar to case  $\gamma = \tau$ . □
2. Similar to 1. □
3. Recall  $\phi(\gamma, \chi_1 \Rightarrow \chi_2) = \phi(\gamma, \chi_1 \Rightarrow \chi_2)$ . Then, by Defn. 6,  
 $\phi(\gamma, \chi_1 \Rightarrow \chi_2) = \phi(\gamma, \neg(\chi_1 \wedge \neg\chi_2))$ . Then, by 1,  $\phi(\gamma, \chi_1 \Rightarrow \chi_2) = \neg\phi(\gamma, \chi_1 \wedge \neg\chi_2)$ . Then, by 2,  
 $\phi(\gamma, \chi_1 \Rightarrow \chi_2) = \neg(\phi(\gamma, \chi_1) \wedge \phi(\gamma, \neg\chi_2))$ . Then, by 1,  
 $\phi(\gamma, \chi_1 \Rightarrow \chi_2) = \neg(\phi(\gamma, \chi_1) \wedge \neg\phi(\gamma, \chi_2))$ . Then, by Defn. 6,  $\phi(\gamma, \chi_1) \Rightarrow \phi(\gamma, \chi_2)$ . □

*Proof (of Lem. 76).* By Defn. 14:

- **Base:**  $\gamma = \tau$ .  
By Defn. 37,  $\phi(\tau, \chi) = \chi$ . Then,  $\phi(\gamma, \chi) = \chi$ .
- **Base:**  $\gamma = q.y := e$ , for some  $q, y, e$ .  
Recall  $\text{write}(\gamma) \cap \text{read}(\chi) = \emptyset$ . Then,  $\text{write}(q.y := e) \cap \text{read}(\chi) = \emptyset$ . Then, by Defn. 15,  
 $\{q.y\} \cap \text{read}(\chi) = \emptyset$ . Then, by Defn. 9,  $\{q.y\} \cap \text{rx}(\chi) = \emptyset$ . Then,  $q.y \notin \text{rx}(\chi)$ . Then, by Lem. 1,  
 $\chi[e/q.y] = \chi$ . Then, by Defn. 37,  $\phi(q.y := e, \chi) = \chi$ . Then,  $\phi(\gamma, \chi) = \chi$ .
- **Base:**  $\gamma = p.e \rightarrow q.y$ , for some  $p, q, y, e$ . Similar to case “ $\gamma = q.y := e$ , for some  $q, y, e$ ”.
- **Base:**  $\gamma = i_R^R$ , for some  $R$ . Similar to case  $\gamma = \tau$ . □

*Proof (of Lem. 77).* By Defn. 14:

- **Base:**  $\gamma_1 = \tau$ .  
Recall  $\phi(\gamma_2, \chi) = \phi(\gamma_2, \chi)$ . Then, by Defn. 37,  $\phi(\tau, \phi(\gamma_2, \chi)) = \phi(\gamma_2, \phi(\tau, \chi))$ . Then,  
 $\phi(\gamma_1, \phi(\gamma_2, \chi)) = \phi(\gamma_2, \phi(\gamma_1, \chi))$ .
- **Base:**  $\gamma_1 = q_1.y_1 := e_1$ , for some  $q_1, y_1, e_1$ .  
By Defn. 14:
  - **Base:**  $\gamma_2 = \tau$ .  
Recall  $\phi(\gamma_1, \chi) = \phi(\gamma_1, \chi)$ . Then, by Defn. 37,  $\phi(\gamma_1, \phi(\tau, \chi)) = \phi(\tau, \phi(\gamma_1, \chi))$ . Then,  
 $\phi(\gamma_1, \phi(\gamma_2, \chi)) = \phi(\gamma_2, \phi(\gamma_1, \chi))$ .
  - **Base:**  $\gamma_2 = q_2.y_2 := e_2$ , for some  $q_2, y_2, e_2$ .
    - \* Recall  $\text{write}(\gamma_1) \cap \text{write}(\gamma_2) = \emptyset$ . Then,  $\text{write}(q_1.y_1 := e_1) \cap \text{write}(q_2.y_2 := e_2) = \emptyset$ . Then,  
by Defn. 15,  $\{q_1.y_1\} \cap \{q_2.y_2\} = \emptyset$ . Then,  $q_1.y_1 \neq q_2.y_2$ .
    - \* Recall  $\text{read}(\gamma_1) \cap \text{write}(\gamma_2) = \emptyset$  and  $\text{write}(\gamma_1) \cap \text{read}(\gamma_2) = \emptyset$ . Then,  
 $\text{read}(q_1.y_1 := e_1) \cap \text{write}(q_2.y_2 := e_2) = \emptyset$  and  $\text{write}(q_1.y_1 := e_1) \cap \text{read}(q_2.y_2 := e_2) = \emptyset$ .  
Then, by Defn. 15,  $\text{rx}(e_1) \cap \{q_2.y_2\} = \emptyset$  and  $\{q_1.y_1\} \cap \text{rx}(e_2) = \emptyset$ . Then,  $q_2.y_2 \notin \text{rx}(e_1)$   
and  $q_1.y_1 \notin \text{rx}(e_2)$ .
    - \* Recall  $q_1.y_1 \neq q_2.y_2$  and  $q_2.y_2 \notin \text{rx}(e_1)$  and  $q_1.y_1 \notin \text{rx}(e_2)$ . Then, by Lem. 2,  
 $\chi[e_1/q_1.y_1][e_2/q_2.y_2] = \chi[e_2/q_2.y_2][e_1/q_1.y_1]$ . Then, by Defn. 37,  
 $\phi(q_2.y_2 := e_2, \chi[e_1/q_1.y_1]) = \phi(q_1.y_1 := e_1, \chi[e_2/q_2.y_2])$ . Then, by Defn. 37,  
 $\phi(q_2.y_2 := e_2, \phi(q_1.y_1 := e_1, \chi)) = \phi(q_1.y_1 := e_1, \phi(q_2.y_2 := e_2, \chi))$ . Then,  
 $\phi(\gamma_2, \phi(\gamma_1, \chi)) = \phi(\gamma_1, \phi(\gamma_2, \chi))$ .
  - **Base:**  $\gamma_2 = p_2.e_2 \rightarrow q_2.y_2$ , for some  $p_2, q_2, e_2, y_2$ . Similar to case “ $\gamma_2 = q_2.y_2 := e_2$ , for  
some  $q_2, y_2, e_2$ ”.
  - **Base:**  $\gamma_2 = i_{R_2}^{R_2}$ , for some  $R_2$ . Similar to case  $\gamma_2 = \tau$ .

- **Base:**  $\gamma_1 = p_1.e_1 \rightarrow q_1.y_1$ , for some  $p_1, q_1, e_1, y_1$ . Similar to case “ $\gamma_1 = q_1.y_1 := e_1$ , for some  $q_1, y_1, e_1$ ”.
- **Base:**  $\gamma_1 = i_{R_1}^R$ , for some  $R_1$ . Similar to case  $\gamma_1 = \tau$ . □

*Proof (of Lem. 78).* By Defn. 14:

- **Base:**  $\gamma = \tau$ .  
Recall  $\llbracket \chi_1 \rrbracket \subseteq \llbracket \chi_2 \rrbracket$ . Then, by Defn. 37,  $\llbracket \Phi(\tau, \chi_1) \rrbracket \subseteq \llbracket \Phi(\tau, \chi_2) \rrbracket$ . Then,  $\llbracket \Phi(\gamma, \chi_1) \rrbracket \subseteq \llbracket \Phi(\gamma, \chi_2) \rrbracket$ .
- **Base:**  $\gamma = q.y := e$ , for some  $q, y, e$ .  
Recall  $\llbracket \chi_1 \rrbracket \subseteq \llbracket \chi_2 \rrbracket$ . Then, by Lem. 20,  $\llbracket \chi_1[e/q.y] \rrbracket \subseteq \llbracket \chi_2[e/q.y] \rrbracket$ . Then, by Defn. 37,  $\llbracket \Phi(q.y := e, \chi_1) \rrbracket \subseteq \llbracket \Phi(q.y := e, \chi_2) \rrbracket$ . Then,  $\llbracket \Phi(\gamma, \chi_1) \rrbracket \subseteq \llbracket \Phi(\gamma, \chi_2) \rrbracket$ .
- **Base:**  $\gamma = p.e \rightarrow q.y$ , for some  $p, q, y, e$ . Similar to case “ $\gamma = q.y := e$ , for some  $q, y, e$ ”.
- **Base:**  $\gamma = i_R^R$ , for some  $R$ . Similar to case  $\gamma = \tau$ . □

*Proof (of Lem. 79).* Recall  $\mathcal{S} \in \llbracket \Phi(\gamma, \chi) \rrbracket$ . Then, by Defn. 37:

- **Base:**  $\gamma = \tau$  and  $\Phi(\gamma, \tau) = \tau$ .  
Recall  $\mathcal{S} \in \llbracket \Phi(\gamma, \chi) \rrbracket$ . Then, by Defn. 20,  $\text{effect}(\tau, \mathcal{S}) \in \llbracket \Phi(\gamma, \chi) \rrbracket$ . Then, by Defn. 19,  $\text{effect}(\text{con}_{\mathcal{S}}(\tau), \mathcal{S}) \in \llbracket \chi \rrbracket$ . Then,  $\text{effect}(\text{con}_{\mathcal{S}}(\gamma), \mathcal{S}) \in \llbracket \chi \rrbracket$ .
- **Base:**  $\gamma = q.y := e$  and  $\Phi(\gamma, \chi) = \chi[e/q.y]$ , for some  $q, y, e$ .  
Recall  $\mathcal{S} \in \llbracket \Phi(\gamma, \chi) \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \Phi(q.y := e, \chi) \rrbracket$ . Then, by Defn. 37,  $\mathcal{S} \in \llbracket \chi[e/q.y] \rrbracket$ . Then, by Lem. 14,  $\mathcal{S}[\text{eval}_{\mathcal{S}}(e)/q.y] \in \llbracket \chi \rrbracket$ . Then, by Defn. 20,  $\text{effect}(q.y := \text{eval}_{\mathcal{S}}(e), \mathcal{S}) \in \llbracket \chi \rrbracket$ . Then, by Defn. 19,  $\text{effect}(\text{con}_{\mathcal{S}}(q.y := e), \mathcal{S}) \in \llbracket \chi \rrbracket$ . Then,  $\text{effect}(\text{con}_{\mathcal{S}}(\gamma), \mathcal{S}) \in \llbracket \chi \rrbracket$ .
- **Base:**  $\gamma = p.e \rightarrow q.y$  and  $\Phi(\gamma, \chi) = \chi[e/q.y]$ , for some  $p, q, y, e$ . Similar to case “ $\gamma = q.y := e$  and  $\Phi(\gamma, \chi) = \chi[e/q.y]$ , for some  $q, y, e$ ”.
- **Base:**  $\gamma = i_R^R$  and  $\Phi(\gamma, \chi) = \chi$ , for some  $R$ . Similar to case “ $\gamma = \tau$  and  $\Phi(\gamma, \tau) = \tau$ ”. □

*Proof (of Lem. 80).* By Defn. 13:

- **Case:**  $\llbracket \forall(\psi \Rightarrow ((\xi^+ \Rightarrow \Phi(G, \psi)) \wedge (\xi^- \Rightarrow \chi) \wedge \xi^{\equiv})) \rrbracket = \llbracket \text{true} \rrbracket$ .
  - Recall  $\llbracket \Phi(\text{while } \xi|_0 \{ \psi \} G|_{\emptyset}, \chi) \rrbracket \subseteq \llbracket \Phi(\text{while } \xi|_0 \{ \psi \} G|_{\emptyset}, \chi) \rrbracket$ . Then, by Defn. 38,  $\llbracket \Phi(\text{while } \xi|_0 \{ \psi \} G|_{\emptyset}, \chi) \rrbracket \subseteq \llbracket \psi \wedge \forall(\psi \Rightarrow ((\xi^+ \Rightarrow \Phi(G, \psi)) \wedge (\xi^- \Rightarrow \chi) \wedge \xi^{\equiv})) \rrbracket$ . Then, by Defn. 13,  $\llbracket \Phi(\text{while } \xi|_0 \{ \psi \} G|_{\emptyset}, \chi) \rrbracket \subseteq \llbracket \psi \rrbracket \cap \llbracket \forall(\psi \Rightarrow ((\xi^+ \Rightarrow \Phi(G, \psi)) \wedge (\xi^- \Rightarrow \chi) \wedge \xi^{\equiv})) \rrbracket$ . Then, by Lem. 24,  $\llbracket \Phi(\text{while } \xi|_0 \{ \psi \} G|_{\emptyset}, \chi) \rrbracket \subseteq \llbracket \psi \rrbracket \cap \llbracket \psi \Rightarrow ((\xi^+ \Rightarrow \Phi(G, \psi)) \wedge (\xi^- \Rightarrow \chi) \wedge \xi^{\equiv}) \rrbracket$ . Then, by Lem. 19,  $\llbracket \Phi(\text{while } \xi|_0 \{ \psi \} G|_{\emptyset}, \chi) \rrbracket \subseteq \llbracket \psi \rrbracket \cap \llbracket ((\xi^+ \Rightarrow \Phi(G, \psi)) \wedge (\xi^- \Rightarrow \chi) \wedge \xi^{\equiv}) \rrbracket$ . Then,  $\llbracket \Phi(\text{while } \xi|_0 \{ \psi \} G|_{\emptyset}, \chi) \rrbracket \subseteq \llbracket ((\xi^+ \Rightarrow \Phi(G, \psi)) \wedge (\xi^- \Rightarrow \chi) \wedge \xi^{\equiv}) \rrbracket$ .
  - By Lem. 15,  $\llbracket \psi \rrbracket \subseteq \llbracket \text{true} \rrbracket$ . Then,  $\llbracket \psi \rrbracket \subseteq \llbracket \psi \rrbracket \cap \llbracket \text{true} \rrbracket$ . Then,  $\llbracket \psi \rrbracket \subseteq \llbracket \psi \rrbracket \cap \llbracket \forall(\psi \Rightarrow ((\xi^+ \Rightarrow \Phi(G, \psi)) \wedge (\xi^- \Rightarrow \chi) \wedge \xi^{\equiv})) \rrbracket$ . Then, by Defn. 13,  $\llbracket \psi \rrbracket \subseteq \llbracket \psi \wedge \forall(\psi \Rightarrow ((\xi^+ \Rightarrow \Phi(G, \psi)) \wedge (\xi^- \Rightarrow \chi) \wedge \xi^{\equiv})) \rrbracket$ . Then, by Lem. 81,  $\llbracket \Phi(G, \psi) \rrbracket \subseteq \llbracket \Phi(G, \psi \wedge \forall(\psi \Rightarrow ((\xi^+ \Rightarrow \Phi(G, \psi)) \wedge (\xi^- \Rightarrow \chi) \wedge \xi^{\equiv}))) \rrbracket$ . Then, by Defn. 38,  $\llbracket \Phi(G, \psi) \rrbracket \subseteq \llbracket \Phi(G, \Phi(\text{while } \xi|_0 \{ \psi \} G|_{\emptyset}, \chi)) \rrbracket$ . Then, by Defn. 38,  $\llbracket \Phi(G, \psi) \rrbracket \subseteq \llbracket \Phi(G; \text{while } \xi|_0 \{ \psi \} G|_{\emptyset}, \chi) \rrbracket$ . Then, by Lem. 22,  $\llbracket \xi^+ \Rightarrow \Phi(G, \psi) \rrbracket \subseteq \llbracket \xi^+ \Rightarrow \Phi(G; \text{while } \xi|_0 \{ \psi \} G|_{\emptyset}, \chi) \rrbracket$ .
  - Recall  $\llbracket \chi \rrbracket \subseteq \llbracket \chi \rrbracket$ . Then, by Defn. 38,  $\llbracket \chi \rrbracket \subseteq \llbracket \Phi(\text{skip}, \chi) \rrbracket$ . Then, by Lem. 22,  $\llbracket \xi^- \Rightarrow \chi \rrbracket \subseteq \llbracket \xi^- \Rightarrow \Phi(\text{skip}, \chi) \rrbracket$ .
  - Recall  $\llbracket \xi^+ \Rightarrow \Phi(G, \psi) \rrbracket \subseteq \llbracket \xi^+ \Rightarrow \Phi(G; \text{while } \xi|_0 \{ \psi \} G|_{\emptyset}, \chi) \rrbracket$  and  $\llbracket \xi^- \Rightarrow \chi \rrbracket \subseteq \llbracket \xi^- \Rightarrow \Phi(\text{skip}, \chi) \rrbracket$ . Then,  $\llbracket \xi^+ \Rightarrow \Phi(G, \psi) \rrbracket \cap \llbracket \xi^- \Rightarrow \chi \rrbracket \subseteq \llbracket \xi^+ \Rightarrow \Phi(G; \text{while } \xi|_0 \{ \psi \} G|_{\emptyset}, \chi) \rrbracket \cap \llbracket \xi^- \Rightarrow \Phi(\text{skip}, \chi) \rrbracket$ . Then,  $\llbracket \xi^+ \Rightarrow \Phi(G, \psi) \rrbracket \cap \llbracket \xi^- \Rightarrow \chi \rrbracket \cap \llbracket \xi^{\equiv} \rrbracket \subseteq \llbracket \xi^+ \Rightarrow \Phi(G; \text{while } \xi|_0 \{ \psi \} G|_{\emptyset}, \chi) \rrbracket \cap \llbracket \xi^- \Rightarrow \Phi(\text{skip}, \chi) \rrbracket \cap \llbracket \xi^{\equiv} \rrbracket$ . Then, by Defn. 13,  $\llbracket (\xi^+ \Rightarrow \Phi(G, \psi)) \wedge (\xi^- \Rightarrow \chi) \wedge \xi^{\equiv} \rrbracket \subseteq \llbracket (\xi^+ \Rightarrow \Phi(G; \text{while } \xi|_0 \{ \psi \} G|_{\emptyset}, \chi)) \wedge (\xi^- \Rightarrow \Phi(\text{skip}, \chi)) \wedge \xi^{\equiv} \rrbracket$ . Then, by Defn. 38,  $\llbracket (\xi^+ \Rightarrow \Phi(G, \psi)) \wedge (\xi^- \Rightarrow \chi) \wedge \xi^{\equiv} \rrbracket \subseteq \llbracket \Phi(\text{if } \xi|_0 (G; \text{while } \xi|_0 \{ \psi \} G|_{\emptyset})|_{\emptyset} \text{skip}|_{\emptyset}, \chi) \rrbracket$ .

- Recall  $\llbracket \Phi(\mathbf{while} \xi|_0 \{\psi\} G|_\emptyset, \chi) \rrbracket \subseteq \llbracket (\xi^+ \Rightarrow \Phi(G, \psi)) \wedge (\xi^- \Rightarrow \chi) \wedge \xi^\equiv \rrbracket \subseteq \llbracket \Phi(\mathbf{if} \xi|_0 (G; \mathbf{while} \xi|_0 \{\psi\} G|_\emptyset)|_\emptyset \mathbf{skip}|_\emptyset, \chi) \rrbracket$ . Then,  $\llbracket \Phi(\mathbf{while} \xi|_0 \{\psi\} G|_\emptyset, \chi) \rrbracket \subseteq \llbracket \Phi(\mathbf{if} \xi|_0 (G; \mathbf{while} \xi|_0 \{\psi\} G|_\emptyset)|_\emptyset \mathbf{skip}|_\emptyset, \chi) \rrbracket$ .
- **Case:**  $\llbracket \forall(\psi \Rightarrow ((\xi^+ \Rightarrow \Phi(G, \psi)) \wedge (\xi^- \Rightarrow \chi) \wedge \xi^\equiv)) \rrbracket = \llbracket \mathbf{false} \rrbracket$ . Recall  $\emptyset \subseteq \llbracket \Phi(\mathbf{if} \xi|_0 (G; \mathbf{while} \xi|_0 \{\psi\} G|_\emptyset)|_\emptyset \mathbf{skip}|_\emptyset, \chi) \rrbracket$ . Then, by Lem. 15,  $\llbracket \mathbf{false} \rrbracket \subseteq \llbracket \Phi(\mathbf{if} \xi|_0 (G; \mathbf{while} \xi|_0 \{\psi\} G|_\emptyset)|_\emptyset \mathbf{skip}|_\emptyset, \chi) \rrbracket$ . Then,  $\llbracket \forall(\psi \Rightarrow ((\xi^+ \Rightarrow \Phi(G, \psi)) \wedge (\xi^- \Rightarrow \chi) \wedge \xi^\equiv)) \rrbracket \subseteq \llbracket \Phi(\mathbf{if} \xi|_0 (G; \mathbf{while} \xi|_0 \{\psi\} G|_\emptyset)|_\emptyset \mathbf{skip}|_\emptyset, \chi) \rrbracket$ . Then,  $\llbracket \psi \rrbracket \cap \llbracket \forall(\psi \Rightarrow ((\xi^+ \Rightarrow \Phi(G, \psi)) \wedge (\xi^- \Rightarrow \chi) \wedge \xi^\equiv)) \rrbracket \subseteq \llbracket \Phi(\mathbf{if} \xi|_0 (G; \mathbf{while} \xi|_0 \{\psi\} G|_\emptyset)|_\emptyset \mathbf{skip}|_\emptyset, \chi) \rrbracket$ . Then, by Defn. 13,  $\llbracket \psi \wedge \forall(\psi \Rightarrow ((\xi^+ \Rightarrow \Phi(G, \psi)) \wedge (\xi^- \Rightarrow \chi) \wedge \xi^\equiv)) \rrbracket \subseteq \llbracket \Phi(\mathbf{if} \xi|_0 (G; \mathbf{while} \xi|_0 \{\psi\} G|_\emptyset)|_\emptyset \mathbf{skip}|_\emptyset, \chi) \rrbracket$ . Then, by Defn. 38,  $\llbracket \Phi(\mathbf{while} \xi|_0 \{\psi\} G|_\emptyset, \chi) \rrbracket \subseteq \llbracket \Phi(\mathbf{if} \xi|_0 (G; \mathbf{while} \xi|_0 \{\psi\} G|_\emptyset)|_\emptyset \mathbf{skip}|_\emptyset, \chi) \rrbracket$ .  $\square$

*Proof (of Lem. 81).*

1. By Defn. 21:

- **Base:**  $G = \xi$ , for some  $\xi$ . Recall  $\llbracket \chi_1 \rrbracket \subseteq \llbracket \chi_2 \rrbracket$ . Then,  $\llbracket \xi \rrbracket \cap \llbracket \chi_1 \rrbracket \subseteq \llbracket \xi \rrbracket \cap \llbracket \chi_2 \rrbracket$ . Then, by Defn. 13,  $\llbracket \xi \wedge \chi_1 \rrbracket \subseteq \llbracket \xi \wedge \chi_2 \rrbracket$ . Then, by Defn. 38,  $\llbracket \Phi(\xi, \chi_1) \rrbracket \subseteq \llbracket \Phi(\xi, \chi_2) \rrbracket$ . Then,  $\llbracket \Phi(G, \chi_1) \rrbracket \subseteq \llbracket \Phi(G, \chi_2) \rrbracket$ .
- **Base:**  $G = \gamma$ , for some  $\gamma$ . Recall  $\llbracket \chi_1 \rrbracket \subseteq \llbracket \chi_2 \rrbracket$ . Then, by Lem. 78,  $\llbracket \Phi(\gamma, \chi_1) \rrbracket \subseteq \llbracket \Phi(\gamma, \chi_2) \rrbracket$ . Then,  $\llbracket \Phi(G, \chi_1) \rrbracket \subseteq \llbracket \Phi(G, \chi_2) \rrbracket$ .
- **Base:**  $G = \mathbf{skip}$ . Recall  $\llbracket \chi_1 \rrbracket \subseteq \llbracket \chi_2 \rrbracket$ . Then, by Defn. 38,  $\llbracket \Phi(\mathbf{skip}, \chi_1) \rrbracket \subseteq \llbracket \Phi(\mathbf{skip}, \chi_2) \rrbracket$ . Then,  $\llbracket \Phi(G, \chi_1) \rrbracket \subseteq \llbracket \Phi(G, \chi_2) \rrbracket$ .
- **Step:**  $G = G_1 ; G_2$ , for some  $G_1, G_2$ . Recall  $\llbracket \chi_1 \rrbracket \subseteq \llbracket \chi_2 \rrbracket$ . Then, by induction,  $\llbracket \Phi(G_2, \chi_1) \rrbracket \subseteq \llbracket \Phi(G_2, \chi_2) \rrbracket$ . Then, by induction,  $\llbracket \Phi(G_1, \Phi(G_2, \chi_1)) \rrbracket \subseteq \llbracket \Phi(G_1, \Phi(G_2, \chi_2)) \rrbracket$ . Then, by Defn. 38,  $\llbracket \Phi(G_1 ; G_2, \chi_1) \rrbracket \subseteq \llbracket \Phi(G_1 ; G_2, \chi_2) \rrbracket$ . Then,  $\llbracket \Phi(G, \chi_1) \rrbracket \subseteq \llbracket \Phi(G, \chi_2) \rrbracket$ .
- **Step:**  $G = G_1 \parallel G_2$ , for some  $G_1, G_2$ . Recall:
  - **Case:**  $G_1, G_2 \in \mathbb{G}$  and  $G_1 \# G_2$ .
    - \* Recall  $G_1, G_2 \in \mathbb{G}$  and  $G_1 \# G_2$ . Then, by Defn. 38,  $\Phi(G_1 \parallel G_2, \chi_1) = \Phi(G_1, \Phi(G_2, \chi_1))$  and  $\Phi(G_1 \parallel G_2, \chi_2) = \Phi(G_1, \Phi(G_2, \chi_2))$ .
    - \* Recall  $\llbracket \chi_1 \rrbracket \subseteq \llbracket \chi_2 \rrbracket$ . Then, by induction,  $\llbracket \Phi(G_2, \chi_1) \rrbracket \subseteq \llbracket \Phi(G_2, \chi_2) \rrbracket$ . Then, by induction,  $\llbracket \Phi(G_1, \Phi(G_2, \chi_1)) \rrbracket \subseteq \llbracket \Phi(G_1, \Phi(G_2, \chi_2)) \rrbracket$ . Then,  $\llbracket \Phi(G_1 \parallel G_2, \chi_1) \rrbracket \subseteq \llbracket \Phi(G_1 \parallel G_2, \chi_2) \rrbracket$ . Then,  $\llbracket \Phi(G, \chi_1) \rrbracket \subseteq \llbracket \Phi(G, \chi_2) \rrbracket$ .
  - **Case:** Not  $G_1, G_2 \in \mathbb{G}$ , or not  $G_1 \# G_2$ .
    - \* Recall not  $G_1, G_2 \in \mathbb{G}$ , or not  $G_1 \# G_2$ . Then, by Defn. 38,  $\Phi(G_1 \parallel G_2, \chi_1) = \mathbf{false}$  and  $\Phi(G_1 \parallel G_2, \chi_2) = \mathbf{false}$ .
    - \* Recall  $\llbracket \mathbf{false} \rrbracket \subseteq \llbracket \mathbf{false} \rrbracket$ . Then,  $\llbracket \Phi(G_1 \parallel G_2, \chi_1) \rrbracket \subseteq \llbracket \Phi(G_1 \parallel G_2, \chi_2) \rrbracket$ . Then,  $\llbracket \Phi(G, \chi_1) \rrbracket \subseteq \llbracket \Phi(G, \chi_2) \rrbracket$ .
- **Step:**  $G = R.\mathbf{if} \xi G_1 G_2$ , for some  $R, G_1, G_2, \xi$ . Recall  $\llbracket \chi_1 \rrbracket \subseteq \llbracket \chi_2 \rrbracket$ . Then, by induction,  $\llbracket \Phi(G_1, \chi_1) \rrbracket \subseteq \llbracket \Phi(G_1, \chi_2) \rrbracket$  and  $\llbracket \Phi(G_2, \chi_1) \rrbracket \subseteq \llbracket \Phi(G_2, \chi_2) \rrbracket$ . Then, by Lem. 22,  $\llbracket \xi^+ \Rightarrow \Phi(G_1, \chi_1) \rrbracket \subseteq \llbracket \xi^+ \Rightarrow \Phi(G_1, \chi_2) \rrbracket$  and  $\llbracket \xi^- \Rightarrow \Phi(G_2, \chi_1) \rrbracket \subseteq \llbracket \xi^- \Rightarrow \Phi(G_2, \chi_2) \rrbracket$ . Then,  $\llbracket \xi^+ \Rightarrow \Phi(G_1, \chi_1) \rrbracket \cap \llbracket \xi^- \Rightarrow \Phi(G_2, \chi_1) \rrbracket \subseteq \llbracket \xi^+ \Rightarrow \Phi(G_1, \chi_2) \rrbracket \cap \llbracket \xi^- \Rightarrow \Phi(G_2, \chi_2) \rrbracket$ . Then,  $\llbracket \xi^+ \Rightarrow \Phi(G_1, \chi_1) \rrbracket \cap \llbracket \xi^- \Rightarrow \Phi(G_2, \chi_1) \rrbracket \cap \llbracket \xi^\equiv \rrbracket \subseteq \llbracket \xi^+ \Rightarrow \Phi(G_1, \chi_2) \rrbracket \cap \llbracket \xi^- \Rightarrow \Phi(G_2, \chi_2) \rrbracket \cap \llbracket \xi^\equiv \rrbracket$ . Then, by Defn. 13,  $\llbracket (\xi^+ \Rightarrow \Phi(G_1, \chi_1)) \wedge (\xi^- \Rightarrow \Phi(G_2, \chi_1)) \wedge \xi^\equiv \rrbracket \subseteq \llbracket (\xi^+ \Rightarrow \Phi(G_1, \chi_2)) \wedge (\xi^- \Rightarrow \Phi(G_2, \chi_2)) \wedge \xi^\equiv \rrbracket$ . Then, by Defn. 38,  $\llbracket \Phi(R.\mathbf{if} \xi G_1 G_2, \chi_1) \rrbracket \subseteq \llbracket \Phi(R.\mathbf{if} \xi G_1 G_2, \chi_2) \rrbracket$ . Then,  $\llbracket \Phi(G, \chi_1) \rrbracket \subseteq \llbracket \Phi(G, \chi_2) \rrbracket$ .
- **Step:**  $G = R.\mathbf{while} \xi \{\psi\} \hat{G}$ , for some  $R, \hat{G}, \psi, \xi$ . Recall  $\llbracket \chi_1 \rrbracket \subseteq \llbracket \chi_2 \rrbracket$ . Then, by Lem. 22,  $\llbracket \xi^- \Rightarrow \chi_1 \rrbracket \subseteq \llbracket \xi^- \Rightarrow \chi_2 \rrbracket$ . Then,  $\llbracket \xi^+ \Rightarrow \Phi(\hat{G}, \psi) \rrbracket \cap \llbracket \xi^- \Rightarrow \chi_1 \rrbracket \cap \llbracket \xi^\equiv \rrbracket \subseteq \llbracket \xi^+ \Rightarrow \Phi(\hat{G}, \psi) \rrbracket \cap \llbracket \xi^- \Rightarrow \chi_2 \rrbracket \cap \llbracket \xi^\equiv \rrbracket$ . Then, by Defn. 13,  $\llbracket (\xi^+ \Rightarrow \Phi(\hat{G}, \psi)) \wedge (\xi^- \Rightarrow \chi_1) \wedge \xi^\equiv \rrbracket \subseteq \llbracket (\xi^+ \Rightarrow \Phi(\hat{G}, \psi)) \wedge (\xi^- \Rightarrow \chi_2) \wedge \xi^\equiv \rrbracket$ .

- Then, by Lem. 22,  
 $\llbracket \psi \Rightarrow ((\xi^+ \Rightarrow \Phi(\hat{G}, \psi)) \wedge (\xi^- \Rightarrow \chi_1) \wedge \xi^\equiv) \rrbracket \subseteq \llbracket \psi \Rightarrow ((\xi^+ \Rightarrow \Phi(\hat{G}, \psi)) \wedge (\xi^- \Rightarrow \chi_2) \wedge \xi^\equiv) \rrbracket$ .  
 Then, by Lem. 21,  $\llbracket \forall(\psi \Rightarrow ((\xi^+ \Rightarrow \Phi(\hat{G}, \psi)) \wedge (\xi^- \Rightarrow \chi_1) \wedge \xi^\equiv)) \rrbracket \subseteq \llbracket \forall(\psi \Rightarrow ((\xi^+ \Rightarrow \Phi(\hat{G}, \psi)) \wedge (\xi^- \Rightarrow \chi_2) \wedge \xi^\equiv)) \rrbracket$ . Then,  $\llbracket \psi \rrbracket \cap \llbracket \forall(\psi \Rightarrow ((\xi^+ \Rightarrow \Phi(\hat{G}, \psi)) \wedge (\xi^- \Rightarrow \chi_2) \wedge \xi^\equiv)) \rrbracket \subseteq \llbracket \psi \rrbracket \cap \llbracket \forall(\psi \Rightarrow ((\xi^+ \Rightarrow \Phi(\hat{G}, \psi)) \wedge (\xi^- \Rightarrow \chi_1) \wedge \xi^\equiv)) \rrbracket$ . Then, by Defn. 13,  
 $\llbracket \psi \wedge \forall(\psi \Rightarrow ((\xi^+ \Rightarrow \Phi(\hat{G}, \psi)) \wedge (\xi^- \Rightarrow \chi_1) \wedge \xi^\equiv)) \rrbracket \subseteq \llbracket \psi \wedge \forall(\psi \Rightarrow ((\xi^+ \Rightarrow \Phi(\hat{G}, \psi)) \wedge (\xi^- \Rightarrow \chi_2) \wedge \xi^\equiv)) \rrbracket$ . Then, by Defn. 38,  
 $\llbracket \Phi(R.\mathbf{while} \xi \{ \psi \} \hat{G}, \chi_1) \rrbracket \subseteq \llbracket \Phi(R.\mathbf{while} \xi \{ \psi \} \hat{G}, \chi_2) \rrbracket$ . Then,  $\llbracket \Phi(G, \chi_1) \rrbracket \subseteq \llbracket \Phi(G, \chi_2) \rrbracket$ .
- **Step:**  $G = \mathbf{if} \xi|_0 G_1|_{R_1} G_2|_{R_2}$ , for some  $R_1, R_2, G_1, G_2, \xi$ .  
 Recall:
- **Case:**  $R_1 = \emptyset = R_2$ .
    - \* Recall  $R_1 = \emptyset = R_2$ . Then, by Defn. 38,  
 $\Phi(\mathbf{if} \xi|_0 G_1|_{R_1} G_2|_{R_2}, \chi_1) = (\xi^+ \Rightarrow \Phi(G_1, \chi_1)) \wedge (\xi^- \Rightarrow \Phi(G_2, \chi_1)) \wedge \xi^\equiv$  and  
 $\Phi(\mathbf{if} \xi|_0 G_1|_{R_1} G_2|_{R_2}, \chi_2) = (\xi^+ \Rightarrow \Phi(G_1, \chi_2)) \wedge (\xi^- \Rightarrow \Phi(G_2, \chi_2)) \wedge \xi^\equiv$ .
    - \* Recall  $\llbracket \chi_1 \rrbracket \subseteq \llbracket \chi_2 \rrbracket$ . Then, by induction,  $\llbracket \Phi(G_1, \chi_1) \rrbracket \subseteq \llbracket \Phi(G_1, \chi_2) \rrbracket$  and  
 $\llbracket \Phi(G_2, \chi_1) \rrbracket \subseteq \llbracket \Phi(G_2, \chi_2) \rrbracket$ . Then, by Lem. 22,  
 $\llbracket \xi^+ \Rightarrow \Phi(G_1, \chi_1) \rrbracket \subseteq \llbracket \xi^+ \Rightarrow \Phi(G_1, \chi_2) \rrbracket$  and  
 $\llbracket \xi^- \Rightarrow \Phi(G_2, \chi_1) \rrbracket \subseteq \llbracket \xi^- \Rightarrow \Phi(G_2, \chi_2) \rrbracket$ . Then,  
 $\llbracket \xi^+ \Rightarrow \Phi(G_1, \chi_1) \rrbracket \cap \llbracket \xi^- \Rightarrow \Phi(G_2, \chi_1) \rrbracket \subseteq \llbracket \xi^+ \Rightarrow \Phi(G_1, \chi_2) \rrbracket \cap \llbracket \xi^- \Rightarrow \Phi(G_2, \chi_2) \rrbracket$ .  
 Then,  $\llbracket \xi^+ \Rightarrow \Phi(G_1, \chi_1) \rrbracket \cap \llbracket \xi^- \Rightarrow \Phi(G_2, \chi_1) \rrbracket \cap \llbracket \xi^\equiv \rrbracket \subseteq \llbracket \xi^+ \Rightarrow \Phi(G_1, \chi_2) \rrbracket \cap \llbracket \xi^- \Rightarrow \Phi(G_2, \chi_2) \rrbracket \cap \llbracket \xi^\equiv \rrbracket$ . Then, by Defn. 13,  $\llbracket (\xi^+ \Rightarrow \Phi(G_1, \chi_1)) \wedge (\xi^- \Rightarrow \Phi(G_2, \chi_1)) \wedge \xi^\equiv \rrbracket \subseteq \llbracket (\xi^+ \Rightarrow \Phi(G_1, \chi_2)) \wedge (\xi^- \Rightarrow \Phi(G_2, \chi_2)) \wedge \xi^\equiv \rrbracket$ . Then,  
 $\llbracket \Phi(\mathbf{if} \xi|_0 G_1|_{R_1} G_2|_{R_2}, \chi_1) \rrbracket \subseteq \llbracket \Phi(\mathbf{if} \xi|_0 G_1|_{R_1} G_2|_{R_2}, \chi_2) \rrbracket$ . Then,  
 $\llbracket \Phi(G, \chi_1) \rrbracket \subseteq \llbracket \Phi(G, \chi_2) \rrbracket$ .
  - **Case:**  $R_1 = \emptyset \neq R_2$ .
    - \* Recall  $R_1 = \emptyset \neq R_2$ . Then, by Defn. 38,  
 $\Phi(\mathbf{if} \xi|_0 G_1|_{R_1} G_2|_{R_2}, \chi_1) = \Phi(G_2, \chi_1) \wedge \bigwedge \{ \xi^- \upharpoonright r \}_{r \in \text{subj}(\xi) \setminus R_2}$  and  
 $\Phi(\mathbf{if} \xi|_0 G_1|_{R_1} G_2|_{R_2}, \chi_2) = \Phi(G_2, \chi_2) \wedge \bigwedge \{ \xi^- \upharpoonright r \}_{r \in \text{subj}(\xi) \setminus R_2}$ .
    - \* Recall  $\llbracket \chi_1 \rrbracket \subseteq \llbracket \chi_2 \rrbracket$ . Then, by induction,  $\llbracket \Phi(G_2, \chi_1) \rrbracket \subseteq \llbracket \Phi(G_2, \chi_2) \rrbracket$ . Then,  
 $\llbracket \Phi(G_2, \chi_1) \rrbracket \cap \llbracket \bigwedge \{ \xi^- \upharpoonright r \}_{r \in \text{subj}(\xi) \setminus R_2} \rrbracket \subseteq \llbracket \Phi(G_2, \chi_2) \rrbracket \cap \llbracket \bigwedge \{ \xi^- \upharpoonright r \}_{r \in \text{subj}(\xi) \setminus R_2} \rrbracket$ . Then,  
 by Defn. 13,  
 $\llbracket \Phi(G_2, \chi_1) \wedge \bigwedge \{ \xi^- \upharpoonright r \}_{r \in \text{subj}(\xi) \setminus R_2} \rrbracket \subseteq \llbracket \Phi(G_2, \chi_2) \wedge \bigwedge \{ \xi^- \upharpoonright r \}_{r \in \text{subj}(\xi) \setminus R_2} \rrbracket$ . Then,  
 $\llbracket \Phi(\mathbf{if} \xi|_0 G_1|_{R_1} G_2|_{R_2}, \chi_1) \rrbracket \subseteq \llbracket \Phi(\mathbf{if} \xi|_0 G_1|_{R_1} G_2|_{R_2}, \chi_2) \rrbracket$ . Then,  
 $\llbracket \Phi(G, \chi_1) \rrbracket \subseteq \llbracket \Phi(G, \chi_2) \rrbracket$ .
  - **Case:**  $R_1 \neq \emptyset = R_2$ . Similar to case  $R_1 = \emptyset \neq R_2$ .
  - **Case:**  $R_1 \neq \emptyset \neq R_2$ .
    - \* Recall  $R_1 = \emptyset \neq R_2$ . Then, by Defn. 38,  $\Phi(\mathbf{if} \xi|_0 G_1|_{R_1} G_2|_{R_2}, \chi_1) = \mathbf{false}$  and  
 $\Phi(\mathbf{if} \xi|_0 G_1|_{R_1} G_2|_{R_2}, \chi_2) = \mathbf{false}$ .
    - \* Recall  $\llbracket \mathbf{false} \rrbracket \subseteq \llbracket \mathbf{false} \rrbracket$ . Then,  
 $\llbracket \Phi(\mathbf{if} \xi|_0 G_1|_{R_1} G_2|_{R_2}, \chi_1) \rrbracket \subseteq \llbracket \Phi(\mathbf{if} \xi|_0 G_1|_{R_1} G_2|_{R_2}, \chi_2) \rrbracket$ . Then,  
 $\llbracket \Phi(G, \chi_1) \rrbracket \subseteq \llbracket \Phi(G, \chi_2) \rrbracket$ .
- **Step:**  $G = \mathbf{while} \xi|_0 \{ \psi \} \hat{G}|_\emptyset$ , for some  $\hat{G}, \psi, \xi$ . Similar to case “ $G = R.\mathbf{while} \xi \{ \psi \} \hat{G}$ , for some  $R, \hat{G}, \psi, \xi$ ”.  $\square$
2. Recall  $\llbracket \chi_1 \rrbracket = \llbracket \chi_2 \rrbracket$ . Then,  $\llbracket \chi_1 \rrbracket \subseteq \llbracket \chi_2 \rrbracket$  and  $\llbracket \chi_1 \rrbracket \supseteq \llbracket \chi_2 \rrbracket$ . Then, by 1,  $\llbracket \Phi(G, \chi_1) \rrbracket \subseteq \llbracket \Phi(G, \chi_2) \rrbracket$  and  $\llbracket \Phi(G, \chi_1) \rrbracket \supseteq \llbracket \Phi(G, \chi_2) \rrbracket$ . Then,  $\llbracket \Phi(G, \chi_1) \rrbracket = \llbracket \Phi(G, \chi_2) \rrbracket$ .  $\square$

*Proof (of Lem. 82).* By Defn. 21:

- **Base:**  $G = \xi$ , for some  $\xi$ .  
 Recall  $\llbracket \xi \rrbracket \cap \llbracket \chi_1 \rrbracket \cap \llbracket \chi_2 \rrbracket = \llbracket \xi \rrbracket \cap \llbracket \chi_1 \rrbracket \cap \llbracket \xi \rrbracket \cap \llbracket \chi_2 \rrbracket$ . Then, by Defn. 13,  
 $\llbracket \xi \wedge \chi_1 \wedge \chi_2 \rrbracket = \llbracket \xi \wedge \chi_1 \wedge \xi \wedge \chi_2 \rrbracket$ . Then, by Defn. 38,  $\llbracket \Phi(\xi, \chi_1 \wedge \chi_2) \rrbracket = \llbracket \Phi(\xi, \chi_1) \wedge \Phi(\xi, \chi_2) \rrbracket$ .  
 Then,  $\llbracket \Phi(G, \chi_1 \wedge \chi_2) \rrbracket = \llbracket \Phi(G, \chi_1) \wedge \Phi(G, \chi_2) \rrbracket$ .

- **Base:**  $G = \gamma$ , for some  $\gamma$ .  
Recall  $\llbracket \phi(\gamma, \chi_1 \wedge \chi_2) \rrbracket = \llbracket \phi(\gamma, \chi_1 \wedge \chi_2) \rrbracket$ . Then, by Lem. 75,  
 $\llbracket \phi(\gamma, \chi_1 \wedge \chi_2) \rrbracket = \llbracket \phi(\gamma, \chi_1) \wedge \phi(\gamma, \chi_2) \rrbracket$ . Then, by Defn. 38,  
 $\llbracket \phi(\gamma, \chi_1 \wedge \chi_2) \rrbracket = \llbracket \phi(\gamma, \chi_1) \wedge \phi(\gamma, \chi_2) \rrbracket$ . Then,  $\llbracket \phi(G, \chi_1 \wedge \chi_2) \rrbracket = \llbracket \phi(G, \chi_1) \wedge \phi(G, \chi_2) \rrbracket$ .
- **Base:**  $G = \text{skip}$ .  
Recall  $\llbracket \chi_1 \wedge \chi_2 \rrbracket = \llbracket \chi_1 \wedge \chi_2 \rrbracket$ . Then, by Defn. 38,  
 $\llbracket \phi(\text{skip}, \chi_1 \wedge \chi_2) \rrbracket = \llbracket \phi(\text{skip}, \chi_1) \wedge \phi(\text{skip}, \chi_2) \rrbracket$ . Then,  
 $\llbracket \phi(G, \chi_1 \wedge \chi_2) \rrbracket = \llbracket \phi(G, \chi_1) \wedge \phi(G, \chi_2) \rrbracket$ .
- **Step:**  $G = G_1 ; G_2$ , for some  $G_1, G_2$ .  
By induction,  $\llbracket \phi(G_2, \chi_1 \wedge \chi_2) \rrbracket = \llbracket \phi(G_2, \chi_1) \wedge \phi(G_2, \chi_2) \rrbracket$ . Then, by Lem. 81,  
 $\llbracket \phi(G_1, \phi(G_2, \chi_1 \wedge \chi_2)) \rrbracket = \llbracket \phi(G_1, \phi(G_2, \chi_1) \wedge \phi(G_2, \chi_2)) \rrbracket$ . Then, by induction,  
 $\llbracket \phi(G_1, \phi(G_2, \chi_1 \wedge \chi_2)) \rrbracket = \llbracket \phi(G_1, \phi(G_2, \chi_1)) \wedge \phi(G_1, \phi(G_2, \chi_2)) \rrbracket$ . Then, by Defn. 38,  
 $\llbracket \phi(G_1 ; G_2, \chi_1 \wedge \chi_2) \rrbracket = \llbracket \phi(G_1 ; G_2, \chi_1) \wedge \phi(G_1 ; G_2, \chi_2) \rrbracket$ . Then,  
 $\llbracket \phi(G, \chi_1 \wedge \chi_2) \rrbracket = \llbracket \phi(G, \chi_1) \wedge \phi(G, \chi_2) \rrbracket$ .
- **Step:**  $G = G_1 \parallel G_2$ , for some  $G_1, G_2$ .  
Recall:  
  - **Case:**  $G_1, G_2 \in \underline{\mathbb{G}}$  and  $G_1 \# G_2$ .
    - \* Recall  $G_1, G_2 \in \underline{\mathbb{G}}$  and  $G_1 \# G_2$ . Then, by Defn. 38,  
 $\phi(G_1 \parallel G_2, \chi_1 \wedge \chi_2) = \phi(G_1, \phi(G_2, \chi_1 \wedge \chi_2))$  and  $\phi(G_1 \parallel G_2, \chi_1) = \phi(G_1, \phi(G_2, \chi_1))$   
and  $\phi(G_1 \parallel G_2, \chi_2) = \phi(G_1, \phi(G_2, \chi_2))$ .
    - \* By induction,  $\llbracket \phi(G_2, \chi_1 \wedge \chi_2) \rrbracket = \llbracket \phi(G_2, \chi_1) \wedge \phi(G_2, \chi_2) \rrbracket$ . Then, by Lem. 81,  
 $\llbracket \phi(G_1, \phi(G_2, \chi_1 \wedge \chi_2)) \rrbracket = \llbracket \phi(G_1, \phi(G_2, \chi_1) \wedge \phi(G_2, \chi_2)) \rrbracket$ . Then, by induction,  
 $\llbracket \phi(G_1, \phi(G_2, \chi_1 \wedge \chi_2)) \rrbracket = \llbracket \phi(G_1, \phi(G_2, \chi_1)) \wedge \phi(G_1, \phi(G_2, \chi_2)) \rrbracket$ . Then,  
 $\llbracket \phi(G_1 \parallel G_2, \chi_1 \wedge \chi_2) \rrbracket = \llbracket \phi(G_1 \parallel G_2, \chi_1) \wedge \phi(G_1 \parallel G_2, \chi_2) \rrbracket$ . Then,  
 $\llbracket \phi(G, \chi_1 \wedge \chi_2) \rrbracket = \llbracket \phi(G, \chi_1) \wedge \phi(G, \chi_2) \rrbracket$ .
  - **Case:** Not  $G_1, G_2 \in \underline{\mathbb{G}}$ , or not  $G_1 \# G_2$ .
    - \* Recall not  $G_1, G_2 \in \underline{\mathbb{G}}$ , or not  $G_1 \# G_2$ . Then, by Defn. 38,  $\phi(G_1 \parallel G_2, \chi) = \text{false}$   
and  $\phi(G_1 \parallel G_2, \chi_1) = \text{false}$  and  $\phi(G_1 \parallel G_2, \chi_2) = \text{false}$ .
    - \* Recall  $\emptyset = \emptyset \cap \emptyset$ . Then, by Lem. 15,  $\llbracket \text{false} \rrbracket = \llbracket \text{false} \rrbracket \cap \llbracket \text{false} \rrbracket$ . Then, by Defn. 13,  
 $\llbracket \text{false} \rrbracket = \llbracket \text{false} \wedge \text{false} \rrbracket$ . Then,  
 $\llbracket \phi(G_1 \parallel G_2, \chi_1 \wedge \chi_2) \rrbracket = \llbracket \phi(G_1 \parallel G_2, \chi_1) \wedge \phi(G_1 \parallel G_2, \chi_2) \rrbracket$ . Then,  
 $\llbracket \phi(G, \chi_1 \wedge \chi_2) \rrbracket = \llbracket \phi(G, \chi_1) \wedge \phi(G, \chi_2) \rrbracket$ .
- **Step:**  $G = R.\text{if } \xi G_1 G_2$ , for some  $R, G_1, G_2, \xi$ .  
By induction,  $\llbracket \phi(G_1, \chi_1 \wedge \chi_2) \rrbracket = \llbracket \phi(G_1, \chi_1) \wedge \phi(G_1, \chi_2) \rrbracket$  and  
 $\llbracket \phi(G_2, \chi_1 \wedge \chi_2) \rrbracket = \llbracket \phi(G_2, \chi_1) \wedge \phi(G_2, \chi_2) \rrbracket$ . Then, by Lem. 22,  
 $\llbracket \xi^+ \Rightarrow \phi(G_1, \chi_1 \wedge \chi_2) \rrbracket = \llbracket \xi^+ \Rightarrow (\phi(G_1, \chi_1) \wedge \phi(G_1, \chi_2)) \rrbracket$  and  
 $\llbracket \xi^- \Rightarrow \phi(G_2, \chi_1 \wedge \chi_2) \rrbracket = \llbracket \xi^- \Rightarrow (\phi(G_2, \chi_1) \wedge \phi(G_2, \chi_2)) \rrbracket$ . Then, by Lem. 18,  
 $\llbracket \xi^+ \Rightarrow \phi(G_1, \chi_1 \wedge \chi_2) \rrbracket = \llbracket (\xi^+ \Rightarrow \phi(G_1, \chi_1)) \wedge (\xi^+ \Rightarrow \phi(G_1, \chi_2)) \rrbracket$  and  
 $\llbracket \xi^- \Rightarrow \phi(G_2, \chi_1 \wedge \chi_2) \rrbracket = \llbracket (\xi^- \Rightarrow \phi(G_2, \chi_1)) \wedge (\xi^- \Rightarrow \phi(G_2, \chi_2)) \rrbracket$ . Then,  
 $\llbracket \xi^+ \Rightarrow \phi(G_1, \chi_1 \wedge \chi_2) \rrbracket \cap \llbracket \xi^- \Rightarrow \phi(G_2, \chi_1 \wedge \chi_2) \rrbracket = \llbracket (\xi^+ \Rightarrow \phi(G_1, \chi_1)) \wedge (\xi^+ \Rightarrow \phi(G_1, \chi_2)) \rrbracket \cap \llbracket (\xi^- \Rightarrow \phi(G_2, \chi_1)) \wedge (\xi^- \Rightarrow \phi(G_2, \chi_2)) \rrbracket$ . Then,  
 $\llbracket \xi^+ \Rightarrow \phi(G_1, \chi_1 \wedge \chi_2) \rrbracket \cap \llbracket \xi^- \Rightarrow \phi(G_2, \chi_1 \wedge \chi_2) \rrbracket \cap \llbracket \xi^\equiv \rrbracket = \llbracket (\xi^+ \Rightarrow \phi(G_1, \chi_1)) \wedge (\xi^+ \Rightarrow \phi(G_1, \chi_2)) \rrbracket \cap \llbracket (\xi^- \Rightarrow \phi(G_2, \chi_1)) \wedge (\xi^- \Rightarrow \phi(G_2, \chi_2)) \rrbracket \cap \llbracket \xi^\equiv \rrbracket$ . Then, by Defn. 13,  
 $\llbracket (\xi^+ \Rightarrow \phi(G_1, \chi_1 \wedge \chi_2)) \wedge (\xi^- \Rightarrow \phi(G_2, \chi_1 \wedge \chi_2)) \wedge \xi^\equiv \rrbracket = \llbracket (\xi^+ \Rightarrow \phi(G_1, \chi_1)) \wedge (\xi^+ \Rightarrow \phi(G_1, \chi_2)) \wedge (\xi^- \Rightarrow \phi(G_2, \chi_1)) \wedge (\xi^- \Rightarrow \phi(G_2, \chi_2)) \wedge \xi^\equiv \rrbracket$ . Then, by Defn. 38,  
 $\llbracket \phi(R.\text{if } \xi G_1 G_2, \chi_1 \wedge \chi_2) \rrbracket = \llbracket \phi(R.\text{if } \xi G_1 G_2, \chi_1) \wedge \phi(R.\text{if } \xi G_1 G_2, \chi_2) \rrbracket$ . Then,  
 $\llbracket \phi(G, \chi_1 \wedge \chi_2) \rrbracket = \llbracket \phi(G, \chi_1) \wedge \phi(G, \chi_2) \rrbracket$ .
- **Step:**  $G = R.\text{while } \xi \{ \psi \} \hat{G}$ , for some  $R, \hat{G}, \psi, \xi$ .  
By Lem. 18,  $\llbracket \xi^- \Rightarrow (\chi_1 \wedge \chi_2) \rrbracket = \llbracket (\xi^- \Rightarrow \chi_1) \wedge (\xi^- \Rightarrow \chi_2) \rrbracket$ . Then,  
 $\llbracket \xi^- \Rightarrow (\chi_1 \wedge \chi_2) \rrbracket \cap \llbracket (\xi^+ \Rightarrow \phi(\hat{G}, \psi)) \wedge \xi^\equiv \rrbracket = \llbracket (\xi^- \Rightarrow \chi_1) \wedge (\xi^- \Rightarrow \chi_2) \rrbracket \cap \llbracket (\xi^+ \Rightarrow \phi(\hat{G}, \psi)) \wedge \xi^\equiv \rrbracket$ . Then, by Defn. 13,  $\llbracket (\xi^- \Rightarrow (\chi_1 \wedge \chi_2)) \wedge (\xi^+ \Rightarrow \phi(\hat{G}, \psi)) \wedge \xi^\equiv \rrbracket = \llbracket (\xi^- \Rightarrow \chi_1) \wedge (\xi^- \Rightarrow \chi_2) \wedge (\xi^+ \Rightarrow \phi(\hat{G}, \psi)) \wedge \xi^\equiv \rrbracket$ . Then, by Lem. 22,  $\llbracket \psi \Rightarrow ((\xi^- \Rightarrow (\chi_1 \wedge \chi_2)) \wedge (\xi^+ \Rightarrow \phi(\hat{G}, \psi)) \wedge \xi^\equiv) \rrbracket = \llbracket \psi \Rightarrow ((\xi^- \Rightarrow \chi_1) \wedge (\xi^- \Rightarrow \chi_2) \wedge (\xi^+ \Rightarrow \phi(\hat{G}, \psi)) \wedge \xi^\equiv) \rrbracket$ . Then, by Lem. 18,



- **Step:**  $G = \text{while } \xi|_0 \{\psi\} \hat{G}|_\emptyset$ , for some  $\hat{G}, \psi, \xi$ . Similar to case “ $G = R.\text{while } \xi \{\psi\} \hat{G}$ , for some  $R, \hat{G}, \psi, \xi$ ”.  $\square$

*Proof (of Lem. 83).* By Defn. 21:

- **Base:**  $\underline{G} = q.y := e$ , for some  $q, y, e$ .  
Recall  $\text{write}(q.y := e) \cap \emptyset = \emptyset$ . Then, by Lem. 3,  $\text{write}(q.y := e) \cap \text{read}(\text{false}) = \emptyset$ . Then, by Lem. 76,  $\phi(q.y := e, \text{false}) = \text{false}$ . Then,  $\llbracket \phi(q.y := e, \text{false}) \rrbracket = \llbracket \text{false} \rrbracket$ . Then,  $\llbracket \phi(\underline{G}, \text{false}) \rrbracket = \llbracket \text{false} \rrbracket$ .
- **Base:**  $\underline{G} = p.e \rightarrow q.y$ , for some  $p, q, y, e$ . Similar to case “ $\underline{G} = q.y := e$ , for some  $q, y, e$ ”.
- **Base:**  $\underline{G} = \text{skip}$ .  
By Defn. 38,  $\phi(\text{skip}, \text{false}) = \text{false}$ . Then,  $\llbracket \phi(\text{skip}, \text{false}) \rrbracket = \llbracket \text{false} \rrbracket$ . Then,  $\llbracket \phi(\underline{G}, \text{false}) \rrbracket = \llbracket \text{false} \rrbracket$ .
- **Step:**  $\underline{G} = \underline{G}_1 ; \underline{G}_2$ , for some  $\underline{G}_1, \underline{G}_2$ .  
By induction,  $\llbracket \phi(\underline{G}_2, \text{false}) \rrbracket = \llbracket \text{false} \rrbracket$ . Then, by Lem. 81,  $\llbracket \phi(\underline{G}_1, \phi(\underline{G}_2, \text{false})) \rrbracket = \llbracket \phi(\underline{G}_1, \text{false}) \rrbracket$ . Then, by induction,  $\llbracket \phi(\underline{G}_1, \phi(\underline{G}_2, \text{false})) \rrbracket = \llbracket \text{false} \rrbracket$ . Then, by Defn. 38,  $\llbracket \phi(\underline{G}_1 ; \underline{G}_2, \text{false}) \rrbracket = \llbracket \text{false} \rrbracket$ . Then,  $\llbracket \phi(\underline{G}, \text{false}) \rrbracket = \llbracket \text{false} \rrbracket$ .
- **Step:**  $\underline{G} = \underline{G}_1 \parallel \underline{G}_2$ , for some  $\underline{G}_1, \underline{G}_2$ .  
Recall:
  - **Case:**  $\underline{G}_1, \underline{G}_2 \in \mathbb{G}$  and  $\underline{G}_1 \# \underline{G}_2$ .
    - \* Recall  $\underline{G}_1, \underline{G}_2 \in \mathbb{G}$  and  $\underline{G}_1 \# \underline{G}_2$ . Then, by Defn. 38,  $\phi(\underline{G}_1 \parallel \underline{G}_2, \text{false}) = \phi(\underline{G}_1, \phi(\underline{G}_2, \text{false}))$ .
    - \* By induction,  $\llbracket \phi(\underline{G}_2, \text{false}) \rrbracket = \llbracket \text{false} \rrbracket$ . Then, by Lem. 81,  $\llbracket \phi(\underline{G}_1, \phi(\underline{G}_2, \text{false})) \rrbracket = \llbracket \phi(\underline{G}_1, \text{false}) \rrbracket$ . Then, by induction,  $\llbracket \phi(\underline{G}_1, \phi(\underline{G}_2, \text{false})) \rrbracket = \llbracket \text{false} \rrbracket$ . Then,  $\llbracket \phi(\underline{G}_1 \parallel \underline{G}_2, \text{false}) \rrbracket = \llbracket \text{false} \rrbracket$ . Then,  $\llbracket \phi(\underline{G}, \text{false}) \rrbracket = \llbracket \text{false} \rrbracket$ .
  - **Case:** Not  $\underline{G}_1, \underline{G}_2 \in \mathbb{G}$ , or not  $\underline{G}_1 \# \underline{G}_2$ . Then, by Defn. 38,  $\phi(\underline{G}_1 \parallel \underline{G}_2, \chi) = \text{false}$ . Then,  $\llbracket \phi(\underline{G}_1 \parallel \underline{G}_2, \chi) \rrbracket = \llbracket \text{false} \rrbracket$ . Then,  $\llbracket \phi(\underline{G}, \text{false}) \rrbracket = \llbracket \text{false} \rrbracket$ .
- **Step:**  $\underline{G} = R.\text{if } \xi \underline{G}_1 \underline{G}_2$ , for some  $R, \underline{G}_1, \underline{G}_2, \xi$ .
  - By Lem. 27,  $\mathcal{S} \in \llbracket \xi^\equiv \rrbracket$  implies  $\mathcal{S} \in \llbracket \xi^+ \rrbracket$  or  $\mathcal{S} \in \llbracket \xi^- \rrbracket$ , for every  $\mathcal{S}$ . Then,  $\mathcal{S} \in \llbracket \xi^\equiv \rrbracket$  implies  $\mathcal{S} \in \llbracket \xi^+ \rrbracket \cup \llbracket \xi^- \rrbracket$ , for every  $\mathcal{S}$ . Then,  $\llbracket \xi^\equiv \rrbracket \subseteq \llbracket \xi^+ \rrbracket \cup \llbracket \xi^- \rrbracket$ . Then,  $\llbracket \xi^\equiv \rrbracket = \llbracket \xi^\equiv \rrbracket \cap (\llbracket \xi^+ \rrbracket \cup \llbracket \xi^- \rrbracket)$ .
  - By Defn. 38,  $\phi(R.\text{if } \xi \underline{G}_1 \underline{G}_2, \text{false}) = (\xi^+ \Rightarrow \phi(\underline{G}_1, \text{false})) \wedge (\xi^- \Rightarrow \phi(\underline{G}_2, \text{false})) \wedge \xi^\equiv$ . Then,  $\llbracket \phi(R.\text{if } \xi \underline{G}_1 \underline{G}_2, \text{false}) \rrbracket = \llbracket (\xi^+ \Rightarrow \phi(\underline{G}_1, \text{false})) \wedge (\xi^- \Rightarrow \phi(\underline{G}_2, \text{false})) \wedge \xi^\equiv \rrbracket$ . Then,  $\llbracket \phi(\underline{G}, \text{false}) \rrbracket = \llbracket (\xi^+ \Rightarrow \phi(\underline{G}_1, \text{false})) \wedge (\xi^- \Rightarrow \phi(\underline{G}_2, \text{false})) \wedge \xi^\equiv \rrbracket$ . Then, by Defn. 13,  $\llbracket \phi(\underline{G}, \text{false}) \rrbracket = \llbracket \xi^+ \Rightarrow \phi(\underline{G}_1, \text{false}) \rrbracket \cap \llbracket \xi^- \Rightarrow \phi(\underline{G}_2, \text{false}) \rrbracket \cap \llbracket \xi^\equiv \rrbracket$ . Then,  $\llbracket \phi(\underline{G}, \text{false}) \rrbracket = \llbracket \xi^+ \Rightarrow \phi(\underline{G}_1, \text{false}) \rrbracket \cap \llbracket \xi^- \Rightarrow \phi(\underline{G}_2, \text{false}) \rrbracket \cap \llbracket \xi^\equiv \rrbracket \cap (\llbracket \xi^+ \rrbracket \cup \llbracket \xi^- \rrbracket)$ . Then,  $\llbracket \phi(\underline{G}, \text{false}) \rrbracket = (\llbracket \xi^+ \Rightarrow \phi(\underline{G}_1, \text{false}) \rrbracket \cap \llbracket \xi^- \Rightarrow \phi(\underline{G}_2, \text{false}) \rrbracket \cap \llbracket \xi^\equiv \rrbracket \cap \llbracket \xi^+ \rrbracket) \cup (\llbracket \xi^+ \Rightarrow \phi(\underline{G}_1, \text{false}) \rrbracket \cap \llbracket \xi^- \Rightarrow \phi(\underline{G}_2, \text{false}) \rrbracket \cap \llbracket \xi^\equiv \rrbracket \cap \llbracket \xi^- \rrbracket)$ . Then, by Lem. 19,  $\llbracket \phi(\underline{G}, \text{false}) \rrbracket = (\llbracket \phi(\underline{G}_1, \text{false}) \rrbracket \cap \llbracket \xi^- \Rightarrow \phi(\underline{G}_2, \text{false}) \rrbracket \cap \llbracket \xi^\equiv \rrbracket \cap \llbracket \xi^+ \rrbracket) \cup (\llbracket \xi^+ \Rightarrow \phi(\underline{G}_1, \text{false}) \rrbracket \cap \llbracket \phi(\underline{G}_2, \text{false}) \rrbracket \cap \llbracket \xi^\equiv \rrbracket \cap \llbracket \xi^- \rrbracket)$ . Then, by induction,  $\llbracket \phi(\underline{G}, \text{false}) \rrbracket = (\llbracket \text{false} \rrbracket \cap \llbracket \xi^- \Rightarrow \phi(\underline{G}_2, \text{false}) \rrbracket \cap \llbracket \xi^\equiv \rrbracket \cap \llbracket \xi^+ \rrbracket) \cup (\llbracket \xi^+ \Rightarrow \phi(\underline{G}_1, \text{false}) \rrbracket \cap \llbracket \text{false} \rrbracket \cap \llbracket \xi^\equiv \rrbracket \cap \llbracket \xi^- \rrbracket)$ . Then, by Lem. 15,  $\llbracket \phi(\underline{G}, \text{false}) \rrbracket = (\emptyset \cap \llbracket \xi^- \Rightarrow \phi(\underline{G}_2, \text{false}) \rrbracket \cap \llbracket \xi^\equiv \rrbracket \cap \llbracket \xi^+ \rrbracket) \cup (\llbracket \xi^+ \Rightarrow \phi(\underline{G}_1, \text{false}) \rrbracket \cap \emptyset \cap \llbracket \xi^\equiv \rrbracket \cap \llbracket \xi^- \rrbracket)$ . Then,  $\llbracket \phi(\underline{G}, \text{false}) \rrbracket = \emptyset \cup \emptyset$ . Then,  $\llbracket \phi(\underline{G}, \text{false}) \rrbracket = \emptyset$ . Then, by Lem. 15,  $\llbracket \phi(\underline{G}, \text{false}) \rrbracket = \llbracket \text{false} \rrbracket$ .
- **Step:**  $\underline{G} = \text{if } \xi|_0 \underline{G}_1|_{R_1} \underline{G}_2|_{R_2}$ , for some  $R_1, R_2, \underline{G}_1, \underline{G}_2, \xi$ .  
Recall:
  - **Case:**  $R_1 = \emptyset = R_2$ .
    - \* By Lem. 27,  $\mathcal{S} \in \llbracket \xi^\equiv \rrbracket$  implies  $\mathcal{S} \in \llbracket \xi^+ \rrbracket$  or  $\mathcal{S} \in \llbracket \xi^- \rrbracket$ , for every  $\mathcal{S}$ . Then,  $\mathcal{S} \in \llbracket \xi^\equiv \rrbracket$  implies  $\mathcal{S} \in \llbracket \xi^+ \rrbracket \cup \llbracket \xi^- \rrbracket$ , for every  $\mathcal{S}$ . Then,  $\llbracket \xi^\equiv \rrbracket \subseteq \llbracket \xi^+ \rrbracket \cup \llbracket \xi^- \rrbracket$ . Then,  $\llbracket \xi^\equiv \rrbracket = \llbracket \xi^\equiv \rrbracket \cap (\llbracket \xi^+ \rrbracket \cup \llbracket \xi^- \rrbracket)$ .



- \* Recall  $R_1 = \emptyset = R_2$ . Then, by Defn. 38,
 
$$\phi(\text{if } \xi|_0 \underline{G}_1|_{R_1} \underline{G}_2|_{R_2}, \text{false}) = (\xi^+ \Rightarrow \phi(\underline{G}_1, \text{false})) \wedge (\xi^- \Rightarrow \phi(\underline{G}_2, \text{false})) \wedge \xi^\equiv.$$
 Then,
 
$$\llbracket \phi(\text{if } \xi|_0 \underline{G}_1|_{R_1} \underline{G}_2|_{R_2}, \text{false}) \rrbracket = \llbracket (\xi^+ \Rightarrow \phi(\underline{G}_1, \text{false})) \wedge (\xi^- \Rightarrow \phi(\underline{G}_2, \text{false})) \wedge \xi^\equiv \rrbracket.$$
 Then,  $\llbracket \phi(\underline{G}, \text{false}) \rrbracket = \llbracket (\xi^+ \Rightarrow \phi(\underline{G}_1, \text{false})) \wedge (\xi^- \Rightarrow \phi(\underline{G}_2, \text{false})) \wedge \xi^\equiv \rrbracket$ . Then, by Defn. 13,  $\llbracket \phi(\underline{G}, \text{false}) \rrbracket = \llbracket \xi^+ \Rightarrow \phi(\underline{G}_1, \text{false}) \rrbracket \cap \llbracket \xi^- \Rightarrow \phi(\underline{G}_2, \text{false}) \rrbracket \cap \llbracket \xi^\equiv \rrbracket$ . Then,
 
$$\llbracket \phi(\underline{G}, \text{false}) \rrbracket = \llbracket \xi^+ \Rightarrow \phi(\underline{G}_1, \text{false}) \rrbracket \cap \llbracket \xi^- \Rightarrow \phi(\underline{G}_2, \text{false}) \rrbracket \cap \llbracket \xi^\equiv \rrbracket \cap (\llbracket \xi^+ \rrbracket \cup \llbracket \xi^- \rrbracket).$$
 Then,
 
$$\llbracket \phi(\underline{G}, \text{false}) \rrbracket = (\llbracket \xi^+ \Rightarrow \phi(\underline{G}_1, \text{false}) \rrbracket \cap \llbracket \xi^- \Rightarrow \phi(\underline{G}_2, \text{false}) \rrbracket \cap \llbracket \xi^\equiv \rrbracket \cap \llbracket \xi^+ \rrbracket) \cup (\llbracket \xi^+ \Rightarrow \phi(\underline{G}_1, \text{false}) \rrbracket \cap \llbracket \xi^- \Rightarrow \phi(\underline{G}_2, \text{false}) \rrbracket \cap \llbracket \xi^\equiv \rrbracket \cap \llbracket \xi^- \rrbracket).$$
 Then, by Lem. 19,
 
$$\llbracket \phi(\underline{G}, \text{false}) \rrbracket = (\llbracket \phi(\underline{G}_1, \text{false}) \rrbracket \cap \llbracket \xi^- \Rightarrow \phi(\underline{G}_2, \text{false}) \rrbracket \cap \llbracket \xi^\equiv \rrbracket \cap \llbracket \xi^+ \rrbracket) \cup (\llbracket \xi^+ \Rightarrow \phi(\underline{G}_1, \text{false}) \rrbracket \cap \llbracket \phi(\underline{G}_2, \text{false}) \rrbracket \cap \llbracket \xi^\equiv \rrbracket \cap \llbracket \xi^- \rrbracket).$$
 Then, by induction,
 
$$\llbracket \phi(\underline{G}, \text{false}) \rrbracket = (\llbracket \text{false} \rrbracket \cap \llbracket \xi^- \Rightarrow \phi(\underline{G}_2, \text{false}) \rrbracket \cap \llbracket \xi^\equiv \rrbracket \cap \llbracket \xi^+ \rrbracket) \cup (\llbracket \xi^+ \Rightarrow \phi(\underline{G}_1, \text{false}) \rrbracket \cap \llbracket \text{false} \rrbracket \cap \llbracket \xi^\equiv \rrbracket \cap \llbracket \xi^- \rrbracket).$$
 Then, by Lem. 15,  $\llbracket \phi(\underline{G}, \text{false}) \rrbracket = (\emptyset \cap \llbracket \xi^- \Rightarrow \phi(\underline{G}_2, \text{false}) \rrbracket \cap \llbracket \xi^\equiv \rrbracket \cap \llbracket \xi^+ \rrbracket) \cup (\llbracket \xi^+ \Rightarrow \phi(\underline{G}_1, \text{false}) \rrbracket \cap \emptyset \cap \llbracket \xi^\equiv \rrbracket \cap \llbracket \xi^- \rrbracket)$ . Then,  $\llbracket \phi(\underline{G}, \text{false}) \rrbracket = \emptyset \cup \emptyset$ . Then,  $\llbracket \phi(\underline{G}, \text{false}) \rrbracket = \emptyset$ . Then, by Lem. 15,  $\llbracket \phi(\underline{G}, \text{false}) \rrbracket = \llbracket \text{false} \rrbracket$ . Then,  $\llbracket \phi(\underline{G}, \text{false}) \rrbracket = \llbracket \text{false} \rrbracket$ .
  - **Case:**  $R_1 = \emptyset \neq R_2$ . Then, by Defn. 38,
 
$$\phi(\text{if } \xi|_0 \underline{G}_1|_{R_1} \underline{G}_2|_{R_2}, \text{false}) = \phi(\underline{G}_2, \text{false}) \wedge \bigwedge \{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_2}.$$
 Then,
 
$$\llbracket \phi(\text{if } \xi|_0 \underline{G}_1|_{R_1} \underline{G}_2|_{R_2}, \text{false}) \rrbracket = \llbracket \phi(\underline{G}_2, \text{false}) \wedge \bigwedge \{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_2} \rrbracket.$$
 Then,
 
$$\llbracket \phi(\underline{G}, \text{false}) \rrbracket = \llbracket \phi(\underline{G}_2, \text{false}) \wedge \bigwedge \{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_2} \rrbracket.$$
 Then, by Defn. 13,
 
$$\llbracket \phi(\underline{G}, \text{false}) \rrbracket = \llbracket \phi(\underline{G}_2, \text{false}) \rrbracket \cap \llbracket \bigwedge \{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_2} \rrbracket.$$
 Then, by induction,
 
$$\llbracket \phi(\underline{G}, \text{false}) \rrbracket = \llbracket \text{false} \rrbracket \cap \llbracket \bigwedge \{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_2} \rrbracket.$$
 Then, by Lem. 15,
 
$$\llbracket \phi(\underline{G}, \text{false}) \rrbracket = \emptyset \cap \llbracket \bigwedge \{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_2} \rrbracket.$$
 Then,  $\llbracket \phi(\underline{G}, \text{false}) \rrbracket = \emptyset$ . Then, by Lem. 15,  $\llbracket \phi(\underline{G}, \text{false}) \rrbracket = \llbracket \text{false} \rrbracket$ .
  - **Case:**  $R_1 \neq \emptyset = R_2$ . Similar to case  $R_1 = \emptyset \neq R_2$ .
  - **Case:**  $R_1 \neq \emptyset \neq R_2$ . Then, by Defn. 38,  $\phi(\text{if } \xi|_0 \underline{G}_1|_{R_1} \underline{G}_2|_{R_2}, \chi) = \text{false}$ . Then,
 
$$\llbracket \phi(\text{if } \xi|_0 \underline{G}_1|_{R_1} \underline{G}_2|_{R_2}, \chi) \rrbracket = \llbracket \text{false} \rrbracket.$$
 Then,  $\llbracket \phi(\underline{G}, \text{false}) \rrbracket = \llbracket \text{false} \rrbracket$ .
- **Step:**  $\underline{G} = \text{if } \xi|_0 \underline{G}_1|_{\emptyset} \underline{G}_2|_{R_2 \cup \{r\}}$ , for some  $R_2, \underline{G}_1, \underline{G}_2, r, \xi$ .  
 Recall  $\emptyset = \emptyset \neq R_2 \cup \{r\}$ . Then, by Defn. 38,
 
$$\phi(\text{if } \xi|_0 \underline{G}_1|_{\emptyset} \underline{G}_2|_{R_2 \cup \{r\}}, \text{false}) = \phi(\underline{G}_2, \text{false}) \wedge \bigwedge \{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus (R_2 \cup \{r\})}.$$
 Then,
 
$$\llbracket \phi(\text{if } \xi|_0 \underline{G}_1|_{\emptyset} \underline{G}_2|_{R_2 \cup \{r\}}, \text{false}) \rrbracket = \llbracket \phi(\underline{G}_2, \text{false}) \wedge \bigwedge \{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus (R_2 \cup \{r\})} \rrbracket.$$
 Then,
 
$$\llbracket \phi(\underline{G}, \text{false}) \rrbracket = \llbracket \phi(\underline{G}_2, \text{false}) \wedge \bigwedge \{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus (R_2 \cup \{r\})} \rrbracket.$$
 Then, by Defn. 13,
 
$$\llbracket \phi(\underline{G}, \text{false}) \rrbracket = \llbracket \phi(\underline{G}_2, \text{false}) \rrbracket \cap \llbracket \bigwedge \{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus (R_2 \cup \{r\})} \rrbracket.$$
 Then, by induction,
 
$$\llbracket \phi(\underline{G}, \text{false}) \rrbracket = \llbracket \text{false} \rrbracket \cap \llbracket \bigwedge \{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus (R_2 \cup \{r\})} \rrbracket.$$
 Then, by Lem. 15,
 
$$\llbracket \phi(\underline{G}, \text{false}) \rrbracket = \emptyset \cap \llbracket \bigwedge \{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus (R_2 \cup \{r\})} \rrbracket.$$
 Then,  $\llbracket \phi(\underline{G}, \text{false}) \rrbracket = \emptyset$ . Then, by Lem. 15,  $\llbracket \phi(\underline{G}, \text{false}) \rrbracket = \llbracket \text{false} \rrbracket$ .
- **Step:**  $\underline{G} = \text{if } \xi|_0 \underline{G}_1|_{R_1 \cup \{r\}} \underline{G}_2|_{\emptyset}$ , for some  $R_1, \underline{G}_1, \underline{G}_2, r, \xi$ . Similar to case “ $\underline{G} = \text{if } \xi|_0 \underline{G}_1|_{\emptyset} \underline{G}_2|_{R_2 \cup \{r\}}$ , for some  $R_2, \underline{G}_1, \underline{G}_2, r, \xi$ ”. □

*Proof (of Lem. 84).* By Defn. 21:

- **Base:**  $\underline{G} = q.y := e$ , for some  $q, y, e$ .  
 Recall  $\llbracket \phi(\underline{G}, \chi) \rrbracket \neq \emptyset$ . Then,  $\llbracket \phi(q.y := e, \chi) \rrbracket \neq \emptyset$ . Then, by Defn. 38,  $\llbracket \chi[e/q.y] \rrbracket \neq \emptyset$ . Then, Then,  $\mathcal{S} \in \llbracket \chi[e/q.y] \rrbracket$ , for some  $\mathcal{S}$ . Then, by Lem. 14,  $\mathcal{S}[\text{eval}_{\mathcal{S}}(e)/q.y] \in \llbracket \chi \rrbracket$ . Then,  $\llbracket \chi \rrbracket \neq \emptyset$ .
- **Base:**  $\underline{G} = p.e \rightarrow q.y$ , for some  $p, q, y, e$ . Similar to case “ $\underline{G} = q.y := e$ , for some  $q, y, e$ ”.
- **Base:**  $\underline{G} = \text{skip}$ .  
 Recall  $\llbracket \phi(\underline{G}, \chi) \rrbracket \neq \emptyset$ . Then,  $\llbracket \phi(\text{skip}, \chi) \rrbracket \neq \emptyset$ . Then, by Defn. 38,  $\llbracket \chi \rrbracket \neq \emptyset$ .
- **Step:**  $\underline{G} = \underline{G}_1 ; \underline{G}_2$ , for some  $\underline{G}_1, \underline{G}_2$ .  
 Recall  $\llbracket \phi(\underline{G}, \chi) \rrbracket \neq \emptyset$ . Then,  $\llbracket \phi(\underline{G}_1 ; \underline{G}_2, \chi) \rrbracket \neq \emptyset$ . Then, by Defn. 38,  $\llbracket \phi(\underline{G}_1, \phi(\underline{G}_2, \chi)) \rrbracket \neq \emptyset$ . Then, by induction,  $\llbracket \phi(\underline{G}_2, \chi) \rrbracket \neq \emptyset$ . Then, by induction,  $\llbracket \chi \rrbracket \neq \emptyset$ .
- **Step:**  $\underline{G} = \underline{G}_1 \parallel \underline{G}_2$ , for some  $\underline{G}_1, \underline{G}_2$ .  
 Recall  $\llbracket \phi(\underline{G}, \chi) \rrbracket \neq \emptyset$ . Then,  $\llbracket \phi(\underline{G}_1 \parallel \underline{G}_2, \chi) \rrbracket \neq \emptyset$ . Then, by Defn. 38:



- **Base:**  $\underline{G} = q.y := e$ , for some  $q, y, e$ .
  - Recall  $\underline{G} \# \xi$ . Then,  $q.y := e \# \xi$ . Then, by Defn. 23,  $\text{write}(q.y := e) \cap \text{read}(\xi) = \emptyset$ . Then, by Lem. 76,  $\phi(q.y := e, \xi) = \xi$ .
  - Recall  $[\phi(\underline{G}, \chi)] \cap [\xi] \subseteq [\xi]$ . Then,  $[\phi(\underline{G}, \chi)] \cap [\xi] \subseteq [\phi(q.y := e, \xi)]$ . Then, by Defn. 13,  $[\phi(\underline{G}, \chi) \wedge \xi] \subseteq [\phi(q.y := e, \xi)]$ . Then,  $[\phi(\underline{G}, \chi) \wedge \xi] \subseteq [\phi(\underline{G}, \xi)]$ .
- **Base:**  $\underline{G} = p.e \rightarrow q.y$ , for some  $p, q, y, e$ . Similar to case “ $\underline{G} = q.y := e$ , for some  $q, y, e$ ”.
- **Base:**  $\underline{G} = \text{skip}$ .
 

Recall  $[\phi(\underline{G}, \chi)] \cap [\xi] \subseteq [\xi]$ . Then, by Defn. 13,  $[\phi(\underline{G}, \chi) \wedge \xi] \subseteq [\xi]$ . Then, by Defn. 38,  $[\phi(\underline{G}, \chi) \wedge \xi] \subseteq [\phi(\text{skip}, \xi)]$ . Then,  $[\phi(\underline{G}, \chi) \wedge \xi] \subseteq [\phi(\underline{G}, \xi)]$ .
- **Step:**  $\underline{G} = \underline{G}_1 ; \underline{G}_2$ , for some  $\underline{G}_1, \underline{G}_2$ .
  - Recall  $\underline{G} \# \xi$ . Then,  $\underline{G}_1 ; \underline{G}_2 \# \xi$ . Then, by Lem. 38,  $\underline{G}_1 \# \xi$  and  $\underline{G}_2 \# \xi$ .
  - Recall  $\underline{G}_1 \# \xi$ . Then, by induction,  $[\phi(\underline{G}_1, \phi(\underline{G}_2, \chi)) \wedge \xi] \subseteq [\phi(\underline{G}_1, \xi)]$ . Then, by Defn. 38,  $[\phi(\underline{G}_1 ; \underline{G}_2, \chi) \wedge \xi] \subseteq [\phi(\underline{G}_1, \xi)]$ . Then,  $[\phi(\underline{G}, \chi) \wedge \xi] \subseteq [\phi(\underline{G}_1, \xi)]$ . Then, by Defn. 13,  $[\phi(\underline{G}, \chi)] \cap [\xi] \subseteq [\phi(\underline{G}_1, \xi)]$ . Then,  $[\phi(\underline{G}, \chi)] \cap [\xi] \subseteq [\phi(\underline{G}, \chi)] \cap [\phi(\underline{G}_1, \xi)]$ . Then, by Defn. 13,  $[\phi(\underline{G}, \chi) \wedge \xi] \subseteq [\phi(\underline{G}, \chi) \wedge \phi(\underline{G}_1, \xi)]$ .
  - Recall  $\underline{G}_2 \# \xi$ . Then, by induction,  $[\phi(\underline{G}_2, \chi) \wedge \xi] \subseteq [\phi(\underline{G}_2, \xi)]$ . Then, by Lem. 81,  $[\phi(\underline{G}_1, \phi(\underline{G}_2, \chi) \wedge \xi)] \subseteq [\phi(\underline{G}_1, \phi(\underline{G}_2, \xi))]$ . Then, by Lem. 82,  $[\phi(\underline{G}_1, \phi(\underline{G}_2, \chi)) \wedge \phi(\underline{G}_1, \xi)] \subseteq [\phi(\underline{G}_1, \phi(\underline{G}_2, \xi))]$ . Then, by Defn. 38,  $[\phi(\underline{G}_1 ; \underline{G}_2, \chi) \wedge \phi(\underline{G}_1, \xi)] \subseteq [\phi(\underline{G}_1 ; \underline{G}_2, \xi)]$ . Then,  $[\phi(\underline{G}, \chi) \wedge \phi(\underline{G}_1, \xi)] \subseteq [\phi(\underline{G}, \xi)]$ .
  - Recall  $[\phi(\underline{G}, \chi) \wedge \xi] \subseteq [\phi(\underline{G}, \chi) \wedge \phi(\underline{G}_1, \xi)] \subseteq [\phi(\underline{G}, \xi)]$ . Then,  $[\phi(\underline{G}, \chi) \wedge \xi] \subseteq [\phi(\underline{G}, \xi)]$ .
- **Step:**  $\underline{G} = \underline{G}_1 \parallel \underline{G}_2$ , for some  $\underline{G}_1, \underline{G}_2$ .
 

Recall:

  - **Case:**  $\underline{G}_1, \underline{G}_2 \in \mathbb{G}$  and  $\underline{G}_1 \# \underline{G}_2$ .
    - \* Recall  $\underline{G} \# \xi$ . Then,  $\underline{G}_1 \parallel \underline{G}_2 \# \xi$ . Then, by Lem. 38,  $\underline{G}_1 \# \xi$  and  $\underline{G}_2 \# \xi$ .
    - \* Recall  $\underline{G}_1, \underline{G}_2 \in \mathbb{G}$  and  $\underline{G}_1 \# \underline{G}_2$ . Then, by Defn. 38,  $\phi(\underline{G}_1 \parallel \underline{G}_2, \chi) = \phi(\underline{G}_1, \phi(\underline{G}_2, \chi))$  and  $\phi(\underline{G}_1 \parallel \underline{G}_2, \xi) = \phi(\underline{G}_1, \phi(\underline{G}_2, \xi))$ .
    - \* Recall  $\underline{G}_1 \# \xi$ . Then, by induction,  $[\phi(\underline{G}_1, \phi(\underline{G}_2, \chi)) \wedge \xi] \subseteq [\phi(\underline{G}_1, \xi)]$ . Then,  $[\phi(\underline{G}_1 \parallel \underline{G}_2, \chi) \wedge \xi] \subseteq [\phi(\underline{G}_1, \xi)]$ . Then,  $[\phi(\underline{G}, \chi) \wedge \xi] \subseteq [\phi(\underline{G}_1, \xi)]$ . Then, by Defn. 13,  $[\phi(\underline{G}, \chi)] \cap [\xi] \subseteq [\phi(\underline{G}_1, \xi)]$ . Then,  $[\phi(\underline{G}, \chi)] \cap [\xi] \subseteq [\phi(\underline{G}, \chi)] \cap [\phi(\underline{G}_1, \xi)]$ . Then, by Defn. 13,  $[\phi(\underline{G}, \chi) \wedge \xi] \subseteq [\phi(\underline{G}, \chi) \wedge \phi(\underline{G}_1, \xi)]$ .
    - \* Recall  $\underline{G}_2 \# \xi$ . Then, by induction,  $[\phi(\underline{G}_2, \chi) \wedge \xi] \subseteq [\phi(\underline{G}_2, \xi)]$ . Then, by Lem. 81,  $[\phi(\underline{G}_1, \phi(\underline{G}_2, \chi) \wedge \xi)] \subseteq [\phi(\underline{G}_1, \phi(\underline{G}_2, \xi))]$ . Then, by Lem. 82,  $[\phi(\underline{G}_1, \phi(\underline{G}_2, \chi)) \wedge \phi(\underline{G}_1, \xi)] \subseteq [\phi(\underline{G}_1, \phi(\underline{G}_2, \xi))]$ . Then,  $[\phi(\underline{G}_1 \parallel \underline{G}_2, \chi) \wedge \phi(\underline{G}_1, \xi)] \subseteq [\phi(\underline{G}_1 \parallel \underline{G}_2, \xi)]$ . Then,  $[\phi(\underline{G}, \chi) \wedge \phi(\underline{G}_1, \xi)] \subseteq [\phi(\underline{G}, \xi)]$ .
    - \* Recall  $[\phi(\underline{G}, \chi) \wedge \xi] \subseteq [\phi(\underline{G}, \chi) \wedge \phi(\underline{G}_1, \xi)] \subseteq [\phi(\underline{G}, \xi)]$ . Then,  $[\phi(\underline{G}, \chi) \wedge \xi] \subseteq [\phi(\underline{G}, \xi)]$ .
  - **Case:** Not  $\underline{G}_1, \underline{G}_2 \in \mathbb{G}$ , or not  $\underline{G}_1 \# \underline{G}_2$ .
    - \* Recall not  $\underline{G}_1, \underline{G}_2 \in \mathbb{G}$ , or not  $\underline{G}_1 \# \underline{G}_2$ . Then, by Defn. 38,  $\phi(\underline{G}_1 \parallel \underline{G}_2, \chi) = \text{false}$ .
    - \* Recall  $\emptyset \cap [\xi] \subseteq [\phi(\underline{G}, \xi)]$ . Then, by Lem. 15,  $[\text{false}] \cap [\xi] \subseteq [\phi(\underline{G}, \xi)]$ . Then, by Defn. 13,  $[\text{false} \wedge \xi] \subseteq [\phi(\underline{G}, \xi)]$ . Then,  $[\phi(\underline{G}_1 \parallel \underline{G}_2, \chi) \wedge \xi] \subseteq [\phi(\underline{G}, \xi)]$ . Then,  $[\phi(\underline{G}, \chi) \wedge \xi] \subseteq [\phi(\underline{G}, \xi)]$ .
- **Step:**  $\underline{G} = R.\text{if } \hat{\xi} \underline{G}_1 \underline{G}_2$ , for some  $R, \underline{G}_1, \underline{G}_2, \hat{\xi}$ .
 

Recall  $\underline{G} \# \xi$ . Then,  $R.\text{if } \hat{\xi} \underline{G}_1 \underline{G}_2 \# \xi$ . Then, by Lem. 38,  $\underline{G}_1 \# \xi$  and  $\underline{G}_2 \# \xi$ . Then, by induction,  $[\phi(\underline{G}_1, \chi) \wedge \xi] \subseteq [\phi(\underline{G}_1, \xi)]$  and  $[\phi(\underline{G}_2, \chi) \wedge \xi] \subseteq [\phi(\underline{G}_2, \xi)]$ . Then, by Lem. 22,  $[\hat{\xi}^+ \Rightarrow (\phi(\underline{G}_1, \chi) \wedge \xi)] \subseteq [\hat{\xi}^+ \Rightarrow \phi(\underline{G}_1, \xi)]$  and  $[\hat{\xi}^- \Rightarrow (\phi(\underline{G}_2, \chi) \wedge \xi)] \subseteq [\hat{\xi}^- \Rightarrow \phi(\underline{G}_2, \xi)]$ . Then, by Lem. 18,  $[\hat{\xi}^+ \Rightarrow \phi(\underline{G}_1, \chi)] \cap [\hat{\xi}^+ \Rightarrow \xi] \subseteq [\hat{\xi}^+ \Rightarrow \phi(\underline{G}_1, \xi)]$  and  $[\hat{\xi}^- \Rightarrow \phi(\underline{G}_2, \chi)] \cap [\hat{\xi}^- \Rightarrow \xi] \subseteq [\hat{\xi}^- \Rightarrow \phi(\underline{G}_2, \xi)]$ . Then, by Lem. 25,  $[\hat{\xi}^+ \Rightarrow \phi(\underline{G}_1, \chi)] \cap [\xi] \subseteq [\hat{\xi}^+ \Rightarrow \phi(\underline{G}_1, \xi)]$  and  $[\hat{\xi}^- \Rightarrow \phi(\underline{G}_2, \chi)] \cap [\xi] \subseteq [\hat{\xi}^- \Rightarrow \phi(\underline{G}_2, \xi)]$ . Then,  $[\hat{\xi}^+ \Rightarrow \phi(\underline{G}_1, \chi)] \cap [\hat{\xi}^- \Rightarrow \phi(\underline{G}_2, \chi)] \cap [\xi] \subseteq [\hat{\xi}^+ \Rightarrow \phi(\underline{G}_1, \xi)] \cap [\hat{\xi}^- \Rightarrow \phi(\underline{G}_2, \xi)]$ . Then,  $[\hat{\xi}^+ \Rightarrow \phi(\underline{G}_1, \chi)] \cap [\hat{\xi}^- \Rightarrow \phi(\underline{G}_2, \chi)] \cap [\hat{\xi}^{\equiv}] \cap [\xi] \subseteq [\hat{\xi}^+ \Rightarrow \phi(\underline{G}_1, \xi)] \cap [\hat{\xi}^- \Rightarrow \phi(\underline{G}_2, \xi)] \cap [\hat{\xi}^{\equiv}]$ . Then, by Defn. 13,



- **Step:**  $\underline{G} = \text{if } \hat{\xi} \mid_0 \underline{G}_1 \mid_{R_1 \cup \{r\}} \underline{G}_2 \mid_\emptyset$ , for some  $R_1, \underline{G}_1, \underline{G}_2, r, \hat{\xi}$ . Similar to case “ $\underline{G} = \text{if } \hat{\xi} \mid_0 \underline{G}_1 \mid_\emptyset \underline{G}_2 \mid_{R_2 \cup \{r\}}$ , for some  $R_2, \underline{G}_1, \underline{G}_2, r, \hat{\xi}$ ”. □

*Proof (of Lem. 86).* By Defn. 21:

- **Base:**  $\underline{G} = q.y := e$ , for some  $q, y, e$ .  
Recall  $\underline{G} \# \gamma$ . Then,  $q.y := e \# \gamma$ . Then, by Defn. 23,  $\text{read}(q.y := e) \cap \text{write}(\gamma) = \emptyset$  and  $\text{write}(q.y := e) \cap \text{read}(\gamma) = \emptyset$  and  $\text{write}(q.y := e) \cap \text{write}(\gamma) = \emptyset$ . Then, by Lem. 77,  $\phi(q.y := e, \phi(\gamma, \chi)) = \phi(\gamma, \phi(q.y := e, \chi))$ . Then,  $\phi(\underline{G}, \phi(\gamma, \chi)) = \phi(\gamma, \phi(\underline{G}, \chi))$ . Then,  $\llbracket \phi(\underline{G}, \phi(\gamma, \chi)) \rrbracket = \llbracket \phi(\gamma, \phi(\underline{G}, \chi)) \rrbracket$ .
- **Base:**  $\underline{G} = p.e \rightarrow q.y$ , for some  $p, q, y, e$ . Similar to case “ $\underline{G} = q.y := e$ , for some  $q, y, e$ ”.
- **Base:**  $\underline{G} = \text{skip}$ .  
Recall  $\phi(\gamma, \chi) = \phi(\gamma, \chi)$ . Then, by Defn. 38,  $\phi(\text{skip}, \phi(\gamma, \chi)) = \phi(\gamma, \phi(\text{skip}, \chi))$ . Then,  $\llbracket \phi(\underline{G}, \phi(\gamma, \chi)) \rrbracket = \llbracket \phi(\gamma, \phi(\underline{G}, \chi)) \rrbracket$ .
- **Step:**  $\underline{G} = \underline{G}_1 ; \underline{G}_2$ , for some  $\underline{G}_1, \underline{G}_2$ .
  - Recall  $\underline{G} \# \xi$ . Then,  $\underline{G}_1 ; \underline{G}_2 \# \xi$ . Then, by Lem. 38,  $\underline{G}_1 \# \xi$  and  $\underline{G}_2 \# \xi$ .
  - Recall  $\underline{G}_2 \# \xi$ . Then, by induction,  $\llbracket \phi(\underline{G}_2, \phi(\gamma, \chi)) \rrbracket = \llbracket \phi(\gamma, \phi(\underline{G}_2, \chi)) \rrbracket$ . Then, by Lem. 81,  $\llbracket \phi(\underline{G}_1, \phi(\underline{G}_2, \phi(\gamma, \chi))) \rrbracket = \llbracket \phi(\underline{G}_1, \phi(\gamma, \phi(\underline{G}_2, \chi))) \rrbracket$ .
  - Recall  $\underline{G}_1 \# \xi$ . Then, by induction,  $\llbracket \phi(\underline{G}_1, \phi(\gamma, \phi(\underline{G}_2, \chi))) \rrbracket = \llbracket \phi(\gamma, \phi(\underline{G}_1, \phi(\underline{G}_2, \chi))) \rrbracket$ .
  - Recall  $\llbracket \phi(\underline{G}_1, \phi(\underline{G}_2, \phi(\gamma, \chi))) \rrbracket = \llbracket \phi(\underline{G}_1, \phi(\gamma, \phi(\underline{G}_2, \chi))) \rrbracket = \llbracket \phi(\gamma, \phi(\underline{G}_1, \phi(\underline{G}_2, \chi))) \rrbracket$ . Then,  $\llbracket \phi(\underline{G}_1, \phi(\underline{G}_2, \phi(\gamma, \chi))) \rrbracket = \llbracket \phi(\gamma, \phi(\underline{G}_1, \phi(\underline{G}_2, \chi))) \rrbracket$ . Then, by Defn. 38,  $\llbracket \phi(\underline{G}_1 ; \underline{G}_2, \phi(\gamma, \chi)) \rrbracket = \llbracket \phi(\gamma, \phi(\underline{G}_1 ; \underline{G}_2, \chi)) \rrbracket$ . Then,  $\llbracket \phi(\underline{G}, \phi(\gamma, \chi)) \rrbracket = \llbracket \phi(\gamma, \phi(\underline{G}, \chi)) \rrbracket$ .
- **Step:**  $\underline{G} = \underline{G}_1 \parallel \underline{G}_2$ , for some  $\underline{G}_1, \underline{G}_2$ .  
Recall:
  - **Case:**  $\underline{G}_1, \underline{G}_2 \in \mathbb{G}$  and  $\underline{G}_1 \# \underline{G}_2$ .
    - \* Recall  $\underline{G} \# \xi$ . Then,  $\underline{G}_1 \parallel \underline{G}_2 \# \xi$ . Then, by Lem. 38,  $\underline{G}_1 \# \xi$  and  $\underline{G}_2 \# \xi$ .
    - \* Recall  $\underline{G}_2 \# \xi$ . Then, by induction,  $\llbracket \phi(\underline{G}_2, \phi(\gamma, \chi)) \rrbracket = \llbracket \phi(\gamma, \phi(\underline{G}_2, \chi)) \rrbracket$ . Then, by Lem. 81,  $\llbracket \phi(\underline{G}_1, \phi(\underline{G}_2, \phi(\gamma, \chi))) \rrbracket = \llbracket \phi(\underline{G}_1, \phi(\gamma, \phi(\underline{G}_2, \chi))) \rrbracket$ .
    - \* Recall  $\underline{G}_1 \# \xi$ . Then, by induction,  $\llbracket \phi(\underline{G}_1, \phi(\gamma, \phi(\underline{G}_2, \chi))) \rrbracket = \llbracket \phi(\gamma, \phi(\underline{G}_1, \phi(\underline{G}_2, \chi))) \rrbracket$ .
    - \* Recall  $\underline{G}_1, \underline{G}_2 \in \mathbb{G}$  and  $\underline{G}_1 \# \underline{G}_2$ . Then, by Defn. 38,  $\phi(\underline{G}_1 \parallel \underline{G}_2, \chi) = \phi(\underline{G}_1, \phi(\underline{G}_2, \chi))$  and  $\phi(\underline{G}_1 \parallel \underline{G}_2, \phi(\gamma, \chi)) = \phi(\underline{G}_1, \phi(\underline{G}_2, \phi(\gamma, \chi)))$ .
    - \* Recall  $\llbracket \phi(\underline{G}_1, \phi(\underline{G}_2, \phi(\gamma, \chi))) \rrbracket = \llbracket \phi(\underline{G}_1, \phi(\gamma, \phi(\underline{G}_2, \chi))) \rrbracket = \llbracket \phi(\gamma, \phi(\underline{G}_1, \phi(\underline{G}_2, \chi))) \rrbracket$ . Then,  $\llbracket \phi(\underline{G}_1, \phi(\underline{G}_2, \phi(\gamma, \chi))) \rrbracket = \llbracket \phi(\gamma, \phi(\underline{G}_1, \phi(\underline{G}_2, \chi))) \rrbracket$ . Then,  $\llbracket \phi(\underline{G}_1 \parallel \underline{G}_2, \phi(\gamma, \chi)) \rrbracket = \llbracket \phi(\gamma, \phi(\underline{G}_1 \parallel \underline{G}_2, \chi)) \rrbracket$ . Then,  $\llbracket \phi(\underline{G}, \phi(\gamma, \chi)) \rrbracket = \llbracket \phi(\gamma, \phi(\underline{G}, \chi)) \rrbracket$ .
  - **Case:** Not  $\underline{G}_1, \underline{G}_2 \in \mathbb{G}$ , or not  $\underline{G}_1 \# \underline{G}_2$ .
    - \* Recall  $\text{write}(\gamma) \cap \emptyset = \emptyset$ . Then, by Lem. 3,  $\text{write}(\gamma) \cap \text{read}(\text{false}) = \emptyset$ . Then, by Lem. 76,  $\phi(\gamma, \text{false}) = \text{false}$ .
    - \* Recall not  $\underline{G}_1, \underline{G}_2 \in \mathbb{G}$ , or not  $\underline{G}_1 \# \underline{G}_2$ . Then, by Defn. 38,  $\phi(\underline{G}_1 \parallel \underline{G}_2, \chi) = \text{false}$  and  $\phi(\underline{G}_1 \parallel \underline{G}_2, \phi(\gamma, \chi)) = \text{false}$ .
    - \* Recall  $\llbracket \text{false} \rrbracket = \llbracket \text{false} \rrbracket$ . Then,  $\llbracket \text{false} \rrbracket = \llbracket \phi(\gamma, \text{false}) \rrbracket$ . Then,  $\llbracket \phi(\underline{G}_1 \parallel \underline{G}_2, \phi(\gamma, \chi)) \rrbracket = \llbracket \phi(\gamma, \phi(\underline{G}_1 \parallel \underline{G}_2, \chi)) \rrbracket$ . Then,  $\llbracket \phi(\underline{G}, \phi(\gamma, \chi)) \rrbracket = \llbracket \phi(\gamma, \phi(\underline{G}, \chi)) \rrbracket$ .
- **Step:**  $\underline{G} = R.\text{if } \xi \underline{G}_1 \underline{G}_2$ , for some  $R, \underline{G}_1, \underline{G}_2, \xi$ .
  - Recall  $\underline{G} \# \gamma$ . Then,  $R.\text{if } \xi \underline{G}_1 \underline{G}_2 \# \gamma$ . Then, by Lem. 38,  $\xi \# \gamma$  and  $\underline{G}_1 \# \gamma$  and  $\underline{G}_2 \# \gamma$ .
  - Recall  $\xi \# \gamma$ . Then, by Lem. 37,  $\gamma \# \xi$ . Then, by Defn. 23,  $\text{write}(\gamma) \cap \text{read}(\xi) = \emptyset$ . Then, by Lem. 12,  $\text{write}(\gamma) \cap \text{read}(\xi^+) = \emptyset$  and  $\text{write}(\gamma) \cap \text{read}(\xi^-) = \emptyset$  and  $\text{write}(\gamma) \cap \text{read}(\xi^\equiv) = \emptyset$ . Then, by Lem. 76,  $\phi(\gamma, \xi^+) = \xi^+$  and  $\phi(\gamma, \xi^-) = \xi^-$  and  $\phi(\gamma, \xi^\equiv) = \xi^\equiv$ .



- write( $\gamma$ )  $\cap$  read( $\bigwedge\{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_2}$ ) =  $\emptyset$ . Then, by Lem. 76,  
 $\phi(\gamma, \bigwedge\{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_2}) = \bigwedge\{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_2}$ .
- \* Recall  $R_1 = \emptyset \neq R_2$ . Then, by Defn. 38,  
 $\phi(\text{if } \xi|_0 \underline{G}_1|_{R_1} \underline{G}_2|_{R_2}, \chi) = \phi(\underline{G}_2, \chi) \wedge \bigwedge\{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_2}$  and  
 $\phi(\text{if } \xi|_0 \underline{G}_1|_{R_1} \underline{G}_2|_{R_2}, \phi(\gamma, \chi)) = \phi(\underline{G}_2, \phi(\gamma, \chi)) \wedge \bigwedge\{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_2}$ .
  - \* Recall  $\underline{G}_2 \# \gamma$ . Then, by induction,  $\llbracket \phi(\underline{G}_2, \phi(\gamma, \chi)) \rrbracket = \llbracket \phi(\gamma, \phi(\underline{G}_2, \chi)) \rrbracket$ . Then,  
 $\llbracket \phi(\underline{G}_2, \phi(\gamma, \chi)) \rrbracket \cap \llbracket \bigwedge\{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_2} \rrbracket = \llbracket \phi(\gamma, \phi(\underline{G}_2, \chi)) \rrbracket \cap \llbracket \bigwedge\{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_2} \rrbracket$ .  
Then,  $\llbracket \phi(\underline{G}_2, \phi(\gamma, \chi)) \rrbracket \cap \llbracket \bigwedge\{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_2} \rrbracket =$   
 $\llbracket \phi(\gamma, \phi(\underline{G}_2, \chi)) \rrbracket \cap \llbracket \bigwedge\{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_2} \rrbracket$ . Then, by Defn. 13,  
 $\llbracket \phi(\underline{G}_2, \phi(\gamma, \chi)) \wedge \bigwedge\{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_2} \rrbracket = \llbracket \phi(\gamma, \phi(\underline{G}_2, \chi)) \wedge \phi(\gamma, \bigwedge\{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_2}) \rrbracket$ .  
Then, by Lem. 75,  
 $\llbracket \phi(\underline{G}_2, \phi(\gamma, \chi)) \wedge \bigwedge\{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_2} \rrbracket = \llbracket \phi(\gamma, \phi(\underline{G}_2, \chi)) \wedge \bigwedge\{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_2} \rrbracket$ .  
Then,  $\llbracket \phi(\text{if } \xi|_0 \underline{G}_1|_{R_1} \underline{G}_2|_{R_2}, \phi(\gamma, \chi)) \rrbracket = \llbracket \phi(\gamma, \phi(\text{if } \xi|_0 \underline{G}_1|_{R_1} \underline{G}_2|_{R_2}, \chi)) \rrbracket$ . Then,  
 $\llbracket \phi(\underline{G}, \phi(\gamma, \chi)) \rrbracket = \llbracket \phi(\gamma, \phi(\underline{G}, \chi)) \rrbracket$ .
  - **Case:**  $R_1 \neq \emptyset = R_2$ . Similar to case  $R_1 = \emptyset \neq R_2$ .
  - **Case:**  $R_1 \neq \emptyset \neq R_2$ .
    - \* Recall write( $\gamma$ )  $\cap \emptyset = \emptyset$ . Then, by Lem. 3, write( $\gamma$ )  $\cap$  read(false) =  $\emptyset$ . Then, by Lem. 76,  $\phi(\gamma, \text{false}) = \text{false}$ .
    - \* Recall  $R_1 \neq \emptyset \neq R_2$ . Then, by Defn. 38,  $\phi(\text{if } \xi|_0 \underline{G}_1|_{R_1} \underline{G}_2|_{R_2}, \chi) = \text{false}$  and  $\phi(\text{if } \xi|_0 \underline{G}_1|_{R_1} \underline{G}_2|_{R_2}, \phi(\gamma, \chi)) = \text{false}$ .
    - \* Recall  $\llbracket \text{false} \rrbracket = \llbracket \text{false} \rrbracket$ . Then,  $\llbracket \text{false} \rrbracket = \llbracket \phi(\gamma, \text{false}) \rrbracket$ . Then,  
 $\llbracket \phi(\text{if } \xi|_0 \underline{G}_1|_{R_1} \underline{G}_2|_{R_2}, \phi(\gamma, \chi)) \rrbracket = \llbracket \phi(\gamma, \phi(\text{if } \xi|_0 \underline{G}_1|_{R_1} \underline{G}_2|_{R_2}, \chi)) \rrbracket$ . Then,  
 $\llbracket \phi(\underline{G}, \phi(\gamma, \chi)) \rrbracket = \llbracket \phi(\gamma, \phi(\underline{G}, \chi)) \rrbracket$ .
- **Step:**  $\underline{G} = \text{if } \xi|_0 \underline{G}_1|_{\emptyset} \underline{G}_2|_{R_2 \cup \{r\}}$ , for some  $R_2, \underline{G}_1, \underline{G}_2, r, \xi$ .
- Recall  $\underline{G} \# \xi$ . Then,  $\text{if } \xi|_0 \underline{G}_1|_{\emptyset} \underline{G}_2|_{R_2 \cup \{r\}} \# \xi$ .
  - Recall  $\text{if } \xi|_0 \underline{G}_1|_{\emptyset} \underline{G}_2|_{R_2 \cup \{r\}} \# \xi$  and  $\emptyset = \emptyset \neq R_2 \cup \{r\}$ . Then, by Lem. 38,  $\xi \uparrow r \# \gamma$  for every  $r \in \text{subj}(\xi) \setminus (R_2 \cup \{r\})$ , and  $\underline{G}_2 \# \gamma$ .
  - Recall  $\xi \uparrow r \# \gamma$  for every  $r \in \text{subj}(\xi) \setminus (R_2 \cup \{r\})$ . Then, by Lem. 39,  
 $\bigwedge\{(\xi \uparrow r)^-\}_{r \in \text{subj}(\xi) \setminus (R_2 \cup \{r\})} \# \gamma$ . Then, by Lem. 11,  $\bigwedge\{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus (R_2 \cup \{r\})} \# \gamma$ .  
Then, by Lem. 37,  $\gamma \# \bigwedge\{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus (R_2 \cup \{r\})}$ . Then, by Defn. 23,  
write( $\gamma$ )  $\cap$  read( $\bigwedge\{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus (R_2 \cup \{r\})}$ ) =  $\emptyset$ . Then, by Lem. 76,  
 $\phi(\gamma, \bigwedge\{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus (R_2 \cup \{r\})}) = \bigwedge\{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus (R_2 \cup \{r\})}$ .
  - Recall  $\emptyset = \emptyset \neq R_2 \cup \{r\}$ . Then, by Defn. 38,  
 $\phi(\text{if } \xi|_0 \underline{G}_1|_{\emptyset} \underline{G}_2|_{R_2 \cup \{r\}}, \chi) = \phi(\underline{G}_2, \chi) \wedge \bigwedge\{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus (R_2 \cup \{r\})}$  and  
 $\phi(\text{if } \xi|_0 \underline{G}_1|_{\emptyset} \underline{G}_2|_{R_2 \cup \{r\}}, \phi(\gamma, \chi)) = \phi(\underline{G}_2, \phi(\gamma, \chi)) \wedge \bigwedge\{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus (R_2 \cup \{r\})}$ .
  - Recall  $\underline{G}_2 \# \gamma$ . Then, by induction,  $\llbracket \phi(\underline{G}_2, \phi(\gamma, \chi)) \rrbracket = \llbracket \phi(\gamma, \phi(\underline{G}_2, \chi)) \rrbracket$ . Then,  
 $\llbracket \phi(\underline{G}_2, \phi(\gamma, \chi)) \rrbracket \cap \llbracket \bigwedge\{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus (R_2 \cup \{r\})} \rrbracket =$   
 $\llbracket \phi(\gamma, \phi(\underline{G}_2, \chi)) \rrbracket \cap \llbracket \bigwedge\{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus (R_2 \cup \{r\})} \rrbracket$ . Then,  
 $\llbracket \phi(\underline{G}_2, \phi(\gamma, \chi)) \rrbracket \cap \llbracket \bigwedge\{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus (R_2 \cup \{r\})} \rrbracket =$   
 $\llbracket \phi(\gamma, \phi(\underline{G}_2, \chi)) \rrbracket \cap \llbracket \bigwedge\{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus (R_2 \cup \{r\})} \rrbracket$ . Then, by Defn. 13,  
 $\llbracket \phi(\underline{G}_2, \phi(\gamma, \chi)) \wedge \bigwedge\{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus (R_2 \cup \{r\})} \rrbracket =$   
 $\llbracket \phi(\gamma, \phi(\underline{G}_2, \chi)) \wedge \phi(\gamma, \bigwedge\{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus (R_2 \cup \{r\})}) \rrbracket$ . Then, by Lem. 75,  
 $\llbracket \phi(\underline{G}_2, \phi(\gamma, \chi)) \wedge \bigwedge\{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus (R_2 \cup \{r\})} \rrbracket =$   
 $\llbracket \phi(\gamma, \phi(\underline{G}_2, \chi)) \wedge \bigwedge\{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus (R_2 \cup \{r\})} \rrbracket$ . Then,  
 $\llbracket \phi(\text{if } \xi|_0 \underline{G}_1|_{\emptyset} \underline{G}_2|_{R_2 \cup \{r\}}, \phi(\gamma, \chi)) \rrbracket = \llbracket \phi(\gamma, \phi(\text{if } \xi|_0 \underline{G}_1|_{\emptyset} \underline{G}_2|_{R_2 \cup \{r\}}, \chi)) \rrbracket$ . Then,  
 $\llbracket \phi(\underline{G}, \phi(\gamma, \chi)) \rrbracket = \llbracket \phi(\gamma, \phi(\underline{G}, \chi)) \rrbracket$ .
- **Step:**  $\underline{G} = \text{if } \xi|_0 \underline{G}_1|_{R_1 \cup \{r\}} \underline{G}_2|_{\emptyset}$ , for some  $R_1, \underline{G}_1, \underline{G}_2, r, \xi$ . Similar to case  
“ $\underline{G} = \text{if } \xi|_0 \underline{G}_1|_{\emptyset} \underline{G}_2|_{R_2 \cup \{r\}}$ , for some  $R_2, \underline{G}_1, \underline{G}_2, r, \xi$ ”. □

*Proof (of Lem. 87).* Recall  $\checkmark_R(G)$ . Then, by Defn. 27:

- **Base:**  $[\checkmark 1\text{-ACT1}]$ , such that  $G = q.y := e$ , for some  $q, y, e$ .  
By Defn. 21,  $q.y := e \in \underline{\mathbb{G}}$ . Then,  $G \in \underline{\mathbb{G}}$ .

- **Base:**  $[\sqrt{1}\text{-ACT2}]$ . Similar to case  $[\sqrt{1}\text{-ACT1}]$ .
- **Base:**  $[\sqrt{1}\text{-SKIP}]$ . Similar to case  $[\sqrt{1}\text{-ACT1}]$ .
- **Step:**  $[\sqrt{1}\text{-SEQ}]$ , such that  $G = G_1 ; G_2$  and  $\sqrt{R}(G_1)$  and  $\sqrt{R}(G_2)$ , for some  $G_1, G_2$ .
  - Recall  $\llbracket \phi(G, \chi) \rrbracket \neq \emptyset$ . Then,  $\llbracket \phi(G_1 ; G_2, \chi) \rrbracket \neq \emptyset$ . Then, by Defn. 38,  $\llbracket \phi(G_1, \phi(G_2, \chi)) \rrbracket \neq \emptyset$ .
  - Recall  $\sqrt{R}(G_1)$  and  $\llbracket \phi(G_1, \phi(G_2, \chi)) \rrbracket \neq \emptyset$ . Then:
    - \* **Case:**  $\text{subj}(G_1) = R$ .  
Recall  $\sqrt{R}(G_2)$ . Then, by Lem. 47,  $\text{subj}(G_2) \subseteq R$ . Then,  $R \cup \text{subj}(G_2) = R$ . Then,  $\text{subj}(G_1) \cup \text{subj}(G_2) = R$ . Then, by Defn. 24,  $\text{subj}(G_1 ; G_2) = R$ . Then,  $\text{subj}(G) = R$ .
    - \* **Case:**  $G_1 \in \underline{\mathbb{G}}$ .
      - Recall  $G_1 \in \underline{\mathbb{G}}$  and  $\llbracket \phi(G_1, \phi(G_2, \chi)) \rrbracket \neq \emptyset$ . Then, by Lem. 84,  $\llbracket \phi(G_2, \chi) \rrbracket \neq \emptyset$ .
      - Recall  $\sqrt{R}(G_2)$  and  $\llbracket \phi(G_2, \chi) \rrbracket \neq \emptyset$ . Then, by induction:
        - **Case:**  $\text{subj}(G_2) = R$ .  
Recall  $\sqrt{R}(G_1)$ . Then, by Lem. 47,  $\text{subj}(G_1) \subseteq R$ . Then,  $\text{subj}(G_1) \cup R = R$ . Then,  $\text{subj}(G_1) \cup \text{subj}(G_2) = R$ . Then, by Defn. 24,  $\text{subj}(G_1 ; G_2) = R$ . Then,  $\text{subj}(G) = R$ .
        - **Case:**  $G_2 \in \underline{\mathbb{G}}$ .  
Recall  $G_1, G_2 \in \underline{\mathbb{G}}$ . Then, by Defn. 21,  $G_1 ; G_2 \in \underline{\mathbb{G}}$ . Then,  $G \in \underline{\mathbb{G}}$ .
- **Step:**  $[\sqrt{1}\text{-PAR}]$ , such that  $G = G_1 \parallel G_2$  and  $\sqrt{R}(G_1)$  and  $\sqrt{R}(G_2)$ , for some  $G_1, G_2$ .  
Recall  $\llbracket \phi(G, \chi) \rrbracket \neq \emptyset$ . Then,  $\llbracket \phi(G_1 \parallel G_2, \chi) \rrbracket \neq \emptyset$ . Then, by Defn. 38:
  - **Case:**  $\llbracket \phi(G_1, \phi(G_2, \chi)) \rrbracket \neq \emptyset$ .  
Recall  $\sqrt{R}(G_1)$  and  $\llbracket \phi(G_1, \phi(G_2, \chi)) \rrbracket \neq \emptyset$ . Then:
    - \* **Case:**  $\text{subj}(G_1) = R$ .  
Recall  $\sqrt{R}(G_2)$ . Then, by Lem. 47,  $\text{subj}(G_2) \subseteq R$ . Then,  $R \cup \text{subj}(G_2) = R$ . Then,  $\text{subj}(G_1) \cup \text{subj}(G_2) = R$ . Then, by Defn. 24,  $\text{subj}(G_1 \parallel G_2) = R$ . Then,  $\text{subj}(G) = R$ .
    - \* **Case:**  $G_1 \in \underline{\mathbb{G}}$ .
      - Recall  $G_1 \in \underline{\mathbb{G}}$  and  $\llbracket \phi(G_1, \phi(G_2, \chi)) \rrbracket \neq \emptyset$ . Then, by Lem. 84,  $\llbracket \phi(G_2, \chi) \rrbracket \neq \emptyset$ .
      - Recall  $\sqrt{R}(G_2)$  and  $\llbracket \phi(G_2, \chi) \rrbracket \neq \emptyset$ . Then, by induction:
        - **Case:**  $\text{subj}(G_2) = R$ .  
Recall  $\sqrt{R}(G_1)$ . Then, by Lem. 47,  $\text{subj}(G_1) \subseteq R$ . Then,  $\text{subj}(G_1) \cup R = R$ . Then,  $\text{subj}(G_1) \cup \text{subj}(G_2) = R$ . Then, by Defn. 24,  $\text{subj}(G_1 \parallel G_2) = R$ . Then,  $\text{subj}(G) = R$ .
        - **Case:**  $G_2 \in \underline{\mathbb{G}}$ .  
Recall  $G_1, G_2 \in \underline{\mathbb{G}}$ . Then, by Defn. 21,  $G_1 \parallel G_2 \in \underline{\mathbb{G}}$ . Then,  $G \in \underline{\mathbb{G}}$ .
    - **Case:**  $\llbracket \text{false} \rrbracket \neq \emptyset$ . Then, by Lem. 15,  $\emptyset \neq \emptyset$ . Then,  $\text{false}$ .
  - **Step:**  $[\sqrt{1}\text{-IF}]$ , such that  $G = R.\text{if } \xi G_1 G_2$  and  $\sqrt{R}(G_1)$  and  $\sqrt{R}(G_2)$  and  $R = \text{subj}(\xi)$ , for some  $G_1, G_2, \xi$ .  
Recall  $\sqrt{R}(G_1)$  and  $\sqrt{R}(G_2)$ . Then, by Lem. 47,  $\text{subj}(G_1) \subseteq R$  and  $\text{subj}(G_2) \subseteq R$ . Then,  $\text{subj}(G_1) \cup \text{subj}(G_2) \subseteq R$ . Then,  $R \cup \text{subj}(G_1) \cup \text{subj}(G_2) = R$ . Then,  $\text{subj}(\xi) \cup \text{subj}(G_1) \cup \text{subj}(G_2) = R$ . Then, by Defn. 24,  $\text{subj}(R.\text{if } \xi G_1 G_2) = R$ . Then,  $\text{subj}(G) = R$ .
  - **Step:**  $[\sqrt{1}\text{-WHILE}]$ . Similar to case  $[\sqrt{1}\text{-IF}]$ .
  - **Step:**  $[\sqrt{1}\text{-NIF}]$ , such that  $G = \text{if } \xi|_0 G_1|_{R_1} G_2|_{R_2}$  and  $\sqrt{R}(G_1)$  and  $\sqrt{R}(G_2)$  and  $R = \text{subj}(\xi)$ , for some  $R_1, R_2, G_1, G_2, \xi$ .
    - **Case:**  $R_1 = \emptyset = R_2$ .  
Recall  $\sqrt{R}(G_1)$  and  $\sqrt{R}(G_2)$ . Then, by Lem. 47,  $\text{subj}(G_1) \subseteq R$  and  $\text{subj}(G_2) \subseteq R$ . Then,  $\text{subj}(G_1) \cup \text{subj}(G_2) \subseteq R$ . Then,  $R \cup \text{subj}(G_1) \cup \text{subj}(G_2) = R$ . Then,  $\text{subj}(\xi) \cup \text{subj}(G_1) \cup \text{subj}(G_2) = R$ . Then,  $(\text{subj}(\xi) \setminus \emptyset) \cup (\text{subj}(G_1) \setminus \emptyset) \cup (\text{subj}(G_2) \setminus \emptyset) = R$ . Then,  $(\text{subj}(\xi) \setminus (\emptyset \cup \emptyset)) \cup (\text{subj}(G_1) \setminus \emptyset) \cup (\text{subj}(G_2) \setminus \emptyset) = R$ . Then,  $(\text{subj}(\xi) \setminus (R_1 \cup R_2)) \cup (\text{subj}(G_1) \setminus R_2) \cup (\text{subj}(G_2) \setminus R_1) = R$ . Then, by Defn. 24,  $\text{subj}(\text{if } \xi|_0 G_1|_{R_1} G_2|_{R_2}) = R$ . Then,  $\text{subj}(G) = R$ .
    - **Case:**  $R_1 = \emptyset \neq R_2$ .
      - \* Recall  $R_1 = \emptyset \neq R_2$ . Then, by Defn. 38,  
 $\phi(\text{if } \xi|_0 G_1|_{R_1} G_2|_{R_2}, \chi) = \phi(G_2, \chi) \wedge \bigwedge \{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_2}$ .



- \* Recall  $\llbracket \Phi(G, \chi) \rrbracket \neq \emptyset$ . Then,  $\llbracket \Phi(\text{if } \xi|_0 G_1|_{R_1} G_2|_{R_2}, \chi) \rrbracket \neq \emptyset$ . Then,  $\llbracket \Phi(G_2, \chi) \wedge \bigwedge \{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_2} \rrbracket \neq \emptyset$ . Then, by Defn. 13,  $\llbracket \Phi(G_2, \chi) \rrbracket \cap \llbracket \bigwedge \{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_2} \rrbracket \neq \emptyset$ . Then,  $\llbracket \Phi(G_2, \chi) \rrbracket \neq \emptyset$ .
  - \* Recall  $\sqrt{R}(G_2)$  and  $\llbracket \Phi(G_2, \chi) \rrbracket \neq \emptyset$ . Then, by induction:
    - **Case:**  $\text{subj}(G_2) = R$ .
      - Recall  $R \subseteq R$ . Then,  $\text{subj}(\xi) \subseteq R$ . Then,  $\text{subj}(\xi) \setminus (R_1 \cup R_2) \subseteq R$ .
      - Recall  $\sqrt{R}(G_1)$ . Then, by Lem. 47,  $\text{subj}(G_1) \subseteq R$ . Then,  $\text{subj}(G_1) \setminus R_2 \subseteq R$ .
      - Recall  $\text{subj}(\xi) \setminus (R_1 \cup R_2) \subseteq R$  and  $\text{subj}(G_1) \setminus R_2 \subseteq R$ . Then,  $(\text{subj}(\xi) \setminus (R_1 \cup R_2)) \cup (\text{subj}(G_1) \setminus R_2) \subseteq R$ . Then,  $(\text{subj}(\xi) \setminus (R_1 \cup R_2)) \cup (\text{subj}(G_1) \setminus R_2) \cup R = R$ . Then,  $(\text{subj}(\xi) \setminus (R_1 \cup R_2)) \cup (\text{subj}(G_1) \setminus R_2) \cup \text{subj}(G_2) = R$ . Then,  $(\text{subj}(\xi) \setminus (R_1 \cup R_2)) \cup (\text{subj}(G_1) \setminus R_2) \cup (\text{subj}(G_2) \setminus \emptyset) = R$ . Then,  $(\text{subj}(\xi) \setminus (R_1 \cup R_2)) \cup (\text{subj}(G_1) \setminus R_2) \cup (\text{subj}(G_2) \setminus R_1) = R$ . Then, by Defn. 24,  $\text{subj}(\text{if } \xi|_0 G_1|_{R_1} G_2|_{R_2}) = R$ . Then,  $\text{subj}(G) = R$ .
    - **Case:**  $G_2 \in \underline{\mathbb{G}}$ .
      - Recall  $G_2 \in \underline{\mathbb{G}}$  and  $R_2 \neq \emptyset$ . Then, by Defn. 21,  $\text{if } \xi|_0 G_1|_{\emptyset} G_2|_{R_2} \in \underline{\mathbb{G}}$ . Then,  $\text{if } \xi|_0 G_1|_{R_1} G_2|_{R_2} \in \underline{\mathbb{G}}$ . Then,  $G \in \underline{\mathbb{G}}$ .
  - **Case:**  $R_1 \neq \emptyset = R_2$ . Similar to case  $R_1 = \emptyset \neq R_2$ .
  - **Case:**  $R_1 \neq \emptyset \neq R_2$ .
    - \* Recall  $R_1 \neq \emptyset \neq R_2$ . Then, by Defn. 38,  $\Phi(\text{if } \xi|_0 G_1|_{R_1} G_2|_{R_2}, \chi) = \text{false}$ .
    - \* Recall  $\llbracket \Phi(G, \chi) \rrbracket \neq \emptyset$ . Then,  $\llbracket \Phi(\text{if } \xi|_0 G_1|_{R_1} G_2|_{R_2}, \chi) \rrbracket \neq \emptyset$ . Then,  $\llbracket \text{false} \rrbracket \neq \emptyset$ . Then, by Lem. 15,  $\emptyset \neq \emptyset$ . Then, **false**.
- **Step:**  $[\sqrt{1}\text{-NWHILE}]$ . Similar to case  $[\sqrt{1}\text{-IF}]$ . □

*Proof (of Lem. 88).*

1. Recall  $G \downarrow$ . Then, by Defn. 28:
  - **Base:**  $[\downarrow 1\text{-SKIP}]$ , such that  $G = \text{skip}$ .
    - Recall  $\llbracket \chi \rrbracket \subseteq \llbracket \chi \rrbracket$ . Then, by Defn. 38,  $\llbracket \Phi(\text{skip}, \chi) \rrbracket = \llbracket \chi \rrbracket$ . Then,  $\llbracket \Phi(G, \chi) \rrbracket \subseteq \llbracket \chi \rrbracket$ .
  - **Step:**  $[\downarrow 1\text{-SEQ}]$ , such that  $G = G_1 ; G_2$  and  $G_1 \downarrow$  and  $G_2 \downarrow$ , for some  $G_1, G_2$ .
    - Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(G_1 ; G_2)$ . Then, by Defn. 27,  $\sqrt{R}(G_1)$  and  $\sqrt{R}(G_2)$ .
    - Recall  $\sqrt{R}(G_1)$  and  $G_1 \downarrow$ . Then, by induction,  $\llbracket \Phi(G_1, \Phi(G_2, \chi)) \rrbracket \subseteq \llbracket \Phi(G_2, \chi) \rrbracket$ .
    - Recall  $\sqrt{R}(G_2)$  and  $G_2 \downarrow$ . Then, by induction,  $\llbracket \Phi(G_2, \chi) \rrbracket \subseteq \llbracket \chi \rrbracket$ .
    - Recall  $\llbracket \Phi(G_1, \Phi(G_2, \chi)) \rrbracket \subseteq \llbracket \Phi(G_2, \chi) \rrbracket \subseteq \llbracket \chi \rrbracket$ . Then,  $\llbracket \Phi(G_1, \Phi(G_2, \chi)) \rrbracket \subseteq \llbracket \chi \rrbracket$ . Then, by Defn. 38,  $\llbracket \Phi(G_1 ; G_2, \chi) \rrbracket \subseteq \llbracket \chi \rrbracket$ . Then,  $\llbracket \Phi(G, \chi) \rrbracket \subseteq \llbracket \chi \rrbracket$ .
  - **Step:**  $[\downarrow 1\text{-PAR}]$ , such that  $G = G_1 \parallel G_2$  and  $G_1 \downarrow$  and  $G_2 \downarrow$ , for some  $G_1, G_2$ .
    - **Case:**  $G_1, G_2 \in \underline{\mathbb{G}}$  and  $G_1 \# G_2$ .
      - \* Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(G_1 \parallel G_2)$ . Then, by Defn. 27,  $\sqrt{R}(G_1)$  and  $\sqrt{R}(G_2)$ .
      - \* Recall  $\sqrt{R}(G_1)$  and  $G_1 \downarrow$ . Then, by induction,  $\llbracket \Phi(G_1, \Phi(G_2, \chi)) \rrbracket \subseteq \llbracket \Phi(G_2, \chi) \rrbracket$ .
      - \* Recall  $\sqrt{R}(G_2)$  and  $G_2 \downarrow$ . Then, by induction,  $\llbracket \Phi(G_2, \chi) \rrbracket \subseteq \llbracket \chi \rrbracket$ .
      - \* Recall  $G_1, G_2 \in \underline{\mathbb{G}}$  and  $G_1 \# G_2$ . Then, by Defn. 38,  $\Phi(G_1 \parallel G_2, \chi) = \Phi(G_1, \Phi(G_2, \chi))$ .
      - \* Recall  $\llbracket \Phi(G_1, \Phi(G_2, \chi)) \rrbracket \subseteq \llbracket \Phi(G_2, \chi) \rrbracket \subseteq \llbracket \chi \rrbracket$ . Then,  $\llbracket \Phi(G_1, \Phi(G_2, \chi)) \rrbracket \subseteq \llbracket \chi \rrbracket$ . Then,  $\llbracket \Phi(G_1 \parallel G_2, \chi) \rrbracket \subseteq \llbracket \chi \rrbracket$ . Then,  $\llbracket \Phi(G, \chi) \rrbracket \subseteq \llbracket \chi \rrbracket$ .
    - **Case:** Not  $G_1, G_2 \in \underline{\mathbb{G}}$ , or not  $G_1 \# G_2$ .
      - \* Recall not  $G_1, G_2 \in \underline{\mathbb{G}}$ , or not  $G_1 \# G_2$ . Then, by Defn. 38,  $\Phi(G_1 \parallel G_2, \chi) = \text{false}$ .
      - \* Recall  $\llbracket \text{false} \rrbracket \subseteq \llbracket \chi \rrbracket$ . Then,  $\llbracket \Phi(G_1 \parallel G_2, \chi) \rrbracket \subseteq \llbracket \chi \rrbracket$ . Then,  $\llbracket \Phi(G, \chi) \rrbracket \subseteq \llbracket \chi \rrbracket$ .
  - **Step:**  $[\downarrow 1\text{-NIF}]$ , such that  $G = \text{if } \xi|_0 G_1|_{R_1} G_2|_{R_2}$  and  $\text{subj}(\xi) = R_1 \cup R_2$ , and  $R_1 \neq \emptyset$  implies  $G_1 \downarrow$ , and  $R_2 \neq \emptyset$  implies  $G_2 \downarrow$ , for some  $R_1, R_2, G_1, G_2, \xi$ .
    - By Defn. 38:
      - **Case:**  $R_1 = \emptyset = R_2$ .
        - \* Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(\text{if } \xi|_0 G_1|_{R_1} G_2|_{R_2})$ . Then, by Defn. 27,  $R = \text{subj}(\xi)$ .

- \* Recall  $\sqrt{R}(G)$ . Then, by Lem. 47,  $R \neq \emptyset$ . Then,  $\text{subj}(\xi) \neq \emptyset$ . Then,  $R_1 \cup R_2 \neq \emptyset$ . Then,  $\emptyset \cup \emptyset \neq \emptyset$ . Then, **false**.
  - **Case:**  $\phi(\text{if } \xi|_0 G_1|_{R_1} G_2|_{R_2}, \chi) = \phi(G_2, \chi) \wedge \bigwedge \{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_2}$  and  $R_1 = \emptyset \neq R_2$ .
    - \* Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(\text{if } \xi|_0 G_1|_{R_1} G_2|_{R_2})$ . Then, by Defn. 27,  $\sqrt{R}(G_2)$ .
    - \* Recall  $R_2 \neq \emptyset$ . Then,  $G_2 \downarrow$ .
    - \* Recall  $\sqrt{R}(G_2)$  and  $G_2 \downarrow$ . Then, by induction,  $\llbracket \phi(G_2, \chi) \rrbracket \subseteq \llbracket \chi \rrbracket$ . Then,  $\llbracket \phi(G_2, \chi) \rrbracket \cap \llbracket \bigwedge \{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_2} \rrbracket \subseteq \llbracket \chi \rrbracket$ . Then, by Defn. 13,  $\llbracket \phi(G_2, \chi) \wedge \bigwedge \{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_2} \rrbracket \subseteq \llbracket \chi \rrbracket$ . Then,  $\llbracket \phi(\text{if } \xi|_0 G_1|_{R_1} G_2|_{R_2}, \chi) \rrbracket \subseteq \llbracket \chi \rrbracket$ . Then,  $\llbracket \phi(G, \chi) \rrbracket \subseteq \llbracket \chi \rrbracket$ .
  - **Case:**  $\phi(\text{if } \xi|_0 G_1|_{R_1} G_2|_{R_2}, \chi) = \phi(G_1, \chi) \wedge \bigwedge \{\xi^+ \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_1}$  and  $R_1 \neq \emptyset = R_2$ . Similar to case “ $\phi(\text{if } \xi|_0 G_1|_{R_1} G_2|_{R_2}, \chi) = \phi(G_2, \chi) \wedge \bigwedge \{\xi^- \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_2}$  and  $R_1 = \emptyset \neq R_2$ ”.
  - **Case:**  $\phi(\text{if } \xi|_0 G_1|_{R_1} G_2|_{R_2}, \chi) = \text{false}$ . By Lem. 15,  $\llbracket \text{false} \rrbracket \subseteq \llbracket \chi \rrbracket$ . Then,  $\llbracket \phi(\text{if } \xi|_0 G_1|_{R_1} G_2|_{R_2}, \chi) \rrbracket \subseteq \llbracket \chi \rrbracket$ . Then,  $\llbracket \phi(G, \chi) \rrbracket \subseteq \llbracket \chi \rrbracket$ . □
2. – Recall  $\sqrt{R}(\mathcal{G})$ . Then, by Defn. 27,  $\mathcal{G} = \{G\}$  and  $\sqrt{R}(G)$ , for some  $G$ .
    - Recall  $\mathcal{G} \downarrow$ . Then,  $\{G\} \downarrow$ . Then, by Defn. 29,  $G \downarrow$ .
    - Recall  $\sqrt{R}(G)$  and  $G \downarrow$ . Then, by 1,  $\llbracket \phi(G, \chi) \rrbracket \subseteq \llbracket \chi \rrbracket$ . Then, by Defn. 38,  $\llbracket \phi(\{G\}, \chi) \rrbracket \subseteq \llbracket \chi \rrbracket$ . Then,  $\llbracket \phi(\mathcal{G}, \chi) \rrbracket \subseteq \llbracket \chi \rrbracket$ . □
  3. – Recall  $(\mathcal{G}, \mathcal{S}) \downarrow$ . Then, by Defn. 30,  $\mathcal{G} \downarrow$ .
    - Recall  $\sqrt{R}(\mathcal{G})$  and  $\mathcal{G} \downarrow$ . Then, by 2,  $\llbracket \phi(\mathcal{G}, \chi) \rrbracket \subseteq \llbracket \chi \rrbracket$ .
    - Recall  $\mathcal{S} \in \llbracket \phi(\mathcal{G}, \chi) \rrbracket \subseteq \llbracket \chi \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \chi \rrbracket$ . □

*Proof (of Lem. 89).*

1. Recall  $G \xrightarrow{\xi, \gamma} G'$ . Then, by Defn. 31:
  - **Base:**  $[\rightarrow 1\text{-ACT}]$ , such that  $G' = \text{skip}$ . Recall  $\sqrt{R}(G)$ . Then, by Lem. 47,  $R \neq \emptyset$ . Then, by  $[\sqrt{1}\text{-SKIP}]$ ,  $\sqrt{R}(\text{skip})$ . Then,  $\sqrt{R}(G')$ .
  - **Base:**  $[\rightarrow 1\text{-IF1}]$ , such that  $G = \hat{R}.\text{if } \hat{\xi} G_1 G_2$  and  $G' = G_1$ , for some  $\hat{R}, G_1, G_2, \hat{\xi}$ . Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(\hat{R}.\text{if } \hat{\xi} G_1 G_2)$ . Then, by Defn. 27,  $\sqrt{R}(G_1)$ . Then,  $\sqrt{R}(G')$ .
  - **Base:**  $[\rightarrow 1\text{-IF2}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
  - **Base:**  $[\rightarrow 1\text{-WHILE1}]$ , such that  $G = \hat{R}.\text{while } \hat{\xi} \{\psi\} \hat{G}$  and  $G' = \hat{G}$ ;  $\hat{R}.\text{while } \hat{\xi} \{\psi\} \hat{G}$ , for some  $\hat{R}, \hat{G}, \psi, \hat{\xi}$ .
    - Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(\hat{R}.\text{while } \hat{\xi} \{\psi\} \hat{G})$ .
    - Recall  $\sqrt{R}(\hat{R}.\text{while } \hat{\xi} \{\psi\} \hat{G})$ . Then, by Defn. 27,  $\sqrt{R}(\hat{G})$ .
    - Recall  $\sqrt{R}(\hat{G})$  and  $\sqrt{R}(\hat{R}.\text{while } \hat{\xi} \{\psi\} \hat{G})$ . Then, by  $[\sqrt{1}\text{-SEQ}]$ ,  $\sqrt{R}(\hat{G}; \hat{R}.\text{while } \hat{\xi} \{\psi\} \hat{G})$ . Then,  $\sqrt{R}(G')$ .
  - **Base:**  $[\rightarrow 1\text{-WHILE2}]$ . Similar to case  $[\rightarrow 1\text{-ACT}]$ .
  - **Base:**  $[\rightarrow 1\text{-NIF1}]$ , such that  $0 > 0$ . Then, **false**.
  - **Base:**  $[\rightarrow 1\text{-NIF2}]$ , such that  $G = \text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2}$  and  $G' = \text{if } \hat{\xi}|_0 G_1|_{R_1 \cup \{r\}} G_2|_{R_2}$  and  $\xi = \hat{\xi}^+ \uparrow r$  and  $r \in \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$ , for some  $R_1, R_2, G_1, G_2, r, \hat{\xi}$ . Recall:
    - **Case:**  $R_1 = \emptyset = R_2$ .
      - \* Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(\text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2})$ . Then, by Defn. 27,  $\sqrt{R}(G_1)$  and  $\sqrt{R}(G_2)$  and  $R = \text{subj}(\hat{\xi})$  and  $R_1, R_2 \subseteq \text{subj}(\hat{\xi})$ .
      - \* Recall  $r \in \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$ . Then,  $r \in \text{subj}(\hat{\xi})$ . Then,  $\{r\} \subseteq \text{subj}(\hat{\xi})$ .
      - \* Recall  $R_1 \subseteq \text{subj}(\hat{\xi})$  and  $\{r\} \subseteq \text{subj}(\hat{\xi})$ . Then,  $R_1 \cup \{r\} \subseteq \text{subj}(\hat{\xi})$ .
      - \* Recall  $R_2 = \emptyset$ . Then,  $R_1 \cup \{r\} \neq \emptyset$  implies  $R_2 = \emptyset$ .
      - \* Recall  $R_2 = \emptyset$ . Then,  $R_2 \neq \emptyset$  implies  $R_1 \cup \{r\} = \emptyset$ .
      - \* Recall  $\sqrt{R}(G_1)$  and  $\sqrt{R}(G_2)$  and  $R = \text{subj}(\hat{\xi})$  and  $R_1 \cup \{r\} \subseteq \text{subj}(\hat{\xi})$  and  $R_2 \subseteq \text{subj}(\hat{\xi})$ , and  $R_1 \cup \{r\} \neq \emptyset$  implies  $R_2 = \emptyset$ , and  $R_2 \neq \emptyset$  implies  $R_1 \cup \{r\} = \emptyset$ . Then, by  $[\sqrt{1}\text{-NIF}]$ ,  $\sqrt{R}(\text{if } \hat{\xi}|_0 G_1|_{R_1 \cup \{r\}} G_2|_{R_2})$ . Then,  $\sqrt{R}(G')$ .
    - **Case:**  $R_1 = \emptyset \neq R_2$ .

- \* Recall  $R_1 = \emptyset \neq R_2$ . Then, by Defn. 38,
 
$$\phi(\text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2}, \chi) = \phi(G_2, \chi) \wedge \bigwedge \{\hat{\xi}^- \uparrow \hat{r}\}_{\hat{r} \in \text{subj}(\hat{\xi}) \setminus R_2}.$$
- \* Recall  $r \in \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$ . Then,  $r \in \text{subj}(\hat{\xi}) \setminus R_2$ . Then,  $r \in \text{subj}(\hat{\xi}) \setminus R_2 \subseteq \text{subj}(\hat{\xi})$ . Then, by Lem. 29,
 
$$\llbracket \bigwedge \{\hat{\xi}^- \uparrow \hat{r}\}_{\hat{r} \in \text{subj}(\hat{\xi}) \setminus R_2} \rrbracket \cap \llbracket \hat{\xi}^+ \uparrow r \rrbracket = \llbracket \text{false} \rrbracket.$$
- \* Recall  $\llbracket \phi(G, \chi) \wedge \xi \rrbracket \neq \emptyset$ . Then,  $\llbracket \phi(\text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2}, \chi) \wedge (\hat{\xi}^+ \uparrow \hat{r}) \rrbracket \neq \emptyset$ . Then,  $\llbracket \phi(G_2, \chi) \wedge \bigwedge \{\hat{\xi}^- \uparrow r\}_{r \in \text{subj}(\hat{\xi}) \setminus R_2} \wedge (\hat{\xi}^+ \uparrow \hat{r}) \rrbracket \neq \emptyset$ . Then, by Defn. 13,  $\llbracket \phi(G_2, \chi) \rrbracket \cap \llbracket \bigwedge \{\hat{\xi}^- \uparrow r\}_{r \in \text{subj}(\hat{\xi}) \setminus R_2} \rrbracket \cap \llbracket \hat{\xi}^+ \uparrow \hat{r} \rrbracket \neq \emptyset$ . Then,  $\llbracket \phi(G_2, \chi) \rrbracket \cap \llbracket \text{false} \rrbracket \neq \emptyset$ . Then, by Lem. 15,  $\llbracket \phi(G_2, \chi) \rrbracket \cap \emptyset \neq \emptyset$ . Then,  $\emptyset \neq \emptyset$ . Then, **false**.
- **Case:**  $R_1 \neq \emptyset = R_2$ . Similar to case  $R_1 = \emptyset = R_2$ .
- **Case:**  $R_1 \neq \emptyset \neq R_2$ .
  - \* Recall  $R_1 \neq \emptyset \neq R_2$ . Then, by Defn. 38,  $\phi(\text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2}, \chi) = \text{false}$ .
  - \* Recall  $\llbracket \phi(G, \chi) \wedge \xi \rrbracket \neq \emptyset$ . Then,  $\llbracket \phi(\text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2}, \chi) \wedge \xi \rrbracket \neq \emptyset$ . Then,  $\llbracket \text{false} \wedge \xi \rrbracket \neq \emptyset$ . Then, by Defn. 13,  $\llbracket \text{false} \rrbracket \cap \llbracket \xi \rrbracket \neq \emptyset$ . Then, by Lem. 15,  $\emptyset \cap \llbracket \xi \rrbracket \neq \emptyset$ . Then,  $\emptyset \neq \emptyset$ . Then, **false**.
- **Base:**  $[\rightarrow 1\text{-NIF3}]$ . Similar to case  $[\rightarrow 1\text{-NIF2}]$ .
- **Step:**  $[\rightarrow 1\text{-SEQ1}]$ , such that  $G = G_1 ; G_2$  and  $G' = G'_1 ; G_2$  and  $G_1 \xrightarrow{\xi, \gamma} G'_1$ , for some  $G_1, G'_1, G_2$ .
  - Recall  $\checkmark_R(G)$ . Then,  $\checkmark_R(G_1 ; G_2)$ . Then, by Defn. 27,  $\checkmark_R(G_1)$  and  $\checkmark_R(G_2)$ .
  - Recall  $\llbracket \phi(G, \chi) \wedge \xi \rrbracket \neq \emptyset$ . Then,  $\llbracket \phi(G_1 ; G_2, \chi) \wedge \xi \rrbracket \neq \emptyset$ . Then, by Defn. 38,  $\llbracket \phi(G_1, \phi(G_2, \chi)) \wedge \xi \rrbracket \neq \emptyset$ .
  - Recall  $\checkmark_R(G_1)$  and  $\llbracket \phi(G_1, \phi(G_2, \chi)) \wedge \xi \rrbracket \neq \emptyset$  and  $G_1 \xrightarrow{\xi, \gamma} G'_1$ . Then, by induction,  $\checkmark_R(G'_1)$ .
  - Recall  $\checkmark_R(G'_1)$  and  $\checkmark_R(G_2)$ . Then, by  $[\checkmark 1\text{-SEQ}]$ ,  $\checkmark_R(G'_1 ; G_2)$ . Then,  $\checkmark_R(G')$ .
- **Step:**  $[\rightarrow 1\text{-SEQ2}]$ , such that  $G = G_1 ; G_2$  and  $G' = G_1 ; G'_2$  and  $\text{subj}(G_1) \cap (\text{subj}(\xi) \cup \text{subj}(\gamma)) = \emptyset$  and  $G_2 \xrightarrow{\xi, \gamma} G'_2$ , for some  $G_1, G_2, G'_2$ .
  - Recall  $\checkmark_R(G)$ . Then,  $\checkmark_R(G_1 ; G_2)$ . Then, by Defn. 27,  $\checkmark_R(G_1)$  and  $\checkmark_R(G_2)$ .
  - Recall  $\llbracket \phi(G, \chi) \wedge \xi \rrbracket \neq \emptyset$ . Then,  $\llbracket \phi(G_1 ; G_2, \chi) \wedge \xi \rrbracket \neq \emptyset$ . Then, by Defn. 38,  $\llbracket \phi(G_1, \phi(G_2, \chi)) \wedge \xi \rrbracket \neq \emptyset$ .
  - Recall  $\checkmark_R(G_1)$  and  $\llbracket \phi(G_1, \phi(G_2, \chi)) \wedge \xi \rrbracket \neq \emptyset$ . Then, by Lem. 87:
    - \* **Case:**  $\text{subj}(G_1) = R$ .
      - Recall  $G \xrightarrow{\xi, \gamma} G'$  and  $\checkmark_R(G)$ . Then, by Lem. 58,  $\text{subj}(\gamma) \neq \emptyset$ . Then,  $\text{subj}(\xi) \cup \text{subj}(\gamma) \neq \emptyset$ .
      - Recall  $G \xrightarrow{\xi, \gamma} G'$ . Then, by Lem. 53,  $\text{subj}(\xi) \cup \text{subj}(\gamma) \cup \text{subj}(G') \subseteq \text{subj}(G)$ . Then,  $\text{subj}(\xi) \cup \text{subj}(\gamma) \subseteq \text{subj}(G)$ .
      - Recall  $\checkmark_R(G)$ . Then, by Lem. 47,  $\text{subj}(G) \subseteq R$ .
      - Recall  $\emptyset \neq \text{subj}(\xi) \cup \text{subj}(\gamma) \subseteq \text{subj}(G) \subseteq R$ . Then,  $\emptyset \neq \text{subj}(\xi) \cup \text{subj}(\gamma) \subseteq R$ . Then,  $\emptyset \neq \text{subj}(\xi) \cup \text{subj}(\gamma) \subseteq \text{subj}(G_1)$ . Then,  $\text{subj}(G_1) \cap (\text{subj}(\xi) \cup \text{subj}(\gamma)) \neq \emptyset$ . Then, **false**.
    - \* **Case:**  $G_1 \in \mathbb{G}$ .
      - Recall  $\text{subj}(G_1) \cap (\text{subj}(\xi) \cup \text{subj}(\gamma)) = \emptyset$ . Then,  $(\text{subj}(G_1) \cap \text{subj}(\xi)) \cup (\text{subj}(G_1) \cap \text{subj}(\gamma)) = \emptyset$ . Then,  $\text{subj}(G_1) \cap \text{subj}(\xi) = \emptyset$ . Then, by Lem. 41,  $G_1 \# \xi$ .
      - Recall  $G_1 \in \mathbb{G}$  and  $G_1 \# \xi$ . Then,  $\llbracket \phi(G_1, \phi(G_2, \chi)) \wedge \xi \rrbracket \subseteq \llbracket \phi(G_1, \xi) \rrbracket$ . Then, by Defn. 13,  $\llbracket \phi(G_1, \phi(G_2, \chi)) \rrbracket \cap \llbracket \xi \rrbracket \subseteq \llbracket \phi(G_1, \xi) \rrbracket$ . Then,  $\llbracket \phi(G_1, \phi(G_2, \chi)) \rrbracket \cap \llbracket \xi \rrbracket \subseteq \llbracket \phi(G_1, \phi(G_2, \chi)) \rrbracket \cap \llbracket \phi(G_1, \xi) \rrbracket$ . Then, by Defn. 13,  $\llbracket \phi(G_1, \phi(G_2, \chi)) \wedge \xi \rrbracket \subseteq \llbracket \phi(G_1, \phi(G_2, \chi)) \wedge \phi(G_1, \xi) \rrbracket$ . Then, by Lem. 82,  $\llbracket \phi(G_1, \phi(G_2, \chi)) \wedge \xi \rrbracket \subseteq \llbracket \phi(G_1, \phi(G_2, \chi) \wedge \xi) \rrbracket$ .
      - Recall  $\emptyset \neq \llbracket \phi(G_1, \phi(G_2, \chi)) \wedge \xi \rrbracket \subseteq \llbracket \phi(G_1, \phi(G_2, \chi) \wedge \xi) \rrbracket$ . Then,  $\emptyset \neq \llbracket \phi(G_1, \phi(G_2, \chi) \wedge \xi) \rrbracket$ .
      - Recall  $G_1 \in \mathbb{G}$  and  $\emptyset \neq \llbracket \phi(G_1, \phi(G_2, \chi) \wedge \xi) \rrbracket$ . Then, by Lem. 84,  $\llbracket \phi(G_2, \chi) \wedge \xi \rrbracket \neq \emptyset$ .

- Recall  $\sqrt{R}(G_2)$  and  $\llbracket \phi(G_2, \chi) \wedge \xi \rrbracket \neq \emptyset$  and  $G_2 \xrightarrow{\xi, \gamma} G'_2$ . Then, by induction,  $\sqrt{R}(G'_2)$ .
- Recall  $\sqrt{R}(G_1)$  and  $\sqrt{R}(G'_2)$ . Then, by  $[\sqrt{1}\text{-SEQ}]$ ,  $\sqrt{R}(G_1 ; G'_2)$ . Then,  $\sqrt{R}(G')$ .
- **Step:**  $[\rightarrow 1\text{-PAR1}]$ , such that  $G = G_1 \parallel G_2$  and  $G' = G'_1 \parallel G_2$  and  $G_1 \xrightarrow{\xi, \gamma} G'_1$ , for some  $G_1, G'_1, G_2$ .  
By Defn. 38:
  - **Case:**  $\phi(G_1 \parallel G_2, \chi) = \phi(G_1, \phi(G_2, \chi))$ .
    - \* Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(G_1 \parallel G_2)$ . Then, by Defn. 27,  $\sqrt{R}(G_1)$  and  $\sqrt{R}(G_2)$  and  $\text{chan}(G_1) \cap \text{chan}(G_2) = \emptyset$ .
    - \* Recall  $\llbracket \phi(G, \chi) \wedge \xi \rrbracket \neq \emptyset$ . Then,  $\llbracket \phi(G_1 \parallel G_2, \chi) \wedge \xi \rrbracket \neq \emptyset$ . Then,  $\llbracket \phi(G_1, \phi(G_2, \chi)) \wedge \xi \rrbracket \neq \emptyset$ .
    - \* Recall  $\sqrt{R}(G_1)$  and  $\llbracket \phi(G_1, \phi(G_2, \chi)) \wedge \xi \rrbracket \neq \emptyset$  and  $G_1 \xrightarrow{\xi, \gamma} G'_1$ . Then, by induction,  $\sqrt{R}(G'_1)$ .
    - \* Recall  $G_1 \xrightarrow{\xi, \gamma} G'_1$ . Then, by Lem. 54,  $\text{chan}(G'_1) \subseteq \text{chan}(G_1)$ .
    - \* Recall  $\text{chan}(G_1) \cap \text{chan}(G_2) = \emptyset$  and  $\text{chan}(G'_1) \subseteq \text{chan}(G_1)$ . Then,  $\text{chan}(G'_1) \cap \text{chan}(G_2) = \emptyset$ .
    - \* Recall  $\sqrt{R}(G'_1)$  and  $\sqrt{R}(G_2)$  and  $\text{chan}(G'_1) \cap \text{chan}(G_2) = \emptyset$ . Then, by  $[\sqrt{1}\text{-PAR}]$ ,  $\sqrt{R}(G'_1 \parallel G_2)$ . Then,  $\sqrt{R}(G')$ .
  - **Case:**  $\phi(G_1 \parallel G_2, \chi) = \text{false}$ .  
Recall  $\emptyset \cap \llbracket \xi \rrbracket = \emptyset$ . Then, by Lem. 15,  $\llbracket \text{false} \rrbracket \cap \llbracket \xi \rrbracket = \emptyset$ . Then, by Defn. 13,  $\llbracket \text{false} \wedge \xi \rrbracket = \emptyset$ . Then,  $\llbracket \phi(G_1 \parallel G_2, \chi) \wedge \xi \rrbracket = \emptyset$ . Then,  $\llbracket \phi(G, \chi) \wedge \xi \rrbracket = \emptyset$ . Then, **false**.
- **Step:**  $[\rightarrow 1\text{-PAR2}]$ , such that  $G = G_1 \parallel G_2$  and  $G' = G_1 \parallel G'_2$  and  $G_2 \xrightarrow{\xi, \gamma} G'_2$ , for some  $G_1, G_2, G'_2$ .  
By Defn. 38:
  - **Case:**  $\phi(G_1 \parallel G_2, \chi) = \phi(G_1, \phi(G_2, \chi))$  and  $G_1, G_2 \in \mathbb{G}$  and  $G_1 \# G_2$ .
    - \* Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(G_1 \parallel G_2)$ . Then, by Defn. 27,  $\sqrt{R}(G_1)$  and  $\sqrt{R}(G_2)$  and  $\text{chan}(G_1) \cap \text{chan}(G_2) = \emptyset$ .
    - \* Recall  $G_1 \# G_2$  and  $G_2 \xrightarrow{\xi, \gamma} G'_2$ . Then, by Lem. 52,  $G_1 \# \xi$ .
    - \* Recall  $G_1 \in \mathbb{G}$  and  $G_1 \# \xi$ . Then,  $\llbracket \phi(G_1, \phi(G_2, \chi)) \wedge \xi \rrbracket \subseteq \llbracket \phi(G_1, \xi) \rrbracket$ . Then, by Defn. 13,  $\llbracket \phi(G_1, \phi(G_2, \chi)) \rrbracket \cap \llbracket \xi \rrbracket \subseteq \llbracket \phi(G_1, \xi) \rrbracket$ . Then,  $\llbracket \phi(G_1, \phi(G_2, \chi)) \wedge \xi \rrbracket \subseteq \llbracket \phi(G_1, \phi(G_2, \chi)) \rrbracket \cap \llbracket \phi(G_1, \xi) \rrbracket$ . Then, by Defn. 13,  $\llbracket \phi(G_1, \phi(G_2, \chi)) \wedge \xi \rrbracket \subseteq \llbracket \phi(G_1, \phi(G_2, \chi)) \wedge \phi(G_1, \xi) \rrbracket$ . Then, by Lem. 82,  $\llbracket \phi(G_1, \phi(G_2, \chi)) \wedge \xi \rrbracket \subseteq \llbracket \phi(G_1, \phi(G_2, \chi) \wedge \xi) \rrbracket$ .
    - \* Recall  $\emptyset \neq \llbracket \phi(G_1, \phi(G_2, \chi)) \wedge \xi \rrbracket \subseteq \llbracket \phi(G_1, \phi(G_2, \chi) \wedge \xi) \rrbracket$ . Then,  $\emptyset \neq \llbracket \phi(G_1, \phi(G_2, \chi) \wedge \xi) \rrbracket$ .
    - \* Recall  $G_1 \in \mathbb{G}$  and  $\emptyset \neq \llbracket \phi(G_1, \phi(G_2, \chi) \wedge \xi) \rrbracket$ . Then, by Lem. 84,  $\llbracket \phi(G_2, \chi) \wedge \xi \rrbracket \neq \emptyset$ .
    - \* Recall  $\sqrt{R}(G_2)$  and  $\llbracket \phi(G_2, \chi) \wedge \xi \rrbracket \neq \emptyset$  and  $G_2 \xrightarrow{\xi, \gamma} G'_2$ . Then, by induction,  $\sqrt{R}(G'_2)$ .
    - \* Recall  $G_2 \xrightarrow{\xi, \gamma} G'_2$ . Then, by Lem. 54,  $\text{chan}(G'_2) \subseteq \text{chan}(G_2)$ .
    - \* Recall  $\text{chan}(G_1) \cap \text{chan}(G_2) = \emptyset$  and  $\text{chan}(G'_2) \subseteq \text{chan}(G_2)$ . Then,  $\text{chan}(G_1) \cap \text{chan}(G'_2) = \emptyset$ .
    - \* Recall  $\sqrt{R}(G_1)$  and  $\sqrt{R}(G'_2)$  and  $\text{chan}(G_1) \cap \text{chan}(G'_2) = \emptyset$ . Then, by  $[\sqrt{1}\text{-PAR}]$ ,  $\sqrt{R}(G_1 \parallel G'_2)$ . Then,  $\sqrt{R}(G')$ .
  - **Case:**  $\phi(G_1 \parallel G_2, \chi) = \text{false}$ .  
Recall  $\emptyset \cap \llbracket \xi \rrbracket = \emptyset$ . Then, by Lem. 15,  $\llbracket \text{false} \rrbracket \cap \llbracket \xi \rrbracket = \emptyset$ . Then, by Defn. 13,  $\llbracket \text{false} \wedge \xi \rrbracket = \emptyset$ . Then,  $\llbracket \phi(G_1 \parallel G_2, \chi) \wedge \xi \rrbracket = \emptyset$ . Then,  $\llbracket \phi(G, \chi) \wedge \xi \rrbracket = \emptyset$ . Then, **false**.
- **Step:**  $[\rightarrow 1\text{-NIF4}]$ , such that  $G = \text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2}$  and  $G' = \text{if } \hat{\xi}|_0 G'_1|_{R_1} G_2|_{R_2}$  and  $G_1 \xrightarrow{\xi, \gamma} G'_1$ , for some  $R_1, R_2, G_1, G'_1, G_2, \hat{\xi}$ .
  - Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(\text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2})$ . Then, by Defn. 27,  $\sqrt{R}(G_1)$  and  $\sqrt{R}(G_2)$  and  $R = \text{subj}(\hat{\xi})$  and  $R_1, R_2 \subseteq \text{subj}(\hat{\xi})$ , and  $R_1 \neq \emptyset$  implies  $R_2 = \emptyset$ , and  $R_2 \neq \emptyset$  implies  $R_1 = \emptyset$ .
  - Recall  $\sqrt{R}(G_1)$  and  $G_1 \xrightarrow{\xi, \gamma} G'_1$ . Then, by induction,  $\sqrt{R}(G'_1)$ .

- Recall  $\sqrt{R}(G_1)$  and  $\sqrt{R}(G_2)$  and  $R = \text{subj}(\xi)$  and  $R_1, R_2 \subseteq \text{subj}(\hat{\xi})$ , and  $R_1 \neq \emptyset$  implies  $R_2 = \emptyset$ , and  $R_2 \neq \emptyset$  implies  $R_1 = \emptyset$ . Then, by  $[\sqrt{1}\text{-NIF}]$ ,  $\sqrt{R}(\text{if } G'_1|_{R_1} G_2|_{R_2})$ . Then,  $\sqrt{R}(G')$ .
- **Step:**  $[\rightarrow 1\text{-NIF5}]$ . Similar to case  $[\rightarrow 1\text{-NIF4}]$ .
- **Step:**  $[\rightarrow 1\text{-NWHILE}]$ , such that  $G = \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset$  and  $\text{if } \hat{\xi}|_0 (\hat{G}; \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) |_\emptyset \text{skip}|_\emptyset \xrightarrow{\xi, \gamma} G'$ , for some  $\hat{G}, \psi, \hat{\xi}$ .
  - Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(\text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset)$ . Then, by Lem. 46,  $\sqrt{R}(\text{if } \hat{\xi}|_0 (\hat{G}; \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) |_\emptyset \text{skip}|_\emptyset)$ .
  - By Lem. 80,  $\llbracket \Phi(\text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset, \chi) \rrbracket \subseteq \llbracket \Phi(\text{if } \hat{\xi}|_0 (\hat{G}; \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) |_\emptyset \text{skip}|_\emptyset, \chi) \rrbracket$ . Then,  $\llbracket \Phi(G, \chi) \rrbracket \subseteq \llbracket \Phi(\text{if } \hat{\xi}|_0 (\hat{G}; \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) |_\emptyset \text{skip}|_\emptyset, \chi) \rrbracket$ . Then,  $\llbracket \Phi(G, \chi) \rrbracket \cap \llbracket \xi \rrbracket \subseteq \llbracket \Phi(\text{if } \hat{\xi}|_0 (\hat{G}; \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) |_\emptyset \text{skip}|_\emptyset, \chi) \rrbracket \cap \llbracket \xi \rrbracket$ . Then, by Defn. 13,  $\llbracket \Phi(G, \chi) \wedge \xi \rrbracket \subseteq \llbracket \Phi(\text{if } \hat{\xi}|_0 (\hat{G}; \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) |_\emptyset \text{skip}|_\emptyset, \chi) \wedge \xi \rrbracket$ .
  - Recall  $\emptyset \neq \llbracket \Phi(G, \chi) \wedge \xi \rrbracket \subseteq \llbracket \Phi(\text{if } \hat{\xi}|_0 (\hat{G}; \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) |_\emptyset \text{skip}|_\emptyset, \chi) \wedge \xi \rrbracket$ . Then,  $\emptyset \neq \llbracket \Phi(\text{if } \hat{\xi}|_0 (\hat{G}; \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) |_\emptyset \text{skip}|_\emptyset, \chi) \wedge \xi \rrbracket$ .
  - Recall  $\sqrt{R}(\text{if } \hat{\xi}|_0 (\hat{G}; \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) |_\emptyset \text{skip}|_\emptyset)$  and  $\llbracket \Phi(\text{if } \hat{\xi}|_0 (\hat{G}; \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) |_\emptyset \text{skip}|_\emptyset, \chi) \wedge \xi \rrbracket \neq \emptyset$  and  $\text{if } \hat{\xi}|_0 (\hat{G}; \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) |_\emptyset \text{skip}|_\emptyset \xrightarrow{\xi, \gamma} G'$ . Then, by induction,  $\sqrt{R}(G')$ .  $\square$
- 2. – Recall  $\sqrt{R}(G)$ . Then, by Defn. 27,  $G = \{G\}$  and  $\sqrt{R}(G)$ , for some  $G$ .
  - Recall  $\llbracket \Phi(G, \chi) \wedge \xi \rrbracket \neq \emptyset$ . Then,  $\llbracket \Phi(\{G\}, \chi) \wedge \xi \rrbracket \neq \emptyset$ . Then, by Defn. 38,  $\llbracket \Phi(G, \chi) \wedge \xi \rrbracket \neq \emptyset$ .
  - Recall  $G \xrightarrow{\xi, \gamma} G'$ . Then,  $\{G\} \xrightarrow{\xi, \gamma} G'$ . Then, by Defn. 32,  $G' = \{G'\}$  and  $G \xrightarrow{\xi, \gamma} G'$ , for some  $G'$ .
  - Recall  $\sqrt{R}(G)$  and  $\llbracket \Phi(G, \chi) \wedge \xi \rrbracket \neq \emptyset$  and  $G \xrightarrow{\xi, \gamma} G'$ . Then, by 1,  $\sqrt{R}(G')$ . Then, by  $[\sqrt{2}]$ ,  $\sqrt{R}(\{G'\})$ . Then,  $\sqrt{R}(G')$ .  $\square$
- 3. – Recall  $S \in \llbracket \Phi(G, \chi) \wedge \xi \rrbracket$ . Then,  $\llbracket \Phi(G, \chi) \wedge \xi \rrbracket \neq \emptyset$ .
  - Recall  $(G, S) \xrightarrow{\gamma} (G', S')$ . Then, by Defn. 33,  $G \xrightarrow{\xi, \gamma} G'$ , for some  $\gamma$ .
  - Recall  $\sqrt{R}(G)$  and  $\llbracket \Phi(G, \chi) \wedge \xi \rrbracket \neq \emptyset$  and  $G \xrightarrow{\xi, \gamma} G'$ . Then, by 2,  $\sqrt{R}(G')$ .  $\square$

*Proof (of Lem. 90).* Recall  $G \xrightarrow{\xi, \gamma} G'$ . Then, by Defn. 31:

- **Base:**  $[\rightarrow 1\text{-ACT}]$ , such that  $G' = \text{skip}$ .  
By Defn. 21,  $\text{skip} \in \underline{\mathbb{G}}$ . Then,  $G' \in \underline{\mathbb{G}}$ .
- **Base:**  $[\rightarrow 1\text{-IF1}]$ , such that  $G = \hat{R}.\text{if } \hat{\xi} G_1 G_2$  and  $G' = G_1$ , for some  $\hat{R}, G_1, G_2, \hat{\xi}$ .  
Recall  $G \in \underline{\mathbb{G}}$ . Then,  $\hat{R}.\text{if } \hat{\xi} G_1 G_2 \in \underline{\mathbb{G}}$ . Then, by Defn. 21,  $G_1 \in \underline{\mathbb{G}}$ . Then,  $G' \in \underline{\mathbb{G}}$ .
- **Base:**  $[\rightarrow 1\text{-IF2}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .
- **Base:**  $[\rightarrow 1\text{-WHILE1}]$ , such that  $G = \hat{R}.\text{while } \hat{\xi} \{\psi\} \hat{G}$ , for some  $\hat{R}, \hat{G}, \psi, \hat{\xi}$ .  
By Defn. 21, not  $\hat{R}.\text{while } \hat{\xi} \{\psi\} \hat{G} \in \underline{\mathbb{G}}$ . Then, not  $G \in \underline{\mathbb{G}}$ . Then,  $\text{false}$ .
- **Base:**  $[\rightarrow 1\text{-WHILE2}]$ . Similar to case  $[\rightarrow 1\text{-WHILE1}]$ .
- **Base:**  $[\rightarrow 1\text{-NIF1}]$ , such that  $0 > 0$ . Then,  $\text{false}$ .
- **Base:**  $[\rightarrow 1\text{-NIF2}]$ , such that  $G = \text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2}$  and  $G' = \text{if } \hat{\xi}|_0 G_1|_{R_1 \cup \{\hat{r}\}} G_2|_{R_2}$  and  $\xi = \hat{\xi}^+ \uparrow \hat{r}$  and  $\hat{r} \in \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$ , for some  $R_1, R_2, G_1, G_2, \hat{\xi}, \hat{r}$ .  
Recall  $G \in \underline{\mathbb{G}}$ . Then,  $\text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2} \in \underline{\mathbb{G}}$ . Then, by Defn. 21:
  - **Case:**  $G_1, G_2 \in \underline{\mathbb{G}}$ . Then, by Defn. 21,  $\text{if } \hat{\xi}|_0 G_1|_{R_1 \cup \{\hat{r}\}} G_2|_{R_2} \in \underline{\mathbb{G}}$ . Then,  $G' \in \underline{\mathbb{G}}$ .
  - **Case:**  $G_2 \in \underline{\mathbb{G}}$  and  $R_1 = \emptyset \neq R_2$ .
    - \* Recall  $R_1 = \emptyset \neq R_2$ . Then, by Defn. 38,  $\Phi(\text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2}, \chi) = \Phi(G_2, \chi) \wedge \bigwedge \{\hat{\xi}^- \uparrow r\}_{r \in \text{subj}(\hat{\xi}) \setminus R_2}$ .
    - \* Recall  $\hat{r} \in \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$ . Then,  $\hat{r} \in \text{subj}(\hat{\xi}) \setminus R_2$ . Then,  $\hat{r} \in \text{subj}(\hat{\xi}) \setminus R_2 \subseteq \text{subj}(\hat{\xi})$ . Then, by Lem. 29,  $\llbracket \bigwedge \{\hat{\xi}^- \uparrow r\}_{r \in \text{subj}(\hat{\xi}) \setminus R_2} \rrbracket \cap \llbracket \hat{\xi}^+ \uparrow \hat{r} \rrbracket = \llbracket \text{false} \rrbracket$ .
    - \* Recall  $\llbracket \Phi(G, \chi) \wedge \xi \rrbracket = \llbracket \Phi(G, \chi) \wedge \xi \rrbracket$ . Then,  $\llbracket \Phi(G, \chi) \wedge \xi \rrbracket = \llbracket \Phi(\text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2}, \chi) \wedge (\hat{\xi}^+ \uparrow \hat{r}) \rrbracket$ . Then,  $\llbracket \Phi(G, \chi) \wedge \xi \rrbracket = \llbracket \Phi(G_2, \chi) \wedge \bigwedge \{\hat{\xi}^- \uparrow r\}_{r \in \text{subj}(\hat{\xi}) \setminus R_2} \wedge (\hat{\xi}^+ \uparrow \hat{r}) \rrbracket$ . Then, by Defn. 13,  $\llbracket \Phi(G, \chi) \wedge \xi \rrbracket = \llbracket \Phi(G_2, \chi) \rrbracket \cap \llbracket \bigwedge \{\hat{\xi}^- \uparrow r\}_{r \in \text{subj}(\hat{\xi}) \setminus R_2} \rrbracket \cap \llbracket \hat{\xi}^+ \uparrow \hat{r} \rrbracket$ . Then,  $\llbracket \Phi(G, \chi) \wedge \xi \rrbracket = \llbracket \Phi(G_2, \chi) \rrbracket \cap \llbracket \text{false} \rrbracket$ . Then, by Lem. 15,  $\llbracket \Phi(G, \chi) \wedge \xi \rrbracket = \llbracket \Phi(G_2, \chi) \rrbracket \cap \emptyset$ . Then,  $\llbracket \Phi(G, \chi) \wedge \xi \rrbracket = \emptyset$ .

- **Case:**  $G_1 \in \underline{\mathbb{G}}$  and  $R_2 = \emptyset$ .  
Recall  $G_1 \in \underline{\mathbb{G}}$ . Then, by Defn. 21, **if**  $\hat{\xi}|_0 G_1|_{R_1 \cup \{\hat{r}\}} G_2|_{\emptyset} \in \underline{\mathbb{G}}$ . Then,  
**if**  $\hat{\xi}|_0 G_1|_{R_1 \cup \{\hat{r}\}} G_2|_{R_2} \in \underline{\mathbb{G}}$ . Then,  $G' \in \underline{\mathbb{G}}$ .
- **Base:**  $[\rightarrow 1\text{-NIF3}]$ . Similar to case  $[\rightarrow 1\text{-NIF2}]$ .
- **Step:**  $[\rightarrow 1\text{-SEQ1}]$ , such that  $G = G_1 ; G_2$  and  $G' = G'_1 ; G_2$  and  $G_1 \xrightarrow{\xi, \gamma} G'_1$ , for some  $G_1, G'_1, G_2$ .
  - Recall  $G \in \underline{\mathbb{G}}$ . Then,  $G_1 ; G_2 \in \underline{\mathbb{G}}$ . Then, by Defn. 21,  $G_1, G_2 \in \underline{\mathbb{G}}$ .
  - Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(G_1 ; G_2)$ . Then, by Defn. 27,  $\sqrt{R}(G_1)$ .
  - Recall  $G_1 \in \underline{\mathbb{G}}$  and  $\sqrt{R}(G_1)$  and  $G_1 \xrightarrow{\xi, \gamma} G'_1$ . Then, by induction:
    - \* **Case:**  $G'_1 \in \underline{\mathbb{G}}$ .  
Recall  $G'_1, G_2 \in \underline{\mathbb{G}}$ . Then, by Defn. 21,  $G'_1 ; G_2 \in \underline{\mathbb{G}}$ . Then,  $G' \in \underline{\mathbb{G}}$ .
    - \* **Case:**  $\llbracket \phi(G_1, \phi(G_2, \chi)) \wedge \xi \rrbracket = \emptyset$ . Then, by Defn. 38,  $\llbracket \phi(G_1 ; G_2, \chi) \wedge \xi \rrbracket = \emptyset$ . Then,  
 $\llbracket \phi(G, \chi) \wedge \xi \rrbracket = \emptyset$ .
- **Step:**  $[\rightarrow 1\text{-SEQ2}]$ , such that  $G = G_1 ; G_2$  and  $G' = G_1 ; G'_2$  and  $\text{subj}(G_1) \cap (\text{subj}(\xi) \cup \text{subj}(\gamma)) = \emptyset$  and  $G_2 \xrightarrow{\xi, \gamma} G'_2$ , for some  $G_1, G_2, G'_2$ .
  - Recall  $G \in \underline{\mathbb{G}}$ . Then,  $G_1 ; G_2 \in \underline{\mathbb{G}}$ . Then, by Defn. 21,  $G_1, G_2 \in \underline{\mathbb{G}}$ .
  - Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(G_1 ; G_2)$ . Then, by Defn. 27,  $\sqrt{R}(G_2)$ .
  - Recall  $G_2 \in \underline{\mathbb{G}}$  and  $\sqrt{R}(G_2)$  and  $G_2 \xrightarrow{\xi, \gamma} G'_2$ . Then, by induction:
    - \* **Case:**  $G'_2 \in \underline{\mathbb{G}}$ .  
Recall  $G_1, G'_2 \in \underline{\mathbb{G}}$ . Then, by Defn. 21,  $G_1 ; G'_2 \in \underline{\mathbb{G}}$ . Then,  $G' \in \underline{\mathbb{G}}$ .
    - \* **Case:**  $\llbracket \phi(G_2, \chi) \wedge \xi \rrbracket = \emptyset$ .
      - Recall  $\text{subj}(G_1) \cap (\text{subj}(\xi) \cup \text{subj}(\gamma)) = \emptyset$ . Then,  $\text{subj}(G_1) \cap \text{subj}(\xi) = \emptyset$ . Then, by Lem. 41,  $G_1 \# \xi$ .
      - Recall  $\llbracket \phi(G_2, \chi) \wedge \xi \rrbracket = \emptyset$ . Then, by Lem. 15,  $\llbracket \phi(G_2, \chi) \wedge \xi \rrbracket = \llbracket \text{false} \rrbracket$ . Then, by Lem. 81,  $\llbracket \phi(G_1, \phi(G_2, \chi) \wedge \xi) \rrbracket = \llbracket \phi(G_1, \text{false}) \rrbracket$ .
      - Recall  $G_1 \in \underline{\mathbb{G}}$ . Then, by Lem. 83,  $\llbracket \phi(G_1, \text{false}) \rrbracket = \llbracket \text{false} \rrbracket$ .
      - Recall  $G_1 \in \underline{\mathbb{G}}$  and  $G_1 \# \xi$ . Then, by Lem. 85,  $\llbracket \phi(G_1, \phi(G_2, \chi)) \wedge \xi \rrbracket \subseteq \llbracket \phi(G_1, \xi) \rrbracket$ . Then, by Defn. 13,  $\llbracket \phi(G_1, \phi(G_2, \chi)) \rrbracket \cap \llbracket \xi \rrbracket \subseteq \llbracket \phi(G_1, \phi(G_2, \chi)) \rrbracket \cap \llbracket \phi(G_1, \xi) \rrbracket$ . Then, by Defn. 13,  $\llbracket \phi(G_1, \phi(G_2, \chi)) \wedge \xi \rrbracket \subseteq \llbracket \phi(G_1, \phi(G_2, \chi)) \wedge \phi(G_1, \xi) \rrbracket$ . Then, by Defn. 38,  $\llbracket \phi(G_1 ; G_2, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(G_1, \phi(G_2, \chi)) \wedge \phi(G_1, \xi) \rrbracket$ . Then, by Lem. 82,  $\llbracket \phi(G, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(G_1, \phi(G_2, \chi)) \wedge \phi(G_1, \xi) \rrbracket$ . Then,  $\llbracket \phi(G, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(G_1, \phi(G_2, \chi) \wedge \xi) \rrbracket$ . Then,  $\llbracket \phi(G, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(G_1, \text{false}) \rrbracket$ . Then,  $\llbracket \phi(G, \chi) \wedge \xi \rrbracket \subseteq \llbracket \text{false} \rrbracket$ . Then, by Lem. 15,  $\llbracket \phi(G, \chi) \wedge \xi \rrbracket \subseteq \emptyset$ . Then,  $\llbracket \phi(G, \chi) \wedge \xi \rrbracket = \emptyset$ .
  - **Step:**  $[\rightarrow 1\text{-PAR1}]$ , such that  $G = G_1 \parallel G_2$ .  
By Defn. 38:
    - **Case:**  $G_1, G_2 \in \underline{\mathbb{G}}$ . Then, by Defn. 21,  $G_1 \parallel G_2 \in \underline{\mathbb{G}}$ . Then,  $G \in \underline{\mathbb{G}}$ .
    - **Case:**  $\phi(G_1 \parallel G_2, \chi) = \text{false}$ .  
Recall  $\emptyset \cap \llbracket \xi \rrbracket = \emptyset$ . Then, by Lem. 15,  $\llbracket \text{false} \rrbracket \cap \llbracket \xi \rrbracket = \emptyset$ . Then, by Defn. 13,  $\llbracket \text{false} \wedge \xi \rrbracket = \emptyset$ . Then,  $\llbracket \phi(G_1 \parallel G_2, \chi) \wedge \xi \rrbracket = \emptyset$ . Then,  $\llbracket \phi(G, \chi) \wedge \xi \rrbracket = \emptyset$ .
  - **Step:**  $[\rightarrow 1\text{-PAR2}]$ . Similar to case  $[\rightarrow 1\text{-PAR1}]$ .
  - **Step:**  $[\rightarrow 1\text{-NIF4}]$ , such that  $G = \mathbf{if} \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2}$  and  $G' = \mathbf{if} \hat{\xi}|_0 G'_1|_{R_1} G_2|_{R_2}$  and  $G_1 \xrightarrow{\xi, \gamma} G'_1$  and  $\text{subj}(\xi) \cup \text{subj}(\gamma) \subseteq R_1$ , for some  $R_1, R_2, G_1, G_2, \hat{\xi}$ .
    - Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(\mathbf{if} \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2})$ . Then, by Defn. 27,  $\sqrt{R}(G_1)$ .
    - Recall  $G \xrightarrow{\xi, \gamma} G'$  and  $\sqrt{R}(G)$ . Then, by Lem. 58,  $\text{subj}(\gamma) \neq \emptyset$ .
    - Recall  $\text{subj}(\xi) \cup \text{subj}(\gamma) \subseteq R_1 \setminus R_2$ . Then,  $\text{subj}(\gamma) \subseteq R_1 \setminus R_2$ . Then,  $\text{subj}(\gamma) \subseteq R_1$ .
    - Recall  $\text{subj}(\gamma) \neq \emptyset$  and  $\text{subj}(\gamma) \subseteq R_1$ . Then,  $R_1 \neq \emptyset$ .
    - Recall  $G \in \underline{\mathbb{G}}$ . Then, **if**  $\hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2} \in \underline{\mathbb{G}}$ . Then, by Defn. 21:
      - \* **Case:**  $G_1, G_2 \in \underline{\mathbb{G}}$ .  
Recall  $G_1 \in \underline{\mathbb{G}}$  and  $\sqrt{R}(G_1)$  and  $G_1 \xrightarrow{\xi, \gamma} G'_1$ . Then, by induction:
        - **Case:**  $G'_1 \in \underline{\mathbb{G}}$ .  
Recall  $G'_1, G_2 \in \underline{\mathbb{G}}$ . Then, by Defn. 21, **if**  $\hat{\xi}|_0 G'_1|_{R_1} G_2|_{R_2} \in \underline{\mathbb{G}}$ . Then,  $G' \in \underline{\mathbb{G}}$ .

· **Case:**  $\llbracket \phi(G_1, \chi) \wedge \xi \rrbracket = \emptyset$ .

Recall:

· **Case:**  $R_2 = \emptyset$ .

Recall  $R_1 \neq \emptyset = R_2$ . Then, by Defn. 38,

$\phi(\text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2}, \chi) = \phi(G_1, \chi) \wedge \bigwedge \{\xi^+ \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_1}$ . Then,

$\llbracket \phi(\text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2}, \chi) \rrbracket = \llbracket \phi(G_1, \chi) \wedge \bigwedge \{\xi^+ \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_1} \rrbracket$ . Then, by Defn. 13,  $\llbracket \phi(\text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2}, \chi) \rrbracket = \llbracket \phi(G_1, \chi) \rrbracket \cap \llbracket \bigwedge \{\xi^+ \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_1} \rrbracket$ .

Then,

$\llbracket \phi(\text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2}, \chi) \rrbracket \cap \llbracket \xi \rrbracket = \llbracket \phi(G_1, \chi) \rrbracket \cap \llbracket \bigwedge \{\xi^+ \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_1} \rrbracket \cap \llbracket \xi \rrbracket$ .

Then, by Defn. 13,

$\llbracket \phi(\text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2}, \chi) \wedge \xi \rrbracket = \llbracket \phi(G_1, \chi) \wedge \xi \rrbracket \cap \llbracket \bigwedge \{\xi^+ \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_1} \rrbracket$ . Then,

$\llbracket \phi(G, \chi) \wedge \xi \rrbracket = \llbracket \phi(G_1, \chi) \wedge \xi \rrbracket \cap \llbracket \bigwedge \{\xi^+ \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_1} \rrbracket$ . Then,

$\llbracket \phi(G, \chi) \wedge \xi \rrbracket = \emptyset \cap \llbracket \bigwedge \{\xi^+ \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_1} \rrbracket$ . Then,  $\llbracket \phi(G, \chi) \wedge \xi \rrbracket = \emptyset$ .

· **Case:**  $R_2 \neq \emptyset$ .

Recall  $R_1 \neq \emptyset \neq R_2$ . Then, by Defn. 38,  $\phi(\text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2}, \chi) = \text{false}$ .

Then,  $\llbracket \phi(\text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2}, \chi) \rrbracket = \llbracket \text{false} \rrbracket$ . Then,  $\llbracket \phi(G, \chi) \rrbracket = \llbracket \text{false} \rrbracket$ . Then, by Lem. 15,  $\llbracket \phi(G, \chi) \rrbracket = \emptyset$ .

\* **Case:**  $R_1 = \emptyset$ .

Recall  $R_1 \neq \emptyset$ . Then,  $\emptyset \neq \emptyset$ . Then,  $\text{false}$ .

\* **Case:**  $G_1 \in \underline{\mathbb{G}}$  and  $R_2 = \emptyset$ .

Recall  $G_1 \in \underline{\mathbb{G}}$  and  $\sqrt{R}(G_1)$  and  $G_1 \xrightarrow{\xi, \gamma} G'_1$ . Then, by induction:

· **Case:**  $G'_1 \in \underline{\mathbb{G}}$ .

Recall  $G'_1 \in \underline{\mathbb{G}}$  and  $R_1 \neq \emptyset$ . Then, by Defn. 21,  $\text{if } \hat{\xi}|_0 G'_1|_{R_1} G_2|_{\emptyset} \in \underline{\mathbb{G}}$ . Then,

$\text{if } \hat{\xi}|_0 G'_1|_{R_1} G_2|_{R_2} \in \underline{\mathbb{G}}$ . Then,  $G' \in \underline{\mathbb{G}}$ .

· **Case:**  $\llbracket \phi(G_1, \chi) \wedge \xi \rrbracket = \emptyset$ .

Recall  $R_1 \neq \emptyset = R_2$ . Then, by Defn. 38,

$\phi(\text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2}, \chi) = \phi(G_1, \chi) \wedge \bigwedge \{\xi^+ \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_1}$ . Then,

$\llbracket \phi(\text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2}, \chi) \rrbracket = \llbracket \phi(G_1, \chi) \wedge \bigwedge \{\xi^+ \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_1} \rrbracket$ . Then, by

Defn. 13,  $\llbracket \phi(\text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2}, \chi) \rrbracket = \llbracket \phi(G_1, \chi) \rrbracket \cap \llbracket \bigwedge \{\xi^+ \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_1} \rrbracket$ . Then,

$\llbracket \phi(\text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2}, \chi) \rrbracket \cap \llbracket \xi \rrbracket = \llbracket \phi(G_1, \chi) \rrbracket \cap \llbracket \bigwedge \{\xi^+ \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_1} \rrbracket \cap \llbracket \xi \rrbracket$ .

Then, by Defn. 13,

$\llbracket \phi(\text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2}, \chi) \wedge \xi \rrbracket = \llbracket \phi(G_1, \chi) \wedge \xi \rrbracket \cap \llbracket \bigwedge \{\xi^+ \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_1} \rrbracket$ . Then,

$\llbracket \phi(G, \chi) \wedge \xi \rrbracket = \llbracket \phi(G_1, \chi) \wedge \xi \rrbracket \cap \llbracket \bigwedge \{\xi^+ \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_1} \rrbracket$ . Then,

$\llbracket \phi(G, \chi) \wedge \xi \rrbracket = \emptyset \cap \llbracket \bigwedge \{\xi^+ \uparrow r\}_{r \in \text{subj}(\xi) \setminus R_1} \rrbracket$ . Then,  $\llbracket \phi(G, \chi) \wedge \xi \rrbracket = \emptyset$ .

– **Step:**  $[\rightarrow 1\text{-NIF5}]$ . Similar to case  $[\rightarrow 1\text{-NIF4}]$ .

– **Step:**  $[\rightarrow 1\text{-NWHILE}]$ . Similar to case  $[\rightarrow 1\text{-WHILE1}]$ .  $\square$

*Proof (of Lem. 91).*

1. Recall  $G \xrightarrow{\xi, \gamma} G'$ . Then, by Defn. 31:

– **Base:**  $[\rightarrow 1\text{-ACT}]$ , such that  $G = \gamma$  and  $G' = \text{skip}$  and  $\xi = \text{true}$ .

By Lem. 15,  $\llbracket \phi(\gamma, \chi) \rrbracket \subseteq \llbracket \text{true} \rrbracket$ . Then,  $\llbracket \phi(\gamma, \chi) \rrbracket \cap \llbracket \text{true} \rrbracket = \llbracket \phi(\gamma, \chi) \rrbracket$ . Then, by Defn. 13,

$\llbracket \phi(\gamma, \chi) \wedge \text{true} \rrbracket = \llbracket \phi(\gamma, \chi) \rrbracket$ . Then, by Defn. 38,  $\llbracket \phi(\gamma, \chi) \wedge \text{true} \rrbracket = \llbracket \phi(\gamma, \phi(\text{skip}, \chi)) \rrbracket$ .

Then,  $\llbracket \phi(G, \chi) \wedge \xi \rrbracket = \llbracket \phi(\gamma, \phi(G', \chi)) \rrbracket$ . Then,  $\llbracket \phi(G, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G', \chi)) \rrbracket$ .

– **Base:**  $[\rightarrow 1\text{-IF1}]$ , such that  $G = \hat{R}.\text{if } \hat{\xi} G_1 G_2$  and  $G' = G_1$  and  $\xi = \hat{\xi}^+$  and  $\gamma = 1_{\text{subj}(\hat{\xi})}^{\hat{R}}$ , for some  $\hat{R}, G_1, G_2, \hat{\xi}$ .

Recall  $\llbracket \phi(G, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(G, \chi) \wedge \xi \rrbracket$ . Then,  $\llbracket \phi(G, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\hat{R}.\text{if } \hat{\xi} G_1 G_2, \chi) \wedge \hat{\xi}^+ \rrbracket$ .

Then, by Defn. 38,  $\llbracket \phi(G, \chi) \wedge \xi \rrbracket \subseteq \llbracket (\hat{\xi}^+ \Rightarrow \phi(G_1, \chi)) \wedge (\hat{\xi}^- \Rightarrow \phi(G_2, \chi)) \wedge \hat{\xi}^{\equiv} \wedge \hat{\xi}^+ \rrbracket$ .

Then, by Defn. 13,  $\llbracket \phi(G, \chi) \wedge \xi \rrbracket \subseteq \llbracket \hat{\xi}^+ \Rightarrow \phi(G_1, \chi) \rrbracket \cap \llbracket \hat{\xi}^- \Rightarrow \phi(G_2, \chi) \rrbracket \cap \llbracket \hat{\xi}^{\equiv} \rrbracket \cap \llbracket \hat{\xi}^+ \rrbracket$ .

Then, by Lem. 19,  $\llbracket \phi(G, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(G_1, \chi) \rrbracket \cap \llbracket \hat{\xi}^- \Rightarrow \phi(G_2, \chi) \rrbracket \cap \llbracket \hat{\xi}^{\equiv} \rrbracket \cap \llbracket \hat{\xi}^+ \rrbracket$ . Then,

$\llbracket \phi(G, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(G_1, \chi) \rrbracket$ . Then, by Defn. 37,  $\llbracket \phi(G, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(1_{\text{subj}(\hat{\xi})}^{\hat{R}}, \phi(G_1, \chi)) \rrbracket$ .

Then,  $\llbracket \phi(G, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G', \chi)) \rrbracket$ .

– **Base:**  $[\rightarrow 1\text{-IF2}]$ . Similar to case  $[\rightarrow 1\text{-IF1}]$ .







- Recall  $\sqrt{R}(G_1)$  and  $G_1 \xrightarrow{\xi, \gamma} G'_1$ . Then, by induction,  $\llbracket \phi(G_1, \phi(G_2, \chi)) \wedge \xi \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G'_1, \phi(G_2, \chi))) \rrbracket$ . Then, by Defn. 38,  $\llbracket \phi(G_1; G_2, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G'_1; G_2, \chi)) \rrbracket$ . Then,  $\llbracket \phi(G, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G', \chi)) \rrbracket$ .
- **Step:** [ $\rightarrow$ 1-SEQ2], such that  $G = G_1; G_2$  and  $G' = G'_1; G'_2$  and  $\text{subj}(G_1) \cap (\text{subj}(\xi) \cup \text{subj}(\gamma)) = \emptyset$  and  $G_2 \xrightarrow{\xi, \gamma} G'_2$ , for some  $G_1, G_2, G'_2$ .  
Recall:
  - **Case:**  $\llbracket \phi(G_1, \phi(G_2, \chi)) \rrbracket = \emptyset$ .  
Recall  $\emptyset \subseteq \llbracket \phi(\gamma, \phi(G', \chi)) \rrbracket$ . Then,  $\emptyset \cap \llbracket \xi \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G', \chi)) \rrbracket$ . Then,  $\llbracket \phi(G_1, \phi(G_2, \chi)) \rrbracket \cap \llbracket \xi \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G', \chi)) \rrbracket$ . Then, by Defn. 13,  $\llbracket \phi(G_1, \phi(G_2, \chi)) \wedge \xi \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G', \chi)) \rrbracket$ . Then, by Defn. 38,  $\llbracket \phi(G_1; G_2, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G', \chi)) \rrbracket$ . Then,  $\llbracket \phi(G, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G', \chi)) \rrbracket$ .
  - **Case:**  $\llbracket \phi(G_1, \phi(G_2, \chi)) \rrbracket \neq \emptyset$ .
    - \* Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(G_1; G_2)$ . Then, by Defn. 27,  $\sqrt{R}(G_1)$  and  $\sqrt{R}(G_2)$ .
    - \* Recall  $\sqrt{R}(G_1)$  and  $\llbracket \phi(G_1, \phi(G_2, \chi)) \rrbracket \neq \emptyset$ . Then, by Lem. 87:
      - **Case:**  $\text{subj}(G_1) = R$ .
        - Recall  $G_2 \xrightarrow{\xi, \gamma} G'_2$ . Then, by Lem. 53,  $\text{subj}(\xi) \cup \text{subj}(\gamma) \cup \text{subj}(G'_2) \subseteq G_2$ . Then,  $\text{subj}(\xi) \cup \text{subj}(\gamma) \subseteq G_2$ .
        - Recall  $\sqrt{R}(G_2)$ . Then, by Lem. 47,  $\text{subj}(G_2) \subseteq R \neq \emptyset$ .
        - Recall  $\text{subj}(\xi) \cup \text{subj}(\gamma) \subseteq G_2 \subseteq R$ . Then,  $\text{subj}(\xi) \cup \text{subj}(\gamma) \subseteq R$ .
        - Recall  $G_2 \xrightarrow{\xi, \gamma} G'_2$  and  $\sqrt{R}(G_2)$ . Then, by Lem. 58,  $\text{subj}(\gamma) \neq \emptyset$ . Then,  $\text{subj}(\xi) \cup \text{subj}(\gamma) \neq \emptyset$ .
        - Recall  $\text{subj}(\xi) \cup \text{subj}(\gamma) \subseteq R$  and  $\text{subj}(\xi) \cup \text{subj}(\gamma) \neq \emptyset$  and  $R \neq \emptyset$ . Then,  $R \cap (\text{subj}(\xi) \cup \text{subj}(\gamma)) \neq \emptyset$ . Then,  $\text{subj}(G_1) \cap (\text{subj}(\xi) \cup \text{subj}(\gamma)) \neq \emptyset$ . Then,  $\emptyset \neq \emptyset$ . Then, **false**.
      - **Case:**  $G_1 \in \underline{\mathbb{G}}$ .
        - Recall  $G_1 \in \underline{\mathbb{G}}$ . Then, by Defn. 21,  $\underline{G}_1 = G_1$ , for some  $\underline{G}_1$ .
        - Recall  $\text{subj}(G_1) \cap (\text{subj}(\xi) \cup \text{subj}(\gamma)) = \emptyset$ . Then,  $\text{subj}(\underline{G}_1) \cap (\text{subj}(\xi) \cup \text{subj}(\gamma)) = \emptyset$ . Then,  $\text{subj}(\underline{G}_1) \cap \text{subj}(\xi) = \emptyset$  and  $\text{subj}(\underline{G}_1) \cap \text{subj}(\gamma) = \emptyset$ . Then, by Lem. 41,  $\underline{G}_1 \# \xi$  and  $\underline{G}_1 \# \gamma$ .
        - Recall  $\underline{G}_1 \# \xi$ . Then, by Lem. 85,  $\llbracket \phi(\underline{G}_1, \phi(G_2, \chi)) \wedge \xi \rrbracket \subseteq \llbracket \phi(\underline{G}_1, \xi) \rrbracket$ . Then, by Defn. 13,  $\llbracket \phi(\underline{G}_1, \phi(G_2, \chi)) \rrbracket \cap \llbracket \xi \rrbracket \subseteq \llbracket \phi(\underline{G}_1, \xi) \rrbracket$ . Then,  $\llbracket \phi(\underline{G}_1, \phi(G_2, \chi)) \wedge \xi \rrbracket \subseteq \llbracket \phi(\underline{G}_1, \phi(G_2, \chi)) \rrbracket \cap \llbracket \phi(\underline{G}_1, \xi) \rrbracket$ . Then, by Defn. 13,  $\llbracket \phi(\underline{G}_1, \phi(G_2, \chi)) \wedge \xi \rrbracket \subseteq \llbracket \phi(\underline{G}_1, \phi(G_2, \chi)) \wedge \phi(\underline{G}_1, \xi) \rrbracket$ . Then, by Lem. 82,  $\llbracket \phi(\underline{G}_1, \phi(G_2, \chi)) \wedge \xi \rrbracket \subseteq \llbracket \phi(\underline{G}_1, \phi(G_2, \chi) \wedge \xi) \rrbracket$ .
        - Recall  $\sqrt{R}(G_2)$  and  $G_2 \xrightarrow{\xi, \gamma} G'_2$ . Then, by induction,  $\llbracket \phi(G_2, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G'_2, \chi)) \rrbracket$ . Then, by Lem. 81,  $\llbracket \phi(\underline{G}_1, \phi(G_2, \chi) \wedge \xi) \rrbracket \subseteq \llbracket \phi(\underline{G}_1, \phi(\gamma, \phi(G'_2, \chi))) \rrbracket$ .
        - Recall  $\underline{G}_1 \# \gamma$ . Then, by Lem. 86,  $\llbracket \phi(\underline{G}_1, \phi(\gamma, \phi(G'_2, \chi))) \rrbracket = \llbracket \phi(\gamma, \phi(\underline{G}_1, \phi(G'_2, \chi))) \rrbracket$ .
        - Recall  $\llbracket \phi(\underline{G}_1, \phi(G_2, \chi)) \wedge \xi \rrbracket \subseteq \llbracket \phi(\underline{G}_1, \phi(G_2, \chi) \wedge \xi) \rrbracket \subseteq \llbracket \phi(\underline{G}_1, \phi(\gamma, \phi(G'_2, \chi))) \rrbracket$ . Then,  $\llbracket \phi(\underline{G}_1, \phi(G_2, \chi)) \wedge \xi \rrbracket \subseteq \llbracket \phi(\underline{G}_1, \phi(\gamma, \phi(G'_2, \chi))) \rrbracket$ . Then,  $\llbracket \phi(\underline{G}_1, \phi(G_2, \chi)) \wedge \xi \rrbracket \subseteq \llbracket \phi(\gamma, \phi(\underline{G}_1, \phi(G'_2, \chi))) \rrbracket$ . Then, by Defn. 38,  $\llbracket \phi(\underline{G}_1; G_2, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\gamma, \phi(\underline{G}_1; G'_2, \chi)) \rrbracket$ . Then,  $\llbracket \phi(G_1; G_2, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G_1; G'_2, \chi)) \rrbracket$ . Then,  $\llbracket \phi(G, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G', \chi)) \rrbracket$ .
  - **Step:** [ $\rightarrow$ 1-PAR1], such that  $G = G_1 \parallel G_2$  and  $G' = G'_1 \parallel G_2$  and  $G_1 \xrightarrow{\xi, \gamma} G'_1$ , for some  $G_1, G'_1, G_2$ .  
Recall:
    - **Case:**  $G_1, G_2 \in \underline{\mathbb{G}}$  and  $G_1 \# G_2$ .
      - \* Recall  $G_1, G_2 \in \underline{\mathbb{G}}$ . Then, by Defn. 21,  $G_1 \parallel G_2 \in \underline{\mathbb{G}}$ . Then,  $G \in \underline{\mathbb{G}}$ .
      - \* Recall  $G \in \underline{\mathbb{G}}$  and  $\sqrt{R}(G)$  and  $G \xrightarrow{\xi, \gamma} G'$ . Then, by Lem. 90:
        - **Case:**  $G' \in \underline{\mathbb{G}}$ .
          - Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(G_1 \parallel G_2)$ . Then, by Defn. 27,  $\sqrt{R}(G_1)$ .

- Recall  $G_1, G_2 \in \underline{\mathbb{G}}$  and  $G_1 \# G_2$ . Then, by Defn. 38,  $\phi(G_1 \parallel G_2, \chi) = \phi(G_1, \phi(G_2, \chi))$ .
- Recall  $G' \in \underline{\mathbb{G}}$ . Then,  $G'_1 \parallel G_2 \in \underline{\mathbb{G}}$ . Then, by Defn. 21,  $G'_1 \in \underline{\mathbb{G}}$ .
- Recall  $G_1 \# G_2$  and  $G_1 \xrightarrow{\xi, \gamma} G'_1$ . Then, by Lem. 52,  $G'_1 \# G_2$ .
- Recall  $G'_1, G_2 \in \underline{\mathbb{G}}$  and  $G'_1 \# G_2$ . Then, by Defn. 38,  $\phi(G'_1 \parallel G_2, \chi) = \phi(G'_1, \phi(G_2, \chi))$ .
- Recall  $\sqrt{R}(G_1)$  and  $G_1 \xrightarrow{\xi, \gamma} G'_1$ . Then, by induction,  $\llbracket \phi(G_1, \phi(G_2, \chi)) \wedge \xi \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G'_1, \phi(G_2, \chi))) \rrbracket$ . Then,  $\llbracket \phi(G_1 \parallel G_2, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G'_1 \parallel G_2, \chi)) \rrbracket$ . Then,  $\llbracket \phi(G, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G', \chi)) \rrbracket$ .
- **Case:**  $\llbracket \phi(G, \chi) \wedge \xi \rrbracket = \emptyset$ .  
Recall  $\emptyset \subseteq \llbracket \phi(\gamma, \phi(G', \chi)) \rrbracket$ . Then,  $\llbracket \phi(G, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G', \chi)) \rrbracket$ .
- **Case:** Not  $G_1, G_2 \in \underline{\mathbb{G}}$ , or not  $G_1 \# G_2$ .
  - \* Recall  $G_1, G_2 \in \underline{\mathbb{G}}$  and  $G_1 \# G_2$ . Then, by Defn. 38,  $\phi(G_1 \parallel G_2, \chi) = \text{false}$ .
  - \* Recall  $\emptyset \subseteq \llbracket \phi(\gamma, \phi(G', \chi)) \rrbracket$ . Then, by Lem. 15,  $\llbracket \text{false} \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G', \chi)) \rrbracket$ . Then,  $\llbracket \phi(G_1 \parallel G_2, \chi) \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G', \chi)) \rrbracket$ . Then,  $\llbracket \phi(G, \chi) \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G', \chi)) \rrbracket$ .
- **Step:**  $[\rightarrow 1\text{-PAR2}]$ , such that  $G = G_1 \parallel G_2$  and  $G' = G_1 \parallel G'_2$  and  $G_2 \xrightarrow{\xi, \gamma} G'_2$ , for some  $G_1, G_2, G'_2$ .  
Recall:
  - **Case:**  $G_1, G_2 \in \underline{\mathbb{G}}$  and  $G_1 \# G_2$ .
    - \* Recall  $G_1, G_2 \in \underline{\mathbb{G}}$ . Then, by Defn. 21,  $G_1 \parallel G_2 \in \underline{\mathbb{G}}$ . Then,  $G \in \underline{\mathbb{G}}$ .
    - \* Recall  $G \in \underline{\mathbb{G}}$  and  $\sqrt{R}(G)$  and  $G \xrightarrow{\xi, \gamma} G'$ . Then, by Lem. 90:
      - **Case:**  $G' \in \underline{\mathbb{G}}$ .
        - Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(G_1 \parallel G_2)$ . Then, by Defn. 27,  $\sqrt{R}(G_2)$ .
        - Recall  $G_1 \# G_2$  and  $G_2 \xrightarrow{\xi, \gamma} G'_2$ . Then, by Lem. 52,  $G_1 \# \xi$  and  $G_1 \# \gamma$  and  $G_1 \# G'_2$ .
        - Recall  $G_1 \# \xi$  and  $G_1 \in \underline{\mathbb{G}}$ . Then, by Lem. 85,  $\llbracket \phi(G_1, \phi(G_2, \chi)) \wedge \xi \rrbracket \subseteq \llbracket \phi(G_1, \xi) \rrbracket$ . Then, by Defn. 13,  $\llbracket \phi(G_1, \phi(G_2, \chi)) \rrbracket \cap \llbracket \xi \rrbracket \subseteq \llbracket \phi(G_1, \xi) \rrbracket$ . Then,  $\llbracket \phi(G_1, \phi(G_2, \chi)) \rrbracket \cap \llbracket \xi \rrbracket \subseteq \llbracket \phi(G_1, \phi(G_2, \chi)) \rrbracket \cap \llbracket \phi(G_1, \xi) \rrbracket$ . Then, by Defn. 13,  $\llbracket \phi(G_1, \phi(G_2, \chi)) \wedge \xi \rrbracket \subseteq \llbracket \phi(G_1, \phi(G_2, \chi)) \wedge \phi(G_1, \xi) \rrbracket$ . Then, by Lem. 82,  $\llbracket \phi(G_1, \phi(G_2, \chi)) \wedge \xi \rrbracket \subseteq \llbracket \phi(G_1, \phi(G_2, \chi) \wedge \xi) \rrbracket$ .
        - Recall  $\sqrt{R}(G_2)$  and  $G_2 \xrightarrow{\xi, \gamma} G'_2$ . Then, by induction,  $\llbracket \phi(G_2, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G'_2, \chi)) \rrbracket$ . Then, by Lem. 81,  $\llbracket \phi(G_1, \phi(G_2, \chi) \wedge \xi) \rrbracket \subseteq \llbracket \phi(G_1, \phi(\gamma, \phi(G'_2, \chi))) \rrbracket$ .
        - Recall  $G_1 \# \gamma$  and  $G_1 \in \underline{\mathbb{G}}$ . Then, by Lem. 86,  $\llbracket \phi(G_1, \phi(\gamma, \phi(G'_2, \chi))) \rrbracket = \llbracket \phi(\gamma, \phi(G_1, \phi(G'_2, \chi))) \rrbracket$ .
        - Recall  $G_1, G_2 \in \underline{\mathbb{G}}$  and  $G_1 \# G_2$ . Then, by Defn. 38,  $\phi(G_1 \parallel G_2, \chi) = \phi(G_1, \phi(G_2, \chi))$ .
        - Recall  $G' \in \underline{\mathbb{G}}$ . Then,  $G_1 \parallel G'_2 \in \underline{\mathbb{G}}$ . Then, by Defn. 21,  $G'_2 \in \underline{\mathbb{G}}$ .
        - Recall  $G_1, G'_2 \in \underline{\mathbb{G}}$  and  $G_1 \# G'_2$ . Then, by Defn. 38,  $\phi(G_1 \parallel G'_2, \chi) = \phi(G_1, \phi(G'_2, \chi))$ .
        - Recall  $\llbracket \phi(G_1, \phi(G_2, \chi)) \wedge \xi \rrbracket \subseteq \llbracket \phi(G_1, \phi(G_2, \chi) \wedge \xi) \rrbracket \subseteq \llbracket \phi(G_1, \phi(\gamma, \phi(G'_2, \chi))) \rrbracket$ . Then,  $\llbracket \phi(G_1, \phi(G_2, \chi)) \wedge \xi \rrbracket \subseteq \llbracket \phi(G_1, \phi(\gamma, \phi(G'_2, \chi))) \rrbracket$ . Then,  $\llbracket \phi(G_1, \phi(G_2, \chi)) \wedge \xi \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G_1, \phi(G'_2, \chi))) \rrbracket$ . Then, by Defn. 38,  $\llbracket \phi(G_1 \parallel G_2, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G_1 \parallel G'_2, \chi)) \rrbracket$ . Then,  $\llbracket \phi(G, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G', \chi)) \rrbracket$ .
      - **Case:**  $\llbracket \phi(G, \chi) \wedge \xi \rrbracket = \emptyset$ .  
Recall  $\emptyset \subseteq \llbracket \phi(\gamma, \phi(G', \chi)) \rrbracket$ . Then,  $\llbracket \phi(G, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G', \chi)) \rrbracket$ .
    - **Case:** Not  $G_1, G_2 \in \underline{\mathbb{G}}$ , or not  $G_1 \# G_2$ .
      - \* Recall  $G_1, G_2 \in \underline{\mathbb{G}}$  and  $G_1 \# G_2$ . Then, by Defn. 38,  $\phi(G_1 \parallel G_2, \chi) = \text{false}$ .
      - \* Recall  $\emptyset \subseteq \llbracket \phi(\gamma, \phi(G', \chi)) \rrbracket$ . Then, by Lem. 15,  $\llbracket \text{false} \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G', \chi)) \rrbracket$ . Then,  $\llbracket \phi(G_1 \parallel G_2, \chi) \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G', \chi)) \rrbracket$ . Then,  $\llbracket \phi(G, \chi) \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G', \chi)) \rrbracket$ .

- **Step:**  $[\rightarrow 1\text{-NIF4}]$ , such that  $G = \mathbf{if} \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2}$  and  $G' = \mathbf{if} \hat{\xi}|_0 G'_1|_{R_1} G_2|_{R_2}$  and  $G_1 \xrightarrow{\xi, \gamma} G'_1$  and  $\text{subj}(\xi) \cup \text{subj}(\gamma) \subseteq R_1 \setminus R_2$ , for some  $R_1, R_2, G_1, G'_1, G_2, \xi$ .

Recall:

- **Case:**  $R_1 = \emptyset$ .
    - \* Recall  $\text{subj}(\gamma) \subseteq R_1$ . Then,  $\text{subj}(\gamma) \subseteq \emptyset$ . Then,  $\text{subj}(\gamma) = \emptyset$ .
    - \* Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(\mathbf{if} \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2})$ . Then, by Defn. 27,  $\sqrt{R}(G_1)$ .
    - \* Recall  $G_1 \xrightarrow{\xi, \gamma} G'_1$  and  $\sqrt{R}(G_1)$ . Then, by Lem. 58,  $\text{subj}(\gamma) \neq \emptyset$ . Then,  $\emptyset \neq \emptyset$ . Then, **false**.
  - **Case:**  $R_1 \neq \emptyset = R_2$ .
    - \* Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(\mathbf{if} \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2})$ . Then, by Defn. 27,  $\sqrt{R}(G_1)$ .
    - \* By Lem. 9,  $\text{subj}(\hat{\xi} \uparrow r) = \{r\}$  for every  $r \in \text{subj}(\hat{\xi})$ . Then,  $\text{subj}(\hat{\xi} \uparrow r) = \{r\}$  for every  $r \in \text{subj}(\hat{\xi}) \setminus R_1$ . Then,  $\bigcup \{\text{subj}(\hat{\xi} \uparrow r)\}_{r \in \text{subj}(\hat{\xi}) \setminus R_1} = \bigcup \{\{r\}\}_{r \in \text{subj}(\hat{\xi}) \setminus R_1}$ .
    - \* Recall  $\text{subj}(\xi) \cup \text{subj}(\gamma) \subseteq R_1 \setminus R_2$ . Then,  $\text{subj}(\xi) \cup \text{subj}(\gamma) \subseteq R_1$ . Then,  $\text{subj}(\gamma) \subseteq R_1$ . Then,  $\text{subj}(\gamma) \cap (\text{subj}(\hat{\xi}) \setminus R_1) = \emptyset$ . Then,  $\text{subj}(\gamma) \cap \{r \mid r \in \text{subj}(\hat{\xi}) \setminus R_1\} = \emptyset$ . Then,  $\text{subj}(\gamma) \cap \bigcup \{\{r\}\}_{r \in \text{subj}(\hat{\xi}) \setminus R_1} = \emptyset$ . Then,  $\text{subj}(\gamma) \cap \bigcup \{\text{subj}(\hat{\xi} \uparrow r)\}_{r \in \text{subj}(\hat{\xi}) \setminus R_1} = \emptyset$ . Then, by Lem. 13,  $\text{subj}(\gamma) \cap \bigcup \{\text{subj}(\hat{\xi} \uparrow r)^+\}_{r \in \text{subj}(\hat{\xi}) \setminus R_1} = \emptyset$ . Then, by Lem. 11,  $\text{subj}(\gamma) \cap \bigcup \{\text{subj}(\hat{\xi}^+ \uparrow r)\}_{r \in \text{subj}(\hat{\xi}) \setminus R_1} = \emptyset$ . Then, by Defn. 10,  $\text{subj}(\gamma) \cap \text{subj}(\bigwedge \{\hat{\xi}^+ \uparrow r\}_{r \in \text{subj}(\hat{\xi}) \setminus R_1}) = \emptyset$ . Then, by Lem. 41,  $\gamma \# \bigwedge \{\hat{\xi}^+ \uparrow r\}_{r \in \text{subj}(\hat{\xi}) \setminus R_1}$ . Then, by Defn. 23,  $\text{write}(\gamma) \cap \text{read}(\bigwedge \{\hat{\xi}^+ \uparrow r\}_{r \in \text{subj}(\hat{\xi}) \setminus R_1}) = \emptyset$ . Then, by Lem. 76,  $\phi(\gamma, \bigwedge \{\hat{\xi}^+ \uparrow r\}_{r \in \text{subj}(\hat{\xi}) \setminus R_1}) = \bigwedge \{\hat{\xi}^+ \uparrow r\}_{r \in \text{subj}(\hat{\xi}) \setminus R_1}$ .
    - \* Recall  $R_1 \neq \emptyset = R_2$ . Then, by Defn. 38,  $\phi(\mathbf{if} \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2}, \chi) = \phi(G_1, \chi) \wedge \bigwedge \{\hat{\xi}^+ \uparrow r\}_{r \in \text{subj}(\hat{\xi}) \setminus R_1}$  and  $\phi(\mathbf{if} \hat{\xi}|_0 G'_1|_{R_1} G_2|_{R_2}, \chi) = \phi(G'_1, \chi) \wedge \bigwedge \{\hat{\xi}^+ \uparrow r\}_{r \in \text{subj}(\hat{\xi}) \setminus R_1}$ .
    - \* Recall  $\sqrt{R}(G_1)$  and  $G_1 \xrightarrow{\xi, \gamma} G'_1$ . Then, by induction,  $\llbracket \phi(G_1, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G'_1, \chi)) \rrbracket$ . Then,  $\llbracket \phi(G_1, \chi) \wedge \xi \rrbracket \cap \llbracket \bigwedge \{\hat{\xi}^+ \uparrow r\}_{r \in \text{subj}(\hat{\xi}) \setminus R_1} \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G'_1, \chi)) \rrbracket \cap \llbracket \bigwedge \{\hat{\xi}^+ \uparrow r\}_{r \in \text{subj}(\hat{\xi}) \setminus R_1} \rrbracket$ . Then,  $\llbracket \phi(G_1, \chi) \wedge \xi \rrbracket \cap \llbracket \bigwedge \{\hat{\xi}^+ \uparrow r\}_{r \in \text{subj}(\hat{\xi}) \setminus R_1} \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G'_1, \chi)) \rrbracket \cap \llbracket \bigwedge \{\hat{\xi}^+ \uparrow r\}_{r \in \text{subj}(\hat{\xi}) \setminus R_1} \rrbracket$ . Then, by Defn. 13,  $\llbracket \phi(G_1, \chi) \wedge \xi \wedge \bigwedge \{\hat{\xi}^+ \uparrow r\}_{r \in \text{subj}(\hat{\xi}) \setminus R_1} \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G'_1, \chi)) \wedge \bigwedge \{\hat{\xi}^+ \uparrow r\}_{r \in \text{subj}(\hat{\xi}) \setminus R_1} \rrbracket$ . Then, by Lem. 82,  $\llbracket \phi(G_1, \chi) \wedge \xi \wedge \bigwedge \{\hat{\xi}^+ \uparrow r\}_{r \in \text{subj}(\hat{\xi}) \setminus R_1} \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G'_1, \chi) \wedge \bigwedge \{\hat{\xi}^+ \uparrow r\}_{r \in \text{subj}(\hat{\xi}) \setminus R_1}) \rrbracket$ . Then,  $\llbracket \phi(\mathbf{if} \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2}, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\gamma, \phi(\mathbf{if} \hat{\xi}|_0 G'_1|_{R_1} G_2|_{R_2}, \chi)) \rrbracket$ . Then,  $\llbracket \phi(G, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G', \chi)) \rrbracket$ .
  - **Case:**  $R_1 \neq \emptyset \neq R_2$ .
    - \* Recall  $R_1 \neq \emptyset \neq R_2$ . Then, by Defn. 38,  $\phi(\mathbf{if} \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2}, \chi) = \mathbf{false}$ .
    - \* Recall  $\emptyset \subseteq \llbracket \phi(\gamma, \phi(G', \chi)) \rrbracket$ . Then, by Lem. 15,  $\llbracket \mathbf{false} \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G', \chi)) \rrbracket$ . Then,  $\llbracket \phi(\mathbf{if} \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2}, \chi) \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G', \chi)) \rrbracket$ . Then,  $\llbracket \phi(G, \chi) \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G', \chi)) \rrbracket$ .
- **Step:**  $[\rightarrow 1\text{-NIF5}]$ . Similar to case  $[\rightarrow 1\text{-NIF4}]$ .
- **Step:**  $[\rightarrow 1\text{-NWHILE}]$ , such that  $G = \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset$  and  $\mathbf{if} \hat{\xi}|_0 (\hat{G}; \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \mathbf{skip}|_\emptyset \xrightarrow{\xi, \gamma} G'$ , for some  $\hat{G}, \psi, \hat{\xi}$ .
- By Lem. 80,  $\llbracket \phi(\mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset, \chi) \rrbracket \subseteq \llbracket \phi(\mathbf{if} \hat{\xi}|_0 (\hat{G}; \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \mathbf{skip}|_\emptyset, \chi) \rrbracket$ . Then,  $\llbracket \phi(G, \chi) \rrbracket \subseteq \llbracket \phi(\mathbf{if} \hat{\xi}|_0 (\hat{G}; \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \mathbf{skip}|_\emptyset, \chi) \rrbracket$ . Then,  $\llbracket \phi(G, \chi) \rrbracket \cap \llbracket \xi \rrbracket \subseteq \llbracket \phi(\mathbf{if} \hat{\xi}|_0 (\hat{G}; \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \mathbf{skip}|_\emptyset, \chi) \rrbracket \cap \llbracket \xi \rrbracket$ . Then, by Defn. 13,  $\llbracket \phi(G, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\mathbf{if} \hat{\xi}|_0 (\hat{G}; \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \mathbf{skip}|_\emptyset, \chi) \wedge \xi \rrbracket$ .
  - Recall  $\sqrt{R}(G)$ . Then,  $\sqrt{R}(\mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset)$ . Then, by Lem. 46,  $\sqrt{R}(\mathbf{if} \hat{\xi}|_0 (\hat{G}; \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \mathbf{skip}|_\emptyset)$ .
  - Recall  $\sqrt{R}(\mathbf{if} \hat{\xi}|_0 (\hat{G}; \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \mathbf{skip}|_\emptyset)$  and  $\mathbf{if} \hat{\xi}|_0 (\hat{G}; \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \mathbf{skip}|_\emptyset \xrightarrow{\xi, \gamma} G'$ . Then, by induction,  $\llbracket \phi(\mathbf{if} \hat{\xi}|_0 (\hat{G}; \mathbf{while} \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \mathbf{skip}|_\emptyset, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G', \chi)) \rrbracket$ .

- Recall
 
$$\llbracket \phi(G, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\text{if } \hat{\xi}|_0 (\hat{G}; \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_0) \text{ skip}|_0, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G', \chi)) \rrbracket.$$
 Then,  $\llbracket \phi(G, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G', \chi)) \rrbracket$ .  $\square$
- 2. – Recall  $\sqrt{R}(\mathcal{G})$ . Then, by Defn. 27,  $\mathcal{G} = \{G\}$  and  $\sqrt{R}(G)$ , for some  $G$ .
  - Recall  $\mathcal{G} \xrightarrow{\xi, \gamma} \mathcal{G}'$ . Then,  $\{G\} \xrightarrow{\xi, \gamma} \mathcal{G}'$ . Then, by Defn. 32,  $\mathcal{G}' = \{G'\}$  and  $G \xrightarrow{\xi, \gamma} G'$ , for some  $G'$ .
  - Recall  $\sqrt{R}(G)$  and  $G \xrightarrow{\xi, \gamma} G'$ . Then, by 1,  $\llbracket \phi(G, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G', \chi)) \rrbracket$ . Then, by Defn. 38,  $\llbracket \phi(\{G\}, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\gamma, \phi(\{G'\}, \chi)) \rrbracket$ . Then,  $\llbracket \phi(\mathcal{G}, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\gamma, \phi(\mathcal{G}', \chi)) \rrbracket$ .  $\square$
- 3. – Recall  $(\mathcal{G}, \mathcal{S}) \xrightarrow{\dot{\gamma}} (\mathcal{G}', \mathcal{S}')$ . Then, by Defn. 33,  $\mathcal{S}' = \text{effect}(\dot{\gamma}, \mathcal{S})$  and  $\mathcal{G} \xrightarrow{\xi, \gamma} \mathcal{G}'$  and  $\mathcal{S} \in \llbracket \xi \rrbracket$  and  $\dot{\gamma} = \text{cons}_{\mathcal{S}}(\gamma)$ , for some  $\xi, \gamma$ .
  - Recall  $\mathcal{S} \in \llbracket \phi(\mathcal{G}, \chi) \rrbracket$  and  $\mathcal{S} \in \llbracket \xi \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \phi(\mathcal{G}, \chi) \rrbracket \cap \llbracket \xi \rrbracket$ . Then, by Defn. 13,  $\mathcal{S} \in \llbracket \phi(\mathcal{G}, \chi) \wedge \xi \rrbracket$ .
  - Recall  $\sqrt{R}(\mathcal{G})$  and  $\mathcal{G} \xrightarrow{\xi, \gamma} \mathcal{G}'$ . Then, by 2,  $\llbracket \phi(\mathcal{G}, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\gamma, \phi(\mathcal{G}', \chi)) \rrbracket$ .
  - Recall  $\mathcal{S} \in \llbracket \phi(\mathcal{G}, \chi) \wedge \xi \rrbracket \subseteq \llbracket \phi(\gamma, \phi(\mathcal{G}', \chi)) \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \phi(\gamma, \phi(\mathcal{G}', \chi)) \rrbracket$ . Then, by Lem. 79,  $\text{effect}(\text{cons}_{\mathcal{S}}(\gamma), \mathcal{S}) \in \llbracket \phi(\mathcal{G}', \chi) \rrbracket$ . Then,  $\text{effect}(\dot{\gamma}, \mathcal{S}) \in \llbracket \phi(\mathcal{G}', \chi) \rrbracket$ . Then,  $\mathcal{S}' \in \llbracket \phi(\mathcal{G}', \chi) \rrbracket$ .  $\square$

*Proof (of Lem. 92).* Recall  $(\mathcal{G}, \mathcal{S}) \xrightarrow{\dot{\Gamma}}^* (\mathcal{G}^\dagger, \mathcal{S}^\dagger)$ . Then, by Defn. 36:

- **Base:**  $[\rightarrow^*3\text{-BASE}]$ , such that  $\mathcal{G} = \mathcal{G}^\dagger$  and  $\mathcal{S} = \mathcal{S}^\dagger$ .
  - Recall  $\sqrt{R}(\mathcal{G})$ . Then,  $\sqrt{R}(\mathcal{G}^\dagger)$ .
  - Recall  $\mathcal{S} \in \llbracket \phi(\mathcal{G}, \chi) \rrbracket$ . Then,  $\mathcal{S}^\dagger \in \llbracket \phi(\mathcal{G}^\dagger, \chi) \rrbracket$ .
- **Step:**  $[\rightarrow^*3\text{-STEP}]$ , such that  $(\mathcal{G}, \mathcal{S}) \xrightarrow{\dot{\Gamma}}^* (\mathcal{G}^*, \mathcal{S}^*) \xrightarrow{\pi} (\mathcal{G}^\dagger, \mathcal{S}^\dagger)$  and  $\pi \in \dot{\Gamma}$ .
  - Recall  $\sqrt{R}(\mathcal{G})$  and  $\mathcal{S} \in \llbracket \phi(\mathcal{G}, \chi) \rrbracket$  and  $(\mathcal{G}, \mathcal{S}) \xrightarrow{\dot{\Gamma}}^* (\mathcal{G}^*, \mathcal{S}^*)$ . Then, by induction,  $\sqrt{R}(\mathcal{G}^*)$  and  $\mathcal{S}^* \in \llbracket \phi(\mathcal{G}^*, \chi) \rrbracket$ .
  - Recall  $\pi \in \dot{\Gamma}$ . Then,  $\pi = \dot{\gamma}$ , for some  $\dot{\gamma}$ .
  - Recall  $(\mathcal{G}^*, \mathcal{S}^*) \xrightarrow{\pi} (\mathcal{G}^\dagger, \mathcal{S}^\dagger)$ . Then,  $(\mathcal{G}^*, \mathcal{S}^*) \xrightarrow{\dot{\gamma}} (\mathcal{G}^\dagger, \mathcal{S}^\dagger)$ .
  - Recall  $\sqrt{R}(\mathcal{G}^*)$  and  $\mathcal{S}^* \in \llbracket \phi(\mathcal{G}^*, \chi) \rrbracket$  and  $(\mathcal{G}^*, \mathcal{S}^*) \xrightarrow{\dot{\gamma}} (\mathcal{G}^\dagger, \mathcal{S}^\dagger)$ . Then, by Lem. 89,  $\sqrt{R}(\mathcal{G}^\dagger)$ .
  - Recall  $\sqrt{R}(\mathcal{G}^*)$  and  $\mathcal{S}^* \in \llbracket \phi(\mathcal{G}^*, \chi) \rrbracket$  and  $(\mathcal{G}^*, \mathcal{S}^*) \xrightarrow{\dot{\gamma}} (\mathcal{G}^\dagger, \mathcal{S}^\dagger)$ . Then, by Lem. 91,  $\mathcal{S}^\dagger \in \llbracket \phi(\mathcal{G}^\dagger, \chi) \rrbracket$ .  $\square$

*Proof (of Lem. 93).*

1. Recall  $\sqrt{R}(G)$ . Then, by Defn. 27:
  - **Base:**  $[\checkmark 1\text{-ACT1}]$ , such that  $G = q.y := e$ , for some  $q, y, e$ .
    - By Defn. 4,  $\text{eval}_{\mathcal{S}}(\text{true}) = \text{true}$ . Then,  $\mathcal{S} \in \{\hat{\mathcal{S}} \mid \text{eval}_{\hat{\mathcal{S}}}(\text{true}) = \text{true}\}$ . Then, by Defn. 13,  $\mathcal{S} \in \llbracket \text{true} \rrbracket$ . Then,  $\xi = \text{true}$  and  $\mathcal{S} \in \llbracket \xi \rrbracket$ , for some  $\xi$ .
    - By  $[\rightarrow 1\text{-ACT}]$ ,  $q.y := e \xrightarrow{\text{true}, q.y := e} \text{skip}$ . Then,  $G \xrightarrow{\xi, q.y := e} \text{skip}$ . Then,  $G' = \text{skip}$  and  $\gamma = q.y := e$  and  $G \xrightarrow{\xi, \gamma} G'$ , for some  $G', \gamma$ .
  - **Base:**  $[\checkmark 1\text{-ACT2}]$ . Similar to case  $[\checkmark 1\text{-ACT1}]$ .
  - **Base:**  $[\checkmark 1\text{-SKIP}]$ , such that  $G = \text{skip}$ .  
By  $[\downarrow 1\text{-SKIP}]$ ,  $\text{skip} \downarrow$ . Then,  $G \downarrow$ .
  - **Step:**  $[\checkmark 1\text{-SEQ}]$ , such that  $G = G_1 ; G_2$  and  $\sqrt{R}(G_1)$  and  $\sqrt{R}(G_2)$ , for some  $G_1, G_2$ .
    - Recall  $\mathcal{S} \in \llbracket \phi(G, \chi) \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \phi(G_1 ; G_2, \chi) \rrbracket$ . Then, by Defn. 38,  $\mathcal{S} \in \llbracket \phi(G_1, \phi(G_2, \chi)) \rrbracket$ .
    - Recall  $\sqrt{R}(G_1)$  and  $\mathcal{S} \in \llbracket \phi(G_1, \phi(G_2, \chi)) \rrbracket$ . Then, by induction:
      - \* **Case:**  $G_1 \downarrow$ .
        - Recall  $\sqrt{R}(G_1)$  and  $G_1 \downarrow$ . Then, by Lem. 88,  $\llbracket \phi(G_1, \phi(G_2, \chi)) \rrbracket \subseteq \llbracket \phi(G_2, \chi) \rrbracket$ .
        - Recall  $\mathcal{S} \in \llbracket \phi(G_1, \phi(G_2, \chi)) \rrbracket \subseteq \llbracket \phi(G_2, \chi) \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \phi(G_2, \chi) \rrbracket$ .
        - Recall  $\sqrt{R}(G_2)$  and  $\mathcal{S} \in \llbracket \phi(G_2, \chi) \rrbracket$ . Then, by induction:
          - **Case:**  $G_2 \downarrow$ .  
Recall  $G_1 \downarrow$  and  $G_2 \downarrow$ . Then, by  $[\downarrow 1\text{-SEQ}]$ ,  $G_1 ; G_2 \downarrow$ . Then,  $G \downarrow$ .

- **Case:**  $G_2 \xrightarrow{\xi, \gamma} G'_2$  and  $\mathcal{S} \in \llbracket \xi \rrbracket$ , for some  $G'_2$ , for some  $\xi, \gamma$ .
  - Recall  $G_1 \downarrow$  and  $\sqrt{R}(G_1)$ . Then, by Lem. 48,  $\text{subj}(G_1) = \emptyset$ .
  - Recall  $\emptyset \cap (\text{subj}(\xi) \cup \text{subj}(\gamma)) = \emptyset$ . Then,  $\text{subj}(G_1) \cap (\text{subj}(\xi) \cup \text{subj}(\gamma)) = \emptyset$ .
  - Recall  $\text{subj}(G_1) \cap (\text{subj}(\xi) \cup \text{subj}(\gamma)) = \emptyset$  and  $G_2 \xrightarrow{\xi, \gamma} G'_2$ . Then, by  $[\rightarrow 1\text{-SEQ2}]$ ,  $G_1 ; G_2 \xrightarrow{\xi, \gamma} G_1 ; G'_2$ . Then,  $G \xrightarrow{\xi, \gamma} G_1 ; G'_2$ . Then,  $G' = G_1 ; G'_2$  and  $G \xrightarrow{\xi, \gamma} G'$ , for some  $G'$ .
- \* **Case:**  $G_1 \xrightarrow{\xi, \gamma} G'_1$  and  $\mathcal{S} \in \llbracket \xi \rrbracket$ , for some  $G'_1$ , for some  $\xi, \gamma$ .  
 Recall  $G_1 \xrightarrow{\xi, \gamma} G'_1$ . Then, by  $[\rightarrow 1\text{-SEQ1}]$ ,  $G_1 ; G_2 \xrightarrow{\xi, \gamma} G'_1 ; G_2$ . Then,  $G \xrightarrow{\xi, \gamma} G'_1 ; G_2$ . Then,  $G' = G'_1 ; G_2$  and  $G \xrightarrow{\xi, \gamma} G'$ , for some  $G'$ .
- **Step:**  $[\sqrt{1}\text{-PAR}]$ , such that  $G = G_1 \parallel G_2$  and  $\sqrt{R}(G_1)$  and  $\sqrt{R}(G_2)$ , for some  $G_1, G_2$ .  
 By Defn. 38:
  - **Case:**  $\phi(G_1 \parallel G_2, \chi) = \phi(G_1, \phi(G_2, \chi))$ .
    - \* Recall  $\mathcal{S} \in \llbracket \phi(G, \chi) \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \phi(G_1 \parallel G_2, \chi) \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \phi(G_1, \phi(G_2, \chi)) \rrbracket$ .
    - \* Recall  $\sqrt{R}(G_1)$  and  $\mathcal{S} \in \llbracket \phi(G_1, \phi(G_2, \chi)) \rrbracket$ . Then, by induction:
      - **Case:**  $G_1 \downarrow$ .
        - Recall  $\sqrt{R}(G_1)$  and  $G_1 \downarrow$ . Then, by Lem. 88,  $\llbracket \phi(G_1, \phi(G_2, \chi)) \rrbracket \subseteq \llbracket \phi(G_2, \chi) \rrbracket$ .
        - Recall  $\mathcal{S} \in \llbracket \phi(G_1, \phi(G_2, \chi)) \rrbracket \subseteq \llbracket \phi(G_2, \chi) \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \phi(G_2, \chi) \rrbracket$ .
        - Recall  $\sqrt{R}(G_2)$  and  $\mathcal{S} \in \llbracket \phi(G_2, \chi) \rrbracket$ . Then, by induction:
          - **Case:**  $G_2 \downarrow$ .  
 Recall  $G_1 \downarrow$  and  $G_2 \downarrow$ . Then, by  $[\downarrow 1\text{-PAR}]$ ,  $G_1 \parallel G_2 \downarrow$ . Then,  $G \downarrow$ .
          - **Case:**  $G_2 \xrightarrow{\xi, \gamma} G'_2$  and  $\mathcal{S} \in \llbracket \xi \rrbracket$ , for some  $G'_2$ , for some  $\xi, \gamma$ .  
 Recall  $G_2 \xrightarrow{\xi, \gamma} G'_2$ . Then, by  $[\rightarrow 1\text{-PAR2}]$ ,  $G_1 \parallel G_2 \xrightarrow{\xi, \gamma} G_1 \parallel G'_2$ . Then,  $G \xrightarrow{\xi, \gamma} G_1 \parallel G'_2$ . Then,  $G' = G_1 \parallel G'_2$  and  $G \xrightarrow{\xi, \gamma} G'$ , for some  $G'$ .
        - **Case:**  $G_1 \xrightarrow{\xi, \gamma} G'_1$  and  $\mathcal{S} \in \llbracket \xi \rrbracket$ , for some  $G'_1$ , for some  $\xi, \gamma$ .  
 Recall  $G_1 \xrightarrow{\xi, \gamma} G'_1$ . Then, by  $[\rightarrow 1\text{-PAR1}]$ ,  $G_1 \parallel G_2 \xrightarrow{\xi, \gamma} G'_1 \parallel G_2$ . Then,  $G \xrightarrow{\xi, \gamma} G'_1 \parallel G_2$ . Then,  $G' = G'_1 \parallel G_2$  and  $G \xrightarrow{\xi, \gamma} G'$ , for some  $G'$ .
    - **Case:**  $G_1 \parallel G_2, \chi = \text{false}$ .  
 Recall  $\mathcal{S} \in \llbracket \phi(G, \chi) \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \phi(G_1 \parallel G_2, \chi) \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \text{false} \rrbracket$ . Then, by Lem. 15,  $\mathcal{S} \in \emptyset$ . Then, **false**.
  - **Step:**  $[\sqrt{1}\text{-IF}]$ , such that  $G = R.\text{if } \hat{\xi} G_1 G_2$ , for some  $G_1, G_2, \hat{\xi}$ .  
 Recall  $\mathcal{S} \in \llbracket \phi(G, \chi) \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \phi(R.\text{if } \hat{\xi} G_1 G_2, \chi) \rrbracket$ . Then, by Defn. 38,  $\mathcal{S} \in \llbracket (\hat{\xi}^+ \Rightarrow \phi(G_1, \chi)) \wedge (\hat{\xi}^- \Rightarrow \phi(G_2, \chi)) \wedge \hat{\xi}^\equiv \rrbracket$ . Then, by Defn. 13,  $\mathcal{S} \in \llbracket \hat{\xi}^+ \Rightarrow \phi(G_1, \chi) \rrbracket \cap \llbracket \hat{\xi}^- \Rightarrow \phi(G_2, \chi) \rrbracket \cap \llbracket \hat{\xi}^\equiv \rrbracket$ . Then, by Lem. 27:
    - **Case:**  $\mathcal{S} \in \llbracket \hat{\xi}^+ \rrbracket$ .
      - \* By  $[\rightarrow 1\text{-IF1}]$ ,  $R.\text{if } \hat{\xi} G_1 G_2 \xrightarrow{\hat{\xi}^+, 1R_{\text{subj}(\hat{\xi})}} G_1$ . Then,  $G \xrightarrow{\hat{\xi}^+, 1R_{\text{subj}(\hat{\xi})}} G_1$ . Then,  $G' = G_1$  and  $\xi = \hat{\xi}^+$  and  $\gamma = 1R_{\text{subj}(\hat{\xi})}$  and  $G \xrightarrow{\xi, \gamma} G'$ , for some  $G', \xi, \gamma$ .
      - \* Recall  $\mathcal{S} \in \llbracket \hat{\xi}^+ \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \xi \rrbracket$ .
    - **Case:**  $\mathcal{S} \in \llbracket \hat{\xi}^- \rrbracket$ . Similar to case  $\mathcal{S} \in \llbracket \hat{\xi}^+ \rrbracket$ .
  - **Step:**  $[\sqrt{1}\text{-WHILE}]$ , such that  $G = R.\text{while } \hat{\xi} \{ \psi \} \hat{G}$ , for some  $\hat{G}, \hat{\psi}, \hat{\xi}$ .
    - Recall  $\mathcal{S} \in \llbracket \phi(G, \chi) \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \phi(R.\text{while } \hat{\xi} \{ \psi \} \hat{G}, \chi) \rrbracket$ . Then, by Defn. 38,  $\mathcal{S} \in \llbracket \psi \wedge \forall (\psi \Rightarrow ((\hat{\xi}^+ \Rightarrow \phi(\hat{G}, \psi)) \wedge (\hat{\xi}^- \Rightarrow \chi) \wedge \hat{\xi}^\equiv)) \rrbracket$ . Then, by Defn. 13,  $\mathcal{S} \in \llbracket \psi \rrbracket \cap \llbracket \forall (\psi \Rightarrow ((\hat{\xi}^+ \Rightarrow \phi(\hat{G}, \psi)) \wedge (\hat{\xi}^- \Rightarrow \chi) \wedge \hat{\xi}^\equiv)) \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \psi \rrbracket$  and  $\mathcal{S} \in \llbracket \forall (\psi \Rightarrow ((\hat{\xi}^+ \Rightarrow \phi(\hat{G}, \psi)) \wedge (\hat{\xi}^- \Rightarrow \chi) \wedge \hat{\xi}^\equiv)) \rrbracket$ .
    - By Lem. 24,  $\llbracket \forall (\psi \Rightarrow ((\hat{\xi}^+ \Rightarrow \phi(\hat{G}, \psi)) \wedge (\hat{\xi}^- \Rightarrow \chi) \wedge \hat{\xi}^\equiv)) \rrbracket \subseteq \llbracket \psi \Rightarrow ((\hat{\xi}^+ \Rightarrow \phi(\hat{G}, \psi)) \wedge (\hat{\xi}^- \Rightarrow \chi) \wedge \hat{\xi}^\equiv) \rrbracket$ .
    - Recall  $\mathcal{S} \in \llbracket \forall (\psi \Rightarrow ((\hat{\xi}^+ \Rightarrow \phi(\hat{G}, \psi)) \wedge (\hat{\xi}^- \Rightarrow \chi) \wedge \hat{\xi}^\equiv)) \rrbracket \subseteq \llbracket \psi \Rightarrow ((\hat{\xi}^+ \Rightarrow \phi(\hat{G}, \psi)) \wedge (\hat{\xi}^- \Rightarrow \chi) \wedge \hat{\xi}^\equiv) \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \psi \Rightarrow ((\hat{\xi}^+ \Rightarrow \phi(\hat{G}, \psi)) \wedge (\hat{\xi}^- \Rightarrow \chi) \wedge \hat{\xi}^\equiv) \rrbracket$ .
    - Recall  $\mathcal{S} \in \llbracket \psi \rrbracket$  and  $\mathcal{S} \in \llbracket \psi \Rightarrow ((\hat{\xi}^+ \Rightarrow \phi(\hat{G}, \psi)) \wedge (\hat{\xi}^- \Rightarrow \chi) \wedge \hat{\xi}^\equiv) \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \psi \rrbracket \cap \llbracket \psi \Rightarrow ((\hat{\xi}^+ \Rightarrow \phi(\hat{G}, \psi)) \wedge (\hat{\xi}^- \Rightarrow \chi) \wedge \hat{\xi}^\equiv) \rrbracket$ . Then, by Lem. 19,  $\mathcal{S} \in \llbracket \psi \rrbracket \cap \llbracket (\hat{\xi}^+ \Rightarrow \phi(\hat{G}, \psi)) \wedge (\hat{\xi}^- \Rightarrow \chi) \wedge \hat{\xi}^\equiv \rrbracket$ . Then,  $\mathcal{S} \in \llbracket (\hat{\xi}^+ \Rightarrow \phi(\hat{G}, \psi)) \wedge (\hat{\xi}^- \Rightarrow \chi) \wedge \hat{\xi}^\equiv \rrbracket$ . Then, by Defn. 13,  $\mathcal{S} \in \llbracket \hat{\xi}^+ \Rightarrow \phi(\hat{G}, \psi) \rrbracket \cap \llbracket \hat{\xi}^- \Rightarrow \chi \rrbracket \cap \llbracket \hat{\xi}^\equiv \rrbracket$ . Then, by Lem. 27:
      - \* **Case:**  $\mathcal{S} \in \llbracket \hat{\xi}^+ \rrbracket$ .

- By  $[\rightarrow 1\text{-WHILE1}]$ ,  $R.\mathbf{while} \hat{\xi} \{ \psi \} \hat{G} \xrightarrow{\xi^+, 1_{\text{subj}(\hat{\xi})}} \hat{G}^-; R.\mathbf{while} \hat{\xi} \{ \psi \} \hat{G}$ . Then,  $G \xrightarrow{\xi^+, 1_{\text{subj}(\hat{\xi})}} \hat{G}^-; R.\mathbf{while} \hat{\xi} \{ \psi \} \hat{G}$ . Then,  $G' = \hat{G}^-; R.\mathbf{while} \hat{\xi} \{ \psi \} \hat{G}$  and  $\xi = \hat{\xi}^+$  and  $\gamma = 1_{\text{subj}(\hat{\xi})}^R$  and  $G \xrightarrow{\xi, \gamma} G'$ , for some  $G', \xi, \gamma$ .
  - Recall  $\mathcal{S} \in \llbracket \hat{\xi}^+ \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \xi \rrbracket$ .
  - \* **Case:**  $\mathcal{S} \in \llbracket \hat{\xi}^- \rrbracket$ . Similar to case  $\mathcal{S} \in \llbracket \hat{\xi}^+ \rrbracket$ . □
- **Step:**  $[\checkmark 1\text{-NIF}]$ , such that  $G = \mathbf{if} \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2}$  and  $\checkmark_R(G_1)$  and  $\checkmark_R(G_2)$  and  $R = \text{subj}(\hat{\xi})$  and  $R_1, R_2 \subseteq \text{subj}(\hat{\xi})$ , for some  $R_1, R_2, G_1, G_2, \hat{\xi}$ .
- Recall:
- **Case:**  $R_1 = \emptyset = R_2$ .
    - \* Recall  $\checkmark_R(G)$ . Then, by Lem. 47,  $R \neq \emptyset$ . Then,  $\text{subj}(\hat{\xi}) \neq \emptyset$ . Then,  $r \in \text{subj}(\hat{\xi})$ , for some  $r$ .
    - \* Recall  $R_1 = \emptyset = R_2$ . Then, by Defn. 38,  $\Phi(\mathbf{if} \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2}, \chi) = (\hat{\xi}^+ \Rightarrow \Phi(G_1, \chi)) \wedge (\hat{\xi}^- \Rightarrow \Phi(G_2, \chi)) \wedge \hat{\xi}^\equiv$ .
    - \* Recall  $\mathcal{S} \in \llbracket \Phi(G, \chi) \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \Phi(\mathbf{if} \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2}, \chi) \rrbracket$ . Then,  $\mathcal{S} \in \llbracket (\hat{\xi}^+ \Rightarrow \Phi(G_1, \chi)) \wedge (\hat{\xi}^- \Rightarrow \Phi(G_2, \chi)) \wedge \hat{\xi}^\equiv \rrbracket$ . Then, by Defn. 13,  $\mathcal{S} \in \llbracket \hat{\xi}^+ \Rightarrow \Phi(G_1, \chi) \rrbracket \cap \llbracket \hat{\xi}^- \Rightarrow \Phi(G_2, \chi) \rrbracket \cap \llbracket \hat{\xi}^\equiv \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \hat{\xi}^\equiv \rrbracket$ . Then, by Lem. 27:
      - **Case:**  $\mathcal{S} \in \llbracket \hat{\xi}^+ \upharpoonright r \rrbracket$ .
        - Recall  $r \in \text{subj}(\hat{\xi})$ . Then,  $r \in \text{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset)$ . Then,  $r \in \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$ . Then, by  $[\rightarrow 1\text{-NIF2}]$ ,  $\mathbf{if} \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2} \xrightarrow{\xi^+ \upharpoonright r, 1_{\{r\}}^R} \mathbf{if} \hat{\xi}|_0 G_1|_{R_1 \cup \{r\}} G_2|_{R_2}$ . Then,  $G \xrightarrow{\xi^+ \upharpoonright r, 1_{\{r\}}^R} \mathbf{if} \hat{\xi}|_0 G_1|_{R_1 \cup \{r\}} G_2|_{R_2}$ . Then,  $G' = \mathbf{if} \hat{\xi}|_0 G_1|_{R_1 \cup \{r\}} G_2|_{R_2}$  and  $\xi = \hat{\xi}^+ \upharpoonright r$  and  $\gamma = 1_{\{r\}}^R$  and  $G \xrightarrow{\xi, \gamma} G'$ , for some  $G', \xi, \gamma$ .
        - Recall  $\mathcal{S} \in \llbracket \hat{\xi}^+ \upharpoonright r \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \xi \rrbracket$ .
      - **Case:**  $\mathcal{S} \in \llbracket \hat{\xi}^- \upharpoonright r \rrbracket$ . Similar to case  $\mathcal{S} \in \llbracket \hat{\xi}^+ \upharpoonright r \rrbracket$ .
    - **Case:**  $R_1 = \emptyset \neq R_2$ .
      - \* Recall  $R_1 = \emptyset \neq R_2$ . Then, by Defn. 38,  $\Phi(\mathbf{if} \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2}, \chi) = \Phi(G_2, \chi) \wedge \bigwedge \{ \hat{\xi}^- \upharpoonright r \}_{r \in \text{subj}(\hat{\xi}) \setminus R_2}$ .
      - \* Recall  $\mathcal{S} \in \llbracket \Phi(G, \chi) \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \Phi(G_2, \chi) \wedge \bigwedge \{ \hat{\xi}^- \upharpoonright r \}_{r \in \text{subj}(\hat{\xi}) \setminus R_2} \rrbracket$ . Then, by Defn. 13,  $\mathcal{S} \in \llbracket \Phi(G_2, \chi) \rrbracket \cap \bigcap \{ \llbracket \hat{\xi}^- \upharpoonright r \rrbracket \}_{r \in \text{subj}(\hat{\xi}) \setminus R_2}$ . Then,  $\mathcal{S} \in \llbracket \Phi(G_2, \chi) \rrbracket$ , and  $\mathcal{S} \in \llbracket \hat{\xi}^- \upharpoonright r \rrbracket$  for every  $r \in \text{subj}(\hat{\xi}) \setminus R_2$ .
      - \* Recall:
        - **Case:**  $R_2 = \text{subj}(\hat{\xi})$ .  
Recall  $\checkmark_R(G_2)$  and  $\mathcal{S} \in \llbracket \Phi(G_2, \chi) \rrbracket$ . Then, by induction:
          - **Case:**  $G_2 \downarrow$ .
            - Recall  $R_2 = \text{subj}(\hat{\xi})$ . Then,  $\emptyset \cup R_2 = \text{subj}(\hat{\xi})$ . Then,  $R_1 \cup R_2 = \text{subj}(\hat{\xi})$ .
            - Recall  $R_1 = \emptyset$ . Then,  $R_1 \neq \emptyset$  implies  $G_1 \downarrow$ .
            - Recall  $G_2 \downarrow$ . Then,  $R_2 \neq \emptyset$  implies  $G_2 \downarrow$ .
            - Recall  $R_1 \cup R_2 = \text{subj}(\hat{\xi})$ , and  $R_1 \neq \emptyset$  implies  $G_1 \downarrow$ , and  $R_2 \neq \emptyset$  implies  $G_2 \downarrow$ . Then, by  $[\downarrow 1\text{-NIF}]$ ,  $\mathbf{if} \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2} \downarrow$ . Then,  $G \downarrow$ .
          - **Case:**  $G_2 \xrightarrow{\xi, \gamma} G'_2$  and  $\mathcal{S} \in \llbracket \xi \rrbracket$ , for some  $G'_2$ , for some  $\xi, \gamma$ .
            - Recall  $G_2 \xrightarrow{\xi, \gamma} G'_2$ . Then, by Lem. 53,  $\text{subj}(\xi) \cup \text{subj}(\gamma) \cup \text{subj}(G'_2) \subseteq \text{subj}(G_2)$ . Then,  $\text{subj}(\xi) \cup \text{subj}(\gamma) \subseteq \text{subj}(G_2)$ .
            - Recall  $\checkmark_R(G_2)$ . Then, by Lem. 47,  $\text{subj}(G_2) \subseteq R$ .
            - Recall  $\text{subj}(\xi) \cup \text{subj}(\gamma) \subseteq \text{subj}(G_2) \subseteq R$ . Then,  $\text{subj}(\xi) \cup \text{subj}(\gamma) \subseteq R$ . Then,  $\text{subj}(\xi) \cup \text{subj}(\gamma) \subseteq \text{subj}(\hat{\xi})$ . Then,  $\text{subj}(\xi) \cup \text{subj}(\gamma) \subseteq R_2$ . Then,  $\text{subj}(\xi) \cup \text{subj}(\gamma) \subseteq R_2 \setminus \emptyset$ . Then,  $\text{subj}(\xi) \cup \text{subj}(\gamma) \subseteq R_2 \setminus R_1$ .
            - Recall  $G_2 \xrightarrow{\xi, \gamma} G'_2$  and  $\text{subj}(\xi) \cup \text{subj}(\gamma) \subseteq R_2 \setminus R_1$ . Then, by  $[\rightarrow 1\text{-NIF5}]$ ,  $\mathbf{if} \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2} \xrightarrow{\xi, \gamma} \mathbf{if} \hat{\xi}|_0 G_1|_{R_1} G'_2|_{R_2}$ . Then,  $G \xrightarrow{\xi, \gamma} \mathbf{if} \hat{\xi}|_0 G_1|_{R_1} G'_2|_{R_2}$ . Then,  $G' = \mathbf{if} \hat{\xi}|_0 G_1|_{R_1} G'_2|_{R_2}$  and  $G \xrightarrow{\xi, \gamma} G'$ , for some  $G'$ .
        - **Case:**  $R_2 \neq \text{subj}(\hat{\xi})$ .
          - Recall  $R_2 \subseteq \text{subj}(\hat{\xi})$  and  $R_2 \neq \text{subj}(\hat{\xi})$ . Then,  $\text{subj}(\hat{\xi}) \setminus R_2 \neq \emptyset$ . Then,  $r \in \text{subj}(\hat{\xi}) \setminus R_2$ , for some  $r$ .

- Recall  $r \in \text{subj}(\hat{\xi}) \setminus R_2$ . Then,  $r \in \text{subj}(\hat{\xi}) \setminus (\emptyset \cup R_2)$ . Then,  $r \in \text{subj}(\hat{\xi}) \setminus (R_1 \cup R_2)$ . Then, by  $[\rightarrow 1\text{-NIF3}]$ ,  $\text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2} \xrightarrow{\xi^- \uparrow r, 2\{r\}} \text{if } \hat{\xi}|_0 G_1|_{R_1 \cup \{r\}} G_2|_{R_2}$ . Then,  $G \xrightarrow{\xi^- \uparrow r, 2\{r\}} \text{if } \hat{\xi}|_0 G_1|_{R_1 \cup \{r\}} G_2|_{R_2}$ . Then,  $G' = \text{if } \hat{\xi}|_0 G_1|_{R_1 \cup \{r\}} G_2|_{R_2}$  and  $\xi = \hat{\xi}^- \uparrow r$  and  $\gamma = 2\{r\}$  and  $G \xrightarrow{\xi, \gamma} G'$ , for some  $G', \xi, \gamma$ .
  - Recall  $r \in \text{subj}(\hat{\xi}) \setminus R_2$ . Then,  $\mathcal{S} \in \llbracket \hat{\xi}^- \uparrow r \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \xi \rrbracket$ .
  - **Case:**  $R_1 \neq \emptyset = R_2$ . Similar to case  $R_1 = \emptyset \neq R_2$ .
  - **Case:**  $R_1 \neq \emptyset \neq R_2$ .
    - \* Recall  $R_1 \neq \emptyset \neq R_2$ . Then, by Defn. 38,  $\phi(\text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2}, \chi) = \text{false}$ .
    - \* Recall  $\mathcal{S} \in \llbracket \phi(G, \chi) \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \phi(\text{if } \hat{\xi}|_0 G_1|_{R_1} G_2|_{R_2}, \chi) \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \text{false} \rrbracket$ . Then, by Lem. 15,  $\mathcal{S} \in \emptyset$ . Then, **false**.
- **Step:**  $[\sqrt{1}\text{-NWHILE}]$ , such that  $G = \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset$  and  $R = \text{subj}(\hat{\xi})$ , for some  $\hat{G}, \psi, \hat{\xi}$ .
- Recall  $\sqrt{R}(G)$ . Then, by Lem. 47,  $R \neq \emptyset$ . Then,  $\text{subj}(\hat{\xi}) \neq \emptyset$ . Then,  $r \in \text{subj}(\hat{\xi})$ , for some  $r$ .
  - By Lem. 80,  $\llbracket \phi(\text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset, \chi) \rrbracket \subseteq \llbracket \phi(\text{if } \hat{\xi}|_0 (\hat{G}; \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \text{ skip}|_\emptyset, \chi) \rrbracket$ . Then,  $\llbracket \phi(G, \chi) \rrbracket \subseteq \llbracket \phi(\text{if } \hat{\xi}|_0 (\hat{G}; \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \text{ skip}|_\emptyset, \chi) \rrbracket$ . Then, by Defn. 38,  $\llbracket \phi(G, \chi) \rrbracket \subseteq \llbracket (\hat{\xi}^+ \Rightarrow \phi(\hat{G}; \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset, \chi)) \wedge (\hat{\xi}^- \Rightarrow \phi(\text{skip}, \chi)) \wedge \hat{\xi}^{\equiv} \rrbracket$ . Then, by Defn. 13,  $\llbracket \phi(G, \chi) \rrbracket \subseteq \llbracket \hat{\xi}^+ \Rightarrow \phi(\hat{G}; \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset, \chi) \rrbracket \cap \llbracket \hat{\xi}^- \Rightarrow \phi(\text{skip}, \chi) \rrbracket \cap \llbracket \hat{\xi}^{\equiv} \rrbracket$ . Then,  $\llbracket \phi(G, \chi) \rrbracket \subseteq \llbracket \hat{\xi}^{\equiv} \rrbracket$ .
  - Recall  $\mathcal{S} \in \llbracket \phi(G, \chi) \rrbracket \subseteq \llbracket \hat{\xi}^{\equiv} \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \hat{\xi}^{\equiv} \rrbracket$ . Then, by Lem. 27:
    - \* **Case:**  $\mathcal{S} \in \llbracket \hat{\xi}^+ \uparrow r \rrbracket$ .
      - Recall  $r \in \text{subj}(\hat{\xi})$ . Then,  $r \in \text{subj}(\hat{\xi}) \setminus (\emptyset \cup \emptyset)$ . Then, by  $[\rightarrow 1\text{-NIF2}]$ ,  $\text{if } \hat{\xi}|_0 (\hat{G}; \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \text{ skip}|_\emptyset \xrightarrow{\xi^+ \uparrow r, 1\{r\}} \text{if } \hat{\xi}|_0 (\hat{G}; \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \text{ skip}|_\emptyset$ . Then, by  $[\rightarrow 1\text{-NWHILE}]$ ,  $\text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset \xrightarrow{\xi^+ \uparrow r, 1\{r\}} \text{if } \hat{\xi}|_0 (\hat{G}; \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \text{ skip}|_\emptyset$ . Then,  $G \xrightarrow{\xi^+ \uparrow r, 1\{r\}} \text{if } \hat{\xi}|_0 (\hat{G}; \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \text{ skip}|_\emptyset$ . Then,  $G' = \text{if } \hat{\xi}|_0 (\hat{G}; \text{while } \hat{\xi}|_0 \{\psi\} \hat{G}|_\emptyset) \text{ skip}|_\emptyset$  and  $\xi = \hat{\xi}^+ \uparrow r$  and  $\gamma = 1\{r\}$  and  $G \xrightarrow{\xi, \gamma} G'$ , for some  $G', \xi, \gamma$ .
      - Recall  $\mathcal{S} \in \llbracket \hat{\xi}^+ \uparrow r \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \xi \rrbracket$ .
    - \* **Case:**  $\mathcal{S} \in \llbracket \hat{\xi}^- \uparrow r \rrbracket$ . Similar to case  $\mathcal{S} \in \llbracket \hat{\xi}^+ \uparrow r \rrbracket$ . □
2. – Recall  $\sqrt{R}(G)$ . Then, by Defn. 27,  $\mathcal{G} = \{G\}$  and  $\sqrt{R}(G)$ , for some  $G$ .
- Recall  $\mathcal{S} \in \llbracket \phi(\mathcal{G}, \chi) \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \phi(\{G\}, \chi) \rrbracket$ . Then, by Defn. 13,  $\mathcal{S} \in \llbracket \phi(G, \chi) \rrbracket$ .
  - Recall  $\sqrt{R}(G)$  and  $\mathcal{S} \in \llbracket \phi(G, \chi) \rrbracket$ . Then, by 1:
    - **Case:**  $G \downarrow$ . Then, by  $[\downarrow 2\text{-GLOB}]$ ,  $\{G\} \downarrow$ . Then,  $\mathcal{G} \downarrow$ .
    - **Case:**  $G \xrightarrow{\xi, \gamma} G'$  and  $\mathcal{S} \in \llbracket \xi \rrbracket$ , for some  $G'$ , for some  $\xi, \gamma$ .  
Recall  $G \xrightarrow{\xi, \gamma} G'$ . Then, by  $[\rightarrow 2\text{-GLOB}]$ ,  $\{G\} \xrightarrow{\xi, \gamma} \{G'\}$ . Then,  $\mathcal{G} \xrightarrow{\xi, \gamma} \{G'\}$ . Then,  $\mathcal{G}' = \{G'\}$  and  $\mathcal{G} \xrightarrow{\xi, \gamma} \mathcal{G}'$ , for some  $\mathcal{G}'$ . □
3. Recall  $\sqrt{R}(G)$  and  $\mathcal{S} \in \llbracket \phi(G, \chi) \rrbracket$ . Then, by 2:
- **Case:**  $\mathcal{G} \downarrow$ . Then, by  $[\downarrow 3]$ ,  $(\mathcal{G}, \mathcal{S}) \downarrow$ .
  - **Case:**  $\mathcal{G} \xrightarrow{\xi, \gamma} \mathcal{G}'$  and  $\mathcal{S} \in \llbracket \xi \rrbracket$ , for some  $\xi, \gamma$ , for some  $\mathcal{G}'$ .
    - By Defn. 19,  $\dot{\gamma} = \text{con}_{\mathcal{S}}(\gamma)$ , for some  $\dot{\gamma}$ .
    - Recall  $\mathcal{G} \xrightarrow{\xi, \gamma} \mathcal{G}'$  and  $\mathcal{S} \in \llbracket \xi \rrbracket$  and  $\dot{\gamma} = \text{con}_{\mathcal{S}}(\gamma)$ . Then, by  $[\rightarrow 3]$ ,  $(\mathcal{G}, \mathcal{S}) \xrightarrow{\dot{\gamma}} (\mathcal{G}', \text{effect}(\dot{\gamma}, \mathcal{S}))$ . Then,  $\mathcal{S}' = \text{effect}(\dot{\gamma}, \mathcal{S})$  and  $(\{G\}, \mathcal{S}) \xrightarrow{\dot{\gamma}} (\{G'\}, \mathcal{S}')$ , for some  $\mathcal{S}'$ . □

*Proof (of Thm. 1).* Recall  $\sqrt{R}(G)$  and  $\mathcal{S} \in \llbracket \phi(G, \chi) \rrbracket$  and  $(\mathcal{G}, \mathcal{S}) \xrightarrow{\dot{\gamma}}^* (\mathcal{G}^\dagger, \mathcal{S}^\dagger)$ . Then, by Lem. 92,  $\sqrt{R}(\mathcal{G}^\dagger)$  and  $\mathcal{S}^\dagger \in \llbracket \phi(\mathcal{G}^\dagger, \chi) \rrbracket$ . Then, by Lem. 93,  $(\mathcal{G}^\dagger, \mathcal{S}^\dagger) \downarrow$ , or  $(\mathcal{G}^\dagger, \mathcal{S}^\dagger) \xrightarrow{\dot{\gamma}} (\mathcal{G}^\ddagger, \mathcal{S}^\ddagger)$ , for some  $\mathcal{G}^\ddagger, \mathcal{S}^\ddagger, \dot{\gamma}$ . □

*Proof (of Thm. 2).*

- Recall  $\sqrt{R}(G)$  and  $\mathcal{S} \in \llbracket \phi(G, \chi) \rrbracket$  and  $(\mathcal{G}, \mathcal{S}) \xrightarrow{\dot{\gamma}}^* (\mathcal{G}^\dagger, \mathcal{S}^\dagger)$ . Then, by Lem. 92,  $\sqrt{R}(\mathcal{G}^\dagger)$  and  $\mathcal{S}^\dagger \in \llbracket \phi(\mathcal{G}^\dagger, \chi) \rrbracket$ .
- Recall  $\sqrt{R}(\mathcal{G}^\dagger)$  and  $\mathcal{S}^\dagger \in \llbracket \phi(\mathcal{G}^\dagger, \chi) \rrbracket$  and  $(\mathcal{G}^\dagger, \mathcal{S}^\dagger) \downarrow$ . Then, by Lem. 88,  $\mathcal{S}^\dagger \in \llbracket \chi \rrbracket$ . □



## I Theorem: Operational Equivalence

*Proof (of Lem. 94).* Recall  $L_1 \rightleftharpoons L_2$ . Then, by Defn. 40,  $L_1 \Longrightarrow L^\dagger \Leftarrow L_2$ , for some  $L^\dagger$ . Then,  $L_2 \Longrightarrow L^\dagger \Leftarrow L_1$ , for some  $L^\dagger$ . Then, by  $[\rightleftharpoons]$ ,  $L_2 \rightleftharpoons L_1$ .  $\square$

*Proof (of Lem. 95).*

1. – Recall  $L_1 \rightleftharpoons L_2$ . Then, by Defn. 40,  $L_1 \Longrightarrow L^\dagger$  and  $L^\dagger \Leftarrow L_2$ , for some  $L^\dagger$ .  
– Recall  $L^\dagger \Leftarrow L_2 \downarrow$ . Then, by Lem. 70,  $L^\dagger \downarrow$ .  $\square$
2. – Recall  $L_1 \rightleftharpoons L_2$ . Then, by Defn. 40,  $L_1 \Longrightarrow L^\dagger \Leftarrow L_2$ , for some  $L^\dagger$ .  
– Recall  $L_1 \Longrightarrow L^\dagger \Leftarrow L_2 \xrightarrow{\text{true}, \tau} L'_2$ . Then, by Lem. 70,  $L_1 \Longrightarrow L^\dagger \Leftarrow L'_2$ , for some  $L^\dagger$ .  
Then, by  $[\rightleftharpoons]$ ,  $L_1 \rightleftharpoons L'_2$ .  $\square$
3. – Recall  $L_1 \rightleftharpoons L_2$ . Then, by Defn. 40,  $L_1 \Longrightarrow L^\dagger \Leftarrow L_2$ , for some  $L^\dagger$ .  
– Recall  $L_1 \Longrightarrow L^\dagger \Leftarrow L_2 \xrightarrow{\xi, \lambda} L'_2$ . Then, by Lem. 70,  $L_1 \xrightarrow{\xi, \lambda} L'_1$  and  $L'_1 \Longrightarrow L^\dagger \Leftarrow L'_2$ , for some  $L^\dagger$ , for some  $L'_1$ .  
– Recall  $L'_1 \Longrightarrow L^\dagger \Leftarrow L'_2$ . Then, by  $[\rightleftharpoons]$ ,  $L'_1 \rightleftharpoons L'_2$ .  $\square$

*Proof (of Lem. 96).*

- Recall  $\checkmark_R(\mathcal{G})$ . Then, by Defn. 27,  $\mathcal{G} = \{G\}$  and  $\checkmark_R(G)$ , for some  $G$ .
- Recall  $\mathcal{S} \in \llbracket \Phi(\mathcal{G}, \text{true}) \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \Phi(\{G\}, \text{true}) \rrbracket$ . Then, by Defn. 38,  $\mathcal{S} \in \llbracket \Phi(G, \text{true}) \rrbracket$ .
- By  $[\rightarrow^* 1\text{-BASE}]$ ,  $(G \upharpoonright r) \xrightarrow{\{\text{true}, \tau\}^*} (G \upharpoonright r)$  for every  $r \in R$ . Then, by  $[\Rightarrow 1]$ ,  $(G \upharpoonright r) \Longrightarrow (G \upharpoonright r)$  for every  $r \in R$ . Then, by  $[\rightleftharpoons]$ ,  $(G \upharpoonright r) \rightleftharpoons (G \upharpoonright r)$  for every  $r \in R$ .
- Recall  $\checkmark_R(G)$  and  $\mathcal{S} \in \llbracket \Phi(G, \text{true}) \rrbracket$ , and  $(G \upharpoonright r) \rightleftharpoons (G \upharpoonright r)$  for every  $r \in R$ . Then, by  $[\boxtimes]$ ,  $(\{G\}, \mathcal{S}) \boxtimes (\{G \upharpoonright r\}_{r \in R}, \mathcal{S})$ . Then, by Defn. 26,  $(\{G\}, \mathcal{S}) \boxtimes (\{G\} \upharpoonright R, \mathcal{S})$ . Then,  $(\mathcal{G}, \mathcal{S}) \boxtimes (\mathcal{G} \upharpoonright R, \mathcal{S})$ .  $\square$

*Proof (of Lem. 97).*

1. – Recall  $(\mathcal{G}, \mathcal{S}) \boxtimes (\mathcal{L}, \mathcal{S})$ . Then, by Defn. 41,  $\mathcal{G} = \{G\}$  and  $\mathcal{L} = \{L_r\}_{r \in R}$  and  $\checkmark_R(G)$ , and  $(G \upharpoonright r) \rightleftharpoons L_r$  for every  $r \in R$ , for some  $G, \{L_r\}_{r \in R}$ .  
– Recall  $(\mathcal{G}, \mathcal{S}) \downarrow$ . Then,  $(\{G\}, \mathcal{S}) \downarrow$ . Then, by Defn. 29,  $G \downarrow$  and  $\checkmark_R(G)$ . Then, by Lem. 49,  $(G \upharpoonright r) \downarrow$  for every  $r \in R$ .  
– Recall  $(G \upharpoonright r) \rightleftharpoons L_r$  for every  $r \in R$ . Then, by Lem. 94,  $L_r \rightleftharpoons (G \upharpoonright r)$  for every  $r \in R$ .  
– Recall  $L_r \rightleftharpoons (G \upharpoonright r) \downarrow$  for every  $r \in R$ . Then, by Lem. 95,  $L_r \Longrightarrow L_r^\dagger \downarrow$  for every  $r \in R$ , for some  $\{L_r^\dagger\}_{r \in R}$ .  
– Recall  $L_r \Longrightarrow L_r^\dagger$  for every  $r \in R$ . Then, by Lem. 72,  $\{L_r\}_{r \in R} \Longrightarrow \{L_r^\dagger\}_{r \in R}$ . Then, by Lem. 74,  $(\{L_r\}_{r \in R}, \mathcal{S}) \Longrightarrow (\{L_r^\dagger\}_{r \in R}, \mathcal{S})$ . Then,  $(\mathcal{L}, \mathcal{S}) \Longrightarrow (\{L_r^\dagger\}_{r \in R}, \mathcal{S})$ . Then,  $\mathcal{L}^\dagger = \{L_r^\dagger\}_{r \in R}$  and  $\mathcal{S}^\dagger = \mathcal{S}$  and  $(\mathcal{L}, \mathcal{S}) \Longrightarrow (\mathcal{L}^\dagger, \mathcal{S}^\dagger)$ , for some  $\mathcal{L}^\dagger, \mathcal{S}^\dagger$ .  
– Recall  $L_r^\dagger \downarrow$  for every  $r \in R$ . Then, by  $[\downarrow 2\text{-LOCS}]$ ,  $\{L_r^\dagger\}_{r \in R} \downarrow$ . Then, by  $[\downarrow 3]$ ,  $(\{L_r^\dagger\}_{r \in R}, \mathcal{S}) \downarrow$ . Then,  $(\mathcal{L}^\dagger, \mathcal{S}^\dagger) \downarrow$ .  $\square$
2. – Recall  $(\mathcal{G}, \mathcal{S}) \boxtimes (\mathcal{L}, \mathcal{S})$ . Then, by Defn. 41,  $\mathcal{G} = \{G\}$  and  $\checkmark_R(G)$ , for some  $G$ .  
– Recall  $(\mathcal{G}, \mathcal{S}) \xrightarrow{\tau} (\mathcal{G}', \mathcal{S}')$ . Then,  $(\{G\}, \mathcal{S}) \xrightarrow{\tau} (\mathcal{G}', \mathcal{S}')$ . Then, by Defn. 33,  $\{G\} \xrightarrow{\xi, \gamma} \mathcal{G}'$  and  $\tau = \text{cons}_{\mathcal{S}}(\gamma)$ , for some  $\xi, \gamma$ .  
– Recall  $\{G\} \xrightarrow{\xi, \gamma} \mathcal{G}'$ . Then, by Defn. 32,  $\mathcal{G}' = \{G'\}$  and  $G \xrightarrow{\xi, \gamma} G'$ , for some  $G'$ .  
– Recall  $G \xrightarrow{\xi, \gamma} G'$  and  $\checkmark_R(G)$ . Then, by Lem. 58,  $\gamma \neq \tau$ .  
– Recall  $\tau = \text{cons}_{\mathcal{S}}(\gamma)$ . Then, by Defn. 19,  $\gamma = \tau$ . Then, **false**.  $\square$
3. – Recall  $(\mathcal{G}, \mathcal{S}) \boxtimes (\mathcal{L}, \mathcal{S})$ . Then, by Defn. 41,  $\mathcal{G} = \{G\}$  and  $\mathcal{L} = \{L_r\}_{r \in R}$  and  $\checkmark_R(G)$  and  $\mathcal{S} \in \llbracket \Phi(G, \text{true}) \rrbracket$ , and  $(G \upharpoonright r) \rightleftharpoons L_r$  for every  $r \in R$ , for some  $G, \{L_r\}_{r \in R}$ .  
– Recall  $(\mathcal{G}, \mathcal{S}) \xrightarrow{\dot{\gamma}} (\mathcal{G}', \mathcal{S}')$ . Then,  $(\{G\}, \mathcal{S}) \xrightarrow{\dot{\gamma}} (\mathcal{G}', \mathcal{S}')$ . Then, by Defn. 33,  $\mathcal{S}' = \text{effect}(\dot{\gamma}, \mathcal{S})$  and  $\{G\} \xrightarrow{\xi, \gamma} \mathcal{G}'$  and  $\mathcal{S} \in \llbracket \xi \rrbracket$  and  $\dot{\gamma} = \text{cons}_{\mathcal{S}}(\gamma)$ , for some  $\xi, \gamma$ .  
– Recall  $\{G\} \xrightarrow{\xi, \gamma} \mathcal{G}'$ . Then, by Defn. 32,  $\mathcal{G}' = \{G'\}$  and  $G \xrightarrow{\xi, \gamma} G'$ , for some  $G'$ .  
– Recall  $\mathcal{S} \in \llbracket \Phi(G, \text{true}) \rrbracket$  and  $\mathcal{S} \in \llbracket \xi \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \Phi(G, \text{true}) \rrbracket \cap \llbracket \xi \rrbracket$ . Then, by Defn. 13,  $\mathcal{S} \in \llbracket \Phi(G, \text{true}) \wedge \xi \rrbracket$ .  
– Recall  $\mathcal{S} \in \llbracket \Phi(G, \text{true}) \wedge \xi \rrbracket$ . Then,  $\llbracket \Phi(G, \text{true}) \wedge \xi \rrbracket \neq \emptyset$ .

- Recall  $\sqrt{R}(G)$  and  $\llbracket \phi(G, \text{true}) \wedge \xi \rrbracket \neq \emptyset$  and  $G \xrightarrow{\xi, \gamma} G'$ . Then, by Lem. 89,  $\sqrt{R}(G')$ .
  - Recall  $\sqrt{R}(G)$  and  $G \xrightarrow{\xi, \gamma} G'$ . Then, by Lem. 91,  $\llbracket \phi(G, \text{true}) \wedge \xi \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G', \text{true})) \rrbracket$ .
  - Recall  $\mathcal{S} \in \llbracket \phi(G, \text{true}) \wedge \xi \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G', \text{true})) \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \phi(\gamma, \phi(G', \text{true})) \rrbracket$ . Then, by Lem. 79,  $\text{effect}(\text{con}_{\mathcal{S}}(\gamma), \mathcal{S}) \in \llbracket \phi(G', \text{true}) \rrbracket$ . Then,  $\text{effect}(\dot{\gamma}, \mathcal{S}) \in \llbracket \phi(G', \text{true}) \rrbracket$ .
  - Recall  $G \xrightarrow{\xi, \gamma} G'$ . Then, by Lem. 53,  $\text{subj}(\xi) \cup \text{subj}(\gamma) \cup \text{subj}(G') \subseteq \text{subj}(G)$ . Then,  $\text{subj}(\gamma) \subseteq \text{subj}(G)$ .
  - Recall  $\sqrt{R}(G)$ . Then, by Lem. 47,  $\text{subj}(G) \subseteq R$ .
  - Recall  $\text{subj}(\gamma) \subseteq \text{subj}(G) \subseteq R$ . Then,  $\text{subj}(\gamma) \subseteq R$ .
  - Recall  $\text{subj}(\gamma) \subseteq R$ . Then,  $\text{subj}(\gamma) = R \cap \text{subj}(\gamma)$  and  $R = (R \setminus \text{subj}(\gamma)) \cup \text{subj}(\gamma)$ .
  - Recall  $(G \upharpoonright r) \rightleftharpoons L_r$  for every  $r \in R$ . Then,  $(G \upharpoonright r) \rightleftharpoons L_r$  for every  $r \in R \cap \text{subj}(\gamma)$ . Then,  $(G \upharpoonright r) \rightleftharpoons L_r$  for every  $r \in \text{subj}(\gamma)$ . Then, by Lem. 94,  $L_r \rightleftharpoons (G \upharpoonright r)$  for every  $r \in R \cap \text{subj}(\gamma)$ .
  - Recall  $G \xrightarrow{\xi, \gamma} G'$  and  $\sqrt{R}(G)$ . Then, by Lem. 59,  $(G \upharpoonright r) \xrightarrow{\xi \upharpoonright r, \gamma \upharpoonright r} (G' \upharpoonright r)$  for every  $r \in R \cap \text{subj}(\gamma)$ . Then,  $(G \upharpoonright r) \xrightarrow{\xi \upharpoonright r, \gamma \upharpoonright r} (G' \upharpoonright r)$  for every  $r \in \text{subj}(\gamma)$ .
  - Recall  $L_r \rightleftharpoons (G \upharpoonright r) \xrightarrow{\xi \upharpoonright r, \gamma \upharpoonright r} (G' \upharpoonright r)$  for every  $r \in \text{subj}(\gamma)$ . Then, by Lem. 95,  $L_r \xrightarrow{\xi \upharpoonright r, \gamma \upharpoonright r} L'_r \rightleftharpoons (G' \upharpoonright r)$  for every  $r \in \text{subj}(\gamma)$ , for some  $\{L'_r\}_{r \in \text{subj}(\gamma)}$ .
  - Recall  $(G \upharpoonright r) \rightleftharpoons L_r$  for every  $r \in R$ . Then,  $(G \upharpoonright r) \rightleftharpoons L_r$  for every  $r \in R \setminus \text{subj}(\gamma)$ . Then, by Lem. 94,  $L_r \rightleftharpoons (G \upharpoonright r)$  for every  $r \in R \setminus \text{subj}(\gamma)$ .
  - Recall  $G \xrightarrow{\xi, \gamma} G'$  and  $\sqrt{R}(G)$ . Then, by Lem. 59,  $(G \upharpoonright r) \xrightarrow{\text{true}, \tau} (G' \upharpoonright r)$  for every  $r \in R \setminus \text{subj}(\gamma)$ .
  - Recall  $L_r \rightleftharpoons (G \upharpoonright r) \xrightarrow{\text{true}, \tau} (G' \upharpoonright r)$  for every  $r \in R \setminus \text{subj}(\gamma)$ . Then, by Lem. 95,  $L_r \rightleftharpoons (G' \upharpoonright r)$  for every  $r \in R \setminus \text{subj}(\gamma)$ . Then,  $L'_r = L_r$  and  $L'_r \rightleftharpoons (G' \upharpoonright r)$ , for every  $r \in R \setminus \text{subj}(\gamma)$ , for some  $\{L'_r\}_{r \in R \setminus \text{subj}(\gamma)}$ .
  - Recall  $L'_r \rightleftharpoons (G' \upharpoonright r)$  for every  $r \in \text{subj}(\gamma)$ , and  $L'_r \rightleftharpoons (G' \upharpoonright r)$  for every  $r \in R \setminus \text{subj}(\gamma)$ . Then,  $L'_r \rightleftharpoons (G' \upharpoonright r)$  for every  $r \in (R \setminus \text{subj}(\gamma)) \cup \text{subj}(\gamma)$ . Then,  $L'_r \rightleftharpoons (G' \upharpoonright r)$  for every  $r \in R$ . Then, by Lem. 94,  $(G' \upharpoonright r) \rightleftharpoons L'_r$  for every  $r \in R$ .
  - Recall  $\sqrt{R}(G')$  and  $\text{effect}(\dot{\gamma}, \mathcal{S}) \in \llbracket \phi(G', \text{true}) \rrbracket$ , and  $(G' \upharpoonright r) \rightleftharpoons L'_r$  for every  $r \in R$ . Then, by  $\llbracket \times \rrbracket$ ,  $(\{G'\}, \text{effect}(\dot{\gamma}, \mathcal{S})) \llbracket \times \rrbracket (\{L'_r\}_{r \in R}, \text{effect}(\dot{\gamma}, \mathcal{S}))$ . Then,  $(G', \mathcal{S}') \llbracket \times \rrbracket (\{L'_r\}_{r \in R}, \text{effect}(\dot{\gamma}, \mathcal{S}))$ . Then,  $\mathcal{L}^\blacksquare = \{L'_r\}_{r \in R}$  and  $\mathcal{S}^\blacksquare = \text{effect}(\dot{\gamma}, \mathcal{S})$  and  $(G', \mathcal{S}') \llbracket \times \rrbracket (\mathcal{L}^\blacksquare, \mathcal{S}^\blacksquare)$ , for some  $\mathcal{L}^\blacksquare, \mathcal{S}^\blacksquare$ .
  - Recall  $G \xrightarrow{\xi, \gamma} G'$  and  $\sqrt{R}(G)$ . Then, by Lem. 58,  $\gamma \neq \tau$ .
  - Recall  $L_r \xrightarrow{\xi \upharpoonright r, \gamma \upharpoonright r} L'_r$  for every  $r \in \text{subj}(\gamma) \subseteq R$ , and  $L_r = L'_r$  for every  $r \in R \setminus \text{subj}(\gamma)$ , and  $\gamma \neq \tau$ . Then, by Lem. 68,  $\{L_r\}_{r \in R} \xrightarrow{\bigwedge \{\xi \upharpoonright r\}_{r \in \text{subj}(\gamma)}, \gamma} \{L'_r\}_{r \in R}$ .
  - Recall  $G \xrightarrow{\xi, \gamma} G'$ . Then, by Lem. 53,  $\text{subj}(\xi) \subseteq \text{subj}(\gamma)$ . Then, by Lem. 26,  $\llbracket \xi \rrbracket = \llbracket \bigwedge \{\xi \upharpoonright r\}_{r \in \text{subj}(\gamma)} \rrbracket$ .
  - Recall  $\mathcal{S} \in \llbracket \xi \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \bigwedge \{\xi \upharpoonright r\}_{r \in \text{subj}(\gamma)} \rrbracket$ .
  - Recall  $\{L_r\}_{r \in R} \xrightarrow{\bigwedge \{\xi \upharpoonright r\}_{r \in \text{subj}(\gamma)}, \gamma} \{L'_r\}_{r \in R}$  and  $\mathcal{S} \in \llbracket \bigwedge \{\xi \upharpoonright r\}_{r \in \text{subj}(\gamma)} \rrbracket$  and  $\dot{\gamma} = \text{con}_{\mathcal{S}}(\gamma)$ . Then, by  $\llbracket \rightarrow 3 \rrbracket$ ,  $(\{L_r\}_{r \in R}, \mathcal{S}) \xrightarrow{\dot{\gamma}} (\{L'_r\}_{r \in R}, \text{effect}(\dot{\gamma}, \mathcal{S}))$ . Then  $\mathcal{L}^\dagger = \{L_r\}_{r \in R}$  and  $\mathcal{L}^\ddagger = \{L'_r\}_{r \in R}$  and  $\mathcal{S}^\dagger = \mathcal{S}$  and  $\mathcal{S}^\ddagger = \text{effect}(\dot{\gamma}, \mathcal{S})$  and  $(\mathcal{L}^\dagger, \mathcal{S}^\dagger) \xrightarrow{\dot{\gamma}} (\mathcal{L}^\ddagger, \mathcal{S}^\ddagger)$ , for some  $\mathcal{L}^\dagger, \mathcal{L}^\ddagger, \mathcal{S}^\dagger, \mathcal{S}^\ddagger$ .
  - By  $\llbracket \rightarrow^* 3\text{-BASE} \rrbracket$ ,  $(\{L_r\}_{r \in R}, \mathcal{S}) \xrightarrow{\{\tau\}}^* (\{L_r\}_{r \in R}, \mathcal{S})$  and  $(\{L'_r\}_{r \in R}, \text{effect}(\dot{\gamma}, \mathcal{S})) \xrightarrow{\{\tau\}}^* (\{L'_r\}_{r \in R}, \text{effect}(\dot{\gamma}, \mathcal{S}))$ . Then, by  $\llbracket \Rightarrow 3 \rrbracket$ ,  $(\{L_r\}_{r \in R}, \mathcal{S}) \Rightarrow (\{L_r\}_{r \in R}, \mathcal{S})$  and  $(\{L'_r\}_{r \in R}, \text{effect}(\dot{\gamma}, \mathcal{S})) \Rightarrow (\{L'_r\}_{r \in R}, \text{effect}(\dot{\gamma}, \mathcal{S}))$ . Then,  $(\mathcal{L}, \mathcal{S}) \Rightarrow (\mathcal{L}^\dagger, \mathcal{S}^\dagger)$  and  $(\mathcal{L}^\ddagger, \mathcal{S}^\ddagger) \Rightarrow (\mathcal{L}^\blacksquare, \mathcal{S}^\blacksquare)$ .  $\square$
4. – Recall  $(\mathcal{G}, \mathcal{S}) \llbracket \times \rrbracket (\mathcal{L}, \mathcal{S})$ . Then, by Defn. 41,  $\mathcal{G} = \{G\}$  and  $\mathcal{L} = \{L_r\}_{r \in R}$  and  $\sqrt{R}(G)$ , and  $(G \upharpoonright r) \rightleftharpoons L_r$  for every  $r \in R$ , for some  $G, \{L_r\}_{r \in R}$ .
- Recall  $\sqrt{R}(G)$ . Then, by Lem. 57:
    - **Case:**  $G \downarrow$ .
      - \* By  $\llbracket \rightarrow^* 3\text{-BASE} \rrbracket$ ,  $(\{G\}, \mathcal{S}) \xrightarrow{\{\tau\}}^* (\{G\}, \mathcal{S})$ . Then, by  $\llbracket \Rightarrow 3 \rrbracket$ ,  $(\{G\}, \mathcal{S}) \Rightarrow (\{G\}, \mathcal{S})$ . Then,  $(\mathcal{G}, \mathcal{S}) \Rightarrow (\{G\}, \mathcal{S})$ . Then,  $\mathcal{G}^\dagger = \{G\}$  and  $\mathcal{S}^\dagger = \mathcal{S}$  and  $(\mathcal{G}, \mathcal{S}) \Rightarrow (\mathcal{G}^\dagger, \mathcal{S}^\dagger)$ , for some  $\mathcal{G}^\dagger, \mathcal{S}^\dagger$ .
      - \* Recall  $G \downarrow$ . Then, by  $\llbracket \downarrow 2\text{-GLOB} \rrbracket$ ,  $\{G\} \downarrow$ . Then, by  $\llbracket \downarrow 3 \rrbracket$ ,  $(\{G\}, \mathcal{S}) \downarrow$ . Then,  $(\mathcal{G}^\dagger, \mathcal{S}^\dagger) \downarrow$ .
    - **Case:**  $G \xrightarrow{\xi, \gamma} G'$ , for some  $G', \xi, \gamma$ .

- \* Recall  $G \xrightarrow{\xi, \gamma} G'$  and  $\checkmark_R(G)$ . Then, by Lem. 58,  $\text{subj}(\gamma) \neq \emptyset$ . Then,  $\hat{r} \in \text{subj}(\gamma)$ , for some  $\hat{r}$ .
  - \* Recall  $G \xrightarrow{\xi, \gamma} G'$ . Then, by Lem. 53,  $\text{subj}(\xi) \cup \text{subj}(\gamma) \cup \text{subj}(G') \subseteq \text{subj}(G)$ . Then,  $\text{subj}(\gamma) \subseteq \text{subj}(G)$ .
  - \* Recall  $\checkmark_R(G)$ . Then, by Lem. 47,  $\text{subj}(G) \subseteq R$ .
  - \* Recall  $\hat{r} \in \text{subj}(\gamma) \subseteq \text{subj}(G) \subseteq R$ . Then,  $\hat{r} \in R$ .
  - \* Recall  $(\mathcal{L}, \mathcal{S}) \downarrow$ . Then,  $(\{L_r\}_{r \in R}, \mathcal{S}) \downarrow$ . Then, by Defn. 30,  $\{L_r\}_{r \in R} \downarrow$ . Then, by Defn. 29,  $L_r \downarrow$  for every  $r \in R$ .
  - \* Recall  $\hat{r} \in R$ . Then,  $L_{\hat{r}} \downarrow$ .
  - \* Recall  $\hat{r} \in R$ . Then,  $(G \upharpoonright \hat{r}) \Rightarrow \Leftarrow L_{\hat{r}}$ . Then, by Lem. 94,  $L_{\hat{r}} \Rightarrow \Leftarrow (G \upharpoonright \hat{r})$ .
  - \* Recall  $\hat{r} \in R$  and  $\hat{r} \in \text{subj}(\gamma)$ . Then,  $\hat{r} \in R \cap \text{subj}(\gamma)$ .
  - \* Recall  $G \xrightarrow{\xi, \gamma} G'$  and  $\checkmark_R(G)$  and  $\hat{r} \in R \cap \text{subj}(\gamma)$ . Then, by Lem. 59,  $(G \upharpoonright \hat{r}) \xrightarrow{\xi \upharpoonright \hat{r}, \gamma \upharpoonright \hat{r}} (G' \upharpoonright \hat{r})$ .
  - \* Recall  $L_{\hat{r}} \Rightarrow \Leftarrow (G \upharpoonright \hat{r}) \xrightarrow{\xi \upharpoonright \hat{r}, \gamma \upharpoonright \hat{r}} (G' \upharpoonright \hat{r})$ . Then, by Lem. 95,  $L_{\hat{r}} \xrightarrow{\xi \upharpoonright \hat{r}, \gamma \upharpoonright \hat{r}} L'_{\hat{r}}$ , for some  $L'_{\hat{r}}$ .
  - \* Recall  $L_{\hat{r}} \downarrow$  and  $L_{\hat{r}} \xrightarrow{\xi \upharpoonright \hat{r}, \gamma \upharpoonright \hat{r}} L'_{\hat{r}}$ . Then, by Lem. 55,  $\gamma \upharpoonright \hat{r} = \tau$ .
  - \* Recall  $\hat{r} \in \text{subj}(\gamma)$ . Then, by Lem. 33,  $\text{subj}(\gamma \upharpoonright \hat{r}) = \{\hat{r}\}$ . Then,  $\text{subj}(\tau) = \{\hat{r}\}$ . Then, by Defn. 16,  $\emptyset = \{\hat{r}\}$ . Then, **false**.  $\square$
5. – Recall  $(\mathcal{G}, \mathcal{S}) \boxtimes (\mathcal{L}, \mathcal{S})$ . Then, by Defn. 41,  $\mathcal{G} = \{G\}$  and  $\mathcal{L} = \{L_r\}_{r \in R}$  and  $\checkmark_R(G)$  and  $\mathcal{S} \in \llbracket \phi(G, \text{true}) \rrbracket$ , and  $(G \upharpoonright r) \Rightarrow \Leftarrow L_r$  for every  $r \in R$ , for some  $G, \{L_r\}_{r \in R}$ .
- Recall  $(\mathcal{L}, \mathcal{S}) \xrightarrow{\xi, \gamma} (\mathcal{L}', \mathcal{S}')$ . Then,  $(\{L_r\}_{r \in R}, \mathcal{S}) \xrightarrow{\xi, \gamma} (\mathcal{L}', \mathcal{S}')$ . Then, by Defn. 33,  $\{L_r\}_{r \in R} \xrightarrow{\xi, \gamma} \mathcal{L}'$  and  $\tau = \text{con}_{\mathcal{S}}(\gamma)$ , for some  $\xi, \gamma$ .
  - Recall  $\tau = \text{con}_{\mathcal{S}}(\gamma)$ . Then, by Defn. 19,  $\gamma = \tau$ .
  - Recall  $\{L_r\}_{r \in R} \xrightarrow{\xi, \tau} \mathcal{L}'$ . Then,  $\{L_r\}_{r \in R} \xrightarrow{\xi, \tau} \mathcal{L}'$ . Then, by Defn. 32,  $\mathcal{L}' = \{L'_r\}_{r \in R}$  and  $\{L_r\}_{r \in R} \xrightarrow{\xi, \tau} \{L'_r\}_{r \in R}$ , for some  $\{L'_r\}_{r \in R}$ .
  - Recall  $\{L_r\}_{r \in R} \xrightarrow{\xi, \tau} \{L'_r\}_{r \in R}$ . Then, by Lem. 67,  $L_{\hat{r}} \xrightarrow{\xi, \tau} L'_{\hat{r}}$  and  $\hat{r} \in R$ , and  $L_r = L'_r$  for every  $r \in R \setminus \{\hat{r}\}$ , for some  $\hat{r}$ .
  - Recall  $\hat{r} \in R$ . Then,  $(G \upharpoonright \hat{r}) \Rightarrow \Leftarrow L_{\hat{r}}$ .
  - Recall  $L_{\hat{r}} \xrightarrow{\xi, \tau} L'_{\hat{r}}$  and  $\gamma = \tau$ . Then, by Lem. 50,  $\xi = \text{true}$ .
  - Recall  $L_{\hat{r}} \xrightarrow{\xi, \tau} L'_{\hat{r}}$ . Then,  $L_{\hat{r}} \xrightarrow{\text{true}, \tau} L'_{\hat{r}}$ .
  - Recall  $(G \upharpoonright \hat{r}) \Rightarrow \Leftarrow L_{\hat{r}} \xrightarrow{\text{true}, \tau} L'_{\hat{r}}$ . Then, by Lem. 95,  $(G \upharpoonright \hat{r}) \Rightarrow \Leftarrow L'_{\hat{r}}$ .
  - Recall  $(G \upharpoonright r) \Rightarrow \Leftarrow L_r$  for every  $r \in R$ . Then,  $(G \upharpoonright r) \Rightarrow \Leftarrow L_r$  for every  $r \in R \setminus \{\hat{r}\}$ . Then,  $(G \upharpoonright r) \Rightarrow \Leftarrow L'_r$  for every  $r \in R \setminus \{\hat{r}\}$ .
  - Recall  $\hat{r} \in R$ . Then,  $R = (R \setminus \{\hat{r}\}) \cup \{\hat{r}\}$ .
  - Recall  $(G \upharpoonright \hat{r}) \Rightarrow \Leftarrow L'_{\hat{r}}$ , and  $(G \upharpoonright r) \Rightarrow \Leftarrow L'_r$  for every  $r \in R \setminus \{\hat{r}\}$ . Then,  $(G \upharpoonright r) \Rightarrow \Leftarrow L'_r$  for every  $r \in (R \setminus \{\hat{r}\}) \cup \{\hat{r}\}$ . Then,  $(G \upharpoonright r) \Rightarrow \Leftarrow L'_r$  for every  $r \in R$ .
  - Recall  $\checkmark_R(G)$  and  $\mathcal{S} \in \llbracket \phi(G, \text{true}) \rrbracket$ , and  $(G \upharpoonright r) \Rightarrow \Leftarrow L'_r$  for every  $r \in R$ . Then, by  $\llbracket \boxtimes \rrbracket$ ,  $\{G\} \boxtimes \{L'_r\}_{r \in R}$ . Then,  $\mathcal{G} \boxtimes \mathcal{L}'$ .  $\square$
6. – Recall  $(\mathcal{G}, \mathcal{S}) \boxtimes (\mathcal{L}, \mathcal{S})$ . Then, by Defn. 41,  $\mathcal{G} = \{G\}$  and  $\mathcal{L} = \{L_r\}_{r \in R}$  and  $\checkmark_R(G)$  and  $\mathcal{S} \in \llbracket \phi(G, \text{true}) \rrbracket$ , and  $(G \upharpoonright r) \Rightarrow \Leftarrow L_r$  for every  $r \in R$ , for some  $G, \{L_r\}_{r \in R}$ .
- Recall  $(\mathcal{L}, \mathcal{S}) \xrightarrow{\dot{\gamma}} (\mathcal{L}', \mathcal{S}')$ . Then,  $(\{L_r\}_{r \in R}, \mathcal{S}) \xrightarrow{\dot{\gamma}} (\mathcal{L}', \mathcal{S}')$ . Then, by Defn. 33,  $\mathcal{S}' = \text{effect}(\dot{\gamma}, \mathcal{S})$  and  $\{L_r\}_{r \in R} \xrightarrow{\xi_1, \dot{\gamma}} \mathcal{L}'$  and  $\mathcal{S} \in \llbracket \xi_1 \rrbracket$  and  $\dot{\gamma} = \text{con}_{\mathcal{S}}(\gamma)$ , for some  $\xi_1, \gamma$ .
  - Recall  $\{L_r\}_{r \in R} \xrightarrow{\xi_1, \dot{\gamma}} \mathcal{L}'$ . Then, by Defn. 32,  $\mathcal{L}' = \{L'_{1r}\}_{r \in R}$  and  $\{L_r\}_{r \in R} \xrightarrow{\xi_1, \dot{\gamma}} \{L'_{1r}\}_{r \in R}$ , for some  $\{L'_{1r}\}_{r \in R}$ .
  - Recall  $\{L_r\}_{r \in R} \xrightarrow{\xi_1, \dot{\gamma}} \{L'_{1r}\}_{r \in R}$ . Then, by Lem. 66,  $\text{subj}(\dot{\gamma}) \subseteq R$ . Then,  $\text{subj}(\gamma) = R \cap \text{subj}(\dot{\gamma})$  and  $R = (R \setminus \text{subj}(\dot{\gamma})) \cup \text{subj}(\dot{\gamma})$ .
  - Recall  $(G \upharpoonright r) \Rightarrow \Leftarrow L_r$  for every  $r \in R$ . Then,  $(G \upharpoonright r) \Rightarrow \Leftarrow L_r$  for every  $r \in R \cap \text{subj}(\dot{\gamma})$ . Then,  $(G \upharpoonright r) \Rightarrow \Leftarrow L_r$  for every  $r \in \text{subj}(\dot{\gamma})$ .
  - Recall  $\dot{\gamma} \neq \tau$ . Then,  $\text{con}_{\mathcal{S}}(\gamma) \neq \tau$ . Then, by Defn. 19,  $\gamma \neq \tau$ .
  - Recall  $\{L_r\}_{r \in R} \xrightarrow{\xi_1, \dot{\gamma}} \{L'_{1r}\}_{r \in R}$  and  $\gamma \neq \tau$ . Then, by Lem. 67,  $L_r \xrightarrow{\xi_{1r}, \gamma \upharpoonright r} L'_{1r}$  for every  $r \in \text{subj}(\dot{\gamma})$ , and  $\xi_1 = \bigwedge \{\xi_{1r}\}_{r \in \text{subj}(\dot{\gamma})}$ , and  $L_r = L'_{1r}$  for every  $r \in R \setminus \text{subj}(\dot{\gamma})$ , for some  $\{\xi_{1r}\}_{r \in \text{subj}(\dot{\gamma})}$ .
  - Recall  $(G \upharpoonright r) \Rightarrow \Leftarrow L_r \xrightarrow{\xi_{1r}, \gamma \upharpoonright r} L'_{1r}$ , for every  $r \in \text{subj}(\dot{\gamma})$ . Then, by Lem. 95,  $(G \upharpoonright r) \xrightarrow{\xi_{1r}, \gamma \upharpoonright r} L'_{2r} \Rightarrow \Leftarrow L'_{1r}$ , for every  $r \in \text{subj}(\dot{\gamma})$ , for some  $\{L'_{2r}\}_{r \in \text{subj}(\dot{\gamma})}$ .

- Recall  $(G \upharpoonright r) \xrightarrow{\xi_{1r}, \gamma \upharpoonright r} L'_{2r}$  for every  $r \in \text{subj}(\gamma)$  and  $\sqrt{R}(G)$  and  $\gamma \neq \tau$ . Then, by Lem. 64,  $G \xrightarrow{\xi_2, \gamma} G'$ , and  $L'_{2r} = G' \upharpoonright r$  for every  $r \in \text{subj}(\gamma)$ , and  $\xi_{1r} = \xi_2 \upharpoonright r$  for every  $r \in \text{subj}(\gamma)$ , for some  $G', \xi_2$ .
- Recall  $G \xrightarrow{\xi_2, \gamma} G'$ . Then, by Lem. 53,  $\text{subj}(\xi_2) \subseteq \text{subj}(\gamma)$ . Then, by Lem. 26,  $\llbracket \xi_2 \rrbracket = \llbracket \bigwedge \{ \xi_2 \upharpoonright r \}_{r \in \text{subj}(\gamma)} \rrbracket$ . Then,  $\llbracket \xi_2 \rrbracket = \llbracket \bigwedge \{ \xi_{1r} \}_{r \in \text{subj}(\gamma)} \rrbracket$ . Then,  $\llbracket \xi_2 \rrbracket = \llbracket \xi_1 \rrbracket$ .
- Recall  $\mathcal{S} \in \llbracket \xi_1 \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \xi_2 \rrbracket$ .
- Recall  $\mathcal{S} \in \llbracket \phi(G, \text{true}) \rrbracket$  and  $\mathcal{S} \in \llbracket \xi_2 \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \phi(G, \text{true}) \rrbracket \cap \llbracket \xi_2 \rrbracket$ . Then, by Defn. 13,  $\mathcal{S} \in \llbracket \phi(G, \text{true}) \wedge \xi_2 \rrbracket$ .
- Recall  $\mathcal{S} \in \llbracket \phi(G, \text{true}) \wedge \xi_2 \rrbracket$ . Then,  $\llbracket \phi(G, \text{true}) \wedge \xi_2 \rrbracket \neq \emptyset$ .
- Recall  $\sqrt{R}(G)$  and  $\llbracket \phi(G, \text{true}) \wedge \xi_2 \rrbracket \neq \emptyset$  and  $G \xrightarrow{\xi_2, \gamma} G'$ . Then, by Lem. 89,  $\sqrt{R}(G')$ .
- Recall  $\sqrt{R}(G)$  and  $G \xrightarrow{\xi_2, \gamma} G'$ . Then, by Lem. 91,  $\llbracket \phi(G, \text{true}) \wedge \xi_2 \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G', \text{true})) \rrbracket$ .
- Recall  $\mathcal{S} \in \llbracket \phi(G, \text{true}) \wedge \xi_2 \rrbracket \subseteq \llbracket \phi(\gamma, \phi(G', \text{true})) \rrbracket$ . Then,  $\mathcal{S} \in \llbracket \phi(\gamma, \phi(G', \text{true})) \rrbracket$ . Then, by Lem. 79,  $\text{effect}(\text{con}_{\mathcal{S}}(\gamma), \mathcal{S}) \in \llbracket \phi(G', \text{true}) \rrbracket$ . Then,  $\text{effect}(\dot{\gamma}, \mathcal{S}) \in \llbracket \phi(G', \text{true}) \rrbracket$ .
- Recall  $L'_{2r} \Rightarrow_{\neq} L'_{1r}$  for every  $r \in \text{subj}(\gamma)$ . Then,  $(G' \upharpoonright r) \Rightarrow_{\neq} L'_{1r}$  for every  $r \in \text{subj}(\gamma)$ .
- Recall  $(G \upharpoonright r) \Rightarrow_{\neq} L_r$  for every  $r \in R$ . Then,  $(G \upharpoonright r) \Rightarrow_{\neq} L_r$  for every  $r \in R \setminus \text{subj}(\gamma)$ . Then, by Lem. 94,  $L_r \Rightarrow_{\neq} (G \upharpoonright r)$  for every  $r \in R \setminus \text{subj}(\gamma)$ .
- Recall  $G \xrightarrow{\xi_2, \gamma} G'$  and  $\sqrt{R}(G)$ . Then, by Lem. 59,  $(G \upharpoonright r) \xrightarrow{\text{true}, \tau} (G' \upharpoonright r)$  for every  $r \in R \setminus \text{subj}(\gamma)$ .
- Recall  $L_r \Rightarrow_{\neq} (G \upharpoonright r) \xrightarrow{\text{true}, \tau} (G' \upharpoonright r)$  for every  $r \in R \setminus \text{subj}(\gamma)$ . Then, by Lem. 95,  $L_r \Rightarrow_{\neq} (G' \upharpoonright r)$  for every  $r \in R \setminus \text{subj}(\gamma)$ . Then, by Lem. 94,  $(G' \upharpoonright r) \Rightarrow_{\neq} L_r$  for every  $r \in R \setminus \text{subj}(\gamma)$ . Then,  $(G' \upharpoonright r) \Rightarrow_{\neq} L'_{1r}$  for every  $r \in R \setminus \text{subj}(\gamma)$ .
- Recall  $(G' \upharpoonright r) \Rightarrow_{\neq} L'_{1r}$  for every  $r \in \text{subj}(\gamma)$ , and  $(G' \upharpoonright r) \Rightarrow_{\neq} L'_{1r}$  for every  $r \in R \setminus \text{subj}(\gamma)$ . Then,  $(G' \upharpoonright r) \Rightarrow_{\neq} L'_{1r}$  for every  $r \in (R \setminus \text{subj}(\gamma)) \cup \text{subj}(\gamma)$ . Then,  $(G' \upharpoonright r) \Rightarrow_{\neq} L'_{1r}$  for every  $r \in R$ .
- Recall  $\sqrt{R}(G')$  and  $\text{effect}(\dot{\gamma}, \mathcal{S}) \in \llbracket \phi(G', \text{true}) \rrbracket$ , and  $(G' \upharpoonright r) \Rightarrow_{\neq} L'_{1r}$  for every  $r \in R$ . Then, by  $\llbracket \bowtie \rrbracket$ ,  $(\{G'\}, \text{effect}(\dot{\gamma}, \mathcal{S})) \bowtie (\{L'_{1r}\}_{r \in R}, \text{effect}(\dot{\gamma}, \mathcal{S}))$ . Then,  $(\{G'\}, \text{effect}(\dot{\gamma}, \mathcal{S})) \bowtie (\mathcal{L}', \mathcal{S}')$ . Then,  $\mathcal{G}^{\blacksquare} = \{G'\}$  and  $\mathcal{S}^{\blacksquare} = \text{effect}(\dot{\gamma}, \mathcal{S})$  and  $(\mathcal{G}^{\blacksquare}, \mathcal{S}^{\blacksquare}) \bowtie (\mathcal{L}', \mathcal{S}')$ , for some  $\mathcal{G}^{\blacksquare}, \mathcal{S}^{\blacksquare}$ .
- Recall  $G \xrightarrow{\xi_2, \gamma} G'$ . Then, by  $\llbracket \rightarrow 2\text{-GLOB} \rrbracket$ ,  $\{G\} \xrightarrow{\xi_2, \gamma} \{G'\}$ .
- Recall  $\{G\} \xrightarrow{\xi_2, \gamma} \{G'\}$  and  $\mathcal{S} \in \llbracket \xi_2 \rrbracket$  and  $\dot{\gamma} = \text{con}_{\mathcal{S}}(\gamma)$ . Then, by  $\llbracket \rightarrow 3 \rrbracket$ ,  $(\{G\}, \mathcal{S}) \xrightarrow{\dot{\gamma}} (\{G'\}, \text{effect}(\dot{\gamma}, \mathcal{S}))$ . Then,  $\mathcal{G}^{\dagger} = \{G\}$  and  $\mathcal{G}^{\ddagger} = \{G'\}$  and  $\mathcal{S}^{\dagger} = \mathcal{S}$  and  $\mathcal{S}^{\ddagger} = \text{effect}(\dot{\gamma}, \mathcal{S})$  and  $(\mathcal{G}^{\dagger}, \mathcal{S}^{\dagger}) \xrightarrow{\dot{\gamma}} (\mathcal{G}^{\ddagger}, \mathcal{S}^{\ddagger})$ , for some  $\mathcal{G}^{\dagger}, \mathcal{G}^{\ddagger}, \mathcal{S}^{\dagger}, \mathcal{S}^{\ddagger}$ .
- By  $\llbracket \rightarrow * 3\text{-BASE} \rrbracket$ ,  $(\{G\}, \mathcal{S}) \xrightarrow{\{\tau\}^*} (\{G\}, \mathcal{S})$  and  $(\{G'\}, \text{effect}(\dot{\gamma}, \mathcal{S})) \xrightarrow{\{\tau\}^*} (\{G'\}, \text{effect}(\dot{\gamma}, \mathcal{S}))$ . Then, by  $\llbracket \Rightarrow 3 \rrbracket$ ,  $(\{G\}, \mathcal{S}) \Rightarrow (\{G\}, \mathcal{S})$  and  $(\{G'\}, \text{effect}(\dot{\gamma}, \mathcal{S})) \Rightarrow (\{G'\}, \text{effect}(\dot{\gamma}, \mathcal{S}))$ . Then,  $(\mathcal{G}, \mathcal{S}) \Rightarrow (\mathcal{G}^{\dagger}, \mathcal{S}^{\dagger})$  and  $(\mathcal{G}^{\ddagger}, \mathcal{S}^{\ddagger}) \Rightarrow (\mathcal{G}^{\blacksquare}, \mathcal{S}^{\blacksquare})$ .  $\square$

*Proof (of Thm. 3).* Recall  $\sqrt{R}(\mathcal{G})$  and  $\mathcal{S} \in \llbracket \phi(\mathcal{G}, \text{true}) \rrbracket$ . Then, by Lem. 96,  $(\mathcal{G}, \mathcal{S}) \bowtie (\mathcal{G} \upharpoonright R, \mathcal{S})$ . Then, by Cor. 1,  $(\mathcal{G}, \mathcal{S}) \approx (\mathcal{G} \upharpoonright R, \mathcal{S})$ .  $\square$