Advanced Research in Mathematics and Computer Science; Doctoral Conference in Mathematics, Informatics and Education [MIE 2014] Proceedings

Advanced Research in Mathematics and Computer Science

ST. KLIMENT OHRIDSKI UNIVERSITY PRESS

Advanced Research in Mathematics and Computer Science

Doctoral Conference
in Mathematics, Informatics
and Education
[MIE 2014]
Sofia, Bulgaria, 2014, Proceedings

St. Kliment Ohridski University Press
Sofia • 2014

ISBN 978-954-07-3759-1

Preface

The conference proceedings are result from the doctoral conference MIE 2014: Doctoral Conference in Mathematics, Informatics and Education, held in September 23–24, 2014, at Sofia, Bulgaria. This conference was organised as part of the activities in the project under the Human resources development scheme in Bulgaria, aiming to support doctoral students, post doctoral lecturers and young scientists.

MIE 2014 is a doctoral student conference for researchers in Mathematics and Computer Science to connect with international research communities for the worldwide dissemination and sharing of research ideas and results.

Three coherently interrelated tracks were arranged in the two-days conference including Mathematics, Informatics, and Technology enhanced learning. Young researchers and post-doctoral students participated in paper presentations, doctoral student consortia and panel discussions under the themes of the conference tracks.

More information about the conference is available on the conference web site: `http://mie.uni-sofia.bg/`.

The papers in the proceedings are organised in three sections, according to the corresponding track.

September 2014

Peter Sloep
Program Committee Chair
MIE2014

Organization

MIE2014 is organized by the Faculty of Mathematics and Informatics, Sofia University "St. Kliment Ohridski".

Committees

Program Chair

Peter Sloep, Open University, The Netherlands

Referred Papers Track Chairs

- Alexandra Soskova, Sofia University, Bulgaria (Mathematics)
- Ivan Koychev, Sofia University, Bulgaria (Informatics)
- Pavel Boytchev, Sofia University, Bulgaria (Technology Enhanced Learning)

Program Committee

- Galia Angelova, Bulgarian Academy of Science, Bulgaria
- Marusia Bojkova, Sofia University, Bulgaria
- Boyan Bontchev, Sofia University, Bulgaria
- Tatiana Chernogorova, Sofia University, Bulgaria
- Yannis Dimitriadis, University of Valiadolid, Spain
- Christo Dichev, Winston-Salem State University, USA
- Vladimir Dimitrov, Sofia University, Bulgaria
- Milena Dobreva, University of Strathclyde, UK
- Gabriella Dodero, Free University of Bozen-Bolzano, Italy
- Asen Dontchev, University of Michigan, USA
- Hendrik Drachsler, Open University, The Netherlands
- Jon Dron, Athabaska University, Canada
- Martin Ebner, Technical University Graz, Austria
- Hristo Ganchev, Sofia University, Bulgaria
- Pando Georgiev, University of Florida, USA
- Vassil Georgiev, Sofia University, Bulgaria
- Olga Georgieva, Sofia University, Bulgaria
- Alexander Grigorov, Sofia University, Bulgaria
- Elissaveta Gourova, Sofia University, Bulgaria
- Sylvia Ilieva, Sofia University, Bulgaria
- Stefan Ivanov, Sofia University, Bulgaria
- Jordan Jordanov, Sofia University, Bulgaria
- Francisco José García Peñalvo, University of Salamanka, Spain
- Kalinka Kaloyanova, Sofia University, Bulgaria

- Valentin Kisimov, University of National and World Economy, Bulgaria
- Piet Kommers, University of Twente, The Netherlands
- Rob Koper, Open University, The Netherlands
- Evgeniy Krastev, Sofia University, Bulgaria
- Miloš Kravčík, Aachen University, Germany
- Barry Cooper, Leeds University, UK
- Mart Laanpere, Tallinn University, Estonia
- Plamen Mateev, Sofia University, Bulgaria
- Leda Minkova, Sofia University, Bulgaria
- Preslav Nakov, Bulgarian Academy of Science, Bulgaria
- Geno Nikolov, Sofia University, Bulgaria
- Maria Nisheva, Sofia University, Bulgaria
- Toshio Okamoto, UEC, Japan
- Edward Omey, KU Leuven, Belgium
- Radoslav Pavlov, Bulgarian Academy of Science, Bulgaria
- Gergana Petrova, Texas A&M University, USA
- Nedyu Popivanov, Sofia University, Bulgaria
- Atanas Radenski, Chapman University, USA
- Petko Ruskov, Sofia University, Bulgaria
- Simos Retalis, University of Piraeus, Greece
- Julian Revalski, Bulgarian Academy of Science, Bulgaria
- Dimitar Skordev, Sofia University, Bulgaria
- Slavi Stoyanov, Open University, The Netherlands
- Stefaan Ternier, Open University, The Netherlands
- Tinko Tinchev, Sofia University, Bulgaria
- Julita Vasileva, University of Saskatchewan, Canada
- Wim Westera, Open University, The Netherlands
- Vladimir Zanev, Columbus State University, USA
- Goran Velinov, Ss. Cyril and Methodius University in Skopje, FYROM

Local Organizing Committee

Chair

Krassen Stefanov

Members

- Stefka Blizneva
- Atanas Georgiev
- Pencho Mihnev
- Eliza Stefanova
- Temenuzhka Zafirova-Maltcheva

Table of Contents

IV Technology Enhanced Learning

Part I

Mathematics

On Certain Asymptotically Optimal Quadrature Formulae

Ana Avdzhieva and Geno Nikolov

Faculty of Mathematics and Informatics, Sofia University "St. Kliment Ohridski",
5, James Bourchier Blvd., 1164 Sofia, Bulgaria
{aavdzhieva,geno}@fmi.uni-sofia.bg

Abstract. We construct certain quadrature formulae, which are asymptotically optimal in the Sobolev classes W_p^4, $p = \infty$ and $p = 2$. Sharp error bounds for these quadrature formulae are given.

Keywords: quadrature formulae, asymptotically optimal quadrature formulae, spline functions, Peano kernels, Euler–MacLaurin summation formulae

1 Introduction

We study quadrature formulae of the type

$$Q[f] = \sum_{i=1}^{n} a_i\, f(x_i), \quad 0 \le x_1 < x_2 < \cdots < x_n \le 1, \tag{1.1}$$

that serve as an estimate for the definite integral

$$I[f] := \int_0^1 f(x)\, dx. \tag{1.2}$$

Throughout this paper π_k will stand for the set of algebraic polynomials of degree not exceeding k.

The classical approach for construction of quadrature formulae is based on the concept of *algebraic degree of precision.* The quadrature formula (1.1) is said to have algebraic degree of precision m (in short, $\mathrm{ADP}(Q) = m$), if its remainder

$$R[Q; f] := I[f] - Q[f]$$

vanishes whenever $f \in \pi_m$, and $R[Q; f] \neq 0$ when f is a polynomial of degree $m + 1$.

The ADP-concept is justified by the Weierstrass theorem about the density of algebraic polynomials in the space of continuous functions on compacts. The pursuit of quadrature formulae (1.1) with the highest possible ADP leads to the well-known quadrature formulae of Gauss, Radau and Lobatto. The latter are uniquely determined by having ADP $2n - 1$, $2n - 2$ and $2n - 3$, respectively, where, in addition, the Radau quadrature formula has one fixed node being endpoint of the integration interval, and the Lobatto quadrature formula has two fixed nodes at the ends of the integration interval.

An alternative concept for evaluation of the quality of quadrature formulae emerged in the forties of the 20th century, namely, the concept of optimality in a given class of functions. Its founders are A. Kolmogorov, A. N. Sard and S. M. Nikolskii. Let us briefly describe the setting of optimal quadrature formulae in a given class of functions.

For a given normed linear space X of functions defined in $[0,1]$, let $\mathcal{E}(Q,X)$ be the largest possible error of a quadrature formula Q of the form (1.1) in the unit ball of X,

$$\mathcal{E}(Q,X) := \sup_{\|f\|_X \le 1} |R[Q;f]|.$$

We look for the best possible choice of the coefficients $\{a_i\}$ and the nodes $\{x_i\}$, and set

$$\mathcal{E}_n(X) := \inf_Q \mathcal{E}(Q,X).$$

If the quantity $\mathcal{E}_n(X)$ is attained for a quadrature formula $Q^{\rm opt}$ of the form (1.1), then $Q^{\rm opt}$ is said to be *optimal quadrature formula* of the type (1.1) in the space X. Of particular interest is the case when X is some of the Sobolev classes of functions

$$\begin{aligned} \widetilde{W}_p^r &:= \{f \in C^{r-1}[0,1],\ f - \text{1-periodic},\ f^{(r-1)} \ \text{abs. cont.},\ \|f\|_p < \infty\}, \\ W_p^r[0,1] &:= \{f \in C^{r-1}[0,1],\ f^{(r-1)} \ \text{abs. cont.},\ \|f\|_p < \infty\}, \end{aligned}$$

where

$$\|f\|_p := \Big(\int_0^1 |f(t)|^p\,dt\Big)^{1/p}, \quad \text{if} \ \ 1 \le p < \infty, \quad \text{and} \quad \|f\|_\infty = \sup_{t\in(0,1)} \text{vrai}\,|f(t)|.$$

For the sake of brevity, henceforth we shall write shortly W_p^r instead of $W_p^r[0,1]$. It is well-known that in all periodic Sobolev classes $\widetilde{W}_p^r$ the rectangles quadrature formula and its translates are the only optimal quadrature formulae. This is a result of Zhensykbaev [13], special cases have been obtained earlier by Motornii [9], and Ligun [8]. The existence and uniqueness of optimal quadrature formulae in the non-periodic Sobolev spaces W_p^r is equivalent to existence and uniqueness of monosplines of degree r of minimal L_q-deviation $(1/p + 1/q = 1)$. This result is due to Zhensykbaev [14] and, in a more general setting, for quadrature formulae involving derivatives of the integrand, to Bojanov [2, 3]. Obviously, $\mathcal{E}_n(\widetilde{W}_p^r) \le \mathcal{E}_n(W_p^r)$, and it is known that (see Brass [5]) for $1 < p \le \infty$,

$$\lim_{n\to\infty} \frac{\mathcal{E}_n(\widetilde{W}_p^r)}{\mathcal{E}_n(W_p^r)} = 1.$$

A drawback of the optimality concept is that, in general, the explicit form of the optimal quadrature formulae is unknown, a fact that vitiates their importance from practical point of view. In particular, except for some special cases of

$r = 1$ and $r = 2$, the optimal quadrature formulae in the non-periodic Sobolev spaces W_p^r are unknown.

The way out of this situation is to step back from the requirement for optimality, and to look for quadrature formulae which are nearly optimal. A sequence $\{Q_n\}$ of quadrature formulae is said to be asymptotically optimal in the function class X, if

$$\lim_{n\to\infty} \frac{\mathcal{E}(Q_n, X)}{\mathcal{E}_n(X)} = 1$$

(here, Q_n is supposed to be a quadrature formula with n nodes).

It has been shown in [7] that the Gauss-type quadrature formulae associated with spaces of spline functions with equidistant knots are asymptotically optimal in the non-periodic Sobolev classes W_p^r. The existence and uniqueness of such Gauss-type quadrature formulae is equivalent to the fundamental theorem of algebra for monosplines satisfying zero boundary conditions was settled in [6]. This fact was a motivation for investigation of such quadratures. Algorithms for the construction along with sharp error estimates of the Gauss-type quadrature formulae associated with spaces of linear and parabolic spline functions were proposed in [10] and [12] (see also [11] for the case of cubic splines with double equidistant knots). Recently, an algorithm for the construction of Gaussian quadrature formulae associated with spaces of cubic splines with equidistant knots was proposed in [1].

It should be noted that the complexity of the algorithms for the construction of Gauss-type quadrature formulae associated with spaces of spline functions with equidistant knots increases with increasing of the degree (that is, of r in W_p^r). For $r \geq 3$ such quadratures are constructed only numerically. This requires high accuracy computations especially when the number of the nodes is large. An additional difficulty causes the fact that the mutual location of the spline knots and the quadratures nodes is unknown.

In the present paper we propose an alternative approach for the construction of asymptotically optimal quadrature formulae in the Sobolev spaces W_p^4. Our approach is based on two Euler–MacLaurin-type summation formulae, in which derivatives are replaced by formulae for numerical differentiation. An advantage of our quadrature formulae, besides their asymptotical optimality, is the explicit form of their weights and nodes. In fact, the set of nodes of each of our quadrature formulae consists of either those of the compound trapezium or those of the compound midpoint quadratures, to which are added at most four additional nodes.

The paper is organized as follows. Section 2 provides some requisites, including the Peano kernel representation of linear functionals, the Bernoulli polynomials, monosplines and numbers, the Euler–MacLaurin-type expansion formulae, and the error representation of the compound trapezium and midpoint quadrature formulae in the periodic Sobolev classes $\widetilde{W}_p^r$. In Section 3 we describe our approach for the construction of asymptotically optimal quadrature formulae in the non-periodic Sobolev classes W_p^4, $p = \infty$ and $p = 2$. These formulae are given in Sect. 4, along with their sharp error estimates. Section 5 contains some concluding remarks.

2 Requisites

2.1 Spline Functions and Peano Kernels of Linear Functionals

A spline function of degree $r-1$ ($r \in \mathbb{N}$) with knots $x_1 < x_2 < \cdots < x_n$ is called every function $s(t)$, satisfying the requirements

1) $s(t)_{|t\in(x_i,x_{i+1})} \in \pi_{r-1}$, $i = 0, \ldots, n$,
2) $s(t) \in C(\mathbb{R})$,

where $x_0 := -\infty$ and $x_{n+1} := \infty$. The set $S_{r-1}(x_1, \ldots, x_n)$ of spline functions of degree $r-1$ with knots $x_1 < x_2 < \cdots < x_n$ is a linear space of dimension $n+r$, and a basis for $S_{r-1}(x_1, \ldots, x_n)$ is given by the functions

$$\{1, t, \ldots, t^{r-1}, (t-x_1)_+^{r-1}, \ldots, (t-x_n)_+^{r-1}\},$$

where $u_+(t)$ is defined by

$$u_+(t) = \max\{t, 0\}, \quad t \in \mathbb{R}.$$

If $\mathcal{L}$ is a linear functional defined on $C[0,1]$ which vanishes on π_s, then by a classical result of Peano, for $r \in \mathbb{N}$, $1 \le r \le s+1$ and $f \in W_1^r$, $\mathcal{L}$ admits the integral representation

$$\mathcal{L}[f] = \int_0^1 K_r(t) f^{(r)}(t)\, dt, \quad \text{where} \quad K_r(t) = \mathcal{L}\left[\frac{(\cdot - t)_+^{r-1}}{(r-1)!}\right], \quad t \in [0,1].$$

In the case when $\mathcal{L}$ is the remainder $R[Q;\cdot]$ of a quadrature formula Q with algebraic degree of precision s, the function $K_r(t) = K_r(Q;t)$ is referred to as the r-th Peano kernel of Q. For Q as in (1.1), explicit representations for $K_r(Q;t)$, $t \in [0,1]$, are

$$K_r(Q;t) = \frac{(1-t)^r}{r!} - \frac{1}{(r-1)!} \sum_{i=1}^n a_i (x_i - t)_+^{r-1}, \tag{2.1}$$

$$K_r(Q;t) = (-1)^r \left[\frac{t^r}{r!} - \frac{1}{(r-1)!} \sum_{i=1}^n a_i (t - x_i)_+^{r-1}\right]. \tag{2.2}$$

If the integrand f belongs to the Sobolev class W_p^r, $(1 \le p \le \infty)$, then from

$$R[Q;f] = \int_0^1 K_r(Q;t) f^{(r)}(t)\, dt$$

and Hölder's inequality one obtains the sharp error estimate

$$|R[Q;f]| \le c_{r,p}(Q) \|f^{(r)}\|_p, \ \text{where } c_{r,p}(Q) = \|K_r(Q;\cdot)\|_q, \ p^{-1} + q^{-1} = 1. \tag{2.3}$$

In other words, we have $\mathcal{E}(Q, W_p^r) = c_{r,p}(Q)$.

$K_r(Q;t)$ is also called a monospline of degree r with simple knots $\{x_i : x_i \in (0,1)\}$. From $K_r(Q;x) = R[Q; (\cdot - x)_+^{r-1}/(r-1)!]$ it is seen that $K_r(Q;x) = 0$ for some $x \in (0,1)$ if and only if Q evaluates to the exact value the integral of the spline function $f(t) = (t-x)_+^{r-1}$. Thus, in order that a quadrature formula Q has *maximal spline degree of precision*, i.e., Q is exact for a space of spline functions of degree $r-1$ with maximal dimension, it is necessary and sufficient that the corresponding monospline $K_r(Q;\cdot)$ has maximal number of zeros in $(0,1)$. Quadrature formulae of the form (1.1) with maximal spline degree of precision are called, analogously to the classical algebraic case, as Gauss, Radau, and Lobatto quadrature formulae, associated with the corresponding spaces of spline functions. Similarly to the classical Gauss-type quadrature formulae, all the nodes of the Gauss-type quadratures associated with spaces of spline functions lie in the integration interval, and all their weights are positive [6, Theorem 7.1].

2.2 Bernoulli Polynomials and Monosplines. Euler–MacLaurin Type Summation Formulae

Recall that the Bernoulli polynomials B_ν are defined recursively by

$$B_0(x) = 1, \quad B_\nu'(x) = B_{\nu-1}(x), \quad \text{and} \quad \int_0^1 B_\nu(t)\,dt = 0, \quad \nu \in \mathbb{N}.$$

In particular, $B_1(x) = x - \dfrac{1}{2}$, $B_2(x) = \dfrac{x^2}{2} - \dfrac{x}{2} + \dfrac{1}{12}$, $B_3(x) = \dfrac{x^3}{6} - \dfrac{x^2}{4} + \dfrac{x}{12}$,

$$B_4(x) = \frac{x^2(1-x)^2}{24} - \frac{1}{720}.$$

The Bernoulli numbers $\mathcal{B}_\nu$ are defined by $\mathcal{B}_\nu = B_\nu(0)/\nu!$.

The notation $\widetilde{B}_\nu(x)$ will stand for the 1-periodic extension of the Bernoulli polynomial $B_\nu(x)$ on $\mathbb{R}$. The functions $\widetilde{B}_\nu(x)$, $\nu = 0, 1, \ldots$, are called Bernoulli monosplines.

Let $n \in \mathbb{N}$ be fixed. Throughout this paper we set

$$x_{k,n} = \frac{k}{n}, \quad k = 0, \ldots, n; \qquad y_{\ell,n} = \frac{2\ell - 1}{2n}, \quad \ell = 1, \ldots, n.$$

Denote by Q_{n+1}^{Tr} and Q_n^{Mi} the n-th compound trapezium (resp., midpoint) quadrature formula, i.e.,

$$Q_{n+1}^{Tr}[f] = \frac{1}{2n}\left(f(x_{0,n}) + f(x_{n,n})\right) + \frac{1}{n}\sum_{k=1}^{n-1} f(x_{k,n}),$$

$$Q_n^{Mi}[f] = \frac{1}{n}\sum_{k=1}^{n-1} f(y_{k,n}).$$

The following summation formulae of Euler–MacLaurin type (adopted for the interval $[0,1]$) are well-known, see, e.g., [5, Satz 98, 99]:

Lemma 1. *Assume that $f \in W_1^s$. Then*

$$\int_0^1 f(x)\,dx = Q_{n+1}^{Tr}[f] - \sum_{\nu=1}^{[\frac{s}{2}]} \frac{\mathcal{B}_{2\nu}}{(2\nu)!}\,\frac{f^{(2\nu-1)}(1)-f^{(2\nu-1)}(0)}{n^{2\nu}} \\ + \frac{(-1)^s}{n^s}\int_0^1 \widetilde{B}_s(nx) f^{(s)}(x)\,dx \quad (2.4)$$

and

$$\int_0^1 f(x)\,dx = Q_n^{Mi}[f] - \sum_{\nu=1}^{[\frac{s}{2}]} \left(1-2^{1-2\nu}\right)\,\frac{\mathcal{B}_{2\nu}}{(2\nu)!}\,\frac{f^{(2\nu-1)}(1)-f^{(2\nu-1)}(0)}{n^{2\nu}} \\ + \frac{(-1)^s}{n^s}\int_0^1 \widetilde{B}_s\left(nx-\frac{1}{2}\right) f^{(s)}(x)\,dx. \quad (2.5)$$

2.3 The Sharp Error Bounds of Q_{n+1}^{Tr} and Q_n^{Mi} in $\widetilde{W}_p^s$

As was already mentioned, the midpoint quadrature formulae $\{Q_n^{Mi}\}_{n=1}^{\infty}$ and their translates are the unique optimal quadrature formulae in the periodic Sobolev classes $\widetilde{W}_p^r$. The trapezium quadrature formulae $\{Q_{n+1}^{Tr}\}_{n=1}^{\infty}$ can also be considered as translates of $\{Q_n^{Mi}\}_{n=1}^{\infty}$, as the values of the integrand at the endpoints are identical. For $f \in \widetilde{W}_p^s$, $1 \le p \le \infty$, the sums in the right-hand sides of (2.4) and (2.5) disappear, due to the periodicity of the integrand. Hence we obtain

$$R[Q_{n+1}^{Tr};f] = \frac{(-1)^s}{n^s}\int_0^1 \left[\widetilde{B}_s(nx)-d\right] f^{(s)}(x)\,dx \quad (2.6)$$

and

$$R[Q_n^{Mi};f] = \frac{(-1)^s}{n^s}\int_0^1 \left[\widetilde{B}_s\left(nx-\frac{1}{2}\right)-d\right] f^{(s)}(x)\,dx, \quad (2.7)$$

where d is an arbitrary constant. Applying Hölder's inequality to (2.6) and (2.7), and taking into account that Q_{n+1}^{Tr} and Q_n^{Mi} are optimal quadrature formulae in $\widetilde{W}_p^s$, we obtain

$$|R[Q_{n+1}^{Tr};f]| \le \mathcal{E}_n(\widetilde{W}_p^s)\,\|f^{(s)}\|_p, \quad |R[Q_n^{Mi};f]| \le \mathcal{E}_n(\widetilde{W}_p^s)\,\|f^{(s)}\|_p,$$

where

$$\mathcal{E}_n(\widetilde{W}_p^s) = \frac{1}{n^s}\inf_d \|B_s-d\|_q =: \|B_s-d_{s,p}\|_q, \quad \frac{1}{p}+\frac{1}{q}=1. \quad (2.8)$$

Some known values of the constant $d_{s,p}$ are (see, e.g., [13])

$$d_{s,p} = \begin{cases} 0 & \text{for odd } s\in\mathbb{N} \text{ and } 1\le p\le\infty, \\ 0 & \text{for all } s\in\mathbb{N} \text{ and } p=2, \\ B_s\left(\frac{1}{4}\right) & \text{for even } s\in\mathbb{N} \text{ and } p=\infty. \end{cases} \quad (2.9)$$

Specifically, for the case $s = 4$, which we are going to study in the non-periodic case, we have

$$\mathcal{E}_n(\widetilde{W}^4_\infty) = \frac{1}{n^4}\,\|B_4(\cdot) - B_4(1/4)\|_1 = \frac{5}{6144\,n^4} \tag{2.10}$$

$$\mathcal{E}_n(\widetilde{W}^4_2) = \frac{1}{n^4}\,\|B_4\|_2 = \frac{1}{240\sqrt{21}\,n^4}. \tag{2.11}$$

3 The Approach

Here we outline briefly our method for construction of asymptotically optimal quadrature formulae in the Sobolev classes W^4_2 and W^4_∞.

The Euler–MacLaurin summation formulae in Lemma 1 in the case $s = 4$ become

$$\int_0^1 f(x)\,dx = Q^{Tr}_{n+1}[f] - \frac{1}{12\,n^2}\,[f'(1) - f'(0)] + \frac{1}{720\,n^4}\,[f'''(1) - f'''(0)] + \frac{1}{n^4}\int_0^1 \widetilde{B}_4(nx) f^{(4)}(x)\,dx \tag{3.1}$$

and

$$\int_0^1 f(x)\,dx = Q^{Mi}_{n}[f] + \frac{1}{24\,n^2}\,[f'(1) - f'(0)] - \frac{7}{5760\,n^4}\,[f'''(1) - f'''(0)] + \frac{1}{n^4}\int_0^1 \widetilde{B}_4\left(nx - \frac{1}{2}\right) f^{(4)}(x)\,dx. \tag{3.2}$$

Formulae (3.1) and (3.2) are needed for derivation of asymptotically optimal quadrature formulae in W^4_2. For the construction of asymptotically optimal quadrature formulae in W^4_∞ (3.1) and (3.2) are rewritten in the form

$$\int_0^1 f(x)\,dx = Q^{Tr}_{n+1}[f] - \frac{1}{12\,n^2}\,[f'(1) - f'(0)] + \frac{3}{2048\,n^4}\,[f'''(1) - f'''(0)] + \frac{1}{n^4}\int_0^1 \left[\widetilde{B}_4(nx) - B_4\left(\frac{1}{4}\right)\right] f^{(4)}(x)\,dx \tag{3.3}$$

and

$$\int_0^1 f(x)\,dx = Q^{Mi}_{n}[f] + \frac{1}{24\,n^2}\,[f'(1) - f'(0)] - \frac{7}{6144\,n^4}\,[f'''(1) - f'''(0)] + \frac{1}{n^4}\int_0^1 \left[\widetilde{B}_4\left(nx - \frac{1}{2}\right) - B_4\left(\frac{1}{4}\right)\right] f^{(4)}(x)\,dx. \tag{3.4}$$

Denote the evaluations of the derivatives of the integrand appearing in (3.1)–(3.4) by

$$D_1[f] := f'(0) = f'(x_{0,n}), \qquad \widetilde{D}_1[f] := f'(1) = f'(x_{n,n}),$$
$$D_3[f] := f'''(0) = f'''(x_{0,n}), \qquad \widetilde{D}_3[f] := f'''(1) = f'''(x_{n,n}).$$

We approximate $D_1[f]$ and $D_3[f]$ by formulae for numerical differentiation $D_{1,*}[f]$ and $D_{3,*}[f]$, which are exact for $f \in \pi_3$ and involve evaluations of f at four points near the left end of the integration interval. Then $\widetilde{D}_1[f]$ and $\widetilde{D}_3[f]$ are approximated by the reflected formulae $D_{1,*}[f]$ and $D_{3,*}[f]$, which are denoted by $\widetilde{D}_{1,*}[f]$ and $\widetilde{D}_{3,*}[f]$, respectively.

We choose four points $(0 \leq)\, t_0 < t_1 < t_2 < t_3\, (< 1/2)$ arbitrarily with the only requirement that $t_3 = O(n^{-1})$ as $n \to \infty$, and set $\widetilde{t}_{n-k} = 1 - t_k$, $k = 0, 1, 2, 3$. For $j \in \{1, 3\}$, let

$$D_{j,*}[f] = \alpha_j f(t_0) + \beta_j f(t_1) + \gamma_j f(t_2) + \delta_j f(t_3)$$

and

$$\widetilde{D}_{j,*}[f] = -\big(\alpha_j f(\widetilde{t}_n) + \beta_j f(\widetilde{t}_{n-1}) + \gamma_j f(\widetilde{t}_{n-2}) + \delta_j f(\widetilde{t}_{n-3})\big)$$

be the unique formulae which approximate $D_j[f]$ and $\widetilde{D}_j[f]$, respectively, and satisfy

$$R_j[f] := D_j[f] - D_{j,*}[f] = 0 \quad \text{and} \quad \widetilde{R}_j[f] := \widetilde{D}_j[f] - \widetilde{D}_{j,*}[f] = 0, \quad f \in \pi_3.$$

If $f \in W_1^4$, then by Peano's representation theorem,

$$R_j[f] = \int_0^1 K_4(R_j; t) f^{(4)}(t)\, dt, \quad \widetilde{R}_j[f] = \int_0^1 K_4(\widetilde{R}_j; t) f^{(4)}(t)\, dt, \quad j = 1, 3,$$

and, in addition, the Peano kernels $K_4(R_j; \cdot)$ and $K_4(\widetilde{R_j}; \cdot)$ satisfy

$$K_4(R_j; t) \equiv 0 \text{ for } t \in [t_3, 1], \quad K_4(\widetilde{R_j}; t) \equiv 0 \text{ for } t \in [\widetilde{t}_{n-3}, 1]. \tag{3.5}$$

Let us illustrate our approach for construction of asymptotically optimal quadrature formulae in W_2^4 on the basis of representation (3.1). To this end, we consider the quadrature formula

$$Q_{n+s,*}^{Tr}[f] := Q_{n+1}^{Tr}[f] - \frac{1}{12\, n^2}\left(\widetilde{D}_{1,*} - D_{1,*}\right)[f] + \frac{1}{720\, n^4}\left(\widetilde{D}_{3,*} - D_{3,*}\right)[f]. \tag{3.6}$$

Clearly, $Q_{n+s,*}^{Tr}$ is a symmetric quadrature formula with $n + s$ nodes, with $s - 1$ being the number of the nodes in $\{t_j\}_{j=0}^3 \cup \{\widetilde{t}_j\}_{j=n-3}^n$ which do not belong to $\{x_{k,n}\}_{k=0}^n$. On using (3.1), we represent the remainder of $Q_{n+s,*}^{Tr}$ in the form

$$\begin{aligned} R[Q_{n+s,*}^{Tr}; f] &= -\frac{\widetilde{R}_1[f] - R_1[f]}{12\, n^2} + \frac{\widetilde{R}_3[f] - R_3[f]}{720\, n^4} + \frac{1}{n^4}\int_0^1 \widetilde{B}_4(nt) f^{(4)}(t)\, dt \\ &= \int_0^1 K_4(Q_{n+s,*}^{Tr}; t) f^{(4)}(t)\, dt. \end{aligned}$$

Taking into account (3.5), we conclude that

$$K_4(Q_{n+s,*}^{Tr}; t) = \frac{1}{n^4}\, \widetilde{B}_4(nt) \quad \text{for} \quad t \in (t_3, 1 - t_3), \tag{3.7}$$

i.e., $K_4(Q^{Tr}_{n+s,*};\cdot)$ coincides, except in the intervals $[0,t_3]$ and $[1-t_3,1]$ (which have length $t_3=O(n^{-1})$), with the kernel of least L_2 deviation in the 1-periodic class $\widetilde{W}^4_2$. If $f\in W^4_2$, then (see (2.9)) the error constant $c_{4,2}(Q^{Tr}_{n+s,*})$ satisfies

$$\begin{aligned}
c_{4,2}(Q^{Tr}_{n+s,*})^2 &= \|K_4(Q^{Tr}_{n+s,*};\cdot)\|_2^2\\
&= 2\int_0^{t_3}\left[K_4(Q^{Tr}_{n+s,*};t)\right]^2\,dt+\frac{1}{n^8}\int_{t_3}^{1-t_3}\left[\widetilde{B}_4(nt)\right]^2\,dt\\
&< 2\int_0^{t_3}\left[K_4(Q^{Tr}_{n+s,*};t)\right]^2\,dt+\frac{1}{n^8}\|B_4\|_2^2\\
&= \mathcal{E}_n(\widetilde{W}^4_2)^2(1+O(n^{-1})),
\end{aligned}$$

which implies the asymptotical optimality of $\{Q^{Tr}_{n+s,*}\}$ in W^4_2.

Making use of (3.2), one can also construct quadrature formulae $\{Q^{Mi}_{n+s,*}\}$ which are asymptotically optimal in W^4_2. These formulae are defined by

$$Q^{Mi}_{n+s,*}[f]=Q^{Mi}_n[f]+\frac{1}{24\,n^2}\left(\widetilde{D}_{1,*}-D_{1,*}\right)[f]-\frac{7}{5760\,n^4}\left(\widetilde{D}_{3,*}-D_{3,*}\right)[f],\tag{3.8}$$

where s is the number of nodes in $\{t_j\}_{j=0}^3\cup\{\widetilde{t}_j\}_{j=n-3}^n$ which do not belong to $\{y_{k,n}\}_{k=1}^n$. The Peano kernel of $Q^{Mi}_{n+s,*}$ satisfies

$$K_4(Q^{Mi}_{n+s,*};t)=\frac{1}{n^4}\,\widetilde{B}_4(nt-1/2)\quad\text{for}\quad t\in(t_3,1-t_3),\tag{3.9}$$

whence the asymptotical optimality of $\{Q^{Mi}_{n+s,*}\}$ in W^4_2 is justified in the same way as that of $\{Q^{Tr}_{n+s,*}\}$.

The construction of asymptotically optimal quadrature formulae in W^4_∞ proceeds in a similar fashion. Namely, we consider the quadrature formulae

$$\widetilde{Q}^{Tr}_{n+s,*}[f]:=Q^{Tr}_{n+1}[f]-\frac{1}{12\,n^2}\left(\widetilde{D}_{1,*}-D_{1,*}\right)[f]+\frac{3}{2048\,n^4}\left(\widetilde{D}_{3,*}-D_{3,*}\right)[f]\tag{3.10}$$

and

$$\widetilde{Q}^{Mi}_{n+s,*}[f]=Q^{Mi}_n[f]+\frac{1}{24\,n^2}\left(\widetilde{D}_{1,*}-D_{1,*}\right)[f]-\frac{7}{6144\,n^4}\left(\widetilde{D}_{3,*}-D_{3,*}\right)[f].\tag{3.11}$$

On using of (3.3) and (3.4), we deduce that for $t\in(t_3,1-t_3)$ the Peano kernels of the quadrature formulae (3.10) and (3.11) coincide with the kernel of least L_1-deviation in the 1-periodic case, namely,

$$K_4(\widetilde{Q}^{Tr}_{n+s,*};t)=\frac{1}{n^4}\left[\widetilde{B}_4(nt)-B_4(1/4)\right],\quad t\in(t_3,1-t_3),\tag{3.12}$$

$$K_4(\widetilde{Q}^{Mi}_{n+s,*};t)=\frac{1}{n^4}\left[\widetilde{B}_4(nt-1/2)-B_4(1/4)\right],\quad t\in(t_3,1-t_3).\tag{3.13}$$

This fact easily implies the asymptotical optimality of the quadrature formulae (3.10) and (3.11) in W^4_∞. More details are given in the next section.

4 Asymptotically Optimal Quadrature Formulae

4.1 Quadrature Formulae Based on Q_{n+1}^{Tr}

1. With the choice of the nodes $\{t_0, t_1, t_2, t_3\} = \{x_{0,n}, x_{1,n}, x_{2,n}, x_{3,n}\}$, the formula for numerical differentiation approximating $D_1[f] = f'(x_{0,0})$ is

$$D_{1,1}[f] = \frac{n}{6}\left[-11f(x_{0,n}) + 18f(x_{1,n}) - 9f(x_{2,n}) + 2f(x_{3,n})\right],$$

and the formula for numerical differentiation approximating $D_3[f] = f'''(x_{0,0})$ is

$$D_{3,1}[f] = n^3\left[-f(x_{0,n}) + 3f(x_{1,n}) - 3f(x_{2,n}) + f(x_{3,n})\right].$$

The corresponding formulae $\widetilde{D}_{1,1}$ and $\widetilde{D}_{3,1}$ approximating $\widetilde{D}_1[f] = f'(x_{n,n})$ and $\widetilde{D}_3[f] = f'''(x_{n,n})$, respectively, are obtained by reflection, i.e.,

$$\widetilde{D}_{1,1}[f] = \frac{n}{6}\left[11f(x_{n,n}) - 18f(x_{n-1,n}) + 9f(x_{n-2,n}) - 2f(x_{n-3,n})\right]$$

and

$$\widetilde{D}_{3,1}[f] = n^3\left[f(x_{n,n}) - 3f(x_{n-1,n}) + 3f(x_{n-2,n}) - f(x_{n-3,n})\right].$$

For $n \geq 8$, the resulting quadrature formula (3.6) is

$$Q_{n+1,1}^{Tr}[f] := \sum_{k=0}^{n} A_{k,n} f(x_{k,n}), \tag{4.1}$$

where the weights $\{A_{k,n}\}_{k=0}^{n}$ are given by

$$\begin{aligned}
A_{0,n} &= A_{n,n} = \frac{251}{720n}, & A_{1,n} &= A_{n-1,n} = \frac{299}{240n}, \\
A_{2,n} &= A_{n-2,n} = \frac{211}{240n}, & A_{3,n} &= A_{n-3,n} = \frac{739}{720n}, \\
A_{k,n} &= \frac{1}{n}, & & 4 \leq k \leq n-4.
\end{aligned} \tag{4.2}$$

Next, we compute the error constant $c_{4,2}(Q_{n+1,1}^{Tr})$. Since $Q_{n+1,1}^{Tr}$ is symmetric and taking into account (3.7), we have

$$\begin{aligned}
c_{4,2}(Q_{n+1,1}^{Tr})^2 &= \int_0^1 \left[K_4(Q_{n+1,1};t)\right]^2\,dt \\
&= 2\int_0^{x_{3,n}} \left[K_4(Q_{n+1,1};t)\right]^2\,dt + \frac{1}{n^8}\int_{x_{3,n}}^{x_{n-3,n}} \left[\widetilde{B}_4(nt)\right]^2\,dt.
\end{aligned}$$

The second summand is evaluated as follows:

$$\begin{aligned}
\frac{1}{n^8}\int_{x_{3,n}}^{x_{n-3,n}} \left[\widetilde{B}_4(nt)\right]^2\,dt &= \frac{1}{n^8}\sum_{k=3}^{n-4}\int_{x_{k,n}}^{x_{k+1,n}} \left[\widetilde{B}_4(nt)\right]^2\,dt \\
&= \frac{1}{n^9}\sum_{k=3}^{n-4}\int_0^1 \left[B_4(t)\right]^2\,dt = \frac{n-6}{n^9}\,\|B_4\|_2^2.
\end{aligned}$$

To evaluate the first summand, we write $K_4(Q_{n+1,1};t)$ in the form given by (2.2), and then perform change of the variable $t = u/n$ to obtain

$$\begin{aligned}\int_0^{x_{3,n}} \left[K_4(Q_{n+1,1};t)\right]^2\,dt &= \int_0^{x_{3,n}} \left[\frac{t^4}{24} - \frac{1}{6}\sum_{k=0}^{2} A_{k,n}(t-x_{k,n})_+^3\right]^2 dt\\ &= \frac{1}{n^9}\int_0^3 \left[\frac{u^4}{24} - \frac{251}{4320}u^3 - \frac{299}{1440}(u-1)_+^3 - \frac{211}{1440}(u-2)_+^3\right]^2 du\\ &= \frac{95051}{435456000\,n^9} = \frac{95051}{360\,n^9}\|B_4\|_2^2\end{aligned}$$

(we have used Wolfram Mathematica for this calculation). Hence, we obtain

$$c_{4,2}(Q_{n+1,1}^{Tr}) = \frac{\|B_4\|_2}{n^4}\left(1+\frac{19009}{36\,n}\right)^{1/2} = \frac{1}{240\sqrt{21}\,n^4}\left(1+\frac{19009}{36\,n}\right)^{1/2},$$

which implies the asymptotical optimality of $\{Q_{n+1,1}^{Tr}\}$ in W_2^4.

With the same choice of formulae for numerical differentiation, the asymptotically optimal quadrature formulae in W_∞^4 obtained by (3.10) are

$$\widetilde{Q}_{n+1,1}^{Tr}[f] = \sum_{k=0}^{n} \widetilde{A}_{k,n} f(x_{k,n}), \tag{4.3}$$

where the weights $\{\widetilde{A}_{k,n}\}_{k=0}^{n}$ are given by

$$\begin{aligned} &\widetilde{A}_{0,n} = \widetilde{A}_{n,n} = \frac{6427}{18432n}, && \widetilde{A}_{1,n} = \widetilde{A}_{n-1,n} = \frac{2551}{2048n},\\ &\widetilde{A}_{2,n} = \widetilde{A}_{n-2,n} = \frac{1801}{2048n}, && \widetilde{A}_{3,n} = \widetilde{A}_{n-3,n} = \frac{18917}{18432n},\\ &\widetilde{A}_{k,n} = \frac{1}{n}, && 4 \le k \le n-4. \end{aligned} \tag{4.4}$$

For the evaluation of the error constant $c_{4,\infty}(\widetilde{Q}_{n+1,1}^{Tr})$ we make use of (3.12)

$$\begin{aligned} c_{4,\infty}(\widetilde{Q}_{n+1,1}^{Tr}) &= \int_0^1 \left|K_4(\widetilde{Q}_{n+1,1}^{Tr};t)\right| dt\\ &= 2\int_0^{x_{3,n}} \left|K_4(\widetilde{Q}_{n+1,1}^{Tr};t)\right| dt + \frac{1}{n^4}\int_{x_{3,n}}^{x_{n-3,n}} \left|\widetilde{B}_4(nt) - B_4(1/4)\right| dt. \end{aligned}$$

The first term is evaluated with the help of Wolfram Mathematica

$$\begin{aligned} &2\int_0^{x_{3,n}} \left|K_4(\widetilde{Q}_{n+1,1}^{Tr};t)\right| dt\\ &\qquad = \frac{2}{n^5}\int_0^3 \left|\frac{u^4}{24} - \frac{6427}{110592}u^3 - \frac{2551}{12288}(u-1)_+^3 - \frac{1801}{12288}(u-2)_+^3\right| du\\ &\qquad = \frac{0.03834415637472169}{n^5}, \end{aligned}$$

while the second summand equals

$$\frac{n-6}{n^5}\int_0^1 |B_4(t)-B_4(1/4)|\,dt=\frac{5(n-6)}{6144\,n^5}.$$

Hence,

$$c_{4,\infty}(\widetilde{Q}^{Tr}_{n+1,1})=\frac{5}{6144n^4}\left(1+\frac{41.11729935325838}{n}\right),$$

which shows the asymptotical optimality of $\{\widetilde{Q}^{Tr}_{n+1,1}\}$ in W^4_∞.

We present below briefly two other pairs of asymptotically optimal quadrature formulae, obtained from Q^{Tr}_{n+1} with other formulae for numerical differentiation.

2. If $\{t_0,t_1,t_2,t_3\}=\{x_{0,n},x_{1,3n},x_{2,3n},x_{1,n}\}$, then the formulae for numerical differentiation approximating D_1 and D_3 are

$$D_{1,2}[f]=\frac{n}{2}\left[-11f(x_{0,n})+18f(x_{1,3n})-9f(x_{2,3n})+2f(x_{1,n})\right]$$

and

$$D_{3,2}[f]=27n^3\left[-f(x_{0,n})+3f(x_{1,3n})-3f(x_{2,3n})+f(x_{1,n})\right].$$

The quadrature formula (3.6) in this case is

$$Q^{Tr}_{n+5,2}[f]=\sum_{k=1}^{n+5}A_{k,n+5}f(\tau_{k,n}), \tag{4.5}$$

where the nodes are given by

$$\begin{aligned} &\tau_{k,n+5}=x_{k-1,3n}, && 1\le k\le 3\\ &\tau_{k,n+5}=x_{k-3,n}, && 4\le k\le n+2\\ &\tau_{k,n+5}=x_{2n-5+k,3n}, && n+3\le k\le n+5 \end{aligned} \tag{4.6}$$

and the weights are given by

$$\begin{aligned} &A_{1,n+5}=A_{n+5,n+5}=\frac{19}{240n}, && A_{2,n+5}=A_{n+4,n+5}=\frac{251}{240n},\\ &A_{3,n+5}=A_{n+3,n+5}=\frac{51}{80n}, && A_{4,n+5}=A_{n+2,n+5}=-\frac{21}{80n},\\ &A_{k,n+5}=\frac{1}{n}, && 5\le k\le n+1. \end{aligned}$$

The error constant of (4.5) in W^4_2 is

$$c_{4,2}(Q^{Tr}_{n+5,2})=\frac{1}{240\sqrt{21}\,n^4}\left(1-\frac{19183}{14580n}\right)^{\frac12}.$$

The corresponding asymptotically optimal in W_∞^4 quadrature formulae (3.10) are

$$\widetilde{Q}_{n+5,2}^{Tr}[f] = \sum_{k=1}^{n+5} \widetilde{A}_{k,n+5} f(\tau_{k,n}), \tag{4.7}$$

where the nodes are given by (4.6) and the weights are:

$$\begin{aligned}
\widetilde{A}_{1,n+5} &= \widetilde{A}_{n+5,n+5} = \frac{499}{6144n}, & \widetilde{A}_{2,n+5} &= \widetilde{A}_{n+4,n+5} = \frac{1293}{2048n},\\
\widetilde{A}_{3,n+5} &= \widetilde{A}_{n+3,n+5} = -\frac{525}{2048n}, & \widetilde{A}_{4,n+5} &= \widetilde{A}_{n+2,n+5} = \frac{6413}{6144n},\\
\widetilde{A}_{k,n+5} &= \frac{1}{n}, & & 5 \le k \le n+1.
\end{aligned}$$

For the error constant of (4.7) in W_∞^4 is

$$c_{4,\infty}(\widetilde{Q}_{n+5,2}^{Tr}) = \frac{5}{6144n^4}\left(1 - \frac{1.112482888448233}{n}\right).$$

3. If $\{t_0, t_1, t_2, t_3\} = \{x_{0,n}, x_{1,2n}, x_{1,n}, x_{2,n}\}$, then the formulae for numerical differentiation approximating D_1 and D_3 are

$$D_{1,3}[f] = \frac{n}{6}\left[-21f(x_{0,n}) + 32f(x_{1,2n}) - 12f(x_{1,n}) + f(x_{2,n})\right],$$

and

$$D_{3,3}[f] = n^3\left[-6f(x_{0,n}) + 16f(x_{1,2n}) - 12f(x_{1,n}) + 2f(x_{2,n})\right].$$

The quadrature formula (3.6) in this case is

$$Q_{n+3,3}^{Tr}[f] = \sum_{k=1}^{n+3} A_{k,n} f(\tau_{k,n+3}), \tag{4.8}$$

where the weights are given by

$$\begin{aligned}
A_{1,n+3} &= A_{n+3,n+3} = \frac{13}{60n}, & A_{2,n+3} &= A_{n+2,n+3} = \frac{19}{45n},\\
A_{3,n+3} &= A_{n+1,n+3} = \frac{91}{90n}, & A_{4,n+3} &= A_{n,n+3} = \frac{17}{20n},\\
A_{k,n} &= \frac{1}{n}, & & 5 \le k \le n-1,
\end{aligned}$$

and the nodes by

$$\begin{aligned}
\tau_{1,n+3} &= x_{0,n}, & \tau_{n+3,n+3} &= x_{n,n},\\
\tau_{2,n+3} &= x_{1,2n}, & \tau_{n+2,n+3} &= x_{2n-1,2n},\\
\tau_{k,n+3} &= x_{k-2,n}, & & 3 \le k \le n+1.
\end{aligned} \tag{4.9}$$

The error constant of the quadrature formula (4.8) in W_2^4 is

$$c_{4,2}(Q_{n+3,3}^{Tr}) = \frac{1}{240\sqrt{21}\, n^4}\left(1+\frac{5749}{480\, n}\right)^{\frac{1}{2}},$$

hence $\{Q_{n+3,3}^{Tr}\}$ are asymptotically optimal in W_2^4.

The quadrature formula (3.10) in this case is

$$\widetilde{Q}_{n+3,3}^{Tr}[f] = \sum_{k=0}^{n} \widetilde{A}_{k,n+3} f(\tau_{k,n+3}), \tag{4.10}$$

where the nodes are given by (4.9), and the weights are

$$\begin{aligned}
\widetilde{A}_{1,n+3} &= \widetilde{A}_{n+3,n+3} = \frac{667}{3072n}, & \widetilde{A}_{2,n+3} &= \widetilde{A}_{n+2,n+3} = \frac{485}{1152n},\\
\widetilde{A}_{3,n+3} &= \widetilde{A}_{n+1,n+3} = \frac{1307}{1536n}, & \widetilde{A}_{4,n+3} &= \widetilde{A}_{n,n+3} = \frac{9317}{9216n},\\
A_{k,n} &= \frac{1}{n}, & & 5 \le k \le n-1.
\end{aligned}$$

The error constant of (4.10) in the Sobolev space W_∞^4 is

$$c_{4,\infty}(\widetilde{Q}_{n+3,3}^{Tr}) = \frac{5}{6144\, n^4}\left(1+\frac{3.01035171820917}{n}\right),$$

therefore $\{\widetilde{Q}_{n+3,3}^{Tr}\}$ are asymptotically optimal in W_∞^4.

4.2 Quadrature Formulae Based on Q_n^{Mi}

1. With the choice of the nodes $\{t_0, t_1, t_2, t_3\} = \{x_{0,n}, y_{1,n}, y_{2,n}, y_{3,n}\}$, the formulae for numerical differentiation approximating D_1 and D_3 are:

$$D_{1,1}[f] = n\left[-\frac{45}{15}f(x_{0,n}) + \frac{15}{4}f(y_{1,n}) - \frac{5}{6}f(y_{2,n}) + \frac{3}{20}f(y_{3,n})\right],$$

and

$$D_{3,1}[f] = n^3\left[-\frac{16}{5}f(x_{0,n}) + 6f(y_{1,n}) - 4f(y_{2,n}) + \frac{16}{5}f(y_{3,n})\right].$$

The quadrature formula (3.8) in this case is

$$Q_{n+2,1}^{Mi}[f] := \sum_{k=1}^{n+2} A_{k,n+2} f(\tau_{k,n}), \tag{4.11}$$

where the weights are given by:

$$\begin{aligned}
A_{1,n+2} &= A_{n+2,n=2} = \frac{223}{1800n}, & A_{2,n+2} &= A_{n+1,n+2} = \frac{817}{960n},\\
A_{3,n+2} &= A_{n,n+2} = \frac{1483}{1440n}, & A_{4,n+2} &= A_{n-1,n+2} = \frac{4777}{4800n},\\
A_{k,n} &= \frac{1}{n}, & & 5 \le k \le n-2,
\end{aligned} \tag{4.12}$$

and the nodes are:

$$\begin{aligned} &\tau_{1,n+2} = x_{0,n}, && \tau_{n+2,n+2} = x_{n,n}, \\ &\tau_{2,n+2} = y_{1,n}, && \tau_{n+1,n+2} = y_{n-1,n}, \\ &\tau_{k,n+2} = y_{k-1,n}, && 3 \le k \le n. \end{aligned} \tag{4.13}$$

The error constant of the quadrature formula (4.11) in W_2^4 is

$$c_{4,2}(Q^{Mi}_{n+2,1}) = \frac{1}{240\sqrt{21}\,n^4}\left(1 + \frac{515557}{57600\,n}\right)^{\frac{1}{2}},$$

therefore $\{Q^{Mi}_{n+2,1}\}$ are asymptotically optimal in W_2^4.

The corresponding asymptotically optimal in W_∞^4 quadrature formulae (3.11) are

$$\widetilde{Q}^{Mi}_{n+2,1}[f] := \sum_{k=1}^{n+2} \widetilde{A}_k f(\tau_{k,n}), \tag{4.14}$$

where the nodes are given by 4.13 and the weights are given by:

$$\begin{aligned} &\widetilde{A}_{1,n+2} = \widetilde{A}_{n+2,n+2} = \frac{143}{1152n}, && \widetilde{A}_{2,n+2} = \widetilde{A}_{n+1,n+2} = \frac{871}{1024n}, \\ &\widetilde{A}_{3,n+2} = \widetilde{A}_{n,n+2} = \frac{4747}{4608n}, && \widetilde{A}_{4,n+2} = \widetilde{A}_{n-1,n+2} = \frac{1019}{1024n}, \\ &\widetilde{A}_{k,n} = \frac{1}{n}, && 5 \le k \le n-2. \end{aligned} \tag{4.15}$$

The error constant of (4.2) in the Sobolev space W_∞^4 is

$$c_{4,\infty}(\widetilde{Q}^{Mi}_{n+2,1}) = \frac{5}{6144\,n^4}\left(1 + \frac{2.434934207865606}{n}\right).$$

2. If $\{t_0, t_1, t_2, t_3\} = \{y_{1,n}, y_{2,n}, y_{3,n}, y_{4,n}\}$, then the formulae for numerical differentiation approximating D_1 and D_3 are:

$$D_{1,2}[f] = \frac{n}{24}\left[-71f(y_{1,n}) + 141f(y_{2,n}) - 93f(y_{3,n}) + 23f(y_{4,n})\right],$$

and

$$D_{3,2}[f] = n^3\left[-f(y_{1,n}) + 3f(y_{2,n}) - 3f(y_{3,n}) + f(y_{4,n})\right].$$

The quadrature formula (3.8) in this case is

$$Q^{Mi}_{n,2}[f] = \sum_{k=1}^{n} A_k f(y_{k,n}), \tag{4.16}$$

where the weights are given by:

$$\begin{aligned} &A_{1,n} = A_{n,n} = \frac{6463}{5760n}, && A_{2,n} = A_{n-1,n} = \frac{1457}{1920n}, \\ &A_{3,n} = A_{n-2,n} = \frac{741}{640n}, && A_{4,n} = A_{n-3,n} = \frac{5537}{5760n}, \\ &A_{k,n} = \frac{1}{n}, && 5 \le k \le n-4. \end{aligned} \tag{4.17}$$

The error constant of the quadrature formula (4.16) in W_2^4 is

$$c_{4,2}(Q_{n,2}^{Mi}) = \frac{1}{240\sqrt{21}\, n^4}\left(1+\frac{59264646673}{2211840\, n}\right)^{\frac{1}{2}},$$

hence $\{Q_{n,2}^{Mi}\}$ are asymptotically optimal in W_2^4.

The corresponding asymptotically optimal in W_∞^4 quadrature formulae (3.11) are

$$\widetilde{Q}_{n,2}^{Tr}[f] = \sum_{k=1}^{n} \widetilde{A}_k f(y_{k,n}), \tag{4.18}$$

the weights and the error constant in the Sobolev space W_∞^4 are given by:

$$\begin{aligned} &\widetilde{A}_{1,n} = \widetilde{A}_{n,n} = \frac{20683}{18432n}, && \widetilde{A}_{2,n} = \widetilde{A}_{n-1,n} = \frac{4661}{6144n},\\ &\widetilde{A}_{3,n} = \widetilde{A}_{n-2,n} = \frac{7115}{6144n}, && \widetilde{A}_{4,n} = \widetilde{A}_{n-3,n} = \frac{17717}{18432n},\\ &\widetilde{A}_{k,n} = \frac{1}{n}, && 5 \le k \le n-4, \end{aligned} \tag{4.19}$$

$$c_{4,\infty}(\widetilde{Q}_{n,2}^{Mi}) = \frac{5}{6144\, n^4}\left(1+\frac{281.276041666667}{n}\right).$$

3. With the choice of the nodes $\{t_0, t_1, t_2, t_3\} = \{x_{0,n}, y_{1,n}, x_{1,n}, y_{2,n}\}$, the formulae for numerical differentiation approximating D_1 and D_3 are:

$$D_{1,3}^{Mi} = \frac{n}{3}\left[-11f(x_{0,n}) + 18f(y_{1,n}) - 9f(x_{1,n}) + 2f(y_{2,n})\right]$$

and

$$D_{3,3}^{Mi} = n^3\left[-8f(x_{0,n}) + 24f(y_{1,n}) - 24f(x_{1,n}) + 8f(y_{2,n})\right].$$

The asymptotically optimal in the Sobolev space W_2^4 quadrature formulae (3.8) are

$$Q_{n+4,3}^{Mi}[f] = \sum_{k=1}^{n+4} A_k f(\tau_{k,n}), \tag{4.20}$$

where the weights are given by:

$$\begin{aligned} &A_{1,n+4} = A_{n+4,n+4} = \frac{103}{720n}, && A_{2,n+4} = A_{n+3,n+4} = \frac{187}{240n},\\ &A_{3,n+4} = A_{n+2,n+4} = \frac{23}{240n}, && A_{4,n+4} = A_{n+1,n+4} = \frac{707}{720n},\\ &A_{k,n} = \frac{1}{n}, && 5 \le k \le n, \end{aligned} \tag{4.21}$$

and the nodes are

$$\begin{aligned}
\tau_{1,n+4} &= x_{0,n}, & \tau_{n+4,n+4} &= x_{n,n}, \\
\tau_{2,n+4} &= y_{1,n}, & \tau_{n+3,n+4} &= y_{n,n}, \\
\tau_{3,n+4} &= x_{1,n}, & \tau_{n+2,n+4} &= x_{n-1,n}, \\
\tau_{4,n+4} &= y_{2,n}, & \tau_{n+1,n+4} &= y_{n-1,n}, \\
\tau_{k,n+2} &= y_{k-1,n}, & &5 \le k \le n.
\end{aligned} \tag{4.22}$$

The error constant of the quadrature formula (4.20) in W_2^4 is

$$c_{4,2}(Q_{n+4,3}^{Mi}) = \frac{1}{240\sqrt{21}\,n^4}\left(1 + \frac{4723973}{69120\,n}\right)^{\frac{1}{2}}.$$

The asymptotically optimal in the Sobolev space W_∞^4 quadrature formulae (3.11) are

$$\widetilde{Q}_{n+4,3}^{Mi}[f] = \sum_{k=1}^{n+4} \widetilde{A}_k f(\tau_{k,n}), \tag{4.23}$$

where the nodes are given by (4.22) and the coefficients are:

$$\begin{aligned}
\widetilde{A}_{1,n+4} &= A_{n+4,n+4} = \frac{331}{2304n}, & \widetilde{A}_{2,n+4} &= \widetilde{A}_{n+3,n+4} = \frac{199}{256n}, \\
\widetilde{A}_{3,n+4} &= \widetilde{A}_{n+2,n+4} = \frac{25}{256n}, & \widetilde{A}_{4,n+4} &= \widetilde{A}_{n+1,n+4} = \frac{2261}{2304n}, \\
\widetilde{A}_{k,n} &= \frac{1}{n}, & &5 \le k \le n.
\end{aligned} \tag{4.24}$$

The error constant of the quadrature formula (4.20) in W_∞^4 is

$$c_{4,\infty}(\widetilde{Q}_{n+4,3}^{Mi}) = \frac{5}{6144\,n^4}\left(1 + \frac{3.9039188023044886}{n}\right).$$

5 Concluding Remarks

We have constructed certain quadrature formulae, which are asymptotically optimal in the Sobolev classes W_p^4, $p = \infty$ and $p = 2$. Their weights and nodes are explicitly given, and their sharp error constants in W_p^4, $p = \infty$ and $p = 2$ are evaluated. Our approach makes use of two Euler–MacLaurin-type summation formulae, in which the values of the derivatives at the end-points are replaced by approximate formulae for numerical differentiation. Figure 1 and 2 depict the graphs of the Peano kernels for the second variant of the trapezium based asymptotically optimal quadrature formulae in $\widetilde{W}_\infty^4$ and the first option of the midpoint based asymptotically optimal quadrature formulae in W_2^4, obtained with $n = 20$.

Notice that all the quadrature formulae, constructed in Sect. 4, are symmetrical, due to our choice of the formulae for numerical differentiation. Various other asymptotically optimal quadrature formulae can be constructed, including non-symmetrical ones, by applying other formulae for numerical differentiation.

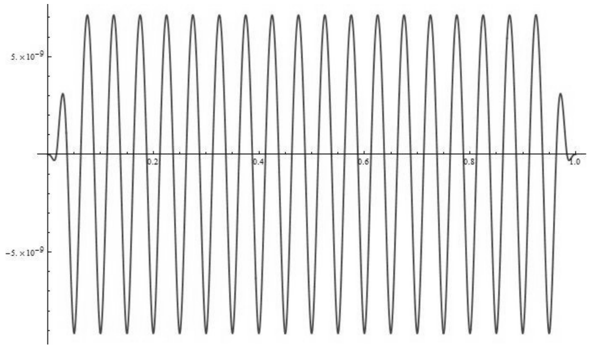

Fig. 1. Graph of the Peano kernel $K_4(\widetilde{Q}^{Tr}_{n+5,2};t), n=20$.

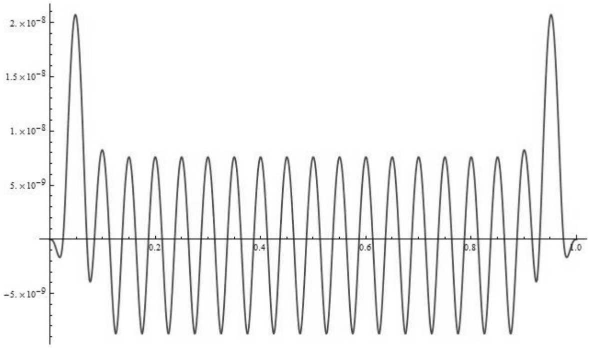

Fig. 2. Graph of the Peano kernel $K_4(Q^{Mi}_{n+2,1};t)$, $n=20$.

Acknowledgements. This work was supported by the European Social Fund through the Human Resource Development Operational Programme under contract BG051PO001-3.3.06-0052 (2012/2014). The authors are also supported by Sofia University Research Fund through Contract 111/2014.

References

1. Avdzhieva, A., Nikolov, G.: Numerical computation of Gaussian quadrature formulae for spaces of cubic splines with equidistant knots. In: Slavova, A., Georgiev, Kr. (eds.) BGSIAM'12, Proc. of the 7th Meeting of the Bulgarian Section of SIAM, ISNM: 1314–7145, Sofia, pp. 28–38 (2012)
2. Bojanov, B.D.: Uniqueness of the monosplines of least deviation. In: Numerische Integration (G. Hämmerlin, ed.), ISNM 45, Birkhäuser, Basel, pp. 67–97 (1979)
3. Bojanov, B.D.: Existence and characterization of monosplines of least L_p deviation. In: Sendov, Bl., Vačov, D. (eds.), Constructive Function Theory'77, pp. 249–268. BAN, Sofia, (1980)
4. Bojanov, B.D.: Uniqueness of the optimal nodes of quadrature formulae. Math. Comput. 36, 525–546 (1981)

5. Braß, H.: Quadraturverfahren. Vandenhoeck & Ruprecht, Göttingen (1977)
6. Karlin, S., Micchelli, C.A.: The fundamental theorem of algebra for monosplines satisfying boundary conditions. Israel J. Math. 11, 405–451 (1972)
7. Köhler, P., Nikolov, G.P.: Error bounds for Gauss type quadrature formulae related to spaces of splines with equidistant knots. J. Approx. Theory 81(3), 368–388 (1995)
8. Ligun, A.A.: Exact inequalities for splines and best quadrature formulas for certain classes of functions. Mat. Zametki 19, 913–926 (1976) (in Russian); English Translation in: Math. Notes 19, 533–541 (1976)
9. Motornii, V.P.: On the best quadrature formula of the form $\sum p_k f(x_k)$ for some classes of differentiable periodic functions. Izv. Akad. Nauk SSSR Ser. Mat. 38, 583–614 (1974) (in Russian); English Translation in: Math. USSR Izv. 8, 591–620 (1974)
10. Nikolov, G.: Gaussian quadrature formulae for splines. In: Hämmerlin, G., Brass, H. (eds.) Numerische Integration, IV, ISNM Vol. 112, pp. 267–281. Birkhäuser, Basel (1993)
11. Nikolov, G.: On certain definite quadrature formulae. J. Comput. Appl. Math. 75, 329–343 (1996)
12. Nikolov, G., Simian, C.: Gauss-type quadrature formulae for parabolic splines with equidistant knots. In: Gautschi, W., Mastroianni, G., Rassias, Th.M. (eds.) Approximation and Computation (In Honor of Gradimir V. Milovanovic), Springer Optimization and its Applications, pp. 207–229. Springer Verlag, Berlin–Heidelberg–New York (2010)
13. Zhensykbaev, A.: Best quadrature formulae for some classes of periodic differentiable functions. Izv. Akad. Nauk SSSR Ser. Mat. 41 (1977) (in Russian); English Translation in: Math. USSR Izv. 11, 1055–1071 (1977)
14. Zhensykbaev, A.: Monosplines and optimal quadrature formulae for certain classes of non-periodic functions. Anal. Math. 5, 301–331 (1979) (in Russian).

On a Generalized Delaporte Distribution

Meglena D. Lazarova[1,2]

[1] University of National and World Economy,
Studentski grad “Hristo Botev”, 8th December Blvd., 1700 Sofia, Bulgaria
meglena.laz@unwe.bg
[2] Faculty of Mathematics and Informatics, Sofia University “St. Kliment Ohridski”,
5, James Bourchier Blvd., 1164 Sofia, Bulgaria

Abstract. In this paper we consider a new mixed Pólya–Aeppli distribution with shifted gamma mixing distribution. We analyze additionally the interpretation as a compound Delaporte distribution with geometric compounding distribution and called it an Inflated-parameter Delaporte distribution (I-Delaporte). Then we give the relation between the defined distribution and well known non-central negative binomial distribution. Probability mass function, recursion formulas and some properties of the defined distribution are given.

Keywords: I-Delaporte distribution, mixed distribution, compound distribution

1 Introduction

The Inflated-parameter negative binomial distribution (INB) was introduced in Minkova [3] as a compound negative binomial distribution with geometric compounding distribution. If N has an INB distribution, it has the form $N = X_1 + \cdots + X_Y$, where $X_1, X_2, \ldots$ are independent, geometrically distributed as the r.v. X with probability mass function (PMF)

$$P(X = i) = (1 - \rho)\rho^{i-1}, \quad i = 1, 2, \ldots \tag{1}$$

and probability generating function (PGF)

$$\psi_1(s) = Es^X = \frac{(1 - \rho)s}{1 - \rho s}. \tag{2}$$

The counting random variable Y is independent of X and has a negative binomial distribution with parameters r and π, $NB(r, \pi)$.

The second distribution analyzed in Minkova [3] is the Pólya–Aeppli distribution. It is a compound Poisson with geometric compounding distribution, given in (1) and (2).

In this paper we analyze the convolution of INB distribution and Pólya–Aeppli distribution. It is a Generalized Delaporte distribution and was defined in Minkova [2]. We call it I-Delaporte distribution. The I-Delaporte distribution is a generalization of the non-central negative binomial distribution, $NNBD$, defined by Ong and Lee [5], and developed in [4, 6]. The $NNBD$ is a convolution of independent negative binomial and Pólya–Aeppli distribution.

In Section 2 we obtain the PGF of the I-Delaporte distribution. The PMF, recursion formulas and some properties are given in Sect. 3. The moments and Fisher index of dispersion are obtained in Sect. 4.

2 Probability Generating Function

The Delaporte distribution is given in [1] as a counting distribution in risk models. It is a sum of independent negative binomial distribution (NBD), and Poisson distribution. In this section we define a generalized Delaporte distribution, which is a mixed Pólya–Aeppli distribution with shifted Gamma mixing distribution.

Suppose the random variable N with a given parameter $\lambda > 0$ has a Pólya–Aeppli distribution, i.e.,

$$P(N = k \mid \lambda) = \begin{cases} e^{-\lambda}, & k = 0, \\ e^{-\lambda} \sum_{i=1}^{k} \binom{k-1}{i-1} \frac{[\lambda(1-\rho)]^i}{i!} \rho^{k-i}, & k = 1, 2, \dots, \end{cases} \tag{3}$$

where $\rho \in [0, 1)$ is a parameter. We use the notation $N \sim PA(\lambda, \rho)$. The mean of N is $E(N) = \frac{\lambda}{1-\rho}$. The probability generating function (PGF) of the Pólya–Aeppli distribution with given parameter λ is given by

$$\psi_N(s \mid \lambda) = e^{-\lambda(1-\psi_1(s))}, \tag{4}$$

where $\psi_1(s)$ is the PGF of the compounding distribution given in (2).

Let the mixing distribution be a shifted Γ-distribution with density function given by

$$g(\lambda) = \frac{\beta^r}{\Gamma(r)} (\lambda - \alpha)^{r-1} e^{-\beta(\lambda-\alpha)}, \quad \beta > 0, \ \lambda > \alpha. \tag{5}$$

The parameter α can be interpreted as a basic risk, see Grandell [1].

Mixing the parameter λ in (4) with the mixing distribution (5) we obtain the following PGF

$$\begin{aligned} \psi_N(s) &= \left[\frac{\pi}{1-(1-\pi)\psi_1(s)}\right]^r e^{-\alpha(1-\psi_1(s))} \\ &= \left[\frac{\pi(1-\rho s)}{1-(1-\pi(1-\rho))s}\right]^r e^{-\alpha\left(1-\frac{(1-\rho)s}{1-\rho s}\right)}, \end{aligned} \tag{6}$$

where $\pi = \frac{\beta}{1+\beta}$.

Remark 1. In the case $\rho = 0$, the compounding variable X degenerates at point one and the distribution of N coincides with the Delaporte distribution, Grandell [1].

This motivates the following definition, see [2].

Definition 1. *The random variable N with PGF* (6) *is referred to as an Inflated-parameter Delaporte distribution* (*I-Delaporte distribution*).

3 Probability Mass Function

The unconditional probability mass function of the I-Delaporte distribution is the following:

$$P(N=m)=\int_{\alpha}^{\infty} P(N=m\mid\lambda)\frac{\beta^r}{\Gamma(r)}(\lambda-\alpha)^{r-1}e^{-\beta(\lambda-\alpha)}\,d\lambda. \tag{7}$$

Calculating the integral in (7) gives the following probability mass function

$$P(N=m)=\begin{cases}\left(\frac{\beta}{\beta+1}\right)^r e^{-\alpha}, & m=0,\\ \left[r(1-\rho)\frac{1}{\beta+1}+\alpha(1-\rho)\right]e^{-\alpha}, & m=1,\\ \left[\sum\limits_{i=1}^{m}\binom{m-1}{i-1}\binom{r+i-1}{i}\left((1-\rho)\frac{1}{\beta+1}\right)^i\rho^{m-i}\right. & \\ +\sum\limits_{i=1}^{m}\binom{m-1}{i-1}\frac{[\alpha(1-\rho)]^i}{i!}\rho^{m-i} & \\ +\sum\limits_{i=1}^{m-1}\sum\limits_{j=1}^{i}\binom{i-1}{j-1}\frac{[\alpha(1-\rho)]^j}{j!}\rho^{i-j} & \\ \left.\times\sum\limits_{k=1}^{m-i}\binom{m-i-1}{k-1}\binom{r+k-1}{k}\left((1-\rho)\frac{1}{\beta+1}\right)^k\rho^{m-i-k}\right]e^{-\alpha}, & m=2,3,\dots\end{cases}$$

From (6) it follows that the random variable N is a sum $N=N_1+N_2$, where N_1 and N_2 are independent, N_1 has a PMF

$$P(N_1=m)=\begin{cases}\left(\frac{\beta}{\beta+1}\right)^r, & m=0\\ \left(\frac{\beta}{\beta+1}\right)^r\sum\limits_{i=1}^{m}\binom{m-1}{i-1}\binom{r+i-1}{i}\left[(1-\rho)\frac{1}{\beta+1}\right]^i\rho^{m-i}, & m=1,2,\dots\end{cases} \tag{8}$$

and $N_2\sim PA(\alpha,\rho)$.

The random variable in (8) has an Inflated-parameter negative binomial distribution with parameters $\frac{\beta}{1+\beta}$, ρ and r, see Minkova [3]. We say shortly I-negative binomial and use the notation $N_1\sim INB\left(r,\frac{\beta}{1+\beta},\rho\right)$.

Remark 2. The random variable N can be represented as a compound Delaporte distribution, i.e., $N=X_1+\cdots+X_{N_0}$, where N_0 has a Delaporte distribution and X_i, $i=1,2,\dots$, are independent, geometrically distributed with parameter $1-\rho$ random variables, independent of N_0.

According the idea of Ong and Lee [5], we can interpret the distribution in the following way. Suppose that the PGF of the r.v. has the form

$$\left[\frac{\pi}{1-(1-\pi)\psi_1(s)}\right]^T,$$

where $T=r+V$, $r>0$ is a constant and V has a Poisson distribution with parameter α. Then the resulting PGF is given by (6).

Proposition 1. *The PMF of the I-Delaporte distribution satisfies the following recursions*

$$\begin{aligned}
p_i &= \left[2\rho + 1 - \pi(1-\rho) + \frac{(1-\rho)(r(1-\pi)+\alpha) - 2\rho - (1-\pi(1-\rho))}{i}\right] p_{i-1} \\
&\quad - \left[\rho(\rho + 2(1-\pi(1-\rho))) + \frac{(1-\rho)[r(1-\pi)\rho + \alpha(1-\pi(1-\rho))]}{i}\right] p_{i-2} \\
&\quad + \rho^2\left(1 - \frac{3}{i}\right)(1-\pi(1-\rho))p_{i-3}, \quad i = 3, 4, \ldots, \\
p_2 &= \left[2\rho + 1 - \pi(1-\rho) + \frac{(1-\rho)(r(1-\pi)+\alpha) - 2\rho - (1-\pi(1-\rho))}{2}\right] p_1 \\
&\quad - (1-\rho)\left[r(1-\pi)\rho + \alpha(1-\pi(1-\rho))\right] p_0, \\
p_1 &= (1-\rho)\left[r(1-\pi) + \alpha\right] p_0, \\
p_0 &= \pi^r e^{-\alpha}.
\end{aligned}$$

Proof. Upon substituting $s = 0$ in the PGF in (6) we obtain the initial value p_0. Differentiation in (6) leads to

$$\psi'(s) = \frac{1-\rho}{1-\rho s}\left[\frac{r(1-\pi)}{1-(1-\pi(1-\rho))s} + \frac{\alpha}{1-\rho s}\right]\psi(s), \tag{9}$$

where $\psi(s) = \sum_{i=0}^{\infty} p_i s^i$ and $\psi'(s) = \sum_{i=0}^{\infty}(i+1)p_{i+1}s^i$. The recursions are obtained by equating the coefficients of s^i on both sides for fixed $i = 0, 1, 2, \ldots$. □

Corollary 1. *The PMF of the I-Delaporte distribution satisfies the following recursions*

$$\begin{aligned}
ip_i &= (1-\rho)[r(1-\pi) + (i-1)\rho]p_{i-1} \\
&\quad + (1-\rho)\sum_{m=0}^{i-2}[r(1-\pi)(1-\pi(1-\rho))^{i-1-m} + \alpha\rho^{i-1-m}]p_m, \quad i = 2, 3, \ldots, \\
p_1 &= (1-\rho)[r(1-\pi) + \alpha]p_0, \\
p_0 &= \pi^r e^{-\alpha}.
\end{aligned}$$

Proof. The required recursions are obtained again from (9). □

4 Moments

The mean and the variance of the I-Delaporte distribution are given by

$$E(N) = \left(\alpha + \frac{r}{\beta}\right)\frac{1}{1-\rho}$$

and

$$Var(N) = \left[\alpha(1+\rho) + \frac{r((1+\rho)\beta+1)}{(1-\rho)\beta^2}\right]\frac{1}{1-\rho}.$$

For the Fisher index we obtain

$$FI(N(t)) = \frac{1+\rho}{1-\rho} + \frac{r-\alpha\beta^2\rho(1+\rho)}{(1-\rho)\beta(r+\alpha\beta)}.$$

If $\alpha\beta^2\rho(1+\rho) > r$, the I-Delaporte distribution is under-dispersed related to Pólya–Aeppli distribution. In the case of $\alpha\beta^2\rho(1+\rho) < r$, it is over-dispersed.

5 Concluding remark

In this paper we have analyzed the I-Delaporte distribution with PMF, PGF, some properties, recursion formulas and moments.

Acknowledgments. This work was supported by the European Social Fund through the Human Resource Development Operational Programme under contract BG051PO001-3.3.06-0052 (2012/2014).

References

1. Grandell, J.: Mixed Poisson Processes. Chapman & Hall, London (1997)
2. Minkova, L.D.: Distributions in Insurance Risk Models. Doctor of Science Thesis (2012). Available on `http://www.fmi.uni-sofia.bg/`
3. Minkova, L.D.: A generalization of the classical discrete distributions. Commun. Statist. – Theory and Methods 31(6), 871–888 (2002)
4. Ong, S.H., Chakraborty, S., Imoto, T., Shimizu, K.: Generalizations of non central negative binomial, Charlier series distributions, and their extensions by Lagrange expansion. Commun. Statist. – Theory and Methods 41(4), 571–590 (2012)
5. Ong, S.H., Lee, P.A.: The non-central negative binomial distribution. Biom. J. 21, 611–628 (1979)
6. Ong, S.H., Shimizu, K.: A discrete distribution arising as a solution of a linear difference equation: extension of the non central negative binomial distribution. Commun. Statist. – Theory and Methods 38(6), 927–938 (2009)

Pólya–Aeppli Risk Model with Two Lines of Business

Krasimira Y. Kostadinova[1,2]

[1] Faculty of Mathematics and Informatics, Shumen University "K. Preslavsky",
115, Universitetska St, 9700 Shumen, Bulgaria
kostadinova@shu-bg.net
[2] Faculty of Mathematics and Informatics, Sofia University "St. Kliment Ohridski",
5, James Bourchier Blvd., 1164 Sofia, Bulgaria

Abstract. In this paper we define a risk model with two lines of business with Pólya–Aeppli counting processes and call this model double Pólya–Aeppli model. We define an exponential martingale related to this risk model and obtain the corresponding martingale approximation. The case of exponentially distributed claims is analyzed.

Keywords: Pólya–Aeppli process, ruin probability, double Pólya–Aeppli risk model, martingale approximation
AMS subject classifications: 60K10; 62P05.

1 Introduction

Assume that the risk model of an insurance company, called risk process $\{Z(t), t \geq 0\}$ is given by

$$Z(t) = ct - \sum_{i=1}^{M(t)} X_i - \sum_{i=1}^{N(t)} Y_i, \quad \left(\sum_1^0 = 0 \right). \tag{1}$$

Here c is a positive real constant representing the risk premium rate. The sequences $\{X_i\}_{i=1}^{\infty}$ and $\{Y_i\}_{i=1}^{\infty}$ of mutually independent random variables with the corresponding distribution functions F_X and F_Y with $F_X(0) = 0$ and $F_Y(0) = 0$ are independent of the counting processes $M(t)$, $t \geq 0$ and $N(t)$, $t \geq 0$. The processes $M(t)$ and $N(t)$ are interpreted as the number of first and second type of claims to the insurance company during the interval $[0, t]$. We suppose that the counting processes are Pólya–Aeppli distributed with the corresponding parameters λ_i and ρ_i. We use the notations $M(t) \sim PA(\lambda_1, \rho_1)$ and $N(t) \sim PA(\lambda_2, \rho_2)$. Denote by $S_1(t) = \sum_{i=1}^{M(t)} X_i$ and $S_2(t) = \sum_{i=1}^{N(t)} Y_i$ the corresponding accumulated claim processes of the business lines. The risk model with a single type of business and Pólya–Aeppli distributed counting process is introduced and analyzed in Minkova [5].

In this paper, in Sect. 2 we give the Pólya–Aeppli process. The double Pólya–Aeppli risk model is introduced in Sect. 3. In Section 4 we define an exponential martingale, related to double Pólya–Aeppli risk model. The martingale approach to the defined risk model is given in Sect. 5. Finally, in Sect. 6 we consider the case of exponentially distributed claims. An estimation of the Lundberg exponent is obtained.

2 Pólya–Aeppli Process

The Pólya–Aeppli distribution is a generalization of the classical $Po(\lambda)$ distribution, by adding a new parameter ρ, see [3]. It appears in [4] as a compound Poisson distribution. The additional parameter ρ is called an inflation parameter. The Pólya–Aeppli process as a generalization of the Poisson process was defined in [5]. The characterization is given in [1].

We will suppose that $M(t)$ is described by the Pólya–Aeppli distribution with mean function $\frac{\lambda_1}{1-\rho_1}t$, i.e.

$$P(M(t)=n)=\begin{cases} e^{-\lambda_1 t}, & n=0,\\ e^{-\lambda_1 t}\sum_{i=1}^{n}\binom{n-1}{i-1}\frac{[\lambda_1(1-\rho_1)t]^i}{i!}\rho_1^{n-i}, & n=1,2,\dots. \end{cases} \tag{2}$$

We use the notation $M(t)\sim PA(\lambda_1 t,\rho_1)$. For the process $N(t)$ we suppose that $N(t)\sim PA(\lambda_2 t,\rho_2)$.

The probability generating functions (PGF) of $M(t)$ and $N(t)$ are given by

$$\psi_{M(t)}(s)=e^{-\lambda_1 t\left(1-\frac{(1-\rho_1)s}{1-\rho_1 s}\right)} \quad\text{and}\quad \psi_{N(t)}(s)=e^{-\lambda_2 t\left(1-\frac{(1-\rho_2)s}{1-\rho_2 s}\right)}.$$

3 The Double Pólya–Aeppli Risk Model

We consider the risk process $Z(t)$, defined by (1), where $M(t)$ and $N(t)$ are independent Pólya–Aeppli processes, independent of the corresponding claim sizes X_i, $i=1,2,\dots$ and Y_i, $i=1,2,\dots$. We call the process (1) a double Pólya–Aeppli risk model. For the classical risk model see [2] and [6]. Let $E(X)=\mu$ and $E(Y)=\nu$ and suppose that the means and variances are finite. The relative safety loading θ is defined by

$$\theta=\frac{c}{\frac{\lambda_1}{1-\rho_1}\mu+\frac{\lambda_2}{1-\rho_2}\nu}-1, \tag{3}$$

and in the case of positive safety loading $\theta>0$, we have $c>\frac{\lambda_1\mu}{1-\rho_1}+\frac{\lambda_2\nu}{1-\rho_2}$.

Denote by

$$\tau(u)=\inf\{t>0, u+Z(t)\le 0\}$$

the time to ruin of a company having initial capital u. We let $\tau=\infty$, if for all $t>0$, $u+Z(t)>0$.

The probability of ruin in the infinite horizon case is

$$\Psi(u)=P(\tau(u)<\infty)$$

and in the finite horizon case

$$\Psi(u,t)=P(\tau(u)\le t).$$

4 Martingales for Double Pólya–Aeppli Risk Model

Let us denote by $LS_X(r) = \int_0^\infty e^{-rx}\, dF_X(x)$ the Laplace–Stieltjes transform (LS-transform) of any random variable X with distribution function $F_X(x)$.

Lemma 1. *For the double Pólya–Aeppli risk model*

$$Ee^{-rZ(t)} = e^{-g(r)t},$$

where

$$g(r) = rc + \lambda_1 \frac{1 - LS_X(-r)}{1 - \rho_1 LS_X(-r)} + \lambda_2 \frac{1 - LS_Y(-r)}{1 - \rho_2 LS_Y(-r)}. \tag{4}$$

Proof. Let us consider the random sums $S_1(t)$ and $S_2(t)$ from the right hand side of (1). $S_1(t)$ is a compound Pólya–Aeppli process and the LS-transform is given by

$$LS_{S_1(t)}(r) = E(e^{-rS_1(t)}) = \psi_{M(t)}(LS_X(r)) = e^{-\lambda_1 t \frac{1-LS_X(r)}{1-\rho_1 LS_X(r)}}.$$

Then, for the LS-transform of $Z(t)$ we have the following

$$\begin{aligned}
LS_{Z(t)}(r) &= Ee^{-rZ(t)} = Ee^{-r[ct - S_1(t) - S_2(t)]} = e^{-rct} Ee^{rS_1(t)} Ee^{rS_2(t)} \\
&= e^{-rct} \psi_{M(t)}(LS_X(-r)) \psi_{N(t)}(LS_Y(-r)) \\
&= e^{-rct} e^{-\left(\lambda_1 \frac{1-LS_X(-r)}{1-\rho_1 LS_X(-r)} + \lambda_2 \frac{1-LS_Y(-r)}{1-\rho_2 LS_Y(-r)}\right)t} = e^{-g(r)t},
\end{aligned}$$

where $g(r)$ is given by (4) and $\psi_{M(t)}$ and $\psi_{N(t)}$ are the corresponding PGFs of $M(t)$ and $N(t)$. $\square$

From the martingale theory we get the following

Lemma 2. *For all $r \in \mathbb{R}$ the process*

$$W(t) = e^{-rZ(t) + g(r)t}, \quad t \geq 0 \tag{5}$$

is an $\mathcal{F}_t^Z$-martingale, provided that $LS_X(-r) < \infty$ and $LS_Y(-r) < \infty$.

Proof. For $v \leq t$ and Lemma 1 we have

$$\begin{aligned}
E(W(t)|\mathcal{F}_v^Z) &= E\left[e^{-rZ(t)+g(r)t} | \mathcal{F}_v^Z\right] \\
&= E\left[e^{-rZ(v)+g(r)v} e^{-r(Z(t)-Z(v))+g(r)(t-v)} | \mathcal{F}_v^Z\right] \\
&= W(v) E\left[e^{-r(Z(t)-Z(v))} e^{g(r)(t-v)}\right] \\
&= W(v) e^{-g(r)(t-v)} e^{g(r)(t-v)} = W(v),
\end{aligned}$$

and then (5). $\square$

5 Martingale Approach to the Pólya–Aeppli Risk Model

Using the martingale properties of $W(t)$, we will give some useful inequalities for the ruin probability. For the martingale approach in the classical case see [7].

Proposition 1. *Let $r > 0$. For the ruin probabilities of the double Pólya–Aeppli risk model we have the following results*

i) $\Psi(u,t) \le e^{-ru} \sup_{0\le s\le t} e^{-g(r)s}$, $0 \le t < \infty$.

ii) $\Psi(u) \le e^{-ru} \sup_{s\ge 0} e^{-g(r)s}$.

iii) If the Lundberg exponent R exists, then R is a strictly positive solution of

$$c(1-\rho_1 LS_X(-r))(1-\rho_2 LS_Y(-r))r + \lambda_1(1-LS_X(-r))(1-\rho_2 LS_Y(-r)) \\ + \lambda_2(1-LS_Y(-r))(1-\rho_1 LS_X(-r)) = 0 \quad (6)$$

and $\Psi(u) \le e^{-Ru}$.

Proof. i) Let $t_0 < \infty$. We apply the martingale stopping time theorem to $W(t)$ in (5), and obtain that

$$\begin{aligned} 1 = W_0 = EW_{t_0\wedge\tau} &= E[W_{t_0\wedge\tau}, \tau \le t] + E[W_{t_0\wedge\tau}, \tau > t] \\ &\ge E[W_{t_0\wedge\tau}, \tau \le t] = E[e^{-rZ(\tau)+g(r)\tau} | \tau \le t] P(\tau \le t) \\ &\ge e^{ru} E[e^{g(r)\tau} | \tau \le t] P(\tau \le t). \end{aligned}$$

The last inequality leads to

$$P(\tau \le t) \le \frac{e^{-ru}}{E[e^{g(r)\tau} | \tau \le t]}.$$

Then, the statement i) follows from the above relation.

ii) Taking $t \to \infty$ in i), we obtain the statement ii).

iii) According to the interpretation of the Lundberg exponent (see for example [6]), it is a constant R, such that the process $E(e^{-RZ(t)})$ is a martingale. Following Lemma 1, the constant R is a positive root of the equation $g(r) = 0$, where $g(r)$ is given in (4). The equation can be rewritten as (6). □

6 Exponentially Distributed Claims

Suppose that X is exponentially distributed with mean μ, and Y is exponential with mean ν, i.e., $F_X(x) = 1 - e^{-\frac{x}{\mu}}$ and $F_Y(x) = 1 - e^{-\frac{x}{\nu}}$, $x \ge 0$. For the LS-transform we have

$$LS_X(-r) = \frac{1}{1-\mu r} \quad \text{and} \quad LS_Y(-r) = \frac{1}{1-\nu r}.$$

Then, for the function $g(r)$ in (4) in this case we obtain

$$g(r) = cr - \lambda_1 \frac{\mu r}{1 - \rho_1 - \mu r} - \lambda_2 \frac{\nu r}{1 - \rho_2 - \nu r}.$$

The equation $g(r) = 0$ has a zero root $r_1 = 0$. Then, for the equation

$$c\mu\nu r^2 + [(\lambda_1 + \lambda_2)\mu\nu - c\mu(1 - \rho_2) - c\nu(1 - \rho_1)]r + c(1 - \rho_1)(1 - \rho_2) - \lambda_1\mu(1 - \rho_2) - \lambda_2\nu(1 - \rho_1) = 0 \quad (7)$$

we obtain two positive roots. Denote the equation (7) as $f(r) = 0$. It follows from (7) that $f(0) = c(1-\rho_1)(1-\rho_2)-\lambda_1\mu(1-\rho_2)-\lambda_2\nu(1-\rho_1) > 0$. Consequently, the equation (7) has two positive roots. From $f'(r) = 0$, we obtain the following inequality for the largest root R of equation (7)

$$R > \frac{1}{2}\left[\frac{1 - \rho_1}{\mu} + \frac{1 - \rho_2}{\nu} - \frac{\lambda_1 + \lambda_2}{c}\right].$$

7 Concluding Remarks

In this paper we have introduced a risk model with two lines of business, where the counting processes are independent Pólya–Aeppli processes. For this model we have obtained a martingale approach and the corresponding estimation of the ruin probability. In the case of exponentially distributed claims an estimation of the Lundberg exponent is given.

Acknowledgments. The author is thankful to the anonymous reviewer and the editor for making some useful comments and suggestions. This work was supported by the European Social Fund through the Human Resource Development Operational Programme under contract BG051PO001-3.3.06-0052 (2012/2014) and by grant RD-08-230/2014 of Shumen University, Bulgaria.

References

1. Chukova, S., Minkova, L.D.: Characterization of the Pólya–Aeppli process. Stochastic Analysis and Applications 31, 590–599 (2013)
2. Grandell, J.: Aspects of Risk Theory. Springer-Verlag, New York (1991)
3. Johnson, N.L., Kemp, A.W., Kotz, S.: Univariate Discrete Distributions. Wiley Series in Probability and Mathematical Statistics, 3 edn. (2005)
4. Minkova, L.D.: A generalization of the classical discrete distributions. Commun. Statist. – Theory and Methods 31(6), 871–888 (2002)
5. Minkova, L.D.: The Pólya–Aeppli process and ruin problems. J. Appl. Math. Stoch. Anal. 3, 221–234 (2004)
6. Rolski, T., Schmidl, H., Schmidt, V., Teugels, J.: Stochastic Processes for Insurance and Finance. John Wiley & Sons (1999)
7. Schmidli, H.: Martingales and insurance risk. In: 8th International Summer School on Probability Theory and Mathematical Statistics. pp. 155–188 (1996)

Multitype Branching Processes with Bivariate Multinomial Offspring Distribution – Bayesian Approach

Ana Staneva

[1] Faculty of Mathematics and Informatics, Sofia University "St. Kliment Ohridski",
5, James Bourchier Blvd., 1164 Sofia, Bulgaria
[2] Technical University of Sofia, Bulgaria
anastaneva@gmail.com

Abstract. In the present work we consider the class of discrete time multitype branching processes with two types of particles. We suppose that the offspring distribution is multinomial, as a special case of the Bivariate Power Series Distributions. The estimation of the individual parameters is an important part of the statistical inference for the branching processes. One of the recent approaches in the branching estimation theory is the Bayesian estimation. We find the conjugate distributions for the multinomial distribution and for its reparametrized power series form and study the posterior estimators.

1 Introduction

The branching processes form an important subclass of the stochastic processes with numerous applications in different fields of theory and practice, many of them involving multitype modeling. Generally speaking, there is a number of objects, often called particles, cells, individuals, which, according to some probabilistic law, reproduce (or "branch"), giving birth to some non-negative number of offspring, who form the next generation, and die out. They can be of multiple types and may have different locations in space. Their evolution and generation may be independent or according to certain probabilistic laws.

In the present article we consider a model of a two type discrete time branching process with trinomial offspring distibution. This type of processes is a subclass of the multitype processes, introduced and handled in the pioneering works of Kolmogorov and Dmitriev [19] and Kolmogorov and Sevastyanov [20] from 1947. The number of offspring of the individuals in these processes are modeled as independent and identically distributed random variables, as within and between generations.

Since 1947 there is an impressive number of work in the area of branching processes theory and applications (see f.e. the books of Asmussen and Herring [1], Athreya and Ney [2], Harris [13], Jagers [15], Sevastyanov [25], Yakovlev and Yanev [30], etc).

The statistical estimation of the process' characteristics like the mean number of offspring, the criticality of the process, the offspring distribution and others, is an important issue in their study. Some of the most resent approaches devoted to the statistical inference for branching processes can be found in González et al. [12]. The work of Jacob [14] gives a comprehensive overview of the theoretical and statistical methods used in epidemiology. The importance of simulation, computing and more flexible statistical procedures can also be traced in González et al. [11].

As in other fields of statistics, there are different approaches for estimation – parametric, nonparametric and semiparametric settings. Recent papers considering the nonparametric estimation in multitype branching processes are those of Yakovlev and Yanev [31, 32] and Yakovlev et al. [29] focusing on applications in cell biology. In [11] the authors consider the Bayesian nonparametric estimation of multitype branching processes. On the other hand, as part of the parametric approach one can use the exact offspring distribution in a specified parametric family like the multivariate power series. The point estimation – classical, robustified, or bayesian, in the multivariate power series family is also of interest in itself. In the class of the univariate power series offspring distributions some topics of the parametric estimation are considered in Stoimenova and Yanev [27] and of the robust parametric estimation – in Stoimenova [28]. The multivariate power series enjoy a strong interest in many papers. In [10] a bisexual model with population size dependent mating with the trinomial offspring distribution is considered. In [3] there is an example how the trinomial distribution can be applied to analyse and model the human population in India. Although there are numerous examples for applications of the multivariate power series, their implementation and estimation in the branching processes is still subject of interest and research. In the present paper we focus on the Bayesian parametric approach for estimation of two-type branching processes with trinomial offspring distribution.

2 Two Type Branching Processes

Let us consider a multitype branching processes with two types of particles

$$Z(n+1) = (Z_1(n+1), Z_2(n+1)),$$

where $Z(n+1)$ denotes the size of the $(n+1)$-st generation, which may be split into two groups according to the type of the particle. Consequently $Z(n+1)$ is a bivariate vector $(Z_1(n+1), Z_2(n+1))$, where $Z_k(n+1)$ is the number of individuals of type k in the $(n+1)$-st generation.

On the other hand, the individuals of type k in the $(n+1)$-st generation are obtained as a sum of the numbers of offspring of type k of the individuals living in the previous (n-th) generation (the so-called "branching property")

$$Z_k(n+1) = \sum_{s=1}^{Z_1(n)} X_{is}^k(n) + \sum_{s=1}^{Z_2(n)} X_{is}^k(n), \quad k = 1, 2,$$

where $X_{is}^k(n)$ denotes the number of children of type k of the s-th particle of type i in the $(n+1)$-st generation.

Let

$$Z_k(n,(i,j)) = \sum_{s=1}^{Z_k(n)} I_{\{X_{ks}(n)=(i,j)\}}$$

be the number of particles in the n-th generation with exactly (i,j) offspring. Here $I_{\{X_{ks}(n)=(i,j)\}}$ is the indicator variable for the event that a particle of type k has i children of type 1 and j children of type 2.

The following basic relations hold

$$Z_k(n) = \sum_{(i,j)} Z_k(n,(i,j)).$$

Furthermore, if $Z_s^k(n+1)$ denotes the number of particles of type s in the $(n+1)$-th generation with father of type k, it can be easily seen that

$$Z_1(n+1) = Z_1^1(n+1) + Z_1^2(n+1) = \sum_i iZ_1(n,(i,j)) + \sum_i iZ_2(n,(i,j))$$

and

$$Z_2(n+1) = Z_2^1(n+1) + Z_2^2(n+1) = \sum_j jZ_1(n,(i,j)) + \sum_j jZ_2(n,(i,j)).$$

Let p_{ij}^k denote the probability that an individual of type k has (i,j) offspring.

In the statistical inference of branching processes usually the following three sampling schemes are used:

- One observes the entire family tree up to the N-th generation (the number of offspring of each particle):

$$\tilde{\tilde{\mathcal{J}}}(N) = \{X_{is}(n) : s = 1,2,\ldots,Z_i(n);\ i = 1,2;\ n = 0,\ldots,N-1\}.$$

- Number of particles with $(i,j) \in J_k$ offspring is known, where J_k is the support of the offspring distribution:

$$\tilde{\mathcal{J}}(N) = \{Z_k(n,(i,j)) : (i,j) \in J_k;\ k = 1,2;\ n = 0,\ldots,N-1\}.$$

- One can observe the generation sizes only:

$$\mathcal{J}(N) = \{Z(0),\ldots,Z(N)\}.$$

3 The Multinomial Offspring Distribution

Suppose that we observe n independent trials where each trial has $d+1$ possible outcomes $\{0,1,\ldots,d\}$ with probabilities $\{p_0,p_1,\ldots,p_d\}$ respectively, where $\sum_{i=0}^{d} p_i = 1$. Let us denote by $\theta = (p_0,p_1,\ldots,p_d)$.

Let $X_0, X_1, \ldots, X_d$ be random variables describing the number of the outcomes in the categories $\{0, 1, \ldots, d\}$ for n trials. Then the joint p.d.f. of $(X_0, X_1, \ldots, X_d)$ is ([26])

$$f_\theta(x_0, x_1, \ldots, x_d) = \frac{n!}{\prod\limits_{i=0}^{d} x_i!} \prod_{i=0}^{d} (p_i)^{x_i} I_B(x_0, x_1, \ldots, x_d), \tag{1}$$

where $\sum\limits_{i=0}^{d} p_i = 1$, $B = \{(x_0, x_1, \ldots, x_d) \mid x_i \geq 0,\ i = 0, \ldots, d,\ \sum\limits_{i=0}^{d} x_i = n\}$ and I_B is the indicator of the set B.

Definition 1 ([26]). *The parametric family* $\{f_\theta : \theta \in \Theta\}$, *where*

$$\Theta = \{\theta = (p_0, p_1, \ldots, p_d) : 0 < p_i < 1,\ \sum_{i=0}^{d} p_i = 1\} \subset R$$

is called a multinomial family.

The distribution of $(X_0, X_1, \ldots, X_d)$ *is called a multinomial distribution.*

The multinomial family belongs to the natural exponential family with natural parameter

$$\eta = (\log p_0, \log p_1, \ldots, \log p_d).$$

In fact, if we introduce notation

$$h(x) = \frac{n!}{\prod\limits_{i=0}^{d} x_i!} I_B(x), \quad x = (x_0, x_1, \ldots, x_d),$$

then we obtain that

$$f_\theta(x) = \exp\{\eta^T x\} h(x), \quad x \in R^{d+1}.$$

This representation of the exponential family is not of full rank.

However, since $\sum\limits_{i=0}^{d} X_i = n$ and $\sum\limits_{i=0}^{d} p_i = 1$, we can use the reparametrization

$$\eta = \left(\log \frac{p_1}{1 - \sum\limits_{i=1}^{d} p_i}, \ldots, \log \frac{p_d}{1 - \sum\limits_{i=1}^{d} p_i} \right), \quad \nu(\eta) = -n \log \left(1 - \sum_{i=1}^{d} p_i \right).$$

Hence the following form of the multinomial family can be obtained

$$f_\theta(x) = \exp\{\eta^T x - \nu(\eta)\} h(x), \quad x = (x_1, \ldots, x_d) \in R^d.$$

The latter is the natural exponential family form of full rank.

It can be also easily seen that after a suitable reparametrization the multinomial distribution belongs to the multivariate power series distribution family. This family is a natural generalization of the univariate power series distribution family and a subclass of the the multivariate discrete exponential family, hence inheriting its properties for the moments, cumulants, covarances, additiveness and so on. There are many sources concerning the properties and applications of the multivariate power series distributions. Among them we mention the pioneering papers of Khatri [17], Patil [23], Gerstenkorn [4] and the thorough books on discrete multivariate distributions of Johnson et al. [16] and discrete bivariate distributions of Kocherlakota [21].

Let us denote by

$$p_{ij}^k = P(X_{ks}(n) = (i,j)), \quad k = 1,2;\ n = 0,1,\dots;\ s = 1,2,\dots,Z(t-1)$$

the bivariate joint distribution (offspring distribution, offspring law) of the random vector $X_{ks}(n)$.

The bivariate power series family can be described in terms of the offspring distribution in the following way:

$$p_{i,j}^k = \frac{a_k(i,j)\theta_{1k}^i\theta_{2k}^j}{A_k(\theta_{1k},\theta_{2k})} \quad i,j \in N_0; \quad k = 1,2, \tag{2}$$

where $(\theta_{1k},\theta_{2k}) \in \Theta \subset R_2^+$ is the vector of positive parameters from the parameter space Θ_k, $a_k(i,j) > 0$ is nonnegative real-valued function of the random vector values, which may depend on some parameters, but does not depend on θ_{1k} and θ_{2k}, and

$$A_k(\theta_1,\theta_2) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_k(i,j)\theta_1^i\theta_2^j. \tag{3}$$

We recall that the function $A_k(\theta_{1k},\theta_{2k})$ is called defining function of the distribution. Note that the form (3) is the second-order Taylor expansion of the scalar-valued function of more than one variable $A_k(\theta_{1k},\theta_{2k})$ in a bivariate power series form. The coefficient $a_k(i,j)$ in the expansion is called a coefficient function.

Also, the trinomial distribution (the multinomial distribution (1) in the bivariate case) can be expressed in the following form

$$p_{i,j}^k = \frac{n!}{i!j!(n-i-j)!}p_{1k}^i p_{2k}^j (1-p_{1k}-p_{2k})^{(n-i-j)}, \tag{4}$$

where p_{1k} and p_{2k} are the parameters from (1) for the type k particle.

And according Khatri [18] after the reparametrization

$$\theta_{ik} = \frac{p_{ik}}{1-p_{1k}-p_{2k}},$$

or equivalently,

$$p_{1k} = \frac{\theta_{1k}}{1+\theta_{1k}+\theta_{2k}},\ p_{2k} = \frac{\theta_{2k}}{1+\theta_{1k}+\theta_{2k}},\ 1-p_{1k}-p_{2k} = \frac{1}{1+\theta_{1k}+\theta_{2k}}$$

one derives the bivariate power series offspring distribution form (2)

$$p_{i,j}^{k}=\frac{\frac{n!}{i!j!(n-i-j)!}\theta_{1k}^{i}\theta_{2k}^{j}}{(1+\theta_{1k}+\theta_{2k})^{n}},$$

where the function $A_k(\theta_{1k},\theta_{2k})=(1+\theta_{1k}+\theta_{2k})^n$.

4 The Likelihood Function

In this section we obtain the likelihood function in the case of trinomial offspring distribution and derive the maximum likelihood estimators (MLE) for the parameters p_{ik} and θ_{ik}.

When we consider the sampling scheme $\widetilde{\mathcal{J}}(N)$, the corresponding likelihood function has the form

$$L(\widetilde{\mathcal{J}}(N)\mid\boldsymbol{\theta})=\prod_{k=1}^{2}\prod_{l=1}^{N-1}\prod_{(i,j)}(p_{ij}^{k})^{Z_k(l,(i,j))}.$$

Let us now assume that the offspring distribution belongs to the multinomial family (4). Then

$$L(\widetilde{\mathcal{J}}(N)\mid\boldsymbol{\theta})$$
$$=\prod_{k=1}^{2}\prod_{l=1}^{N-1}\prod_{(i,j)}\left(\frac{n!}{i!j!(n-i-j)!}p_{1k}^{i}p_{2k}^{j}(1-p_{1k}-p_{2k})^{n-i-j}\right)^{Z_k(l,(i,j))}$$

Hence

$$L(\widetilde{\mathcal{J}}(N)\mid\boldsymbol{\theta})=\prod_{k=1}^{2}\prod_{l=1}^{N-1}\frac{\left(\prod_{(i,j)}\frac{n!}{i!.j!.(n-i-j)!}\right).p_{1k}^{\sum\limits_{(i,j)}i.Z_k(l,(i,j))}.p_{2k}^{\sum\limits_{(i,j)}j.Z_k(l,(i,j))}}{(1-p_{1k}-p_{2k})^{\sum\limits_{(i,j)}(i+j-n).Z_k(l,(i,j))}}$$
$$=\prod_{k=1}^{2}\prod_{l=1}^{N-1}C_k(l).p_{1k}^{Z_1^k(l+1)}.p_{2k}^{Z_2^k(l+1)}\,(1-p_{1k}-p_{2k})^{n.Z_k(l)-Z_1^k(l+1)-Z_2^k(l+1)}$$
$$=\prod_{k=1}^{2}C_k.p_{1k}^{\sum\limits_{l=1}^{N}Z_1^k(l)}.p_{2k}^{\sum\limits_{l=1}^{N}Z_2^k(l)}\,(1-p_{1k}-p_{2k})^{\left(\sum\limits_{l=0}^{N-1}n.Z_k(l)-Z_k(l+1)\right)}.$$

From the conditions $\partial L/\partial p_{1k}=0$, $\partial L/\partial p_{2k}=0$ we find the MLE for the probabilities p_{1k} and p_{2k}

$$\widehat{p}_{1k}=\frac{\sum\limits_{l=1}^{N}Z_1^k(l)}{\sum\limits_{l=0}^{N-1}nZ_k(l)},\quad \widehat{p}_{2k}=\frac{\sum\limits_{l=1}^{N}Z_2^k(l)}{\sum\limits_{l=0}^{N-1}nZ_k(l)}.$$

It is well known that the relationship between the MLE's $\hat{p}_{ik}$ and $\hat{\theta}_{ik}$ is the same as the relationship between p_{ik} and the natural parameter θ_{ik}.

Consequently

$$\hat{\theta}_{ik} = \frac{\hat{p}_{ik}}{1 - \hat{p}_{1k} - \hat{p}_{2k}} = \frac{\sum\limits_{l=1}^{N} Z_i^k(l)}{\sum\limits_{l=0}^{N-1} Z_k(l)}.$$

5 The Parametric Bayes

The Bayesian approach to statistical design and analysis is effective and practical alternative to the frequentist one.

Gonzalez et al. [11] apply the non-parametric Baysian approach to multitype branching processes. In [5, 7, 8] some aspects of multinomial distribution are investigated. Rufo, Perez and Martin [24] discuss the Bayesian estimation for the multinomial family distribution from the exponential family point of view.

In the present work we use the parametric Bayesian approach, applying for the trinomial offspring distribution in multitype branching processes.

In this case the likelihood function can be written in the following form:

$$L(\widetilde{\mathcal{J}}(n) \mid \boldsymbol{p}) = \prod_{k=1}^{2} C_k p_{1k}^{\sum\limits_{l=1}^{N} Z_1^k(l)} p_{2k}^{\sum\limits_{l=1}^{N} Z_2^k(l)} (1 - p_{1k} - p_{2k})^{\left(\sum\limits_{l=0}^{N-1} n.Z_k(l) - Z_k(l+1)\right)},$$

where C_k does not depend on the unknown parameters.

For a conjugate prior distribution of the random vector $\boldsymbol{p}_k = (p_{1k}, p_{2k}, 1 - p_{1k} - p_{2k})$ we choose the Dirichlet distribution $Dirichlet(\alpha_{1k}, \alpha_{2k}, \alpha_{3k})$.

Consequently the prior distribution is

$$\begin{aligned}\pi_k(p_{1k}, p_{2k}; \alpha_{1k}, \alpha_{2k}, \alpha_{3k})& \\ = \frac{\Gamma(\alpha_{1k} + \alpha_{2k} + \alpha_{3k})}{\Gamma(\alpha_{1k})\Gamma(\alpha_{2k})\Gamma(\alpha_{3k})} &\cdot p_{1k}^{\alpha_{1k}-1} \cdot p_{2k}^{\alpha_{2k}-1} \cdot (1 - p_{2k} - p_{3k})^{\alpha_{3k}-1}.\end{aligned}$$

Since we suppose that the two types of particles reproduce independently, we suppose for their joint prior distribution that

$$\pi(\boldsymbol{p}; \boldsymbol{\alpha}) = \pi_1(\boldsymbol{p}_1; \boldsymbol{\alpha}_1) \cdot \pi_2(\boldsymbol{p}_2; \boldsymbol{\alpha}_2).$$

Hence the posterior distribution of the multinomial parameters is:

$$\begin{aligned}f(\boldsymbol{p} \mid \widetilde{\mathcal{J}}) &= \frac{\pi(\boldsymbol{p}; \boldsymbol{\alpha}).L(\widetilde{\mathcal{J}} \mid \boldsymbol{p})}{\int \pi(\boldsymbol{p}; \boldsymbol{\alpha}).L(\widetilde{\mathcal{J}} \mid \boldsymbol{p})} \,\mathrm{d}\boldsymbol{p} \propto \pi(\boldsymbol{p}; \boldsymbol{\alpha}).L(\widetilde{\mathcal{J}} \mid \boldsymbol{p}) \\ &\propto \prod_{k=1}^{2} \frac{\Gamma(\alpha_{1k} + \alpha_{2k} + \alpha_{3k})}{\Gamma(\alpha_{1k})\Gamma(\alpha_{2k})\Gamma(\alpha_{3k})} .p_{1k}^{\alpha_{1k}-1} .p_{2k}^{\alpha_{2k}-1} .(1 - p_{2k} - p_{3k})^{\alpha_{3k}-1}\end{aligned}$$

$$\times \prod_{k=1}^{2} p_{1k}^{\sum\limits_{l=1}^{N} Z_1^k(l)} . p_{2k}^{\sum\limits_{l=1}^{N} Z_2^k(l)} (1 - p_{1k} - p_{2k})^{\left(\sum\limits_{l=0}^{N-1} n.Z_k(l) - Z_k(l+1)\right)}$$

$$\propto \prod_{k=1}^{2} p_{1k}^{\sum\limits_{l=1}^{N} Z_1^k(l)+\alpha_{1k}-1} . p_{2k}^{\sum\limits_{l=1}^{N} Z_2^k(l)+\alpha_{2k}-1} (1 - p_{1k} - p_{2k})^{\left(\sum\limits_{l=0}^{N-1} n.Z_k(l) - Z_k(l+1)+\alpha_{3k}-1\right)}.$$

Consequently

$$f(\boldsymbol{p} \mid \widetilde{\mathcal{J}}) \sim Dirichlet\left(\sum_{l=1}^{N} Z_1^k(l) + \alpha_{1k}\ , \sum_{l=1}^{N} Z_2^k(l) + \alpha_{2k}, \sum_{l=0}^{N-1} (n.Z_k(l) - Z_k(l+1)) + \alpha_{3k}\right).$$

Therefore

$$E[p_{ik} \mid \widetilde{\mathcal{J}}] = \frac{\sum\limits_{l=1}^{N} Z_i^k(l) + \alpha_{ik}}{K_k}, \quad i = 1, 2 \tag{5}$$

and

$$\mathrm{Var}[p_{ik} \mid \widetilde{\mathcal{J}}] = \frac{\left(\sum\limits_{l=1}^{N} Z_i^k(l) + \alpha_{ik}\right)\left(K_k - \left(\sum\limits_{l=1}^{N} Z_i^k(l) + \alpha_{ik}\right)\right)}{K_k^2(K_k + 1)}, \quad i = 1, 2, \tag{6}$$

where

$$K_k = \sum_{l=1}^{N} Z_1^k(l) + \alpha_{1k} + \sum_{l=1}^{N} Z_2^k(l) + \alpha_{2k} + \sum_{l=0}^{N-1} (n.Z_k(l) - Z_k(l+1)) + \alpha_{3k},$$

$$K_k = \alpha_{1k} + \alpha_{2k} + \alpha_{3k} + \sum_{l=0}^{N-1} n.Z_k(l)$$

and $k = 1, 2$. We would like to notice that the conjugate prior distribution for the bivariate power series offspring distribution can be chosen in the following form

$$\pi_k(\boldsymbol{\theta}_k) \propto \frac{\theta_{1k}^{\alpha_{1k}} \theta_{2k}^{\alpha_{2k}}}{(1 + \theta_{1k} + \theta_{2k})^{n\beta_k}}.$$

So, regardless of the distribution, the information necessary for the priori distributions is equivalent to the knowledge of three parameters

$$(\alpha_{1k}, \alpha_{2k}, \alpha_{3k}) \quad \text{or} \quad (\alpha_{1k}, \alpha_{2k}, \beta_k).$$

6 Simulation

Now, we will simulate the branching process with two type of particles, which has a trinomial offspring distribution. This branching process is simulated for 20 generations. Two cases are considered when the prior distribution of the probabilities is a Dirichlet distribution with parameters $\alpha_1 = (1.1, 2.1, 1.3)$, $\alpha_2 = (1.25, 1.81, 1.14)$ and $\alpha_1 = (0.6, 0.3, 0.4)$, $\alpha_2 = (0.5, 0.7, 0.6)$, respectively. In the first case the results of the simulation are shown in Table 1 and in the second case – in Table 2. For this purpose one can use the language for Statistical Data Analysis **R**.

One could compare the posterior expectation of the parameters $\hat{p}_{ik}$

$$\hat{p}_{11} = 0.2210526, \quad \hat{p}_{12} = 0.4473684, \quad \hat{p}_{21} = 0.003506671, \quad \hat{p}_{22} = 0.609656175$$

calculated in the **R** script according (5) to the probabilities p_{ik}

$$p_{11} = 0.3842388, \quad p_{12} = 0.4449498, \quad p_{21} = 0.005182268, \quad p_{22} = 0.6336251$$

that are used in the **R** script when the family tree is generated. One can see that the posterior expectation $\hat{p}_{ik}$ is close to the probability p_{ik}.

The graphics of the population with two type particles in case 1 are shown in Fig. 1.

Table 1. Population of two type BP with trinomial distribution. The prior distribution of p_i is $Dirichlet(\alpha_i)$, where $\alpha_1 = (1.1, 2.1, 1.3)$ and $\alpha_2 = (1.25, 1.81, 1.14)$, $i = 1, 2$

Generation	$Z_1^1(l)$	$Z_1^2(l)$	$Z_1(l)$	$Z_2^1(l)$	$Z_2^2(l)$	$Z_2(l)$
1	0	0	1	0	0	1
2	0	0	0	2	2	4
3	0	0	0	0	6	6
4	0	0	0	0	6	6
5	0	0	0	0	10	10
6	0	0	0	0	11	11
7	0	0	0	0	13	13
8	0	0	0	0	16	16
9	0	0	0	0	18	18
10	0	0	0	0	20	20
11	0	0	0	0	20	20
12	0	0	0	0	25	25
13	0	0	0	0	32	32
14	0	0	0	0	36	36
15	0	0	0	0	48	48
16	0	1	1	0	51	51
17	1	0	1	0	69	69
18	0	0	0	1	90	91
19	0	0	0	0	106	106
20	0	1	1	0	132	132

Table 2. Population of two type BP with trinomial distribution. The prior distribution of p_i is $Dirichlet(\alpha_i)$, where $\alpha_1 = (0.6, 0.3, 0.4)$ and $\alpha_2 = (0.5, 0.7, 0.6)$, $i = 1, 2$

Generation	$Z_1^1(l)$	$Z_1^2(l)$	$Z_1(l)$	$Z_2^1(l)$	$Z_2^2(l)$	$Z_2(l)$
1	0	0	1	0	0	1
2	2	0	2	0	2	2
3	4	1	5	0	3	3
4	10	1	11	0	4	4
5	22	1	23	0	6	6
6	46	1	47	0	7	7
7	94	3	97	0	10	10
8	189	1	190	0	14	14
9	376	2	378	0	20	20
10	751	10	761	0	23	23
11	1509	4	1513	0	35	35
12	2998	9	3007	0	50	50
13	5958	9	5967	0	66	66
14	11822	19	11841	3	86	89
15	23439	31	23470	3	117	120
16	46482	27	46509	8	164	172
17	92122	49	92171	13	231	244
18	182550	67	182617	29	338	367
19	361602	125	361727	70	468	538
20	716437	125	716562	99	726	825

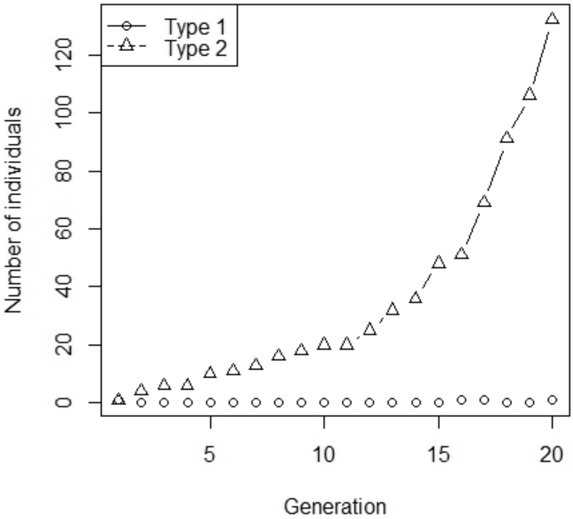

Fig. 1.

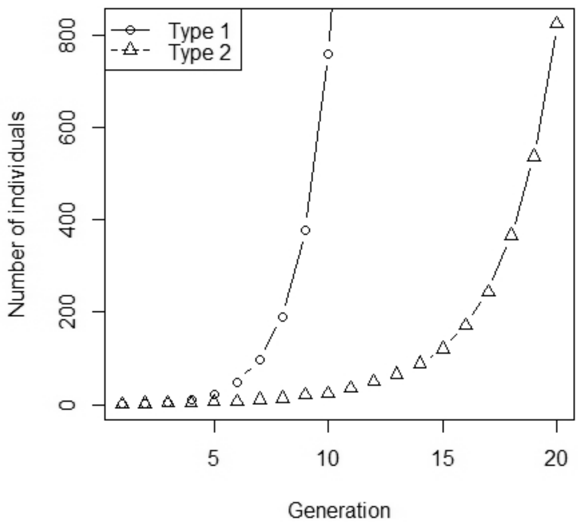

Fig. 2.

The second experiment shows the same results.

Again the obtaining posteriors of the probability is too close to the initially generated:

$$\hat{p}_{11} = 0.9902369063, \quad \hat{p}_{12} = 0.0001543808, \quad \hat{p}_{21} = 0.136982, \quad \hat{p}_{22} = 0.669160,$$
$$p_{11} = 0.9901949, \quad p_{12} = 0.0001423653, \quad p_{21} = 0.1348626, \quad p_{22} = 0.6506443.$$

The graphics of two type population in case 2 are shown in Fig. 2.

To obtain the above results the following source code in **R** can be used:

```
rdirichlet<-function(n,a)
{
  l<-length(a);
  x<-matrix(rgamma(l*n,a),ncol=l,byrow=TRUE);
  sm<-x%*%rep(1,l);
  x/as.vector(sm);
}
N=20;
a=1;b=1;
mm <- matrix(0, nrow=N, ncol=7);
mm[1,1] = 1;
mm[1,4] = a;
mm[1,7] = b;
alpha1=c(0.6,0.3,0.4)
s1=alpha1[1]+alpha1[2]+alpha1[3]
alpha2=c(0.5,0.7,0.6)
s2=alpha2[1]+alpha2[2]+alpha2[3]
no=2;
```

```
suma=matrix(c(alpha1[1],alpha1[2],alpha2[1],alpha2[2])
            ,nrow=2,ncol=2)
p1=rdirichlet(1, alpha1)
p2=rdirichlet(1, alpha2)
p1
p2
for(iteration in 2:N){
  a_new=rmultinom(a,no,p1)
  b_new=rmultinom(b,no,p2)
  a_sum = rowSums(a_new)
  b_sum = rowSums(b_new)
  a = a_sum[1]+b_sum[1] #first type
  suma[1,1]=suma[1,1]+a_sum[1]
  suma[2,1]=suma[2,1]+b_sum[1]
  s1=s1+no*a
  b = a_sum[2]+b_sum[2] #second type
  suma[1,2]=suma[1,2]+a_sum[2]
  suma[2,2]=suma[2,2]+b_sum[2]
  s2=s2+no*b
  mm[iteration,1]=iteration
  mm[iteration,2]=a_sum[1]#Z_1^1
  mm[iteration,3]=b_sum[1]#Z_1^2
  mm[iteration,4]=a   #Z_1
  mm[iteration,5]=a_sum[2]#Z_2^1
  mm[iteration,6]=b_sum[2]#Z_2^2
  mm[iteration,7]=b   #Z_2
  }
K1=s1-no*a+1
K2=s2-no*b+1
posteriorExp=c(suma[1,1]/K1,suma[1,2]/K1,suma[2,1]/K2,suma[2,2]/K2)
posteriorExp
df=data.frame(mm)
names(df)[1]="iteration";
names(df)[2]="Aa";
names(df)[3]="Ba";
names(df)[4]="A" ;
names(df)[5]="Ab";
names(df)[6]="Bb";
names(df)[7]="B" ;
df$iteration=as.numeric(df$iteration);
df$Aa=as.numeric(df$Aa);
df$Ab=as.numeric(df$Ab);
df$A=as.numeric(df$A);
df$Ba=as.numeric(df$Ba);
df$Bb=as.numeric(df$Bb);
df$B=as.numeric(df$B);
xrange=range(df$iteration);
yrange=range(df$B);
plot(a, b, xlim =xrange, ylim = yrange, type="p",
     xlab = "Generation", ylab = "Number of individuals");
```

```
plotchar <- seq(20, 20, 2);
legend("topleft",legend=c("Type 1","Type 2"),
      col=gray(0),lwd=1,lty=c(1,2),pch=c(1,2));
for (i in 1:N) {
 lines(df$iteration, df$A, type="b",lwd=1, col=gray(0), pch=1);}
 yrange=range(df$B);
for (i in 1:N) {
 lines(df$iteration, df$B,type="b",lwd=1, col=gray(0),   pch=2);}
```

The function `rdirichlet()` is copied from [33].

Acknowledgements. This work is supported by the European Social Fund through the Human Resource Development Operational Programme under contract BG051PO001-3.3.06-0052 (2012/2014) and is partially supported by the financial funds allocated to the Sofia University "St. Kl. Ohridski", Grant No 012/ 2014.

References

1. Asmussen, S., Herring, H.: Branching Processes. Birkhauser, Boston, (1983)
2. Athreya, K.B., Ney, P.E.: Branching Processes. Springer-Verlag, Berlin (1972)
3. Chamarbagwala, R., Ranger, M.: A multinomial model of fertility choice and offspring sex-ratios in India. The Journal of Development Studies 46(3), 417–438 (2010)
4. Gerstenkorn, T.: On multivariate power series distributions. Revue Roumaine de Mathématiques Pures et Appliquées 26, 247–266 (1981)
5. González, M., Gutiérrez, C., Martínez, R., del Puerto, I.M.: In: Controlled Branching Processes: Applications in Biology (Proc. 59th ISI World Statistics Congress, Session STS033), pp. 1750–1755. Hong Kong (2013)
6. González, M., Gutiérrez, C., Martínez, R., del Puerto, I.M.: Bayesian inference for controlled branching processes through MCMC and ABC methodologies. RACSAM 107(2), 459–473 (2013)
7. González, M., Martínez R., Mota M.: Multitype population-size-dependent branching process with dependent offspring. J. Math. Sci. 132(5), 610-613 (2006)
8. González, M., Mart ınez R., Mota M.: A Note on the extinction problem for controlled multitype branching processes. Pliska Stud. Math. Bulgar. 17, 85–96 (2005)
9. González, M., Martínez, R., Mota, M., del Puerto, I.M.: Gibbs Sampler in the Context of Bisexual Branching Processes. `http://www.cirm.univ-mrs.fr/videos/2007/exposes/10a/gonzalez.pdf`
10. González, M., Martínez, R., Mota, M.: Bisexual branching processes to model extinction conditions for Y-linked genes. J. Theor. Biology 258(3), 478–488 (2009)
11. González, M., Martín, J., Martínez, R., Mota, M.: Non-parametric Bayesian estimation for multitype branching processes through simulation-based methods. Comp. Stat. & Data Analysis 52(3), 1281–1291 (2008)
12. González, M., del Puerto I.M., Martínez, R., Mota, M.: Workshop on Branching Processes and Their Applications. Lecture Notes in Statistics, vol. 197. Springer, Heidelberg (2010)
13. Harris, T.E.: The Theory of Branching Processes. Springer-Verlag, Berlin (1963)

14. Jacob, C.: Branching processes: their role in epidemiology. Int. J. Environ. Res. Public Health 7, 1186–1204 (2010)
15. Jagers, P. Branching Processes with Biological Applications. Wiley, London (1975)
16. Johnson, N., Kotz, S., Balakrishnan, N.: Discrete Multivariate Distributions. Wiley (1997)
17. Khatri, C.G.: On multivariate contagious distributions. Sankhyia, Ser. B 33, 197–216 (1971)
18. Khatri, C.G.: On certain properties of power-series distributions. Biometrika 46(3/4), 486–490 (1959)
19. Kolmogorov, A.N., Dmitriev, N.A.: Branching random processes. Dokl. Akad. Nauk (Proc. Acad. Sci. USSR) 56, 7–10 (1947) (in Russian)
20. Kolmogorov, A.N., Sevastyanov, B.A.: Calculation of final probabilities of branching random processes. Dokl. Akad. Nauk (Proc. Acad. Sci. USSR) 56, 783–786 (1947) (in Russian)
21. Kocherlakota, S., Kocherlakota, K.: Bivariate Discrete Distributions. Marcel Dekker, Inc., New York (1992)
22. Daskalova, N.: Algoritmi ot tip EM za statistichesko oceniavane v razkloniavasti se stochastichni procesi. Ph.D. Disertacia, BAN (2012) (in Bulgarian)
23. Patil, G.P.: On multivariate generalized power series distribution and its applications to the multinomial and negative multinomial. In: Patil, G.P. (ed.) Clussical und Contugious Discrete Distributions (Proc. of Int. Symp. at McGill University on August 15–20, 183–194. Statistical Publishing Society, Calcutta; Pergamon Press, Oxford (1965)
24. Rufo, M.J., Pérez, C.J., Martín, J.: Bayesian analysis of finite mixtures of multinomial and negative-multinomial distributions. Comp. Stat. & Data Analysis 51(11), 5452-5466 (2007)
25. Sevastyanov, B.A.: Branching Processes. Nauka, Moscow (1971) (in Russian)
26. Shao, J.: Mathematical Statistics (Springer Texts in Statistics), 2nd edn. Springer, New York (2007)
27. Stoimenova, V., Yanev, N.: Parametric estimation in branching processes with an increasing random number of ancestors. Pliska Stud. Math. Bulgar. 17, 295–312 (2005)
28. Stoimenova, V.: Robust parametric estimation of branching processes with random number of ancestors. Serdica Math. J. 31(3), 243–262 (2005)
29. Yakovlev, A.Y., Stoimenova, V.K., Yanev, N.M.: Branching processes as models of progenitor cell populations and estimation of the offspring distributions. J. Am. Stat. Assoc. 103, 1357–1366 (1998)
30. Yakovlev, A.Yu., Yanev, N.M.: Transient Processes in Cell Proliferation Kinetics. Springer Verlag, Berlin (1989)
31. Yakovlev A.Yu. , Yanev, N.M.: Relative frequencies in multitype branching processes. Ann. Appl. Probab. 19(1), 1–14 (2009)
32. Yakovlev A.Yu., Yanev, N.M.: Limiting distributions in multitype branching processes. Stochastic Analysis and Applications 28(6), 1040–1060 (2010)
33. `https://stat.ethz.ch/pipermail/r-help/2000-December/009561.html`

Part II

Informatics

Performance Study of SQL and NoSQL Solutions for Analytical Loads

Hristo Kyurkchiev and Emanuela Mitreva

Faculty of Mathematics and Informatics, Sofia University "St. Kliment Ohridski",
5, James Bourchier Blvd., 1164 Sofia, Bulgaria
hkyurkchiev@fmi.uni-sofia.bg, emitreva@gmail.com

Abstract. Data mining has become more popular in recent years and with it the trend of analyzing transactional data straight from the DBMS. This trend is provoking the research into read-optimized database solutions. One branch of such research – column orientation claims significant improvement in read performance in comparison with row oriented DBMS. Another branch is the growing number of NoSQL solutions. Having previously conducted research between the traditional relational DBMS and column-stores we take on studying the performance of NoSQL DBMSs in the face of MongoDB with the goal of comparing the three approaches. The first step in this is examining their data models. We then use our previously developed benchmark to measure each DBMS's performance. Evaluating the results we draw conclusions about each DBMS's suitability and main advantages over the other.

Keywords: database systems, relational databases, data warehouses, NoSQL, column stores, C-store, MongoDB, performance evaluation, analytical queries

1 Introduction

Mining transactional databases has become increasingly popular in recent years. Thus the research into database management systems (DBMS), which can provide fast performance for analytical queries, has great significance to the business world. One such DBMS is Abadi's Vertica [7], the commercialization of C-Store [2, 3, 13]. It is a column store (a DBMS, which stores data in columns, rather than rows), which provides both the standard SQL language for querying databases, and the performance needed for effective data mining.

Since several performance studies [1–3, 5, 13] have been carried out using C-Store and a custom build row store DBMS and some comparing C-Store and Vertica [7] there was little information on how these databases fare against commercial grade RDBMS solutions such as Oracle. Furthermore, most of the results were achieved by using either an implementation of the TCP-H [1, 5, 7, 13] or the Star schema benchmark [3], a benchmark commissioned [9] by Stonebraker, one of Vertica's creators. This posed some reasonable doubts about the validity of the results.

Thus in previous research [6] we aimed at filling this gap by researching how Vertica compares to Oracle – a commercial grade row store. We found out that by using Vertica instead of Oracle one achieves significant performance gains, which however, are not in the same order of magnitude as previously suggested [16].

Stepping on these findings we decided to widen the research scope and as a logical next step to include other alternatives to traditional relational DBMS in this comparison. The natural choice for such solutions is the modern NoSQL databases [12], as little information about how they fair against the relational world as far as online analytical processing (OLAP) is concerned.

2 Choosing a NoSQL Solution for the Comparison

In order to make an educated choice of which NoSQL solution is the most appropriate for a data warehousing solution one has to critically look into their characteristics. And although they share one thing in common – not being relational – they are very different considering almost all other aspects [12]. Thus the first step in choosing would be to split them in homogenous categories and analyze each of them sequentially.

One way to categorize the NoSQL databases is by the way they store data:

- key-value stores – they save the data as key-value pairs (having the ability to search by the key and supports only indices on the key) [10];
- column-oriented databases – they store the data in the form of one extendable column of closely related data [10];
- graph stores – they keep the data in graph structure – by using nodes and relations [8];
- document stores – as their name suggests, they save the data as documents as either using XML or JSON [14].

Key-value stores and column-oriented databases do not offer the necessary flexibility and expressivity needed in a data warehousing solution, thus are not suitable for our experiment. This leaves graph and document stores. And while both are valid options and graph stores are more expressive and flexible [4], we chose MongoDB, a document store, for the comparison because of its simplicity, scalability, community support, provision of easy data access and performance [11].

3 Architectural Overview

3.1 MongoDB Data Model

MongoDB was created with the idea of handling large amounts of data, which is even suggested by its name (huMONGOus) [11].

Logical and Physical Database Structure. In MongoDB the data is stored as either XML or binary JSON (BJSON) [14]. The databases in it are schema-less – they do not require defining the structure of the data prior to its importing, but are created the moment some data is stored in them. Instead of traditional relations or tables, they have the so-called collections, which are not stored in rows and column, but rather in JSON dictionaries (Fig. 1). MongoDB allows for great flexibility – the number and type of the attributes or fields, as they are referred to, can vary significantly between the records, which provides for easy extensions to the database, but at the same time makes it easy for mistakes to get inserted into the structure. Another feature is documents' (records) nesting, which is unlimited in level, but can prove to introduce difficulties for DML queries.

```
{
        "_id" : 0,
        "name" : "John Doe",
        "scores" : [
                {
                        "score" : 1.24345355,
                        "type" : "exam"
                },
                {
                        "score" : 12.4983573,
                        "type" : "quiz"
                },
                {
                        "score" : 34.2434535,
                        "type" : "homework"
                }
          ]
}
```

Fig. 1. Sample JSON document

Read Optimization. MongoDB provides a set of features to optimize read performance – query optimizer, indices, parallelization, etc. It supports various types of indices – single field indices, compound indices, multikey indices, geospatial indices, text indices, hashed indices [15].

3.2 Compared to Vertica and Oracle

There are both similarities and differences between MongoDB, Oracle, and Vertica. A comparison of the main ones is presented in Table 1.

As seen from the table above, there are several significant differences between the three systems, starting with the physical data model, which has significant impact over the other features of the DBMSs, such as replication, sharding, etc.

Table 1. Basic comparison between MongoDB, Vertica and Oracle

Characteristic	MongoDB	Vertica	Oracle
Data storage	XML or (B)JSON	Columns	Tables, rows
CAP	Consistency and Partition tolerance	Consistency and Availability	Consistency and Availability
ACID rule	No	Yes	Yes
Transactions	No	Yes	Yes
JOINs	No	Yes	Yes
Indices	Supports single and compound indices on every level of the JSON	Does not support indices at all, uses projections for optimization	Supports single and compound indices on all columns
Replication	Yes	Yes	Yes
Sharding	Yes	Yes	Yes

Another significant difference, especially when data warehousing is concerned, is that MongoDB does not support joins [15]. The work around for this is to store all the data in one single collection by pre-joining the tables and using nesting if necessary. This can be considered, as a denormalization in the eyes of the relational world and in most cases is prone to leaving the data in an inconsistent state at some moment. Of course, data can be stored in several separate collections and joined by the source code of the application that is using it. Validating whether this limitation results a performance overhead on the part of MongoDB is something we would be discussing in the later sections of the study.

On the similarities front, all systems follow the CAP theorem [10], which states that only two of the three (Consistency, Availability and Persistence) can be supported. For MongoDB these are consistency and partitioning, while for Vertica and Oracle they are consistency and availability [14]. These characteristics result in Oracle and Vertica's support for the ACID properties and transactions and MongoDB's lack thereof.

Based on this analysis, our hypothesis is that MongoDB would be on par with Oracle and Vertica, when making DML queries (insert, update, delete), but would be less efficient when it comes to OLAP processing due to its lack of join support and possible code overhead.

4 Experiment Setup

In this section the main aspects of the experiment such as the hardware and software, the database schema, the benchmark, and the measuring tools are discussed. Only a part of the original database schema is presented, as it is proprietary information.

4.1 Setup Description

In order for the comparison to be valid the same setup as in the original performance study [6] is used. For completeness, we have included a brief outline bellow.

Hardware Setup. The DBMSs were run as virtual machines on VMWare ESXi, each equipped with two CPU cores (2.4 GHz each), 6 GB of RAM and 16 GB of storage.

Database Setup. No specific tweaks have been performed on each database to improve performance, except the ones, which are implicit or completely trivial:

- For Oracle, Oracle 11g SE One was used with implicit indices on all of the surrogate keys, as well as explicit indices on all of the foreign keys.
- HP Vertica Community Edition was used as the Vertica instance with the compulsory super projection on each table.
- The latest version of MongoDB (2.6.1) was installed on server with indices on the same columns as the ones in Oracle.

Database Schema. The same schema as the one used in previous research [6] is employed here as well – three dimension tables and one fact table. Not changing the schema proved to be the cause of some issues that we will discuss later on, when we present the results, due to the fact that MongoDB does not support joins.

The data used is from a travelling agency and concerns autobus. It includes autobus data ($\sim$ 50 records), station data ($\sim$ 5000 records), trips data ($\sim$ 100 000 records), and stops data ($\sim$ 1 750 000 records), which has been transferred to MongoDB without any changes.

4.2 Benchmark

For the results to be comparable to the ones previously measured for Oracle and Vertica the same benchmark suggested in previous research was used [6]. Of course, since the queries were originally written in SQL and MongoDB does not support SQL for querying data, they had to be rewritten in JavaScript – the native query language of MongoDB. Since the elementary queries are straightforward to translate and it would take too much space to include all of the others, only a model approach is presented here. Essentially the rewriting of each join is done by first reducing the *Trips* table based on its ID (no actual reducing done) so that the appropriate computation can be done (MongoDB does not support computations inside an aggregate function). Then performing the necessary aggregations on this newly made collection. And finally "*joining*" it with the other tables as required by using a couple of MapReduce calls. The resulting collection was then used in a simple query to get the end result. Later on in the study, it will become obvious that it is not the actual "*joining*" that is slowing the process down, but rather the preparation of the first and largest collection – *Trips*. Another trade-off with this approach for translating the SQL queries into MongoDB ones is that in order to make the needed result set, on which the actual query will be executed, additional temporary collections are created – a copy of the *Trips* collection, prepared to be joined by the other collection. This not only takes time to prepare and implement, but it also takes a great deal of resources, which after the query is performed, should be cleared.

To make things worse, appropriate mapping and reduce functions should be prepared and applied in a specific order to get the desired result. This makes the usage of MongoDB more than error prone and hard on the developer. We should note, however, that the idea behind not including joins in MongoDB is based on the idea of using nested document structures instead, which we are not using in order to remain true to the original schema and experiment setup.

In addition to this disadvantage we also were unable to rewrite the fourth flight of queries using this or any other approach due to MongoDB's query language being limited and due to the queries requiring joining of relations with different kinds of relationships between them (e.g. one-to-many and many-to-many). Thus this flight was not implemented and run on top of MongoDB.

5 Performance Comparison

Aqua Fold's Aqua Data Studio was used to measure the response time and Fig. 2 and 3 were generated using Shield UI. All queries have been run multiple times and the results were averaged so that any differences are smoothened. It should be noted that for response times of less than 1 s the measured results varied significantly on MongoDB.

5.1 General Queries Performance Analysis

The result patterns seen in Fig. 2 for the DML statements are on par with Oracle and Vertica with some peculiarities, e.g. the result for the insert in stations and the delete of an autobus, which however, can most likely be attributed to measurement error. As expected all DML statements take longer on the MongoDB DMBS than on Oracle and Vertica and while we see some uniformity in the results of the other two DBMSs as far as the same kind of statement is performed on a different relation, with MongoDB this is not the case – the results are very unpredictable.

The second part of this flight consists of general select statements. In contrast to the DML statements, here there is a very distinct pattern, which matches the one we saw in Vertica in the previous study [6]. Again, the results show that MongoDB is significantly slower than the other two systems, with the difference reaching 11 fold. It is interesting to note that there is one occasion when MongoDB outperforms, although with a narrow gap, both Oracle and Vertica and this is when all stations are selected. This difference is most likely due to the size of the relation.

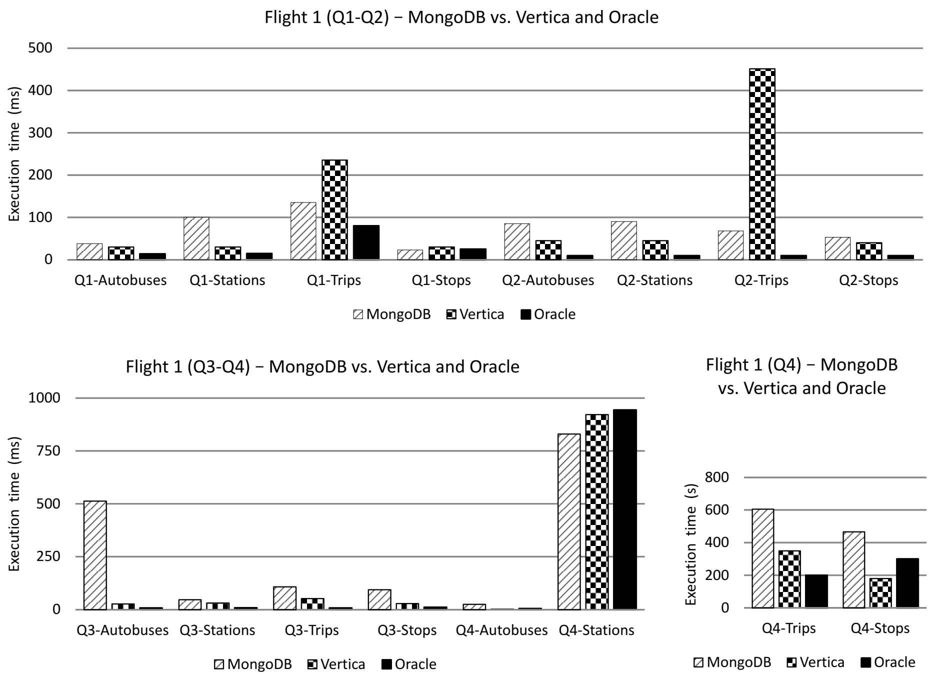

Fig. 2. General queries results

5.2 Analytical Queries Performance Analysis

We ran the analytical queries (flight two and three) several times performing some optimizations with each subsequent run (see Fig. 3).

After the initial tests for the second flight, which showed about 25 times slower responses than Oracle and close to 85 times slower than Vertica, corrections were made on the map functions used for the queries by adding some preliminary filtering on the data, which is inserted in the temporary collection.

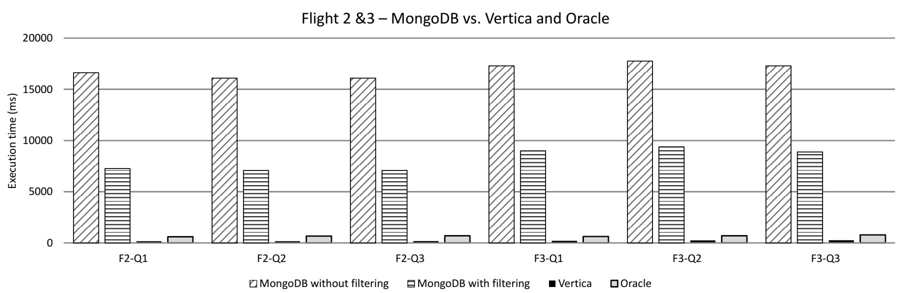

Fig. 3. Flight 2 and 3 results

And since we have already proven that the DML operations are slower than the other two DBMSs, this slowness is propagated to the other queries due to the necessity of building temporary collections (especially if no filtering is done on the data).

The initial test results show that if there is no filtering applied to the data used to build the new collection upon, it took around 16–17 s for the query to be executed. If the data is filtered beforehand (thus limiting the number of inserts to be made during the MapReduce functions), the query execution time is decreased to 7–8 s (the time is cut by 50%).

These results were still more than unsatisfactory – 10 times slower than Oracle and 30 times slower than Vertica for a NoSQL solution, designed to handle large amounts of data such as MongoDB.

The next step in measuring the performance was to make a new denormalized collection by nesting the other three collections (*Autobuses*, *Stops*, and *Stations*) within it. Having just one collection allows the skipping of the MapReduce functions and achieving the result with a simple non-join query. This resulted in execution times of 2–3 s. While it still cannot compare with the results of Oracle and Vertica, but was of 8–9 times better than the initial test results for MongoDB.

The situation with the third flight of queries is similar to the second one. The difference here is that we have one more additional collection in the join. This proves to not have such a great effect on the query execution times as they are increased with just 1 s to 17–18 s without filtering and a little over 8 s with filtering, which makes it again more than 10 times slower that Oracle and 30 times slower than Vertica. Apparently, as noted previously the biggest performance hit is due to creation of the temporary collection in which the records from *Trips* are inserted after the first MapReduce functions are executed.

6 Conclusion

During the work with MongoDB we have discovered that it is very easy to work with as far as data loading is concerned. This, however, was not the case for querying data. It provides a limited language, which expressive properties are far behind the ones of the well-known and widely used SQL language. The main omission – the lack of joins and their substitution with denormalized relations, although appropriate for the data warehousing solutions and providing some performance benefits, make for a bigger database and are not a subject of our study as we aim at looking into transactional databases, used for analytical purposes. Comparing the performance results of MongoDB with the previously recorded times of Oracle and Vertica, show that it cannot match them in any of the tested scenarios. What is more, it lags significantly behind. Thus we can conclude that for analytical queries Vertica still remains the best choice among the three. Another contender can be the graph databases, which we intend on studying in future research.

Acknowledgments. This paper is supported by the European Social Fund through the Human Resource Development Operational Programme under contract BG051PO001-3.3.06-0052 (2012/2014) and by the Sofia University "St. Kliment Ohridski" SRF under Contract 43/2013.

References

1. Abadi, D. et al.: Integrating Compression and Execution in Column-Oriented Database Systems. In: Proc. 2006 ACM SIGMOD Int. Conf. Manag. data – SIGMOD'06. pp. 671 (2006)
2. Abadi, D.J. et al.: Column-Oriented Database Systems. In: Proc. VLDB '09. pp. 1664–1665 (2009)
3. Abadi, D.J., Madden, S.R.: Column-Stores vs. Row-Stores?: How Different Are They Really? In: SIGMOD'08. pp. 967–980 (2008)
4. Angles, R., Gutierrez, C.: Survey of graph database models. ACM Comput. Surv. 40(1), 1–39 (2008)
5. Harizopoulos, S. et al.: Performance Tradeoffs in Read-Optimized Databases. In: Proc. VLDB Endowment. pp. 487–498 (2006)
6. Kyurkchiev, H., Kaloyanova, K.: Performance Study of Analytical Queries of Oracle and Vertica. In: Proc. 7th ISGT Int. Conf. pp. 127 – 139 2013)
7. Lamb, A. et al.: The Vertica Analytic Database?: C-Store 7 Years Later. In: Proc. VLDB Endowment. pp. 1790–1801 (2012)
8. Mitreva, E., Kaloyanova, K.: NoSQL Solutions to Handle Big Data. In: Proc. Doctoral Conference in MIE. pp. 77–86 (2013).
9. Neil, B.O. et al.: The Star Schema Benchmark and Augmented Fact Table Indexing Outline of Talk. In: Nambiar, R., Poess, M. (eds.) Performance Evaluation and Benchmarking. LNCS, vol. 5895, pp. 237–252. Springer, Heidelberg (2009)
10. Padhy, R.P. et al.: RDBMS to NoSQL: Reviewing some next-generation non-relational database's. Int. J. Adv. Eng. Sci. Technol. 11, 15–30 (2011)
11. Redmond, E., Wilson, J.R.: Seven Databases in Seven Weeks. The Pragmatic Bookshelf (2012)
12. Scholz, J.: Coping with Dynamic, Unstructured Data Sets – NoSQL: a Buzzword or a Savior? In: Schrenk, M., Popovich, V.V., Zeile, P. Proc. REAL CORP 2011 May, 1–9 (2011)
13. Stonebraker, M. et al.: C-store: a Column-Oriented DBMS. In: Proc. 31st VLDB Conference. pp. 553–564 (2005)
14. Zaki, A.K.: NoSQL Databases?: New millenium database for big data, big users, cloud computing and its security challenges. 403–409 (2014)
15. MongoDB Manual. `http://docs.mongodb.org/manual/`
16. The Vertica® Analytic Database Technical Overview White Paper (2010)

Classification of Methods for Description of Textures – a Review

Veselina Vetova-Pavlova and Pavel Boytchev

Faculty of Mathematics and Informatics, Sofia University "St. Kliment Ohridski",
5, James Bourchier Blvd., 1164 Sofia, Bulgaria
{vvetova,boytchev}@fmi.uni-sofia.bg
http://www.fmi.uni-sofia.bg

Abstract. Texture description is an important part of many image processing applications. The diversity of possible texture patterns leads to creation of a large variety of different methods for texture description. At the same time there is no official definition for the notion of texture, neither classification of the methods. This paper examines the basic characteristics of texture, as well as several different classifications. A partial classification is proposed, based on these characteristics, that includes a large number of methods. Example methods are provided for each of the classes.

Keywords: texture analysis, texture descriptors, descriptor classification

1 Introduction

Many images, especially the natural once, contain regions that are characterized not by certain value of brightness but by repetitive or quasi-repetitive patterns of brightness called texture. It is well known that human visual system perceives images that contain textures [1]. Texture recognition is an important part of object recognition as the texture usually defines the material, as well as the shape of the object [48]. Thus texture analysis, and texture description in particular, is an important part of many image processing applications. Four major issues of texture analysis can be identified [33]:

– Feature extraction: computing a characteristic of a digital image that describes its texture properties;
– Texture discrimination: partitioning an image that contains various textures into regions, each containing a perceptually homogeneous texture;
– Texture classification: determine to which of a finite number of classes a homogeneous texture region belongs;
– Extracting the shape from texture: reconstructing the geometry of a 3D surface using the texture information.

Due to the large diversity of texture patterns it is hard to create a universal method for texture descriptions that defines uniquely each possible texture.

Instead, a confusing wide range of empirical and semi-empirical approaches exists [21], each proposing different descriptors [33, 40], usually based on mathematical methods. Each method provides a way for identification of textures that possess certain visual features by measuring these features. Therefore, depending on the tasks that an application has, usually a set of methods is designed, that corresponds to the characteristics of the textures that will be processed [23]. Unfortunately, the great variety of proposed methods for texture description complicates significantly the choice of the most appropriate set of methods. Various classifications were proposed by different authors [1, 10, 14, 34, 51, 56], in order to solve this problem. However, there is neither formal definition for texture nor a formal classification of the methods for texture description yet [36]. In result the classifications are often inconsistent and incomplete.

This paper analyzes a number of texture definitions and descriptions, proposed by different authors, and identifies several basic texture characteristics. Then, based on these characteristics, existing classifications of methods for texture description are reviewed and compared. In result, a partial but consistent classification of compatible classes of methods is proposed, that generalizes the classifications, included in the research. Finally, well known methods for texture description are offered as examples for each of the classes.

2 The Nature of Texture

To be able to model textures it is essential to understand their nature as well as the mechanism employed by the human visual system in recognizing them. Since the 1960s various experiments were conducted in the area [2, 25–27, 32]. However, this mechanism is still not well understood [21]. At the same time there is an enormous variety of different textures and the notion of texture remains relatively unclear. Numerous definitions were proposed, but there is no agreement on a complete definition [34].

The majority of the authors agree that the basic property of textures is the repetition or quasi-repetition of a pattern or patterns [20, 36]. Some definitions connect this property directly with characteristics of the pixels like brightness [19, 22, 44] or a wider area of local human perceptual or statistical characteristics [23, 47, 62]. Others consider textures as build by elements, often called texels or primitives, whose spatial arrangement creates the repetition [16, 18, 29, 39, 50]. Some authors view textures as complex multi-level structures, organized in a hierarchical matter, combining the two earlier approaches and introducing the property scale of texture [20, 63].

Another important problem to consider, when describing textures, is the random aspect. The size, shape, color or other characteristics of the texture elements may vary over the image region that contains it, as long as the main repetitiveness remains [36, 39].

Some authors prefer to define textures as descriptors that provide measures for certain properties such as smoothness or regularity [14, 42]. Several features of the texture can be used to describe it in quantitative terms – coarseness, homogeneity, orientation and spatial relationships [1]. The majority of texture discriminators are based on one a subset of these properties. These features are usually independent of size, shape, position, orientation and average brightness of the textured region [1, 34]. However, in most of the cases they depend strongly on the scale of the texture.

Based on the spatial relationships between the primitives textures can be divided to strong (pattern) or weak (random) [1, 56]. Depending on the size of their primitives there are micro and macro textures. This classification corresponds directly to the coarseness of the texture [1]. Textures can be also oriented or isotropic depending on the level of orientation they exhibit [23]. In the end the properties that textures exhibit are usually strongly connected to the way these textures are obtained. Depending on the source, textures can be divided to natural and artificial [40]. As the name implies, natural textures present natural images and usually have a certain degree of randomness. The properties of the artificial textures depend on the methods, used to generate them and therefore those methods can be used to choose effective discriminators for the textures.

3 Classification of Methods for Texture Description

A variety of different classifications can be found in literature [1, 10, 14, 34, 51, 56]. By generalizing those classifications, we propose a partial but consistent classification that includes four major classes of methods – statistical, spectral, model-base and structural. Further, the proposed classification may be used, in future, as a base for detailed comparison of the different methods included or as a tool for choosing the most appropriate methods for certain task of texture analysis.

Since textures exhibit certain degree of order it is natural to use their statistical properties to describe them. Methods, based on these properties, are usually called statistical and form a large and well known class that is mentioned by many authors [10, 14, 16, 33, 34, 51, 56]. Statistical methods do not take in consideration the hierarchical placement of the texture elements in the space [33]. Instead, they measure properties like smoothness, roughness and grainy [14].

Various experiments has shown that human brain performs frequency analysis when perceiving visual information [4, 12]. Methods based on frequency analysis form the class of spectral methods [10, 14]. Some authors refer to this group of methods as signal processing methods [51] or transform methods [33]. Most of the methods in the class process the resulting filtered image in order to obtain different characteristics, such as orientation.

Certain textures exhibit similar characteristics as some mathematical models. In that case the model parameters can be used to characterize the textures. In addition the models can be used to synthesize analogous artificial textures. These methods are classified as model-based [33, 51] and are especially useful for natural textures, which usually contain certain amount of noise [51, 61].

Another class of methods, often mentioned by the authors, is the class of structural methods [10, 14, 16, 33, 51]. Those methods consider the texture as a set of well-defined primitives, that have a certain rule of placement over the space [16]. The placement rule can be very complex and in many cases hierarchical [33]. By extracting the primitives of a texture and then formulating the placement rule, the texture can be described completely. Those methods are appropriate for highly regular, usually artificial textures [51].

There is a number of methods that can not be classified in either of the classes described, like the Watershed Transform Method [56] or the Vector Dispersion Method [36]. The vast variety of possible texture patterns often requires a specific method to be created for solving particular problem or for methods to employ original techniques that were not used in texture analysis before [20]. Such methods can not be classified effectively at the moment. However, if they prove effective, they can be the beginning of new classes of methods for texture description.

Statistical Methods. One of the simplest methods for texture description is to use the statistical properties of gray-scale histogram [10, 14]. Some well-known characteristics of that type are the standard deviation, variance, entropy, skewness and kurtosis. During the years, the list of different histogram descriptors have increased significantly [16, 24, 54]. Obviously in most of the cases several different descriptors will be needed in order to describe certain texture. The main disadvantage of these measures is that they do not provide information about the relative position of the pixels with respect of each other [14]. It is possible two completely different textures to have similar histogram statistics.

The gray-scale co-occurrence matrices are considered one of the most popular statistical methods for texture description [34]. They were first introduced by Haralick [17] and even today are included in many resent papers on texture analysis [5, 6, 46, 59]. GSCM provides information about the relative positions of pixels having particular gray-level values as well as the direction between them [38]. Numerous statistical characteristics as contrast, homogeneity or entropy can be calculated over the GSCM to produce texture discriminators [36].

Spectral Methods. The most prominent spectral method is the Fourier transform. As several experiments indicate, human visual system decomposes textured regions to their frequency and orientation components [4, 12]. Thus the Fourier spectrum is very suitable for describing directionality in periodic and quasi-periodic 2d textures. After Fourier transform, prominent peaks in the spectrum will give the principle direction of the texture, while the location of the peaks in the frequency plane will give the fundamental spatial period of the patterns [14]. Since the 1960s, various experiments were conducted in the area [7, 30, 43]. Recently the Fourier transform is included as a part of more complex description methods [3, 15].

Other transformations can be used in order to yield different characteristics for analysis. Wavelet transformations are especially popular as they allow localization both in time and frequency [34, 38, 49, 57]. In order to localize the analysis in the spacial domain, Gabor transformation can be used [38, 41, 51].

Model-Based Methods. Natural textures usually exhibit some noise-like properties although they are essentially different from noise. Therefore it is natural to model such textures using two-dimensional random fields [40, 61]. Then the texture can be described by the properties of the random field. One of the most popular random fields to use as a model of texture is the Markov random field [9, 55] and in particular – Gauss–Markov random field [28] and Gibbs random field [13]. A significant number of resent methods are based on them [11, 35, 45, 52, 58, 60].

Many natural textures exhibit self-similarity on different scales [8, 51]. Such textures can be modeled as fractals and different parameters such as fractal dimension, can be used to describe the texture. Analysis of remotely sensed images are one particular area of texture analysis where that type of methods proved to be very effective [31, 53].

Structural Methods. In syntactic based methods the placement rule is described as a grammar. Then the texture can be represented as a string [14]. These methods are part of the syntactic texture analysis and can be used in order to represent some spatially and temporally semantic pattern models found in images [37].

4 Conclusion

The notion of texture remains vague. It is generally accepted that the most notorious property of textures is that they exhibit certain repetitive or quasi-repetitive pattern for one or several human perceptual characteristics. Textures can be described in quantitative terms by their coarseness, homogeneity, orientation and spatial relationships. Due to the diversity of textures, a large number of different methods for texture description were created. It is extremely difficult to classify all methods since many of them were created in order to solve particular problems. However four major classes can be identified that include significant part of the methods for texture description – statistical, spectral, model-based and structural methods.

Acknowledgments. This work was supported by the European Social Fund through the Human Resource Development Operational Programme under contract BG051PO001-3.3.06-0052 (2012/2014).

References

1. Acharya, T., Ajoy K.R.: Image Processing Priciples and Applications. Wiley and Sons (2005)
2. Bergen, J.R., Edward H.A.: Early vision and texture perception. Nature 333(6171), 363–364 (1988)

3. Bittkau, K., et al.: Modeling of light scattering properties from surface profile in thin-film solar cells by Fourier transform techniques. Thin Solid Films 519(19), 6538–6543 (2011)
4. Campbell, F.W., Robson, J.G.: Application of Fourier analysis to the visibility of gratings. J. Physiology 197, 551–566 (1968)
5. Chai, H.Y., Wee, L.K., Swee, T.T., Salleh, S.H., Ariff, A.K.: Gray-level co-occurrence matrix bone fracture detection. Am. J. Appl. Sci. 8(1), 26–32 (2011)
6. Champion, I., Germain, C., Da Costa, J.P., Alborini, A., Dubois-Fernandez, P.: Retrieval of forest stand age from SAR image texture for varying distance and orientation values of the gray level co-occurrence matrix. Geoscience and Remote Sensing Letters, IEEE 11(1), 5–9 (2014)
7. Coggins, J.M., Jain, A.K.: A spatial filtering approach to texture analysis. Pattern Recognition Letters 3(3), 195–203 (1985)
8. Costa, A.F., Humpire-Mamani, G., Traina, A.J.M.: An efficient algorithm for fractal analysis of textures. In: Graphics, Patterns and Images (SIBGRAPI), 25th SIBGRAPI Conference, pp. 39–46. IEEE (2012)
9. Cross, G.R., Jain, A.K.: Markov random field texture models. IEEE Trans. Pattern Analysis and Machine Intelligence PAMI-5(1), 25–39 (1983)
10. Dougherty, G.: Digital Image Processing for Medical Applications. Prentice Hall (2009)
11. Espinoza M.D., Gleich, D., Datcu, M.: Evaluation of Bayesian despeckling and texture extraction methods based on Gauss-Markov and auto-binomial Gibbs random fields: application to TerraSAR-X data. IEEE Trans. Geoscience and Remote Sensing 50(5), 2001–2025 (2012)
12. Georgeson, M.A.: Spatial Fourier Analysis and Human Vision. In: Sutherland, N.S. (ed.) Tutorial Essays in Psychology: A Guide to Recent Advances (Vol. 2, Ch. 2). Lawrence Earlbaum Associates, Hillsdale, N.J. (1979)
13. Gimel'farb, G.L.: Image textures and Gibbs random fields (Vol. 16). Kluwer Academic Publishers, Dordrecht, The Netherlands (1999)
14. Gonzalez, R.C., Woods, R.E.: Digital Image Processing. Prentice Hall (2001)
15. Guo, Y., Zhao, G., Pietikäinen, M.: Texture classification using a linear configuration model based descriptor. In: Proc. BMVC, pp. 119.1–119.10 (2011)
16. Haralick, R.M.: Statistical and structural approaches to texture. Proc. IEEE 67, 786–804 (1979)
17. Haralick, R.M., Shanmugam, K., Dinstein I.H.: Textural features for image classification. IEEE Trans. Systems, Man and Cybernetics 6, 610–621 (1973)
18. Hawkins, J.K.: Textural properties for pattern recognition. In: Lipkin, B.C., Rosenfeld, A. (eds.) Picture Processing and Psychopictorics, pp. 347–370. Academic Press, New York (1970)
19. IEEE Standard 601.4-1990: IEEE Standard Glossary of Image Processing and Pattern Recognition Terminology. IEEE Press (1990)
20. Jahne, B.: Practical Handbook on Image Processing for Scientific and Technical Applications. CRC Press (2004)
21. Jahne, B.: Digital Image Processing. Springer (2002)
22. Jain, R., Kasturi, R., Schunk, B.G.: Machine Vision. McGraw-Hill (1995)
23. Jan, J.: Medical Image Processing, Reconstruction and Restoration: Concepts and Methods. CRC Press (2010)
24. Jiang, Y.: Texture description based on multiresolution moments of image histograms. Optical Engineering 47(3):037005 (2008)
25. Julesz, B.: Visual pattern discrimination. IRE Trans. Information Theory 8(2), 84–92 (1962)

26. Julesz, B.: Experiments in the visual perception of texture. Scientific American 232, 34–43 (1975)
27. Julesz, B.: Textons, the elements of texture perception, and their interactions. Nature 290(5802), 91–97 (1981)
28. Krishnamachari, S., Chellappa, R.: Multiresolution Gauss–Markov random field models for texture segmentation. IEEE Trans. Image Processing 6(2), 251–267 (1997)
29. Landau, S.I.: Webster Illustrated Contemporary Dictionary, Doubleday (1982)
30. Lendaris, G.G., Stanley, G.L.: Diffraction pattern sampling for automatic pattern recognition. Proc. IEEE 58(2), 198–216 (1970)
31. Luan, H., Wang, X., Tian, Q., Wei, C., Wei, X., Zhan, X.: Comparison of methods of fractal texture extraction for high-resolution remotely sensed images. In: Earth Observation and Remote Sensing Applications (EORSA), 2nd Int. Workshop, pp. 201–205. IEEE (2012)
32. Malik, J., Perona, P.: Preattentive texture discrimination with early vision mechanisms. JOSA A 7(5), 923–932 (1990)
33. Materka, A., Strzelecki, M.: Texture analysis methods – a review. In: Technical University of Łódź, Institute of Electronics, COST B11 Report, Brussels, pp. 9–11 (1998)
34. Nixon, M.S., Aguado, A.S.: Feature Extraction and Image Processing. Newnes (2002)
35. Odstrcilik, J., Kolar, R., Jan, J., Gazarek, J., Kuna, Z., Vodakova, M.: Analysis of retinal nerve fiber layer via Markov random fields in color fundus images. In: Systems, Signals and Image Processing (IWSSIP), 19th Int. Conf., pp. 504–507. IEEE (2012)
36. Parker, J.R.: Algorithms for Image Processing and Computer Vision. Wiley and Sons (2011)
37. Pedro, R.W.D., Nunes, F.L., Machado-Lima, A.: Using grammars for pattern recognition in images: A systematic review. ACM Computing Surveys (CSUR) 46(2), 26 (2013)
38. Petrou, M., Sevilla P.G.: Image Processing Dealing with Texture. Wiley and Sons (2006)
39. Pickett, R.M.: Visual analysis of texture in the detection and recognition of objects. In: Lipkin, B.C., Rosenfeld, A. (eds.) Picture Processing and Psychopictorics, pp. 289–308. Academic Press, New York (1970)
40. Pratt, W.K.: Digital Image Processing. Wiley and Sons (2007)
41. Rahman, M.H., Pickering, M., Kundu, D.: Efficient rotation invariant Gabor descriptors for texture classification. In: Informatics, Electronics and Vision (ICIEV), 2012 Int. Conf., pp. 661–666. IEEE (2012)
42. Richards, W., Polit, A.: Texture matching. Kybernetic, 16(3), 155–162 (1974)
43. Rosenfeld, A.: Automatic recognition of basic terrain types from aerial photographs. Photogrammic Engineering 28(1), 115–132 (1962)
44. Russ, J.C.: The Image Processing Handbook. CRC Press (2010)
45. Seetharaman, K., Palanivel, N.: Texture characterization, representation, description, and classification based on full range Gaussian Markov random field model with Bayesian approach. Int. J. Image and Data Fusion 4(4), 342–362 (2013)
46. Sharma, M., Singh, S.: Evaluation of texture methods for image analysis. In: 7th Australian and New Zealand Intelligent Information Systems Conference, pp. 117–121. IEEE (2001)
47. Sklansky, J.: Image Segmentation and Feature Extraction. IEEE Trans. Systems, Man, and Cybernetics, SMC-8, pp. 237–247 (1978)

48. Snowden, R., Snowden, R.J., Thompson, P., Troscianko, T.: Basic Vision: An Introduction to Visual Perception. Oxford University Press (2012)
49. Soulard, R., Carré, P.: Quaternionic wavelets for texture classification, Pattern Recognition Letters 32(13), 1669–1678 (2011)
50. Tamura, H., Mori, S., Yamawaki, Y.: Textural features corresponding to visual perception. IEEE Trans. Systems, Man, and Cybernetics, SMC-8, 460–473 (1978)
51. Tuceryan, M.: Texture Analysis. In: Chen, C.H., Pau, L.F., Wang, P.S.P. (eds.) Handbook of Pattern Recognition and Computer Vision. Imperial College Press, London (2010)
52. Vácha, P., Haindl, M., Suk, T.: Colour and rotation invariant textural features based on Markov random fields. Pattern Recognition Letters 32(6), 771–779 (2011)
53. Weng, Q.: Fractal analysis of satellite-detected urban heat island effect. Photogrammetric Engineering and Remote Sensing 69(5), 555–566 (2003)
54. Weszka, J.: A comparative study of texture measures for terrain classification. IEEE Trans. Systems, Man, and Cybernetics, SMC-6, 269–285 (1976)
55. Won, C.S., Derin, H.: Unsupervised segmentation of noisy and textured images using Markov random fields. CVGIP: Graphical Models and Image Processing 54(4), 308–328 (1992)
56. Wu, Q., Merchant, F.A., Castleman, K.R.: Microscope Image Processing. Academic Press (2010)
57. Xu, Y., Yang, X., Ling, H., Ji, H.: A new texture descriptor using multifractal analysis in multi-orientation wavelet pyramid. In: Computer Vision and Pattern Recognition (CVPR), 2010 IEEE Conference, pp. 161–168. IEEE (2010)
58. Xu, X., Guo, Y., Wang, Z.: Cloud image detection based on Markov random field. J. Electronics (China) 29(3–4), 262–270 (2012)
59. Yang, J., Guo, J.: Image texture feature extraction method based on regional average binary gray level difference co-occurrence matrix. In: 2011 Int. Conf. on Virtual Reality and Visualization (ICVRV), pp. 239–242. IEEE (2011)
60. Yang, L., Lu, R.S., Liu, Z.J., Lei, L.Q.: Surface roughness extraction by gibbs random fields of laser speckle pattern texture. Advanced Materials Research 629, 515–520 (2013)
61. Yap, K., Guan, L., Perry S.W., Wong H.: Adaptive Image Processing A Computational Intelligence Perspective. CRC Press (2010)
62. Zhou, H., Wu, J., Zhang, J.: Digital Image Processing: Part II. Bookboon(2010)
63. Zucker, S.W., Kant, K.: Multiple-level representations for texture discrimination. In: Proc. IEEE Conf. Pattern Recognition and Image Processing, Dallas, TX, pp. 609–614. IEEE (1981)

Part III

Business Informatics

Declarative Representation of a Business Process Model with Business Rules

Maria Semerdjieva and Evgeniy Krastev

Faculty of Mathematics and Informatics, Sofia University "St. Kliment Ohridski",
5, James Bourchier Blvd., 1164 Sofia, Bulgaria
{mtutanova,eck}@fmi.uni-sofia.bg

Abstract. The declarative representation of a business process model (BPM) makes use of a vocabulary and a set of business rules to describe the control-flow, data and resource aspects of a business process modeling. Most of the existing formal languages with declarative approach to process modeling address part of these aspects. A common problem for such languages is the usage of non-standard vocabularies that complicates the interchange of rules models between platforms. The EM-BrA2CE Framework extends the OMG standard for SBVR and offers a unifying approach for declarative representation of a BPM. It provides a highly expressive Structured English language with standard constructs supporting all the aspects of business process modeling. The internal representation of the vocabulary contents and the business rules of this framework are not aligned with existing OMG standards. Certain constructs, representing concepts for time duration and message passing events are undeveloped. These deficiencies limit substantially the scope of implementations of this framework. The purpose of this paper is to present a first-version meta-model that integrates the declarative BPM representation built with the Em-BrA2CE framework with business rules standards RDF(S), OWL and RIF. A simple case study illustrates the main hypothesis that a declarative representation of a business process model may be achieved on the basis of Semantic-web technology-oriented modeling.

Keywords: Business Process Model representation, Business Rules, SBVR, RDFS, RDF, OWL

Introduction

One of the widely used definitions for a business process is [1]: *A business process can be defined as a collection of activities that takes one or more kinds of input, and creates an output that is of value to the customer*. Initially, BPMs have been control-flow-based, procedural process models, following the concept of Rule independence introduced by the Semantics of Business Vocabulary and Business Rules (SBVR) standard of OMG [2]. Accordingly, the Business vocabulary and the Business rules are specified in procedural process models independently of business processes and events. This way, however, the procedural models became difficult to adapt to changes that might occur at different stages of the software project lifecycle.

A shift from the paradigm of procedural models toward declarative process modeling has been initiated with the introduction of the ConDec language [3]. Unlike a procedural BPM, the declarative procedural model takes explicitly into consideration the so called business concerns like cycle time and other time characteristics, costs of the resources, constraints imposed on the execution of activities and so on. This kind of declarative information about the activities of a business process that are necessary, obligatory, forbidden or possible to undertake is expressed the best way in terms of rule-based business process modeling languages [4]. However, only few business rule specification languages are human understandable. The EM-BrA2CE language overcomes this deficiency by extending the SBVR vocabulary with constructs to handle directly process related concepts as agent, activity, events and deontic assignments, explicitly defining authorizations in interactions between business partners. On the other side there are no tools that support this language. This paper considers an approach to transform the natural language expressions for business rules in EM-BrA2CE into OMG standard representations for which there exist automatable information system support.

Representation of a Business Process Model with Business Rules

EM-BrA2CE provides a unifying framework for declarative business process modeling using business rules, agents, activities, concepts and events [5]. It specifies an extension to the SBVR Structured English (SBVRSE) vocabulary [2], where appropriate process-related concepts for agent, activity, events and deontic assignments have been added. The events represent 12 transitions in the state of an activity and two of transitions in the state a business fact. For instance, business rules in EM-BrA2CE most frequently use predefined in the vocabulary names for states of an activity like started, completed, created, skipped and assigned to represent an activity.

Thus, the declarative representation of the BPM in Fig. 1 allows describing the state transitions of an activity in an explicit relation to business and non-business concerns given in the form of business rules and business facts. The state transition type of an activity depends on its current state. On the other hand, it is allowed only an agent (or the so called service provider) assigned to an activity to change the state of that activity provided that none of the business concerns is violated.

Definition 1. *State space is the set of the discrete states an activity type that may be imposed by an agent or the business concerns associated to that activity type.*

Hence, business concerns impose an additional constraint to the admissible movements along the state space of an activity type during a business process execution.

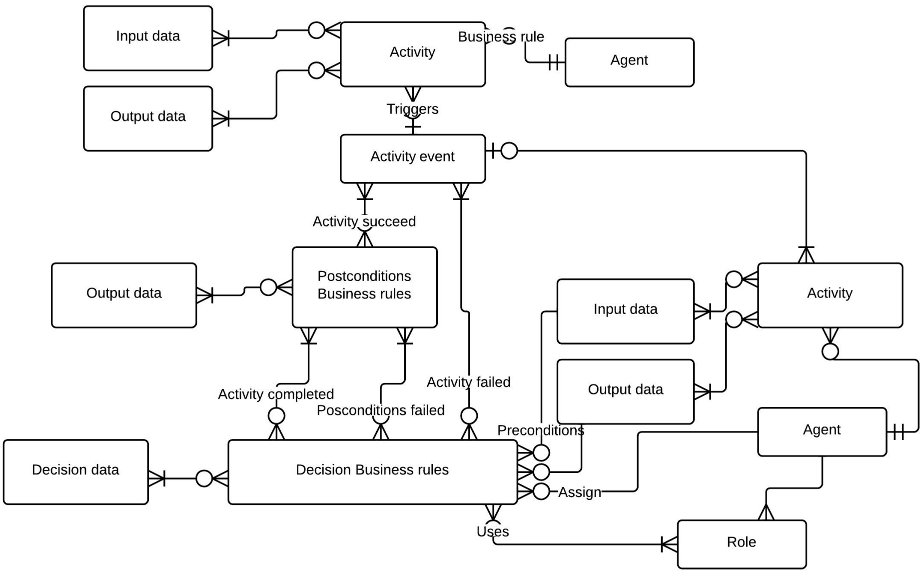

Fig. 1. Business process model transitions related to activity state transitions and process data

A representational analysis [6] over the modeling languages for business processes and business rules allows viewing the 16 different business rule types introduced in EM-BrA2CE from two points of view:

– Structural business rules that describe the domain model of a given BPM, described usually in Entity Relationship diagrams or class diagrams. These rules describe business and processing data, as well as, the organizational structure in the form of human and machine roles that are responsible for the execution of particular activities.
– Operative business rules related to activity state transitions of behavioral and guidance aspect. Business rules that define the control-flow aspect of a BPM belong to this category. These rules describe the activities and their execution order. Pre-conditions, post-conditions and production rules are some typical examples.

Although EM-BrA2CE is a powerful framework for describing BPMs in a declarative way, there are almost no tools that support visual process diagrams or model verification techniques. One major reason for this limited support is the lack of appropriate representation of the vocabulary concepts and the business rules in a standard language. In this paper we present an approach to describe the structural business rules in terms of the Resource Description Framework (RDF), RDF schema (RDFS) [7] and Web Ontology Language (OWL) [8]. Respectively, the operative business rules we represent in terms of Rule Interchange Format (RIF) standard [9], which is compatible with RDFS and OWL.

Case Study

A standard request to handle credit card requests is frequently used in the literature as a proof-of-concept [2, 10] and thus serves as a basis for comparing different approached in BPM. The following Credit card request process illustrates the concept for representing a business process with business rules. Depending on the assets of the customer in the bank and the amount of a requested credit, the credit card request can be directly approved or sent to Credit manager for evaluation. The Credit card manager has to determine *whether* to approve or deny credit card request. The Credit card request can be directly approved in following situations:

- The status of the bank assets of the customer is referred to as good and the amount of credit request is below 1000.
- The status of the bank assets of the customer is referred to as medium and the amount of credit request is below 500.

The Credit card manager has to determine whether to approve or deny credit card request in the following situations:

- The status of the bank assets of the customer is referred to as low.
- The status of the bank assets of the customer is referred to as good and the amount of credit request is greater than 1000.
- The status of the bank assets of the customer is referred to as medium and the amount of credit request is greater than 500.

The declarative representation of this sample business process with business rules involves the following tasks:

1. Create the Vocabulary

In accordance with the Em-BrA2CE framework the vocabulary for a process includes the terms for entity data types, activity types and roles. The terms in the Vocabulary for the Credit card request process are denoted in bold in the description of the activity types, the entity data types and the respective roles involved in this process.

- **Entity data types:**
  (a) The `customerStatus` can have following state: `good`, `medium` and `low`.
  (b) A `customer` has `customerStatus`.
  (c) A `creditRequestEntity` has `customer` and `amount`.
  (d) An `endActivity` has `responseStatus` as `outputData`.
  (e) A `responseStatus` can be `approved` or `denied`.
- **Activity types:** `creditCardRequest` activity, `judgeCreditCard` activity and `endActivity` activity.
- **Roles:** `customer` and `CreditManager`.

Note, that the `endActivity` does not follow all the logic of an `Activity`. The execution of `endActivity` initializes `outputData` and sets the `endActivity` state to completed. Thus, when the Credit card request process is completed then the process output can be retrieved using `outputData` of `endActivity`.

2. Create a Declarative Representation of the Business Process

Here we illustrate the representation of structural business rules in terms of RDFS and OWL, as well as, the representation of business rules related to activity state transitions and data in terms of RIF.

2.1. Structural Business Rules

Since it is difficult to provide the representation of all the structural rules, we focus on the representation of some typical cases of structural rules. Let us consider the structural business rules related to the `creditCardRequest` activity.

Rule 1. The `creditCardRequest` activity is a subclass of `Activity`.

Rule 2. The property `outputData` of the `creditCardRequest` activity can have values in the range of `creditRequestEntity`.

Figure 2 demonstrates how to represent the `creditCardRequest` business rule as a subclass of class `Activity` by means of RDFS and OWL. In a similar way an instance of the `creditCardRequest` activity can be represented in Fig. 3.

OWL and RDFS allow representing authorization business rules as it is shown in Fig. 4.

The following structural business rules 3–4 can be expressed with RDFS and OWL similarly to the representation of the authorization rules in Fig. 4.

```
<owl:Class rdf:ID="creditCardRequest">
   <rdfs:label>Credit card request activity</rdfs:label>
   <rdfs:subClassOf rdf:resource="{\#}Activity"/>
   <rdfs:subClassOf>
     <owl:Restriction>
      <owl:onProperty rdf:resource="{\#}outputData"/>
      <owl:toClass rdf:resource="{\#}creditRequestEntity"/>
    </owl:Restriction>
   </rdfs:subClassOf>
</owl:Class>
```

Fig. 2. Extending properties and business rules from superclass `Activity`

```
<creditRequest rdf:ID="creditR1331">
  <rdfs:label>creditR1331</rdfs:label>
  <rdfs:comment>credit request is activity, performed by
                customer</rdfs:comment>
  <status ><xsd:string rdf:value="started"/></status>
  <assignee rdf:resource = "#JohnUUID122"/>
  <outputData rdf:resource="#creditRequestDoc2332"/>
</creditRequest>
```

Fig. 3. Representing an instance of an activity

```
<rdfs:subClassOf>
     <owl:Restriction>
      <owl:onProperty rdf:resource="#assignee"/>
      <owl:toClass rdf:resource="#customer"/>
     </owl:Restriction>
</rdfs:subClassOf>
```

Fig. 4. Representing Authorization business rules with OWL and RDFS

Rule 3. The `judgeCreditCard` Activity has `outputData` of type `responseStatus`.

Rule 4. The `judgeCreditCard` Activity has `assignee` of type `CreditManager`.

2.2. Operative Business Rules Related to Activity State Transitions

The transitions in the state space are determined by the business rules. The transitions in the state space of the Credit card request process are displayed on Fig. 5. Every instance of this process follows a trajectory in this state space.

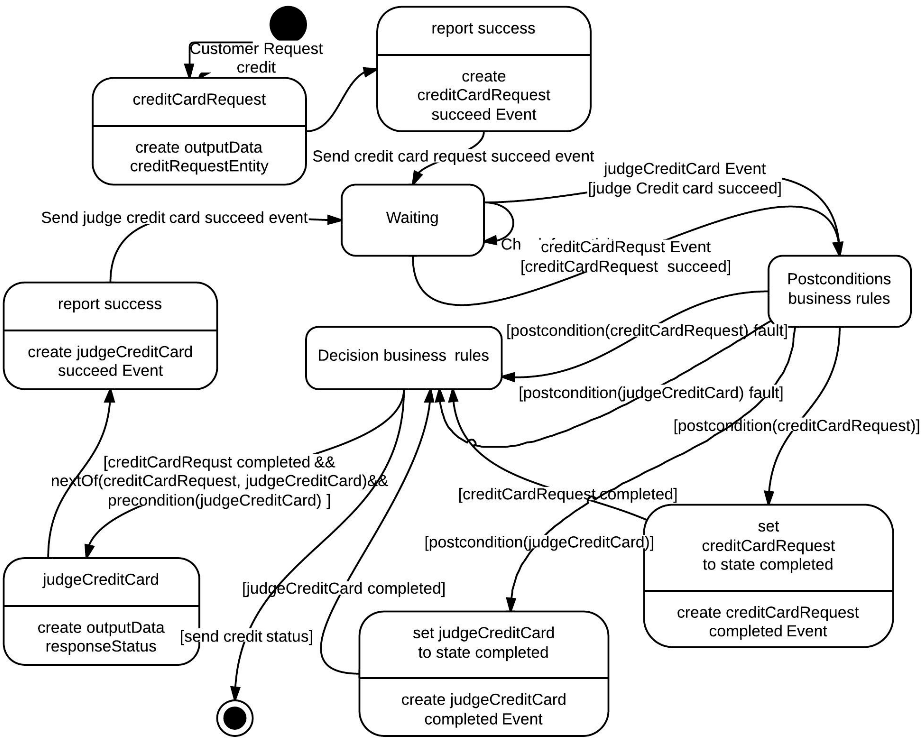

Fig. 5. UML state diagram of the Credit card request process

Figure 5 displays all the structural rules, as well as, the business rules related to activity state transitions.

Activity state transitions are defined in terms of pre-conditions, post-conditions and productivity rules, which involve logic of "triples", i.e. a statement about resources in the form subject-predicate-object expression [11]. Therefore, unlike the structural business rules, it is more convenient to represent the state transitional rules in terms of RIF instead with RDFS and OWL, while retaining the compatibility among these representations.

The business process starts with the `creditCardRequest` activity and it has no preconditions. After the process gets enacted, the `creditCardRequest` activity receives `assignee` of type `Requester` and state `assigned`. Let the `creditCardRequest` activity post-conditions be defined with the following business rules:

Rule 5. A `creditCardRequest` has a `postcondition` (here true is omitted), if its `outputData` of type `creditRequestEntity` is a `validCreditRequestEntity`. (Note, that the following fact follows: "*`outputData` for `creditCardRequest` has type `creditRequestEntity`*" from Rule 2 in Sect. 2.1.)

Rule 6. A `creditRequestEntity` is a `validCreditRequestEntity`, if `creditRequestEntity` has amount and `creditRequestEntity` has `customer`, which is `assignee` to `creditCardRequest`.

The RIF representation for these two rules is as follows:

```
Forall ?A ?D (
   postcondition(?A) :-And(?A[rdf:type ->
                       crediCardRequest] ?A[:outputData -> D]
                               validCreditRequestEntity(D)) )
Forall ?creditCardRequest ?creditRequestEntity (
       validCreditRequestEntity(?creditCardRequest,
                                ?creditRequestEntity) :-
       Exists ?y ?c And(?creditRequestEntity[:amount -> ?y]
                        ?creditRequestEntity[:customer -> ?c]
                            creditCardRequest[:assignee -> ?c]))
```

Similarly, the RIF representation for transitions between activities is defined with a business rule of the following kind

Rule 7. A `judgeCreditCard` activity follows `creditCardRequest` activity.

which may be expressed as follows:

```
Forall ?creditCardRequest ?judgeCreditCard(
           nextOf (?creditCardRequest ,?judgeCreditCard):-
        And (?creditCardRequest[rdf:type -> creditCardRequest]
             ?judgeCreditCard[rdf:type -> judgeCreditCard]))
```

The `judgeCreditCard` activity has preconditions and postconditions that are expressed by the following rules:

Rule 8 (pre-condition). To start `judgeCreditCard` it is necessary that `inputData` has type `creditRequestEntity` with `customerStatus` **good**, when the **amount** is greater or equal to **1000** or `customerStatus` is **medium**, when the **amount** is greater or equal to **500** or `customerStatus` is **low**.

Rule 9 (post-condition). `outputData` of `judgeCreditCard` must exist.

These rules and the rest of the rules of this case study related to activity state transitions are also represented in RIF standard.

Conclusion

In this paper we present a first-version meta-model that integrates the declarative BPM representation built with the Em-BrA2CE framework with business rules standards RDF(S), OWL and RIF. This proves the convergence of business process modeling and business rules on the basis of Semantic-web technology-oriented modeling. Further research is planned for parsing, visualization and verification of the thus obtained model.

Acknowledgement. This work was supported by the European Social Fund through the Human Resource Development Operational Program under contracts BG051PO001-4.3.04-0018 (2013/2014), BG051PO001-3.1.08-0010 (2012/2013).

References

1. Hammer, M., Champy, J.: Reengineering the Corporation: A Manifesto for Business Revolution (3rd edn.). Nicholas Brealey Publishing (2001)
2. Semantics of Business Vocabulary and Business Rules (SBVR) Version 1.0. Specification. `http://www.omg.org/spec/SBVR/1.0/` (2008) (accessed Aug 10, 2014)
3. Pesic, M., van der Aalst, W.M.P.: A declarative approach for flexible business processes management. In: Proc. 2006 Int. Conf. on Business Process Management Workshops, pp. 169–180. Springer-Verlag Berlin, Heidelberg (2006)
4. Goedertier, S., Vanthienen, J.: Rule-based business process modeling and execution. In: Proc. IEEE EDOC Workshop on Vocabularies Ontologies and Rules for the Enterprise (VORTE 2005). CTIT Workshop Proceeding Series (ISSN 0929-0672). Enschede (2005)
5. Goedertier, S. et. al.: Goedertier, S., Haesen, R., Vanthienen, J.: EM-BrA2CE v0.1: A vocabulary and execution model for declarative business process modeling. Katholieke Universiteit Leuven, Faculty of Economics and Applied Economics, DTEW – KBI_0728 Series (2007)
6. zur Muehlen, M., Indulska, M.: Modeling languages for business processes and business rules: A representational analysis. Information Systems 35(4), 379–390 (2010)
7. Linked Data Tools: Tutorial 4: Introducing RDFS & OWL. `http://www.linkeddatatools.com/introducing-rdfs-owl` (August 2014) (accessed Aug 10, 2014)

8. Hitzler, P. et. al. (eds.): OWL 2 Web Ontology Language. Primer (2nd edn.), W3C Recommendation (11 December 2012). `http://www.w3.org/2012/pdf/REC-owl2-primer-20121211.pdf` (accessed Aug 10, 2014)
9. Kifer, M.: Rule Interchange Format: The Framework. In: Calvanese, D., Lausen, G. (eds.) Web Reasoning and Rule Systems. LNCS, vol. 5341, pp. 1–11. Springer, Heidelberg (2008)
10. Steen, B.: Generation of optimal business processes from business rules. University of Twente, Master Thesis (2009)
11. de Bruijn, J., Welty, C.: RIF RDF and OWL Compatibility, W3C Recommendation (2nd edn.), 5 February 2013. `http://www.w3.org/2013/pdf/REC-rif-rdf-owl-20130205.pdf` (accessed Aug 10, 2014)

Business Model for Successful Public-Private Partnership between ICT Company and University

Alexander Ivanov[1], Ivaylo Ivanov[1], and Krassen Stefanov[2]

[1] Hewlet Packard Delivery Center, Sofia, Bulgaria
{aivanov,Ivaylo.Ivanov}@hp.com
[2] Faculty of Mathematics and Informatics, Sofia University "St. Kliment Ohridski",
5, James Bourchier Blvd., 1164 Sofia, Bulgaria
krassen@fmi.uni-sofia.bg

Abstract. Outsourcing is an important instrument to gain competitive advantage. In recent years Bulgaria started to address attention of big companies as an outsourcing destination. In this paper we present new business model of partnership between University and big international ICT company. We outline the concept of this business model and show why it is beneficiary for both sides.

Keywords: outsourcing, BPO, ICT, education, knowledge management

1 Introduction

Business Process Outsourcing is seen as one of the flagships of the industrial revolution [2, 4, 5, 7–10]. Various international sources [6, 8, 11–13] show that Bulgaria is becoming an attractive outsourcing destination, especially from American companies. Bulgaria is a country with low production costs and taxes compared to other European economies, which in many cases is crucial for attracting capital investments. An advantage is also the skilled workers, working at some of the most competitive pay levels in Europe. According to recent studies [14, 15] Bulgarian outsourcing market is estimated at about 150 million, currently employed in this sector are more than 14 000 people. From the other hand, as a potential risk for the development of outsourcing in Bulgaria is given education system and, in particular, analysts doubt the potential of Bulgarian higher education market to supply the companies in the industry with required work force in the expected timelines, in an effort to address the challenging expectations from the business. To be able to meet this challenge in recent years, outsourcing has been identified as a strategic priority of the Republic of Bulgaria. In this regard within the past six years new business model of partnership has been developed between Sofia University (SU) and the largest employer in Bulgaria – Hewlett-Packard (HP), which operates very successfully so far. The purpose of this article is to outline the concept of this business model, which could be considered as a possible solution to the challenges, which the business and universities in Bulgaria are facing in the following highly relevant area for both sides – attracting of motivated students and employees. Main subject of this article is an analysis of the results achieved so far, the mutual benefits to the participants in this partnership and the future direction of development of this business model.

2 HP Business Case

Hewlett Packard covers more than $1/4$ in the Bulgarian outsourcing market with it is over 4000 employees working in the Global Delivery Center in Business Park Sofia. Other major companies in the market are C3, IBM, CSC, Comverse, etc.

HP Global Delivery Center officially opened in early 2006 as primarily location for the outsourcing services of IT support for HP customers in the Europe, Middle East and Africa. Bulgaria was chosen after a deep analysis of the market among 14 other competing countries for this investment. HP held a detailed market analysis of the labor market for more than six months upfront taking the final decision.

Critical element for the success of the center is the way it is going to attract talents. From another side finding motivated students and providing them with education on latest technologies is among the topics with critical importance for Sofia University too. With an aim to address all that, at the end of 2007, HP launched a special training program at Sofia University. The newly created educational program was based on the leading standards for IT education used at Stanford and Massachusetts Institute of Technology universities. This was within the core of the new business model for the delivery of an effective IT education within the university, subject to the current article. In essence this business model is in fact partnership model between the business – HP and the University, in an effort to address their common goals – to ensure that University students are educated on the latest IT technologies. The required laboratory equipment for ensuring smooth delivery of the classes in these courses, as well as supplying the right lectors was also among the commitments made by HP.

3 Benefits of the Implementation of the New Business Model for Effective University Teaching in ICT

While working for the last 6 years to achieve the stated goals, both sides have shaped the following positive aspects of the business in terms of this model:

- Opportunity for advancement and training of the best HP technical experts to become teachers and instructors.
- Certification of the best university teachers as instructors in their respective fields.
- Ability to invest in self-training and continuous improvement of the qualification of certified instructors due to their free access to all certified courses in technology that are certified.

- Opportunities for in-house training of staff on certification courses by certified instructors – high positive effect on cost, compared to external training in certification centers.
- Free education quotas for HP employees in the university courses.
- Ensuring to the company clients high quality service by professionals what possess multi technology, including instructor's certifications.
- Periodic training required for new staff and selecting new hires among the university elite.
- Ability to quickly and effectively train staff to areas where standard university education does not offer adequate answer.
- A much better quality in the selection of new staff – instructors have the chance to monitor the work of the new candidates in all semester instead of judgment to be made after one or two hiring interviews.
- Obtaining reliable and motivated employees, while in the same time attrition with this selection of personnel is practically negligible.
- Cooperation with the University is mutually beneficial, rather than traditional forms of collaboration – recurrent funding of schools without clear vision what should be the outcome from this investment.

In terms of the phases of development of this business model – the courses started with a single discipline in 2007, to reach 11 disciplines now. In total, over 300 students and staff have gone through the program, and all they provided quite positive feedback afterwards.

4 Perspectives for the University and Students as a Result of the Implementation of the New Business Model

In the bottom of every successful partnership is Win-Win situation (model of partnership which guarantees success for all involved parties). The main problem for IT companies (in this case HP) is related to the increased need of IT specialists in specific areas, where it is extremely difficult to find them on the free labor market. This problem was addressed and the solution was found in this successful partnership. The main focus in this chapter is the other side of the partnership – the University with its teachers and students.

Finding proper people for open positions is not an easy job for IT companies. But this is even harder for university professors and lectors in Information and Communication Technologies (ICT). It is dynamic field where something which is actual and widely used today is already outdated and replaced tomorrow. From the other side, to rely only on learning theoretical basics of ICT (like logic of programming and computer architecture) is simply not enough for the students who want to see practical realization of their knowledge. To be in line with increasing business demands, university must open new courses every semester which brings additional difficulties for professors in their preparation. From the other side companies like HP, have proven specialists in specific technologies which use this knowledge on daily basis. Most of these specialists are involved as lectors in internal trainings and mentoring for new employees (every company have initial training needed to prepare the new employees not only for specific tasks and processes in the company but also in new and "exotic" technologies). This can easily evolve in involvement of these experienced employees as lectors also in FMI courses. This can assure more lectors who are ready to lead University courses and also relief of the pressure from professors and lectors which are from regular University staff. Big part of the courses which are part of Bachelor and Master degree programs include process and technologies which not only needs years of preparation, but also serious investments in trainings and certification, which in many cases are not possible for Universities.

Involvement of highly skilled and trained HP employees in specific technologies in University training programs is one way this problem to be solved in very short time (no needs to wait years of preparation for regular University staff of lectors) and with almost no investments. In addition to their excellent technical knowledge this employees brings their work experience, which is extremely helpful for the students. This adds additional interest and it is one of the reasons this bachelor and master degree programs in FMI to be so successful. Giving examples from real situations from ICT environments of well-known worldwide companies brings additional motivation for the students – they see practical realization on what they learn in theory. Giving examples is very powerful tool – students understand what advantage they have when they go to free labor market with their specific knowledge and skills. Involving HP employees as lectors in FMI not only improves quality of education, but also helps university to attract additional candidates for University Bachelor and Master Degree programs. ICT also allows university to increase the number of students in different programs – something which is not possible if University rely only on their own lectors. University lectors from other side exchange their experience with lectors from HP, which increase their knowledge and qualification in latest technologies. Every semester new courses are included in University programs, which give a student a choice to select courses which basically cover almost all variety of technologies used in ICT business at the moment. Every new course includes short introduction of the content, which help for student orientation. Student efforts are concentrated in concrete technologies widely used in ICT corporations specialized in software development and support. This not only assures quick realization of the students but also improve University rating. At the moment interest to this Bachelor and Master Degree programs is extremely high. This includes also interest from employees from different ICT companies which want to become experts in different technologies and start new career or simply want to extend their current technical knowledge. Today there are many HP employees which are also students in Master Degree programs in FMI. This increased the scope of potential students in FMI and also helped to promote ICT education in Bulgaria. Such type of relationship is not very popular in Universities in Europe providing education in ICT technologies. This put Bulgaria and FMI specifically as one of the pioneers in this area.

5 Comparing the Model with Other Existing Partnerships

Back in 2006 HP donated equipment and started cooperation programs not only to FMI, but also to Technical University and NBU in Sofia.

Compared to FMI, where HP lectors lead courses as part of Bachelor and Master degree program for a whole semester, in Technical University (TU) and New Bulgarian University (NBU) they lead courses which are not part of regular University program or organize short events like presentation or workshop on specific predefined topics.

For example in TU, HP conducted Unix training for two weeks in last two consecutive years. In NBU in last year two workshops were provided – Lotus Domino and Project Management.

There are certain advantages for HP to provide such trainings. HP lectors are less involved, since normally such courses continue less than a full semester and final assessment is not mandatory. At same time using these single events on interesting and modern topics, they can present latest technologies to many students. From other side the students can see the latest technologies presented in real life examples which can motivate them to invest more time to develop their skills in these technologies and may be later continue with their careers in HP.

There is also a negative side of providing such trainings. Student interest is not the same – participation in such trainings is additional effort for them since these courses are not part of regular University program. When training is for only one or two week, there is not enough time to practical testing on what have been learned in theory. Often such trainings require a lot of advertisement to assure needed participation. After this workshops and additional courses, less than 10 students apply for HP positions.

The comparison in between the reviewed different partnership models clearly identifies the model practiced in FMI as by far the better among the rest. Within the main reasons why is the fact sheet at the end of the article in the Conclusion. Few are also highlighted here to support this statement as evidences: From low to medium interest towards the HP courses in TU and NBU to the huge interest among FMI students. From over 100 people hired in HP from FMI to more than ten times lower result in the other universities.

As evidence of the successful implementation of the model HP was announced in 2012 as preferred employer among students who completed their education and are about to start their first job in the respective field [3]. For Bulgaria the study included more than 9000 respondent's students of which over 2300 in the ICT sector.

6 Risks and Mitigation

Such a partnership might introduce some risks for both parties [1, 3, 4], which are subject of review in this section.

Risks for the University are:

- Accepting donations and entering in partnership like that one could be classified as sponsorship, which might impact the independence and fair judgment of the university leadership team. Whenever going forward they need to take decisions which are impacting the donating company – in this case that is HP. This has been settled with well formulated contract where the interest of the university is protected with clear rights and responsibilities of the partnering sides. The University has the full right to deal with the donated equipment for any education purposes they like, without any additional obligations or restrictions requested from HP.
- Partnering with only one company could really make the university vulnerable to additional influence from this external business, as an only source of additional investments. Within FMI situation currently, however the University has well defined partnership not only with HP, but also with other leading global ICT vendors like Microsoft, IBM, VMWare and Cisco. This way diversity among the external partners is secured.
- Allowing external trainings in the university curriculum could lead into building specific knowledge, which is in interest of only the external company in this case HP. So the students could be forced into joining HP as they do not have any other interested employer in their newly added skill during their education in FMI. This risk was mitigated in the way how courses are being selected and added in the program. Currently both sides could come up with proposal of courses, which to be added in the future training program. The final approval call always has the University though, so this way the relevance of the training content is ensured.
- Donating such an expensive and rare hardware could put the University in situation to pay quite an expensive fee to maintain the hardware in good shape in out of warranty period. This has been addressed by offering extended warranty by HP and significant discounts when ordering replacement parts.

Risks for the business (HP) are:

- The donated equipment to not be used for its purpose – education. This has been settled in the mutual agreement document signed with the university.
- The company management could be put under attack if the investment is made in only one University. Like why selecting this particular University among all the rest and could be a ground for further speculations. This has been settled by delivering one and the same donated hardware to three of the biggest universities in Bulgaria – FMI, TU and NBU Sofia. Sofia has been the obvious choice for the investment as the main city where the business operates.

7 Conclusion

With numbers below we can summarize what has been achieved in last three years (2010–2013) of partnership:

- more than 300 HP employees and thousands of students are trained form HP trainers;
- 37 full semester courses were provided with involvement of 15 HP trainers;
- over 100 students from FMI are already HP employees. Some of them are promoted on key positions for the company;
- HP had invested more than million in HW equipment for University computer rooms and labs used for education of students.

Future plans to extend this partnership include adding additional trainings/courses covering all technologies used in ICT business. Expectations are not only HP employees to help in education programs in FMI, but also involve HP management in open discussion with students presenting them the trends in ICT business which can additionally help students in their choice and orientation.

Acknowledgments. This work was supported by the European Social Fund through the Human Resource Development Operational Programme under contract BG051PO001-3.3.06-0052 (2012/2014).

References

1. Bahli, B., Rivard, S: Validating measures of information technology outsourcing risk factors. Omega 33(2), 175–187 (2005)
2. Blinder, A.: Offshoring: The Next Industrial Revolution. Foreign Affairs 85(2), 113–128 (2006)
3. Chorafas, D.N: Outsourcing, Insourcing and IT for Enterprise Management. Palgrave Macmillan, New York (2003)
4. Cummings, J.L., Teng, B.S: Transferring R&D knowledge: the key factors affecting knowledge transfer success. J. Eng. Technology Management 20, 39–68 (2003)
5. Greco, J: Outsourcing: The new partnership. J. Business Strategy 18(4), 48–54 (1997)
6. Hansen, M.T., Løvås, B: How do multinational companies leverage technological competencies? Moving from single to interdependent explanations. Strategic Management J. 25(8–9), 801–822 (2004)
7. Khan, N., Fitzgerald, G: Dimensions of outsourcing business models. J. Inf. Techn. Cases Appl. 6(3), 35–40 (2004)
8. Pfannenstein, L., Tsai, R.: Offshore outsourcing: Current and future effects on American IT industry. Information Systems Management 21(4), 72–80 (2004)
9. Quinn, J.B: Strategic outsourcing. Sloan Management Review 40(4), 23–26 (1999)
10. Zack, M.: The strategic advantage of knowledge and learning. Int. J. Intellectual Capital and Learning 2(1), 1–20 (2005)
11. Gaille, B.: Most attractive outsourcing destination for US companies. Personal blog at: `http://brandongaille.com/26-us-outsourcing-statistics-and-trends/` (2013)
12. Kearney, A.T.: Global Services Location Index. Available at `http://www.atkearney.com/documents/10192/f062cfd8-ee98-4312-ae4f-0439afc10880` (2011)

13. Trendence Institut©: Trendence Graduate Barometer 2012, Bulgarian Engineering/IT Edition. Available at http://www.trendence.com/fileadmin/trendence/content/Unternehmen/Rankings/Embargo-ENG_IT/trendence_Bulgaria_ENG_2012.pdf (2012)
14. InvestBulgaria Agency: Business Process Outsourcing in Bulgaria. Country report. Available at: http://www.investbg.government.bg/files/useruploads/files/statichni_stranici/broshuri_sectors/business_process_outsourcing/bpo_en-1.pdf (2014)
15. German Outsourcing Association: Central & Eastern Europe: A Colorful outsourcing landscape. The Outsourcing Journal Q2/3 (2013). Available at http://www.outsourcing-journal.org/Outsourcing_CEE_2013/index.html

Algorithm for Strategy Modeling Information System – Typology Strategies for Technology New Ventures Definitions

Sia Tsolova

Faculty of Mathematics and Informatics, Sofia University "St. Kliment Ohridski",
5, James Bourchier Blvd., 1164 Sofia, Bulgaria
s.valentinova@gmail.com

Abstract. The process of creating of algorithm for creation of Strategy modeling supporting e-system is connected with identifying the process of strategy management and strategy modeling and their adaptation for the specifics of technology new ventures, as well as the identification of the key typology strategies and their characteristics, which will be used as a basis in the implementation of such system. A definition of typology strategies to be incorporated in such e-system is implemented and tested through a qualitative research amongst 121 Bulgarian technology new ventures (start-up companies). The key findings and characteristics are introduced, as well as the process, specifics and the place of defining typology strategies in the entire process of strategy management e-system's creation.

Keywords: technology new ventures, strategy, entrepreneurship, strategy modeling, start-up companies, strategy management, e-management system

1 Introduction

Strategy modeling is a complex and currently accepted as a very intuitive process for technology new ventures, which, however, is very important for the future success of each technology new venture. Only a small number of technological innovations turn into successful technological companies, but together with technological skills, another ingredient is necessary for the new technology ventures to succeed, which is a successful strategy identification and modeling [2]. This process complexity could be highly simplified [1] through the creation of an e-system supporting the strategy modeling and strategic management. However, such system is yet not available, and the processes, together with the typological strategies and their success and failure characteristics, are not yet adapted for the technological start-up companies. This article presents the current results from a research on creating such strategy management e-system. The presented results are part from a qualitative research amongst 121 Bulgarian technology start-up companies and technology entrepreneurs in Bulgaria. After the research the results were processed with IBM SPSS 19 [3].

1.1 Place of the Typology Strategies Identification and Usage in the Entire Process of Strategy Modeling Algorithm for Creating of Strategy Management e-System

The research on identification of typology strategies is part of the research process of creating strategic management e-system, which needs algorithm for its implementation. The creation of such algorithm includes (Fig. 1): (1) research on the process of strategic management [4] and its adaptation for the needs and specifics of the technology entrepreneurial team, already implemented by the author, (2) research on the tools [6] and (3) methodologies for strategic management and strategic modeling [5] and their alignment with the redesigned process and needs of the technology new ventures, also already implemented by the author, and (4) identification of the basic types of typological strategies for the technology new ventures, together with (5) research on their characteristics and specifics – characteristics and factors for success and strategic threads in front of their development. A final step in the development of such system is (6) unification of all topics of the research in a structured step-by-step procedure of logical connections and calculations, implemented in the current development of the strategic management e-system. All these typological strategies are essential part of the algorithm of strategic modeling and will provide essential information for the companies, following the defined process and methodology, according the algorithm of the system.

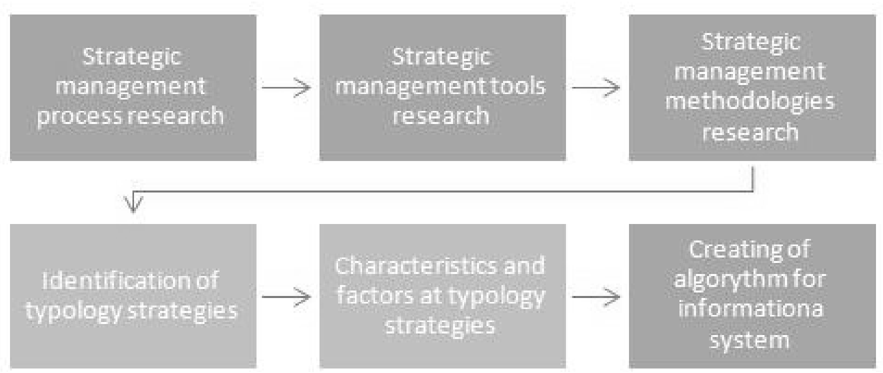

Fig. 1. Steps in the process of research and place of current results

2 Typology Strategies for Technology New Ventures

2.1 Classification Factors of the Typological Strategies for Technology New Ventures

After a research on over 30 types of strategies with their corresponding classifications, a combination of factors from some articles and two different tools in company's management process was identified (product-market matrix according innovations of products and Porter's generic strategies according market scope), which resulted in a modified classification for strategic development, proposed by the author for the Bulgarian market, which was followed by a research implemented amongst 121 Bulgarian start-up companies and technology entrepreneurs. The categorisation in the research is implemented according the company's types of technological capabilities for innovation of the company, market scope and market maturity.

$$\mathrm{NVTS} = f(\mathrm{IC}, \mathrm{MS}, \mathrm{MM}),$$

where NVTS is the new venture typological strategy, IC is the company's innovation capabilities, MS is the market scope and MM is the market maturity. Each of these variables has two values, which defines a total of eight typological strategies, described further in this article.

Figure 2 presents the classification of the typological strategies, according the market scope variable.

Technology products for local market		
New (emerging) market	3	4
Existing market	1	2
	Low innovative capabilities	High innovative capabilities

Technology products for global market		
New (emerging) market	3	4
Existing market	1	2
	Low innovative capabilities	High innovative capabilities

Fig. 2. Developed classification of typology strategies for technology new ventures

The suitability of factors for classification of strategies for technology new ventures typological strategies was confirmed by an implemented by the author research, which results lead to implementation of the basic typological strategies for technology new ventures.

2.2 Definitions of Typological Strategies Based on Implemented Research

The basic types of typological strategies, which together with their characteristics will be implemented in the current development of the strategic management e-system, are Companis: (1) Innovators developing products/services for local new/emerging market; (2) Innovators developing products/services for local existing market; (3) Innovators developing products/services for global new/emerging market; (4) Innovators developing products/services for global existing market; (5) Company follower developing products/services for local new/emerging market; (6) Company follower developing products/services for local existing market; (7) Company follower developing products/services for global new/emerging market; (8) Company follower developing products/services for global existing market. These types of strategies have proved to be connected with clear results on their success and thread factors and allow typological description of the strategies.

Company Innovator Developing Products/Services for Local New/Emerging Market. Strategy modeling for company innovator developing products/services for local new/emerging market: technology new ventures with such typological strategy offer innovative technological products on a local market, but in field which is new or emerging for the corresponding market, i.e. market is new or emerging.

The implemented research showed that basic factors of success are (1) uniqueness of the product at the market (81% support from the research), (2) additional serviced for the offered product/service (67%); (3) ease of access to the application of the product/service (57%) and (4) high quality of the product (52%), which all correspond to the described typological strategy and chosen factors of classification. Technological new ventures, which are included in this typological strategy are offering innovative products/services on the local market, which are new for the corresponding market, although they may be available in some similar form on different markets in other countries/continents. The results from the research also support the innovation management descriptions of the first steps for each technological company, introducing new and innovative products at certain markets. This, once again, suitability of the chosen factors of classification.

Companies with such characteristics and strategy have average growth potential, due to the size of the market, average market insecurity and average research and developments needs from investments. They usually have local technological partners and the priority in front of the company and the research and development of the product is product quality.

Again, as a confirmation, according the research, the basic strategic threads are: (1) lack of information for the application of the product (62%); (2) emerging of competitive product at lower cost (52%); (3) lack of ease of access to the product/service (48%). The full list of raw data, together with the description of the research process, are subject to another article, but there we use this data, in order to show the concept of the corresponding typological strategy and also to confirm the adaptability of the results with the theoretical basics of innovations development.

Company Innovator Developing Products/Services for Local Existing Market. Company innovator developing products/services for local existing market are usually offering products – substitution of global and more expensive products at the local market. Such products are developed by the technology new venture and have several advantages – localisation, lower price and smaller scope of the features (for example: accounting software produced by Bulgarian companies). Such companies have the technological capacity to introduce innovative products at the market, but they choose a developed field of business, while the first typological strategy is pointed towards new and emerging technological fields.

The basic characteristics of this strategy are: (1) According technology, market and product perspective; (2) Market maturity: existing market; (3) Technological capabilities: innovator; (4) Target market: local; (5) Growth potential: low; (6) Risk for company's realisation at the market: low; (7) Research and development investing: average; (8) Research and development priority: lower price; (9) Technological partners: local; (10) Internationalisation level: low.

The implemented research showed that basic factors of success are: (1) Ease of access and application of the product: 93%; (2) Lower product price: 71%; (3) Sufficiently high level of the product: 71%; (4) Additional services for the product: 57%; (5) Product uniqueness at the market: 57%. The strategic threads in the implemented research are: (1) Lack of information for products applicability and work: 79%; (2) Emerging of competitive product at lower price: 64%.

Company Innovator Developing Products/Services for Global New/ Emerging Market. Company innovator developing products/services for global new/emerging market offers innovative products at the global new/emerging market. This is a strategy which involves the highest level of innovative capacity, investments and highest in difficulty for implementation, as well as highest in risk strategy, which however has highest growth potential. This strategy has all typical for new product on new market characteristics from Innovation management theory, which were confirmed also by the implemented research.

The basic characteristics of this strategy are: (1) According technology, market and product perspective; (2) Market maturity: new/emerging market; (3) Technological capabilities: innovator; (4) Target market: global; (5) Growth potential: very high; (6) Risk for company's realisation at the market: very high; (7) Research and development investing: high; (8) Research and development priority: time of entering at the market; (9) Technological partners: global; (10) Internationalisation level: high.

The implemented research showed that basic factors of success are: (1) Product uniqueness at the market (85%); (2) Ease of access and application of the product (77%); (3) High quality of the product (62%). The strategic threads in the implemented research are: (1) Lack of ease of access and application of the product (69%); (2) Lack of information about product work and application (62%); (3) Availability of a competitive product with higher quality, even at a higher price (54%).

Company Innovator Developing Products/Services for Global Existing Market. Company innovator developing products/services for global existing market is offering products on existing field with a lot of competitiveness in it. In these cases company's typological strategy is offering unique narrowly specialised products, which will give the company opportunity for competitive advantage, offering specialised products at high price, which could also be lower from the wide specialised products with a lot of functions in them. These is connected with high innovative capacity of the companies and also new technologies, but mainly with high level of competence on the developed narrowly specialised product, functionalities, ease of use and product quality.

The basic characteristics of this strategy are: (1) According technology, market and product perspective; (2) Market maturity: existing market; (3) Technological capabilities: innovator; (4) Target market: global; (5) Growth potential: high; (6) Risk for company's realisation at the market: average; (7) Research and development investing: high; (8) Research and development priority: product quality; (9) Technological partners: global; (10) Internationalisation level: high.

The implemented research showed that basic factors of success are: (1) High quality of the product (75%); (2) Product uniqueness at the market (67%); (3) Lower price of the product (in comparison with the wide functioning products – this was not specified, which justifies the results 50:50%) (50%). The strategic threads in the implemented research are: (1) Lack of information about the work and application of the product (58%); (2) Lack of ease of access to the product (50%).

Company Follower Developing Products/Services for Local Existing Market. Company follower developing products/services for local new/emerging market is a representative of the traditional small and medium enterprises, which are following existing good practices in business. Here the typological strategy has all traditional elements of competitive advantage described at marketing field, but such companies have low level of innovations and low level of research and development investments.

The basic characteristics of this strategy are: (1) According technology, market and product perspective; (2) Market maturity: existing market; (3) Technological capabilities: follower; (4) Target market: local; (5) Growth potential: low; (6) Risk for company's realisation at the market: low; (7) Research and development investing: low; (8) Research and development priority: low price with lower quality, additional services at slightly higher price or high price with high quality – traditional competitive strategies – according Porter's generic strategies; (9) Technological partners: local; (10) Internationalisation level: low.

The implemented research showed that basic factors of success are: (1) Lower price of the product (61%); (2) Additional services for the product/service (61%); (3) High quality of the product (55%). These results prove the traditional type of competitive theories in this typological strategy. The strategic threads in the implemented research are: (1) Availability of competitive product at lower price (50%); (2) Lack of information about the work and application of the product (39%).

Company Follower Developing Products/Services for Local New/ Emerging Market. Company follower developing products/services for local new/emerging market is introducing products at local market on a new or emerging field, but since the company's technological capacity is not enough to introduce innovative products, developed by itself, such companies usually introduce products of other companies at the local market, by localising, translating and promoting them, they become representatives of global or foreign products.

The basic characteristics of this strategy are: (1) According technology, market and product perspective; (2) Market maturity: new/emerging market; (3) Technological capabilities: follower; (4) Target market: local; (5) Growth potential: average; (6) Risk for company's realisation at the market: average; (7) Research and development investing: average; (8) Research and development priority: lowering the price; (9) Technological partners: local; (10) Internationalisation level: low.

The implemented research showed that basic factors of success are: (1) Product uniqueness at the market (50%); (2) High quality of the product (50%); (3) Ease of access and application of the product (50%). A pattern is already visible by the results, but this category needs a little more data by accessing more companies of this type with the research in future. The strategic threads in the implemented research are: (1) Availability of competitive product at lower price (60%); (2) Insufficient resource supplement of the company (40%).

Company Follower Developing Products/Services for Global New/ Existing Market. Company follower developing products/services for global new/existing market is a company, which offers products for the global new or emerging market, but is usually not the innovator of new ideas globally. Such companies, however, have the capacity to quickly duplicate innovative products and bring them to market, which is always connected with availability of higher technological capacity than that of the other types of follower at the typological strategies.

The basic characteristics of this strategy are: (1) According technology, market and product perspective; (2) Market maturity: new/emerging market; (3) Technological capabilities: follower; (4) Target market: global; (5) Growth potential: very high; (6) Risk for company's realisation at the market: high; (7) Research and development investing: average; (8) Research and development priority: time for entering the market; (9) Technological partners: global; (10) Internationalisation level: high.

The implemented research showed that basic factors of success are: (1) Ease of access and application of the product (75%); (2) Product uniqueness at the market (50%); (3) High quality of the product (50%); (4) Additional services for the product/service(50%). A pattern is already visible by the results, but more results are needed in future. The strategic threads in the implemented research are: (1) Lack of information about the work and application of the product (90%); (2) Slowing the speed of new elements integrations in the product (75%);

(3) Lack of ease of access to the product (50%); (4) Changes in law or standards (50%).

Company Follower Developing Products/Services for Global Existing Market. Company follower developing products/services for global existing market – offering different varieties of existing products at the global market, following good practices. Such companies follow the traditional theories of competitive advantage, but at global scale.

The basic characteristics of this strategy are: (1) According technology, market and product perspective; (2) Market maturity: existing market; (3) Technological capabilities: follower; (4) Target market: global; (5) Growth potential: high; (6) Risk for company's realisation at the market: average; (7) Research and development investing: average; (8) Research and development priority: lowering the price; (9) Technological partners: global; (10) Internationalisation level: high.

The implemented research showed that basic factors of success are: (1) High quality of the product (56%); (2) Additional services for the product/service (56%); (3) Ease of access and application of the product (56%); (4) Conformity with existing standards at the market (44%). A pattern is already visible by the results, but more results are needed in future. The strategic threads in the implemented research are: (1) Availability of competitive product at lower price (56%); (2) Lack of information about the work and application of the product (56%); (3) Slowing the speed of new elements integrations in the product (75%).

3 Conclusion and Following Work

Integrating a the results from this research in an interactive e-system, aiming the support of strategic modeling and strategic management of technology new ventures, together with integration in the system of the entire process and methodology for strategy management, which are previously adapted to the activity of technology new ventures is the overall goal of the research. The goal of this research was identifying the different typological strategies, identifying their factors of success and strategic threads, as well as verifying the classification of the strategies for technology new ventures. The clear results and their conformity with the theoretical basic directions in innovation management, as well as the interpretation of the answers are giving us a positive result from the implemented research. The article shows the quantitative outcomes from the implemented qualitative research, which are a valuable part in the process of development of the strategic management e-system.

Acknowledgments This work was supported by the European Social Fund through the Human Resource Development Operational Programme under contract BG051PO001-3.3.06-0052 (2012/2014).

References

1. Tsolova, S.: Model for strategy of technology new ventures. Vanguard Scientific Instruments in Management (ISSN 1314-0582) 2(7), 294–308 (2013) (in Bulgarian)
2. Yankov, B.: Overview of Success Prediction Models for New Ventures. In: Int. Conf. Automatics and Informatics'12 (ISSN 1313-1850), pp. 13–16 (2012)
3. Tsolova, S.: Algorithm for strategy modeling implemented by e-governance system – results from research. In: Proc. 6th Int. Sci. Conf. "E-governance", Sozopol 2014, vol. 1, pp. 224–232 (2014) (in Bulgarian)
4. Harvard Business Essentials: Strategic Management. Harvard Business Press, Harvard (2006)
5. Kaplan, R., Norton, D.: The Balanced Scorecard: Translating Strategy into Action. Harvard Business School Press, Harvard (2001)
6. Haberberg, A., Rieple, A.: Strategic Management: Theory and Application. Oxford University Press, Oxford (2008)

Information System for Forecasting the Success of Bulgarian Start-up Companies

Boyan Yankov and Nikolay Vitanov

Faculty of Mathematics and Informatics, Sofia University "St. Kliment Ohridski",
5, James Bourchier Blvd., 1164 Sofia, Bulgaria
boian_iankov@abv.bg, nikolay.vitanov@gmail.com

Abstract. An Information System (IS) for predicting the success of Bulgarian start-up companies has been designed and developed. The IS is based on a model for success prediction derived from a quantitative research of 137 Bulgarian start-up companies. Entrepreneurs fill in a survey and based on a prediction model the IS estimates the chances of success of their start-up and shows a graphical decision tree with the factors that led to the result.

Keywords: technology entrepreneurship, start-up companies, new ventures, modeling, prediction software, NVP

1 Introduction

Predicting the success of new companies increases the efficiency of the venture creation process, minimizes the risks and resources spent and increases the returns. Unfortunately there are no success prediction models and software tools developed for the specifics of the Bulgarian start-up companies [1].

An Information system (IS) for forecasting the success of Bulgarian start-up companies would be useful to entrepreneurs, business owners, business incubators, university start-up centers, business consultants, venture capitalists and investors to predict the success probability for the new companies and to identify the possible strengths and weaknesses.

1.1 Theoretical Model for New Venture Success Prediction

After an analysis of 42 success prediction models [1], a pattern introduced by Sandberg in 1986 [2] has been identified. It has been adopted and improved in later models. Previous studies of Bulgarian start-ups success factors and business models have been examined [3, 4]. An improved and adapted for Bulgarian companies venture success prediction model has been proposed by the author [5]. The model is presented with the formula:

$$\mathrm{NVP} = f(\mathrm{E}, \mathrm{IS}, \mathrm{BS}, \mathrm{R}),$$

where NVP is the new venture performance, E is the entrepreneur, IS is the industry structure, BS is the business strategy and R represents the available

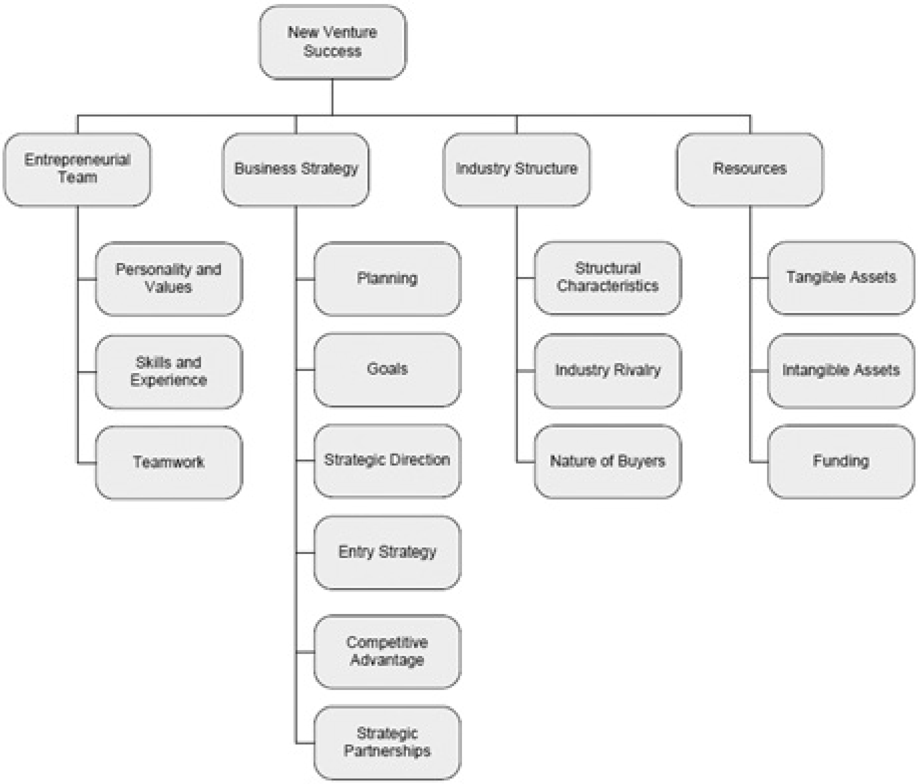

Fig. 1. New venture success prediction model proposed by the author

resources. Each of the main categories in the company success prediction model is decomposed into subcategories [5] as it is shown in Fig. 1.

1.2 Quantitative Research and Modeling

The new venture success prediction model has been validated with the help of a quantitative research [6] in the form of in-depth inquiries of 137 start-up companies in Bulgaria. The quantitative research is based on the CRISP-DM [7] methodology for data mining projects using the IBM SPSS Modeler [8] and WEKA software products. The software products generate start-up company success models in the form of decision trees based on the theoretical model and dataset.

The decision trees start with the most important success predictors and split the cases into groups (represented by nodes) depending on the responses. The process continues until the case reaches an end (leaf) node which indicates the predicted value of the target – the company success. Based on the generation and validation setting of the tree generation algorithms, the successfully recog-

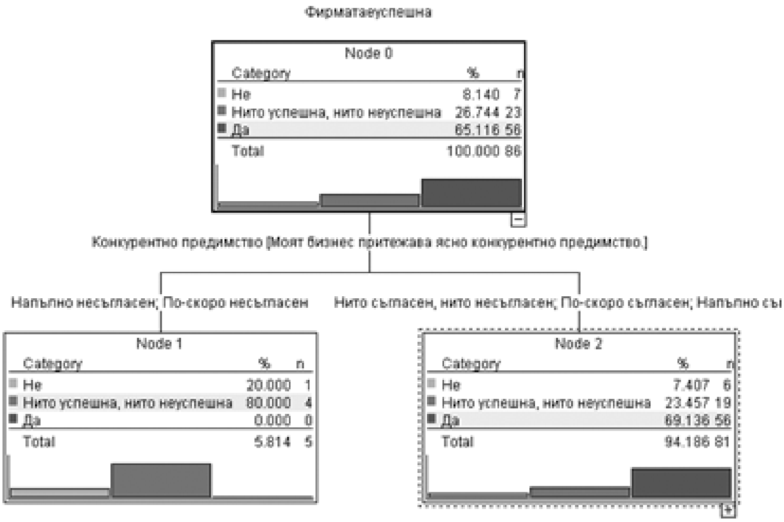

Fig. 2. Nodes from a generated decision tree using IBM SPSS Modeler

nized cases vary from 75% to 91.86%. A part of a generated decision tree using IBM SPSS Modeler is shown in Fig. 2. The figure indicates that the most important predictor for the company success is the presence of a clear competitive advantage.

2 Existing Software Applications for Success Prediction of Start-up Companies

Based on internet research, three similar tools for evaluation of the chances of success or failure of a start-up company has been identified. The tools rely on input data from a survey.

The most advanced tool is called “Odds of Success Calculator” [9] and was created by the “StartupNation” company. The purpose of this calculator is to determine the chances of success of companies. The tool is a web application and is freely accessible on the Internet. It works by assessing eight key factors about the company:

1. About how much debt and equity capital has been provided to your business?
2. How long do you think it would take to obtain additional funding for your business?
3. How often do you evaluate your cash flow status or plan?
4. What is your degree of business planning?
5. What is the approximate annual growth rate of your market?
6. How many years of management experience do you or your team have?
7. How many years of industry experience do you or your team have?

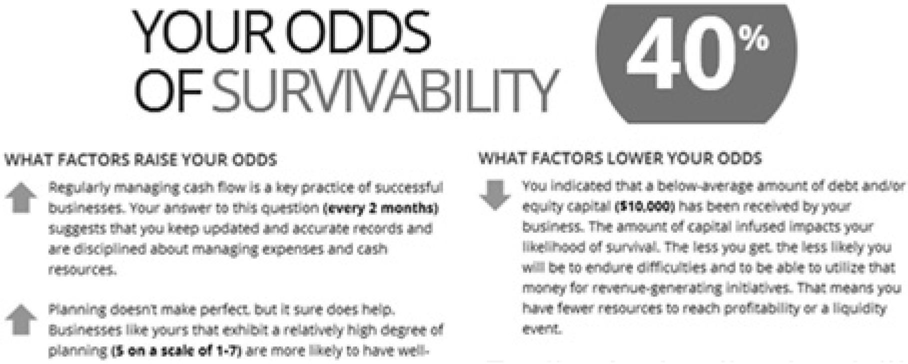

Fig. 3. Odds of Success Calculator Results

8. Over what timeframe do you want to know your chances of survival?

The main advantage of this tool is that it is open and easy to use due to the few factors used to assess the success (Fig. 3). On the other hand, due to the little information that the user must enter to describe their business, creating an accurate forecast does not seem very reliable. The results are not based on a scientific research but on data about businesses from North America provided by the EquityNet Crowdfunding Platform.

Another application for predicting the success of start-up found in the study is the "Startup Compass" [10]. The application assesses technology start-up companies. It uses a questionnaire and produces an evaluation of the company according to its type and stage of development (Fig. 4). The results are displayed as various graphics evaluating the current company on the basis of other companies using the tool. The idea of the creators of this app is to serve as a personal assessment of a business and use it to more easily detect potential problems, clarify priorities, to support the process of shaping the entrepreneurial team and to easily measure the progress of a company.

The assessment criteria of Startup Compass are based on a research of over 600 start-up companies from the Silicon Valley. The main factors are:

- customers,
- product/service,
- management team,
- business model,
- funding sources.

The software uses data from various Internet resources: Quickbooks, Salesforce.com, Stripe, etc.

The last application that will be discussed is called "Blueprint start-up success calculator" [11]. The application was created by the Australian company "Think Blueprint" and it is meant to be a tool for complete business planning, including tools for creating business plans, risk assessment and outlining the main activities

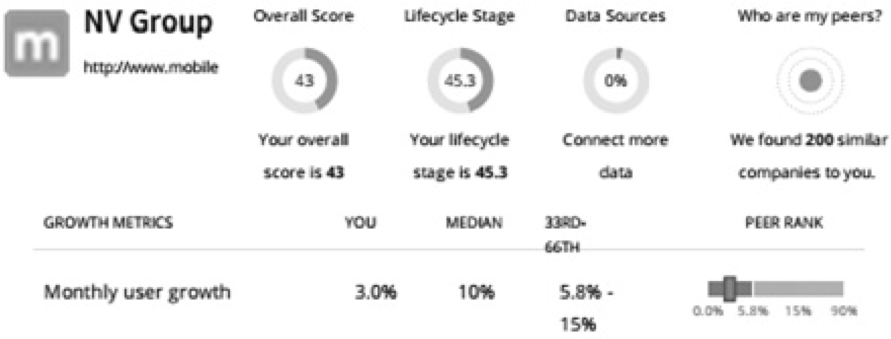

Fig. 4. Startup Compass Results

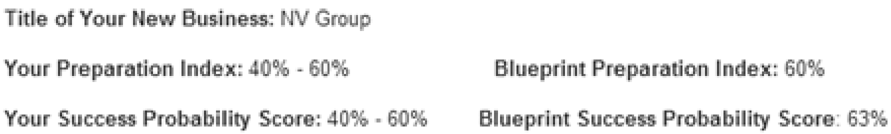

Fig. 5. Blueprint Start-up Success Calculator Results

in the creation of a new company. Some of its features are free to use. The software uses a brief questionnaire to obtain data about the start-up company. The results for the chances of success are sent via email as it is shown in Fig. 5.

The workflow of the application cannot be traced and no details are given about the calculated score. This application has the most basic functionality among all examined in the study.

3 Information System for Forecasting the Success of Bulgarian Start-up Companies

3.1 Forecasting Algorithm

The main objective of the IS is to automatically predict the chances of success of start-ups. The data flow of the IS is illustrated in Fig. 6. The prediction is done in four steps, assuming that the IS will be used by a company owner with the help of a business consultant who will guide him through the process and will help him interpret the results and take actions:

1. Company owner fills in a survey in Google Drive.
2. Business consultant start the IS.
3. In the IS, the business consultant selects the survey data and a prediction model and starts the prediction.

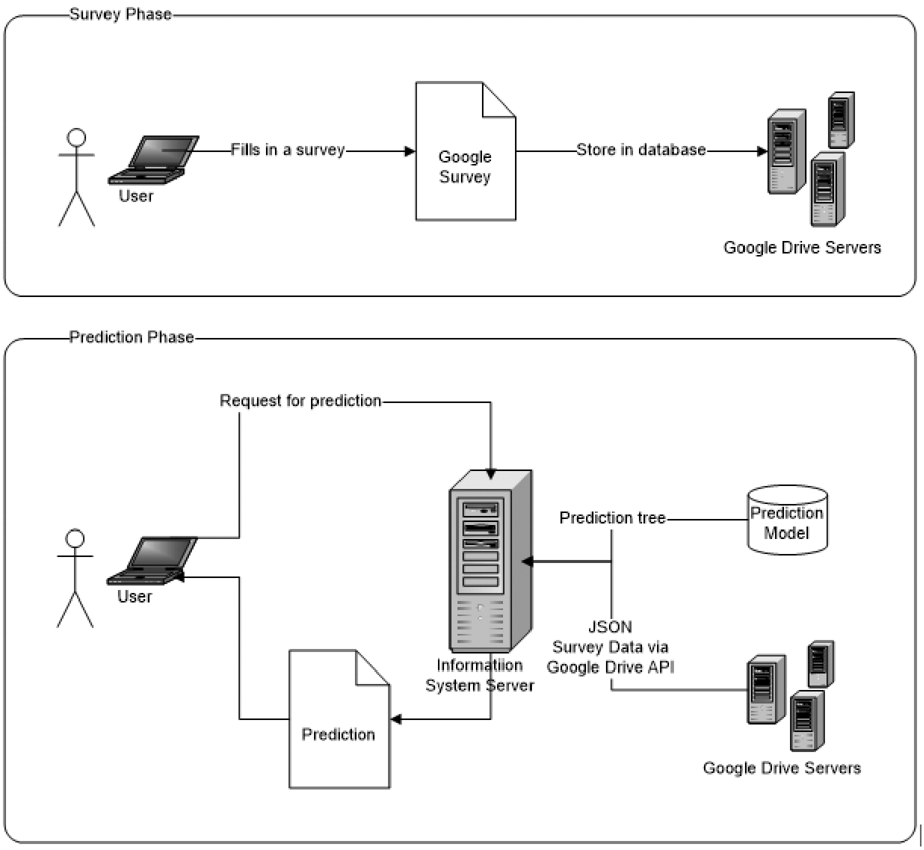

Fig. 6. Data Flow Diagram of the IS

4. The business consultant and the company owner analyze the prediction result and the success factors.

After the users of the IS fill in a survey, the IS reads the survey data with the help of Google APIs and prepares the data. The IS also reads the prediction model which is a decision tree. Then the survey data is interpreted as nodes of the decision tree. The rules from the decision tree are applied consequently to generate a success prediction result. The result is in the form of a success probability rate and a graphic of a personalized tree which illustrates the success factors compared with the user data and the success probability of each node (Fig. 7).

The IS will evolve as more companies use it and the database grows. It supports various decision tree models from IBM SPSS Modeler and WEKA.

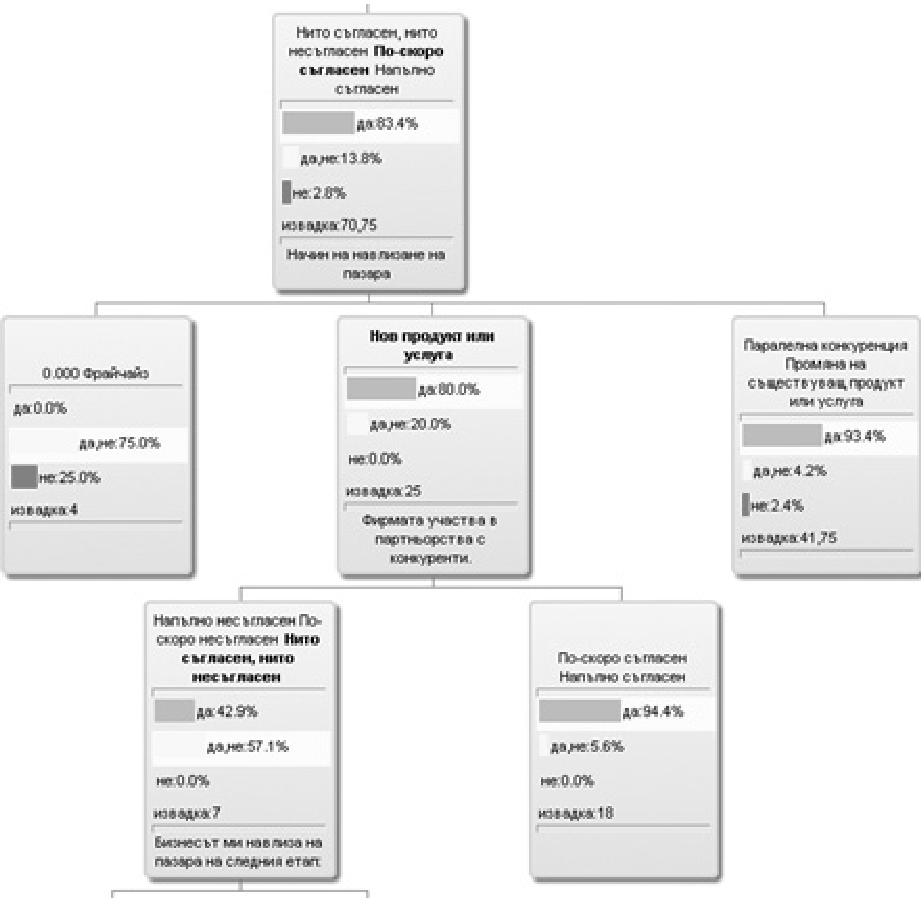

Fig. 7. Example of a Personalized Decision Tree

3.2 Technologies

The IS was developed using the principles of OOP in PHP programming language [12] and MySQL database. The Smarty framework is used to separate the logic layer from the presentation layer. The application is browser-based and the front-end is developed in xHTML, CSS and JavaScript. The IS needs several third party libraries and APIs for login, access to the survey data, visualization, AJAX calls, etc.:

- Google Disk API – used to access survey data on Google Drive,
- Google Spreadsheet API – used to access, sort, group and paginate survey data on Google Drive,
- Google Visualization API – used to visualize the decision trees,
- Smarty – a PHP template engine,
- jQuery – a JavaScript library.

4 Conclusion and Future Work

Interviews with five Bulgarian company owners indicate that they would use the IS for predicting the success of their companies and for indicating possibilities for improvement. Most of them would trust the prediction result because it is rationalized with a decision tree. Although they are not ready to pay for using the software, they would try it if it was free.

The IS predicts the success of Bulgarian start-up companies with the accuracy of the prediction model. The model is based on a qualitative research of 137 companies. The prediction accuracy would increase if more companies used the IS and the database increases. This is an opportunity for constructing better prediction models. The IS is designed to work with any decision tree classification model, which opens the possibility to extend the success predictions to companies outside Bulgaria.

Acknowledgments. This work was supported by the European Social Fund through the Human Resource Development Operational Programme under contract BG051PO001-3.3.06-0052 (2012/2014).

References

1. Yankov, B.: Overview of Success Prediction Models for New Ventures. In: Int. Conf. Automatics and Informatics'12, ISSN 1313-1850, pp. 13-16 (2012)
2. Sandberg, W.R.: New Venture Performance: The Role of Strategy and Industry Structure. Lexington Books, Lexington, MA (1986)
3. Tsolova, S.: Model for strategy of technology new ventures. Vanguard Scientific Instruments in Management (ISSN 1314-0582) 2(7), pp. 294–308 (2013) (in Bulgarian)
4. Manev, I.M., Manolova, T.S., Gyoshev, B.S., Harkins, J.A.: Social capital and strategy effectiveness: an empirical study of entrepreneurial ventures in a transition economy. Modern Competition 36(6), 57–70 (2012)
5. Yankov, B.: A Model for Predicting the Success of New Ventures. In: Int. Sci. Conf. "e-Governance", 2013, ISSN 1313-8774, pp. 128–135 (2013)
6. Yankov, B., Haralampiev, K., Ruskov, P.: Start-up companies predictive models analysis. Vanguard Scientific Instruments in Management (ISSN 1314-0582) 2(7), pp. 275–285 (2013)
7. Chapman, P., Clinton, J., Kerber, R., Khabaza, T., Reinartz, T., Shearer, C., Wirth, R.: CRISP-DM 1.0: Step-by-step data mining guide. CRISP-DM consortium, `http://www.crisp-dm.org` (2000)
8. IBM Corporation. IBM SPSS Modeler 15 User's Guide (2012)
9. Odds of Success Calculator. `https://www.equitynet.com/crowdfunding-tools/startup-risk-calculator.aspx`
10. Startup Compass. `https://www.compass.co/`
11. Blueprint start-up success calculator. `http://q1.prototype.thinkblueprint.com.au/new-business-success-calculator/`
12. PHP: Hypertext Preprocessor, `http://php.net/`

Part IV

Technology Enhanced Learning

An Integrative Approach to Effective Learning in Mathematics

Tanya Tonova

Faculty of Mathematics and Informatics, Sofia University “St. Kliment Ohridski”,
5, James Bourchier Blvd., 1164 Sofia, Bulgaria
ttonova@fmi.uni-sofia.bg

Abstract. This article explores the opportunities provided by the cultural and educational field of Mathematics, Informatics and Information Technology to achieve effective learning. To this end, characteristics of effective learning have been formulated. The examples have been used to demonstrate certain strategies and their application in solving problematic situations. The examples place the accent on combined knowledge in algebra, geometry, stereometry, mathematical analysis and the use of IT in solving them. It also touches on the teachers' responsibility both for the knowledge and the metacognition of their students.

Keywords: effective learning, characteristics of effective learning, strategies, combined knowledge, isomorphic problems, algorithms, heuristics, computer simulation, visualization of knowledge

1 Characteristics of Effective Learning

Many scientific fields, schools and theories also study the effective learning in the subsequent generation, presenting, structuring, deriving, sharing and use of knowledge. A goal of the effective learning is to give opportunity to students to classify ideas or objects, place them in a different order, make assessments and extrapolate data. In addition, they should be able to weigh proofs and draw up conclusions, make appropriate analogies, think not only analytically, but also critically and creatively [1]. Students must **fully** understand what they are being taught. To do this, they must learn how to think thoroughly of the school material presented to them. Moreover, they must comprehend their own thinking and implement **self-control** on their own cognitive activity, i.e., to achieve a certain metacognition. A key learning tool is solving a problematic situation (mathematical problem). There are two main approaches to solving the problem. The first is to follow a clear and fixed set of steps which guarantees the problem solving, called algorithm. The second is to use heuristics – an informal, intuitive strategy. Heuristics provides efficiency, although at the expense of security.

2 Opportunities for Implementing Effective Learning

The cultural and educational field of Mathematics, Informatics and Information Technology using appropriate methods is a rich field for effective learning due to the opportunity for integrity of the three schools in this field. Effective learning is based on experimental freedom. In the field of Mathematics, the computer algebra and dynamic geometry systems ensure to both teachers and their students the opportunity to experiment in the conditions of the problematic situation. Informatics builds up algorithmic thinking which, in turn, is the basis of one of the two main approaches to solving mathematical problems.

Information technology implements computer simulations which are a valuable tool for heuristic experiments. Simulation is an excellent means for "what if" scenario studies in an interactive form, since they can present complex problems in an easily comprehensible way.

3 Strategies to achieve effective learning

3.1 The "Isomorphism" Strategy

Isomorphism is a basic term in Mathematics, but it also has a place and importance in the methods for teaching Maths. Certain problems can be difficult for students, although they already know how to solve them without realizing it. But they can be assisted by presenting the respective problem (problematic situation) in a different form. Such somewhat different verbal and/or visual forms of the problem are called problem isomorphs – problems that have the same structure but different forms of expression [1]. For instance, the problem "Find the number of roots of the equation $x^4-2\sqrt{5}x^2+x+5-\sqrt{5}=0$" has as its isomorph "Find the number of roots of the parametric equation $x^4-2ax+x+a^2-a=0$ where $a=\sqrt{5}$.", which makes a serious and significant interpretation of the original clause. The strategy for creation of different isomorphic forms of a certain problem supports the student's efforts, contributes to their success and, in this sense, is a strategy for effective learning.

Example 1 ([2]). *Construct the roots of the quadratic equation*
a) $x^2-ax+b^2=0$*;*
b) $x^2+px-q^2=0$*.*

The *first* isomorph of the problem "Construct the roots of the quadratic equation $x^2-ax+b^2=0$" is obtained if it is re-formulated into the form: "Construct the roots of the quadratic equation $x^2-ax+b^2=0$, if

Case 1. $a>0$;

Case 2. $a<0$."

The *second* isomorph of the problem in the first case is obtained when, using the Vieta's formulas, it is re-formulated into the form:

"Construct segments with lengths $x_1>0$ and $x_2>0$, so that $x_1+x_2=a$ and $x_1x_2=b^2$". This problem is a verbal isomorph of the first case and it is much more likely that students are familiar with it and they are willing to seek its solution.

The *third* isomorph is: "Construct the projections of x_1 and x_2 on the sides of a right triangle with hypotenuse with length a and height with length b^2." This isomorph is a problem already studied and its solution is a practical application of already acquired knowledge.

Formulating one or more isomorphs of a certain problem is a successful strategy for effective learning because it builds up the *integrity* of knowledge. To be more specific, in the reviewed problem, the effect lies in the unity of knowledge about the quadratic equation and its roots, and the metric relationships within a right triangle.

The second case of the problem is analogous to the first one, but the two roots are negative and the segments that must be constructed are $-x_1 > 0$ and $-x_2 > 0$.

To apply the same strategy in solving sub-clauses b), we have to analyze the differences between the two sub-clauses (the similarities are obvious). While in sub-clauses a) the roots have equal signs ($b^2 > 0$), in sub-clauses b) the signs of the roots are different ($-p^2 < 0$). The first isomorph generates the cases $p > 0$ and $p < 0$. The second isomorph connects, through the Vieta's formulas, the lengths of the sought segments with the given coefficients of the quadratic equation. The third isomorph is "Construct a secant through the center of a circle with radius $|p|$ and through a point at which the length of the tangent to the circle is q^2." Once again, this isomorph is an isolated case of a problem which has already been studied, and the lengths of its secant and external part are the absolute values of the sought roots.

The constructions in the problem made with the use of a dynamic geometry programme, are displayed in Fig. 1. The diagram is dynamic in terms of the change in values of coefficients of the given quadratic equations. It gives opportunity:

- to observe the universality of the solution in terms of the different values of coefficients;
- to establish additional facts and assertions outside the specifics of the problem;
- to update and activate available knowledge in a new context.

Solving an algebra problem using a geometry approach implemented with the help of dynamic geometry builds up the **integrity** of students' knowledge in algebra, geometry and IT – one of the characteristics of effective learning. As a result, the constructions visualize:

- the existence and number of the roots;
- the geometric meaning of the coefficients;
- the metric dependencies in a right triangle;
- the metric dependencies in a circle;
- the computer simulation of a problematic situation;
- the logical relationships that lead to a cognitive effect in several directions.

For the students to achieve the level of effective learning, it is important that their teachers understand the concept of isomorphism of problems. A given mathematical problem can often be presented (1) verbally, (2) algebraically,

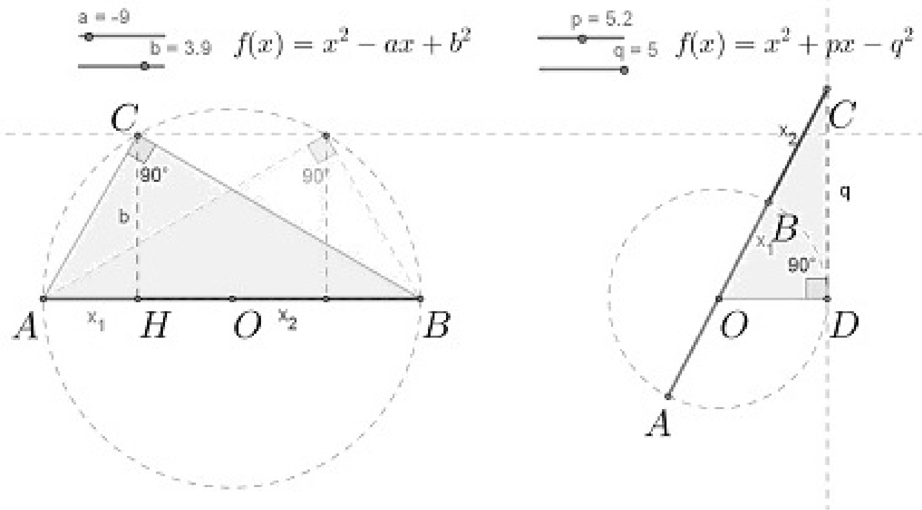

Fig. 1. Construction of the roots of a quadratic equation

(3) geometrically and, thanks to IT, (4) dynamically as well. In these cases, the problem is the same, only its presentation is different. However, students react differently to the different presentations. This helps the students find a presentation that works best for them. The ability to express a given problem under different forms is a sign that the teacher and the student understand it completely.

3.2 The "Control, Self-Control" Strategy

In his book "How to Solve It" [3], György Pólya formulates four phases that make up the process of solving a random mathematical problem: (1) Understand the problem, (2) Devise a plan, (3) Carry out the plan, (4) Review the solution. The fourth phase proposed by Pólya is also commented in [5], as its action covers the entire course of problem solving. Thus, the phase "Review the solution" turns into the strategy "Control, Self-Control" which is used by the student to learn how to control every step of solving a given problem.

Example 2 ([4]). *Given is a regular tetrahedron $ABCD$ with an edge length 1. Through the middle K of its base edge AB, a section has been constructed parallel to the lateral edge BD. Express the perimeter of the obtained section through the length of segment DM, where point M is the piercing point of edge CD with the section's plane. Find the highest and lowest value of the section's perimeter.*

To solve this problem, the strategy of isomorphic problems is inappropriate. The goal and means are clear, and the efficiency lies in "breaking" the solution into separate consecutive steps and implementing control in the transition from one step to another.

These steps are:

(1) Construct the section $KNML$, where $KL \parallel DB$ and $NM \parallel DB$.
(2) Introduce the variable $x = DM$, $x \in [0;1]$ and construct the function $P(x) = \dfrac{3}{2} - x + \sqrt{4x^2 - 2x + 1}$, which expresses the section's perimeter.
(3) Find the derivative $P'(x) = \dfrac{4x - 1 - \sqrt{4x^2 - 2x + 1}}{\sqrt{4x^2 - 2x + 1}}$ and determine the only extremum $P_{\min}\left(\dfrac{1}{2}\right) = 2$.
(4) Determine the value of $P(x)$ at the ends of the definitional set $P(0) = \dfrac{5}{2}$ and $P(1) = \dfrac{1}{2} + \sqrt{3}$.
(5) Finally, find the lowest value of section 2 and the highest value $\dfrac{5}{2}$.

In implementing the separate "fragmented" steps, the effective learning requires the application of the **self-control** strategy. This strategy has allowed the person solving the problem to find a peculiarity of the section. At $M \equiv D$, the section is the lateral side ABD with a perimeter 3. Therefore, the function which provides the values of the section's perimeter, is a discontinuous function

$$P(x) = \begin{cases} \dfrac{3}{2} - x + \sqrt{4x^2 - 2x + 1}, & 0 < x \leq 1, \\ 3, & x = 0. \end{cases}$$

From a geometric point of view, the function $P(x)$ makes a leap at $x = 0$, because the triangle FKL "lies" on the lateral side ABD, thus becoming a part of the section, while at any other value of x, it is outside the body (Fig. 2).

The cognitive effect of the problem lies in combining knowledge in stereometry and mathematical analysis which has lead to the construction of a discontinuous function thanks to the strategy for effective learning – self-control. The **control** implemented both in carrying out the separate steps and deriving conclusions is a sign for effective learning. It is appropriate to animate the constructions in the problem and to change the position of the "moving" point in order to achieve a fuller understanding of the problem as a result of visualizing the process of changing the section and its face, respectively.

4 Future Research

Effective learning is most often associated with assessing the students' achievement, i.e., if results are good, learning is effective, otherwise it is not. This is not the fairest and most beneficial approach towards the student. The above strategies and their use in particular problematic situations give us hope that effective learning may be an element of the method for teaching Maths, especially

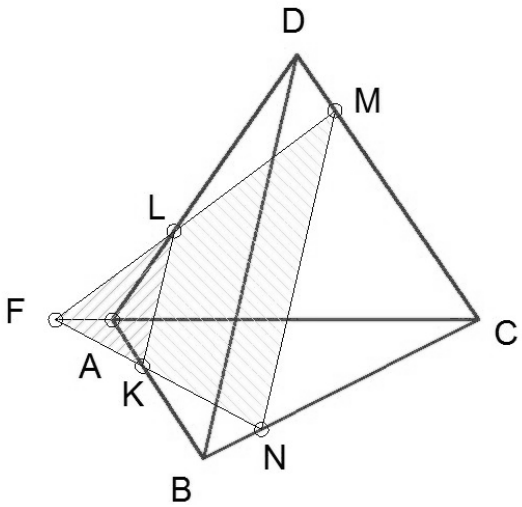

Fig. 2. Section of a regular tetrahedron with the given plane

considering the opportunities for integrative approach derived from Mathematics, Informatics and Information Technologies. This opens up a wide field for future experimental observations and scientific summarizations based on an integrative approach accompanied by effective teaching and acquiring mastery in mathematics.

5 Conclusion

The reviewed examples give us ground to believe that, to achieve effective learning, it is appropriate to combine the knowledge and skills in Mathematics, Informatics and Information Technology to solve problems using relevant strategies, such as formulation of isomorphic problems, breaking into small steps, self-control and visualization. Having command of different strategies enables the students to organize their thoughts and apply their knowledge [5]. Teachers are responsible both for the knowledge of their students and their metacognition at the "mastery of strategies" level at least, without which even the smartest and hard-working students face the risk of not coping with the assigned tasks.

Acknowledgements. The presented research has received funding from the National Project for Supporting Doctoral Students and Young Scientists in Mathematics and Computer Science at Sofia University "St. Kliment Ohridski" and is partially supported by the science research project at Sofia University "St. Kliment Ohridski", Contract No. 126/08.05.2014.

References

1. Sterbberg R., Williams W.: Educational Psychology. Iztok-Zapad Publishing House, Sofia (2014) (Translation to Bulgarian)
2. Aleksandrov, I.I.: Collection of Geometric Construction Problems (with Solutions), 18th edn. Uchpedgiz, Moscow (1950) (in Russian)
3. Pólya G.: How to Solve It. Narodna Prosveta Publishing House, Sofia (1972) (Translation to Bulgarian)
4. Ryzhik, V.: A problem as a problem. Quantum 4, 38–39 (1981) (in Russian).
5. Tonov I.: Heuristics – Science, Art, Craft (in Bulgarian). http://www.fmi.uni-sofia.bg/habil_disert_trudove/habilitacionni_trudove_papka/resolveuid/152587535096940e9def51e6daac8f1b

‘The Lost Energy’ and How to Collect It with weSPOT Tools

Dafinka Miteva, Elitsa Peltekova, and Eliza Stefanova

Faculty of Mathematics and Informatics, Sofia University “St. Kliment Ohridski”,
5, James Bourchier Blvd., 1164 Sofia, Bulgaria
{dafinca,epeltekova,eliza}@fmi.uni-sofia.bg

Abstract. This paper presents an inquiry-based learning research project called ‘The Lost Energy’ created and conducted in accordance with the educational model which uses the technology environment of the Working Environment with Social and Personal Open Tools (weSPOT) project. Nowadays, living in times of information ‘boom’ and unstoppable technology development, traditional teaching methods are not sufficient to attract students’ attention, to satisfy their thirst for knowledge and discoveries. Always with a mobile device in hand, young people expend their energy surfing Internet, ‘networking’ in social websites, playing games, communicating. ‘The Lost Energy’ is an example of this endless energy expenditure being captured and directed to creative scientific study, answering the question whether it is possible to catch the mechanical energy generated by people in public places and transform it into a source of energy. The aim of this study is to test the weSPOT tools, to find their weaknesses and to suggest improvements.

Keywords: weSPOT, inquiry-based learning, mobile teaching, mobile learning, educational technology

1 Introduction

The pervasiveness of mobile devices in the daily life of the 21st century generation is growing steadily, so the introduction of the mobile devices and their use for educational purposes in the schools is expected and unavoidable. weSPOT is an ongoing international project (`http://portal.ou.nl/en/web/wespot`) dedicated to introducing to the schools the working environment with social and personal open tools, supporting an/the inquiry-based learning approach, including research by means of mobile devices. In order to be really useful, it is very important for the environment and the toolkit to take into account the teacher’s and the students’ point of view. the research described aims to find answers to the following questions: (1) Are the weSPOT tools appropriate to support inquiry-based learning, to catch ‘the lost energy of students’ by means of new technologies;(2) Are these tools intuitive enough and easy to use by the teacher (in case not to waste his/her energy)?

‘The lost energy’ inquiry-based experiment could be seen as an example of: (1) how a teacher could use the weSPOT environment and tools to bring students to make up for the lost energy (thinking about places where people-generated mechanical energy is concentrated) and to seek solutions to the problem of how to collect it; (2) how the students' expenditure of energy on their mobile devices, mainly in social networks and games, can be collected and can be used for educational purposes.

To answer the research questions and to discuss the mentioned above, the paper is structured as follows. The next section briefly presents the weSPOT model with its phases and the weSPOT tools. The 'Lost energy' scenario for inquiry-based education and its context are described in the third section. The teacher challenges and tools limitations that were observed are presented in section four and five respectively.

2 weSPOT Model and Tools

Based on extensive pedagogical research, the weSPOT project develops an inquiry-based educational model as well as a set of technology tools for guiding students in their investigations [1]. 'The lost energy' approach was tested in the context of the model and the tools.

2.1 The weSPOT Educational Model

The weSPOT model [2] considers inquiry-based research as a process that goes through six key stages during its life-cycle and describes them as phases (Fig. 1) with recommended tasks for teachers and students:

- Question/Hypothesis – for definition of the scientific assertion
- Operationalisation – for planning the research methods
- Data Collection – for gathering scientific facts and evidence

Fig. 1. The weSPOT Model

– Data Analysis – for surveying (observing) collected data
– Discussion/Interpretation – for debating the research results
– Communication – for sharing final results with all interested parties

2.2 The weSPOT Inquiry Tools

The inquiry workflow engine is the core component of the weSPOT learning environment. It is a frame in which the weSPOT model is realized. The environment is built upon the leading open-source social networking platform Elgg distinguished as the best one in 2008 (`http://www.elgg.org/about.php`). It represents an educational environment requiring user authentication and enabling teachers and students to share information according to predefined authorization. The front-end weSPOT application allows users to create accounts and make friends, to define inquiries with a set of available components and invite collaborators to become its members (to form a community), to collect mobile data, analyze and discuss them, as well as exchange the results.

By using web plug-ins the weSPOT environment integrates the following popular collaboration tools as its components:

The ARLearn component, which is used for mobile collection of data for inquiry. Its primary design is as a toolkit for mobile learning games. Students can learn and complete real-life tasks while being on a field trip [3] with a smart phone or a tablet and visiting special places shown on the map. They can answer different types of questions, take and upload pictures, video and audio objects.

The weSPOT inquiry offers a component for collecting mobile data integrated with the mobile learning tool ARLearn (Fig. 2).

Fig. 2. Collect Data by ARLearn Mobile Application

The MindMeister's mind mapping component is a visual thinking tool that helps structuring information and generate new ideas. A mind map is a graphical way to represent ideas and concepts [4]. Teachers and students can create mind map diagrams for structuring key concepts, outlining the steps of tasks completion and summarizing the whole information. Maps could be managed, shared to be accessible and editable by inquiry users simultaneously anytime. In the frame of the weSPOT inquiry engine this component has an important role for generating hypotheses, proposing methods for research, etc.

The FCA (Formal concept analysis) component is based on a web-based graphic tool which allows the teacher to load and edit an existing one or to create a new knowledge domain with a set of objects and their relatively common attributes. Based on principles of the Formal Concept Analysis these concepts are represented as a mapping of attributes (e.g. "even") onto objects (e.g. the infinite set of even numbers). The Tool consists of two main parts, the Editor View and the Lattice View [5]. In an additional window an interactive taxonomy diagram or a full lattice map of the domain can be maximized. In the frame of the weSPOT inquiry engine it could be used by teacher to present students' relations between concepts in the domain, to support them with corresponding materials, to help learners to understand better through visualization.

Learning Analytics and Reflection & Awareness Environment (LARA) is a graphic analysis component visualizing the students' and teachers' activities grouped by the weSPOT educational model phases. The detailed Learning Analysis dashboard shows all users' data publications as five different-sized dots, according to the rating stars of each element and all the event details for a selected item – title, sender, type of data and content. Finally, the conclusions, based on the research done, are drawn.

The weSPOT environment in general and the components listed above are used by the authors to describe 'The Lost Energy' inquiry-based learning scenario and on this base to make a conclusion about the teacher challenges in the process of use of weSPOT tools, their limitations at the moment of the experiment, as well as to recommend improvements, so that it will finally be possible for teachers to apply it in order 'to use the student energy'.

3 The Lost Energy – an inquiry-based learning scenario in weSPOT

The main goal of 'The Lost Energy' scenario is to make a preliminary examination of the inquiry itself before its test within a secondary education context, eight grade students (aged 14–16 years) in Physics classes in the National High School of Mathematics "Acad. Lyubomir Chakalov", Sofia, Bulgaria. 'The lost energy' idea for inquiry-based learning is suggested by the Physics teacher from the school where a real test is planned after the examination of the inquiry. The scenario presents the teachers' viewpoint and the students' viewpoint in terms of using the weSPOT framework and the weSPOT integrated tools.

The inquiry-based learning scenario aims to implement a personal provocative approach [6] for students and to ask them for the answer to the question whether 'it is possible to catch the mechanical energy generated by people in public places and transform it into a source of electrical energy'. The present scenario presumes that during the process of searching for the answer students would use part of the energy they expend on their mobile devices every day.

On the basis of the teachers' idea, the authors performed activities which should be done by a teacher. As a beginning an inquiry (Fig. 3) was created in the weSPOT environment. It was named 'The lost energy' ('Изгубената енергия' is the original name in Bulgarian). The six phases of the inquiry which describes the scenario are presented below.

Fig. 3. weSPOT inquiry

In the first phase “Create the questions and hypothesis” ‘the lost energy’ idea for the generation of energy from the mechanical movements and where it could be found in Sofia was described by authors from teacher’s point of view.

The second step (“Plan the method”) was the creation of the mind map, which is exactly MindMeister’s mind mapping component. Here both the teacher and students defined on the map the locations of busy places in Sofia where the examination took place: the five shopping malls in Sofia and the points in the malls where the people traffic is biggest – escalator, entrances/exits, shops offering sales, playgrounds, etc were selected.

The third phase (“Collect the data”) uses the mobile learning tool ARLearn – a component for collecting mobile data which is integrated in the inquiry. In this phase the teacher presented the five examined malls (addresses, working hours, and websites), and the students (equipped with tablets and mobile phones with Android OS) went around the five shopping malls and recorded a video of the busiest places in it.

The next two phases “Analyze the data” and “Discuss the findings”, according to the scenario, are expected to be conducted mainly by the students and because of that no additional blocks are added at that stage. The Sixth phase “Communicate the results” is the stage where both students and teacher draw the conclusions from the examination.

In the process of the description and completion of the scenario by means of the weSPOT some challenges faced the teachers and system limitations have been identified. They are presented in the next two sections below.

4 Teacher’s Challenges

The experience from the development of the scenario described above shows that it could be expected that the teachers would encounter some challenges.

As a person who leads students during the inquiry learning process, the teacher has to formulate a set of appropriate guiding questions to direct students in proving or rejecting a conjecture. Traditional teaching often makes teachers inclined to ask questions, the answers to which are already known or there is not a requirement to prove the validity of the answer. Because of that, the first challenge which teachers could face applying the inquiry-based weSPOT model is how to formulate the ‘right’ questions to provoke students to use their energy in looking and proving the answer.

Another task that the teacher has to deal with is the design of mobile data collection tasks. It could become a real challenge for the teachers, because (s)he expects the design and the front-end interface of the application to be clear and user-friendly, easy for handling from inexperienced users. Many tools, including ARLearn, are intuitive for computer scientists, but not for others.

The current weSPOT environment functionality put a real challenge for the tutor to organize a team working on the same inquiry. At the time of the current study the environment did not allow: 1) more than one teacher to lead an inquiry; 2) in the frame of one inquiry to form groups working on one question, but different hypothesis, and so on. To solve these problems, the teacher could create several inquiries based on the same question, but this makes his work hard foreseeable, hard to plan.

The *ball ultimately remains in the hands of the teacher* also with relation to the decision which component at which phase of model to be visible. From the teachers' point of view, it should be free choice, but when an inquiry is defined with some limitations, they should be applied by default and only when the teacher decides to change them. The version of the environment used makes it unpredictable for the teacher which components will be visible for the wider public, and which will be private.

5 Tools Limitation

During the experimental inquiry system several limitations were identified which can be roughly divided into two categories: (1) External technology tools integration into the framework. (2) Environmental limitations due to already developed functionalities.

5.1 Limitation in Integration of External Tools

This category examines the problems caused by incompletely developed data exchange functionality. An Example of such a problem is the communication between the mobile data toolkit ARLearn and the MindMeister's map object sharing.

Mobile tasks definition is a one-directional tool: from weSPOT to ARLearn. When creating a new weSPOT inquiry, the system automatically generates an ARLearn game and 'run' (game-like assignment) with the same name like the inquiry. Adding a data collection task in the weSPOT component creates a corresponding object into a mobile game with the same data collection type for the answer selected: pictures, video, audio, text, numeric data. ARLearn properties like visualizing tasks on the map, action dependencies for appearing and disappearing of messages, roles and sharing the play with other people can be improved. The main problem is that each correction made from the weSPOT inquiry deletes any changes set in ARLearn as well as previously created games in ARLearn cannot be 'imported' in weSPOT, any improvements made outside are not accepted by the weSPOT environment. Hence the suggestion is to work on/develop better integration between the weSPOT inquiry and the ARLearn tool, adding a feature for bidirectional data transfer (communication).

Another integration limitation is the mind map's sharing feature. All inquiry members can create their own maps and make them visible for other weSPOT members but the editing option is only allowed for the map's creator. The proper object sharing is a key factor for good teamwork and students' brainstorming. The suggestion is that these characteristics be elaborated.

5.2 Limitations in the Tools Functionalities

This type of constraint is caused mainly by the fact that the components and their functionality are still in the process of development which prevents their usage as intended.

One such limitation is the invitation to become a member of inquiry. It is visible only in the weSPOT system after a successful login, there is no e-mail notification. Sometimes receiving frequent notifications is considered as 'annoying' spam and browsers are set to automatically filter messages directly to a special folder named "spam". But e-mails remain the best way to inform someone about an event, to draw someone's attention to accept an inquiry invitation.

Another limitation at the time of the experiment was that the ARLearn tool was developed only for the Android operating system (OS); the iOS version is under development. These two OS are the most used all over the world [7].

During the experiment the weSPOT inquiry was developed so that it was managed by one supervisor. In such case there is a problem in teaching interdisciplinary subjects in which students need to be guided by more than one teacher on various topics. In this case every teacher needs rights to manage the relevant tools. Currently the weSPOT inquiry provides only two user roles: one inquiry owner and inquiry members. The owner is entitled to change the inquiry properties, to add new weSPOT components to the inquiry and to configure them. The inquiry members' access is limited only to adding elements to an already created component. Based on 'The lost energy', it is good to have more than one instructor in each inquiry conducted by more than one teacher. Even more, rights for editing components to be delegated to students, giving them the chance for more active participation in the whole educational process.

From the perspective of the scenario given, there are also other types of limitations in the functionality of the data collection component of the weSPOT environment. Currently the ARLearn tool allows the teacher to define action dependencies - when to show or to hide a task according to the student's activities, and how long a task will be available for completion; the instructor can even choose the option to show a countdown timer. But the teacher cannot ensure that students will collect the required data in a particular time interval. The problem has two parts. First, the students can upload data but it is possible that the content is irrelevant. Second, in our case, if students want to compare the rush-hour traffic in different Sofia MALLs it is reasonable that will they make video clips at the same predefined time intervals to achieve more authoritative results. Another case when time might be important is when a few groups of students work on the same tasks competitively. This requires that the active time period for mobile tasks be more flexible.

Image Slider limitations could also be listed briefly. This component allows the teacher to create a slideshow in two formats FlexSlider and s3Slider with up to five images published in the Internet with an option for user-defined links to them. This unit is not designed to accept user uploaded photos. The only option for source property is a URL. Image Slider would be a more attractive tool if students could upload their own photos made during a field trip.

In addition to the limitations, there is some unexpected 'behavior' of the weSPOT environment. Such behavior could be observed when new components are added to the inquiry. For example, it is not expected when defining an inquiry with closed membership type in which users must be explicitly invited to participate, the newly added components to it to be suggested by the system to be publicly accessed as default option. It makes more sense if newly created objects inherit the admission property of the inquiry they belong to, and closed inquiry components are normally available for the members of the inquiry only. Switching the inquiry type between predefined types: 'public', 'logged in users' and 'inquiry' is intuitively expected to be followed by changing the default access method for all inquiry tools.

It is difficult to decide in which phase of the weSPOT model the LARA component should be added. It is also expected that it would show activities filtering them by their usefulness.

The aforementioned findings and reported problems are some of the first contributions to the improvement of the working environment created in the frame of weSPOT and enabling students to catch the 'lost energy'.

6 Conclusion

'The lost energy' is a pressing issue of our time – literally and figuratively. both mechanical energy and 'students' energy regularly gets "lost". The idea is collected students' energy to help in finding places and methods for collecting mechanical energy generated every day, and everywhere on busy places, places concentrated with many people.

Thanks to tests and reports from the presented inquiry-based sample 'Lost energy' were found some limitations of weSPOT tools. They were reported to the weSPOT developers and some of them are already fixed. The framework is still underdevelopment and needs to be improved, but discussed research shows that with common efforts of the researchers, developers and educators the 'Lost energy' of students could be caught by means of inquiry-based learning via weSPOT tools.

Acknowledgements. The presented research has received funding from the National Project for Supporting Doctoral Students and Young Scientists in Mathematics and Computer Science at Sofia University "St. Kliment Ohridski" and co-financed by the European Social Fund through the Human Resource Development Operational Programme under contract BG051PO001-3.3.06-0052 (2012/2014) and partially supported by funds of the European Community's Seventh Framework Programme (2007–2013) under grant agreement No. 318499 – project weSPOT.

References

1. Protopsaltis, A., Hetzner, S., Held, P., Seitlinger, P., Bedek, M., Kopeinik, S., Rusman, E., Firssova, O., Specht, M., Haimala, F., Kikis-Papadaki, K., Okada,

A., Mikroyannidis, A., Scott, P.: Delivarable D2.3.1: Pedagogical and Diagnostic Framework. Retrieved June 10th, 2014, http://portal.ou.nl/documents/7822028/3b2c3110-98bf-4686-b2bd-5756fcab93f1
2. Mikroyannidis, A.: Delivarable D3.1: Inquiry workflow engine. Retrieved June 10th, 2014, http://portal.ou.nl/documents/7822028/fc221e4e-55ab-4ec2-ba08-477232947700
3. Littlejohn, A., Pegler, C.: Preparing for Blended E-learning. Great Britain: Routledge (2009). ISBN 978-0-415-40361-0
4. What is Mind Mapping? (and How to Get Started Immediately). Retrieved July 23rd, 2014, http://litemind.com/what-is-mind-mapping/
5. Santos, J., Charleer, S., Klerkx, J., Kopeinik, S., Georgiev, A.: D5.2 Inquiry Performance Analytics. Retrieved July 23rd, 2014, http://portal.ou.nl/documents/7822028/e0e256fc-3fd4-4087-a7a9-f602b9be7f2e
6. Mikroyannidis, A., Okada, A., Scott, P., Rusman, E., Specht, M., Stefanov, K., Boytchev, P., Protopsaltis, A., Held, P., Hetzner, S., Kikis-Papadakis, K., Chaimala, F.: weSPOT: A Personal and Social Approach to Inquiry-Based Learning. Journal of Universal Computer Science 19(14), 2093–2111 (2012). Retrieved June 4th, 2014 http://www.jucs.org/jucs_19_14/wespot_a_personal_and/jucs_19_14_2093_2111_mikroyannidis.pdf
7. Gartner Says Annual Smartphone Sales Surpassed Sales of Feature Phones for the First Time in 2013. Retrieved June 12th 2014, http://www.gartner.com/newsroom/id/2665715

How to Include IT in Education of Children with SNE

Temenuzhka Zafirova-Malcheva

Faculty of Mathematics and Informatics, Sofia University "St. Kliment Ohridski", 5, James Bourchier Blvd., 1164 Sofia, Bulgaria
tzafirova@fmi.uni-sofia.bg

Abstract. The paper discusses the problems in teacher training posed by integration of children with special educational needs (SEN) in mainstream school. It presents the course "IT in education of children with SEN" for students from program "Mathematics and Informatics" in Faculty of Mathematics and Informatics of Sofia University "St. Kliment Ohridski", who are preparing to be teachers in mathematics, informatics and information technologies in secondary school. The course provides a basic knowledge of working with children with SEN and developing appropriate educational materials and software for these children. The paper presents the process of development of student's projects and the results of its work.

Keywords: IT in education of children with SEN, educational software development, special needs education, teacher training

1 Introduction

There are many benefits of using computer technologies in the education of children with special educational needs (SEN). They make easier access to educational materials and to education as whole. It is easier to communicate through them. Different forms of education enable students with SEN to learn regardless of place and time. Various types of assistive technologies can help them to overcome a number of problems arising from different disabilities and make students with SEN equal participants in the educational process. The use of technologies gives a number of advantages especially in the process of "Inclusion"[1], which takes place in Bulgaria. Research of British Educational Communications and Technology Agency (BECTA) makes review of many scientific researches in this area which shows that the "Technology can help these children overcome many of their communication difficulties, so they can be included in lessons, and access a wider curriculum. For example, access devices can help learners with physical difficulties to use a computer, and enable them to access the same curriculum as their peers. Software designed to meet a student's particular needs can also help to motivate him or her. For some students technology may be the only way

[1] Process of integration of children with SEN in the common educational environment

to ensure they can make their thoughts and needs known. For them, access to appropriate ICT-based solutions provides perhaps the only chance of participating in society and realizing their full potential." and conclude that "ICT usage in schools to support students with SEN can enable learners to communicate, participate in lessons, and learn more effectively" [1] Successful implementation of inclusive education requires a number of conditions as accessible environment, well-trained teachers and appropriate educational materials. Most of teachers in Bulgarian schools are not adequately prepared to work with children with SEN. They do not know their needs and problems and do not know how to use technologies effectively in education of children with SEN. Other problem in Bulgaria is lack of enough appropriate educational materials and software for children with SEN and the inability of teachers to create such materials. This raises the need for proper training of teachers to work with children with SEN, to use IT in children's education and to create appropriate educational materials. Of course these problems cannot be solved at once. A small piece of puzzle of the decision which will be present in this paper is the course "Information technologies in education of children with special educational needs" ("IT in education of children with SEN"). This course is designed for students from program "Mathematics and Informatics" in Faculty of Mathematics and Informatics of Sofia University "St. Kliment Ohridski", who are preparing to be teachers in mathematics, informatics and information technologies in secondary school.

Mentioning already several times terms *educational software* and *children with SEN* this is the place to clarify what we mean using this terms in the paper.

According to Bulgarian legislation "children and students with special educational needs are children and students with various disabilities – sensory, physical, mental (mental retardation), multiple disabilities; speech or language impairments; learning disabilities" [2].

In his book "Choosing and Using Educational Software: A Teachers' Guide", David Squires and Ann McDougall define educational software as *"any software that is used in an educational context, whether or not it was specifically designed for educational use"* [3]. Considered in the context of *selection* of software for educational purposes this definition fully meets the teacher's needs. However, when talking about *design and development* of educational software, this definition does not fully meet the context in which is used, so in this paper by educational software we will understand "*software with specific educational content, in which development is set specific learning objectives*". This definition of the term educational software reflects the pedagogical principles and instructional design methods applied in development of this type of software.

2 IT in Education of Children with SEN – the Course

The course "IT in education of children with SEN" is interdisciplinary. On the one hand includes basic knowledge in the field of special needs. Besides mastering a set of medical terminology it requires building basic knowledge on the causes of the different types of disabilities and the problems they pose. In addition, the work with these children requires knowledge and application of different approaches. On the other hand, it is focused on use of information technologies in education of children with SEN therefore it includes knowledge and skills related to the design and development of appropriate educational software and materials for these children.

2.1 Objectives

From a pedagogical perspective the main objectives of the course is students to learn who the children with SEN are. To know the main problems which arise in the various disability groups especially in interactions with the computer and to identify and implement various solutions to overcome these problems. On the other hand students have to acquire practical knowledge and skills to prepare specific educational materials and software suitable for these children.

2.2 Structure

Based to these objectives the course was structured in two main parts, the first involve the acquisition of basic knowledge for children with SEN, including:

- national policy for the application of ICT in the education of children with SEN – study of legal documents related to the rights of people with special needs and use of ICT in education in Bulgaria;
- children with SEN – types of disabilities and learning problems caused by them;
- main stages in computer education of children with SEN – what, when and by what means is studied;
- computer as a goal of education for children with SEN – problems and solutions in work with computer;
- technologies to support children with SEN – application of various types of assistive technologies to access and use the computer system from users with SEN.

The second part was related to the practical application of this knowledge in the development of interactive educational materials or software designed to solve a specific problem for a specific SEN group of children. It includes:

- educational software for children with SEN – types of educational software; stages and criteria for development;
- Power Point and Visual Basic for Applications as tools for development of interactive educational software and materials.

2.3 Context

The course is conducted over the past three years. Its duration is one semester within 30 hours (two hours per week). The course was attended by 38 students from program “Mathematics and Informatics” – 23 women and 15 men.

All learning materials are provided to students through e-learning system Moodle used in Sofia University. Every student access the system through their own account. The course structured in Moodle gives students access to course materials – lectures, presentations, examples, exercises, tests and assignments. It also provides an opportunity for communication between students and teacher outside the mandatory classes.

The first part of the course is related to the acquisition of basic knowledge in the area of SEN and is structured mainly in the form of lectures and discussions.

The second part presents the main steps for design of educational software for children with SEN and includes final task for development and presentation of project. For this task students are divided into groups of two or three. They have to accomplish a project which includes design and development of an educational software application for children with SEN.

3 The Project

The final project that students have to prepare has two parts:

- in the first part students must identify the topic of their project using the knowledge acquired in the first part of the course. For this purpose they must determine: the users group for which the software is designed, the problems that it will solve and its learning objectives;
- in the second part they have to design and develop a prototype of the software. As the course is quite short so it does not include specific software tools (programming language or environment) that students must use when develop their software. They have the freedom to choose the tools that will use in its work. However, for those who are not sure what tool to use, the course offers several practical classes to work with Power Point and Visual Basic for Applications, through which they can create interactive presentations that are sufficient to create a well-functioning software prototype.

3.1 Software Topic, Users and Objectives

The first part of the project requires only a textual description. Therefore, it is realized in a course in Moodle with test that students can fill out many times (Fig. 1). The teacher can give feedback and if there is need for improvements students can complete the test again (Fig. 2). They cannot continue to the next part of the project, unless the teacher has not approved the first part of software description.

The test contains five questions from essay type in which students must describe:

- project title;
- team members;
- user group for which the project is designed (the type of disability and age of children);

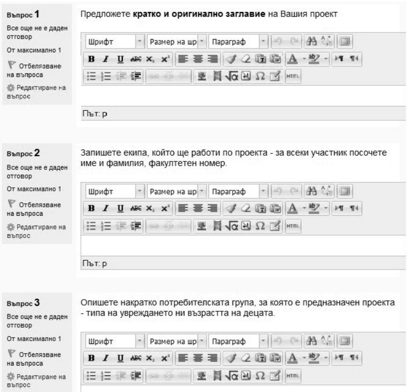

Fig. 1. General structure of the test

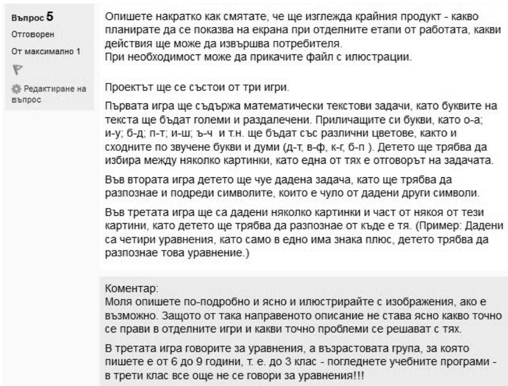

Fig. 2. Part of the test with feedback

– problems of the user group, which project must solve and its objectives;
– how the final product will look like – what will be displayed on the screens in the different stages of the work, what actions user will be able to do;
– if necessary, students can attach additional file.

In "Guidelines for the Design of Educational Software"[2] developed by American Dental Association the first step in the design of educational software is to define the pedagogical issues [4]. This is a very important step in the development of educational software especially when it is designed for children with SEN. It defines the first steps in the software design and is related to determination of topic, user's group and objectives of the software. The easiest way to do this is to answer to next three questions about the software:

– Who is it for?
– What problems will solve?
– What are its learning objectives?

All this of course applies to educational software in general, thing that distinguishes educational software for children with SEN, are the specifics of the user group.

When design educational software for children with SEN answer to the first question is much more complex and extensive. Before answering it we need to know the answer to another question: Who are children with SEN? This term covers a very large group of children with various types of disability, needs and problems that they cause. It is not so homogeneous user group and its problems and needs are diverse and different. They can vary even to children with the same type of disability, but with different degrees or areas of impairment. This

[2] The "Guidelines for the Design of Educational Software" is developed by American Dental Association and is approved in 2002 as an American National Standard by the American National Standards Institute.

is one of the challenges in the development of educational software for children with SEN – it is impossible to develop universal software that helps children with any type of disability to learn to read for example. Because reasons for problems with reading can be very different and depend on the type of disability (for example visually impaired children and children with dyslexia[3]). Therefore, in development software for children with SEN first thing that must be done is to determine type and degree of disability of children to whom software will be designed and to recognize the specific problems which it arises.

The second main thing that should be determined after defined user group (type of disability) is to define which problems, caused by disability, will be in the focus of the software. Their determination is the starting point for defining the learning objectives of the software, as they derive from the problem that should be solved. This is the last step of the pedagogical part of design – to determine learning objectives of the software, based on focus problems.

3.2 Software Design and Development

Software topic, the problem that will solve and its learning objectives define the general framework of the software. Here begins an essential part of software design and development. Students must define:

- what will be the type of software that will be developed;
- in what situation will put the users (software scenario);
- what will be the learning content of the software;
- what problems users need to solve;
- how it will interact with the system – challenges, fun, competitions;
- what will be the level of difficulty – will there be different levels;

Software has to contain two parts – first one educational and the second to check the acquired knowledge. In the educational part depending on the purpose of the software and the problems that it solves, students must represent appropriately the learning content. In the second part they have to choose proper learning activities in the form of the different games, questions, etc., by which to check whether users have understood and learned presented knowledge.

To be effective, to work properly and to perform its goals, the different components the software must comply with certain criteria. First the content must meet the intellectual potential of children, to be appropriate to their age and type of disability. Of course it must be clear and precise, actual, well-structured and in a reasonable volume. Very important for children with SEN is the information that is presented to be appropriately illustrated. Tasks must have different levels of difficulty and to build gradually knowledge of children, in order to avoid

[3] Problems with visually impaired children are associated with limited ability to see the printed text, while children with dyslexia have problem to interpret and process already seen printed text. In the first case the problem is sensory, in the second – not.

failures and disappointments. To avoid demotivation activities and tasks should be various.

These are only a small part of the requirements that students must comply with the development of their projects. The course discusses in detail the criteria which educational software for children with SEN must meet. Very useful in this aspect are individual consultations that students obtain in the process of working on their projects.

4 Results

As a result of work in the course, during the three years in which it conducts, students created 21 projects of educational software for children with different types of disability. Most of them have a high level and cover the criteria for qualitative and effective software. Figures 3–6 present some of these projects.

Fig. 3. “The opposites” is an educational game that helps children to understand the concept of opposite by appropriate illustrations

5 Conclusion

“Inclusion” of children with SEN in education is part of their socialization. This is a long and difficult journey, and we are still in its beginning. But giving adequate training of future teachers is a big step on it. Knowledge and skills obtained during the course “IT in education of children with SEN” give the students practical experience that will be useful in the occupation for which they are preparing. During the course students have the opportunity to visit a specialized center for children with cerebral palsy, where they really can face the children and feel their problems, and to consult with experts. One of the lectures is presented together with specialist which working with hearing impaired children. This further expands the horizons of students and enriches their knowledge.

Fig. 4. "Prepositions" helps children to learn prepositions for a place. It is especially helpful for children with problems in orientation. The software has two parts educational – in which prepositions are explained with illustrative examples – and a game. Design uses very suitable illustrations and has excellent feedback

Fig. 5. "Animals" is an educational game that introduces different animal's species – domestic, wild, aquatic, birds and insects. The game includes more than 50 animals. Each animal is represented by an image, text that is duplicated with audio and video, with sound that the animal makes. The knowledge is exercised in 5 different games. The software is very rich, appropriately illustrated and with excellent feedback

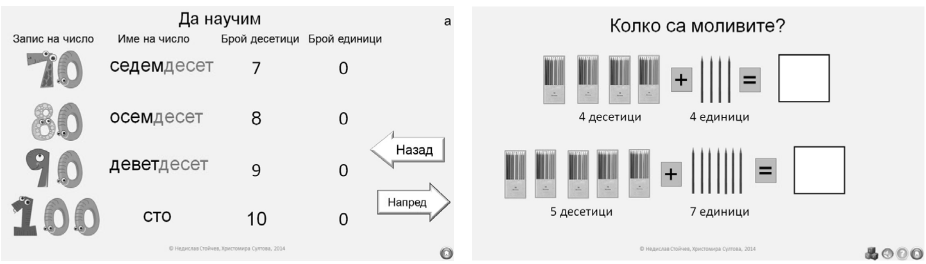

Fig. 6. "Numbers from 20 to 100" is an educational software designed for children with dyslexia, which aims are to help children to learn the names of numbers from 20 to 100 and recording them with words, to compare two numbers, to arranged numbers by size

Acknowledgements. The study was carried out using the European Social Fund through the Human Resource Development Operational Programme under contract BG051PO001-3.3.06-0052 (2012/2014).

References

1. BECTA: What the research says about ICT supporting special educational needs (SEN) and inclusion. BECTA ICT Research (2003). http://www.education.gov.uk/publications/eOrderingDownload/15009MIG2791.pdf (21 June 2014)
2. Bulgarian Ministry of Education and Science: Ordinance No 1 dated 23 January 2009 for the education of children with special educational needs and/or chronic diseases (in Bulgarian). http://www.lex.bg/bg/laws/ldoc/2135617694 (20 June 2014)
3. Squires, D., McDougall A.: Choosing and Using Educational Software: A Teachers' Guide. Routledge (1994) ISBN-10 0750703067. http://www.google.bg/books?hl=en{\&}lr={\&}id=HxSHVkB9epMC{\&}oi=fnd{\&}pg=PR8{\&}dq=educational+software{\&}ots=vXnEsY4r6h{\&}sig=RaG9StaW218PUqslFLZTR4dcTOI{\&}redir{_}esc=y{\#}v=onepage{\&}q=educational{\%}20software{\&}f=false (20 June 2014)
4. American Dental Association: Guidelines for the Design of Educational Software. Created: January 12, 1999. Revised: 08/09/2012. http://www.dental.pitt.edu/informatics/edswstd/ (20 June 2014)

Author Index