The Coastal Business Journal

Volume 15 | Number 1 Article 1

May 2016

Estimating Modified Duration and Convexity for Income Properties

Frank Handforth

Eugene M. Bland
Texas A&M University-Corpus Christi

Neil F. Riley
Francis Marion University

The Coastal Business Journal

ISSN: 2163-9280 Spring 2016
Volume 15, Number 1

ESTIMATING MODIFIED DURATION AND CONVEXITY FOR INCOME PROPERTIES

Frank Handforth, CFA, Financial Consultant
Eugene M Bland, CFA, CTP, Texas A&M University – Corpus Christi
Neil F. Riley, Francis Marion University

ABSTRACT

Some institutions hold income producing properties directly as part of an investment portfolio. These properties are subject to interest rate risk. The two widely used measures of interest rate risk are modified duration and convexity. These measures can be difficult to calculate directly for income properties when net operating income and cash flows are not constant or when they are functions of the underlying yield. A numerical technique using sensitivity analysis is developed to provide estimates of modified duration and convexity. These estimates can then be used, along with corresponding values for debt, to determine the modified duration and convexity of the equity of the property. This can be extended to a portfolio of properties. The measurement of the interest rate risk of the equity of a portfolio of properties, using conventional interest rate risk measures, has a number of useful applications.

INTRODUCTION

Some institutions, such as large pension funds, hold income producing properties directly or as co-investments as part of an investment portfolio. For a funded pension plan, these investment assets are used to meet the pension liability payments. Funded pension plans can have considerable exposure to interest rate risk. Fund managers attempt to manage this risk by deploying strategies to match the risk of the assets and liabilities and thereby immunize the plans (Redington 1952). In order to deploy an immunization strategy, some measure of the interest rate risk is required. Investment in income producing properties competes in the capital markets with investments in other assets. As the general level of interest rates rises and falls, the prices of income properties adjust so that their returns remain competitive with the returns from investments in other assets. While considerable research effort has been devoted to studying the interest rate risk of fixed income securities such as bonds, less work done has been done in this regard for income properties. Some difficulty lies in the calculation and specification of the risk measures. This paper proposes an approximation technique which could be used to estimate two measures of interest rate risk for income properties. These measures could be used by pension fund managers and other portfolio managers to better manage the interest rate risk of their plans and portfolios.

The Coastal Business Journal

ISSN: 2163-9280 Spring 2016
Volume 15, Number 1

DURATION AND CONVEXITY

Macaulay (1938) developed the concept of interest rate risk being linked to the duration of time that a security is outstanding, where the remaining length of time to each future cash flow is weighted by the fractional contribution of that discounted future cash flow to the total present value of the security. This measure of interest rate risk is known as duration or Macaulay duration and it is measured in years. It is a linear measure of risk.

Samuelson (1945) used a different approach to measure interest rate risk by calculating the semi-elasticity of the price of a security with respect to its yield. This semi-elasticity is generally negative, and it is multiplied by -1 and expressed as a positive number. The two different approaches result in similar, but not identical, measures of interest rate risk. To distinguish between them, this second approach is known as modified duration. It is also a linear measure of interest rate risk. See Appendix A for the relationship between these two measures.

$$D^* = -\frac{\partial V}{\partial y} \cdot \frac{1}{V} \tag{1}$$

where
D^* modified duration
V present value of a security
y market yield (discount rate used the calculate the present value of the security)

Figure 1 illustrates the present value of a security as a function of market yield. This curve is not linear. Both the value of the security and the slope of this curve change with yield so that modified duration is not constant. In Figure 1, the value of the security as a function of yield has been estimated using modified duration, where the modified duration has been calculated at the point of tangency. Modified duration is linear and this estimate is a straight line. However, the value function is not linear, and an improved estimate for larger changes in yield can be obtained by using the second term of a Taylor series expansion.

$$\frac{\partial V}{V} = \frac{\partial V}{\partial y} \cdot \frac{1}{V}(\partial y) + \frac{1}{2}\frac{\partial^2 V}{\partial y^2} \cdot \frac{1}{V}(\partial y)^2 + remainder \tag{2}$$

The second term of the expansion leads to the concept of a convexity measure C and it approximately measures the change in value due to convexity for a change in the underlying yield.

The Coastal Business Journal

ISSN: 2163-9280 Spring 2016
Volume 15, Number 1

$$C = \frac{\partial^2 V}{\partial y^2} \frac{1}{V} \qquad (3)$$

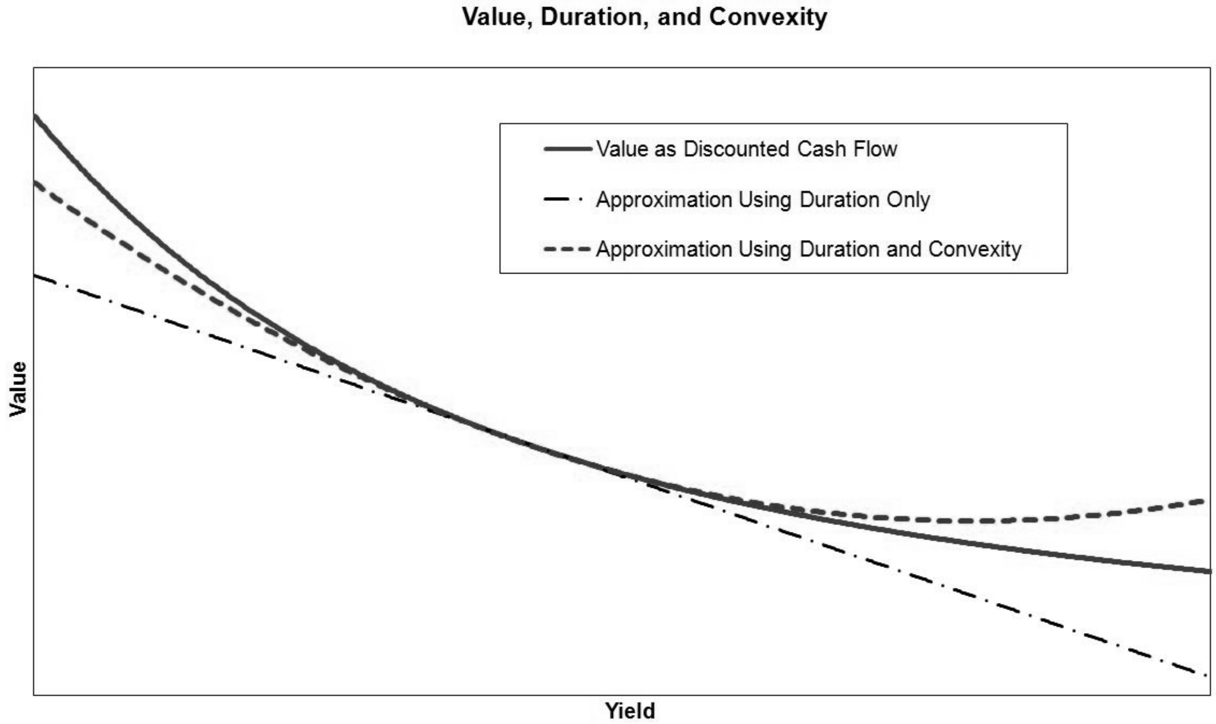
where
C convexity measure

Since both the value of the security and rate of change of the slope of this curve change with yield, the convexity measure is not constant. In Figure 1, the value of the security as a function of yield has also been estimated using both modified duration and convexity, where these have been calculated at the point of tangency. This third curve is a parabola and provides an improved estimate over the estimate using modified duration alone. Figure 1 illustrates that a higher value for modified duration would result in a steeper negative slope for the straight line indicating more variability in value for changes in yield, and a lower value would result in less variability. Figure 1 also illustrates that convexity is good, in the sense that more convexity results in a higher value, and less convexity results in a lower value, for a change in yield.

The Coastal Business Journal

ISSN: 2163-9280 Spring 2016
Volume 15, Number 1

Figure 1



INCOME PRODUCING PROPERTIES

Peyton (2009) pointed out that before the advent of easy computing, the value of a property V was often quickly determined by the capitalization of the net operating income NOI. Net operating income is a property's yearly gross income less operating expenses. Gross income includes both rental income and all other income such as income from coin vending machines, coin laundry, parking, and so on. Operating expenses include those expenses related to the operation and maintenance of a property such as repairs, insurance, utilities, property taxes, supplies and management. Operating expenses do not include debt service (mortgage loan interest and repayment of principal), amortization of pre-paid interest (loan points), depreciation, income taxes, and capital expenditures. The capitalization rate or cap rate $y_{CapRate}$ is obtained from market sales transactions. Peyton (2009) describes how the cap rate embodies different aspects of real estate pricing risk such as macroeconomic, credit, and commercial real estate factors, and how the cap rate also includes any expected growth in the stream of cash flow from the property. However, in another sense, the capitalization model is limited in that it implies that the property is similar to a perpetuity. But this model is easy to use, and modified duration and convexity can be readily calculated.

$$V = \frac{NOI}{y_{Cap\ Rate}} \tag{4}$$

where NOI is the net operating income during year 1.

$$D^* = \frac{1}{y_{Cap\ Rate}} \tag{5}$$

$$C = \frac{2}{y^2_{Cap\ Rate}} \tag{6}$$

Brown (2000) uses several approaches to measure property risk. He develops a modified duration-based beta of a property's return with respect to a property market index, and he compares these with time-varying betas. Next, he develops an expression for modified duration where the net income operating is fixed for a finite period of time with the property reversion at the market price. Then, he offers a simplified Macaulay duration expression for a valuation model where net operating income is growing and rent is reviewed annually. He explains that it could be problematic to find the inflation flow through rate or how the growth rate of the net operating income changes with respect to the underlying rate used to discount. This Macaulay duration measure has been adjusted to give the modified duration expression below.

$$D^* = \frac{1}{y-g} \cdot \left[1 - \frac{\partial g}{\partial y}\right] \tag{7}$$

where

D^* modified duration for a simplified growth model
y rate used to discount cash flow
g growth rate for net operating income
$\partial g/\partial y$ inflation flow through rate (a measure of the change of the growth rate for the net operating income with respect to the rate used to discount)

When growth in net operating income is introduced, the simplicity of the income capitalization model disappears. If the income capitalization approach is used but net operating income is a function of the capitalization rate, the measures of modified duration and convexity based on the income capitalization approach in equations 5 and 6 might not be accurate. If a lease were to have a consumer price index adjustment, then the net operating income could vary with changes in the CPI. Further, if the assumption were made that, in general, rates (CPI, capitalization rate, discount rate, and so on) experience parallel moves, then in the above example, NOI would be a function of the capitalization rate. With growth in net operating income, the calculations of modified duration and convexity become more complicated and they could be difficult to model mathematically. Refer to Appendix B for these calculations.

Another valuation approach could be used such as discounted cash flow model.

$$V = \sum_{i=1}^{n} \frac{CF_i}{(1+y)^i} \tag{8}$$

where

CF_i cash flow at time i

n holding period

y rate used to discount the cash flow

However, if the cash flow is a function of yield, again the calculations of modified duration and convexity become more complicated could be difficult to model mathematically. Refer to Appendix B for these calculations.

Today, managers of income properties can access sophisticated software for cash flow analysis and valuation of real estate income properties. Some software can incorporate the details of each lease, as well as make forecasts under different scenarios. Some organizations acquire software from external vendors while others develop their own in-house. Modified duration D* and convexity C could be estimated using information from a sensitivity analysis using a well-constructed spreadsheet created to evaluate the property based on its cash flow. See Appendix A for the details of the numerical techniques that could be used.

EXPANSION TO EQUITY POSITION AND A PORTFOLIO OF PROPERTIES

Since the value V of an income property can be considered to be the sum of its debt Debt and equity E, the value of the equity can be calculated by taking the difference between the value of the property and the (market) value of its debt. Extending this idea, the interest rate risk of a property is equal to the interest rate risk of the debt and the interest rate risk of the equity, so that the interest rate risk of the equity can be estimated as the difference between the interest rate risk of the property and its debt. See Appendix C for a detailed explanation of the calculations.

$$D_E^* = \frac{1}{w_E} D_V^* - \frac{w_{Debt}}{w_E} D_{Debt}^* \tag{9}$$

where

D^*_E modified duration of the equity

D^*_V modified duration of the property

ISSN: 2163-9280 Spring 2016
Volume 15, Number 1

D^*_{Debt} modified duration of the debt
w_E E/V, weight of equity calculated using market values for the property and debt
w_{Debt} Debt / V, weight of debt calculated using market values for the property and debt

In a similar manner, the convexity of the equity position can also be calculated.

$$C_E = \frac{1}{w_E} C_V - \frac{w_{Debt}}{w_E} C_{Debt} \tag{10}$$

C_E convexity of the equity
C_V convexity of the property
C_{Debt} convexity of the debt

Modified duration and convexity calculations can be extended to the equity position in a portfolio of properties.

$$D^*_{E\,p} = \sum_{i=1}^{n} w_{E\,i} D^*_{E\,i} \tag{11}$$

and

$$C_{E\,p} = \sum_{i=1}^{n} w_{E\,i} C_{E\,i} \tag{12}$$

where
$w_{E\,i}$ E_i / E_p, the equity weight of the ith property in the portfolio
$D^*_{E\,i}$ modified duration of the equity of the ith property in the portfolio
$D^*_{E\,P}$ modified duration of the equity of the portfolio
$C_{E\,i}$ convexity of the equity of the ith property in the portfolio
$C_{E\,P}$ convexity of the equity of the portfolio
N number of properties in the portfolio

A method of estimating the modified duration and convexity of income properties has been developed in this article. Handforth (2004) developed a methodology for calculating the modified duration and convexity of mortgage debt. With this information, the modified duration and convexity of the equity of an income property can be determined. This knowledge of the value for modified duration allows the opportunity to match the duration of the equity position with the forecast holding period, offering an attempt at immunization. Estimates for modified duration and convexity permit a quantification of the interest rate risk exposure of a portfolio.

The Coastal Business Journal

ISSN: 2163-9280 Spring 2016
Volume 15, Number 1

$$\frac{\partial E_P}{E_P} \approx -D^*_{EP}\left(\partial y\right) + \frac{1}{2}C_{EP}\left(\partial y\right)^2 \qquad (13)$$

This quantification can provide an avenue for hedging the risk. A pension plan manager could use this information to better match assets and liabilities. An additional possible use might be the management of interest rate risk by using sensitivity analysis to determine how variables influence modified duration and convexity. Variables such as the debt-to-equity ratio and the amortization period of debt can be managed within some relevant range. Other debt terms and characteristics of leases might be negotiable as well. This might lead to better managed interest rate risk.

SUMMARY

Some institutions hold income producing properties directly as part of an investment portfolio. These properties are subject to interest rate risk. The two widely used measures of interest rate risk are modified duration and convexity. These measures can be difficult to calculate directly for income properties when net operating income and cash flows are not constant or when they are functions of the underlying yield. A numerical technique using sensitivity analysis is developed to provide estimates of modified duration and convexity. These estimates can then be used, along with corresponding values for debt, to determine the modified duration and convexity of the equity of the property. This can be extended to a portfolio of properties. The measurement of the interest rate risk of the equity of a portfolio of properties, using conventional interest rate risk measures, has a number of useful applications.

APPENDIX A

Macaulay duration or duration D is the remaining length of time that a security is outstanding, where the remaining length of time to each future cash flow is weighted by the fractional contribution of that discounted future cash flow to the total present value of the security.

$$D = \left(\frac{1\,CF_1}{\left(1+y\right)^1} + \frac{2\,CF_2}{\left(1+y\right)^2} + \frac{3\,CF_3}{\left(1+y\right)^3} + \ldots + \frac{N \cdot CF_N}{\left(1+y\right)^N}\right)\frac{1}{V} \qquad \text{(A-1)}$$

where
D Macaulay duration or duration

CF_i cash flow at time i
y yield-to-maturity
N number of periods
V present value of the security

Modified duration D^* is a measure of the sensitivity of the value of a security to changes in the underlying yield or discount rate. It is closely related to Macaulay duration D.

$$D^* = -\frac{\partial V}{\partial y} \cdot \frac{1}{V} = \frac{1}{(1+y)} D \qquad \text{(A-2)}$$

where
D^* modified duration

Fabozzi (2013) discusses duration and convexity and outlines the derivation of the numerical methods which can be used to approximate modified duration and convexity measures. The first derivative of the value function with respect to yield, $\partial V/\partial y$, is the slope of the value curve, and it could be approximated at an evaluation point (y_0,V_0) by evaluating the value function equidistant either side of the evaluation point (y_0,V_0) at points (y_1,V_1) and (y_2,V_2) and using this information to calculate an estimate of the slope $\Delta V/\Delta y$.

$$\frac{\partial V}{\partial y} \approx \frac{\Delta V}{\Delta y} = \frac{V_2 - V_1}{y_2 - y_1} \qquad \text{(A-3)}$$

The resulting approximation to this first derivative could then be used to estimate the modified duration at (y_0,V_0). The sensitivity of the security's value to a small change in market yield away from its current valuation point can be expressed as a function of the modified duration and the small change in market yield.

$$\frac{\partial V}{V} = -D^* \partial y \qquad \text{(A-4)}$$

Since the value function is not linear, an improved estimate for larger changes in yield can be obtained by using the second term of a Taylor series expansion.

ISSN: 2163-9280 Spring 2016
Volume 15, Number 1

$$\frac{\partial V}{V} = \frac{\partial V}{\partial y}\cdot\frac{1}{V}(\partial y) + \frac{1}{2}\frac{\partial^2 V}{\partial y^2}\cdot\frac{1}{V}(\partial y)^2 + remainder \tag{A-5}$$

The second term of the expansion is a convexity measure C and it approximately measures the change in value due to convexity for a change in the underlying yield. Convexity can be approximated using a second degree Taylor series expansion and the following numerical technique.

$$V_1 - V_0 \approx \frac{\partial V}{\partial y}\cdot(\partial y) + \frac{1}{2}\frac{\partial^2 V}{\partial y^2}\cdot(\partial y)^2 \tag{A-6}$$

$$\frac{\partial V}{V} \approx -D^*(\partial y) + \frac{1}{2}C(\partial y)^2 \tag{A-7}$$

$$\partial V \approx -D^* V(\partial y) + \frac{1}{2}CV(\partial y)^2 \tag{A-8}$$

Select y_1 and y_2 on opposite sides of y_0 and equal distance from y_0 so that

$$(y_1 - y_0) = -(y_2 - y_0) \tag{A-9}$$

then

$$V_1 - V_0 \approx -D^* V_0(y_1 - y_0) + \frac{1}{2}CV_0\cdot(y_1 - y_0)^2 \tag{A-10}$$

and

$$V_2 - V_0 \approx -D^* V_0(y_2 - y_0) + \frac{1}{2}CV_0\cdot(y_2 - y_0)^2 \tag{A-11}$$

Add the above two equations and simplify with

The Coastal Business Journal

ISSN: 2163-9280 Spring 2016
Volume 15, Number 1

$$D^* V_0 (y_1 - y_0) \quad = \quad - D^* V_0 (y_2 - y_0) \tag{A-12}$$

and

$$(y_1 - y_0)^2 \quad = \quad [-(y_2 - y_0)]^2 \tag{A-13}$$

so that

$$C \quad = \quad \frac{V_1 + V_2 - 2V_0}{V_0 (y_1 - y_0)^2} \tag{A-14}$$

These approximations for modified duration D* and convexity C can be combined to provide a better estimate of the interest rate risk than using modified duration alone.

$$\frac{\partial V}{V} \quad \approx \quad - D^* (\partial y) \quad + \quad \frac{1}{2} C (\partial y)^2 \tag{A-15}$$

APPENDIX B

For the income capitalization model, to calculate modified duration D^* with growth in net operating income,

$$\frac{\partial V}{\partial y} \quad = \quad \frac{\partial \; NOI \;\; / \; y}{\partial \; y} \tag{B-1}$$

where

V value

y capitalization rate

NOI net operating income during year 1.

ISSN: 2163-9280 Spring 2016
Volume 15, Number 1

$$\frac{\partial V}{\partial y} = -\frac{NOI}{y^2} + \frac{\partial\, NOI}{\partial\, y}\cdot\frac{1}{y} \tag{B-2}$$

$$D^* = \frac{1}{y} - \frac{\partial\, NOI}{\partial\, y}\cdot\frac{1}{NOI} \tag{B-3}$$

$$\frac{\partial^2 V}{\partial y^2} = \frac{2NOI}{y^3} - 2\frac{\partial NOI}{\partial\, y}\cdot\frac{1}{y^2} + \frac{\partial^2\ NOI}{\partial\, y^2}\cdot\frac{1}{y} \tag{B-4}$$

$$C = \frac{2}{y^2} - 2\frac{\partial\, NOI}{\partial\, y}\cdot\frac{1}{NOI\ \ y} + \frac{\partial^2\ NOI}{\partial\, y^2}\cdot\frac{1}{NOI} \tag{B-5}$$

If the first derivative of NOI is zero, then the estimate for D* in equation B-3 reduces to the earlier form in equation 5; and if the first and second derivatives of NOI are zero, then the estimate for C in equation B-5 also reduces to the earlier form in equation 6.

For the discounted cash flow model, to calculate modified duration D^* with growth in net operating income,

$$V = \sum_{i=1}^{n}\frac{CF_i}{(1+y)^i} \tag{B-6}$$

where
CF_i cash flow at time i
n holding period
y rate used to discount the cash flow

$$\frac{\partial V}{\partial y} = \sum_{i=1}^{n} -\frac{i\,CF_i}{(1+y)^{i+1}} + \sum_{i=1}^{n}\frac{1}{(1+y)^i}\frac{\partial CF_i}{\partial y} \tag{B-7}$$

$$D^* = -\frac{\partial V}{\partial y}\cdot\frac{1}{V} \tag{B-8}$$

$$\frac{\partial^2 V}{\partial y^2} = \sum_{i=1}^{n} -\frac{i(i+1)CF_i}{(1+y)^{i+2}} + \sum_{i=1}^{n}\frac{-i}{(1+y)^{i+1}}\frac{\partial CF_i}{\partial y} + \sum_{i=1}^{n}\frac{1}{(1+y)^i}\frac{\partial^2\ CF_i}{\partial y^2} \tag{B-9}$$

ISSN: 2163-9280

Spring 2016
Volume 15, Number 1

$$C = \frac{\partial^2 V}{\partial y^2}\frac{1}{V} \qquad \text{(B-10)}$$

APPENDIX C

The interest rate risk of the equity in an income property can be calculated in the following manner. Market values for the property and debt are used.

$$V = Debt + E \qquad \text{(C-1)}$$

$$\frac{\partial V}{\partial y} = \frac{\partial Debt}{\partial y} + \frac{\partial Equity}{\partial y} \qquad \text{(C-2)}$$

Substituting the modified duration of the property D^*_V, modified duration of the debt D^*_{Debt}, and the modified duration of the equity D^*_E

$$-D^*_V V = -D^*_{Debt} Debt + -D^*_E E \qquad \text{(C-3)}$$

Dividing each side by the value of the property V and substituting the weight for debt w_{Debt} = Debt / V and the debt for w_E = E / V

$$D^*_V = w_{Debt} D^*_{Debt} + w_E D^*_E \qquad \text{(C-4)}$$

$$D^*_E = \frac{1}{w_E} D^*_V - \frac{w_{Debt}}{w_E} D^*_{Debt} \qquad \text{(C-5)}$$

In a similar manner, the convexity of the equity position can be calculated.

$$C_E = \frac{1}{w_E} C_V - \frac{w_{Debt}}{w_E} C_{Debt} \qquad \text{(C-6)}$$

REFERENCES

The Coastal Business Journal

ISSN: 2163-9280 Spring 2016
Volume 15, Number 1

Brown, G. (2000). Duration and risk. *Journal of Real Estate Research*, *20*(3), 337-356.

Fabozzi, F. (2013). *Bond Markets, Analysis and Strategies*.

Handforth, F. (2004). Duration and convexity of mortgages in the context of real estate investment analysis. *Journal of Real Estate Portfolio Management*, *10*(3), 187-202.

Macaulay, F. R. (1938). Some theoretical problems suggested by the movements of interest rates, bond yields and stock prices in the United States since 1856. *NBER Books*.

Peyton, M. S. (2009). Capital markets impact on commercial real estate cap rates: A practitioner's view. *Journal of Portfolio Management*, *35*(5), 38-49.

Redington, F. M. (1952). Review of the principles of life-office valuations. *Journal of the Institute of Actuaries (1886-1994)*, *78*(3), 286-340.

Samuelson, P. A. (1945). The effect of interest rate increases on the banking system. *The American Economic Review*, *35*(1), 16-27.

ABOUT THE AUTHORS

Frank Handforth has a Ph.D. degree in Finance from the University of Mississippi. He holds the Chartered Financial Analyst designation and he is a Licensed Real Estate Appraiser. He received the Texas A&M University System's Teaching Excellence Award. His 2004 paper on "Duration and Convexity of Mortgages in the Context of Real Estate Investment Analysis" won the Real Estate Finance Prize from the American Real Estate Society.

Eugene Bland earned a Ph.D. in Finance from the University of Mississippi. He is Certified in Financial Management, and he is a Certified Treasury Professional. He holds the Chartered Financial Analyst designation. He is a Licensed Real Estate Appraiser and a Licensed Real Estate Broker. Dr. Bland is a Professor of Finance at Texas A&M University – Corpus Christi where he has worked since 2003.

Neil F. Riley, Ph.D., is a Professor of Finance at Francis Marion University and holds the Wells Fargo Bank of South Carolina Chair in Financial Management. He has been at Francis Marion University since 1991 and is a Full Professor of Finance. He has published numerous articles in the fields of business valuation, investments, and corporate finance.

The Coastal Business Journal

ISSN: 2163-9280 Spring 2016
Volume 15, Number 1