Integral representation of moderate cohomology

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Integral representation of moderate cohomology [(*)]

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Abstract. — We make the classical Dickenstein–Sessa canonical representation in local moderate cohomology explicit by an integral formula. We also provide a similar representation of the higher local moderate cohomology groups.

Résumé. — Nous faisons la représentation canonique classique de Dickenstein–Sessa dans la cohomologie modérée locale explicite par une formule intégrale. Nous fournissons également une représentation similaire des groupes de cohomologie modérés locaux supérieurs.

1. Introduction

Let M be a complex N-dimensional manifold, $Y \subset M$ an analytic subset of pure codimension κ, and μ a $\bar{\partial}$-closed current on M with support in Y. In general there are local obstructions to solving $\bar{\partial}\tau = \mu$ for a current τ with support in Y. For instance, the current of integration, $[Y]$, along Y has locally no $\bar{\partial}$-potential with support in Y. Such local obstructions constitute the moderate cohomology sheaf.

The following fundamental result was proved by Dickenstein and Sessa in [17]: If Y is a locally complete intersection and μ has bidegree (p, κ), then there is locally a unique decomposition

$$\mu = \nu + \bar{\partial}\tau, \tag{1.1}$$

where ν is a "locally residual current" with support in Y and τ is a current with support in Y. Hence, the moderate cohomology sheaf is isomorphic to the sheaf of locally residual currents.

That ν is a locally residual current here means that ν locally can be written $\nu = \xi \wedge R^f$, where ξ is a holomorphic p-form and R^f is the Coleff–Herrera product, [15],

$$R^f = \bar{\partial}(1/f_1) \wedge \cdots \wedge \bar{\partial}(1/f_\kappa),$$

associated with a tuple $f = (f_1, \ldots, f_\kappa)$ of holomorphic functions such that $Y = \{f_1 = \cdots = f_\kappa = 0\}$. The uniqueness of the decomposition (1.1) means that if also $\mu = \nu' + \bar{\partial}\tau'$ then $\nu = \nu'$ and $\bar{\partial}\tau = \bar{\partial}\tau'$. The known proofs of the Dickenstein–Sessa decomposition are implicit and so it is a natural problem to find ν explicitly given μ. The main objective of this paper is to find ν by an integral formula.

Given μ the current ν is unique. However, the factorization $\nu = \xi \wedge R^f$ is not. Any R^f as above works as long as $g\mu = 0$ for all g in the ideal $\langle f_1, \ldots, f_\kappa \rangle$. Thus, given μ and such a choice of R^f, our problem is to find ξ.

From now on $\mathcal{J}$ is a coherent ideal sheaf with zero set Y and Y need not be a locally complete intersection. In this context Björk's formalization of locally residual currents is very useful. Following [14, Section 6.2], a (p, κ)-current w is a *Coleff–Herrera current* with support in Y if it is $\bar{\partial}$-closed, satisfies $\overline{\mathcal{J}_Y} w = 0$, where $\mathcal{J}_Y := \sqrt{\mathcal{J}}$, and has the *standard extension property* (SEP) with respect to Y. That w has the SEP with respect to Y means roughly speaking that it has no mass concentrated on proper analytic subsets of Y, see Section 2.1 below for details. The Coleff–Herrera currents with support in Y and the locally residual currents as described above are the same if Y is a locally complete intersection.

Suppose now that $\mathcal{J}\mu = 0$; notice that for any given current with support in Y this holds locally for some choice of $\mathcal{J}$ since currents have locally finite order. Then, there is a unique decomposition (1.1), where ν is a Coleff–Herrera current with support in Y such that $\mathcal{J}\nu = 0$ and where τ can be chosen so that $\mathcal{J}\tau = 0$; see, e.g., [6] and [14], cf. also [18]. It is proved in [6] that any such ν can be written (in a non-unique way) locally as $\xi \cdot R_\kappa$. Here, and in what follows, R_κ is the part of bidegree $(0, \kappa)$ of the Andersson–Wulcan current, [9], associated to a Hermitian locally free resolution of $\mathscr{O}_M/\mathcal{J}$ (cf. Section 2.2 below); R_κ takes values in a vector bundle E and ξ is a holomorphic p-form with values in the dual bundle E^*. If $\mathcal{J} = \langle f_1, \ldots, f_\kappa \rangle$ is a locally complete intersection, then the Koszul complex of $f_1, \ldots, f_\kappa$ is a free resolution of $\mathscr{O}_M/\mathcal{J}$ and the associated current then equals R^f, cf. Example 2.1.

THEOREM 1.1. — *Let $M \subset \mathbb{C}^N$ be a pseudoconvex domain and let $M' \Subset M$. There is an integral kernel $P(\zeta, z)$ with the following properties:*

(i) $\zeta \mapsto P(\zeta, z)$ *is a holomorphic p-form in M' with values in E^*;*

(ii) $z \mapsto P(\zeta, z)$ *is a smooth compactly supported $(N - p, N - \kappa)$-form in M;*

(iii) *if μ is a $\bar{\partial}$-closed (p, κ)-current in M with $\mathcal{J}\mu = 0$ then, in M', ν in (1.1) is given by*

$$\nu = \widetilde{\mathscr{P}}\mu := R_\kappa \cdot \int_z P(\zeta, z) \wedge \mu(z).$$

The integral means the action of μ on the test form $z \mapsto P(\zeta, z)$. The kernel $P(\zeta, z)$ is explicitly constructed given a free resolution of $\mathscr{O}_M / \mathcal{J}$, see Section 3 below.

One application of Theorem 1.1 is to the problem of factorizing cycles. It is shown in [15] that if $\mathcal{J} = \langle f_1, \ldots, f_\kappa \rangle$ is a complete intersection and Z is the corresponding fundamental cycle, i.e., $Z = \sum_j m_j Y_j$ where m_j are certain multiplicities and Y_j are the irreducible components of Y, then $[Z] = R^f \wedge \mathrm{d}f_\kappa \wedge \cdots \wedge \mathrm{d}f_1/(2\pi i)^\kappa$. This was globalized to locally complete intersections by Demailly and Passare, [16], and further generalized by Andersson in [3]. Recently, Lärkäng and Wulcan, [22], proved a formula similar to $[Z] = R^f \wedge \mathrm{d}f_\kappa \wedge \cdots \wedge \mathrm{d}f_1/(2\pi i)^\kappa$ for the fundamental cycle of a quite general complex subspace. If Z is any cycle with $|Z| = Y$ we obtain

$$[Z] = R_\kappa \cdot \int_z P(\zeta, z) \wedge [Z]. \tag{1.2}$$

Indeed, by the decomposition (1.1) and Theorem 1.1, (1.2) holds modulo a term $\bar{\partial}\tau$. However, $[Z]$ already is a Coleff–Herrera current with support in Y so the uniqueness of the decomposition (1.1) implies that $\bar{\partial}\tau = 0$.

The Dickenstein–Sessa result gives a canonical representation of the moderate cohomology sheaf associated with Y in degree κ, which is the lowest degree with non-vanishing cohomology. In Section 4, Theorems 4.2 and 4.3, we give a representation of the moderate cohomology sheaf in degree q with $q > \kappa$.

Acknowledgments

2. Pseudomeromorphic currents and weighted integral formulas

2.1. Pseudomeromorphic currents

In one complex variable z it is elementary to see that the principal value current $1/z^m$ exists and can be defined, e.g., as the limit as $\epsilon \to 0$ in the sense of currents of $\chi(|h(z)|^2/\epsilon)/z^m$, where h is a non-trivial holomorphic function (or tuple) vanishing at $z = 0$ and χ is a smooth regularization of the characteristic function of $[1, \infty) \subset \mathbb{R}$; for the rest of the paper χ will denote such a function. The current $1/z^m$ can also be defined as the value at $\lambda = 0$ of the analytic continuation of the current-valued function $\lambda \mapsto |h(z)|^{2\lambda}/z^m$. It follows that the *residue current* $\bar{\partial}(1/z^m)$ can be computed as the limit of $\bar{\partial}\chi(|h(z)|^2/\epsilon)/z^m$ or as the value at $\lambda = 0$ of $\lambda \mapsto \bar{\partial}|h(z)|^{2\lambda}/z^m$. Since tensor products of currents are well-defined we can form the current

$$\tau = \bar{\partial}\frac{1}{z_1^{m_1}} \wedge \cdots \wedge \bar{\partial}\frac{1}{z_r^{m_r}} \wedge \frac{\gamma(z)}{z_{r+1}^{m_{r+1}} \cdots z_M^{m_d}} \tag{2.1}$$

in $\mathbb{C}^d$, where $m_1, \ldots, m_r$ are positive integers, $m_{r+1}, \ldots, m_d$ are nonnegative integers, and γ is a smooth compactly supported form. Notice that τ is anti-commuting in the residue factors $\bar{\partial}(1/z_j^{m_j})$ and commuting in the principal value factors $1/z_k^{m_k}$. We say that a current of the form (2.1) is an *elementary pseudomeromorphic current*. A current μ on a complex manifold M is *pseudomeromorphic* if and only if μ is a locally finite sum of currents of the form $\pi_*\tau$, where τ is of the form (2.1) and π is a holomorphic map from a neighborhood of supp γ to M, see [11, Theorem 2.15]. Currents on reduced complex spaces are also defined, see [19]. A current μ on a reduced pure-dimensional complex space X is pseudomeromorphic if and only if there is a modification $\pi \colon X' \to X$ with X' smooth and a pseudomeromorphic current μ' on X' such that $\mu = \pi_*\mu'$, [11, Theorem 2.15]. This yields the subsheaf $\mathscr{PM}_X$ of the sheaf of germs of currents on any reduced pure-dimensional complex space X. Notice that, since $\bar{\partial}$ maps an elementary pseudomeromorphic current to a sum of such currents, $\bar{\partial}$ maps $\mathscr{PM}$ to itself. Moreover, if X and Z are reduced pure-dimensional complex spaces and $\mu \in \mathscr{PM}(X)$, then $\mu \otimes 1 \in \mathscr{PM}(X \times Z)$, see [8, Section 2]. Below, we will omit "$\otimes 1$" and write, e.g., $\mu(x)$ to denote on what variables a current depends.

DIMENSION PRINCIPLE ([10, Corollary 2.4], [8, Proposition 2.3]). — *Let X be a reduced pure-dimensional complex space, let $\mu \in \mathscr{PM}(X)$, and assume that μ has support contained in a subvariety $V \subset X$. If $h \in \mathscr{O}_X$ vanishes on V then $\bar{h}\mu = \mathrm{d}\bar{h} \wedge \mu = 0$. Moreover, if μ has bidegree $(*, q)$ and $\mathrm{codim}_X V > q$, then $\mu = 0$.*

Pseudomeromorphic currents can be "restricted" to analytic (or constructible) subsets: Let $\mu \in \mathscr{PM}(X)$, let $V \subset X$ be an analytic subset, and set $V^c := X \setminus V$. Then the restriction of μ to the open subset V^c has a natural pseudomeromorphic extension $\mathbf{1}_{V^c}\mu$ to X. In [10], $\mathbf{1}_{V^c}\mu$ is defined as the value at 0 of the analytic continuation of the current-valued function $\lambda \mapsto |h|^{2\lambda}\mu$, where h is any holomorphic tuple with zero set V; $\mathbf{1}_{V^c}\mu$ can also be defined as $\lim_{\epsilon \to 0} \chi(|h|^2 v/\epsilon)\mu$, where v is any smooth strictly positive function, see [11, Lemma 2.6], cf. also [21, Lemma 6]. The current $\mathbf{1}_V\mu := \mu - \mathbf{1}_{V^c}\mu$ thus is a pseudomeromorphic current with support contained in V, and if supp $\mu \subset V$, then $\mathbf{1}_V\mu = \mu$. Moreover, see [11, Section 2.2], if V and W are any constructible subsets then $\mathbf{1}_V\mathbf{1}_W\mu = \mathbf{1}_{V \cap W}\mu$. A current $\mu \in \mathscr{PM}(X)$ has the standard extension property (SEP) with respect to an analytic subsets $V \subset X$ if $\mathbf{1}_W\mu = 0$ for all germs of analytic subsets $W \subset X$ such that $\operatorname{codim}_V W \cap V > 0$.

Recall that a current on X is said to be semi-meromorphic if it is a principal value current of the form α/f, where α is a smooth form and f is a holomorphic function or section of a line bundle such that f does not vanish identically on any component of X. Following [8], see also [11, Section 4], we say that a current a on X is *almost semi-meromorphic* if there is a modification $\pi \colon X' \to X$ and a semi-meromorphic current α/f on X' such that $a = \pi_*(\alpha/f)$; if f takes values in $L \to X'$ we need also α to take values in $L \to X'$. If a is almost semi-meromorphic on X, then the smallest Zariski-closed set outside of which a is smooth has positive codimension and is denoted $ZSS(a)$, the *Zariski-singular support* of a, see [11, Section 4].

For proofs of the statements in this paragraph we refer to [11, Section 4], see also [8, Section 2]. Let a be an almost semi-meromorphic current on X and let $\mu \in \mathscr{PM}(X)$. Then there is a unique pseudomeromorphic current T on X coinciding with $a \wedge \mu$ outside of $ZSS(a)$ and such that $\mathbf{1}_{ZSS(a)}T = 0$. If h is a holomorphic tuple, or section of a Hermitian vector bundle, such that $\{h = 0\} = ZSS(a)$, then $T = \lim_{\epsilon \to 0} \chi(|h|^2/\epsilon)a \wedge \mu$; henceforth we will write $a \wedge \mu$ in place of T. One defines $\bar{\partial}a \wedge \mu$ so that Leibniz' rule holds, i.e., $\bar{\partial}a \wedge \mu := \bar{\partial}(a \wedge \mu) - (-1)^{\deg a}a \wedge \bar{\partial}\mu$. If μ is almost semi-meromorphic then $a \wedge \mu$ is almost semi-meromorphic and, in fact, $a \wedge \mu = (-1)^{\deg a \deg \mu}\mu \wedge a$.

The sheaf $\mathscr{CH}_Y^p$ of Coleff–Herrera currents on a complex manifold with support in an analytic subset Y of pure codimension κ was introduced by Björk in the way described in the introduction. It is proved in [5] that $\mathscr{CH}_Y^p$ equals the subsheaf of $\mathscr{PM}^{p,\kappa}$ of germs of $\bar{\partial}$-closed currents with support in Y.

2.2. Currents associated to generically exact complexes

Let M, Y, and $\mathcal{J}$ be as in the introduction. Suppose that we have a complex

$$0 \longrightarrow E_m \xrightarrow{f_m} \cdots \xrightarrow{f_1} E_0 \longrightarrow 0$$

of holomorphic Hermitian vector bundles, with E_0 being the trivial line bundle, such that the associated sheaf complex $(\mathcal{O}(E_\bullet), f_\bullet)$ is a resolution of $\mathcal{O}_M/\mathcal{J}$. The bundle $E := \bigoplus_j E_j$ gets a natural superstructure by setting $E^+ := \bigoplus_j E_{2j}$ and $E^- := \bigoplus_j E_{2j+1}$. Following [9] we define pseudomeromorphic currents U and R with values in $\operatorname{End}(E)$ associated with the Hermitian complex $(E_\bullet, f_\bullet)$. Notice that $\operatorname{End}(E)$ gets an induced superstructure and so spaces of forms and currents with values in E or $\operatorname{End}(E)$ get superstructures as well. Let $f := \bigoplus_j f_j$ and set $\nabla := f - \bar\partial$, which then becomes an odd mapping on spaces of forms or currents with values in E such that $\nabla^2 = 0$; notice that ∇ induces an odd mapping $\nabla_{\operatorname{End}}$ on $\operatorname{End}(E)$-valued forms or currents such that $\nabla_{\operatorname{End}}^2 = 0$. Outside of Y, $(E_\bullet, f_\bullet)$ is pointwise exact and we let $\sigma_k \colon E_{k-1} \to E_k$ be the pointwise minimal inverse of f_k. Set $\sigma := \sigma_1 + \sigma_2 + \cdots$ and let $u := \sigma + \sigma\bar\partial\sigma + \sigma(\bar\partial\sigma)^2 + \cdots$. Notice that $u = \sum_{0 \leqslant \ell < k} u_k^\ell$, where $u_k^\ell := \sigma_k \bar\partial\sigma_{k-1} \cdots \bar\partial\sigma_{\ell+1}$ is a smooth $\operatorname{Hom}(E_\ell, E_k)$-valued $(0, k - \ell - 1)$-form outside of Y. One can show that $\nabla_{\operatorname{End}} u = \operatorname{Id}_E$. The form u can be exended as a current across Y by setting

$$U := \lim_{\epsilon \to 0} \chi(|F|^2/\epsilon)u, \tag{2.2}$$

where F is a (non-trivial) holomorphic tuple vanishing on Y, see, e.g., [9, Section 2]. As with u we will write $U = \sum_{0 \leqslant \ell < k} U_\ell^k$, where now U_ℓ^k is a $\operatorname{Hom}(E_\ell, E_k)$-valued $(0, k - \ell - 1)$-current; in fact, U is an almost semi-meromorphic current, cf., e.g., [11]. The current R is defined by $\nabla_{\operatorname{End}} U = \operatorname{Id}_E - R$ and hence R is supported on Y and $fR - \bar\partial R = \nabla_{\operatorname{End}} R = 0$. Notice that R is an almost semi-meromorphic current plus $\bar\partial$ of such a current. One can check that

$$R = \lim_{\epsilon \to 0} \left(1 - \chi(|F|^2/\epsilon)\right) \operatorname{Id}_E + \bar\partial\chi(|F|^2/\epsilon) \wedge u. \tag{2.3}$$

We write $R = \sum_{0 \leqslant \ell < k} R_k^\ell$, where R_k^ℓ is a $\operatorname{Hom}(E_\ell, E_k)$-valued $(0, k - \ell)$-current. Since E_0 is the trivial line bundle we have $\operatorname{Hom}(E_0, E_k) \simeq E_k$ and we may identify R_k^0 with an E_k-valued current; sometimes we just write R_k for R_k^0.

Recall that $\kappa = \operatorname{codim} Y$. It is proved in [9] that $R = R_\kappa^0 + R_{\kappa+1}^0 + \cdots$ and that a holomorphic function g is in $\mathcal{J}$ if and only if the E-valued current Rg vanishes.

Example 2.1. — Suppose that $\mathcal{J} = \langle f_1, \cdots, f_\kappa \rangle$ is a complete intersection. Let E_1 be the trivial bundle of rank κ with trivial metric and let $e_1, \ldots, e_\kappa$ be an orthonormal holomorphic frame. Set $E_k = \Lambda^k E_1$ and let the map $E_k \to E_{k-1}$ be interior multiplication with $\sum_j f_j e_j^*$. We get the Koszul complex of $f_1, \ldots, f_\kappa$; this is exact since $\mathcal{J}$ is a complete intersection. The map σ is exterior multiplication by $\sum_j \bar{f}_j e_j / |f|^2$. The resulting current R is the Coleff–Herrera product R^f times $e_1 \wedge \cdots \wedge e_\kappa$, see [2].

2.3. Weighted integral formulas

We apply Andersson's method, [1], of generating weighted integral formulas in a domain $D \subset \mathbb{C}^N$. To begin with, suppose that $k(\zeta, z)$ is an integrable $(N, N-1)$-form in $D \times D$ and $p(\zeta, z)$ is a smooth (N, N)-form in $D \times D$ such that

$$\bar{\partial} k(\zeta, z) = [\Delta^D] - p(\zeta, z), \tag{2.4}$$

where $[\Delta^D]$ is the current of integration along the diagonal. Applying (2.4) to test forms of the form $\psi_\epsilon(z) \wedge \varphi(\zeta)$, where ψ_ϵ is an approximate identity and φ is a test form in D, one obtains the integral formula

$$\varphi(z) = \bar{\partial}_z \int_{D_\zeta} k(\zeta, z) \wedge \varphi(\zeta) + \int_{D_\zeta} k(\zeta, z) \wedge \bar{\partial}\varphi(\zeta) + \int_{D_\zeta} p(\zeta, z) \wedge \varphi(\zeta) \tag{2.5}$$

for all $z \in D$ by letting $\epsilon \to 0$.

Following [1], to find such k and p let $\eta = (\eta_1, \ldots, \eta_N)$ be a holomorphic tuple in $D \times D$ that defines the diagonal and let Λ_η be the exterior algebra spanned by $\Lambda^{0,1} T^*(D \times D)$ and the $(1,0)$-forms $d\eta_1, \ldots, d\eta_N$. On forms with values in Λ_η interior multiplication with $2\pi i \sum \eta_j \partial / \partial \eta_j$, denoted δ_η, is defined; set $\nabla_\eta = \delta_\eta - \bar{\partial}$. Let s be a smooth $(1,0)$-form in Λ_η such that $|s| \lesssim |\eta|$ and $|\eta|^2 \lesssim |\delta_\eta s|$ and let $B = \sum_{k=1}^N s \wedge (\bar{\partial}s)^{k-1} / (\delta_\eta s)^k$. It is proved in [1] that then $\nabla_\eta B = 1 - [\Delta^D]$. Identifying terms of top degree we see that $\bar{\partial} B_{N,N-1} = [\Delta^D]$ and so (2.4) is satisfied with $k(\zeta, z) = B_{N,N-1}$ and $p(\zeta, z) = 0$. For instance, if we take $s = \partial |\zeta - z|^2$ and $\eta = \zeta - z$, then the resulting B is sometimes called the full Bochner–Martinelli form and the term of top degree is the classical Bochner–Martinelli kernel.

Let $D_1, D_2 \subset D$. A smooth section $g(\zeta, z) = g_0 + \cdots + g_N$ of Λ_η over $D_1 \times D_2$ such that $\nabla_\eta g = 0$ in $D_1 \times D_2$ and $g_0(z, z) = 1$ for $z \in D' := D_1 \cap D_2$ is called a *weight* with respect to $D_1 \times D_2$; g_j is the sum of the terms of g of bidegree (j, j). Notice that the exterior product of two weights again is a weight (with respect to a suitable set). If g is a weight with respect to

$D_1 \times D_2$, then it follows that $\nabla_\eta(g \wedge B) = g - g \wedge [\Delta^D]$ in $D_1 \times D_2$ and, identifying terms of bidegree $(N, N-1)$, we get that

$$\bar{\partial}(g \wedge B)_{N,N-1} = [\Delta^D] - g_N \qquad (2.6)$$

in $D_1 \times D_2$. It follows that if φ is smooth with compact support in D_1, then (2.5), with $k(\zeta, z) = (g(\zeta, z) \wedge B)_{N,N-1}$ and $p(\zeta, z) = g_N(\zeta, z)$, holds in D_2, and vice versa.

Example 2.2. — Let $D \Subset \mathbb{C}^N$ be pseudoconvex and let $K \subset D$ be a holomorphically convex compact subset. Let ρ be a smooth compactly supported function in D that is 1 in a neighborhood of K. One can find a smooth form $\widetilde{s}(\zeta, z) = \widetilde{s}_1(\zeta, z)\mathrm{d}\eta_1 + \cdots + \widetilde{s}_N(\zeta, z)\mathrm{d}\eta_N$, defined for z in a neighborhood of supp $\bar{\partial}\rho$ and ζ in a neighborhood $\widetilde{D}$ of K, such that $\zeta \mapsto \widetilde{s}(\zeta, z)$ is holomorphic and $\delta_\eta \widetilde{s} = 1$. Then

$$g(\zeta, z) := \rho(z) - \bar{\partial}\rho(z) \wedge \sum_{k=1}^{N} \widetilde{s} \wedge (\bar{\partial}\widetilde{s})^{k-1},$$

is a weight with respect to $\widetilde{D} \times D$ that depends holomorphically on ζ and has compact support in D_z.

If D is a ball centered at 0 and $\eta = \zeta - z$ one can take $\widetilde{s} = (2\pi i)^{-1}\bar{z} \cdot \mathrm{d}\eta/(\zeta \cdot \bar{z} - |z|^2)$.

Example 2.3. — Let $D \Subset \mathbb{C}^N$ be pseudoconvex and assume that there is a free resolution $(\mathscr{O}(E_\bullet), f_\bullet)$ of $\mathscr{O}_D/\mathscr{J}$ in D. Let $U = U(\zeta)$ and $R = R(\zeta)$ be associated currents, let U^ϵ and R^ϵ be the regularizations given by (2.2) and (2.3), respectively, and let $U_k^{\epsilon,\ell}$ and $R_k^{\epsilon,\ell}$ be the parts taking values in $\mathrm{Hom}(E_\ell, E_k)$. By [4, Proposition 5.3] we can, for $\ell \leqslant k$, find *Hefer morphisms* H_k^ℓ, which are holomorphic sections of $\Lambda_\eta \otimes \mathrm{Hom}(E^\zeta, E^z)$, such that H_k^ℓ is a holomorphic $k - \ell$-form with values in $\mathrm{Hom}(E_k^\zeta, E_\ell^z)$ and

$$H_k^k \restriction_{\Delta_D} = \mathrm{Id}_{E_k} \quad \text{and} \quad \delta_\eta H_k^\ell = H_{k-1}^\ell f_k - f_{\ell+1}(z)H_k^{\ell+1}, \ k > \ell,$$

where $f_k = f_k(\zeta)$. One can check that then

$$G^\epsilon := \sum_{k \geqslant 0} H_k^0 R_k^{\epsilon,0} + f_1(z) \sum_{k \geqslant 1} H_k^1 U_k^{\epsilon,0},$$

is a weight with respect to $D \times D$ for all $\epsilon > 0$; cf. [4], [8], and [9].

In the case of the Koszul complex, see Example 2.1, the Hefer morphisms can be chosen as follows. We use the notation of Example 2.1. Let $h_j = h_j(\zeta, z)$ be holomorphic $(1,0)$-forms such that $\delta_\eta h_j = f_j(\zeta) - f_j(z)$ and set $h = \sum_j h_j \wedge e_j^*$. Then interior multiplication by $h^{\wedge k - \ell}/(k - \ell)!$ define Hefer morphisms H_k^ℓ.

3. Integral operators associated to an ideal sheaf

Let $D \Subset \mathbb{C}^N$ be pseudoconvex, let $\mathcal{J} \subset \mathscr{O}_D$ be a coherent sheaf of ideals in D with $Y = Z(\mathcal{J})$ of pure codimension κ, and let $D' \Subset D$. Let $g(\zeta, z)$ be any weight with respect to $D' \times D$ such that $z \mapsto g(\zeta, z)$ has compact support in some $D'' \Subset D$ for all ζ. In D'' there is a free resolution $(\mathscr{O}(E_\bullet), f_\bullet)$ of $\mathscr{O}_D / \mathcal{J}$, associated currents $U = U(\zeta)$ and $R = R(\zeta)$, and, moreover, in $D'' \times D''$ we can find associated Hefer morphisms H_k^ℓ. Then $G^\epsilon = HR^\epsilon + f_1(z)HU^\epsilon$ of Example 2.3 is a weight with respect to $D'' \times D''$. It follows that $G^\epsilon \wedge g$ is a weight with respect to $D' \times D$ and has compact support in D_z.

Notice, in view of Example 2.2, that we may choose the weight g so that it contains no $d\bar{\zeta}$-differentials and $\zeta \mapsto g(\zeta, z)$ is holomorphic; we then say that g is holomorphic in ζ.

LEMMA 3.1. — *If φ is a test form in D', then for all $\epsilon > 0$ and all $z \in D$*

$$\varphi(z) = \bar{\partial}_z \int_\zeta (HR^\epsilon \wedge g \wedge B)_{N,N-1} \wedge \varphi(\zeta) + \int_\zeta (HR^\epsilon \wedge g \wedge B)_{N,N-1} \wedge \bar{\partial}\varphi(\zeta)$$
$$+ \int_\zeta (HR^\epsilon \wedge g)_{N,N} \wedge \varphi(\zeta) + \phi\psi,$$

where ϕ is a section of $\mathcal{J}$ and ψ is some test form in D. The integrals on the right-hand side are test forms in D.

Proof. — Since $G^\epsilon \wedge g = HR^\epsilon \wedge g + f_1(z)HU^\epsilon \wedge g$ is a weight with respect to $D' \times D$ and the entries of f_1 are sections of $\mathcal{J}$, it follows from Section 2.3 that the claimed equality holds in D. Since $z \mapsto g(\zeta, z)$ has compact support in D, ψ as well as the integrals on the right-hand side are test forms in D. $\square$

Let μ be an arbitrary current in D. Since z and $\zeta - z$ are independent variables in $D \times D$, the tensor product $B \wedge \mu(z)$ is well-defined, cf. [20, Theorem 5.1.1]. Let $\pi \colon D_\zeta \times D_z \to D_\zeta$ be the natural projection. Then, since $z \mapsto g(\zeta, z)$ has compact support,

$$\pi_* (HR^\epsilon \wedge g \wedge B)_{N,N-1} \wedge \mu(z) \quad \text{and} \quad \pi_* (HR^\epsilon \wedge g)_{N,N} \wedge \mu(z)$$

are well-defined current in D'_ζ. For notational convenience we will often write $\int_z \tau$ instead of $\pi_* \tau$ for a current τ in $D \times D$.

LEMMA 3.2. — *Let μ be a current in D such that $\mathcal{J}\mu = 0$. Then, for all $\epsilon > 0$,*

$$\mu(\zeta) = \int_z (HR^\epsilon \wedge g \wedge B)_{N,N-1} \wedge \bar{\partial}\mu(z) + \bar{\partial}_\zeta \int_z (HR^\epsilon \wedge g \wedge B)_{N,N-1} \wedge \mu(z)$$
$$+ \int_z (HR^\epsilon \wedge g)_{N,N} \wedge \mu(z) \quad (3.1)$$

holds in D'.

Proof. — Let φ be a test form in D'. The action of the current $\int_z (HR^\epsilon \wedge g \wedge B)_{N,N-1} \wedge \bar\partial\mu(z)$ on $\varphi(\zeta)$ is by definition the action of $(HR^\epsilon \wedge g \wedge B)_{N,N-1} \wedge \bar\partial\mu(z)$ on $\varphi(\zeta) \otimes 1$ and this equals, by [20, Theorem 5.1.1], the action of $\bar\partial\mu(z)$ on $\int_\zeta (HR^\epsilon \wedge g \wedge B)_{N,N-1} \wedge \varphi(\zeta)$. Hence,

$$
\left(\int_z (HR^\epsilon \wedge g \wedge B)_{N,N-1} \wedge \bar\partial\mu(z) \right) . \varphi(\zeta)
$$
$$
= \pm\mu(z) . \left(\bar\partial_z \int_\zeta (HR^\epsilon \wedge g \wedge B)_{N,N-1} \wedge \varphi(\zeta) \right).
$$

Similar formulas hold for the second and third term on the right-hand side of (3.1) and the lemma thus follows from Lemma 3.1. □

DEFINITION 3.3. — *For a current μ in D we let $\widetilde{\mathscr{P}}\mu$ be the current in D' given by*

$$
\widetilde{\mathscr{P}}\mu(\zeta) = \int_z (HR \wedge g)_{N,N} \wedge \mu(z).
$$

Notice, since $R = R(\zeta)$, that $(HR \wedge g)_{N,N} \wedge \mu(z)$ is well-defined as a tensor product in $D' \times D$. Notice also that $\widetilde{\mathscr{P}}$ maps arbitrary currents in D to pseudomeromorphic currents in D' annihilated by $\mathcal{J}$.

The operator $\widetilde{\mathscr{P}}$ of Theorem 1.1 (and Theorem 4.3 below) corresponds to a choice of weight g such that $\zeta \mapsto g(\zeta, z)$ is holomorphic, but in the definition above we do not require this.

Example 3.4. — Suppose that $D \subset \mathbb{C}^N$ is a ball and that $\mathcal{J} = \langle f_1, \dots, f_\kappa \rangle$ is a complete intersection. Using the Koszul complex we get, see Example 2.1 and the notation there, that $R = R^f \wedge e_1 \wedge \cdots \wedge e_\kappa$. Moreover, in view of the paragraph after Example 2.3 we get $HR = h_1 \wedge \cdots \wedge h_\kappa \wedge R^f / \kappa!$. Taking g as in Example 2.2 with $\widetilde{s} = (2\pi i)^{-1} \bar{z} \cdot d\eta / (\zeta \cdot \bar{z} - |z|^2)$ and $\eta = \zeta - z$ we then get

$$
\widetilde{\mathscr{P}}\mu = R^f \wedge \int_z \frac{h_1 \wedge \cdots \wedge h_\kappa}{\kappa!} \wedge \bar\partial\rho(z) \wedge \frac{(\bar{z} \cdot d\eta) \wedge (d\bar{z} \cdot d\eta)^{N-\kappa-1}}{(2\pi i(\zeta \cdot \bar{z} - |z|^2))^{N-\kappa}} \wedge \mu(z).
$$

Notice that, for degree reasons, $\widetilde{\mathscr{P}}\mu = 0$ unless μ has degree $(*, \kappa)$.

PROPOSITION 3.5. — *Suppose that the weight g in the definition of $\widetilde{\mathscr{P}}$ depends holomorphically on ζ. Then, for any (p,q)-current μ in D, $\widetilde{\mathscr{P}}\mu$ is of the form $\xi \cdot R_q^0$, where*

$$
\xi = \Phi(\mu) = \int_z H_q^0 \wedge g_{N-q} \wedge \mu(z)
$$

is a holomorphic p-form with values in E_q^*. Moreover, if $\mathcal{J}\mu = 0$ then $\widecheck{\mathscr{P}}\bar{\partial}\mu = \bar{\partial}\widecheck{\mathscr{P}}\mu$ and $\Phi(\bar{\partial}\mu) = f_{q+1}^*\Phi(\mu)$, where f_{q+1}^* is the transpose of the map $f_{q+1}\colon E_{q+1} \to E_q$.

For the rest of this section we will fix frames for the trivial E_j-bundles and associated dual frames for the E_j^*'s; sections of the E_j's will be represented by column vectors, sections of the E_j^*'s by row vectors, and maps between bundles by matrices. Notice that E_0 is assumed to be the trivial line bundle so that $\mathrm{Hom}(E_j, E_0) \simeq E_j^*$. Recall that $f_k = f_k(\zeta)$. To prove Proposition 3.5 we will need

LEMMA 3.6. — *Letting* $(\cdot)^*$ *denote matrix transpose we have*

$$f_{q+1}^*(H_q^0)^* \wedge g_{N-q} = \bar{\partial}\big((H_{q+1}^0)^* \wedge g_{N-q-1}\big) + (f_1(z)H_{q+1}^1)^* \wedge g_{N-q}.$$

Proof. — This is verified by the following computation.

$$
\begin{aligned}
f_{q+1}^*(H_q^0)^* \wedge g_{N-q} &= (H_q^0 f_{q+1})^* \wedge g_{N-q} \\
&= (\delta_\eta H_{q+1}^0 + f_1(z)H_{q+1}^1)^* \wedge g_{N-q} \\
&= (\delta_\eta H_{q+1}^0 \wedge g_{N-q})^* + (f_1(z)H_{q+1}^1)^* \wedge g_{N-q} \\
&= (\pm H_{q+1}^0 \wedge \delta_\eta g_{N-q})^* + (f_1(z)H_{q+1}^1)^* \wedge g_{N-q} \\
&= (\pm H_{q+1}^0 \wedge \bar{\partial} g_{N-q-1})^* + (f_1(z)H_{q+1}^1)^* \wedge g_{N-q} \\
&= \bar{\partial}(H_{q+1}^0 \wedge g_{N-q-1})^* + (f_1(z)H_{q+1}^1)^* \wedge g_{N-q},
\end{aligned}
$$

where the second equality follows from the properties of the Hefer morphisms (see Example 2.3), the forth since $0 = \delta_\eta(H_{q+1}^0 \wedge g_{N-q})$ for degree reasons, the fifth since g is a weight, and the sixth since H_{q+1}^0 is holomorphic. $\square$

Proof of Proposition 3.5. — Since g is holomorphic in ζ and in particular contains no $d\bar{\zeta}$-differentials, it follows for degree reasons that

$$\widecheck{\mathscr{P}}\mu(\zeta) = \int_z H_q^0 R_q^0 \wedge g_{N-q} \wedge \mu(z) = \pm(R_q^0)^* \int_z (H_q^0)^* \wedge g_{N-q} \wedge \mu(z).$$

Since also H is holomorphic in ζ the first statement of the proposition follows. For the rest of this proof we will write R_j instead of R_j^0.

Assume that $\mathcal{J}\mu = 0$. To see that $\widetilde{\mathscr{P}}\bar{\partial}\mu = \bar{\partial}\widetilde{\mathscr{P}}\mu$ we compute:

$$
\widetilde{\mathscr{P}}\bar{\partial}\mu = \int_z H^0_{q+1} R_{q+1} \wedge g_{N-q-1} \wedge \bar{\partial}\mu(z)
$$

$$
= \pm R^*_{q+1} \int_z \bar{\partial}\big((H^0_{q+1})^* \wedge g_{N-q-1}\big) \wedge \mu(z)
$$

$$
= \pm R^*_{q+1} \int_z f^*_{q+1}(H^0_q)^* \wedge g_{N-q} \wedge \mu(z)
$$

$$
\mp R^*_{q+1} \int_z (f_1(z)H^1_{q+1})^* \wedge g_{N-q} \wedge \mu(z)
$$

$$
= \pm(f_{q+1}R_{q+1})^* \int_z (H^0_q)^* \wedge g_{N-q} \wedge \mu(z)
$$

$$
= \bar{\partial}\left(R^*_q \int_z (H^0_q)^* \wedge g_{N-q} \wedge \mu(z) \right) = \bar{\partial}\widetilde{\mathscr{P}}\mu,
$$

where the second equality holds since $R = R(\zeta)$ is independent of z, the third by Lemma 3.6, the forth since (the entries of) $f_1(z)$ annihilate $\mu(z)$, and the fifth since $\nabla_{\mathrm{End}}R = 0$ (see Section 2.2) and H and g are holomorphic in ζ.

Moreover, in view of Lemma 3.6, we get that

$$
\Phi(\bar{\partial}\mu)^* = \int_z \bar{\partial}\big((H^0_{q+1})^* \wedge g_{N-q-1}\big) \wedge \mu(z)
$$

$$
= f^*_{q+1} \int_z (H^0_q)^* \wedge g_{N-q} \wedge \mu(z) + \int_z \big(f_1(z)H^1_{q+1}\big)^* \wedge g_{N-q} \wedge \mu(z)
$$

$$
= f^*_{q+1}\Phi(\mu)^*
$$

since (the entries of) $f_1(z)$ annihilate $\mu(z)$. $\qquad\square$

Let $\mu \in \mathscr{PM}(D)$. Since B is almost semi-meromorphic in $D \times D$, the product $B \wedge \mu(z)$ is a well-defined pseudomeromorphic current in $D \times D$, in view of Section 2.1; by the uniqueness in [20, Theorem 5.1.1], $B \wedge \mu(z)$ coincides with the tensor product of B and μ (z and $\zeta - z$ are independent variables). From Section 2.2, R is an almost semi-meromorphic current plus $\bar{\partial}$ of such a current. Thus, by Section 2.1, $R \wedge B \wedge \mu(z)$ is a well-defined pseudomeromorphic current in $D' \times D$ that can be defined as $\lim_{\epsilon \to 0} R^\epsilon \wedge B \wedge \mu(z)$, where R^ϵ is the regularization of R given by (2.3). Even though $R^0_j = 0$ for $j < \kappa$ it may be the case that $\lim_{\epsilon \to 0} R^{0,\epsilon}_j \wedge B \wedge \mu(z) \neq 0$. Still, the support of $R \wedge B \wedge \mu(z)$ is contained in $Y \times \mathrm{supp}\,\mu$. To see this it suffices, in view of (2.3) and Section 2.1, to see that $\mathbf{1}_{Y \times D}B \wedge \mu(z) = 0$. Since B is smooth outside of the diagonal $\Delta \subset D \times D$ it is clear that $\mathrm{supp}\,\mathbf{1}_{Y \times D}B \wedge \mu(z) \subset \Delta$. Moreover, $ZSS(B) \subset \Delta$ and so $\mathbf{1}_\Delta B \wedge \mu(z) = 0$. Hence,

$$
\mathbf{1}_{Y \times D}B \wedge \mu(z) = \mathbf{1}_\Delta \mathbf{1}_{Y \times D}B \wedge \mu(z) = \mathbf{1}_{Y \times D}\mathbf{1}_\Delta B \wedge \mu(z) = 0.
$$

DEFINITION 3.7. — *For a pseudomeromorphic current μ in D we let $\widetilde{\mathscr{K}}\mu$ be the pseudomeromorphic current in D' given by*

$$\widetilde{\mathscr{K}}\mu(\zeta) = \int_z (HR \wedge g \wedge B)_{N,N-1} \wedge \mu(z).$$

As in the definition of the $\widetilde{\mathscr{P}}$-operators, we do not require g to be holomorphic in ζ, but the $\widetilde{\mathscr{K}}$-operator of Theorem 4.3 corresponds to such a choice.

Notice that, since supp $R \wedge B \wedge \mu(z) \subset Y \times D$, $\widetilde{\mathscr{K}}$ maps pseudomeromorphic currents in D to pseudomeromorphic currents in D' with support contained in Y. We do not know whether or not $\mathcal{J}\widetilde{\mathscr{K}}\mu = 0$ for a general pseudomeromorphic μ.

The following proposition follows from Lemma 3.2 by letting $\epsilon \to 0$.

PROPOSITION 3.8. — *For any $\mu \in \mathscr{PM}^{p,q}(D)$ such that $\mathcal{J}\mu = 0$ we have $\mu_{\upharpoonright_{D'}} = \bar{\partial}\widetilde{\mathscr{K}}\mu + \widetilde{\mathscr{K}}\bar{\partial}\mu + \widetilde{\mathscr{P}}\mu.$*

Let $\mathscr{CH}^p_{\mathcal{J}}$ be the subsheaf of $\mathscr{CH}^p_Y$ of currents annihilated by $\mathcal{J}$.

PROPOSITION 3.9. — *Let $\widetilde{\mathscr{P}}$ be an operator associated to $\mathcal{J}$ and corresponding to a weight g such that $\zeta \mapsto g(\zeta, z)$ is holomorphic. If $\nu \in \mathscr{CH}^p_{\mathcal{J}}(D)$, then $\widetilde{\mathscr{P}}\nu = \nu_{\upharpoonright_{D'}}$. Moreover, if μ is any $\bar{\partial}$-closed (p, κ)-current such that $\mathcal{J}\mu = 0$, then $\widetilde{\mathscr{P}}\mu \in \mathscr{CH}^p_{\mathcal{J}}(D')$ and $\widetilde{\mathscr{P}}\mu$ is the current ν in (1.1).*

Proof. — By Proposition 3.8, $\nu_{\upharpoonright_{D'}} = \bar{\partial}\widetilde{\mathscr{K}}\nu + \widetilde{\mathscr{P}}\nu$. However, $\widetilde{\mathscr{K}}\nu$ is a pseudomeromorphic $(p, \kappa-1)$-current with support contained in Y and must thus vanish in view of the Dimension principle.

For the second statement, notice that $\widetilde{\mathscr{P}}\mu$ is a pseudomeromorphic (p, κ)-current annihilated by $\mathcal{J}$ (in particular with support in Y) and, by Proposition 3.5, $\bar{\partial}\widetilde{\mathscr{P}}\mu = 0$. Thus, in view of Section 2.1, $\widetilde{\mathscr{P}}\mu$ is a section of $\mathscr{CH}^p_{\mathcal{J}}$. Now consider the decomposition (1.1) and recall that $\mathcal{J}\nu = 0$ and that we may assume that $\mathcal{J}\tau = 0$. By the first part of the proof and Proposition 3.5

$$\widetilde{\mathscr{P}}\mu = \widetilde{\mathscr{P}}\nu + \widetilde{\mathscr{P}}\bar{\partial}\tau = \nu + \bar{\partial}\widetilde{\mathscr{P}}\tau.$$

Together with (1.1) this gives that $\mu - \widetilde{\mathscr{P}}\mu = \bar{\partial}(\tau - \widetilde{\mathscr{P}}\tau)$, and so by uniqueness of (1.1), $\widetilde{\mathscr{P}}\mu = \nu$. $\qquad\square$

Proof of Theorem 1.1. — Setting $P(\zeta, z) := H^0_\kappa \wedge g_{N-\kappa}$, where g is a weight depending holomorphically on ζ, Theorem 1.1 follows from Propositions 3.5 and 3.9. $\qquad\square$

4. Higher local moderate cohomology

Let $\mathscr{C}_{\mathcal{J}}^{p,q}$ be the sheaf of (p,q)-currents in M annihilated by $\mathcal{J}$. Notice that we have a complex $(\mathscr{C}_{\mathcal{J}}^{p,\bullet}, \bar{\partial})$. The Dickenstein–Sessa decomposition (1.1) implies that there is a canonical isomorphism $\mathscr{H}^{\kappa}(\mathscr{C}_{\mathcal{J}}^{p,\bullet}, \bar{\partial}) \simeq \mathscr{CH}_{\mathcal{J}}^{p}$, where, as above, $\mathscr{CH}_{\mathcal{J}}^{p}$ is the subsheaf of $\mathscr{CH}_{Y}^{p}$ of currents annihilated by $\mathcal{J}$. The map $\mathscr{CH}_{\mathcal{J}}^{p} \to \mathscr{H}^{\kappa}(\mathscr{C}_{\mathcal{J}}^{p,\bullet}, \bar{\partial})$ is induced by the inclusion $\mathscr{CH}_{\mathcal{J}}^{p} \hookrightarrow \mathscr{C}_{\mathcal{J}}^{p,\kappa}$ and the inverse map $\mathscr{H}^{\kappa}(\mathscr{C}_{\mathcal{J}}^{p,\bullet}, \bar{\partial}) \to \mathscr{CH}_{\mathcal{J}}^{p}$ is induced by our operator $\widetilde{\mathscr{P}}$. It is well-known that $\mathscr{H}^{q}(\mathscr{C}_{\mathcal{J}}^{p,\bullet}, \bar{\partial}) = 0$ for $q < \kappa$, see, e.g., [13]. Our objective now is to give a description of the higher cohomology $\mathscr{H}^{q}(\mathscr{C}_{\mathcal{J}}^{p,\bullet}, \bar{\partial})$, $q > \kappa$.

Remark 4.1. — It is well-known that the cohomology sheaves $\mathscr{H}^{q}(\mathscr{C}_{\mathcal{J}}^{p,\bullet}, \bar{\partial})$ are isomorphic to $\mathscr{E}xt^{q}(\mathscr{O}_{M}/\mathcal{J}, \Omega_{M}^{p})$, where Ω_{M}^{p} is the sheaf of holomorphic p-forms on M.

THEOREM 4.2. — *There are fine sheaves of currents $\mathscr{B}_{\mathcal{J}}^{p,q} \subset \mathscr{C}_{\mathcal{J}}^{p,q}$, $p \geqslant 0$, $q \geqslant \kappa$, such that $\bar{\partial}$ maps $\mathscr{B}_{\mathcal{J}}^{p,q}$ to $\mathscr{B}_{\mathcal{J}}^{p,q+1}$ and the inclusion $\mathscr{B}_{\mathcal{J}}^{p,q} \subset \mathscr{C}_{\mathcal{J}}^{p,q}$ induces an isomorphism $\mathscr{H}^{q}(\mathscr{B}_{\mathcal{J}}^{p,\bullet}, \bar{\partial}) \simeq \mathscr{H}^{q}(\mathscr{C}_{\mathcal{J}}^{p,\bullet}, \bar{\partial})$. Setting $\mathscr{B}_{\mathcal{J}}^{p,q} = 0$ for $q < \kappa$ this holds for all q. Moreover, $\overline{\sqrt{\mathcal{J}}}\mathscr{B}_{\mathcal{J}}^{p,q} = d\overline{\sqrt{\mathcal{J}}} \wedge \mathscr{B}_{\mathcal{J}}^{p,q} = 0$.*

The point is that $\mathscr{B}_{\mathcal{J}}^{p,q}$ is much smaller and less singular than $\mathscr{C}_{\mathcal{J}}^{p,q}$; cf. Example 4.5 below. Moreover, $\mathscr{B}_{\mathcal{J}}^{p,q}$ is a concretely defined subsheaf of $\mathscr{PM}_{M}^{p,q}$, and the kernel of $\bar{\partial}$ in $\mathscr{B}_{\mathcal{J}}^{p,\kappa}$ is $\mathscr{CH}_{\mathcal{J}}^{p}$.

Theorem 4.2 shows that the natural inclusion of complexes $(\mathscr{B}_{\mathcal{J}}^{p,\bullet}, \bar{\partial}) \hookrightarrow (\mathscr{C}_{\mathcal{J}}^{p,\bullet}, \bar{\partial})$ is a quasi-isomorphism, i.e., an isomorphism on cohomology. Our final result provides in particular an explicit projection operator $\widetilde{\mathscr{P}} : (\mathscr{C}_{\mathcal{J}}^{p,\bullet}, \bar{\partial}) \to (\mathscr{B}_{\mathcal{J}}^{p,\bullet}, \bar{\partial})$ giving the inverse of this quasi-isomorphism. Let $\mathscr{C}^{p,q}$ be the sheaf of (p,q)-currents in M.

THEOREM 4.3. — *Let $M \subset \mathbb{C}^{N}$ be pseudoconvex and let $M' \Subset M$. There is an integral operator $\widetilde{\mathscr{P}} : \mathscr{C}^{p,q}(M) \to \mathscr{B}_{\mathcal{J}}^{p,q}(M')$ giving a quasi-isomorphism of complexes $(\mathscr{C}_{\mathcal{J}}^{p,\bullet}, \bar{\partial}) \to (\mathscr{B}_{\mathcal{J}}^{p,\bullet}, \bar{\partial})$. Moreover, there is an integral operator $\widetilde{\mathscr{K}} : \mathscr{B}_{\mathcal{J}}^{p,q}(M) \to \mathscr{B}_{\mathcal{J}}^{p,q-1}(M')$ such that $\mu_{\restriction M'} = \bar{\partial}\widetilde{\mathscr{K}}\mu + \widetilde{\mathscr{K}}\bar{\partial}\mu + \widetilde{\mathscr{P}}\mu$ for any $\mu \in \mathscr{B}_{\mathcal{J}}^{p,q}(M)$.*

In analogy with [8, Definition 7.1], [23, Definition 4.1], and [24, Section 6.2] we make the following definition.

DEFINITION 4.4. — *A (p,q)-current μ on M is a section of $\mathscr{B}_{\mathcal{J}}^{p,q}$ over an open set $\mathcal{U} \subset M$ if for each $x \in \mathcal{U}$ there is a neighborhood D of x such*

that $\mu_{\upharpoonright D}$ is a finite sum of currents of the form

$$\xi_m \wedge \widetilde{\mathscr{K}_m}(\cdots \xi_1 \wedge \widetilde{\mathscr{K}_1}(\xi_0 \cdot R) \cdots), \tag{4.1}$$

where R is a current corresponding to a Hermitian free resolution $(\mathscr{O}(E_\bullet), f_\bullet)$ of $\mathscr{O}/\mathcal{J}$ in D, ξ_0 is a smooth form with values in $\bigoplus_q E_q^*$, ξ_j is a smooth form for $j \geqslant 1$, and the $\widetilde{\mathscr{K}_j}$'s are operators as in Definition 3.7.

It is clear from the definition that $\mathscr{B}_{\mathcal{J}} := \bigoplus_{p,q} \mathscr{B}_{\mathcal{J}}^{p,q}$ is a module over the sheaf $\bigoplus_{p,q} \mathscr{E}_M^{p,q}$ of smooth forms on M and that it is closed under $\widetilde{\mathscr{K}}$-operators. Moreover, $\mathscr{B}_{\mathcal{J}} \subset \mathscr{PM}$ and any section of $\mathscr{B}_{\mathcal{J}}$ has support in Y. Hence, in view of Section 2.1, if $\mu \in \mathscr{B}_{\mathcal{J}}^{p,\kappa}$ is $\bar{\partial}$-closed then $\mu \in \mathscr{CH}_Y^p$. Notice also, by Section 3, that a $\widetilde{\mathscr{P}}$-operator associated with $\mathcal{J}$ maps an arbitrary current to a section of $\mathscr{B}_{\mathcal{J}}$.

Example 4.5. — Let $i \colon Y \hookrightarrow M$, suppose that Y is smooth and that $\mathcal{J}$ is the radical ideal sheaf of Y. We claim that then $\mathscr{B}_{\mathcal{J}}^{N,q} = i_* \mathscr{E}_Y^{N-\kappa, q-\kappa}$. Given the claim, we see that

$$0 \longrightarrow \mathscr{B}_{\mathcal{J}}^{N,\kappa} \stackrel{\bar{\partial}}{\longrightarrow} \cdots \stackrel{\bar{\partial}}{\longrightarrow} \mathscr{B}_{\mathcal{J}}^{N,N} \longrightarrow 0 \tag{4.2}$$

is i_* of the Dolbeault complex of smooth $(\dim Y, \bullet)$-forms on Y. Moreover, since, by Theorem 4.2, the kernel of $\bar{\partial}$ at $\mathscr{B}_{\mathcal{J}}^{N,\kappa}$ is $\mathscr{CH}_{\mathcal{J}}^N$ it follows that $i_* \Omega_Y^{\dim Y} = \mathscr{CH}_{\mathcal{J}}^N$.

The claim may be verified locally so we assume that M is the unit ball in $\mathbb{C}^N$ with coordinates $(z; w)$ such that $Y = \{z = 0\}$ and $\mathcal{J} = \langle z \rangle$. For notational convenience we will also assume that $\kappa = 1$. Let φ be a smooth $(N-1, q-1)$-form on Y and let $\widetilde{\varphi}$ be any smooth extension of φ to M. Then $i_* \varphi = \widetilde{\varphi} \wedge [Y]$. From the free resolution $0 \to \mathscr{O} \stackrel{z}{\to} \mathscr{O}$ of $\mathscr{O}/\mathcal{J}$ we get the current $R = \bar{\partial}(1/z)$ and so, by the Poincaré–Lelong formula, $\widetilde{\varphi} \wedge [Y] = i \widetilde{\varphi} \wedge \mathrm{d}z \wedge R/2\pi$. Hence, $i_* \mathscr{E}_Y^{N-\kappa, q-\kappa} \subset \mathscr{B}_{\mathcal{J}}^{N,q}$.

For the opposite inclusion, let μ be a section of $\mathscr{B}_{\mathcal{J}}^{N,q}$; we may assume that μ is of the form (4.1). We use induction over m in (4.1) to see that $\mu = \xi \wedge [Y]$ for some smooth ξ; notice that such a ξ exists if there is a continuous ξ', which is smooth along Y, such that $\mu = \xi' \wedge [Y]$. If $(\mathscr{O}(E_\bullet), f_\bullet)$ is any free Hermitian resolution of $\mathscr{O}/\mathcal{J}$ with rank $E_0 = 1$, then the associated residue current is of the form $\alpha \wedge \bar{\partial}(1/z)$, where α is a smooth $(0,*)$-form with values in $\bigoplus_j E_j$, see [9, Theorem 4.4]. Hence, if μ is of the form (4.1) with $m = 0$, then $\mu = \xi_0 \cdot \alpha \wedge \bar{\partial}(1/z)$, where ξ_0 is a smooth $(N,*)$-form with values in $\bigoplus_j E_j^*$. Writing $\xi_0 \cdot \alpha = \xi \wedge dz$, for some scalar-valued smooth form ξ, we get $\xi_0 \cdot \alpha \wedge \bar{\partial}(1/z) = -2\pi i \xi \wedge [Y]$, and the induction start follows. Now, if

$\widetilde{\mathcal{K}}$ is an integral operator as in Definition 3.7, then

$$\widetilde{\mathcal{K}}(\xi \wedge [Y]) = \int_{z \in Y} (HR \wedge g \wedge B)_{N,N-1} \wedge \xi(z) = R \cdot \int_{z \in Y} k(\zeta, z) \wedge \xi(z),$$

where $R = R(\zeta)$ is of the form $\alpha(\zeta) \wedge \bar{\partial}(1/\zeta)$ as above, and $k(\zeta, z)$ is a kernel in $M \times M$ which is $\mathcal{O}(1/|\zeta - z|^{2(N-1)-1})$. It follows that $\zeta \mapsto \int_{z \in Y} k(\zeta, z) \wedge \xi(z)$ is a continuous $(N, *)$-form with values in $\bigoplus_j E_j^*$ and we write it as $\psi(\zeta) \wedge d\zeta$. Moreover, in view of [8, Lemma 6.2], $\psi_{\restriction_Y}$ is smooth. We get

$$\widetilde{\mathcal{K}}(\xi \wedge [Y]) = \alpha \wedge \bar{\partial}(1/\zeta) \cdot \psi \wedge d\zeta = \pm 2\pi i \alpha \cdot \psi \wedge [Y],$$

and the induction step follows.

Remark 4.6. — For not necessarily smooth Y and any $\mathcal{J}$ of pure codimension κ and zero set Y, Andersson and Lärkäng introduce a notion of holomorphic top-degree forms on the possibly non-reduced complex space $Y_{\mathcal{J}} = (Y, \mathcal{O}_M/\mathcal{J})$, see [7, Section 5]; see [12] for the reduced case. Via i_* this notion precisely corresponds to $\mathcal{CH}_{\mathcal{J}}^N$. By analogy it is reasonable to think of (4.2) as a certain Dolbeault complex for $Y_{\mathcal{J}}$ with the cohomology at $\mathcal{B}_{\mathcal{J}}^{N,\kappa}$ being the holomorphic top-degree forms on $Y_{\mathcal{J}}$ and $\widetilde{\mathcal{P}}$ as a projection operator onto these forms.

To prove Theorem 4.2 (and 4.3) we may assume that M and M' are as in Theorem 4.3. Let $M'' \Subset D$ be a suitable neighborhood of $\overline{M}'$ similar to the beginning of Section 3. Let

$$0 \longrightarrow \mathcal{O}(E_m) \xrightarrow{f_m} \cdots \xrightarrow{f_1} \mathcal{O}(E_0) \longrightarrow \mathcal{O}/\mathcal{J} \longrightarrow 0$$

be a Hermitian free resolution of $\mathcal{O}/\mathcal{J}$ in M'' and let U and R be the associated currents, see Section 2.2. Dualizing and tensoring with $\Omega^p := \Omega_M^p$ we get the complex

$$0 \longleftarrow \mathcal{O}(E_m^*) \otimes \Omega^p \xleftarrow{f_m^* \otimes \mathrm{Id}} \cdots \xleftarrow{f_1^* \otimes \mathrm{Id}} \mathcal{O}(E_0) \otimes \Omega^p \longleftarrow 0. \qquad (4.3)$$

It is well-known that the cohomology of this complex is isomorphic to $\mathcal{E}xt^{\bullet}(\mathcal{O}_M/\mathcal{J}, \Omega^p)$. Recall that $\mathcal{C}^{p,q}$ is a stalk-wise injective $\mathcal{O}_M$-module by a well-known theorem of Malgrange and consider the resolution

$$0 \longrightarrow \Omega^p \longrightarrow \mathcal{C}^{p,0} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{C}^{p,N} \longrightarrow 0$$

of Ω^p. Applying the functor $\mathcal{H}om(\mathcal{O}_M/\mathcal{J}, -)$ and noticing that $\mathcal{H}om(\mathcal{O}_M/\mathcal{J}, \mathcal{C}^{p,q}) \simeq \mathcal{C}_{\mathcal{J}}^{p,q}$ we get the complex

$$0 \longrightarrow \mathcal{C}_{\mathcal{J}}^{p,0} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{C}_{\mathcal{J}}^{p,N} \longrightarrow 0. \qquad (4.4)$$

By standard homological algebra, the cohomology of (4.4) is naturally isomorphic to the cohomology of (4.3). Following [6] this isomorphism can be realized using the current R as follows. For convenience we will write just f_k^*

instead of $f_k^* \otimes \text{Id}$. If ξ is a holomorphic p-form with values in E_q^* such that $f_{q+1}^* \xi = 0$, then since $fR - \bar{\partial}R = 0$ (see Section 2.2)

$$\bar{\partial}(\xi \cdot R_q) = \pm \xi \cdot \bar{\partial}R_q = \pm \xi \cdot f_{q+1}R_{q+1} = \pm f_{q+1}^* \xi \cdot R_{q+1} = 0. \qquad (4.5)$$

Similarly, if $\xi = f_q^* \xi'$, then $\xi \cdot R_q = 0$. Thus the map $\mathcal{O}(E_p^*) \otimes \Omega^p \to \mathscr{C}_{\mathcal{J}}^{p,q}$ given by $\xi \mapsto \xi \cdot R_q$ induces a map on cohomology and it turns out to be the natural isomorphism. Notice that $\xi \cdot R_q$ is a section of $\mathscr{B}_{\mathcal{J}}^{p,q}$.

Proof of Theorems 4.2 and 4.3. — We have already noticed that any $\widetilde{\mathscr{P}}$-operator associated to $\mathcal{J}$ maps arbitrary currents to sections of $\mathscr{B}_{\mathcal{J}}$ and that any $\widetilde{\mathscr{K}}$-operator associated to $\mathcal{J}$ maps sections of $\mathscr{B}_{\mathcal{J}}$ to sections of $\mathscr{B}_{\mathcal{J}}$. As $\mathscr{B}_{\mathcal{J}} \subset \mathscr{PM}$ and sections of $\mathscr{B}_{\mathcal{J}}$ have support in Y it follows from the Dimension principle that $\overline{\sqrt{\mathcal{J}}}\mathscr{B}_{\mathcal{J}} = \mathrm{d}\overline{\sqrt{\mathcal{J}}} \wedge \mathscr{B}_{\mathcal{J}} = 0$.

Let us temporarily assume that $\mathcal{J}\mathscr{B}_{\mathcal{J}}^{p,q} = 0$ and show how Theorems 4.2 and 4.3 follow. Then the kernel of $\bar{\partial}$ in $\mathscr{B}_{\mathcal{J}}^{p,\kappa}$ is $\mathscr{CH}_{\mathcal{J}}^p$ and, by Proposition 3.8, we have $\mu = \bar{\partial}\widetilde{\mathscr{K}}\mu + \widetilde{\mathscr{K}}\bar{\partial}\mu + \widetilde{\mathscr{P}}\mu$ for any μ in $\mathscr{B}_{\mathcal{J}}$ if $\widetilde{\mathscr{P}}$ and $\widetilde{\mathscr{K}}$ are constructed using the same R, H, and g. Assume henceforth that g depends holomorphically on ζ.

To show that $\bar{\partial}$ maps $\mathscr{B}_{\mathcal{J}}$ to itself it suffices to show that if μ is of the form (4.1) (with $\xi_m = 1$) then $\bar{\partial}\mu$ is a section of $\mathscr{B}_{\mathcal{J}}$; we will use induction over m to see this. The case $m = 0$ follows since $\nabla_{\mathrm{End}}R = 0$. Indeed, then $\mu = \xi \cdot R$ for some smooth ξ and a computation similar to (4.5) gives

$$\bar{\partial}(\xi \cdot R) = \bar{\partial}\xi \cdot R \pm f^*\xi \cdot R.$$

If $m > 0$ we write $\mu = \widetilde{\mathscr{K}}_m \mu'$, where μ' is of the form (4.1) with m replaced by $m-1$. By Proposition 3.8 we get

$$\bar{\partial}\mu = \bar{\partial}\widetilde{\mathscr{K}}_m \mu' = \mu' - \widetilde{\mathscr{K}}_m \bar{\partial}\mu' - \widetilde{\mathscr{P}}_m \mu',$$

and since $\bar{\partial}\mu'$ is a section of $\mathscr{B}_{\mathcal{J}}$ by the induction hypothesis, it follows that $\bar{\partial}\mu$ is a section of $\mathscr{B}_{\mathcal{J}}$.

To see that the inclusion of complexes $(\mathscr{B}_{\mathcal{J}}^{p,\bullet}, \bar{\partial}) \hookrightarrow (\mathscr{C}_{\mathcal{J}}^{p,\bullet}, \bar{\partial})$ is a quasi-isomorphism, consider the diagram

$$\begin{array}{ccc} \mathscr{H}^q\big(\mathcal{O}(E_\bullet^*) \otimes \Omega^p, \, f_\bullet^*\big) & \xrightarrow{\Psi} & \mathscr{H}^q\big(\mathscr{B}_{\mathcal{J}}^{p,\bullet}, \, \bar{\partial}\big) \\ & \searrow & \downarrow \\ & & \mathscr{H}^q\big(\mathscr{C}_{\mathcal{J}}^{p,\bullet}, \, \bar{\partial}\big), \end{array} \qquad (4.6)$$

where the diagonal map is the natural isomorphism and Ψ is the map $\xi \mapsto \xi \cdot R_q$; cf. [6, Section 7]. It follows that Ψ is injective and that the vertical

map is surjective. Let μ be a $\bar\partial$-closed section of $\mathscr{B}_{\mathcal{J}}^{p,q}$ and suppose that there is a $\tau \in \mathscr{C}_{\mathcal{J}}^{p,q-1}$ such that $\mu = \bar\partial\tau$. Then by Propositions 3.8 and 3.5

$$\mu = \bar\partial\widetilde{\mathscr{K}}\mu + \widetilde{\mathscr{P}}\mu = \bar\partial(\widetilde{\mathscr{K}}\mu + \widetilde{\mathscr{P}}\tau).$$

Since $\widetilde{\mathscr{K}}\mu + \widetilde{\mathscr{P}}\tau$ is in $\mathscr{B}_{\mathcal{J}}$ it follows that the vertical map is injective. Thus, the horizontal map must be surjective and so all maps in (4.6) are isomorphisms.

Notice that $\bar\partial\widetilde{\mathscr{P}}\mu = \widetilde{\mathscr{P}}\bar\partial\mu$ for any $\mu \in \mathscr{B}_{\mathcal{J}}$ by Proposition 3.5. Hence, $\widetilde{\mathscr{P}}$ is a map of complexes $(\mathscr{C}_{\mathcal{J}}^{p,\bullet}, \bar\partial) \to (\mathscr{B}_{\mathcal{J}}^{p,\bullet}, \bar\partial)$. Showing that it induces the inverse of the vertical map in (4.6) is similar to the proof of Proposition 3.9: Let $\mu \in \mathscr{C}_{\mathcal{J}}^{p,q}$ be $\bar\partial$-closed; we must show that $\widetilde{\mathscr{P}}\mu$ is a $\bar\partial$-closed section of $\mathscr{B}_{\mathcal{J}}^{p,q}$ such that $\mu - \widetilde{\mathscr{P}}\mu \in \bar\partial\mathscr{C}_{\mathcal{J}}^{p,q-1}$. In view of Proposition 3.5 the first part of this is clear. The vertical map in (4.6) is an isomorphism and so there is a $\bar\partial$-closed $\nu \in \mathscr{B}_{\mathcal{J}}^{p,q}$ and a $\tau \in \mathscr{C}_{\mathcal{J}}^{p,q-1}$ such that (1.1) holds. Then, by Proposition 3.8, $\nu = \widetilde{\mathscr{P}}\nu + \bar\partial\widetilde{\mathscr{K}}\nu$ and so by Proposition 3.5 we get

$$\widetilde{\mathscr{P}}\mu = \widetilde{\mathscr{P}}\nu + \widetilde{\mathscr{P}}\bar\partial\tau = \nu - \bar\partial\widetilde{\mathscr{K}}\nu + \bar\partial\widetilde{\mathscr{P}}\tau$$

by applying $\widetilde{\mathscr{P}}$ to (1.1). Subtracting this from (1.1) we get

$$\mu - \widetilde{\mathscr{P}}\mu = \bar\partial(\tau - \widetilde{\mathscr{P}}\tau + \widetilde{\mathscr{K}}\nu).$$

To conclude the proof of Theorems 4.2 and 4.3 it remains to show that $\mathcal{J}\mu = 0$ for any section μ of $\mathscr{B}_{\mathcal{J}}^{p,q}$. We may assume that μ is of the form (4.1). In view of Section 2.1, the product

$$T_{m,q} := R_{k_m}(z_m) \wedge B_{\ell_m}(z_m, z_{m-1}) \wedge \cdots \wedge B_{\ell_1}(z_1, z_0) \wedge R_q(z_0), \quad k_j + \ell_j \leqslant N,$$

is a well-defined pseudomeromorphic current in $M_{z_0}'' \times \cdots \times M_{z_m}''$; here the R's are currents corresponding to Hermitian resolutions of $\mathscr{O}/\mathcal{J}$ and the B's are Bochner–Martinelli type forms as in Section 2.3; R_{k_j} is a component of the part of the jth R-current of bidegree $(0, k_j)$ and B_{ℓ_j} is the part of the jth Bochner–Martinelli type form of bidegree $(\ell_j, \ell_j - 1)$. In view of Definition 3.7, μ is a sum of push-forwards, under maps $M_{z_0}'' \times \cdots \times M_{z_m}'' \to M_{z_m}''$, of $T_{m,q}$-currents times smooth forms, and therefore it is sufficient to show that $\mathcal{J}T_{m,q} = 0$ where $\mathcal{J} = \mathcal{J}(z_m)$; we will do this by double induction over m and q. Since $T_{0,q} = R_q$ it is clear that $\mathcal{J}T_{0,q} = 0$ for all q. Fix now $m > 0$ and notice that $T_{m,q}$ has bidegree $(\sum_{j=1}^m \ell_j, \sum_{j=1}^m k_j + \ell_j - m + q)$ and that $\sum_{j=1}^m k_j + \ell_j - m + q \leqslant mN - m + q$. From the paragraph preceding Definition 3.7 it follows that the support of $T_{m,q}$ is contained in $Y \times \cdots \times Y$ ($m + 1$ copies). Moreover, if $z_{s+1} \neq z_s$ for some $0 \leqslant s \leqslant m - 1$, then $B_{\ell_{s+1}}(z_{s+1}, z_s)$ is smooth and so $T_{m,q}$ is a smooth form times the tensor product of two currents $T_{s,q}$ and $T_{m-s-1,k_{s+1}}$. By induction over m we have $\mathcal{J}T_{m-s-1,k_{s+1}} = 0$. It follows that the support of $\mathcal{J}T_{m,q}$ is contained in

$\{z_0 = \cdots = z_m\} \cap Y \times \cdots \times Y \simeq Y$, which has codimension $mN + \kappa$ in $M''_{z_0} \times \cdots \times M''_{z_m}$. Thus, by the Dimension principle, $\mathcal{J}T_{m,q} = 0$ if $mN - m + q < mN + \kappa$, i.e., if $q < \kappa + m$. In particular, $\mathcal{J}T_{m,\kappa} = 0$, which is the induction start for showing $\mathcal{J}T_{m,q} = 0$ by induction over q. Let $R = R(z_0) = R_\kappa(z_0) + R_{\kappa+1}(z_0) + \cdots$ be the R-current associated to the Hermitian free resolution $(\mathcal{O}(E_\bullet), f_\bullet)$ as in Section 2.2. From (the proof of) [9, Theorem 4.4] it follows that outside of the set Y_k where f_k does not have optimal rank, there is a smooth $\mathrm{Hom}(E_k, E_{k+1})$-valued $(0,1)$-form $\alpha_k = \alpha_k(z_0)$ such that $R_k = \alpha_k R_{k-1}$. Moreover, since $(\mathcal{O}(E_\bullet), f_\bullet)$ is exact, it follows from the Buchsbaum–Eisenbud criterion that $\mathrm{codim}\, Y_k \geqslant k$, $k \geqslant 1$. Hence, for $z_0 \notin Y_{\kappa+1}$ we have $T_{m,\kappa+1} = \alpha_{\kappa+1}T_{m,\kappa}$ and so the support of $\mathcal{J}T_{m,\kappa+1}$ must be contained in the set where $z_0 \in Y_{\kappa+1}$. Since it also has support contained in $\{z_0 = \cdots = z_m\}$ it must in fact have support contained in $\{z_0 = \cdots = z_m\} \cap Y_{\kappa+1} \times \cdots \times Y_{\kappa+1} \simeq Y_{\kappa+1}$, which has codimension $\geqslant mN + \kappa + 1$ in $M''_{z_0} \times \cdots \times M''_{z_m}$. By the Dimension principle, then, $\mathcal{J}T_{m,\kappa+1} = 0$. Continuing in this way we get that $\mathcal{J}T_{m,q} = 0$ for all q. $\qquad\square$

The proof shows that the map Ψ of (4.6) is an isomorphism. The injectivity followed since the diagonal map in (4.6) is an isomorphism which in turn relies on Malgrange's result that $\mathscr{C}^{p,q}$ is stalk-wise injective. However, both surjectivity and injectivity of Ψ can be showed directly using the methods of this paper. To conclude the paper we sketch how this can be done. For the surjectivity of Ψ, let μ be a $\bar{\partial}$-closed section of $\mathscr{B}^{p,q}_{\mathcal{J}}$. Then, by Propositions 3.8 and 3.5, $\mu = \xi \cdot R_q^0 + \bar{\partial}\widetilde{\mathscr{K}}\mu$, where $f_{q+1}^*\xi = 0$, and so the germ of a section of $\mathscr{H}^q(\mathscr{B}^{p,\bullet}_{\mathcal{J}}, \bar{\partial})$ defined by μ is in the image of Ψ.

For the injectivity we will use a new kind of weight in our integral formulas to see that the map Φ, defined in Proposition 3.5, induces a left inverse of Ψ. Notice that Φ indeed induces a map on cohomology since $\Phi(\bar{\partial}\mu) = f_{q+1}^*\Phi(\mu)$ for sections μ of $\mathscr{B}^{p,q}_{\mathcal{J}}$. With the setup of Example 2.3, consider

$$\check{G}_q^\epsilon := \sum_{\ell=0}^{q} R_q^{\ell,\epsilon}(z)H_q^\ell + \sum_{\ell=0}^{q-1} U_q^{\ell,\epsilon}(z)H_{q-1}^\ell f_q + \sum_{\ell=0}^{q} f_{q+1}(z)U_{q+1}^{\ell,\epsilon}(z)H_q^\ell$$
$$=: R_q^\epsilon(z)H_q + U_q^\epsilon(z)H_{q-1}f_q + f_{q+1}(z)U_{q+1}^\epsilon(z)H_q,$$

which is a smooth section of $\Lambda_\eta \otimes \mathrm{Hom}(E_q^\zeta, E_q^z)$ for any $\epsilon > 0$; notice that $U^\epsilon(z)$ and $R^\epsilon(z)$ here depend on z. One can check that $\check{G}_q^\epsilon$ satisfies the properties of being a weight, with the property $\check{G}_{q,0}^\epsilon(z,z) = 1$ construed as $\check{G}_{q,0}^\epsilon(z,z) = \mathrm{Id}_{E_q}$. Let also $g = g(\zeta, z)$ be a suitable weight such that $\zeta \mapsto g(\zeta, z)$ is holomorphic and $z \mapsto g(\zeta, z)$ has compact support, cf. Example 2.2. Identifying sections of the E_j^*'s with row vectors, sections of the E_j's with column vectors, mappings with matrices, and letting $(\cdot)^*$ denote matrix

transpose as in Section 3, we get for any E_q^*-valued holomorphic p-form ξ, in view of Section 2.3, that

$$\xi^*(\zeta) = \int_z \left(\check{G}^\epsilon \wedge g\right)^*_{N,N} \wedge \xi^*(z)$$

$$= \int_z \left(R_q^\epsilon(z) H_q \wedge g\right)^*_{N,N} \wedge \xi^*(z) + \int_z \left(U_q^\epsilon(z) H_{q-1} f_q \wedge g\right)^*_{N,N} \wedge \xi^*(z)$$

$$+ \int_z \left(f_{q+1}(z) U_{q+1}^\epsilon(z) H_q \wedge g\right)^*_{N,N} \wedge \xi^*(z).$$

Notice that the last integral vanishes if $f_{q+1}^* \xi^* = 0$, and that the second last integral is f_q^*-exact. Since $R = R_\kappa^0 + R_{\kappa+1}^0 + \cdots$, it follows that $R_q^\epsilon(z) H_q \to R_q^0(z) H_q^0$ as $\epsilon \to 0$, and so we see that if $f_{q+1}^* \xi^* = 0$, then

$$\xi^*(\zeta) = \int_z \left(R_q^0(z) H_q^0 \wedge g_{N-q}\right)^* \wedge \xi^*(z) + f_q^* \widetilde{\xi}^* = \Phi(\xi R_q^0)^* + f_q^* \widetilde{\xi}^*,$$

where $\widetilde{\xi}$ is an E_{q-1}^*-valued holomorphic p-form. Hence, Φ induces a left inverse of Ψ.

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