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Strongly outer actions of amenable groups on $\mathcal{Z}$-stable nuclear C*-algebras [☆]

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ABSTRACT

Let A be a separable, unital, simple, $\mathcal{Z}$-stable, nuclear C^*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a discrete, countable, amenable group. Suppose that the orbits of the action of G on $T(A)$ are finite and that their cardinality is bounded. We show that the following are equivalent:

(1) α is strongly outer;
(2) $\alpha \otimes \mathrm{id}_{\mathcal{Z}}$ has the weak tracial Rokhlin property.

If G is moreover residually finite, the above conditions are also equivalent to

(3) $\alpha \otimes \mathrm{id}_{\mathcal{Z}}$ has finite Rokhlin dimension (in fact, at most 2).

If $\partial_e T(A)$ is furthermore compact, has finite covering dimension, and the orbit space $\partial_e T(A)/G$ is Hausdorff, we generalize results by Matui and Sato to show that α is cocycle conjugate to $\alpha \otimes \mathrm{id}_{\mathcal{Z}}$, even if α is not strongly outer. In particular, in this case the equivalences above hold for α in place of $\alpha \otimes \mathrm{id}_{\mathcal{Z}}$. In the course of the proof, we develop equivariant versions of complemented partitions of unity and uniform property Γ as technical tools of independent interest.

RÉSUMÉ

Soit A une C*-algèbre nucléaire, unifère, simple, $\mathcal{Z}$-stable, et soit $\alpha\colon G \to \mathrm{Aut}(A)$ une action d'un groupe discret, dénombrable et moyennable. Supposons que les orbites de l'action de G sur $T(A)$ soient finies et que leur cardinalité soit bornée. Nous montrons que les énoncés suivants sont équivalents :

(1) α est fortement extérieur ;
(2) $\alpha \otimes \mathrm{id}_{\mathcal{Z}}$ possède la propriété traciale faible de Rokhlin.

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Si en plus G est résiduellement fini, les conditions ci-dessus sont également équivalentes à :

(3) $\alpha \otimes \mathrm{id}_{\mathcal{Z}}$ a dimension de Rokhlin finie.

Si en plus $\partial_e T(A)$ est compact et de dimension finie, et l'espace des orbites $\partial_e T(A)/G$ est Hausdorff, nous généralisons les résultats de Matui et Sato pour montrer que α est conjugué par cocycles à $\alpha \otimes \mathrm{id}_{\mathcal{Z}}$, même si α n'est pas fortement exterieur. En particulier, dans ce cas, les équivalences ci-dessus sont valables pour α à la place de $\alpha \otimes \mathrm{id}_{\mathcal{Z}}$. Au cours de la preuve, nous développons des versions équivariantes des partitions complétées de l'unité et de la propriété uniforme Γ comme outils techniques d'intérêt indépendant.

Contents

1. Introduction

This paper concerns the structure of group actions by discrete, countable, amenable groups on separable, simple, unital, nuclear, $\mathcal{Z}$-stable C^*-algebras. One of the main themes of research in operator algebra theory in the past several decades has been the Elliott program to classify simple, nuclear, separable C^*-algebras by K-theoretic data. The Elliott program is now essentially complete: simple nuclear separable C^*-algebras which satisfy the Universal Coefficient Theorem and absorb the Jiang-Su algebra $\mathcal{Z}$ ([39]) are classified via the Elliott invariant. Moreover, this classification cannot be extended to the non-$\mathcal{Z}$-stable case without enlarging the invariant and without significant new ideas. The property of $\mathcal{Z}$-stability is a regularity condition for simple C^*-algebras, analogous to the McDuff property for type II_1 factors. We refer the reader to [72] for a recent survey and further references concerning the classification program and Toms-Winter regularity, as a detailed exposition of these topics is beyond the scope of this paper.

The analysis of group actions on operator algebras is a natural and important line of research which has been studied intensively for C^*-algebras as well as in von Neumann algebra theory. It is closely related to the classification program discussed above, particularly via the crossed product construction. Specifically, given the role of $\mathcal{Z}$-stability in the Elliott program (or the related regularity properties in the Toms-Winter conjecture), it is important to understand how robust the class of simple, nuclear, separable, $\mathcal{Z}$-stable C^*-algebras is, in particular with respect to standard constructions such as crossed products. The following is an important open problem in this context.

Problem 1.1. Let A be a separable, simple, nuclear, $\mathcal{Z}$-stable C^*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a discrete countable amenable group G. Find conditions that ensure that $A \rtimes_\alpha G$ is also separable, simple, nuclear, and $\mathcal{Z}$-stable.

In the above setting, nuclearity and separability are always guaranteed since G is amenable and countable. By a celebrated result of Kishimoto [40], $A \rtimes_\alpha G$ is simple whenever α_g is outer for all $g \in G \setminus \{1\}$. The main task, then, is to find general conditions that entail preservation of $\mathcal{Z}$-stability.

The Rokhlin property and its various generalizations form a collection of regularity conditions for group actions on C^*-algebras, whose roots stem from the Rokhlin Lemma in Ergodic Theory. This result was first extended to von Neumann algebras, starting with Connes' work for the case of a single automorphism ([9, Theorem 1.2.5]; see also [70, Chapter XVII, Lemma 2.3]). Connes' result was later generalized by Ocneanu in [55, Section 6.1], and used to prove that all outer actions of a discrete countable amenable group G on the hyperfinite II_1 factor are cocycle conjugate. In the context of actions on C^*-algebras, early works include the studies of cyclic and finite group actions on UHF-algebras by Herman and Jones ([28,29]), and Herman and Ocneanu ([30]), and later for automorphisms on UHF and $\mathrm{A}\mathbb{T}$-algebras by Kishimoto ([41–43]). A connection with permanence of $\mathcal{Z}$-stability was made in [33], where it was shown that for actions of the integers, the reals or compact groups, $\mathcal{Z}$-stability is preserved when passing to the crossed product, provided the action has the Rokhlin property. Other results for Rokhlin actions of compact groups can be found in [18,20,21].

Although the Rokhlin property is relatively common for actions of the integers, there are significant K-theoretic obstructions for finite group actions (and hence actions of groups which have torsion). This was studied in depth by Izumi [37,38] and spurred additional work [61,27]. Attempts to circumvent impediments of this sort led Phillips to introduce the *tracial Rokhlin property* [58], where the projections in the Rokhlin property are assumed to have a leftover which is small in trace. Among other applications, the tracial Rokhlin property has been used in [13] to study fixed point algebras of the irrational rotation algebra A_θ under certain canonical actions of finite cyclic groups.

The tracial Rokhlin property does not bypass the most obvious obstruction to admitting Rokhlin actions: the existence of nontrivial projections. The need to study weaker versions of these properties led to two further generalizations. The first one, called the *weak tracial Rokhlin property*, which replaces projections with positive elements, has been considered in [31,62,51,53,71,23].

A different approach was taken in a paper by the second author, Winter, and Zacharias [34], who introduced the notion of *Rokhlin dimension*. In this formulation, the partition of unity appearing in the Rokhlin property is replaced by a multi-tower partition of unity consisting of positive contractions, the elements of each tower being indexed by the group elements and permuted by the group action. Rokhlin dimension zero then corresponds to the Rokhlin property, but the extra flexibility makes finiteness of the Rokhlin dimension a much more common feature. This notion has primarily been used as a tool to show that various structural properties of interest (such as $\mathcal{Z}$-stability or finite nuclear dimension) pass from an algebra to the crossed product. Rokhlin dimension has been extended and studied for actions of various classes of groups; the generalization which is pertinent for this paper is in work of Szabó, Wu, and Zacharias for residually finite groups ([68]). We refer the reader to [17,19,32,35,24] for further generalizations.

This work focuses on actions on simple C^*-algebras, and aims to improve upon related works by Matui-Sato ([51,53]) and by Liao ([48,49]). We study the relationships between strong outerness, the weak tracial Rokhlin property and finite Rokhlin dimension by showing that they are equivalent in many cases of interest. More specifically, we obtain the following main results.

Theorem A. *Let A be a separable, simple, nuclear, unital, stably finite, C^*-algebra, let G be a countable, discrete, amenable group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Suppose that the orbits of the action induced by α on $T(A)$ are finite and that their cardinality is uniformly bounded. Then the following are equivalent:*

(1) α is strongly outer.
(2) $\alpha \otimes \mathrm{id}_{\mathcal{Z}}$ has the weak tracial Rokhlin property.

When G is residually finite, then the above are also equivalent to:

(3) $\alpha \otimes \mathrm{id}_{\mathcal{Z}}$ has finite Rokhlin dimension.

(4) $\alpha \otimes \mathrm{id}_{\mathcal{Z}}$ *has Rokhlin dimension at most 2.*

This theorem is restated as Theorem 7.8 and proved in Section 7. The definitions of strong outerness, the weak tracial Rokhlin property, and Rokhlin dimension are provided in Section 2.

The first precursor of this result is Theorem 5.5 in [13], where (1) ⇔ (2) is shown under the additional assumptions that A has tracial rank zero, A has a unique tracial state, and G is finite. Matui and Sato proved (1) ⇔ (2) in the case that A is nuclear and has finitely many extreme tracial states, and the group G is elementary amenable ([53, Theorem 3.6]). This was later extended by Wang to all discrete countable amenable groups in [71, Theorem 3.8]. Here, we remove all the smallness assumptions on the extreme points of $T(A)$, although we still have some non-trivial requirements on the size of the orbits for the induced action on $T(A)$. This is made possible by developing an equivariant version of complemented partitions of unity from [6]; see more on this below.

The equivalence (1) ⇔ (3) of Theorem A generalizes Liao's work in [48,49], where a similar result is proved for $\mathbb{Z}^m$-actions, under the additional assumptions that $T(A)$ is a Bauer simplex with finite dimensional extreme boundary, and that the group acts trivially in $T(A)$. Our method of proof differs significantly from Liao's (other than in the use of complemented partitions of unity to remove the topological assumptions on $\partial_e T(A)$), in that we obtain the Rokhlin towers by embedding suitable model actions on dimension drop algebras into the central sequence algebra of A. The advantage of our approach is that it does not require any restrictions on the group: in particular, we are able to treat groups with torsion, as well as groups that are not finitely generated. (The application of property (SI) in [48, Theorem 6.4] makes essential use of the fact that $\mathbb{Z}$ has no torsion, and the same applies for the generalization to $\mathbb{Z}^m$ in [49].)

Our next main theorem improves upon results of Sato from [63] (which in turn generalizes results from [53]). The main theorem in [63] requires for the group to act trivially on the trace space, whereas we can weaken this assumption to a condition on the orbits analogous to the one in Theorem A.

Theorem B *(Theorem 8.5). Let A be a separable, simple, nuclear, stably finite, $\mathcal{Z}$-stable, infinite-dimensional, unital C^*-algebra. Let G be a countable, discrete, amenable group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Suppose that $\partial_e T(A)$ is compact, that $\dim(\partial_e T(A)) < \infty$, that the orbits of the induced action of G on $\partial_e T(A)$ are finite with uniformly bounded cardinality, and that the orbit space $\partial_e T(A)/G$ is Hausdorff. Then α is cocycle conjugate to $\alpha \otimes \mathrm{id}_{\mathcal{Z}}$.*

The conditions on the trace space are met, for instance, when the G-action induced by α on $\partial_e T(A)$ factors through a finite group action. We do not know whether the restriction on the topology of the trace space or the way in which the group acts on it can be relaxed. Some progress has been made in the recent paper [73], where the methods here developed are combined with arguments from topological dynamics to show equivariant $\mathcal{Z}$-stability for actions of $\mathbb{Z}$, assuming that $\partial_e T(A)$ is compact and finite-dimensional.

The arguments in this paper make an essential use of an equivariant version of uniform property Γ and complemented partitions of unity (CPoU). The non-equivariant versions are methods introduced in [6] in order to prove one of the remaining implications of the Toms-Winter conjecture, and were further developed in [7]. Roughly speaking, complemented partitions of unity provide a technique for globalizing properties which occur fiber-wise in the von Neumann algebras associated to the GNS representations of the traces, to properties holding uniformly over all traces; we adapt this method to allow gluing dynamical properties. As this is of independent interest, we record here what is proved in this respect, which can be thought of as a dynamical analogue of [7, Theorem 4.6]. The definitions of the terms in the theorem below appear in the relevant parts of the paper.

Theorem C. *Let A be a separable, simple, nuclear, unital, stably finite, C^*-algebra with no finite-dimensional quotients. Let G be a countable, discrete, amenable group and let $\alpha\colon G \to Aut(A)$ be an action such that the* *induced action on $T(A)$ has finite orbits bounded in size by a uniform constant $M > 0$. Then the following are equivalent:*

(1) (A, α) has uniform property Γ.
(2) (A, α) has complemented partitions of unity with constant M.
(3) For every $n \in \mathbb{N}$ there is a unital embedding of the matrix algebra $M_n \to (A^{\mathcal{U}} \cap A')^{\alpha^{\mathcal{U}}}$.

If A is also $\mathcal{Z}$-stable and simple, then the previous conditions are equivalent to

(4) (A, α) is cocycle conjugate to $(A \otimes \mathcal{Z}, \alpha \otimes \mathrm{id}_{\mathcal{Z}})$.

Theorem C is the combination of Theorem 5.7, which covers the equivalence of the first three conditions, and Theorem 7.6, which adds the last condition.

The paper is organized as follows. In Section 3 and Section 4 we modify ideas from [6] in order to develop equivariant analogues of uniform property Γ and complemented partitions of unity. The main goal in Section 5 is to use techniques from the previous two sections to obtain that, under our assumptions, we can equivariantly embed model actions on the hyperfinite II_1-factor into the central sequence algebra of uniform-tracial ultrapowers. Section 6 is a technical section, devoted to constructing model actions of finite groups which have Rokhlin-type towers on dimension drop algebras; those are needed in order to lift actions from the uniform-tracial central sequence algebra to the norm central sequence algebra. Section 7 contains the proofs of Theorem A and Theorem C. The last section provides a proof of Theorem B.

2. Preliminaries

2.1. Trace norms

For a unital C^*-algebra A, the *trace space* of A, denoted $T(A)$, is the compact subspace of the dual of A (endowed with the weak* topology) consisting of all states τ of A such that $\tau(ab) = \tau(ba)$ for all $a, b \in A$.

We say that a trace $\tau \in T(A)$ is *faithful* if $\tau(a^*a) > 0$ for all $a \in A \setminus \{0\}$. Given a trace $\tau \in T(A)$, the associated *trace seminorm* $\|\cdot\|_{2,\tau}$ on A is given by

$$\|a\|_{2,\tau} = \tau(a^*a)^{1/2},$$

for all $a \in A$. For a closed subset $T \subseteq T(A)$, we set $\|\cdot\|_{2,T} = \sup_{\tau\in T} \|\cdot\|_{2,\tau}$, which is a seminorm on A. We use the abbreviation $\|\cdot\|_{2,u}$ for $\|\cdot\|_{2,T(A)}$. Notice that $\|\cdot\|_{2,T}$ is in fact a norm if T contains at least one faithful trace. This is always the case, for instance, when A is simple and admits a trace.

Let G be a discrete group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Then α naturally induces an action α^* of G on $T(A)$ by affine homeomorphisms,[1] given by $\alpha^*_g(\tau) = \tau \circ \alpha_{g^{-1}}$ for all $g \in G$ and all $\tau \in T(A)$. We say that a trace $\tau \in T(A)$ is α-invariant if $\tau \circ \alpha_g = \tau$ for all $g \in G$, and we denote by $T(A)^\alpha \subseteq T(A)$ the space of α-invariant traces. Note that $T(A)^\alpha$ is always non-empty if G is amenable ([60, Theorem 1.3.1]). Given $\tau \in T(A)$ such that the orbit $G \cdot \tau$ is finite, we set

$$\tau^\alpha := \frac{1}{|G \cdot \tau|} \sum_{\sigma \in G\cdot\tau} \sigma.$$

If $T \subseteq T(A)$ is G-invariant, then α_g is isometric with respect to $\|\cdot\|_{2,T}$ for all $g \in G$.

[1] The action α^* is the restriction of the dual of α to $T(A) \subseteq A^*$, hence the notation.

2.2. Ultrapowers

Let A be a C^*-algebra, let G be a discrete group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. We denote by $\ell^\infty(A)$ the C^*-algebra of all bounded sequences in A with the supremum norm, endowed with the G-action given by pointwise application of α. For a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ (which we will fix throughout), set

$$c_{\mathcal{U}}(A) = \{(a_n) \in \ell^\infty(A)\colon \lim_{n\to\mathcal{U}} \|a_n\| = 0\}.$$

This is a closed, two-sided G-invariant ideal in $\ell^\infty(A)$. We define the *norm ultrapower* of A to be the quotient $A_{\mathcal{U}} = \ell^\infty(A)/c_{\mathcal{U}}(A)$, and denote by $\alpha_{\mathcal{U}}\colon G \to \mathrm{Aut}(A_{\mathcal{U}})$ the induced action. We denote by $\pi_A\colon \ell^\infty(A) \to A_{\mathcal{U}}$ the equivariant quotient map.

Given a closed subset $T \subseteq T(A)$, we set

$$J_T = \{(a_n)_{n\in\mathbb{N}} \in A_{\mathcal{U}}\colon \lim_{n\to\mathcal{U}} \|a_n\|_{2,T} = 0\}.$$

Then J_T is a closed, two-sided, ideal in $A_{\mathcal{U}}$, and it is G-invariant if T is. If $\tau \in T(A)$, we abbreviate $J_{\{\tau\}}$ to J_τ. For $T = T(A)$, we abbreviate $J_{T(A)}$ to J_A, and call it the *trace kernel ideal.* The associated quotient

$$A^{\mathcal{U}} = A_{\mathcal{U}}/J_A$$

is called the *uniform tracial ultrapower* of A. We denote by $\alpha^{\mathcal{U}}\colon G \to \mathrm{Aut}(A^{\mathcal{U}})$ the induced action, and by $\kappa_A\colon A_{\mathcal{U}} \to A^{\mathcal{U}}$ the equivariant quotient map. We abbreviate κ_A to κ whenever the algebra A is clear from the context. Given a subset $S \subseteq A_{\mathcal{U}}$, the commutant of S in $A_{\mathcal{U}}$ is denoted by $A_{\mathcal{U}} \cap S'$, and we use similar notation for subsets of $A^{\mathcal{U}}$. The following useful fact will be used repeatedly; see [45, Proposition 4.5, Proposition 4.6]:

Lemma 2.1. *Let A be a separable C^*-algebra, let G be a discrete group, let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action, let $S \subseteq A_{\mathcal{U}}$ be a separable G-invariant subset, and set $\overline{S} = \kappa(S)$. Then κ restricts to a surjective, equivariant map*

$$\kappa\colon (A_{\mathcal{U}} \cap S', \alpha_{\mathcal{U}}) \to (A^{\mathcal{U}} \cap \overline{S}', \alpha^{\mathcal{U}}).$$

Like the norm ultrapower, the uniform tracial ultrapower of a C^*-algebra A satisfies countable saturation properties that allow us, via reindexing and diagonal arguments, to derive exact statements in $A^{\mathcal{U}}$ from approximations in $\|\cdot\|_{2,T_{\mathcal{U}}(A)}$ or in $\|\cdot\|_{2,T(A)}$. For future reference, we isolate this in the following remark.

Remark 2.2. The notion of saturation is a fundamental and classical concept from model theory, which has also been formalized for C^*-algebras (see [16, Section 4.3], and see [26] for an extension to the equivariant setting). Among operator algebraists, all instances of saturation in the context of ultrapowers are usually reduced to an application of a technical lemma known as Kirchberg's ε-test [44, Lemma A.1]. We also refer to this technical tool when invoking 'countable saturation' in our proofs.

Given a sequence $(\tau_n)_{n\in\mathbb{N}}$ in $T(A)$, there is a trace $\tau \in T(A_{\mathcal{U}})$ given by $\tau(a) = \lim_{n\to\mathcal{U}} \tau_n(a_n)$ whenever $(a_n)_{n\in\mathbb{N}} \in \ell^\infty(A)$ is a representing sequence for a. We call traces of this form *limit traces*, and denote by $T_{\mathcal{U}}(A)$ the set of all limit traces on $A_{\mathcal{U}}$. Since any limit trace vanishes on J_A, with a slight abuse of notation we also regard the elements in $T_{\mathcal{U}}(A)$ as traces over $A^{\mathcal{U}}$. We denote by $T_{\mathcal{U}}^{\alpha}(A)$ the set of all traces in $T_{\mathcal{U}}(A)$ which arise from sequences of traces in $T(A)^\alpha$. (This set should not be confused with the set $T_{\mathcal{U}}(A)^{\alpha_{\mathcal{U}}}$ of G-invariant elements of $T_{\mathcal{U}}(A)$, which may a-priori be larger.)

A straightforward computation shows that for $(a_n)_{n\in\mathbb{N}}$ in $\ell^\infty(A)$ with corresponding class $a \in A_{\mathcal{U}}$, we have $\|a\|_{2,T_{\mathcal{U}}(A)} = 0$ if and only if $\lim_{n\to\mathcal{U}} \|a_n\|_{2,u} = 0$. In particular, this shows that

$$J_A = \{a \in A_{\mathcal{U}} \colon \|a\|_{2,T_{\mathcal{U}}(A)} = 0\}.$$

The ideals $J_{T(A)}$ and $J_{T(A)^\alpha}$ are not equal in general, even if G is amenable. (Take, for example, $A = C(S^1)$ and $G = \mathbb{Z}$ acting on it via irrational rotations.) While the tools that we develop in this work are suitable for studying the quotient $A_{\mathcal{U}}/J_{T(A)^\alpha}$, the kind of conclusions that we are interested in refer to the quotient $A_{\mathcal{U}}/J_A = A^{\mathcal{U}}$. In general, it is not clear how to transfer information from one to the other. The assumptions in our main results concerning the size of the orbits of α^* are used to do this.

Proposition 2.3. *Let A be a C^*-algebra such that $T(A)$ is nonempty, let G be a discrete group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Suppose that the cardinality of the orbits of α^* is uniformly bounded. Then $\|\cdot\|_{2,T(A)^\alpha}$ and $\|\cdot\|_{2,u}$ are equivalent, and in particular $J_A = J_{T(A)^\alpha}$.*

Proof. Given $\tau \in T(A)$, set $\tau^\alpha = \frac{1}{|G\cdot\tau|}\sum_{\sigma\in G\cdot\tau}\sigma$. One readily checks that $\tau^\alpha \in T(A)^\alpha$, and that $\tau(a) \leq |G\cdot\tau|\tau^\alpha(a)$ for all $a \in A_+$. Let $M > 0$ be a uniform bound for the orbits of α^*. Given for $a \in A$, we have

$$\|a\|_{2,T(A)^\alpha} \leq \|a\|_{2,u} \leq M^{1/2}\|a\|_{2,T(A)^\alpha}. \quad \square$$

Given a C^*-algebra B and a G-action $\gamma\colon G \to \mathrm{Aut}(B)$, we write B^γ for the fixed point algebra of γ. The following simple lemma follows from a straightforward reindexation argument, which we omit.

Lemma 2.4. *Let A and B be a separable unital C^*-algebra, let G be a countable discrete group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Suppose there exists a unital homomorphism $\varphi\colon B \to (A_{\mathcal{U}} \cap A')^{\alpha_{\mathcal{U}}}$. Then for any separable subset $S \subseteq (A_{\mathcal{U}})^{\alpha_{\mathcal{U}}}$ there exists a unital homomorphism $\psi\colon B \to (A_{\mathcal{U}} \cap S')^{\alpha_{\mathcal{U}}}$.*

2.3. W^*-ultrapowers

We will also need to use tracial ultrapowers for von Neumann algebras, so we recall this notion as well. Let $(\mathcal{M}, \tau)$ be a tracial von Neumann algebra, and set

$$c_{\mathcal{U},\tau} = \{(a_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathcal{M}) \colon \lim_{n\to\mathcal{U}} \|a_n\|_{2,\tau} = 0\}.$$

We denote by $\mathcal{M}^{\mathcal{U}}$ the quotient $\mathcal{M}^{\mathcal{U}} = \ell^\infty(\mathcal{M})/c_{\mathcal{U},\tau}$, and call it the *$W^*$-ultrapower* of $(\mathcal{M}, \tau)$.

There is unfortunately a notational conflict, since the notation $\mathcal{M}^{\mathcal{U}}$ could mean both the W^*-ultrapower of $(\mathcal{M}, \tau)$, or the uniform tracial ultrapower of $\mathcal{M}$ regarded as a C^*-algebra. Both notations are by now well established in the literature, and we will always make it clear which one we are referring to. In practice, little confusion should arise since we will never consider the uniform tracial ultrapower of a von Neumann algebra.

Denote by π_τ the GNS representation associated to τ. Most tracial von Neumann algebras we deal with in this note are of the form $\pi_\tau(A)''$ for some separable, simple, unital C^*-algebra A and some trace $\tau \in T(A)$. The canonical and implicit choice of faithful, normal trace on $\pi_\tau(A)''$ is (the unique tracial extension of) τ. We will often use the notation $\mathcal{M}_\tau$ to abbreviate $\pi_\tau(A)''$.

2.4. Strong outerness

Let A be a C^*-algebra, and let $\tau \in T(A)$ be a trace. Note that if $\theta \in \mathrm{Aut}(A)$ and τ is left invariant by θ, then θ extends uniquely to a trace-preserving automorphism of $\pi_\tau(A)''$, which we denote by θ^τ.

Definition 2.5. Let A be a simple, unital C^*-algebra with nonempty trace space, and let $\theta \in \mathrm{Aut}(A)$ be an automorphism. We say that θ is *strongly outer* if θ^τ is outer for every $\tau \in T(A)$ satisfying $\tau \circ \theta = \tau$.

An action $\alpha\colon G \to \mathrm{Aut}(A)$ of a discrete group G on A is said to be *strongly outer* if α_g is strongly outer for all $g \in G \setminus \{1\}$.

A strongly outer automorphism is clearly outer. The reverse implication is, however, false. For example, let $A = \bigotimes_{n=1}^{\infty} M_{2^n}$ and let θ be the approximately inner order 2 automorphism of A given as the limit of $\mathrm{Ad}(u_k)$, where $u_k = v_1 \otimes v_2 \otimes \ldots \otimes v_k \otimes 1_{\bigotimes_{n=k+1}^{\infty} M_{2^n}}$ and $v_j = \mathrm{diag}(-1, 1, 1, 1, \ldots, 1)$. One can check that the sequence u_k is Cauchy in the trace norm, and therefore θ^τ is inner, although θ is not inner (see [50]).

2.5. Rokhlin dimension

The notion of Rokhlin dimension was first defined for actions of $\mathbb{Z}$ and finite groups in [34], and later extended to residually finite groups in [68]. A group G is *residually finite* if for every $g \in G \setminus \{e\}$ there is a normal subgroup $H \leq G$ of finite index such that $g \notin H$.

Definition 2.6 *([68, Definition 4.4]).* Let G be a countable, discrete, residually finite group, let A be a separable, unital C^*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Given $d \in \mathbb{N}$, we say that α has *Rokhlin dimension at most d*, written $\dim_{\mathrm{Rok}}(\alpha) \leq d$, if for any normal subgroup $H \leq G$ of finite index, there are positive contractions $f_{\overline{g}}^{(j)} \in A_{\mathcal{U}} \cap A'$, for $j = 0, \ldots, d$ and $\overline{g} \in G/H$, such that:

(1) $(\alpha_{\mathcal{U}})_g(f_{\overline{h}}^{(j)}) = f_{\overline{gh}}^{(j)}$ for all $j = 0, \ldots, d$ and $g \in G$ and $\overline{h} \in G/H$,
(2) $f_{\overline{g}}^{(j)} f_{\overline{h}}^{(j)} = 0$ for all $j = 0, \ldots, d$ and $\overline{g}, \overline{h} \in G/H$ with $\overline{g} \neq \overline{h}$,
(3) $\sum_{j=0}^{d} \sum_{\overline{g} \in G/H} f_{\overline{g}}^{(j)} = 1$.

There is a related notion, called *Rokhlin dimension with commuting towers*, where the elements $f_{\overline{g}}^{(j)}$ are assumed to moreover pairwise commute (see [34, Definition 2.3.b] and [68, Definition 9.2]). We will not deal with this notion here.

We record here an equivalent definition of Rokhlin dimension, which uses approximations instead of ultrapowers.

Proposition 2.7 *([68, Proposition 4.5]). Using the notation from Definition 2.6, we have $\dim_{\mathrm{Rok}}(\alpha) \leq d$ if and only if for any normal subgroup $H \leq G$ of finite index, for any finite subset $G_0 \subseteq G$, for every $\varepsilon > 0$ and for every finite subset $F \subseteq A$, there are positive contractions $f_{\overline{g}}^{(j)} \in A$, for $j = 0, \ldots, d$ and for $\overline{g} \in G/H$, satisfying:*

(a) $\|\alpha_g(f_{\overline{h}}^{(j)}) - f_{\overline{gh}}^{(j)}\| < \varepsilon$ *for all* $j = 0, \ldots, d$, *for all* $g \in G_0$ *and for all* $\overline{h} \in G/H$,
(b) $\|f_{\overline{g}}^{(j)} f_{\overline{h}}^{(j)}\| < \varepsilon$ *for all* $j = 0, \ldots, d$ *and for all* $\overline{g}, \overline{h} \in G/H$ *with* $\overline{g} \neq \overline{h}$,
(c) $\|1 - \sum_{j=0}^{d} \sum_{\overline{g} \in G/H} f_{\overline{g}}^{(j)}\| < \varepsilon$,
(d) $\|a f_{\overline{g}}^{(j)} - f_{\overline{g}}^{(j)} a\| < \varepsilon$ *for all* $a \in F$, *for all* $\overline{g} \in G/H$ *and for all* $j = 0, \ldots, d$.

2.6. The weak tracial Rokhlin property

We begin by recalling some terminology.

Definition 2.8. Let G be a discrete group, and let $\delta > 0$. Given finite subsets $K, S \subseteq G$, we say that S is *(K, δ)-invariant* if

$$\Big|S \cap \bigcap_{g\in K} gS\Big| \geq (1-\delta)|S|.$$

The existence of (K,δ)-invariant subsets of G for every finite set K and $\delta > 0$ is Følner's characterization of amenability.

The next definition goes back to Ocneanu's notion of the Rokhlin property for actions of amenable groups on von Neumann algebras and specifically on the hyperfinite II_1-factor; see [55, Chapter 6].

Definition 2.9. Let G be a discrete, amenable group, let A be a simple, separable unital C^*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. We say that α has the *weak tracial Rokhlin property* if for any finite subset $K \subseteq G$ and any $\delta > 0$, there are $n \in \mathbb{N}$, (K,δ)-invariant finite subsets $S_1, \ldots, S_n \subseteq G$, and positive contractions $f_{\ell,g} \in A_{\mathcal{U}} \cap A'$ for $\ell = 1,\ldots,n$ and $g \in S_\ell$, such that

(1) $(\alpha_{\mathcal{U}})_{gh^{-1}}(f_{\ell,h}) = f_{\ell,g}$ for all $\ell = 1,\ldots,n$ and all $g,h \in S_\ell$,
(2) $f_{\ell,g}f_{k,h} = 0$ for all $\ell,k = 1,\ldots,n$, $g \in S_\ell$, $k \in S_h$, whenever $(\ell,g) \neq (k,h)$,
(3) $1 - \sum_{\ell=1}^{n}\sum_{g\in S_\ell} f_{\ell,g} \in J_A$,
(4) for $\tau \in T_{\mathcal{U}}(A)$, for $\ell = 1,\ldots,n$ and for $g \in S_\ell$, the value of $\tau(f_{\ell,g})$ is independent of τ and g, and is positive.

Definition 2.9 is inspired by Wang's [71, Proposition 2.4], except for item (4), which is inspired by Matui and Sato's [53, Definition 2.5.3]. In particular, our definition of the weak tracial Rokhlin property extends that of Matui-Sato to groups that are not necessarily monotileable.

Remark 2.10. Condition (4) in Definition 2.9 was used in a predecessor of this paper ([22]) to prove that actions with the weak tracial Rokhlin property have equivariant property (SI) whenever the underlying algebra has property (SI), which is needed in the proof of implication (1) ⇒ (3) of Theorem C. In this paper, we rely on the more general results from [67]; thus item (4) above is no longer used to prove the other implications. Nevertheless, we carry out the proof of (1) ⇒ (2) in Theorem C so as to obtain Rokhlin towers also satisfying condition (4), as we believe that such a stronger condition might prove to be useful in future applications.

3. Equivariant uniform property Γ

In the theory of von Neumann algebras, property Γ was originally introduced by Murray and von Neumann ([54]) in order prove the existence of non-hyperfinite II_1-factors. A II_1-factor $\mathcal{M}$ with trace τ has property Γ if its central sequence algebra $\mathcal{M}^{\mathcal{U}} \cap \mathcal{M}'$ is non-trivial. Dixmier later showed that in this case property Γ is equivalent to the requirement that the II_1-factor $\mathcal{M}^{\mathcal{U}} \cap \mathcal{M}'$ is *diffuse* ([12]), that is, for every $n \in \mathbb{N}$ there are orthogonal projections $p_1,\ldots,p_n \in \mathcal{M}^{\mathcal{U}} \cap \mathcal{M}'$ such that $\tau_{\mathcal{M}^{\mathcal{U}}}(p_i) = 1/n$. This latter formulation of property Γ inspired an analogous definition for uniform tracial ultrapowers, which has been recently introduced in [6, Definition 2.1] and systematically studied in [7].

In the next two sections we borrow some of the main ideas in [6] and [7], and adapt them to the equivariant setting. We work with actions of discrete countable groups; for some statements, the group will be assumed to be amenable as well. We start with the definition of uniform property Γ for actions of countable discrete groups on unital separable C^*-algebras.

Definition 3.1. Let G be a countable, discrete group, let A be a unital, separable C^*-algebra with non-empty trace space, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. We say that (A,α) has *uniform property* Γ if for every $n \in \mathbb{N}$ and every $\|\cdot\|_{2,T_{\mathcal{U}}(A)}$-separable subset $S \subseteq A^{\mathcal{U}}$, there are projections $p_1,\ldots,p_n \in (A^{\mathcal{U}} \cap S')^{\alpha^{\mathcal{U}}}$ with

(1) $\sum_{j=1}^{n} p_j = 1$,
(2) $\tau(ap_j) = \frac{1}{n}\tau(a)$ for all $a \in S$, all $\tau \in T_{\mathcal{U}}(A)$ and all $j = 1, \ldots, n$.

When α is the trivial action the definition above coincides with the definition of uniform property Γ as given in [6, Definition 2.1].

Remark 3.2. In Definition 3.1, we could have equivalently required that the projections $p_1, \ldots, p_n$ belong to $(A^{\mathcal{U}} \cap A')^{\alpha^{\mathcal{U}}}$ and only require that condition (2) holds for all $a \in A$; this follows from separability of the sets S considered in Definition 3.1, along with a standard diagonal argument (Lemma 2.4).

In [6, Proposition 2.3], it is shown that unital separable $\mathcal{Z}$-stable C^*-algebras with non-empty trace space have uniform property Γ, since it is possible to embed matrix algebras of arbitrary dimension into the central sequence algebra of the uniform tracial ultrapower. A modification of that argument allows us to show that equivariantly $\mathcal{Z}$-stable actions of discrete countable groups on such algebras have uniform property Γ.

Proposition 3.3. *Let G be a countable, discrete group and let A be a separable, unital, $\mathcal{Z}$-stable C^*-algebra with non-empty trace space. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action and suppose that (A, α) is cocycle conjugate to $(A \otimes \mathcal{Z}, \alpha \otimes id_{\mathcal{Z}})$. Then for any $\|\cdot\|_{2,T_{\mathcal{U}}(A)}$-separable subset $S \subseteq A^{\mathcal{U}}$ and every $n \in \mathbb{N}$, there exists a unital $*$-homomorphism $M_n \to (A^{\mathcal{U}} \cap S')^{\alpha^{\mathcal{U}}}$. In particular (A, α) has uniform property Γ.*

Proof. By Lemma 2.4, we can assume that $S = A$. By Theorem 3.7 in [66], because α is cocycle conjugate to $\alpha \otimes \mathrm{id}_{\mathcal{Z}}$, there is an equivariant embedding $\varphi\colon (\mathcal{Z}, \mathrm{id}_{\mathcal{Z}}) \to (A_{\mathcal{U}} \cap A', \alpha_{\mathcal{U}})$. Note that the image of φ is contained in the fixed point algebra $(A_{\mathcal{U}} \cap A')^{\alpha_{\mathcal{U}}}$. With $\kappa\colon A_{\mathcal{U}} \to A^{\mathcal{U}}$ denoting the canonical equivariant quotient map (see Lemma 2.1), it follows by simplicity of $\mathcal{Z}$ that $\kappa \circ \varphi\colon \mathcal{Z} \to (A^{\mathcal{U}} \cap A')^{\alpha^{\mathcal{U}}}$ is a unital embedding. Since $\mathcal{Z}$ has a unique trace $\tau_{\mathcal{Z}}$, we have for all $z \in \mathcal{Z}$:

$$\|z\|_{2,\tau_{\mathcal{Z}}} = \|\kappa(\varphi(z))\|_{2,T_{\mathcal{U}}(A)}.$$

It follows that $\kappa \circ \varphi$ is $(\|\cdot\|_{2,\tau_{\mathcal{Z}}}$-$\|\cdot\|_{2,T_{\mathcal{U}}(A)})$-contractive. The completion of C^*-norm unit ball of $\mathcal{Z}$ under the norm $\|\cdot\|_{2,\tau_{\mathcal{Z}}}$ is the C^*-norm unit ball of $\pi_{\tau_{\mathcal{Z}}}(\mathcal{Z})'' \cong \mathcal{R}$. As the C^*-norm unit ball of $A^{\mathcal{U}}$ is $\|\cdot\|_{2,T_{\mathcal{U}}(A)}$-complete ([6, Lemma 1.6]), it follows that $\kappa \circ \varphi$ extends to a unital homomorphism $\mathcal{R} \cong \pi_{\tau_{\mathcal{Z}}}(\mathcal{Z})'' \to (A^{\mathcal{U}} \cap A')^{\alpha^{\mathcal{U}}}$. By restriction, there is also a unital homomorphism $\rho\colon M_n \to (A^{\mathcal{U}} \cap A')^{\alpha^{\mathcal{U}}}$.

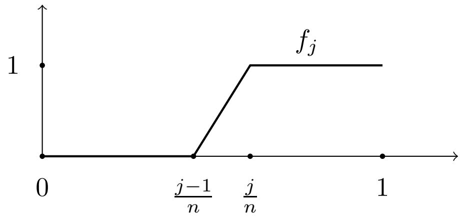
Let $e_1, \ldots, e_n \in M_n$ be the canonical diagonal projections, and set $p_j = \rho(e_j)$ for all $j = 1, \ldots, n$. Then condition (1) in Definition 3.1 is automatically satisfied. To check (2), let $a \in S$, let $j = 1, \ldots, n$, and let $\tau \in T_{\mathcal{U}}(A)$. The map $M_n \to \mathbb{C}$ defined by $b \mapsto \tau(\rho(b)a)$ is a (not necessarily normalized) trace on M_n, hence a multiple of the canonical trace τ_{M_n} on M_n. Taking $b = 1$ we deduce that the multiple is $\tau(a)$, so that $\tau(\rho(b)a) = \tau_{M_n}(b)\tau(a)$. Now taking $b = e_j$, we get $\tau(p_j a) = \frac{1}{n}\tau(a)$, as desired. □

The following proposition is an equivariant version of [6, Lemma 2.4]. It roughly states that, in the presence of uniform property Γ, positive contractions can be replaced by projections when computing tracial values in $A^{\mathcal{U}}$.

Proposition 3.4. *Let G be a countable, discrete group and let A be a separable, unital C^*-algebra with non-empty trace space. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action and suppose that (A, α) has uniform property Γ. Let $S, S_0 \subseteq A^{\mathcal{U}}$ be $\|\cdot\|_{2,T_{\mathcal{U}}(A)}$-separable subsets and let $b \in (A^{\mathcal{U}} \cap S')^{\alpha^{\mathcal{U}}}$ be a positive contraction. Then there exists a projection $p \in (A^{\mathcal{U}} \cap S')^{\alpha^{\mathcal{U}}}$ such that $\tau(ab) = \tau(ap)$ for all $a \in S_0$ and $\tau \in T_{\mathcal{U}}(A)$.*

Proof. We follow closely the proof of Lemma 2.4 in [6]. By countable saturation of ultrapowers (see Remark 2.2), it suffices to find, for every $n \in \mathbb{N}$, a positive contraction $e \in (A^{\mathcal{U}} \cap S')^{\alpha^{\mathcal{U}}}$ with $\|e - e^2\|_{2,T_{\mathcal{U}}(A)} <$

$1/n$, and such that $\tau(ab) = \tau(ae)$ for all $a \in S_0$ and $\tau \in T_{\mathcal{U}}(A)$. Fix $n \in \mathbb{N}$ and let $f_1, \ldots, f_n \in C([0,1])$ be given by the following graph:



Set $\widetilde{S} = S \cup \{b\}$. Using uniform property Γ for (A, α), let $p_1, \ldots, p_n$ be projections in $(A^{\mathcal{U}} \cap \widetilde{S}')^{\alpha^{\mathcal{U}}}$ such that for all $c \in \widetilde{S}$ and all $\tau \in T_{\mathcal{U}}(A)$ we have

$$\sum_{j=1}^{n} p_j = 1 \quad \text{and} \quad \tau(p_j c) = \frac{1}{n}\tau(c). \tag{3.1}$$

Notice that $t = \frac{1}{n}\sum_{j=1}^{n} f_j(t)$ for all $t \in [0,1]$. Set $e = \sum_{j=1}^{n} p_j f_j(b) \in (A^{\mathcal{U}} \cap S')^{\alpha^{\mathcal{U}}}$. Fix $a \in S_0$ and $\tau \in T_{\mathcal{U}}(A)$. Using at the last step that $b = \frac{1}{n}\sum_{j=1}^{n} f_j(b)$, we have

$$\tau(ea) = \sum_{j=1}^{n} \tau(p_j f_j(b) a) \overset{(3.1)}{=} \sum_{j=1}^{n} \frac{1}{n}\tau(f_j(b)a) = \tau(ba).$$

Moreover, using at the second to last step that $\sum_{j=1}^{n} f_j - f_j^2 \leq 1$, we get

$$\tau(e - e^2) = \sum_{j=1}^{n} \tau(p_j(f_j(b) - f_j(b)^2)) \overset{(3.1)}{=} \sum_{j=1}^{n} \frac{1}{n}\tau(f_j(b) - f_j(b)^2) \leq \frac{1}{n}\tau(1) = \frac{1}{n}.$$

Using that $e - e^2$ is a positive contraction (because this is the case for $f_j - f_j^2$ and the p_j's are orthogonal), we conclude that

$$\|e - e^2\|_{2,T_{\mathcal{U}}(A)} = \sup_{\tau \in T_{\mathcal{U}}(A)} \tau((e - e^2)^2) \leq \sup_{\tau \in T_{\mathcal{U}}(A)} \tau(e - e^2) \leq \frac{1}{n},$$

as desired. $\square$

4. Equivariant complemented partitions of unity

In [6] uniform property Γ is used to infer the existence of well-behaved partitions of unity in the central sequence algebra of uniform tracial ultrapowers, for nuclear separable C^*-algebras. Here we introduce the equivariant version of that definition.

Definition 4.1. Let A be a separable, unital C^*-algebra with non-empty trace space, let G be a countable, discrete amenable group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Given $M > 0$, we say that (A, α) has *complemented partitions of unity (CPoU) with constant* M, if for any $\|\cdot\|_{2,T_{\mathcal{U}}(A)}$-separable subset $S \subseteq A^{\mathcal{U}}$, for any $n \in \mathbb{N}$, for any $a_1, \ldots, a_n \in A_+$ and for any $\delta > 0$ with

$$\sup_{\tau \in T(A)^{\alpha}} \min\{\tau(a_1), \ldots, \tau(a_n)\} < \delta,$$

there exist projections $p_1, \ldots, p_n \in (A^{\mathcal{U}} \cap S')^{\alpha^{\mathcal{U}}}$ such that

(1) $\sum_{j=1}^{n} p_j = 1$,
(2) $\tau(a_j p_j) \leq M\delta\tau(p_j)$ for every $\tau \in T_{\mathcal{U}}^{\alpha}(A)$ and $j = 1, \ldots, n$.

We say that (A, α) has *CPoU* if there is a constant $M > 0$ such that (A, α) has CPoU with constant M.

Note that actions as in the above definition always have invariant traces.

Remark 4.2. It is not clear to us whether the above definition is the right one for general amenable group actions. Specifically, one may want to take the supremum over all $T(A)$ in the displayed inequality and require that (2) holds for traces in $T_{\mathcal{U}}(A)$. On the other hand, we will only work with actions for which the cardinality of the orbits of the induced action on $T(A)$ is bounded, and in this case the two versions are equivalent by Proposition 2.3.

The intuition behind the name *complemented partition of unity* is the following (see also the discussion after Definition G in [6]): condition (1) implies that the elements $p_1, \ldots, p_n$, when identified with the functions $\hat{p}_j$ on $T_{\mathcal{U}}(A)$ which send τ to $\tau(p_j)$, form a partition of unity, while condition (2) asserts that $\hat{p}_j$ is approximately subordinate to $1 - \hat{a}_j$, the complement of $\hat{a}_j$. Indeed when $\tau(1 - a_j) = 0$, and hence $\tau(a_j) = 1$, condition (2) forces $\tau(p_j) = \tau(a_j p_j) \leq M\delta\tau(p_j)$, and thus $\tau(p_j) = 0$.

The main differences between our Definition 4.1 and the one in [6, Definition 3.1] are the requirements that $p_1, \ldots p_n$ have to be $\alpha^{\mathcal{U}}$-invariant, the restriction to invariant traces and the presence of the constant M. The motivation for this constant is of technical nature[2]; it will play a role in the proof of Theorem 4.3, where we show that for an action α of an amenable group on a nuclear C^*-algebra A, uniform property Γ implies the existence of invariant CPoU. The proof of Theorem 4.3 is not only inspired by some of the results in [6, Section 3], but also directly uses some of them ([6, Lemma 3.6]). In the proof of Theorem 4.3 we need to uniformly bound the images of some elements in A via traces in $T(A)$, starting from a bound on the images of invariant traces, which is where the constant M appears.

Theorem 4.3. *Let G be a countable, discrete, amenable group, let A be a separable, nuclear, unital C^*-algebra with non-empty trace space and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Suppose that the induced action on $T(A)$ has finite orbits bounded in size by some uniform constant $M > 0$, and that (A, α) has uniform property Γ. Then (A, α) has CPoU with constant M.*

Proof. We fix a $\|\cdot\|_{2,T_{\mathcal{U}}(A)}$-separable subset $S \subseteq A^{\mathcal{U}}$, positive elements $a_1, \ldots, a_n \in A$ and $\delta > 0$ with

$$\sup_{\tau \in T(A)^{\alpha}} \min\{\tau(a_1), \ldots, \tau(a_n)\} < \delta.$$

We divide the proof into two claims. The first one does not require the use of uniform property Γ; it is an equivariant version of Lemma 3.6 in [6].

Claim 4.3.1. *Suppose there exist $t > 0$ and a projection $q \in (A^{\mathcal{U}} \cap A')^{\alpha^{\mathcal{U}}}$ such that $\tau(q) = t$ for all $\tau \in T_{\mathcal{U}}(A)$. Then there are positive contractions $b_1, \ldots, b_n \in (A^{\mathcal{U}} \cap S')^{\alpha^{\mathcal{U}}}$ such that*

(1.a) $\sum_{j=1}^{n} \tau(b_j q) = t$ *for all* $\tau \in T_{\mathcal{U}}(A)$,
(1.b) $\tau(a_j b_j q) \leq M\delta\tau(b_j q)$ *for all* $\tau \in T_{\mathcal{U}}^{\alpha}(A)$ *and for* $j = 1, \ldots, n$.

[2] It is in fact possible to show that (A, α) has CPoU with constant M if and only if it has CPoU with constant 1 (we refer to [15] for a discussion on this). This technical improvement makes no difference in this paper.

Since G is countable, by replacing S with $\bigcup_{g\in G}\alpha_g^{\mathcal{U}}(S)$ we may assume that S is $\alpha^{\mathcal{U}}$-invariant. Fix $\varepsilon>0$ and a finite subset $F\subseteq G$. By saturation of $A^{\mathcal{U}}$ (Remark 2.2), it suffices to find positive contractions $b_1,\dots,b_n\in A^{\mathcal{U}}\cap S'$ satisfying (1.a) and (1.b) and

$$\max_{g\in F}\max_{j=1,\dots,n}\|\alpha_g^{\mathcal{U}}(b_j)-b_j\|_{2,T_{\mathcal{U}}(A)}<\varepsilon.$$

Use amenability of G to find a finite subset $K\subseteq G$ such that

$$\frac{|gK\triangle K|}{|K|}<\varepsilon$$

for all $g\in F$. For each $j=1,\dots,n$, set $a_j'=\frac{1}{|K|}\sum_{k\in K}\alpha_{k^{-1}}(a_j)$, which is a positive element in A. For $\tau\in T(A)^\alpha$, we have $\tau(a_j')=\tau(a_j)$, and in particular

$$\sup_{\tau\in T(A)^\alpha}\min\{\tau(a_1'),\dots,\tau(a_n')\}<\delta.$$

Because $\tau\le M\tau^\alpha$ (see subsection 2.1 where the notation τ^α is introduced), we deduce that

$$\sup_{\tau\in T(A)}\min\{\tau(a_1'),\dots,\tau(a_n')\}<M\delta.$$

Apply [6, Lemma 3.6] to $a_1',\dots,a_n'$ to find positive contractions $b_1',\dots,b_n'\in A^{\mathcal{U}}\cap S'$ satisfying conditions (1.a) and (1.b) in this claim. For $j=1,\dots,n$, set

$$b_j=\frac{1}{|K|}\sum_{k\in K}\alpha_k^{\mathcal{U}}(b_j')\in A^{\mathcal{U}}.$$

Since S is $\alpha^{\mathcal{U}}$-invariant and since $b_1',\dots,b_n'$ commute with S, it follows that $b_1,\dots,b_n$ also commute with S. Moreover, a routine computation shows that for all $g\in F$ and for $j=1,\dots,n$ we have

$$\|\alpha_g^{\mathcal{U}}(b_j)-b_j\|_{2,T_{\mathcal{U}}(A)}\le\frac{|gK\triangle K|}{|K|}<\varepsilon.$$

On the other hand, for every $\tau\in T_{\mathcal{U}}(A)$ and every $j=1,\dots,n$, we have

$$\tau(b_jq)=\frac{1}{|K|}\sum_{k\in K}\tau(\alpha_k^{\mathcal{U}}(b_j')q)=\frac{1}{|K|}\sum_{k\in K}\tau(\alpha_k^{\mathcal{U}}(b_j'q)).$$

Thus, by (1.a) we have $\sum_{j=1}^n\tau(b_jq)=t$.

Let $\tau\in T_{\mathcal{U}}^\alpha(A)$ and $j=1,\dots,n$. In the next computation, we use the fact that q is G-invariant and that $\tau=\tau\circ\alpha_{k^{-1}}^{\mathcal{U}}$ at the second step, and the above displayed equation at the last step in combination with the fact that τ is G-invariant, to get

$$\begin{aligned}\tau(a_jb_jq)&=\frac{1}{|K|}\sum_{k\in K}\tau(a_j\alpha_k^{\mathcal{U}}(b_j')q)\\&=\frac{1}{|K|}\sum_{k\in K}\tau(\alpha_{k^{-1}}(a_j)b_j'q)\\&=\tau\Big(\frac{1}{|K|}\sum_{k\in K}\alpha_{k^{-1}}(a_j)b_j'q\Big)\\&=\tau(a_j'b_j'q)\overset{(1.b)}{\le}M\delta\tau(b_j'q)=M\delta\tau(b_jq).\end{aligned}$$

This proves the claim.

By saturation of ultrapowers (see Remark 2.2), the set I of all numbers $t \in [0,1]$ for which there are orthogonal projections $\widetilde{p}_1, \ldots, \widetilde{p}_n \in (A^{\mathcal{U}} \cap S')^{\alpha^{\mathcal{U}}}$ such that

(i) $\tau(\sum_{j=1}^n \widetilde{p}_j) = t$ for all $\tau \in T_{\mathcal{U}}(A)$,
(ii) $\tau(a_j \widetilde{p}_j) \leq M\delta\tau(\widetilde{p}_j)$ for all $\tau \in T_{\mathcal{U}}^{\alpha}(A)$ and all $j = 1, \ldots, n$,

is closed, and it is clearly non-empty as it contains zero. Let t_0 be the maximal element in this set. Then (A, α) has CPoU if and only if $t_0 = 1$. Set $s_0 = t_0 + \frac{1-t_0}{n}$. Then $t_0 \leq s_0 \leq 1$ and $s_0 = t_0$ if and only if $t_0 = 1$.

Claim 4.3.2. *s_0 belongs to I (and thus $s_0 = t_0 = 1$).*

Let $\widetilde{p}_1, \ldots, \widetilde{p}_n \in (A^{\mathcal{U}} \cap S')^{\alpha^{\mathcal{U}}}$ be projections satisfying (i) and (ii) above for t_0. Set $q = 1 - \sum_{j=1}^n \widetilde{p}_j$, and note that $\tau(q) = 1 - t_0$ for all $\tau \in T_{\mathcal{U}}(A)$. Apply Claim 4.3.1 to $\widetilde{S} = S \cup \{q\}$ in place of S to obtain $b_1, \ldots, b_n \in (A^{\mathcal{U}} \cap \widetilde{S}')^{\alpha^{\mathcal{U}}}$ satisfying conditions (1.a) and (1.b) for $1 - t_0$ in place of t. Use Proposition 3.4 to find projections $p'_1, \ldots, p'_n \in (A^{\mathcal{U}} \cap \widetilde{S}')^{\alpha^{\mathcal{U}}}$ such that

$$\tau(qb_j) = \tau(qp'_j) \quad \text{and} \quad \tau(a_j q b_j) = \tau(a_j q p'_j) \tag{4.1}$$

for all $j = 1, \ldots, n$ and all $\tau \in T_{\mathcal{U}}(A)$. Set

$$T = A \cup S \cup \{q, \widetilde{p}_1, \ldots, \widetilde{p}_n, p'_1, \ldots, p'_n\}.$$

Use uniform property Γ for (A, α) to find orthogonal projections $e_1, \ldots, e_n \in (A^{\mathcal{U}} \cap T')^{\alpha^{\mathcal{U}}}$ which add up to 1 and satisfy

$$\tau(e_j y) = \frac{1}{n}\tau(y) \tag{4.2}$$

for all $\tau \in T_{\mathcal{U}}(A)$, for all $y \in T$ and for all $j = 1, \ldots, n$. For $j = 1, \ldots, n$, set

$$p_j = \widetilde{p}_j + qp'_j e_j \in (A^{\mathcal{U}} \cap S')^{\alpha^{\mathcal{U}}}.$$

Since $q \perp \widetilde{p}_j$, it follows that p_j is a projection. Moreover, $p_j \perp p_k$ if $j \neq k$. For $\tau \in T_{\mathcal{U}}(A)$ we have

$$\begin{aligned} \tau\Big(\sum_{j=1}^n p_j\Big) &\overset{(4.2)}{=} \tau\Big(\sum_{j=1}^n \widetilde{p}_j\Big) + \frac{1}{n}\tau\Big(\sum_{j=1}^n qp'_j\Big) \\ &\overset{(i),(4.1)}{=} t_0 + \frac{1}{n}\sum_{j=1}^n \tau(qb_j) \\ &\overset{(1.a)}{=} t_0 + \frac{1-t_0}{n} = s_0. \end{aligned}$$

In addition, given $\tau \in T_{\mathcal{U}}^{\alpha}(A)$, for $j = 1, \ldots, n$ we have

$$\begin{aligned} \tau(a_j p_j) &= \tau(a_j \widetilde{p}_j) + \tau(a_j q p'_j e_j) \\ &\overset{(4.2)}{=} \tau(a_j \widetilde{p}_j) + \frac{1}{n}\tau(a_j q p'_j) \\ &\overset{(ii),(1.b)}{\leq} M\delta\tau(\widetilde{p}_j) + \frac{M\delta}{n}\tau(qp'_j) \end{aligned}$$

$$= \quad M\delta\tau(\widetilde{p}_j + qp'_j e_j) = M\delta\tau(p_j).$$

It follows that $p_1, \ldots, p_n$ witness the fact that s_0 belongs to I, as desired. This proves the claim and the theorem. □

Invariant CPoUs are the main technical ingredient for the 'local to global' arguments employed to prove the implications (1) ⇒ (2) and (1) ⇒ (3) of Theorem A. This allows us to avoid the machinery involving W^*-bundles developed in [48,49,22] (and in the non-dynamical setting in [1]), which required the assumption that $T(A)$ is a Bauer simplex. Lemma 4.5 asserts that, in the presence of CPoU, if a polynomial identity is (approximately) satisfied in each *individual* tracial completion of (A, α), then the identity is (exactly) satisfied in $(A^{\mathcal{U}}, \alpha^{\mathcal{U}})$. To explicitly describe how this transfer works, we begin by establishing some terminology.

Definition 4.4. Let G be a discrete group. Given a tuple of non-commuting variables $\bar{x} = (x_1, \ldots, x_r)$ and $g \in G$, set $g \cdot \bar{x} = (g \cdot x_1, \ldots, g \cdot x_r)$, which we also regard as a tuple of non-commuting variables. By a *G-$*$-polynomial* in the variables $\bar{x}$ we mean a $*$-polynomial in the variables $\{g \cdot \bar{x} \colon g \in G\}$. Let A be a C^*-algebra and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action. Given a tuple $\bar{x} = (x_1, \ldots, x_r)$, given a G-$*$-polynomial $Q(\bar{x})$, and given a coefficient tuple $\bar{a} = (a_1, \ldots, a_r) \in A^r$, the term $Q(\bar{a})$ is computed by interpreting each $g \cdot x_j$ as $\alpha_g(a_j)$ for $j = 1, \ldots, r$.

Lemma 4.5. *Let A be a separable, unital C^*-algebra with non-empty trace space, let G be a countable, discrete group, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action such that the induced action on $T(A)$ has orbits which are uniformly bounded in size. Assume further that (A, α) has CPoU. For $m \in \mathbb{N}$, let Q_m be a G-$*$-polynomial in $r_m + s_m$ non-commuting variables, and let $(a_j)_{j\in\mathbb{N}}$ be a sequence in A. Suppose that for every $\varepsilon > 0$, for every $n \in \mathbb{N}$ and for every $\tau \in T(A)^\alpha$, there are contractions $w_j^\tau \in \mathcal{M}_\tau$, for $j \in \mathbb{N}$, such that, for $m = 1, \ldots, n$ we have*

$$\|Q_m(\pi_\tau(a_1), \ldots, \pi_\tau(a_{r_m}), w_1^\tau, \ldots, w_{s_m}^\tau)\|_{2,\tau} < \varepsilon.$$

Then there are contractions $w_j \in A^{\mathcal{U}}$, for $j \in \mathbb{N}$, such that

$$Q_m(a_1, \ldots, a_{r_m}, w_1, \ldots, w_{s_m}) \in J_{T(A)^\alpha},$$

for all $m \in \mathbb{N}$.

Proof. Fix $M > 0$ such that (A, α) has CPoU with constant M. Let $\varepsilon > 0$ and let $\ell \in \mathbb{N}$. By saturation of $(A^{\mathcal{U}}, \alpha^{\mathcal{U}})$ (see Remark 2.2), it is sufficient to find contractions $w_j \in A^{\mathcal{U}}$, for $j \in \mathbb{N}$, such that for all $m \leq \ell$ we have

$$\sup_{\tau \in T_{\mathcal{U}}^\alpha(A)} \|Q_m(a_1, \ldots, a_{r_m}, w_1, \ldots, w_{s_m})\|_{2,\tau} < \varepsilon.$$

By assumption, for each $\tau \in T(A)^\alpha$ there are contractions $\widetilde{w}_j^\tau \in \mathcal{M}_\tau$, for $j \in \mathbb{N}$, such that for $m = 1, \ldots, \ell$ we have

$$\|Q_m(\pi_\tau(a_1), \ldots, \pi_\tau(a_{r_m}), \widetilde{w}_1^\tau, \ldots, \widetilde{w}_{s_m}^\tau)\|_{2,\tau}^2 < \frac{\varepsilon^2}{M\ell}.$$

By Kaplansky's density theorem, we can choose contractions $w_j^\tau \in A$ with $\widetilde{w}_j^\tau = \pi_\tau(w_j^\tau)$ for all $j \in \mathbb{N}$. For each $\tau \in T(A)^\alpha$, set

$$b_\tau = \sum_{m=1}^{\ell} \Big| Q_m(a_1, \dots, a_{r_m}, w_1^\tau, \dots, w_{s_m}^\tau) \Big|^2. \tag{4.3}$$

Then

$$\tau(b_\tau) = \sum_{m=1}^{\ell} \| Q_m(a_1, \dots, a_{r_m}, w_1^\tau, \dots, w_{s_m}^\tau) \|_{2,\tau}^2 < \frac{\varepsilon^2}{M}.$$

Use compactness of $T(A)^\alpha$ to find $\tau_1, \dots, \tau_n \in T(A)^\alpha$ such that

$$\sup_{\tau \in T(A)^\alpha} \min\{\tau(b_{\tau_1}), \dots, \tau(b_{\tau_n})\} < \frac{\varepsilon^2}{M}.$$

Set $S = \{w_j^{\tau_1}, \dots, w_j^{\tau_n}, a_j \colon j \in \mathbb{N}\}$. Since (A, α) has CPoU with constant M, there are projections $p_1, \dots, p_n \in (A^{\mathcal{U}} \cap S')^{\alpha^{\mathcal{U}}}$ adding up to 1 such that

$$\tau(b_{\tau_j} p_j) \le \varepsilon^2 \tau(p_j) \tag{4.4}$$

for all $\tau \in T_{\mathcal{U}}^\alpha(A)$ and $j = 1, \dots, n$.

For $k \in \mathbb{N}$, set $w_k = \sum_{j=1}^n p_j w_k^{\tau_j}$. Those are contractions in $A^{\mathcal{U}}$. Fix $j = 1, \dots, n$. Because $p_1, \dots, p_n$ are $\alpha^{\mathcal{U}}$-invariant projections which commute with all elements in S, we have

$$\begin{aligned} \Big| Q_m(a_1, \dots, a_{r_m}, w_1, \dots, w_{s_m}) \Big|^2 &= \sum_{j=1}^n p_j \Big| Q_m(a_1, \dots, a_{r_m}, w_{s_1}^{\tau_j}, \dots, w_{s_m}^{\tau_j}) \Big|^2 \\ &\overset{(4.3)}{\le} \sum_{j=1}^n p_j b_{\tau_j}. \end{aligned}$$

As a consequence, given $\tau \in T_{\mathcal{U}}^\alpha(A)$ we have

$$\begin{aligned} \| Q_m(a_1, \dots, a_{r_m}, w_1, \dots, w_{s_m}) \|_{2,\tau}^2 &= \tau\Big(\Big| Q_m(a_1, \dots, a_{r_m}, w_1, \dots, w_{s_m}) \Big|^2 \Big) \\ &\le \sum_{j=1}^n \tau\Big(p_j \Big| Q_m(a_1, \dots, a_{r_m}, w_{s_1}^{\tau_j}, \dots, w_{s_m}^{\tau_j}) \Big|^2 \Big) \\ &\le \sum_{j=1}^n \tau(p_j b_{\tau_j}) \\ &\overset{(4.4)}{\le} \sum_{j=1}^n \varepsilon^2 \tau(p_j) = \varepsilon^2. \end{aligned}$$

This concludes the proof. $\square$

Remark 4.6. By countable saturation of ultrapowers (see Remark 2.2), the assumptions of Lemma 4.5 are satisfied if (and only if) for every $\tau \in T(A)^\alpha$ there exist contractions w_j^τ in the von Neumann ultrapower $\mathcal{M}_\tau^{\mathcal{U}}$, for $j \in \mathbb{N}$, such that

$$Q_m(\pi_\tau(a_1), \dots, \pi_\tau(a_{r_m}), w_1^\tau, \dots, w_{s_m}^\tau) = 0,$$

for all $m \in \mathbb{N}$. This amounts to saying that the polynomial relations one wishes to realize in $(A^{\mathcal{U}}, \alpha^{\mathcal{U}})$ are *exactly* realized in the tracial ultrapower of every GNS closure.

As pointed out in [6, Remark 4.2.iii], there is a notable difference when employing CPoU in 'local to global' arguments over trace spaces, as opposed to older techniques relying on W^*-bundles and on the assumption that $T(A)$ is a Bauer simplex. Indeed, when $\partial_e T(A)$ is compact, it is enough to consider extreme traces in order to obtain an analogue of Lemma 4.5 (see [1, Lemma 3.18]). Concretely, this allows one to work exclusively with von Neumann algebras of the form $\pi_\tau(A)''$ for $\tau \in \partial_e T(A)$; when A is nuclear, those are always isomorphic to the hyperfinite II_1-factor $\mathcal{R}$. The same applies in the dynamical setting if one furthermore assumes $T(A)^\alpha = T(A)$; see [48], [49] and [63]. This is not the case in our framework, since we want to remove the requirement that $\partial_e T(A)$ is compact (except for Section 8) and we work in a situation where in general $T(A)^\alpha \neq T(A)$; we may even have $T(A)^\alpha \cap \partial_e T(A) = \emptyset$. As a consequence, we are forced to consider all traces in order to apply Lemma 4.5, and thus find approximate solutions in general (tracial) GNS representations, not just factorial ones. The next section provides a concrete example of this approach.

5. Tracial ultrapowers of actions of amenable groups

The main objective of the current section is the following result, which plays a key role in the proofs in both Section 7 and Section 8. Given a group G and a normal subgroup $N \leq G$, throughout the rest of the paper we let $q_N \colon G \to N$ denote the quotient map.

The goal of the present section is to prove the following result.

Theorem 5.1. *Let G be a countable, discrete, amenable group, let A be a separable, simple, unital, stably finite, nuclear C^*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action such that the orbits of the action induced by α on $T(A)$ are finite and that their cardinality is bounded, and assume that (A, α) has CPoU. Let N be a normal subgroup of G such that α_g is strongly outer for all $g \in G \setminus N$, and let $\mu_{G/N} \colon G/N \to Aut(\mathcal{R})$ be an outer action on the hyperfinite II_1 factor $\mathcal{R}$. Then there exists an equivariant, unital embedding*

$$(\mathcal{R}, \mu_{G/N} \circ q_N) \to \left(A^{\mathcal{U}} \cap A', \alpha^{\mathcal{U}}\right).$$

The strategy for proving Theorem 5.1 is as follows. First, we construct equivariant, unital embeddings of $(\mathcal{R}, \mu_{G/N} \circ q_N)$ in the central sequence algebra of the weak closure of each individual invariant trace of A. After that, we apply Lemma 4.5 to glue those embeddings using CPoU, thus obtaining an equivariant, unital embedding $(\mathcal{R}, \mu_{G/N} \circ q_N) \to (A^{\mathcal{U}} \cap A', \alpha^{\mathcal{U}})$, thanks to Proposition 2.3. The first part translates into proving the existence of equivariant embeddings of $(\mathcal{R}, \mu_{G/N} \circ q_N)$ into the central sequence algebras of hyperfinite, not-necessarily factorial type II_1 von Neumann algebras with respect to suitable outer actions; see Theorem 5.4.

We begin with some preliminaries.

Definition 5.2. Let A and B be unital C^*-algebras, let G be a discrete group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$ be actions.

(1) We say that (A, α) and (B, β) are *conjugate* if there exists an isomorphism $\varphi\colon A \to B$ satisfying $\varphi \circ \alpha_g = \beta_g \circ \varphi$ for all $g \in G$. In this case, we say that $\varphi\colon (A, \alpha) \to (B, \beta)$ is an *equivariant isomorphism*.
(2) An α-*cocycle* is a function $u\colon G \to \mathcal{U}(A)$ satisfying $u_{gh} = u_g \alpha_g(u_h)$ for all $g, h \in G$. In this case, we define the *cocycle perturbation* α^u of α to be the action given by $\alpha^u_g = \mathrm{Ad}(u_g) \circ \alpha_g$ for all $g \in G$. (The cocycle condition guarantees that this is indeed an action.)
(3) We say that (A, α) and (B, β) are *cocycle conjugate*, written $(A, \alpha) \cong_{\mathrm{cc}} (B, \beta)$, if there is an α-cocycle u such that (A, α^u) and (B, β) are conjugate.

We say that (A, α) *(tensorially) absorbs* (B, β) if $(A, \alpha) \cong_{\mathrm{cc}} (A \otimes_{\min} B, \alpha \otimes \beta)$.

Similar notions apply to actions on von Neumann algebras, where tensorial absorption is considered with respect to the von Neumann tensor product $\overline{\otimes}$.

Proposition 5.3. *Let $(\mathcal{M}, \tau)$ be a separably representable, type II_1 von Neumann algebra, let G be a countable, discrete, amenable group, let $\gamma\colon G \to \mathrm{Aut}(\mathcal{M}, \tau)$ be an action and let $\delta\colon G \to Aut(\mathcal{R})$ be an action such that $(\mathcal{R}, \delta)$ is cocycle conjugate to $(\overline{\bigotimes}_{n\in\mathbb{N}} \mathcal{R}, \bigotimes_{n\in\mathbb{N}} \delta)$. Suppose that $(\mathcal{M}, \gamma)$ absorbs $(\mathcal{R}, \delta)$. Then there is a unital equivariant homomorphism $(\mathcal{R}, \delta) \to (\mathcal{M}^{\mathcal{U}} \cap \mathcal{M}', \gamma^{\mathcal{U}})$.*

Proof. Observe that if $(\mathcal{N}_0, \gamma_0)$ is cocycle conjugate to $(\mathcal{N}_1, \gamma_1)$, then $(\mathcal{N}_0^{\mathcal{U}} \cap \mathcal{N}_0', \gamma_0^{\mathcal{U}})$ is *conjugate* to $(\mathcal{N}_1^{\mathcal{U}} \cap \mathcal{N}_1', \gamma_1^{\mathcal{U}})$, since the inner automorphisms induced by a γ_0-cocycle act trivially on the central sequence algebra $\mathcal{N}_0^{\mathcal{U}} \cap \mathcal{N}_0'$. It is therefore enough to show that there is a unital map

$$(\mathcal{R}, \delta) \to \big((\mathcal{M}\overline{\otimes}\mathcal{R})^{\mathcal{U}} \cap (\mathcal{M}\overline{\otimes}\mathcal{R})', (\gamma\otimes\delta)^{\mathcal{U}}\big).$$

In turn, it suffices to find a unital map $(\mathcal{R}, \delta) \to (\mathcal{R}^{\mathcal{U}} \cap \mathcal{R}', \delta^{\mathcal{U}})$, since the latter is unitally contained in $\big((\mathcal{M}\overline{\otimes}\mathcal{R})^{\mathcal{U}} \cap (\mathcal{M}\overline{\otimes}\mathcal{R})', (\gamma\otimes\delta)^{\mathcal{U}}\big)$. We use the notation $(\overline{\mathcal{R}}, \bar{\delta})$ to abbreviate $(\overline{\bigotimes}_{n\in\mathbb{N}} \mathcal{R}, \bigotimes_{n\in\mathbb{N}} \delta)$. By assumption $(\mathcal{R}, \delta) \cong_{\mathrm{cc}} (\overline{\mathcal{R}}, \bar{\delta})$, and hence $(\mathcal{R}^{\mathcal{U}} \cap \mathcal{R}', \delta^{\mathcal{U}})$ is conjugate to $(\overline{\mathcal{R}}^{\mathcal{U}} \cap \overline{\mathcal{R}}', \bar{\delta}^{\mathcal{U}})$; thus, it suffices to find a unital map $(\mathcal{R}, \delta) \to \big(\overline{\mathcal{R}}^{\mathcal{U}} \cap \overline{\mathcal{R}}', \bar{\delta}^{\mathcal{U}}\big)$. Let $\varphi_n\colon (\mathcal{R}, \delta) \to (\overline{\mathcal{R}}, \bar{\delta})$ be the equivariant unital inclusion in the n-th tensor factor. Then $(\varphi_n)_{n\in\mathbb{N}}\colon \mathcal{R} \to \overline{\mathcal{R}}^{\mathcal{U}}$ is the desired unital map. □

We recall that an automorphism α of a von Neumann algebra $\mathcal{M}$ is said to be *properly outer* if for every α-invariant central non-zero projection p in $\mathcal{M}$, the restriction of α to $p\mathcal{M}p$ is outer. Notice that if $\mathcal{M}$ is a factor, an automorphism is properly outer if and only if it is outer. An action $\gamma\colon G \to \mathrm{Aut}(\mathcal{M})$ is said to be *properly outer* if γ_g is properly outer for every $g \in G \setminus \{e\}$.

When $(\mathcal{M}, \tau)$ is a II_1-factor, the following is a well-known result of Ocneanu [55]. The version we give here for hyperfinite II_1 von Neumann algebras follows from the classification of actions of discrete amenable groups on semifinite, hyperfinite von Neumann algebras in [65] (see also [64, Section 3]).

Theorem 5.4. *Let $(\mathcal{M}, \tau)$ be a separably representable, hyperfinite, type II_1 von Neumann algebra and let G be a countable, discrete, amenable group. Fix an action $\gamma\colon G \to \mathrm{Aut}(\mathcal{M}, \tau)$ which preserves τ. Let N be a normal subgroup of G such that γ_g is properly outer for all $g \in G \setminus N$ and let $\mu_{G/N}\colon G/N \to Aut(\mathcal{R})$ be an outer action. Then*

$$(\mathcal{M}, \gamma) \cong_{\mathrm{cc}} (\mathcal{M}\overline{\otimes}\mathcal{R}, \gamma \otimes (\mu_{G/N} \circ q_N)).$$

Proof. Denote by $\mathcal{C}$ the center of $\mathcal{M}$, and let (X, ν) be its von Neumann spectrum, so that $(\mathcal{C}, \tau|_{\mathcal{C}}) \cong (L^\infty(X, \nu), \int_X d\nu)$ as tracial von Neumann algebras. Without loss of generality, we assume that (X, ν) is a standard Borel probability space. By [70, Theorem XVI.1.5], there is an isomorphism $\mathcal{M} \cong \mathcal{C}\overline{\otimes}\mathcal{R}$. Therefore, by [69, Corollary IV.8.30], we can identify every $b \in \mathcal{M}$ with a decomposable element $\int_X^{\oplus} b_x d\nu$, where each $b_x \in \mathcal{R}$.

Given an arbitrary action $\alpha\colon G \to \mathrm{Aut}(\mathcal{M}, \tau)$, which will later be taken to be either γ or $\gamma \otimes \mu_{G/N} \circ q_N$, we let σ^α the measurable G-action on (X, ν) induced by α. We recall the notion of the *ancillary groupoid* and *ancillary action* associated to α, as well as the cocycle conjugate invariants introduced in [65]. We refer to [65] for details (see also [64, Section 3] and [36]). There exists a measurable map $\bar{\alpha}\colon X \times G \to \mathrm{Aut}(\mathcal{R})$ such that, given $b = \int_X^{\oplus} b_x d\nu \in \mathcal{M}$,

$$\bar{\alpha}_{x,g}\big(b_{\sigma_g^\alpha(x)}\big) = \alpha_g(b)_x, \tag{5.1}$$

for every $(x, g) \in X \times G$. Denote by $\mathcal{G}_\alpha := X \times G$ the measured transformation groupoid (with unit space X) corresponding to the G-action induced by α on X. Let

$$\mathcal{H}_\alpha := \{(x, g) \in \mathcal{G}_\alpha : \sigma_g(x) = x\}, \quad \text{and} \quad \mathcal{N}_\alpha := \{(x, g) \in \mathcal{H}_\alpha : \bar{\alpha}_{x,g} \text{ is inner}\}.$$

Note that $\mathcal{N}_\alpha$ is an invariant of cocycle conjugacy. Choose a Borel function $U \colon \mathcal{N}_\alpha \to \mathcal{U}(\mathcal{R})$ such that $\bar{\alpha}_n = \mathrm{Ad}(U(n))$ for every $n \in \mathcal{N}_\alpha$. The ways in which the unitaries in the image of U interact with each other and with the rest of the action is described via a relative cohomology class χ_α, which is also a cocycle conjugacy invariant of α (see [64, Section 2] for the precise definition).

We identify $\mathcal{M}$ and $\mathcal{M}\overline{\otimes}\mathcal{R}$ throughout. Note that γ and $\gamma \otimes (\mu_{G/N} \circ q_N)$ induce the same action on the center of $\mathcal{M}$, which we will denote simply by σ. It follows that the groupoids $\mathcal{G}_\gamma$ and $\mathcal{G}_{\gamma\otimes(\mu_{G/N}\circ q_N)}$ are isomorphic, which in turn implies that $\mathcal{H}_\gamma \cong \mathcal{H}_{\gamma\otimes(\mu_{G/N}\circ q_N)}$. It follows then by [64, Theorem p. 324] that the two actions are cocycle conjugate if and only if $\mathcal{N}_\gamma \cong \mathcal{N}_{\gamma\otimes(\mu_{G/N}\circ q_N)}$ and $\chi_\gamma = \chi_{\gamma\otimes(\mu_{G/N}\circ q_N)}$. (The invariant δ does not play a role when $\mathcal{M}$ is finite.)

Claim 5.4.1. *Let $g \in G$ and $x \in X$ satisfy $\sigma_g(x) = x$. (In other words, $(x, g) \in \mathcal{H}_\gamma = \mathcal{H}_{\gamma\otimes(\mu_{G/N}\circ q_N)}$.) Then*

$$(\overline{\gamma \otimes (\mu_{G/N} \circ q_N)})_{x,g} = \bar{\gamma}_{x,g} \otimes (\mu_{G/N} \circ q_N)_g.$$

We show that the two automorphisms are equal on all elementary tensors $b_0 \otimes b_1 \in \mathcal{R}\overline{\otimes}\mathcal{R}$ (which we identify with $\mathcal{R}$). We have

$$\begin{aligned}(\overline{\gamma \otimes (\mu_{G/N} \circ q_N)})_{x,g}(b_0 \otimes b_1) &\overset{(5.1)}{=} (\gamma \otimes (\mu_{G/N} \circ q_N))_g(1_{L^\infty(X)} \otimes b_0 \otimes b_1)_x \\ &= \gamma_g(1_{L^\infty(X)} \otimes b_0)_x \otimes (\mu_{G/N} \circ q_N)_g(b_1) \\ &\overset{(5.1)}{=} \bar{\gamma}_{x,g}(b_0) \otimes (\mu_{G/N} \circ q_N)_g(b_1).\end{aligned}$$

This computation proves the claim.

Claim 5.4.2. *Fix $g \in G \setminus N$. We claim that*

$$\nu\big(\{x \in X \colon (x, g) \in \mathcal{H}_\gamma \text{ and } \overline{\gamma}_{x,g} \text{ is inner}\}\big) = 0.$$

Denote by Y the set in the above displayed equation, and note that Y is Borel. To prove the claim, assume by contradiction that $\nu(Y) > 0$. By [47, Theorem 3.4], there is a non-zero central projection $q \in \mathcal{M}$ such that γ_g is inner on $q\mathcal{M}$. This contradicts the assumption that γ_g is properly outer (since $g \notin N$).

Let $g \in G$. Using Claim 5.4.2, and up to removing a σ-invariant measure zero set from X, we can assume that if there is $x \in X$ such that $(x, g) \in \mathcal{N}_\gamma$ implies $g \in N$. Similarly, it follows from Claim 5.4.1 that if there exists $x \in X$ such that $(x, g) \in \mathcal{N}_{\gamma\otimes(\mu_{G/N}\circ q_N)}$, then $g \in N$. In fact, if $g \in G \setminus N$, then $(\mu_{G/N} \circ q_N)_g$ is outer, which in turn forces $\bar{\gamma}_{x,g} \otimes (\mu_{G/N} \circ q_N)_g$ to be outer as well.

Finally, let $(x, g) \in X \times N$. Using Claim 5.4.1 in the second to last equivalence, we have

$$\begin{aligned}(x, g) \in \mathcal{N}_\gamma &\Leftrightarrow \bar{\gamma}_{x,g} \text{ is inner} \\ &\Leftrightarrow \bar{\gamma}_{x,g} \otimes \mathrm{id}_{\mathcal{R}} = \bar{\gamma}_{x,g} \otimes (\mu_{G/N} \circ q_N)_g \text{ is inner} \\ &\Leftrightarrow (\overline{\gamma \otimes (\mu_{G/N} \circ q_N)})_{x,g} \text{ is inner} \\ &\Leftrightarrow (x, g) \in \mathcal{N}_{\gamma\otimes(\mu_{G/N}\circ q_N)}.\end{aligned}$$

The equality $\chi_\gamma = \chi_{\gamma\otimes(\mu_{G/N}\circ q_N)}$ follows by the definition of the invariant χ ([64, Section 2]). Let $U\colon \mathcal{N}_\gamma \to \mathcal{U}(\mathcal{R})$ be a Borel map such that $\bar{\gamma}_g = \mathrm{Ad}(U(g))$ for every $g \in \mathcal{N}_\gamma$. Note that $(\overline{\gamma\otimes(\mu_{G/N}\circ q_N)})_n = \bar{\gamma}_n \otimes \mathrm{id}_\mathcal{R}$ by Claim 5.4.1. Let $V\colon N \to \mathcal{U}(\mathcal{R}\overline{\otimes}\mathcal{R})$ be given by $V = U \otimes 1_\mathcal{R}$. Then

$$(\overline{\gamma\otimes(\mu_{G/N}\circ q_N)})_g = \mathrm{Ad}(V(g))$$

for every $g \in \mathcal{N}_\gamma$. This finishes the proof. □

We record here the following consequence of Theorem 5.4, which will be needed in Section 8.

Corollary 5.5. *Let $(\mathcal{M},\tau)$ be a separably representable, hyperfinite, type II_1 von Neumann algebra, let G be a countable, discrete, amenable group and let $\gamma\colon G \to \mathrm{Aut}(\mathcal{M},\tau)$ be an action which preserves τ. Then for every $d \in \mathbb{N}$, there is a unital homomorphism $M_d \to (\mathcal{M}^\mathcal{U} \cap \mathcal{M}')^{\gamma^\mathcal{U}}$.*

Proof. By taking $G = N$ in Theorem 5.4, it follows that $(\mathcal{M},\gamma)$ absorbs $(\mathcal{R},\mathrm{id}_\mathcal{R})$ tensorially. By Proposition 5.3, it follows that there exists a unital embedding $\mathcal{R} \to (\mathcal{M}^\mathcal{U} \cap \mathcal{M}')^{\gamma^\mathcal{U}}$. Since there exists a unital homomorphism $M_d \to \mathcal{R}$ for every $d \in \mathbb{N}$, the conclusion follows. □

The following result is the main application of CPoU in this section, and it is last ingredient we need in order to prove Theorem 5.1.

Proposition 5.6. *Let G be a countable, discrete group, let A be a separable, unital C^*-algebra with non-empty trace space, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Suppose that the induced action on $T(A)$ has finite orbits bounded in size by some constant, and that (A,α) has CPoU. Let $\beta\colon G \to \mathrm{Aut}(B)$ be an action of G on a separable, unital C^*-algebra B, and suppose that for every $\tau \in T(A)^\alpha$ there exists an equivariant, unital homomorphism*

$$(B,\beta) \to \left(\mathcal{M}_\tau^\mathcal{U} \cap \mathcal{M}_\tau', (\alpha^\tau)^\mathcal{U}\right).$$

Then there exists an equivariant, unital homomorphism $(B,\beta) \to (A^\mathcal{U} \cap A', \alpha^\mathcal{U})$.

Proof. For each $\tau \in T(A)^\alpha$, fix an equivariant, unital homomorphism

$$\varphi_\tau\colon (B,\beta) \to \left(\mathcal{M}_\tau^\mathcal{U} \cap \mathcal{M}_\tau', (\alpha^\tau)^\mathcal{U}\right).$$

Let $\{a_n\}_{n\in\mathbb{N}}$ be a countable dense subset of the unit ball of A. Let D_1 be the unit disk in $\mathbb{C}$ and set $\mathbb{Q}_1 = \mathbb{Q}[i] \cap D_1$. Let B_0 be a countable dense subset of the unit ball of B containing 1_B, which is invariant under the adjoint operation, multiplication, multiplication by scalars from $\mathbb{Q}_1$, and the operation $(x,y) \mapsto \frac{1}{2}(x+y)$. Since G is countable, we can assume without loss of generality that B_0 is β-invariant. Let $\{b_n\}_{n\in\mathbb{N}}$ be an enumeration of B_0 such that $b_0 = 1_B$. Fix non-commuting variables x, y, z, w, and for $\lambda \in \mathbb{Q}_1$ and $g \in G$ define the following G-$*$-polynomials:

- $Q_{\mathrm{id}}(x) = x - 1$,
- $Q_+(x,y,z) = \frac{1}{2}(x+y) - z$,
- $Q_\times(x,y,z) = xy - z$,
- $Q_*(x,y) = x^* - y$,
- $Q_\lambda(x,y) = \lambda x - y$,
- $Q_g(x,y) = g \cdot x - y$,
- $Q_{[\cdot,\cdot]}(x,w) = xw - wx$.

Clearly, $Q_{\mathrm{id}}(b_n) = 0$ if and only if $n = 0$. Given $n, m \in \mathbb{N}$ there is a unique $k^+_{(n,m)} \in \mathbb{N}$ such that $Q_+(b_n, b_m, b_{k^+_{(n,m)}}) = 0$. Similarly, there is a unique $k^\times_{(n,m)} \in \mathbb{N}$ such that $Q_\times(b_n, b_m, b_{k^\times_{(n,m)}}) = 0$. Analogously, given $n \in \mathbb{N}$, $\lambda \in \mathbb{Q}_1$ and $g \in G$ there are unique $k^*_n, k^\lambda_n, k^g_n \in \mathbb{N}$ such that

$$Q_*(b_n, b_{k^*_n}) = 0, \quad Q_\lambda(b_n, b_{k^\lambda_n}) = 0, \quad Q_g(b_n, b_{k^g_n}) = 0.$$

Since φ_τ is an equivariant, unital homomorphism, we have $Q_{\mathrm{id}}(\varphi_\tau(1_B)) = 0$, moreover for every $n, m \in \mathbb{N}$, $\lambda \in \mathbb{Q}_1$ and $g \in G$ we have

$$Q_+(\varphi_\tau(b_n), \varphi_\tau(b_m), \varphi_\tau(b_{k^+_{(n,m)}})) = 0, \quad Q_\times(\varphi_\tau(b_n), \varphi_\tau(b_m), \varphi_\tau(b_{k^\times_{(n,m)}})) = 0,$$

and

$$Q_*(\varphi_\tau(b_n), \varphi_\tau(b_{k^*_n})) = 0, \quad Q_\lambda(\varphi_\tau(b_n), \varphi_\tau(b_{k^\lambda_n})) = 0, \quad Q_g(\varphi_\tau(b_n), \varphi_\tau(b_{k^g_n})) = 0.$$

Finally, for all $n, m \in \mathbb{N}$ we have

$$Q_{[\cdot,\cdot]}(\varphi_\tau(b_n), \pi_\tau(a_m)) = 0,$$

since $\varphi_\tau(b_n)$ belongs to the relative commutant of $\mathcal{M}_\tau$, and thus commutes with $\pi_\tau(A)$. By Lemma 4.5 (see also Remark 4.6) and Proposition 2.3, we can find contractions $b'_n \in A^{\mathcal{U}}$, for $n \in \mathbb{N}$, such that

(a) $Q_{\mathrm{id}}(b'_0) = 0$, that is $b'_0 = 1_{A^{\mathcal{U}}}$,
(b) for every $n, m \in \mathbb{N}$, we have $Q_+(b'_n, b'_m, b'_{k^+_{(n,m)}}) = 0 = Q_\times(b'_n, b'_m, b'_{k^\times_{(n,m)}})$,
(c) for every $n \in \mathbb{N}$, $\lambda \in \mathbb{Q}_1$ and $g \in G$, we have

$$Q_*(b'_n, b'_{k^*_n}) = Q_\lambda(b'_n, b'_{k^\lambda_n}) = Q_g(b'_n, b'_{k^g_n}) = 0,$$

(d) for all $n, m \in \mathbb{N}$ we have $Q_{[\cdot,\cdot]}(b'_n, \pi_\tau(a_m)) = 0$.

Let $\Phi_0 \colon B_0 \to A^{\mathcal{U}}$ be the map defined by sending b_n to b'_n for all $n \in \mathbb{N}$. This map is unital by (a), it is additive and multiplicative by (b), and it is $*$-preserving, equivariant and $\mathbb{Q}_1$-homogeneous by (c). It can be therefore extended uniquely to a $\mathbb{Q}[i]$-linear, equivariant, unital homomorphism $\Phi_0 \colon \mathrm{span}(B_0) \to A^{\mathcal{U}}$. The image of Φ_0 is contained in $A^{\mathcal{U}} \cap A'$ by (d). Notice that Φ_0 is contractive, since it is contractive on B_0, which is dense in the unit ball of $\mathrm{span}(B_0)$. This, along with the fact that $\mathrm{span}(B_0)$ is dense in B, allows us to extend uniquely Φ_0 to an equivariant, unital homomorphism $(B, \beta) \to \left(A^{\mathcal{U}} \cap A', \alpha^{\mathcal{U}}\right)$, as desired. □

Proof of Theorem 5.1. For every $\tau \in T(A)^\alpha$, we abbreviate $\pi_\tau(A)''$ by $\mathcal{M}_\tau$. Then $(\mathcal{M}_\tau, \tau)$ is a separably representable, hyperfinite, type II_1 von Neumann algebra. Moreover, the dynamical system $(\mathcal{M}, \alpha^\tau)$ absorbs $(\mathcal{R}, \mu_{G/N} \circ q_N)$ by Theorem 5.4, since α^τ_g is properly outer whenever α_g is strongly outer (see, for example, [25, Remark 2.17] or [67, Proposition 5.7]). The dynamical system $(\mathcal{R}, \mu_{G/N} \circ q_N)$ is cocycle conjugate to $(\overline{\bigotimes}_{n\in\mathbb{N}}\mathcal{R}, \bigotimes_{n\in\mathbb{N}}\mu_{G/N} \circ q_N)$ by [55, Theorem 2.6], therefore by Proposition 5.3 there exists an equivariant, unital embedding

$$\varphi_\tau \colon (\mathcal{R}, \mu_{G/N} \circ q_N) \to \left(\mathcal{M}^{\mathcal{U}}_\tau \cap \mathcal{M}'_\tau, (\alpha^\tau)^{\mathcal{U}}\right).$$

Claim 5.6.1. *There exists a unital, monotracial, separable, simple, $\mu_{G/N} \circ q_N$-invariant C^*-subalgebra B of $\mathcal{R}$.*[3]

We recall that being a finite von Neumann algebra, the hyperfinite II$_1$-factor $\mathcal{R}$ has the *strong Dixmier property* (see [2, Definition III.2.5.16, Theorem III.2.5.18]). This means that for every $a \in \mathcal{R}$ the norm-closed convex hull of $\{uau^* \colon u \in \mathcal{U}(\mathcal{R})\}$ intersects the center in exactly one element which, in this case, is necessarily $\tau_{\mathcal{R}}(a)$. Given any $c \in \mathcal{R}$, we can thus find a countable set $W \subseteq \mathcal{U}(\mathcal{R})$ such that the intersection of $\overline{\mathrm{conv}\{ucu^* \colon u \in W\}}^{\|\cdot\|}$ with the center of $\mathcal{R}$ is precisely $\{\tau_{\mathcal{R}}(c)\}$. If $C \subseteq \mathcal{R}$ is a C^*-subalgebra containing $W \cup \{c\}$, then the ideal I in C generated by c contains $\overline{\mathrm{conv}\{ucu^* \colon u \in W\}}^{\|\cdot\|}$. In particular, I contains the scalar $\tau_{\mathcal{R}}(c)$, and thus $I = C$ if c is a non-zero positive element. Furthermore, since all traces on C are constant on $\overline{\mathrm{conv}\{ucu^* \colon u \in W\}}^{\|\cdot\|}$, they all map c to $\tau_{\mathcal{R}}(c)$. Therefore, given a norm-separable C^*-subalgebra $C \subseteq \mathcal{R}$, the strong Dixmier property can be used a countable number of times to obtain a separable, simple, unital, monotracial C^*-algebra B_0 containing $1_{\mathcal{R}}$ such that $C \subseteq B_0 \subseteq \mathcal{R}$. On the other hand, using the fact that G is countable, we can find a separable, $(\mu_{G/N} \circ q_N)$-invariant C^*-algebra B_0' such that $B_0 \subseteq B_0' \subseteq \mathcal{R}$. By iterating this construction we can build an increasing sequence

$$B_0 \subseteq B_0' \subseteq \cdots \subseteq B_n \subseteq B_n' \subseteq \cdots \subseteq \mathcal{R},$$

such that B_n is separable, simple, and monotracial, and such that B_n' is separable, and $\mu_{G/N} \circ q_N$-invariant, for every $n \in \mathbb{N}$. It follows that the inductive limit of this sequence is monotracial, separable, simple, $(\mu_{G/N} \circ q_N)$-invariant and it contains $1_{\mathcal{R}}$. This proves the claim.

Let B be a C^*-subalgebra of $\mathcal{R}$ as in the above claim, and let β denote the restriction of $\mu_{G/N} \circ q_N$ to B. By restricting φ_τ to (B, β), we obtain an equivariant, unital embedding $(B, \beta) \to (\mathcal{M}_\tau^{\mathcal{U}} \cap \mathcal{M}_\tau', (\alpha^\tau)^{\mathcal{U}})$. By Proposition 5.6, there exists an equivariant, unital homomorphism $\Phi \colon (B, \beta) \to (A^{\mathcal{U}} \cap A', \alpha^{\mathcal{U}})$, which is injective as B is simple. Since B has a unique trace τ_B, the map Φ is $(\|\cdot\|_{2,\tau_B}$-$\|\cdot\|_{2,T_{\mathcal{U}}(A)})$-contractive. As moreover the norm-unit ball of $A^{\mathcal{U}}$ is $\|\cdot\|_{2,T_{\mathcal{U}}(A)}$-complete (see [6, Lemma 1.6]), the map Φ can be extended by continuity to an equivariant unital embedding $(\mathcal{R}, \mu_{G/N} \circ q_N) \to (A^{\mathcal{U}} \cap A', \alpha_{\mathcal{U}})$. □

The following is the main result of this section. It is an equivariant version of [7, Theorem 4.6], and summarizes the results of this and the previous section.

Theorem 5.7. *Let A be a separable, unital, nuclear C^*-algebra with non-empty trace space and with no finite-dimensional quotients. Let G be a countable, discrete, amenable group and let $\alpha \colon G \to Aut(A)$ be an action such that the induced action on $T(A)$ has finite orbits bounded in size by a constant $M > 0$. Then the following are equivalent:*

(1) (A, α) has uniform property Γ,
(2) (A, α) has CPoU with constant M,
(3) for every $n \in \mathbb{N}$ there is a unital embedding $M_n \to (A^{\mathcal{U}} \cap A')^{\alpha^{\mathcal{U}}}$.

Proof. (1) ⇒ (2) follows by Theorem 4.3, while (3) ⇒ (1) is an immediate consequence of the definition of uniform property Γ, and both these implications do not require the assumption that A has no finite-dimensional quotients.

[3] This claim is immediate for certain specific outer actions of G/N on $\mathcal{R}$, such as the Bernoulli shifts. On the other hand, and even though any two outer actions of G/N on $\mathcal{R}$ are cocycle conjugate, it is not clear how to obtain the claim for an arbitrary outer action only using that it is true for *some* outer action, due to the 1-cocycle.

(2) $\Rightarrow$ (3). Since A is nuclear and has no finite-dimensional quotients, $\mathcal{M}_\tau$ is a hyperfinite, type II_1 von Neumann algebra for every $\tau \in T(A)$. By Theorem 5.4 and Proposition 5.3, the dynamical system $(\mathcal{R}, \mathrm{id}_{\mathcal{R}})$ embeds unitally into $(\mathcal{M}_\tau^{\mathcal{U}} \cap \mathcal{M}_\tau', (\alpha^\tau)^{\mathcal{U}})$ for every $\tau \in T(A)^\alpha$. Given $n \in \mathbb{N}$, by composing with a unital embedding $M_n \to \mathcal{R}$ we get a unital equivariant embedding $(M_n, \mathrm{id}_{M_n}) \to (\mathcal{M}_\tau^{\mathcal{U}} \cap \mathcal{M}_\tau', (\alpha^\tau)^{\mathcal{U}})$. The conclusion then follows by Proposition 5.6. $\square$

6. Model actions with Rokhlin towers

In this section, we analyze specific model examples of actions of residually finite groups which either have finite Rokhlin dimension or satisfy part of the definition. This is a technical step in our proof of the equivalences of Theorem A (see Theorem 7.8) concerning the Rokhlin dimension of the dynamical system (A, α): it will be shown that under the assumptions of Theorem A, strong outerness of α implies that those model actions can be embedded equivariantly into the central sequence algebra of A.

We begin by computing the Rokhlin dimension of a natural product-type action.

Proposition 6.1. *Let G be a finite group. Set $D = \bigotimes_{n\in\mathbb{N}} \mathcal{B}(\ell^2(G)^{\otimes n} \oplus \mathbb{C})$. Denote by $\lambda\colon G \to \mathcal{U}(\ell^2(G))$ the left regular representation. Define an action $\alpha\colon G \to \mathrm{Aut}(D)$ by $\alpha_g = \bigotimes_{n\in\mathbb{N}} \mathrm{Ad}(\lambda_g^{\otimes n} \oplus 1)$, for all $g \in G$. Then $\dim_{\mathrm{Rok}}(\alpha) = 1$.*

Proof. Given $m \in \mathbb{N}$, set $D_m = \mathcal{B}(\ell^2(G)^{\otimes m} \oplus \mathbb{C})$ and let $\alpha^{(m)}\colon G \to \mathrm{Aut}(D_m)$ be the action given by $\alpha_g^{(m)} = \mathrm{Ad}(\lambda_g^{\otimes m} \oplus 1)$ for all $g \in G$.

Claim 6.1.1. *Let $\varepsilon > 0$ and fix $n_0 \in \mathbb{N}$. Then there exist $m \in \mathbb{N}$ with $m \geq n_0$ and positive contractions $f_g^{(j)} \in D_m$, for $g \in G$ and $j = 0, 1$, satisfying*

(1) $\alpha_g^{(m)}(f_h^{(j)}) = f_{gh}^{(j)}$ for all $g, h \in G$ and for all $j = 0, 1$,
(2) $f_g^{(j)} f_h^{(j)} = 0$ for all $g, h \in G$ with $g \neq h$ and for all $j = 0, 1$,
(3) $\left\| 1 - \sum_{j=0}^{1} \sum_{g\in G} f_g^{(j)} \right\| < \varepsilon$.

For the $\varepsilon > 0$ given, choose $m \in \mathbb{N}$ such that $|G|^{m-1} > 1/\varepsilon$ and also $m \geq n_0$. By Fell's absorption principle ([3, Theorem 2.5.5]), if $\pi\colon G \to \mathcal{U}(\mathcal{H})$ is any finite dimensional representation of G on a separable Hilbert space $\mathcal{H}$, then $\lambda \otimes \pi$ is unitarily equivalent to a direct sum of $\dim(\mathcal{H})$ copies of λ. It follows that $\lambda^{\otimes m}$ is unitarily equivalent to the direct sum of $|G|^{m-1}$ copies of λ, and thus $\lambda^{\otimes m} \oplus 1$ is unitarily equivalent to $(\bigoplus_{k=1}^{|G|^{m-1}} \lambda) \oplus 1$. We fix such an identification for the remainder of the proof.

Since λ contains a copy of the trivial representation 1, there is a unitary representation $\widetilde{\lambda}\colon G \to \mathcal{U}(V)$ such that λ is unitarily equivalent to $1 \oplus \widetilde{\lambda}$. Then $\lambda^{\otimes m} \oplus 1$ is unitarily conjugate to the diagonal representation $\mathrm{diag}(1, \widetilde{\lambda}, 1, \widetilde{\lambda}, 1 \ldots, \widetilde{\lambda}, 1)$, where the trivial representation appears $|G|^{m-1} + 1$ times and $\widetilde{\lambda}$ appears $|G|^{m-1}$ times.

For $g \in G$, let $\delta_g \in \ell^2(G)$ be the corresponding Dirac function, and let $e_g \in \mathcal{B}(\ell^2(G))$ be the projection onto the span of δ_g. By taking a suitable unitary conjugation of the e_g, we find projections $p_g \in \mathcal{B}(\mathbb{C} \oplus V)$ satisfying $\sum_{g\in G} p_g = 1$ and $\mathrm{Ad}(1 \oplus \widetilde{\lambda}_g)(p_h) = p_{gh}$ for all $g, h \in G$. Similarly, let $q_g \in \mathcal{B}(V \oplus \mathbb{C})$, for $g \in G$, be projections satisfying $\sum_{g\in G} q_g = 1$ and $\mathrm{Ad}(\widetilde{\lambda}_g \oplus 1)(q_h) = q_{gh}$ for all $g, h \in G$.

Let $a_0\colon [0, |G|^{m-1}] \to [0, 1]$ be defined as $a_0(x) = \frac{x}{|G|^{m-1}}$, and set $a_1 = 1 - a_0$. For $g \in G$, set

$$f_g^{(0)} = \mathrm{diag}(0, a_0(1)q_g, \ldots, a_0(|G|^{m-1})q_g)$$

and

$$f_g^{(1)} = \mathrm{diag}(a_1(1)p_g, \ldots, a_1(|G|^{m-1})p_g, 0).$$

These are positive contractions satisfying conditions (1), (2) and (3) above, and the claim is proved.

We note that, since G is a finite group, it suffices to take $H = \{e\}$ in Definition 2.6. The claim shows that there exist Rokhlin towers in D satisfying conditions (a), (b) and (c) in Proposition 2.7 for $d = 1$ and $H = \{e\}$. We now explain how to find new towers satisfying these conditions in addition to condition (d). Let $F \subseteq D$ be a finite set. For the $\varepsilon > 0$ given above, find $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, the unital equivariant embedding $\varphi_n \colon (D_n, \alpha^{(n)}) \hookrightarrow (D, \alpha)$ into the n-th coordinate satisfies

$$\|\varphi_n(x)a - a\varphi_n(x)\| \leq \frac{\varepsilon}{2}\|x\|$$

for all $x \in D_n$ and all $a \in F$. Use the claim to find $m \in \mathbb{N}$ with $m \geq n_0$ and positive contractions $f_g^{(j)} \in D_m$, for $g \in G$ and $j = 0, 1$, satisfying conditions (1), (2) and (3) above for the tolerance $\varepsilon/2$. One checks that the positive contractions $\varphi_m(f_g^{(j)}) \in D$, for $g \in G$ and $j = 0, 1$, satisfy conditions (a) through (d) in Proposition 2.7, as desired. It follows that $\dim_{\mathrm{Rok}}(\alpha) \leq 1$.

Finally, $\dim_{\mathrm{Rok}}(\alpha) = 0$ is impossible because the unit of D is not divisible by $|G|$ in $K_0(D)$. $\square$

Using the computation above, we will show that certain canonical actions on dimension drop algebras admit Rokhlin towers that satisfy all the conditions in Definition 2.6 except for centrality. We define these actions next.

Definition 6.2. Let G be a finite group, and let $k \in \mathbb{N}$. We denote by $I_G^{(k)}$ the dimension drop algebra

$$I_G^{(k)} = \left\{ f \in C\Big([0,1], \mathcal{B}(\ell^2(G)^{\otimes k}) \otimes \mathcal{B}(\ell^2(G)^{\otimes k} \oplus \mathbb{C})\Big) \colon \begin{array}{l} f(0) \in \mathcal{B}(\ell^2(G)^{\otimes k}) \otimes 1, \\ f(1) \in 1 \otimes \mathcal{B}(\ell^2(G)^{\otimes k} \oplus \mathbb{C}) \end{array} \right\}.$$

We denote by $\mu_G^{(k)} \colon G \to \mathrm{Aut}(I_G^{(k)})$ the restriction to $I_G^{(k)}$ of the action of G on $C([0,1]) \otimes \mathcal{B}(\ell^2(G)^{\otimes k}) \otimes \mathcal{B}(\ell^2(G)^{\otimes k} \oplus \mathbb{C})$ given by $\mathrm{id}_{C([0,1])} \otimes \mathrm{Ad}(\lambda^{\otimes k}) \otimes \mathrm{Ad}(\lambda^{\otimes k} \oplus 1_{\mathbb{C}})$.

Proposition 6.3. *Let $\varepsilon > 0$, let G be a finite group, and adopt the notation for $(I_G^{(k)}, \mu_G^{(k)})$ from Definition 6.2. Then there exist $k \in \mathbb{N}$ and positive contractions $f_g^{(j)} \in I_G^{(k)}$, for $g \in G$ and $j = 0, 1, 2$, satisfying*

(a) $\big\|(\mu_G^{(k)})_g(f_h^{(j)}) - f_{gh}^{(j)}\big\| < \varepsilon$ *for $j = 0, 1, 2$, and for all $g, h \in G$,*
(b) $\|f_g^{(j)} f_h^{(j)}\| < \varepsilon$ *for $j = 0, 1, 2$, and for all $g, h \in G$ with $g \neq h$,*
(c) $\Big\|1 - \sum_{g \in G} f_g^{(0)} + f_g^{(1)}\Big\| < \varepsilon$.

Proof. By Proposition 6.1 (see specifically the claim in its proof), we can find $k \in \mathbb{N}$ and positive contractions $\widetilde{f}_g^{(j)} \in \mathcal{B}(\ell^2(G)^{\otimes k} \oplus \mathbb{C})$, for $g \in G$ and $j = 0, 1$, satisfying

(i) $\|\mathrm{Ad}(\lambda_g^{\otimes k} \oplus 1)(\widetilde{f}_h^{(j)}) - \widetilde{f}_{gh}^{(j)}\| < \varepsilon$ for all $j = 0, 1$, and for all $g, h \in G$,
(ii) $\|\widetilde{f}_g^{(j)} \widetilde{f}_h^{(j)}\| < \varepsilon$ for all $j = 0, 1$, and for all $g, h \in G$ with $g \neq h$,
(iii) $\Big\|1 - \sum_{g \in G}(\widetilde{f}_g^{(0)} + \widetilde{f}_g^{(1)})\Big\| < \varepsilon$.

For $g \in G$, denote by $e_g \in \mathcal{B}(\ell^2(G))$ the projection onto the span of $\delta_g \in \ell^2(G)$. Then $\mathrm{Ad}(\lambda_g)(e_h) = e_{gh}$ for all $g, h \in G$, and $\sum_{g \in G} e_g = 1$. Regard e_g as an element in $\mathcal{B}(\ell^2(G)^{\otimes k}) \cong \mathcal{B}(\ell^2(G))^{\otimes k}$ via the first factor embedding.

Let $h_0 \in C([0,1])$ denote the inclusion of $[0,1]$ into $\mathbb{C}$, and set $h_1 = 1 - h_0 \in C([0,1])$. For $g \in G$ and $j = 0, 1, 2$, set

$$f_g^{(j)} = \begin{cases} h_0 \widetilde{f}_g^{(j)} & \text{if } j = 0, 1; \\ h_1 e_g & \text{if } j = 2. \end{cases}$$

One checks that conditions (a), (b) and (c) in the statement are satisfied, completing the proof. □

Next, we give a recipe for constructing unital equivariant homomorphisms from $(I_G^{(k)}, \mu_G^{(k)})$; see Theorem 6.7. We do so in a generality greater than necessary, because the proof is not more complicated and in fact the higher level of abstraction makes the argument conceptually clearer.

We need some preparatory facts about $C(X)$-algebras first. The following definition is standard. For a C^*-algebra A, we denote its center by $Z(A)$.

Definition 6.4. Let X be a compact Hausdorff space. A unital $C(X)$*-algebra* is a pair (A, ζ) consisting of a unital C^*-algebra A and a unital homomorphism $\zeta\colon C(X) \to Z(A)$. If A is a unital $C(X)$-algebra and $U \subseteq X$ is open, then $\zeta(C_0(U))A$ is an ideal in A. Given $x \in X$, the *fiber over* x is the quotient

$$A(x) := A/\zeta(C(X \setminus \{x\}))A.$$

For $a \in A$, we write a_x for its image in $A(x)$.

If (A, ζ_A) and (B, ζ_B) are unital $C(X)$-algebras and $\varphi\colon A \to B$ is a homomorphism, we say that φ is a $C(X)$*-homomorphism* if $\varphi \circ \zeta_A = \zeta_B$. If this is the case, then for every $x \in X$, the map φ induces a unital homomorphism between the corresponding fibers, which we denote by $\varphi_x\colon A(x) \to B(x)$.

As is customary, we will usually suppress ζ from the notation for $C(X)$-algebras. Recall that if A is a $C(X)$-algebra and $a \in A$, then the function $x \mapsto \|a_x\|$ is upper-semicontinuous. We assume that the following proposition may well be known, but we were not able to find it in the literature.

Proposition 6.5. *Let X be a compact Hausdorff space, let A and B be unital $C(X)$-algebras, and let $\varphi\colon A \to B$ be a $C(X)$-homomorphism. Then φ is injective (respectively, surjective) if and only if φ_x is injective (respectively, surjective) for all $x \in X$. In particular, φ is an isomorphism if and only if φ_x is an isomorphism for every $x \in X$.*

Proof. We begin with the assertion regarding injectivity. Assume that φ is injective, and fix $x \in X$. Arguing by contradiction, assume that φ_x is not injective, and find $a \in A$ such that $\|a_x\| = 1$ and $\varphi_x(a_x) = 0$. By upper-semicontinuity of the norm function on B, there is an open neighborhood U of x such that $\|\varphi(a)_y\| < 1/2$ for all $y \in U$. Let V be an open neighborhood of x such that $\overline{V} \subseteq U$. Let $f \in C(X)$ be supported on V and such that $f(x) = 1$. Using part (ii) of Lemma 2.1 of [10] at the fourth step, we get

$$1 = \|f(x)a_x\| \leq \|\varphi(fa)\| = \|f\varphi(a)\| \leq \sup_{y \in \overline{V}} \|f(y)\varphi(a)_y\| < 1/2.$$

This contradiction implies that φ_x is injective, as desired.

Conversely, assume that φ_x is injective for all $x \in X$ and let $a \in \ker(\varphi)$. Since $a_x \in \ker(\varphi_x)$ for all $x \in X$, we must have $a_x = 0$ for all $x \in X$, and thus $a = 0$.

We now prove the statement about surjectivity. Assume that φ is surjective, and let $x \in X$. Denote by $\pi_x^A\colon A \to A(x)$ and $\pi_x^B\colon B \to B(x)$ the corresponding quotient maps, and note that $\pi_x^B \circ \varphi = \varphi_x \circ \pi_x^A$. Let

$c \in B(x)$, and find $b \in B$ such that $\pi_x^B(b) = c$. Since φ is surjective, there is $a \in A$ with $\varphi(a) = b$. Then $\varphi_x(a_x) = \varphi(a)_x = c$. Thus φ_x is surjective.

Conversely, assume that φ_x is surjective for all $x \in X$. Let $b \in B$ and let $\varepsilon > 0$. Given $x \in X$, find $c^{(x)} \in A(x)$ such that $\varphi_x(c^{(x)}) = b_x$. Since π_x^A is surjective, we find $a^{(x)} \in A$ such that $a_x^{(x)} = c^{(x)}$. Since the norm function on A is upper-semicontinuous, there exists an open set $U_x \subseteq X$ containing x such that $\|\varphi(a_y^{(x)}) - b_y\| < \varepsilon$ for every $y \in U_x$. Since X is compact, we can find $x_1, \ldots, x_n$ such that $\mathcal{U} = \{U_{x_1}, \ldots, U_{x_n}\}$ covers X. Let $f_1, \ldots, f_n \in C(X)$ be a partition of unity subordinate to $\mathcal{U}$, and set $a = \sum_{j=1}^n f_j a^{(x_j)} \in A$. Then $\|\varphi(a)_x - b_x\| < \varepsilon$ for every $x \in X$, and hence $\|\varphi(a) - b\| < \varepsilon$ by part (ii) in Lemma 2.1 of [10]. Since $\varepsilon > 0$ is arbitrary, we conclude that φ is surjective. $\square$

Remark 6.6. Let $n \in \mathbb{N}$. Recall that the matrix algebra M_n is the universal C^*-algebra generated by elements $\{e_{1,k}\}_{k=1}^n$ satisfying $e_{1,k}e_{1,j} = 0$ when $k \neq 1$, $e_{1,k}e_{1,j}^* = \delta_{k,j}e_{1,1}$ for all $j, k = 1, \ldots, n$ and such that $e_{1,1}$ is a projection. By setting $e_{j,k} = e_{1,j}^* e_{1,k}$ for every $j, k = 1, \ldots, n$, it is well-known that M_n is also the universal C^*-algebra generated by $\{e_{j,k}\}_{1 \leq j,k \leq n}$ with the relations $e_{j,k}^* = e_{k,j}$ and $e_{i,j}e_{k,\ell} = \delta_{j,k}e_{i,\ell}$ for every $i, j, k, \ell = 1, \ldots, n$. We will use freely both representations.

The following is an equivariant version of a well-known characterization of the dimension drop algebra from [59, Proposition 5.1]. As it is not clear how the proof in [59, Proposition 5.1] can be adapted to the equivariant setting, we give here a different and more explicit proof.

Theorem 6.7. *Let G be a finite group, let B be a unital C^*-algebra, and let $\beta\colon G \to \mathrm{Aut}(B)$ be an action. Let $n \in \mathbb{N}$, let $v\colon G \to M_n$ be a unitary representation, and suppose there is a rank-one projection e in M_n such that $v_g e = e$ for all $g \in G$. Suppose that there exist a completely positive contractive equivariant order zero map*

$$\xi\colon\ (M_n, \mathrm{Ad}(v)) \to (B, \beta)$$

and a contraction $s \in B^\beta$ satisfying $\xi(e)s = s$ and $\xi(1) + s^*s = 1$. Let γ be the restriction to $I_{n,n+1}$ of the action of G on $C([0,1]) \otimes M_n \otimes M_{n+1}$ given by

$$\mathrm{id}_{C([0,1])} \otimes \mathrm{Ad}(v) \otimes \mathrm{Ad}(v \oplus 1_{\mathbb{C}}).$$

Then there exists a unital, equivariant homomorphism $\varphi\colon (I_{n,n+1}, \gamma) \to (B, \beta)$.

Proof. Denote by D the universal C^*-algebra generated by the set $\{s, f_{j,k}\colon j, k = 1, \ldots, n\}$ of contractions satisfying:

(R.1) $f_{j,k}^* = f_{k,j}$ for all $j, k = 1, \ldots, n$,
(R.2) $f_{j,k}f_{\ell,m} = \delta_{k,\ell}f_{j,j}f_{j,m}$ for all $j, k, \ell, m = 1, \ldots, n$,
(R.3) $f_{1,1}s = s$,
(R.4) $\sum_{j=1}^n f_{j,j} + s^*s = 1$.

It is clear that B admits a unital homomorphism from D. Most of the proof consists of showing that D is isomorphic to the dimension drop algebra $I_{n,n+1}$, which will then give us a unital homomorphism $\varphi\colon I_{n,n+1} \to B$. The last step will be to show that the map φ can be chosen to be equivariant.

Consider the following matrices in $M_{n(n+1)}$ (each vertical line comes after $n+1$ entries, and the horizontal line appears after n rows):

$$F_{1,1} = \left(\begin{array}{ccccc|ccccc|c|ccccc}
1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\hline
\multicolumn{5}{c|}{\vdots} & \multicolumn{5}{c|}{\vdots} & \vdots & \multicolumn{5}{c}{\vdots} \\
\hline
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0
\end{array}\right)$$

$$F_{1,2} = \left(\begin{array}{ccccc|ccccc|c|ccccc}
0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\hline
\multicolumn{5}{c|}{\vdots} & \multicolumn{5}{c|}{\vdots} & \vdots & \multicolumn{5}{c}{\vdots} \\
\hline
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0
\end{array}\right)$$

$$\vdots$$

$$F_{1,n} = \left(\begin{array}{ccccc|ccccc|c|ccccc}
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 \\
\hline
\multicolumn{5}{c|}{\vdots} & \multicolumn{5}{c|}{\vdots} & \vdots & \multicolumn{5}{c}{\vdots} \\
\hline
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0
\end{array}\right)$$

$$F_{1,n+1} = \left(\begin{array}{ccccc|ccccc|c|ccccc}
0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \\
\hline
\multicolumn{5}{c|}{\vdots} & \multicolumn{5}{c|}{\vdots} & \vdots & \multicolumn{5}{c}{\vdots} \\
\hline
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0
\end{array}\right)$$

The elements $F_{1,j}$, for $j = 1, 2, \ldots, n+1$, are partial isometries satisfying $F_{1,j}F_{1,i} = 0$ when $j \neq 1$, $F_{1,j}F_{1,i}^* = \delta_{i,j}F_{1,1}$ for all $i, j \in \{1, 2, \ldots, n+1\}$, and $\sum_{j=1}^{n+1} F_{1,j}^* F_{1,j} = 1$. Therefore they generate a unital copy of M_{n+1}. We identify $I_{n,n+1}$ with the algebra of continuous functions from $[0,1]$ to $M_{n(n+1)}$ such that $f(1)$ is in the C^*-algebra generated by $\{F_{1,k} \mid k = 1, 2, \ldots, n+1\}$ (which is isomorphic to M_{n+1}) and $f(0)$ is in the commutant of the C^*-algebra generated by $\{F_{1,k} \mid k = 1, 2, \ldots, n+1\}$ (which is isomorphic to M_n).

Denote by $\rho\colon [0,1] \to [0,1]$ the identity function. For $j, k = 1, \ldots, n$, let $\tilde{f}_{j,k} \in I_{n,n+1}$ be the matrix-valued function which, written in block form where each block has size $(n+1) \times (n+1)$, has in its (j,k)-th block the diagonal matrix valued function $\mathrm{diag}(\underbrace{1, 1, \ldots, 1}_{n \text{ times}}, 1-\rho)$, and 0 elsewhere. Let $\tilde{s}$ be the matrix-valued function

$$\tilde{s} = \left(\begin{array}{ccccc|ccccc|c|ccccc}
0 & 0 & \cdots & 0 & \sqrt{\rho} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \sqrt{\rho} & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \sqrt{\rho} \\
\hline
\multicolumn{5}{c|}{\vdots} & \multicolumn{5}{c|}{\vdots} & \vdots & \multicolumn{5}{c}{\vdots} \\
\hline
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0
\end{array}\right),$$

where each vertical dividing line represents $n+1$ entries.

One checks that the functions $\tilde{f}_{j,k}$ for $j,k = 1,\dots,n$ and $\tilde{s}$ satisfy the relations defining D. Fix a unital homomorphism $\pi\colon D \to I_{n,n+1}$ satisfying $\pi(f_{j,k}) = \tilde{f}_{j,k}$ for all $j,k = 1,\dots,n$ and $\pi(s) = \tilde{s}$. We will show that π is an isomorphism.

Claim 6.7.1. *The following identities hold:*

(1) $f_{j,j}f_{j,k} = f_{j,k}f_{k,k}$ *for all* $j,k = 1,\dots,n$.
(2) For $j = 1,\dots,n$, *we have* $sf_{j,1}s = 0$.

The first equality follows from item (R.2) in the list of relations defining D. The following computation establishes the second identity:

$$\begin{aligned}
(sf_{j,1}s)^*(sf_{j,1}s) &= s^*f_{1,j}s^*sf_{j,1}s \\
&\overset{\text{(R.4)}}{=} s^*f_{1,j}\Big(1 - \sum_{i=1}^{n} f_{i,i}\Big)f_{j,1}s \\
&\overset{\text{(R.2)}}{=} s^*f_{1,j}f_{j,1}s - s^*f_{1,j}f_{j,j}f_{j,1}s \\
&\overset{\text{(R.2)}}{=} s^*f_{1,1}^2s - s^*f_{1,1}^3s \overset{\text{(R.3)}}{=} 0.
\end{aligned}$$

This proves the claim.

Set

$$b = s^*s + \sum_{j=1}^{n} f_{j,1}ss^*f_{1,j} = s^*s + ss^* + \sum_{j=2}^{n} f_{j,1}ss^*f_{1,j}. \tag{6.1}$$

Note that b is a positive contraction (since all of the summands in its definition are pairwise orthogonal positive contractions). One checks that $\pi(b) = \rho \cdot 1$, and therefore $\mathrm{sp}(b) = [0,1]$.

Claim 6.7.2. *The element b belongs to the center of D.*

Let $j,k = 1,\dots,n$. Then

$$f_{j,k}b \overset{\text{(R.2)}}{=} f_{j,k}s^*s + f_{j,j}f_{j,1}ss^*f_{1,k} \quad \text{and} \quad bf_{j,k} \overset{\text{(R.2)}}{=} s^*sf_{j,k} + f_{j,1}ss^*f_{1,k}f_{k,k}.$$

We show term by term that both expressions agree, which will imply the claim. For the first terms in both right hand sides, we have

$$f_{j,k}s^*s \overset{(R.4)}{=} f_{j,k}\left(1-\sum_{i=1}^n f_{i,i}\right) \overset{(R.2)}{=} f_{j,k} - f_{j,k}f_{k,k} \overset{(R.2)}{=} f_{j,k} - f_{j,j}f_{j,k}$$
$$\overset{(R.2)}{=} \left(1-\sum_{i=1}^n f_{i,i}\right)f_{j,k} \overset{(R.4)}{=} s^*sf_{j,k}.$$

For the second terms in both right hand sides, we have

$$f_{j,j}f_{j,1}ss^*f_{1,k} \overset{(R.2)}{=} f_{j,1}(f_{1,1}s)s^*f_{1,k} \overset{(R.3)}{=} f_{j,1}ss^*f_{1,k}$$
$$\overset{(R.3)}{=} f_{j,1}s(f_{1,1}s)^*f_{1,k} \overset{(R.2)}{=} f_{j,1}ss^*f_{1,k}f_{k,k}.$$

Likewise, using the fact that $sf_{i,1}s = 0$ for all $i = 1, \dots, n$, proved in Claim 6.7.1, we get

$$sb = ss^*s + s\sum_{i=1}^n f_{i,1}ss^*f_{1,i} = ss^*s. \tag{6.2}$$

Moreover, by Claim 6.7.1, $s^*s^2 = s^*sf_{1,1}s = 0$, which implies $s^{*2}s^2 = 0$ and therefore

$$s^2 = 0, \tag{6.3}$$

which allows us to compute, using (R.2) and (R.3) for the second equality and using (6.3) and (R.3) for the third equality:

$$bs = s^*s^2 + \sum_{i=1}^n f_{i,1}ss^*f_{1,i}s = s^*sf_{1,1}s + f_{1,1}ss^*f_{1,1}s = 0 + ss^*s = sb,.$$

This completes the proof of the claim.

It follows that b endows D with a $C([0,1])$-algebra structure, and the map π is a $C([0,1])$-homomorphism. For $t \in [0,1]$, denote by $D(t)$ and by $I_{n,n+1}(t)$ the induced fibers, and by $\pi_t\colon D(t) \to I_{n,n+1}(t)$ the corresponding unital homomorphism. By Proposition 6.5, it suffices to show that π_t is an isomorphism for all $t \in [0,1]$.

The fiber $I_{n,n+1}(t)$ is isomorphic to M_n if $t = 0$, to M_{n+1} if $t = 1$, and to $M_n \otimes M_{n+1}$ otherwise. Since π_t is unital for all $t \in [0,1]$, it suffices to show that the fibers $D(t)$ are also isomorphic to those corresponding matrix algebras. Denote by $f_{j,k}(t)$, $s(t)$ and $b(t)$ the images of the elements $f_{j,k}$, s and b in the fiber $D(t)$.

Case I: $t = 0$. Here $b(0) = 0$. In particular, as $b(0) \geq s^*(0)s(0)$, it follows that $s(0) = 0$. Therefore, $\{f_{j,k}(0)\}_{j,k=1}^n$ generates $D(0)$, and these are precisely the matrix units of M_n. Thus $D(0) \cong M_n$.

Case II: $t = 1$. Using $b(1) = 1$, we compute

$$\begin{aligned}
s(1)s^*(1) &= s(1)s^*(1)b(1)\\
&\overset{(6.1)}{=} s(1)s^*(1)s^*(1)s(1) + s(1)s^*(1)\sum_{j=1}^n f_{j,1}(1)s(1)s^*(1)f_{1,j}(1)\\
&\overset{(R.3),(6.3)}{=} 0 + (s(1)s^*(1))^2 + s(1)s^*(1)f_{1,1}(1)\sum_{j=2}^n f_{j,1}(1)s(1)s^*(1)f_{1,j}(1)\\
&\overset{(R.2)}{=} (s(1)s^*(1))^2
\end{aligned}$$

Thus $s(1)s^*(1)$ is a projection, so $s(1)$ is a partial isometry. Therefore, $s^*(1)s(1)$ is a projection as well, which is orthogonal to $s(1)s^*(1)$ as $s^*(1)s(1) + s(1)s^*(1) \leq 1$. By (R.4), we have $\sum_{j=1}^n f_{j,j}(1) = 1 - s^*s(1)$.

The summands on the left hand side are pairwise orthogonal, and right hand side is a projection. Therefore, $f_{j,j}(1)$ are pairwise orthogonal projections for $j = 1, 2, \dots, n$. It therefore follows that $f_{j,k}(1)$ are partial isometries, which satisfy $f_{j,k}(1)f_{\ell,m}(1) = \delta_{k,\ell} f_{j,m}(1)$ for all $j, k, \ell, m = 1, \dots, n$.

Thus, the family $\{f_{1,j}(1)\}_{j=1,2,\dots,n} \cup \{s(1)\}$ generates $D(1)$, and it also satisfies the relations from Remark 6.6, which shows that the C^*-algebra they generate is isomorphic to M_{n+1}.

Case III: $t \in (0, 1)$. Here $b(t) = t1$. For $j, k, \ell = 1, \dots, n$, set

$$c_{j,k} = \frac{1}{t - t^2} s(t)f_{1,j}(t)s^*(t)f_{1,k}(t) \quad \text{and} \quad d_\ell = \frac{1}{\sqrt{t}(1-t)} s(t)f_{1,\ell}(t).$$

We verify that these elements satisfy the conditions from Remark 6.6, where we regard $\{c_{j,k}\}_{j,k\leq n}$ as matrix units of the form $\{e_{1,i}\}_{i\leq n^2}$ and each d_ℓ as $e_{1,n^2+\ell}$. To that end, note that

$$\begin{aligned}
ts(t)f_{1,1}(t)^2 s^*(t) &= s(t)f_{1,1}(t)\cdot t1 \cdot f_{1,1}(t)s^*(t) \\
&\overset{(6.1)}{=} s(t)f_{1,1}(t)\Big(s^*(t)s(t) + \sum_{j=1}^{n} f_{j,1}(t)s(t)s^*(t)f_{1,j}(t)\Big)f_{1,1}(t)s^*(t) \\
&\overset{(R.2)}{=} s(t)f_{1,1}(t)s^*(t)s(t)f_{1,1}(t)s^*(t) \\
&\qquad + s(t)f_{1,1}(t)^2 s(t)s^*(t)f_{1,1}(t)^2 s^*(t) \\
&\overset{(R.3)}{=} s(t)f_{1,1}(t)s^*(t)s(t)f_{1,1}(t)s^*(t) + s(t)^2 s^*(t)f_{1,1}(t)^2 s^*(t) \\
&\overset{(6.3)}{=} s(t)f_{1,1}(t)s^*(t)s(t)f_{1,1}(t)s^*(t) + 0 \\
&\overset{(R.4)}{=} s(t)f_{1,1}(t)\Big(1 - \sum_{i=1}^{n} f_{i,i}(t)\Big)f_{1,1}(t)s^*(t) \\
&\overset{(R.2)}{=} s(t)\Big(f_{1,1}(t)^2 - f_{1,1}(t)^3\Big)s^*(t).
\end{aligned}$$

Likewise, we see that

$$ts(t)f_{1,1}(t)s^*(t) = s(t)\Big(f_{1,1}(t) - f_{1,1}(t)^2\Big)s^*(t).$$

Thus

$$s(t)f_{1,1}^2(t)s^*(t) = (1-t)s(t)f_{1,1}(t)s^*(t), \tag{6.4}$$

and therefore

$$\begin{aligned}
s(t)\Big(f_{1,1}(t)^2 - f_{1,1}(t)^3\Big)s^*(t) &= t(1-t)s(t)f_{1,1}(t)s^*(t) \\
&\overset{(R.3)}{=} t(1-t)s(t)f_{1,1}(t)s^*(t)f_{1,1}(t).
\end{aligned} \tag{6.5}$$

We can now verify that the elements $c_{j,k}$ and d_ℓ from above satisfy the conditions from Remark 6.6. For any $j, k, \ell, m = 1, \dots, n$ we have:

$$\begin{aligned}
c_{j,k}c_{\ell,m}^* &= \frac{1}{(t-t^2)^2} s(t)f_{1,j}(t)s^*(t)f_{1,k}(t) \cdot f_{m,1}(t)s(t)f_{\ell,1}(t)s^*(t) \\
&\overset{(R.2)}{=} \delta_{k,m} \frac{1}{(t-t^2)^2} s(t)f_{1,j}(t)s^*(t)\Big(f_{1,1}(t)^2 s(t)\Big)f_{\ell,1}(t)s^*(t)
\end{aligned}$$

$$
\begin{aligned}
&\overset{(R.3)}{=} \delta_{k,m}\frac{1}{(t-t^2)^2}s(t)f_{1,j}(t)s^*(t)s(t)f_{\ell,1}(t)s^*(t)\\
&\overset{(R.4)}{=} \delta_{k,m}\frac{1}{(t-t^2)^2}s(t)f_{1,j}(t)\Big(1-\sum_{i=1}^{n} f_{i,i}(t)\Big)f_{\ell,1}(t)s^*(t)\\
&\overset{(R.2)}{=} \delta_{k,m}\delta_{j,\ell}\frac{1}{(t-t^2)^2}s(t)\Big(f_{1,1}(t)^2-f_{1,1}(t)^3\Big)s^*(t)\\
&\overset{(6.5)}{=} \delta_{k,m}\delta_{j,\ell}\frac{1}{(t-t^2)^2}\cdot t(1-t)s(t)f_{1,1}(t)s^*(t)f_{1,1}(t) = \delta_{k,m}\delta_{j,\ell}c_{1,1}
\end{aligned}
$$

For $j,k = 1,\dots,n$, we have:

$$
\begin{aligned}
d_jd_k^* &= \frac{1}{t(1-t)^2}s(t)f_{1,j}(t)\cdot f_{k1}(t)s^*(t)\\
&\overset{(R.2)}{=} \delta_{j,k}\frac{1}{t(1-t)^2}s(t)f_{1,1}(t)^2s^*(t)\\
&\overset{(6.4)}{=} \delta_{j,k}\frac{1}{t(1-t)^2}\cdot(1-t)s(t)f_{1,1}(t)s^*(t) \overset{(R.3)}{=} \delta_{j,k}c_{1,1}
\end{aligned}
$$

Similarly, for $j,k,\ell = 1,\dots,n$ we have:

$$
\begin{aligned}
c_{j,k}d_\ell^* &= \frac{1}{t^{3/2}(1-t)^2}s(t)f_{1,j}(t)s^*(t)f_{1,k}(t)\cdot f_{\ell,1}(t)s^*(t)\\
&\overset{(R.2)}{=} \delta_{k,\ell}\frac{1}{t^{3/2}(1-t)^2}s(t)f_{1,j}(t)s^*(t)f_{1,1}(t)^2s^*(t)\\
&\overset{(R.3)}{=} \delta_{k,\ell}\frac{1}{t^{3/2}(1-t)^2}s(t)f_{1,j}(t)(s^*(t))^2 \overset{(6.3)}{=} 0
\end{aligned}
$$

and likewise $d_\ell c_{j,k}^* = 0$. By Remark 6.6, it follows that $\{c_{j,k}, d_l\}_{j,k,l=1,\dots,n}$ generates a copy of $M_{n(n+1)}$.

Claim 6.7.3. *The set $\{c_{j,k}, d_l \colon j,k,l = 1,\dots,n\}$ also generates $D(t)$.*

It suffices to show that this family generates $s(t)$ and $\{f_{1,j}(t)\}_{j=1}^n$. Indeed,

$$
\begin{aligned}
\sqrt{t}\sum_{j=1}^{n} c_{j,1}^*d_j &\overset{(R.3)}{=} \frac{1}{t(1-t)^2}\sum_{j=1}^{n} s(t)f_{j,1}(t)s^*(t)s(t)f_{1,j}(t)\\
&\overset{(R.4)}{=} \frac{1}{t(1-t)^2}\sum_{j=1}^{n} s(t)f_{j,1}(t)\Big(1-\sum_{m=1}^{n} f_{m,m}(t)\Big)f_{1,j}(t)\\
&\overset{(R.2)}{=} \frac{1}{t(1-t)^2}\sum_{j=1}^{n} s(t)\Big(f_{j,j}(t)^2-f_{j,j}(t)^3\Big)\\
&\overset{(R.4),(R.2)}{=} \frac{1}{t(1-t)^2}s(t)\Big((1-s^*(t)s(t))^2-(1-s^*(t)s(t))^3\Big)\\
&\overset{(6.2)}{=} \frac{1}{t(1-t)^2}\Big((1-t)^2-(1-t)^3\Big)s(t) = s(t).
\end{aligned}
$$

Given $j,k = 1,\dots,n$, we want to show that $f_{j,k}(t) = \sum_{\ell=1}^n c_{\ell,j}^*c_{\ell,k} + (1-t)d_j^*d_k$. We have

$$\begin{aligned}
\sum_{\ell=1}^{n} c_{\ell,j}^{*} c_{\ell,k} &= \frac{1}{t^2(1-t)^2} \sum_{\ell=1}^{n} f_{j,1}(t) s(t) f_{\ell,1}(t) s^*(t) \cdot s(t) f_{1,\ell}(t) s^*(t) f_{1,k}(t) \\
&\overset{(\mathrm{R.4})}{=} \frac{1}{t^2(1-t)^2} \sum_{\ell=1}^{n} f_{j,1}(t) s(t) f_{\ell,1}(t) \Big(1 - \sum_{m=1}^{n} f_{m,m}(t)\Big) f_{1,\ell}(t) s^*(t) f_{1,k}(t) \\
&\overset{(\mathrm{R.2})}{=} \frac{1}{t^2(1-t)^2} \sum_{\ell=1}^{n} f_{j,1}(t) s(t) (f_{\ell,\ell}(t)^2 - f_{\ell,\ell}(t)^3) s^*(t) f_{1,k}(t) \\
&\overset{(\mathrm{R.4})}{=} \frac{1}{t^2(1-t)^2} f_{j,1}(t) s(t) ((1 - s^*(t)s(t))^2 - (1 - s^*(t)s(t))^3) s^*(t) f_{1,k}(t) \\
&\overset{(6.2)}{=} \frac{1}{t^2(1-t)^2} \cdot t(1-t)^2 f_{j,1}(t) s(t) s^*(t) f_{1,k}(t) \\
&= \frac{1}{t} f_{j,1}(t) s(t) s^*(t) f_{1,k}(t),
\end{aligned}$$

and

$$\begin{aligned}
(1-t) d_j^* d_k &= \frac{1}{t(1-t)} f_{j,1}(t) s^*(t) s(t) f_{1,k}(t) \\
&\overset{(\mathrm{R.2}),\,(\mathrm{R.4})}{=} \frac{1}{t(1-t)} f_{j,1}(t)(1 - f_{1,1}(t)) f_{1,k}(t) \\
&\overset{(\mathrm{R.2})}{=} \frac{1}{t(1-t)} (f_{j,j}(t) f_{j,k}(t) - f_{j,j}(t) f_{j,1}(t) f_{1,k}(t)) \\
&\overset{(\mathrm{R.2}),\,(\mathrm{R.4})}{=} \frac{1}{t(1-t)} (1 - s^*(t)s(t))(f_{j,k}(t) - f_{j,1}(t) f_{1,k}(t)).
\end{aligned}$$

Moreover

$$f_{j,1} s s^* f_{1,j} f_{j,k} \overset{(\mathrm{R.2})}{=} f_{j,1} s s^* f_{1,1} f_{1,k} \overset{(\mathrm{R.3})}{=} f_{j,1} s s^* f_{1,k},$$

and

$$f_{j,1} s s^* f_{1,j} f_{j,1} f_{1,k} \overset{(\mathrm{R.2})}{=} f_{j,1} s s^* f_{1,1}^2 f_{1,k} \overset{(\mathrm{R.3})}{=} f_{j,1} s s^* f_{1,k}.$$

Thus, by definition of b and by the fact that $b(t) = t$ we obtain

$$\begin{aligned}
(1-t) d_j^* d_k &= \frac{1}{t(1-t)} (1 - s^*(t)s(t))(f_{j,k}(t) - f_{j,1}(t) f_{1,k}(t)) \\
&= \frac{1}{t(1-t)} (1 - b(t))(f_{j,k}(t) - f_{j,1}(t) f_{1,k}(t)) \\
&\overset{(\mathrm{R.2})}{=} \frac{1}{t} (f_{j,k}(t) - f_{j,j}(t) f_{j,k}(t)) \\
&\overset{(\mathrm{R.2})}{=} \frac{1}{t} \Big(1 - \sum_{m=1}^{n} f_{m,m}(t)\Big) f_{j,k}(t) \\
&\overset{(\mathrm{R.4})}{=} \frac{1}{t} s^*(t) s(t) f_{j,k}(t).
\end{aligned}$$

Combining those observations, we get

$$\begin{aligned}
\sum_{k=1}^{n} c_{k,1}^* c_{k,j} + (1-t) d_1^* d_j &= \frac{1}{t}(f_{j,1}(t)s(t)s^*(t)f_{1,k}(t) + s^*(t)s(t)f_{j,k}(t)) \\
&\overset{\text{(R.3)}}{=} \frac{1}{t}(f_{j,1}(t)s(t)s^*(t)f_{1,1}f_{1,k}(t) + s^*(t)s(t)f_{j,k}(t)) \\
&\overset{\text{(R.2)}}{=} \frac{1}{t}(f_{j,1}(t)s(t)s^*(t)f_{1,j}f_{j,k}(t) + s^*(t)s(t)f_{j,k}(t)) \\
&\overset{\text{(R.2)}}{=} \frac{1}{t}\Big(s^*(t)s(t) + \sum_{m=1}^{n} f_{m,1}(t)s(t)s^*(t)f_{1,m}(t)\Big) f_{j,k}(t) \\
&= \frac{1}{t} b(t) f_{j,k}(t) = f_{j,k}(t).
\end{aligned}$$

This proves the claim, so have proved that D is isomorphic to $I_{n,n+1}$.

By universality of $I_{n,n+1}$, there is a unique unital homomorphism $\varphi\colon I_{n,n+1} \to B$ satisfying $\varphi(f_{i,j}) = \xi(e_{i,j})$ for all $i, j = 1, \ldots, n$ and $\varphi(s) = s$. (To lighten the notation, we use the same letter s to denote the given element in B and the element s in the universal C^*-algebra $I_{n,n+1}$.) We are left with checking that φ is equivariant.

Note that β leaves $\varphi(I_{n,n+1})$ invariant. Furthermore, using the fact that $v_g e_{1,1} = e_{1,1}$ and $e_{1,j} = e_{1,1}e_{1,j}$, we have $v_g e_{1,j} = e_{1,j}$ for $j = 1, \ldots, n$ and for all $g \in G$. It follows that

$$\beta_g(\varphi(b)) = s^*s + ss^* + \sum_{j=2}^{n} \xi(v_g e_{j,1}) ss^* \xi(e_{1,j} v_g^*)$$

for all $g \in G$, we deduce thus that $\beta_g(\varphi(b)) = \varphi(b)$ for all $g \in G$. Thus, the restriction of β to $\varphi(I_{n,n+1})$ is an action via $C([0,1])$-automorphisms. It follows that it suffices to check equivariance on each fiber.

Let $t \in (0,1)$. Define finite-dimensional Hilbert spaces $\mathcal{H}_0, \mathcal{H}_1$ and $\mathcal{H}_2$, contained in $\varphi(I_{n,n+1})(t)$, via $\mathcal{H}_0 = \operatorname{span}\{\varphi_t(c_{1,1})\}$,

$$\mathcal{H}_1 = \operatorname{span}\{\varphi(c_{j,k})\colon j,k = 1,\ldots,n\} \quad \text{and} \quad \mathcal{H}_2 = \operatorname{span}\{\varphi(d_l)\colon l = 1,\ldots,n\}.$$

Then $\mathcal{H}_0$, $\mathcal{H}_1$ and $\mathcal{H}_2$ are invariant under β. Set $E = \operatorname{span}\{e_{1,1}, e_{1,2}, \ldots, e_{1,n}\}$. Note that there are natural isomorphisms $\mathcal{H}_1 \cong E \otimes E$ and $\mathcal{H}_2 \cong E$, the first one given by identifying $e_{1,j} \otimes e_{1,k}$ with $c_{j,k}$, and the second one given by identifying $e_{1,k}$ with d_k, for $j, k = 1, \ldots, n$. With these identifications, β_t acts as $v_g \otimes v_g$ on $\mathcal{H}_1$ while leaving $\mathcal{H}_0$ fixed, and acts as v_g on $\mathcal{H}_2$. Thus, the action induced by β on the fiber corresponding to some $t \in (0,1)$ is conjugate to $\operatorname{Ad}(v \otimes (v \oplus 1_{\mathbb{C}}))$. The end-cases $t = 0$ and $t = 1$ are verified similarly, thus concluding the proof. □

We will apply Theorem 6.7 in the proof of Theorem 7.8, at the end of next section, to representations v of the form $\lambda^{\otimes k}\colon G \to \mathcal{U}(\ell^2(G)^{\otimes k})$, for $k \in \mathbb{N}$, where λ is the left regular representation.

7. Regularity properties for actions of amenable groups

This section is devoted to the proofs of Theorem A and Theorem C, starting with the latter. The equivalence of item (4) with the other items in Theorem C, which is stated under the assumption of $\mathcal{Z}$-stability, is actually obtained in the presence of equivariant property (SI). This property, which we recall below, is an adaptation to the equivariant setting of Sato's property (SI), which first appeared in [62] and was further used in [52], [51], [53], [63].

Definition 7.1. Let G be a countable, discrete, amenable group, let A be a unital, separable C^*-algebra with non-empty trace space, and let $\alpha\colon G \to \operatorname{Aut}(A)$ be an action. We say that (A, α) has *equivariant property*

(SI) if for any positive contractions $a, b \in (A_\mathcal{U} \cap A')^{\alpha_\mathcal{U}}$ such that $a \in J_A$ and $\sup_{m\in\mathbb{N}} \|1 - b^m\|_{2,T_\mathcal{U}(A)} < 1$, there is $s \in (A_\mathcal{U} \cap A')^{\alpha_\mathcal{U}}$ such that $bs = s$ and $s^*s = a$.

We also need the following analogue of Kirchberg's σ-ideal ([44, Definition 1.5]) in the equivariant setting. For $\mathbb{Z}$-actions, this notion was considered in [48].

Definition 7.2. Let G be a discrete group, let B be a C^*-algebra, let $\beta\colon G \to \mathrm{Aut}(B)$ be an action, and let $J \subseteq B$ be a β-invariant ideal. We say that J is an *equivariant σ-ideal (with respect to β)*, if for every separable, β-invariant subalgebra $C \subseteq B$, there is a positive contraction $x \in (J \cap C')^\beta$ with $xc = c$ for all $c \in C \cap J$.

A σ-ideal is simply an equivariant σ-ideal with respect to the trivial action. The trace kernel ideal J_A is a σ-ideal by [45, Proposition 4.6]. Recall that J_τ (defined in subsection 2.2) is the kernel of the quotient map $\kappa_\tau\colon A_\mathcal{U} \to M_\tau^\mathcal{U}$ (see [45, Theorem 3.3]); it is also a σ-ideal (see [45, Remark 4.7]).

It is easy to see that if a finite group acts on a C^*-algebra, then any σ-ideal is automatically an equivariant σ-ideal: one simply averages the positive contraction in the definition of a σ-ideal to obtain a fixed one. When the group is amenable, one can average over Følner sets to get equivariant σ-ideals in ultrapowers, as we show below.

Proposition 7.3. *Let A be a unital C^*-algebra with non-empty trace space, let G be a countable, discrete, amenable group, let $\alpha\colon G \to \mathrm{Aut}(A)$ be any action. Let $T \subseteq T(A)$ be a G-invariant closed subset. Then $J_T \subseteq A_\mathcal{U}$ is a G-invariant, equivariant σ-ideal in $A_\mathcal{U}$.*

Proof. Abbreviate J_T to J. It is clear that J_T is a G-invariant ideal in $A_\mathcal{U}$. Let $C \subseteq A_\mathcal{U}$ be a separable, $\alpha_\mathcal{U}$-invariant subalgebra. Since J is a σ-ideal in $A_\mathcal{U}$ ([45, Proposition 4.6, Remark 4.7]), there is a positive contraction $x \in J \cap C'$ with $xc = c$ for all $c \in C \cap J$. By Kirchberg's ε-test ([44, Lemma A.1]), it is enough to prove that for every finite subset $K \subseteq G$ and every $\varepsilon > 0$, there is a positive contraction $y \in J \cap C'$ with $\|(\alpha_\mathcal{U})_k(y) - y\| < \varepsilon$ for all $k \in K$ and $yc = c$ for all $c \in C \cap J$.

We fix a finite subset $K \subseteq G$ and $\varepsilon > 0$. Using amenability of G, find a finite subset F of G such that $|kF \triangle F| \leq \frac{\varepsilon}{2}|F|$ for all $k \in K$. Set $y = \frac{1}{|F|}\sum_{g\in F}(\alpha_\mathcal{U})_g(x)$. Since C is $\alpha_\mathcal{U}$-invariant, it follows that $(\alpha_\mathcal{U})_g(x)c = c$ for every $g \in G$ and $c \in C$, therefore $yc = c$ for all $c \in C \cap J$. For $k \in K$, we have

$$\begin{aligned}
\|(\alpha_\mathcal{U})_k(y) - y\| &= \Big\|\frac{1}{|F|}\sum_{g\in F}(\alpha_\mathcal{U})_{kg}(x) - (\alpha_\mathcal{U})_g(x)\Big\| \\
&= \Big\|\frac{1}{|F|}\sum_{g\in F\setminus(kF\cup k^{-1}F)}(\alpha_\mathcal{U})_{kg}(x) - (\alpha_\mathcal{U})_g(x)\Big\| \\
&\leq \frac{1}{|F|}\sum_{g\in F\setminus(kF\cup k^{-1}F)}\|(\alpha_\mathcal{U})_{kg}(x) - (\alpha_\mathcal{U})_g(x)\| \\
&\leq 2\frac{|kF\triangle F|}{|F|} \leq 2\,\frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Since J is an $\alpha_\mathcal{U}$-invariant ideal, the positive contraction y also belongs to J. Finally, it is also easy to check that y commutes with C, since C is also invariant under $\alpha_\mathcal{U}$. This concludes the proof. $\square$

We record the following consequence of Proposition 7.3, which is used repeatedly in the sequel. Recall that $\kappa\colon (A_\mathcal{U} \cap A', \alpha_\mathcal{U}) \to (A^\mathcal{U} \cap A', \alpha^\mathcal{U})$ denotes the canonical quotient map (Lemma 2.1).

Corollary 7.4. *Let A be a unital separable C^*-algebra with non-empty trace space, let G be a countable, discrete, amenable group, let $\alpha\colon G \to \mathrm{Aut}(A)$ be any action. Let B be a separable, nuclear C^*-algebra, let $\beta\colon G \to \mathrm{Aut}(B)$ be an action, and let*

$$\Psi\colon (B, \beta) \to (A^{\mathcal{U}} \cap A', \alpha^{\mathcal{U}})$$

be an equivariant completely positive order zero map. Then there exists an equivariant completely positive order zero map $\Phi\colon (B, \beta) \to (A_{\mathcal{U}} \cap A', \alpha_{\mathcal{U}})$ satisfying $\Psi = \kappa \circ \Phi$.

Proof. By the Choi-Effros Lifting Theorem [5] there is a completely positive map $\Phi_0\colon B \to A_{\mathcal{U}} \cap A'$ such that $\Psi = \kappa \circ \Phi_0$. Let C be a separable, $\alpha_{\mathcal{U}}$-invariant subalgebra of $A_{\mathcal{U}}$ containing $A \cup \Phi_0(B)$. Since J_A is an equivariant σ-ideal by Proposition 7.3, there exists $x \in (J_A \cap C')^{\alpha_{\mathcal{U}}}$ such that $xc = c$ for all $c \in J_A \cap C$. Define $\Phi\colon B \to A_{\mathcal{U}} \cap A'$ by $\Phi(b) = (1-x)\Phi_0(b)(1-x)$ for all $b \in B$. One readily checks that Φ is completely positive, contractive, order zero and equivariant, and that $\kappa \circ \Phi = \kappa \circ \Phi_0 = \Psi$, as required. □

Another useful consequence of Proposition 7.3 is the following dynamical analogue of [45, Theorem 3.3]. We point out that the following lemma also follows from the main result of [14].

Lemma 7.5. *Let A be a separable unital C^*-algebra, let G be a discrete group, let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action and $\tau \in T(A)^\alpha$. Then the quotient maps $\kappa : A_{\mathcal{U}} \to A^{\mathcal{U}}$ and $\kappa_\tau : A_{\mathcal{U}} \to \mathcal{M}_\tau$ restrict to surjective, equivariant maps*

$$\kappa\colon (A_{\mathcal{U}} \cap A')^{\alpha_{\mathcal{U}}} \to (A^{\mathcal{U}} \cap A')^{\alpha^{\mathcal{U}}},$$
$$\kappa_\tau\colon (A_{\mathcal{U}} \cap A')^{\alpha_{\mathcal{U}}} \to (\mathcal{M}_\tau^{\mathcal{U}} \cap \mathcal{M}_\tau')^{(\alpha^\tau)^{\mathcal{U}}}.$$

Proof. We only show the statement for $\kappa : A_{\mathcal{U}} \to A^{\mathcal{U}}$, as the proof for $\kappa_\tau : A_{\mathcal{U}} \to \mathcal{M}_\tau^{\mathcal{U}}$ is analogous. Given $a \in (A^{\mathcal{U}} \cap A')^{\alpha^{\mathcal{U}}}$, there is $b \in A_{\mathcal{U}}$ such that $\kappa(b) = a$. Let $C \subseteq A_{\mathcal{U}}$ be the separable C^*-algebra generated by A and b. By Proposition 7.3 there is $x \in (J_A \cap C')^{\alpha_{\mathcal{U}}}$ such that $xc = c$ for all $c \in J_A \cap C$. This implies that $b' := (1-x)b$ belongs to $(A_{\mathcal{U}} \cap A')^{\alpha_{\mathcal{U}}}$ and it satisfies $\kappa(b') = a$. □

Theorem 7.6. *Let A be a separable, unital, nuclear, C^*-algebra with non-empty trace space and with no finite-dimensional quotients. Let G be a countable, discrete, amenable group and let $\alpha\colon G \to Aut(A)$ be an action such that the induced action on $T(A)$ has finite orbits bounded in size by a constant $M > 0$. Then the following are equivalent:*

(1) (A, α) has uniform property Γ,
(2) (A, α) has CPoU with constant M,
(3) for every $n \in \mathbb{N}$ there is a unital embedding $M_n \to (A^{\mathcal{U}} \cap A')^{\alpha^{\mathcal{U}}}$.

If A is moreover simple and $\mathcal{Z}$-stable, then the above are also equivalent to:

(4) (A, α) is cocycle conjugate to $(A \otimes \mathcal{Z}, \alpha \otimes id_{\mathcal{Z}})$.

Proof. The equivalence of (1), (2) and (3) was already proved in Theorem 5.7, and (4) ⇒ (3) is Proposition 3.3. It remains to show that (3) ⇒ (4). Fix a unital embedding $\Psi\colon M_2 \to (A^{\mathcal{U}} \cap A')^{\alpha^{\mathcal{U}}}$, which we regard as a unital equivariant homomorphism $\Psi\colon (M_2, \mathrm{id}_{M_2}) \to (A^{\mathcal{U}} \cap A', \alpha^{\mathcal{U}})$. By Corollary 7.4, there exists an equivariant completely positive contractive order zero map $\Phi\colon M_2 \to (A_{\mathcal{U}} \cap A')^{\alpha_{\mathcal{U}}}$ making the following diagram commute:

$$\begin{tikzcd} & (A_{\mathcal{U}} \cap A')^{\alpha_{\mathcal{U}}} \arrow[d, "\kappa"] \\ M_2 \arrow[ur, "\Phi"] \arrow[r, "\Psi"'] & (A^{\mathcal{U}} \cap A')^{\alpha^{\mathcal{U}}}. \end{tikzcd}$$

Note that $\tau(\Phi(e_{1,1})^m) = 1/2$ for all $\tau \in T_{\mathcal{U}}(A)$ and for all $m \in \mathbb{N}$. Set $c_1 = \Phi(e_{1,1})$ and $c_2 = \Phi(e_{1,2})$. Then

$$c_1 \geq 0, \quad c_2c_2^* = c_1^2, \quad \text{and} \quad c_1c_2^* = c_2c_1^* = 0.$$

By [67, Theorem B], the dynamical system (A, α) has equivariant property (SI). Using this, fix a contraction $s \in (A_{\mathcal{U}} \cap A')^{\alpha_{\mathcal{U}}}$ satisfying $s^*s = 1 - \sum_{j=1}^{2} c_j^* c_j$ and $c_1 s = s$. By Theorem 6.7, the elements c_1, c_2, s generate a copy of the prime dimension drop algebra

$$I_{2,3} = \{f \in C([0,1], M_2 \otimes M_3) \colon f(0) \in M_2 \otimes 1, f(1) \in 1 \otimes M_3\}$$

so there exists a unital homomorphism $I_{2,3} \to (A_{\mathcal{U}} \cap A')^{\alpha_{\mathcal{U}}}$. By repeatedly using Lemma 2.4, we can find a countable sequence of unital homomorphisms $I_{2,3} \to (A_{\mathcal{U}} \cap A')^{\alpha_{\mathcal{U}}}$ with commuting ranges, and therefore a unital homomorphism $I_{2,3}^{\otimes\infty} \to (A_{\mathcal{U}} \cap A')^{\alpha_{\mathcal{U}}}$. By [11, Theorem 1.1], $\mathcal{Z}$ embeds unitally in $I_{2,3}^{\otimes\infty}$, and in particular, we obtain a unital homomorphism $\mathcal{Z} \to (A_{\mathcal{U}} \cap A')^{\alpha_{\mathcal{U}}}$. This implies, by [66, Theorem 2.6], that α is cocycle conjugate to $\alpha \otimes \mathrm{id}_{\mathcal{Z}}$. □

We now turn our attention to the proof of Theorem A. We briefly describe our strategy for proving (1) ⇒ (2). First, we prove the existence of projections with properties analogous to those in Definition 2.9 for Bernoulli shifts on the hyperfinite II$_1$-factor $\overline{\bigotimes}_{g\in G}\mathcal{R} \cong \mathcal{R}$. Then Proposition 5.3 and Theorem 5.4 are used to show that there are similar projections also in the central sequence algebra of each fiber $\mathcal{M}_\tau$ for $\tau \in T(A)^\alpha$. Using Theorem 5.1 we then perform a 'local to global' argument via CPoU to glue these projections and obtain a Rokhlin tower in $A^{\mathcal{U}} \cap A'$. Finally, we exploit the fact that J_A is an equivariant σ-ideal (Definition 7.2) to lift those towers to $A_{\mathcal{U}} \cap A'$ and conclude the proof.

We start by addressing the first part, namely the existence of projections satisfying conditions analogous to those in Definition 2.9 for the Bernoulli shift on $\mathcal{R}$. We do this in the following proposition, using the tiling result for amenable groups from [8].

Proposition 7.7. *Let G be a countable, discrete, amenable group, and let $\beta_G \colon G \to \mathrm{Aut}(\mathcal{R})$ be the Bernoulli shift acting by left multiplication on the indices of the elementary tensors of $\overline{\bigotimes}_{g\in G}\mathcal{R} \cong \mathcal{R}$. Given a finite set $K \subseteq G$ and $\delta > 0$, there are (K, δ)-invariant, finite sets $S_1, \ldots, S_n \subseteq G$ and projections $p_{\ell,g} \in \mathcal{R}$ for $\ell = 1, \ldots, n$ and $g \in S_\ell$, such that*

(i) $(\beta_G)_{gh^{-1}}(p_{\ell,h}) = p_{\ell,g}$ *for all* $\ell = 1, \ldots, n$ *and all* $g, h \in S_\ell$,
(ii) $p_{\ell,g}p_{k,h} = 0$ *for all* $\ell, k = 1, \ldots, n$, $g \in S_\ell$, $k \in S_h$, *whenever* $(\ell, g) \neq (k, h)$,
(iii) $\sum_{\ell=1}^{n} \sum_{g\in S_\ell} p_{\ell,g} = 1$,
(iv) $\tau_{\mathcal{R}}(p_{\ell,g})$ *is positive and independent of* $g \in S_\ell$.

Proof. Fix a finite set $K \subseteq G$ and $\delta > 0$. Fix an embedding θ of $L^\infty([0,1])$ with the Lebesgue measure in $\mathcal{R}$, and let $\tilde{\beta}_G$ be the Bernoulli shift on $\overline{\bigotimes}_{g\in G}L^\infty([0,1])$. Notice that the embedding θ naturally induces an equivariant embedding

$$\Theta \colon \left(\overline{\bigotimes_{g\in G}} L^\infty([0,1]), \tilde{\beta}_G\right) \to \left(\overline{\bigotimes_{g\in G}} \mathcal{R}, \beta_G\right),$$

where Θ sends, for every $g \in G$, the g-th coordinate of the domain to the corresponding g-th coordinate of the codomain via θ.

Set $X = \prod_{g\in G}[0,1]$ with the product measure. It follows that $\overline{\bigotimes}_{g\in G} L^\infty([0,1]) \cong L^\infty(X)$, and that $\tilde{\beta}_G$ induces an action on X which is easily seen (and well known) to be measure preserving and ergodic. By [8, Theorem 3.6] there are (K,δ)-invariant sets $S_1,\ldots,S_n \subseteq G$ and projections $\tilde{p}_{\ell,g} \in L^\infty(X)$ for $\ell = 1,\ldots,n$ and $g \in S_\ell$ such that

(i)' $(\tilde{\beta}_G)_{gh^{-1}}(\tilde{p}_{\ell,h}) = \tilde{p}_{\ell,g}$ for all $\ell = 1,\ldots,n$ and all $g,h \in S_\ell$,
(ii)' $\tilde{p}_{\ell,g}\tilde{p}_{k,h} = 0$ for all $\ell, k = 1,\ldots,n$, $g \in S_\ell$, $k \in S_h$, whenever $(\ell,g) \neq (k,h)$,
(iii)' $\sum_{\ell=1}^n \sum_{g\in S_\ell} \tilde{p}_{\ell,g} = 1$.

It is clear then that the projections $p_{\ell,g} = \Theta(\tilde{p}_{\ell,g})$ for $\ell = 1,\ldots,n$ and $g \in S_\ell$ satisfy conditions (i), (ii) and (iii) of the statement.

Finally, item (iv) is an automatic consequence of the uniqueness of $\tau_{\mathcal{R}}$ as a (faithful) normal trace on $\mathcal{R}$, which is therefore β_G-invariant. More specifically, for every $\ell = 1,\ldots,n$, by condition (i) and β_G-invariance of $\tau_{\mathcal{R}}$, the value $\tau_{\mathcal{R}}(p_{\ell,g})$ is independent of $g \in S_\ell$. Thus, if $\tau_{\mathcal{R}}(p_{\ell,g}) = 0$ for some $\ell = 1,\ldots,n$ and $g \in S_\ell$, then by faithfulness it follows that $p_{\ell,h} = 0$ for all $h \in S_\ell$. Therefore, up to discarding some of the S_ℓ's, we obtain a family of projections satisfying condition (iv). □

Although (iii) implies (ii) in the statement of the above proposition, we state it explicitly because it is a condition that can be lifted along quotient maps via σ-ideals; see in particular the proof of (2) ⇒ (1) in Theorem 7.8.

For the reader's convenience, we reproduce the statement of Theorem A.

Theorem 7.8. *Let A be a separable, simple, nuclear unital C^*-algebra with non-empty trace space, let G be a countable, discrete, amenable group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Suppose that the orbits of the action induced by α on $T(A)$ are finite and with uniformly bounded cardinality. Then the following are equivalent:*

(1) α is strongly outer,
(2) $\alpha \otimes \mathrm{id}_{\mathcal{Z}}$ has the weak tracial Rokhlin property.

When G is residually finite, then the above statements are also equivalent to:

(3) $\alpha \otimes \mathrm{id}_{\mathcal{Z}}$ has finite Rokhlin dimension,
(4) $\alpha \otimes \mathrm{id}_{\mathcal{Z}}$ has Rokhlin dimension at most 2.

Proof. Note that α is strongly outer if and only if $\alpha \otimes \mathrm{id}_{\mathcal{Z}}$ is strongly outer. By replacing α with $\alpha \otimes \mathrm{id}_{\mathcal{Z}}$, we may assume that $(A,\alpha) \cong_{\mathrm{cc}} (A \otimes \mathcal{Z}, \alpha \otimes \mathrm{id}_{\mathcal{Z}})$.

(2) ⇒ (1). The argument is mostly standard, but we include it since we work with a slightly different notion of the weak tracial Rokhlin property. Fix $g \in G \setminus \{1\}$ and $\tau \in T(A)^{\alpha_g}$. We denote as usual the von Neumann algebra generated by $\pi_\tau(A)$ by $\mathcal{M}_\tau$, and note that α_g extends to an automorphism α_g^τ of $\mathcal{M}_\tau$, even though α does not necessarily induce an action on $\mathcal{M}_\tau$. Fix $K = \{g\}$ and $\delta = 1/2$. Let $S_1,\ldots,S_n$ be a family of $(\{g\},1/2)$-invariant subsets of G, and $f_{\ell,h} \in A_{\mathcal{U}} \cap A'$, for $\ell = 1,\ldots,n$ and $h \in S_\ell$, be contractions as in Definition 2.9. Let $\kappa_\tau\colon A_{\mathcal{U}} \to \mathcal{M}_\tau^{\mathcal{U}}$ be the quotient map (see Lemma 2.1), and set $p_{\ell,h} = \kappa_\tau(f_{\ell,h}) \in \mathcal{M}_\tau^{\mathcal{U}} \cap \mathcal{M}_\tau'$. Since $\sum_{\ell=1}^n \sum_{h\in S_\ell} p_{\ell,h} = 1$, we can assume that $\{p_{1,h}\}_{h\in S_1}$ is a collection of non-zero, pairwise orthogonal (hence distinct) projections. By $(\{g\},1/2)$-invariance of S_1, there exists $h \in S_1$ such that $gh \in S_1$, thus we have

$$(\alpha^\tau)^{\mathcal{U}}_g(p_{1,h}) = (\alpha^\tau)^{\mathcal{U}}_{ghh^{-1}}(p_{1,h}) = p_{1,gh}.$$

It follows that the automorphism $(\alpha^\tau)^{\mathcal{U}}_g$ acts non-trivially on the family $\{p_{1,h}\}_{h\in S_1}$. Therefore $(\alpha^\tau)^{\mathcal{U}}_g$ acts non-trivially on $\mathcal{M}^{\mathcal{U}}_\tau \cap \mathcal{M}'_\tau$, which in turn implies that α^τ_g is outer, as desired.

(1) $\Rightarrow$ (2). Assume that α is strongly outer. Let β_G be the Bernoulli shift on $\mathcal{R}$ (see Proposition 7.7), which is an outer action. Since (A, α) is assumed to be cocycle conjugate to $(A \otimes \mathcal{Z}, \alpha \otimes \mathrm{id}_{\mathcal{Z}})$, we deduce from Theorem 7.6 that (A, α) has CPoU. By Theorem 5.1 (with $N = \{e\}$), there exists a unital, equivariant embedding $\Psi\colon (\mathcal{R}, \beta_G) \to (A^{\mathcal{U}} \cap A', \alpha^{\mathcal{U}})$.

Let $K \subseteq G$ be finite and let $\delta > 0$. Using Proposition 7.7, find (K, δ)-invariant finite subsets $S_1, \dots, S_n$ of G and projections $p_{\ell,g} \in \mathcal{R}$, for $\ell = 1, \dots, n$ and $g \in S_\ell$, satisfying (i) through (iv) in Proposition 7.7. Denote by $\tau_{\mathcal{R}}$ the unique trace on $\mathcal{R}$. By lifting the projections $\Psi(p_{\ell,g})$ along the surjective map $\kappa\colon A_{\mathcal{U}}\cap A' \to A^{\mathcal{U}}\cap A'$, we find positive contractions $e_{\ell,g} \in A_{\mathcal{U}} \cap A'$, for $\ell = 1, \dots, n$ and $g \in S_\ell$, satisfying:

(a) $(\alpha_{\mathcal{U}})_{gh^{-1}}(e_{\ell,h}) - e_{\ell,g} \in J_A$ for $\ell = 1, \dots, n$ and for all $g, h \in S_\ell$,
(b) $e_{\ell,g}e_{k,h} \in J_A$ for $\ell, k = 1, \dots, n$, and for all $g \in S_\ell$, $h \in S_k$, whenever $(\ell, g) \neq (k, h)$,
(c) $1 - \sum_{\ell=1}^{n} \sum_{g\in S_\ell} e_{\ell,g} \in J_A$,
(d) $\tau(e_{\ell,g}) = \tau(\Psi(p_{\ell,g})) = \tau_{\mathcal{R}}(p_{\ell,h}) > 0$ for all $\tau \in T_{\mathcal{U}}(A)$, for $\ell = 1, \dots, n$ and for all $g, h \in S_\ell$.

Let C be a separable G-invariant subalgebra of $A_{\mathcal{U}}$ containing A and the finite set $\{e_{\ell,h} \colon \ell = 1, \dots, n, h \in S_\ell\}$. By Proposition 7.3 there exists a positive contraction $x \in (J_A\cap C')^{\alpha_{\mathcal{U}}}$ satisfying $xc = c$ for all $c \in C\cap J_A$. For $\ell = 1, \dots, n$ and $g \in S_\ell$, we set $f_{\ell,g} = (1-x)e_{\ell,g}(1-x) \in A_{\mathcal{U}} \cap A'$. We claim that these elements satisfy the conditions of Definition 2.9.

Condition (3) in Definition 2.9 is satisfied by item (c) above and because $x \in J_A$. To check (1) in Definition 2.9, let $\ell = 1, \dots, n$ and $g, h \in S_\ell$. Observe that $(1 - x)c = 0$ for every $c \in C \cap J_A$, and use this, along with invariance of x, to get

$$\begin{aligned}(\alpha_{\mathcal{U}})_{gh^{-1}}(f_{\ell,h}) - f_{\ell,g} &= (\alpha_{\mathcal{U}})_{gh^{-1}}((1-x)e_{\ell,h}(1-x)) - (1-x)e_{\ell,g}(1-x)\\ &= (1-x)\left((\alpha_{\mathcal{U}})_{gh^{-1}}(e_{\ell,h}) - e_{\ell,g}\right)(1-x) \overset{(a)}{=} 0.\end{aligned}$$

To check condition (2) in Definition 2.9, let $\ell, k = 1, \dots, n$, let $g \in S_\ell$ and let $h \in S_k$ with $(\ell, g) \neq (k, h)$. We use $[x, e_{\ell,g}] = 0$ at the second step to get

$$f_{\ell,g}f_{k,h} = (1-x)e_{\ell,g}(1-x)^2 e_{k,h}(1-x) = (1-x)e_{\ell,g}e_{k,h}(1-x)^3 \overset{(b)}{=} 0.$$

Finally, for item (4) of Definition 2.9 observe that, given $\tau \in T_{\mathcal{U}}(A)$, for $\ell = 1, \dots, n$ and for all $g \in S_\ell$, we have

$$\begin{aligned}\tau(f_{\ell,g}) &= \tau((1-x)e_{\ell,g}(1-x)) = \tau(e_{\ell,g}) + \tau(e_{\ell,g}x) + \tau(xe_{\ell,g}) + \tau(xe_{\ell,g}x)\\ &= \tau(e_{\ell,g}).\end{aligned}$$

Hence $\tau(f_{\ell,g}) = \tau_{\mathcal{R}}(p_{\ell,g}) > 0$, and this value depends only on ℓ by condition (iv) of Proposition 7.7.

From now on, we assume that G is residually finite.

(3) $\Rightarrow$ (1). Fix $g \in G\setminus\{1\}$ and $\tau \in T(A)^{\alpha_g}$. Abbreviate $\pi_\tau(A)''$ to $\mathcal{M}_\tau$, and let $\kappa_\tau\colon A_{\mathcal{U}}\cap A' \to \mathcal{M}^{\mathcal{U}}_\tau\cap\mathcal{M}'_\tau$ be the equivariant quotient map. Since G is residually finite, there is a finite-index, normal subgroup $H \leq G$ such that $g \notin H$. Using the fact that $d = \dim_{\mathrm{Rok}}(\alpha) < \infty$, find positive contractions $f^{(j)}_{\overline{k}} \in A_{\mathcal{U}} \cap A'$, for $j = 0, \dots, d$ and $\overline{k} \in G/H$, which satisfy, for $j = 0, \dots, d$, for all $g \in G$ and for all $\overline{k}, \overline{k}' \in G/H$ with $\overline{k} \neq \overline{k}'$:

$$f_{\overline{k}}^{(j)} f_{\overline{k}'}^{(j)} = 0, \quad \sum_{j=0}^{d} \sum_{\overline{k} \in G/H} f_{\overline{k}}^{(j)} = 1 \quad \text{and} \quad (\alpha_{\mathcal{U}})_g(f_{\overline{k}}^{(j)}) = f_{\overline{gk}}^{(j)}.$$

Fix $j_0 = 0, \ldots, d$ such that $\kappa_\tau(f_{\overline{k}}^{(j_0)})$ is not zero for some (and hence all) $\overline{k} \in G/H$. Note that the positive contractions $\kappa_\tau(f_{\overline{k}}^{(j_0)})$, for $\overline{k} \in G/H$, are thus pairwise orthogonal and satisfy $(\alpha^\tau)_g^{\mathcal{U}}(\kappa_\tau(f_{\overline{k}}^{(j)})) = \kappa_\tau(f_{\overline{gk}}^{(j)})$. Since $g \notin H$ there is $\overline{k} \in G/H$ such that $\overline{gk} \neq \overline{k}$. In particular, $(\alpha_g^\tau)^{\mathcal{U}}$ acts non-trivially on $\mathcal{M}_\tau^{\mathcal{U}} \cap \mathcal{M}'$, thus α_g^τ cannot be an inner automorphism of $\mathcal{M}_\tau$.

(4) ⇒ (3). This implication is trivial.

(1) ⇒ (4). Let H be a normal subgroup of G of finite index. After identifying $\mathcal{R}$ with the weak closure of $\bigotimes_{n\in\mathbb{N}} \mathcal{B}(\ell^2(G/H))$, denote by $\mu_{G/H}$ the action given by $\bigotimes_{n\in\mathbb{N}} \mathrm{Ad}(\lambda_{G/H})$. Since $(A, \alpha) \cong_{\mathrm{cc}} (A\otimes\mathcal{Z}, \alpha\otimes\mathrm{id}_{\mathcal{Z}})$, by Theorem 7.6 the dynamical system (A, α) has CPoU. As moreover α is a strongly outer action, we can apply Theorem 5.1 to obtain an equivariant, unital homomorphism

$$\Psi\colon (\mathcal{R}, \mu_{G/H} \circ q_H) \to (A^{\mathcal{U}} \cap A', \alpha^{\mathcal{U}}).$$

Fix $\varepsilon > 0$, and let $k \in \mathbb{N}$ be given by Proposition 6.3. Denote by

$$\varphi\colon \Big(\mathcal{B}(\ell^2(G/H)^{\otimes k}), \mathrm{Ad}(\lambda_{G/H}^{\otimes k}) \circ q_H\Big) \to (A^{\mathcal{U}} \cap A', \alpha^{\mathcal{U}})$$

the restriction of Ψ to $\mathcal{B}(\ell^2(G/H)^{\otimes k}) \subseteq \mathcal{R}$. By Corollary 7.4, there exists an equivariant completely positive contractive order zero map

$$\rho\colon \Big(\mathcal{B}(\ell^2(G/H)^{\otimes k}), \mathrm{Ad}(\lambda_{G/H}^{\otimes k}) \circ q_H\Big) \to (A_{\mathcal{U}} \cap A', \alpha_{\mathcal{U}})$$

such that the following diagram commutes:

$$\begin{tikzcd} & A_{\mathcal{U}} \cap A' \arrow[d, "\kappa"] \\ \mathcal{B}(\ell^2(G/H)^{\otimes k}) \arrow[ur, "\rho"] \arrow[r, "\varphi"'] & A^{\mathcal{U}} \cap A'. \end{tikzcd}$$

We denote by $e \in \mathcal{B}(\ell^2(G/H))$ the projection onto the constant functions, and regard $e^{\otimes k}$ as a projection in $\mathcal{B}(\ell^2(G/H)^{\otimes k})$. One can verify that

$$\tau(\rho(e^{\otimes k})^m) = \tau(\varphi(e^{\otimes k})^m) = \tau(\varphi(e^{\otimes k})) = 1/[G : H]^k$$

for all $\tau \in T_{\mathcal{U}}(A)$ and for all $m \in \mathbb{N}$, using that $\varphi(e^{\otimes k})$ is a projection.

By [67, Theorem B], the system (A, α) has equivariant property (SI), so we can pick a contraction $s \in (A_{\mathcal{U}} \cap A')^{\alpha_{\mathcal{U}}}$ satisfying $s^*s = 1 - \rho(1)$ and $\rho(e^{\otimes k})s = s$. By Theorem 6.7, there exists a unital equivariant homomorphism

$$\theta\colon (I_{G/H}^{(k)}, \mu_{G/H}^{(k)} \circ q_H) \to (A_{\mathcal{U}} \cap A', \alpha_{\mathcal{U}}).$$

Let $f_{\overline{g}}^{(j)} \in I_{G/H}^{(k)}$, for $\overline{g} \in G/H$ and $j = 0, 1, 2$, be positive contractions as in the conclusion of Proposition 6.3. Then the positive contractions $\theta(f_{\overline{g}}^{(j)}) \in A_{\mathcal{U}} \cap A'$ satisfy the conditions of Definition 2.6 up to ε. Since $\varepsilon > 0$ is arbitrary, the result follows by saturation of $A_{\mathcal{U}}$ (Remark 2.2). □

8. Equivariant $\mathcal{Z}$-stability

We conclude the paper with the proof of Theorem B, which improves the main result of [63] by allowing the group to act nontrivially on $T(A)$. In the case of integer actions, the recent paper [73] used the techniques in this work to remove the assumptions that we make on the induced action $G \curvearrowright T(A)$.

We recall the following equivalent definition of covering dimension from [46]: a topological space X has *(covering) dimension* $m \in \mathbb{N}$, in symbols $\dim(X) = m$, if m is the minimal value such that every open cover $\mathcal{O}$ of X has a refinement $\mathcal{O}' = \mathcal{O}'_0 \sqcup \cdots \sqcup \mathcal{O}'_m$ such that the elements in $\mathcal{O}'_j$ are pairwise disjoint, for every $j = 0, \ldots, m$. If no such value exists, we say that X has infinite dimension.

Under the assumptions of Theorem B, it is proved in Theorem 7.6 that (A, α) is $(\mathcal{Z}, \mathrm{id}_{\mathcal{Z}})$-stable if and only if for all $d \geq 2$ there exists a unital homomorphism $M_d \to (A_{\mathcal{U}} \cap A')^{\alpha_{\mathcal{U}}}$. The strategy to prove the existence of such homomorphisms is, once again, via a 'local to global' argument over the trace space. More specifically, in Corollary 5.5 we use results from Section 5 to show that for every $\tau \in T(A)^{\alpha}$ and every $d \in \mathbb{N}$, there exists a unital homomorphism $M_d \to (\mathcal{M}_\tau^{\mathcal{U}} \cap \mathcal{M}')^{(\alpha^\tau)^{\mathcal{U}}}$. Then, in Proposition 8.4, we glue these maps together to obtain a unital homomorphism $M_d \to (A^{\mathcal{U}} \cap A')^{\alpha^{\mathcal{U}}}$.

To perform this gluing argument, since in this case we do not assume that (A, α) has CPoU, we rely on the techniques from [45, Section 6, 7], which make essential use of the assumption that $\partial_e T(A)$ is compact and finite-dimensional. Those arguments could also be carried out using Ozawa's work on W^*-bundles in [56] (see for instance [48], [49]), whose role in our setting is replaced by $A^{\mathcal{U}}$. (One can show that $A^{\mathcal{U}}$ is isomorphic to the ultrapower of the W^*-bundle associated to the tracial completion of A when $\partial_e T(A)$ is compact.)

The main 'local to global' argument is contained in Proposition 8.4; it is an equivariant analogue of [45, Proposition 7.4, Lemma 7.5]. Related arguments in the equivariant setting, under the additional assumption that α^* is trivial on $\partial_e T(A)$, can be found in [63], [48], [49].

We start with a preliminary lemma.

Lemma 8.1. *Let A be a unital, separable C^*-algebra such that $\partial_e T(A)$ is compact. Let G be a countable, discrete group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Suppose that the orbits of the induced action α^* of G on $\partial_e T(A)$ are finite with bounded cardinality, and that the orbit space $\partial_e T(A)/G$ is Hausdorff. Then the averaging function $\partial_e T(A) \to T(A)$ mapping $\tau \in \partial_e T(A)$ to $\tau^\alpha = \frac{1}{|G\cdot\tau|}\sum_{\sigma \in G\cdot\tau} \sigma$ is continuous.*

Proof. Since A is separable, both $\partial_e T(A)$ and $T(A)$ are metrizable. It is therefore enough to show that if $(\tau_n)_{n\in\mathbb{N}}$ is a sequence in $\partial_e T(A)$ such that $\tau_n \to \tau$ in $\partial_e T(A)$, then $\tau_n^\alpha \to \tau^\alpha$ in $T(A)$. Fix $\varepsilon > 0$ and $a_1, \ldots, a_k \in A$.

Since the orbits of the G-action on $\partial_e T(A)$ have bounded cardinality, upon passing to a subsequence we can assume without loss of generality that there is $N \in \mathbb{N}$ such that $|G \cdot \tau_n| = N$ for every $n \in \mathbb{N}$. Let G_τ be the stabilizer of τ and let $S = \{s_1, \ldots, s_M\}$ be a set of representatives of left cosets of G_τ with $s_1 = e$. In particular $|G \cdot \tau| = |S| = M$. Let V be an open subset of $\partial_e T(A)$ such that

(i) $\tau \in V$,
(ii) $\alpha^*_{s_j}(V) \cap V = \emptyset$ for every $1 < j \leq M$,
(iii) $|\sigma(\alpha_{s_j}(a_i)) - \tau(\alpha_{s_j}(a_i))| < \varepsilon$, for all $\sigma \in V$, all $j \leq M$ and all $i \leq k$.

Claim 8.1.1. *There exists $m \in \mathbb{N}$ such that $G \cdot \tau_n \subseteq \bigcup_{j=1}^M \alpha^*_{s_j}(V)$ for every $n > m$.*

Suppose this is not the case. Up to taking a subsequence, for every $n \in \mathbb{N}$, let $\sigma_n \in G \cdot \tau_n$ such that $\sigma_n \notin \bigcup_{j=1}^M \alpha^*_{s_j}(V)$. By compactness of $\partial_e T(A)$, up to taking a subsequence, we can assume that σ_n converges to some $\sigma \in \partial_e T(A)$. Since $\partial_e T(A)/G$ is Hausdorff, it follows that $\sigma \in G \cdot \tau \subseteq \bigcup_{j=1}^{M} \alpha^*_{s_j}(V)$, which is a contradiction. This concludes the proof of the claim.

Given $n > m$, let $\{\sigma_{1,n} \dots, \sigma_{K,n}\}$ be the intersection $G \cdot \tau_n \cap V$. Therefore, as each $\alpha^*_{s_j}$ is a homeomorphism, it follows that $G \cdot \tau_n = \bigcup_{j=1}^{M} \alpha^*_{s_j}(G \cdot \tau_n \cap V)$, and that in particular $N = M \cdot K$, so K does not depend on $n > m$. Finally, for all $i \leq k$ and every $n > m$ big enough so that $\tau_n \in V$, we conclude that

$$\begin{aligned} |\tau_n^\alpha(a_i) - \tau^\alpha(a_i)| &= \Big| \frac{1}{M \cdot K} \sum_{j=1}^{M} \sum_{h=1}^{K} \sigma_{h,n}(\alpha_{s_j}(a_i)) - \frac{1}{M} \sum_{j=1}^{M} \tau(\alpha_{s_j}(a_i)) \Big| \\ &= \frac{1}{M} \Big| \frac{1}{K} \sum_{j=1}^{M} \sum_{h=1}^{K} \sigma_{h,n}(\alpha_{s_j}(a_i)) - \frac{1}{K} \sum_{j=1}^{M} \sum_{h=1}^{K} \tau(\alpha_{s_j}(a_i)) \Big| \\ &\leq \frac{1}{M \cdot K} \sum_{j=1}^{M} \sum_{h=1}^{K} |\sigma_{h,n}(\alpha_{s_j}(a_i)) - \tau(\alpha_{s_j}(a_i))| \overset{\text{(iii)}}{<} \varepsilon. \quad \square \end{aligned}$$

We fix some notation for the next couple of propositions.

Notation 8.2. Let A be a unital C^*-algebra such that $\partial_e T(A)$ is compact, let G be a countable group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Recall that α^* denotes the affine action of G on $T(A)$ induced by α; see subsection 2.1. Let $\alpha^{**}\colon G \to \mathrm{Aut}(C(\partial_e T(A)))$ be defined as $\alpha^{**}_g(f) = f \circ \alpha^*_{g^{-1}}$ for every $g \in G$ and all $f \in C(\partial_e T(A))$. Note that α^{**} is the restriction of the double-dual action to $C(\partial_e(T(A))) \subseteq A^{**}$, thus justifying the notation.

Let $\mathcal{T}\colon A \to C(\partial_e T(A))$ be the unital, completely positive map defined as $\mathcal{T}(a)(\tau) = \tau(a)$ for all $a \in A$ and all $\tau \in T(A)$. Then $\mathcal{T}\colon (A, \alpha) \to (C(\partial_e T(A), \alpha^{**})$ is equivariant, since for $g \in G$, $a \in A$ and $\tau \in \partial_e T(A)$ we have:

$$\mathcal{T}(\alpha_g(a))(\tau) = \tau(\alpha_g(a)) = \mathcal{T}(a)(\alpha^*_{g^{-1}}(\tau)) = \alpha^{**}_g(\mathcal{T}(a))(\tau).$$

The following proposition is an equivariant version of [45, Corollary 6.8].

Proposition 8.3. *Let A be a separable, simple, nuclear, infinite-dimensional, unital C^*-algebra. Let G be a countable, discrete, amenable group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Suppose that $\partial_e T(A)$ is compact and nonempty.*

Let $\{f^{(j)}_k \colon 1 \leq j \leq m, 1 \leq k \leq s_j\}$ be an α^{**}-invariant partition of unity in $C(\partial_e T(A))$ such that $f^{(j)}_k f^{(j)}_h = 0$ for all $j = 1, \dots, m$ and all $k, h = 1, \dots, s_j$ with $k \neq h$. Given $\varepsilon > 0$, a compact subset $\Omega \subseteq A$ and a finite subset $G_0 \subseteq G$, there exist positive contractions $c^{(j)}_k \in A$, for $j = 1, \dots, m$, and for $k = 1, \dots, s_j$, such that

(1) $\|c^{(j)}_k b - b c^{(j)}_k\| < \varepsilon$ *for all* $b \in \Omega$,
(2) $\sup_{\tau \in \partial_e T(A)} |f^{(j)}_k(\tau) - \tau(c^{(j)}_k)| < \varepsilon$,
(3) $c^{(j)}_k c^{(j)}_h = 0$ *for all* $h = 1, \dots, s_j$ *such that* $h \neq k$,
(4) $\big\|\alpha_g(c^{(j)}_k) - c^{(j)}_k\big\| < \varepsilon$ *for all* $g \in G_0$,
(5) $\sum_{j=0}^{m} \sum_{k=1}^{s_j} (c^{(j)}_k)^2 \leq 1 + \varepsilon_0$.

Proof. Let $\mathcal{T}\colon A \to C(\partial_e T(A))$ be the unital, completely positive equivariant map from Notation 8.2. By [45, Proposition 6.6], when $\partial_e T(A)$ is closed, the induced map $\mathcal{T}_{\mathcal{U}}\colon A_{\mathcal{U}} \to C(\partial_e T(A))_{\mathcal{U}}$ maps the unit ball of $A_{\mathcal{U}}$ onto the unit ball of $C(\partial_e T(A))_{\mathcal{U}}$. Moreover, denoting by $\mathrm{Mult}(\mathcal{T}_{\mathcal{U}})$ the multiplicative domain of $\mathcal{T}_{\mathcal{U}}$, we have

$$J_A \subseteq \mathrm{Mult}(\mathcal{T}_{\mathcal{U}}) \subseteq J_A + (A_{\mathcal{U}} \cap A'),$$

and the restriction $\mathcal{T}_{\mathcal{U}}\colon \mathrm{Mult}(\mathcal{T}_{\mathcal{U}}) \to C(\partial_e T(A))_{\mathcal{U}}$ is a surjective homomorphism whose kernel is J_A ([45, Proposition 6.6.vi]). Moreover, the unital completely positive map $\mathcal{T}_{\mathcal{U}}\colon (A_{\mathcal{U}}, \alpha_{\mathcal{U}}) \to (C(\partial_e T(A))_{\mathcal{U}}, \alpha_{\mathcal{U}}^{**})$ is equivariant by Notation 8.2.

Let C be the ($\alpha_{\mathcal{U}}^{**}$-invariant) C^*-subalgebra of $C(\partial_e T(A))_{\mathcal{U}}$ generated by the family $\{f_k^{(j)}\}_{1\leq j\leq m, 1\leq k\leq s_j}$. By the previous observations, there is an injective homomorphism $\Phi\colon C \to (A^{\mathcal{U}} \cap A')^{\alpha^{\mathcal{U}}}$ satisfying $\tau(\Phi(f)) = f(\tau)$ for $\tau \in \partial_e T(A)$ and $f \in C$. Regarding Φ as an equivariant map $(C, \mathrm{id}_C) \to (A^{\mathcal{U}} \cap A', \alpha^{\mathcal{U}})$, by Corollary 7.4 there exists a completely positive, contractive, order zero map Ψ such that the following diagram commutes:

$$\begin{tikzcd} & (A_{\mathcal{U}} \cap A')^{\alpha_{\mathcal{U}}} \arrow[d, "\kappa"] \\ C \arrow[ur, "\Psi"] \arrow[r, "\Phi"'] & (A^{\mathcal{U}} \cap A')^{\alpha^{\mathcal{U}}}. \end{tikzcd}$$

The positive contractions $\Psi(f_k^{(j)})$, for $1 \leq j \leq m$ and $1 \leq k \leq s_j$, are pairwise orthogonal, and hence can be lifted to pairwise orthogonal positive contractions $(d_{k,n}^{(j)})_{n\in\mathbb{N}} \in \ell^\infty(A)$, for $1 \leq j \leq m$ and $1 \leq k \leq s_j$. We can thus find $n \in \mathbb{N}$ big enough so that setting $c_k^{(j)} = d_{k,n}^{(j)}$, for all $j = 1, \ldots, m$ and $k = 1, \ldots, s_j$, gives the required elements. $\square$

The following equivariant analogue of [45, Proposition 7.4, Lemma 7.5] extends [49, Corollary 3.2] and [48, Proposition 3.3] to all amenable groups, while at the same time relaxing the assumptions on the induced action $G \curvearrowright T(A)$.

Proposition 8.4. *Let A be a separable, simple, nuclear, infinite-dimensional, unital C^*-algebra, let G be a countable, discrete, amenable group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Suppose that $\partial_e T(A)$ is compact, that $\dim(\partial_e T(A)) < \infty$, that the orbits of the induced action of G on $\partial_e T(A)$ are finite with bounded cardinality, and that the orbit space $\partial_e T(A)/G$ is Hausdorff. Then, for every $d \in \mathbb{N}$, there exists a unital homomorphism $\psi\colon M_d \to (A^{\mathcal{U}} \cap A')^{\alpha^{\mathcal{U}}}$.*

Proof. Fix $d \geq 1$ and let $\tau \in T(A)^\alpha$. By the assumptions on A, the von Neumann algebra $\mathcal{M}_\tau$ is a separably representable, hyperfinite, and type II_1, thus by Corollary 5.5 there is a unital homomorphism

$$\theta_\tau\colon M_d \to (\mathcal{M}_\tau^{\mathcal{U}} \cap \mathcal{M}')^{(\alpha^\tau)^{\mathcal{U}}}.$$

The rest of the proof is divided into two claims, respectively inspired by Proposition 7.4 and Lemma 7.5 in [45].

Claim 8.4.1. *Let $\varepsilon > 0$, let $m \in \mathbb{N}$, let $G_0 \subseteq G$ be finite and let $\Omega \subseteq A$ be compact. Then there exist an open cover $\mathcal{O}$ of $\partial_e T(A)$, and families $\Phi^{(j)} = \{\varphi_1^{(j)}, \ldots, \varphi_{r_j}^{(j)}\}$, for $j = 1, \ldots, m$, consisting of unital completely positive contractive maps $\varphi_k^{(j)}\colon M_d \to A$, for $k = 1, \ldots, r_j$, such that all open sets $V \in \mathcal{O}$ are α^*-invariant, and for every $j = 1, \ldots, m$, and $k \in \{1, \ldots, r_j\}$ we have*

(1.a) $\left\|\varphi_k^{(j)}(a)b - b\varphi_k^{(j)}(a)\right\| < \varepsilon\|a\|$ *for all $a \in M_d$ and for all $b \in \Omega$,*
(1.b) $\left\|\varphi_k^{(j)}(a)\varphi_h^{(i)}(b) - \varphi_h^{(i)}(b)\varphi_k^{(j)}(a)\right\| < \varepsilon\|a\|\|b\|$ *for all $i = 1, \ldots, m$ with $i \neq j$, all $h = 1, \ldots, r_i$, and for all $a, b \in M_d$,*

(1.c) $\left\|\alpha_g(\varphi_k^{(j)}(a)) - \varphi_k^{(j)}(a)\right\| < \varepsilon\|a\|$ *for all* $g \in G_0$*, and for all* $a \in M_d$*,*
(1.d) for every $V \in \mathcal{O}$ *and every* $j = 1, \dots, m$ *there is* $k \in \{1, \dots, r_j\}$ *such that*

$$\sup_{\tau \in V} \left\|\varphi_k^{(j)}(a^*a) - \varphi_k^{(j)}(a)^*\varphi_k^{(j)}(a)\right\|_{2,\tau} < \varepsilon\|a\|^2,$$

for all $a \in M_d$*.*

We call the tuple $(\mathcal{O}; \Phi^{(1)}, \dots, \Phi^{(m)})$ *an* $(\varepsilon, \Omega, G_0)$-commuting covering system*, in analogy with Definition 7.1 in [45].*

To prove the claim, fix $\Omega \subseteq A$ and $G_0 \subseteq G$ as in the statement. Let $\tau \in \partial_e T(A)$ and set

$$\tau^\alpha = \frac{1}{|G \cdot \tau|} \sum_{\sigma \in G \cdot \tau} \sigma \in T(A)^\alpha.$$

By Lemma 7.5 and by the Choi-Effros Lifting Theorem [5], there exists a unital, completely positive, contractive map $\tilde{\theta}$ such that the following diagram commutes:

$$\begin{tikzcd} & (A_\mathcal{U} \cap A')^{\alpha_\mathcal{U}} \arrow[d, "\kappa_\tau"] \\ M_d \arrow[ur, "\tilde{\theta}"] \arrow[r, "\theta_{\tau^\alpha}"'] & (\mathcal{M}_{\tau^\alpha}^\mathcal{U} \cap \mathcal{M}'_{\tau^\alpha})^{(\alpha^\tau)^\mathcal{U}}. \end{tikzcd}$$

Applying the Choi-Effros Lifting Theorem again, we find unital, completely positive maps $\tilde{\theta}_n \colon M_d \to A$, for $n \in \mathbb{N}$, such that $(\widetilde{\theta}_n)_{n\in\mathbb{N}} \colon M_d \to \ell^\infty(A)$ lifts $\tilde{\theta}$.

Let M be an upper bound for the cardinality of the orbits of the G-action α^* on $\partial_e T(A)$. By choosing a map far enough in the sequence, we claim that there exists a unital completely positive map $\varphi_\tau \colon M_d \to A$ satisfying

(a) $\|\varphi_\tau(a)b - b\varphi_\tau(a)\| < \varepsilon\|a\|$ for all $a \in M_d$ and all $b \in F$,
(b) $\|\alpha_g(\varphi_\tau(a)) - \varphi_\tau(a)\| < \varepsilon\|a\|$ for all $g \in G_0$ and all $a \in M_d$,
(c) $\|\varphi_\tau(a^*a) - \varphi_\tau(a)^*\varphi_\tau(a)\|_{2,\tau^\alpha} < \varepsilon\|a\|^2 M^{1/2}$ for all $a \in M_d$.

Conditions (a) and (b) are immediate, since G_0 and F are finite, and the unit ball of M_d is compact. To justify condition (c), note that $\tilde{\theta}(a^*a) - \tilde{\theta}(a)^*\tilde{\theta}(a)$ belongs to J_{τ^α} for all $a \in M_d$, as $\tilde{\theta}$ is itself a lift of the homomorphism θ_{τ^α}.

By compactness of the unit ball of M_d and Lemma 8.1, we can find an open set V'_τ of $\partial_e T(A)$ containing τ such the inequality in (c) holds also when substituting $\|\cdot\|_{2,\tau^\alpha}$ with $\|\cdot\|_{2,\sigma^\alpha}$ for all $\sigma \in V'_\tau$. Set $V_\tau = \bigcup_{g\in G} \alpha_g^*(V'_\tau)$, and note that again for every $\sigma \in V_\tau$ the inequalities in item (c) hold with respect to the $\|\cdot\|_{2,\sigma^\alpha}$-norm. Arguing as in Proposition 2.3, we see that $\sigma \leq M\sigma^\alpha$ for every $\sigma \in \partial_e T(A)$, which in turn implies that

$$\|\varphi_\tau(a^*a) - \varphi_\tau(a)^*\varphi_\tau(a)\|_{2,\sigma} < \varepsilon\|a\|^2,$$

for all $a \in M_d$ and all $\sigma \in V_\tau$.

We finish the proof of the claim by induction on m. When $m = 1$, we cover $\partial_e T(A)$ by the open sets V_τ obtained in the previous paragraph and, by compactness of $\partial_e T(A)$, we can find an integer $r_1 \in \mathbb{N}$ and traces

$\tau_1, \ldots, \tau_{r_1} \in \partial_e T(A)$, such that $\mathcal{O} = \{V_{\tau_1}, \ldots, V_{\tau_{r_1}}\}$ is a cover of $\partial_e T(A)$. Then $\mathcal{O}$ and $\Phi^{(1)} = \{\varphi_{\tau_1}, \ldots, \varphi_{\tau_{r_1}}\}$ satisfy the desired properties.

Assume that we have found an open, α^*-invariant cover $\mathcal{O}'$ and families $\Phi^{(j)}$ for $j = 1, \ldots, m-1$, satisfying the conditions in the statement. Let M_d^1 denote the unit ball of M_d. For every $\tau \in \partial_e T(A)$, find a map $\varphi_\tau \colon M_d \to A$ and an open, α^*-invariant neighborhood W_τ of τ as in the first part of the proof for the finite set

$$\Omega \cup \bigcup_{j=1}^{m-1} \bigcup_{k=1}^{r_j} \varphi_k^{(j)}(M_d^1) \subseteq A$$

instead of Ω. Find an integer $r_m \in \mathbb{N}$ and traces $\tau_1^{(m)}, \ldots, \tau_{r_m}^{(m)} \in \partial_e T(A)$, such that $\{W_{\tau_1^{(m)}}, \ldots, W_{\tau_{r_m}^{(m)}}\}$ is an α^*-invariant cover of $\partial_e T(A)$. Let $\mathcal{O}$ be the family of α^*-invariant open sets of the form $V \cap W_{\tau_k^{(m)}}$, for $k = 1, \ldots, r_m$ and $V \in \mathcal{O}'$, and set $\Phi^{(m)} := \{\varphi_{\tau_1^{(m)}}, \ldots, \varphi_{\tau_{r_m}^{(m)}}\}$. Then $\mathcal{O}$ and $\{\Phi^{(j)}\}_{1 \le j \le m}$ satisfy the desired properties.

Claim 8.4.2. *Let $\varepsilon > 0$, let $m = \dim(\partial_e T(A))$, let $G_0 \subseteq G$ be finite and let $\Omega \subseteq A$ be compact. Set $m = \dim(\partial_e T(A))$. There exist completely positive contractive maps $\psi^{(0)}, \ldots, \psi^{(m)} \colon M_d \to A$ satisfying*

(2.a) $\|\psi^{(j)}(a)b - b\psi^{(j)}(a)\| < \varepsilon$ *for all* $j = 0, \ldots, m$, *for all* $a \in M_d^1$ *and all* $b \in \Omega$,
(2.b) $\|\psi^{(j)}(a)\psi^{(k)}(b) - \psi^{(k)}(b)\psi^{(j)}(a)\| < \varepsilon$ *for all* $j, k = 0, \ldots, m$ *with* $j \neq k$, *and for all* $a, b \in M_d^1$,
(2.c) $\|\alpha_g(\psi^{(j)}(a)) - \psi^{(j)}(a)\| < \varepsilon$ *for all* $g \in G_0$, *for all* $j = 0, \ldots, m$, *and for all* $a \in M_d^1$,
(2.d) $\|\psi^{(j)}(1)\psi^{(j)}(a^*a) - \psi^{(j)}(a)^*\psi^{(j)}(a)\|_{2,u} < \varepsilon$ *for all* $j = 0, \ldots, m$, *and for all* $a \in M_d^1$,
(2.e) $\sup_{\tau \in T(A)} \tau\big(\sum_{j=0}^m \psi^{(j)}(1) - 1\big) < \varepsilon$.

Let $(\mathcal{O}; \Phi^{(1)}, \ldots, \Phi^{(m)})$ be an $(\varepsilon/2, \Omega, G_0)$-commuting covering system given by Claim 8.4.1. Since the orbit space $\partial_e T(A)/G$ is Hausdorff (in addition to compact and second countable), it is metrizable. Moreover since the quotient map $\pi \colon \partial_e T(A) \to \partial_e T(A)/G$ is open and all orbits are finite, π preserves the topological dimension ([57, Proposition 2.16]). In particular, $\dim(\partial_e T(A)/G) = \dim(\partial_e T(A)) = m$. Find a refinement $\mathcal{P}$ of $\pi(\mathcal{O})$ witnessing $\dim(\partial_e T(A)/G) = m$, and let $\mathcal{O}'$ be the collection of preimages of the elements of $\mathcal{P}$. Every open set in $\mathcal{O}'$ is then α^*-invariant, being a preimage of a set via the map π. Since the preimages of disjoint sets are themselves disjoint, we can decompose $\mathcal{O}'$ into finite subsets $\mathcal{O}'_0, \ldots, \mathcal{O}'_m$ such that $\mathcal{O}'_j = \{V_1^{(j)}, \ldots, V_{s_j}^{(j)}\}$ consists of pairwise disjoint open subsets of $\partial_e T(A)$ which are moreover α^*-invariant. It is clear that $\mathcal{O}'$ is a refinement of $\mathcal{O}$, since the latter is composed of invariant open sets.

Let $\{\tilde{f}_k^{(j)} \colon j = 0, \ldots, m, k = 1, \ldots, s_j\} \subseteq C(\partial_e T(A))/G$ be a partition of unity of $\partial_e T(A)/G$ with $\mathrm{supp}(\tilde{f}_k^{(j)}) \subseteq \pi(V_k^{(j)})$ for $j = 0, \ldots, m$ and $k = 1, \ldots, s_j$. For every $j = 0, \ldots, m$ and $k = 1, \ldots, s_j$, define

$$f_k^{(j)} = \tilde{f}_k^{(j)} \circ \pi \in C(\partial_e T(A)).$$

It is immediate that each $f_k^{(j)}$ is α^{**}-invariant. Moreover, the family $\{f_k^{(j)} \colon j = 0, \ldots, m, k = 1, \ldots, s_j\} \subseteq C(\partial_e T(A))$ is also a partition of unity of $\partial_e T(A)$ with $\mathrm{supp}(f_k^{(j)}) \subseteq V_k^{(j)} = \pi^{-1}(\pi(V_k^{(j)}))$ for $j = 0, \ldots, m$ and $k = 1, \ldots, s_j$, since all elements in $\mathcal{O}'$ are α^*-invariant. For a fixed j, the functions $f_1^{(j)}, \ldots, f_{s_j}^{(j)}$ are pairwise orthogonal, since they have disjoint supports.

For every $j \in \{0, \ldots, m\}$ and $k \in \{1, \ldots, s_j\}$, there is a unital completely positive map $\varphi_k^{(j)} \colon M_d \to A$ belonging to $\Phi^{(j)}$ such that conditions (1.a)-(1.d) in Claim 8.4.1 hold. Set

$$K = \max\left\{8 \cdot \max_{0 \le j \le m} s_j^2, 4 \cdot \sum_{j=0}^m s_j\right\} \quad , \quad \varepsilon_0 = \frac{\varepsilon^2}{2K}, \tag{8.1}$$

and

$$\Omega_0 = \Omega \cup \{\varphi_k^{(j)}(a)\colon a \in M_d^1, 0 \le j \le m, 1 \le k \le s_j\}.$$

By Proposition 8.3, there are positive contractions $c_k^{(j)} \in A$, for $j = 0, \ldots, m$ and $k = 1, \ldots, s_j$, satisfying the following for all $j = 0, \ldots m$ and $k = 1, \ldots, s_j$:

(i) $\|bc_k^{(j)} - bc_k^{(j)}\| < \varepsilon_0$ for all $b \in \Omega_0$,
(ii) $\|c_k^{(j)} c_\ell^{(i)} - c_\ell^{(i)} c_k^{(j)}\| < \varepsilon_0$ for all $i = 0, \ldots, m$ and all $\ell = 1, \ldots, s_i$,
(iii) $\sup_{\tau \in \partial_e T(A)} |f_k^{(j)}(\tau) - \tau(c_k^{(j)})| < \varepsilon_0^{1/2}$,
(iv) $c_k^{(j)} c_\ell^{(j)} = 0$ for all $\ell = 1, \ldots, s_j$ such that $\ell \ne k$,
(v) $\|\alpha_g(c_k^{(j)}) - c_k^{(j)}\| < \varepsilon_0$ for all $g \in G_0$,
(vi) $\sum_{j=0}^m \sum_{k=1}^{s_j} (c_k^{(j)})^2 \le 1 + \varepsilon_0$.

Fix $j \in \{0, \ldots, m\}$, and define a linear map $\psi^{(j)}\colon M_d \to A$ by

$$\psi^{(j)}(a) = \sum_{k=1}^{s_j} c_k^{(j)} \varphi_k^{(j)}(a) c_k^{(j)}$$

for all $a \in M_d$. As all $c_k^{(j)}$ are mutually orthogonal positive contractions, it follows that $\psi^{(j)}$ is completely positive and contractive.

It remains to show that $\psi^{(0)}, \ldots, \psi^{(m)}$ satisfy the conditions of the claim. The verification of conditions (2.a) and (2.b) is the same as verification of conditions (i) and (iii) in the proof of [45, Lemma 7.5]. To verify condition (2.c), fix $g \in G_0$, an index $j \in \{0, \ldots, m\}$, and a contraction $a \in M_d$. We have

$$\begin{aligned}
\|\alpha_g(\psi^{(j)}(a)) - \psi^{(j)}(a)\| &\overset{\text{(v)}}{\le} \left\| \sum_{k=1}^{s_j} c_k^{(j)} \left(\alpha_g(\varphi_k^{(j)}(a)) - \varphi_k^{(j)}(a) \right) c_k^{(j)} \right\| + 2 s_j \varepsilon_0 \\
&\overset{\text{(iv)}}{\le} \max_{1 \le k \le s_j} \left\| c_k^{(j)} \left(\alpha_g(\varphi_k^{(j)}(a)) - \varphi_k^{(j)}(a) \right) c_k^{(j)} \right\| + \frac{\varepsilon}{2} \\
&\le \max_{1 \le k \le s_j} \left\| \alpha_g(\varphi_k^{(j)}(a)) - \varphi_k^{(j)}(a) \right\| + \frac{\varepsilon}{2} \overset{\text{(1.c)}}{<} \varepsilon.
\end{aligned}$$

Item (2.d) corresponds to condition (iv) in [45, Lemma 7.5], and can be inferred as follows. Fix $j = 0, \ldots, m$ and $a \in M_d^1$. For $k = 1, \ldots, s_j$, set

$$T_k = (c_k^{(j)})^2 \varphi_k^{(j)}(a^* a) - \varphi_k^{(j)}(a)^* (c_k^{(j)})^2 \varphi_k^{(j)}(a),$$

and note that $\|T_k\| \le 2$. Fix $\tau \in \partial_e T(A)$. Then

$$\begin{aligned}
\|\psi^{(j)}(1)\psi^{(j)}(a^* a) - \psi^{(j)}(a)^* \psi^{(j)}(a)\|_{2,\tau} &\overset{\text{(iv)}}{=} \Big\| \sum_{k=1}^{s_j} c_k^{(j)} T_k c_k^{(j)} \Big\|_{2,\tau} \\
&\overset{\text{(v)}}{=} \max_{k=1,\ldots,s_j} \big\| c_k^{(j)} T_k c_k^{(j)} \big\|_{2,\tau}.
\end{aligned}$$

By the above computation and since $T(A)$ is convex, it is enough to prove that $\big\| c_k^{(j)} T_k c_k^{(j)} \big\|_{2,\tau} < \varepsilon$ for every $k = 1, \ldots, s_j$. Fix $k = 1, \ldots, s_j$. If $\tau \notin V_k^{(j)}$, then

$$\left\|c_k^{(j)} T_k c_k^{(j)}\right\|_{2,\tau} \leq \|T_k\| \|c_k^{(j),2}\|_{2,\tau} \leq 2\tau\big((c_k^{(j)})^4\big)^{1/2} \leq 2\tau(c_k^{(j)})^{1/2} \overset{\text{(iii)}}{<} 2\varepsilon_0 < \varepsilon,$$

as desired. Assume instead that $\tau \in V_k^{(j)}$. Set $D_k^{(j)} := \varphi_k^{(j)}(a^*a) - \varphi_k^{(j)}(a)^*\varphi_k^{(j)}(a)$, which is positive by the Schwarz inequality for completely positive maps ([4, Corollary 2.8]). Then

$$\begin{aligned}
\|c_k^{(j)} T_k c_k^{(j)}\|_{2,\tau} &\overset{\text{(i)}}{\leq} \left\|c_k^{(j),2} D_k^{(j)} c_k^{(j),2}\right\|_{2,\tau} + 3\varepsilon_0 \\
&\leq \left\|(D_k^{(j)})^{1/2} c_k^{(j),2} (D_k^{(j)})^{1/2}\right\|_{2,\tau} + 3\varepsilon_0 \\
&\leq \left\|D_k^{(j)}\right\|_{2,\tau} + 3\varepsilon_0 \overset{\text{(1.d)}}{<} \frac{\varepsilon}{2} + 3\varepsilon_0 < \varepsilon,
\end{aligned}$$

which concludes the proof of (2.d). Item (2.e) is verified in the same way as condition (v) from [45, Lemma 7.5], and we omit the proof. This proves the claim.

By iterating Claim 8.4.2 for larger and larger $\Omega \subseteq A$, $G_0 \subseteq G$, and for smaller and smaller $\varepsilon > 0$, we obtain $m = \dim(\partial_e T(A))$ completely positive, contractive, order zero maps $\psi^{(0)}, \dots, \psi^{(m)} \colon M_d \to (A^{\mathcal{U}} \cap A')^{\alpha^{\mathcal{U}}}$ with commuting images. Moreover, by condition (vi) on the $c_k^{(j)}$, it follows that $\sum_{j=0}^m \psi^{(j)}(1) \leq 1$, which in combination with (2.e) from Claim 8.4.2 yields $\sum_{j=0}^m \psi^{(j)}(1) = 1$. By [45, Lemma 7.6] this gives a unital homomorphism $\psi \colon M_d \to (A^{\mathcal{U}} \cap A')^{\alpha^{\mathcal{U}}}$, as desired. □

For the reader's convenience, we reproduce the statement of Theorem B here.

Theorem 8.5. *Let A be a separable, simple, nuclear, $\mathcal{Z}$-stable unital C^*-algebra with non-empty trace space. Let G be a countable, discrete, amenable group, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action. Suppose that $\partial_e T(A)$ is compact, that $\dim(\partial_e T(A)) < \infty$, that the orbits of the induced action of G on $\partial_e T(A)$ are finite with uniformly bounded cardinality, and that the orbit space $\partial_e T(A)/G$ is Hausdorff. Then α is cocycle conjugate to $\alpha \otimes \mathrm{id}_{\mathcal{Z}}$.*

Proof. By Proposition 8.4, for every $d \in \mathbb{N}$ there exists a unital homomorphism

$$\varphi \colon M_d \to (A^{\mathcal{U}} \cap A')^{\alpha^{\mathcal{U}}}.$$

The conclusion follows by the implication (3) ⇒ (4) in Theorem 7.6. □

We record the following immediate combination of Theorem 8.5 and Theorem 7.8.

Corollary 8.6. *Let A be a separable, simple, nuclear, $\mathcal{Z}$-stable unital C^*-algebra, let G be a countable, discrete, amenable group, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action. Suppose that $\partial_e T(A)$ is compact, that $\dim(\partial_e T(A)) < \infty$, that the orbits of the induced action of G on $\partial_e T(A)$ are finite with uniformly bounded cardinality, and that the orbit space $\partial_e T(A)/G$ is Hausdorff. Then the following are equivalent:*

(1) α is strongly outer.
(2) α has the weak tracial Rokhlin property.
(3) when G is residually finite, α has finite Rokhlin dimension.
(4) when G is residually finite, we have $\dim_{\mathrm{Rok}}(\alpha) \leq 2$.

The difference between the above corollary and Theorem 7.8 is that here we do not assume that α absorbs $\mathrm{id}_{\mathcal{Z}}$; since $\partial_e T(A)$ is compact finite dimensional and $\partial_e T(A)/G$ is Hausdorff, this assumption follows from Theorem 8.5.

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