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Results in Mathematics

Strong Law of Large Numbers for Iterates of Some Random-Valued Functions

Karol Baron and Rafał Kapica

Abstract. Assume $(\Omega, \mathscr{A}, P)$ is a probability space, X is a compact metric space with the σ-algebra $\mathscr{B}$ of all its Borel subsets and $f : X \times \Omega \to X$ is $\mathscr{B} \otimes \mathscr{A}$-measurable and contractive in mean. We consider the sequence of iterates of f defined on $X \times \Omega^{\mathbb{N}}$ by $f^0(x, \omega) = x$ and $f^n(x, \omega) = f\big(f^{n-1}(x, \omega), \omega_n\big)$ for $n \in \mathbb{N}$, and its weak limit π. We show that if $\psi : X \to \mathbb{R}$ is continuous, then for every $x \in X$ the sequence $\left(\frac{1}{n}\sum_{k=1}^{n} \psi\big(f^k(x, \cdot)\big)\right)_{n\in\mathbb{N}}$ converges almost surely to $\int_X \psi d\pi$. In fact, we are focusing on the case where the metric space is complete and separable.

Mathematics Subject Classification. 37H12, 39B12, 60B12, 60F15.

Keywords. Random-valued functions, Iterates, Strong law of large numbers, Convergence in law, Almost sure convergence.

1. Introduction

Fix a probability space $(\Omega, \mathscr{A}, P)$ and a metric space X.

Let $\mathscr{B}$ denote the σ-algebra of all Borel subsets of X. We say that $f : X \times \Omega \to X$ is a *random-valued* function (shortly: an *rv-function*) if it is measurable with respect to the product σ-algebra $\mathscr{B} \otimes \mathscr{A}$. The iterates of such an rv-function are given by

$$f^0(x, \omega_1, \omega_2, \ldots) = x, \quad f^n(x, \omega_1, \omega_2, \ldots) = f\big(f^{n-1}(x, \omega_1, \omega_2, \ldots), \omega_n\big)$$

for $n \in \mathbb{N}$, $x \in X$ and $(\omega_1, \omega_2, \ldots)$ from Ω^∞ defined as $\Omega^{\mathbb{N}}$. Note that $f^n : X \times \Omega^\infty \to X$ is an rv-function on the product probability space $(\Omega^\infty, \mathscr{A}^\infty, P^\infty)$. More exactly, for $n \in \mathbb{N}$ the n-th iterate f^n is $\mathscr{B} \otimes \mathscr{A}_n$-measurable, where $\mathscr{A}_n$ denotes the σ-algebra of all sets of the form

$$\{(\omega_1, \omega_2, \ldots) \in \Omega^\infty : \ (\omega_1, \ldots, \omega_n) \in A\}$$

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with A from the product σ-algebra $\mathscr{A}^n$. See [10, Sec. 1.4], [8].

A result on a.s. convergence of $\big(f^n(x,\cdot)\big)_{n\in\mathbb{N}}$ for X being the unit interval can be found in [10, Sec. 1.4B]. The paper [7] brings theorems on the convergence a.s. and in L^1 of those sequences of iterates in the case where X is a closed subset of a separable Banach lattice. A simple criterion for the convergence in law of $\big(f^n(x,\cdot)\big)_{n\in\mathbb{N}}$ to a random variable independent of $x \in X$ was proved in [1], assuming that X is complete and separable. In [2] it has been strengthened and applied to obtain a weak law of large numbers for iterates of random-valued functions. In the present paper we are interested in a strong law of large numbers. We will be based on the following Brunk-Prokhorov-type theorem, see [11, Theorem 3.3.1] and [6, Corollary 3.1].

(C) Let $(\mathscr{F}_n)_{n\in\mathbb{N}}$ be an increasing sequence of sub-σ-algebras of $\mathscr{A}$ and $(\xi_n)_{n\in\mathbb{N}}$ a sequence of random variables such that ξ_n is $\mathscr{F}_n$-measurable and $\mathbb{E}(\xi_{n+1}|\mathscr{F}_n) = 0$ for each $n \in \mathbb{N}$. If $(a_n)_{n\in\mathbb{N}}$ is an increasing and unbounded sequence of positive reals and

$$\sum_{n=1}^{\infty} \frac{\mathbb{E}\big(|\xi_n|^2\big)}{a_n^2} < \infty,$$

then

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{k=1}^{n} \xi_k = 0 \text{ a.s.}$$

2. A Scheme

Assume X is a metric space and $f : X \times \Omega \to X$ an rv-function.

Lemma 1. *If $\varphi : X \to \mathbb{R}$ is Borel and $\varphi \circ f^n(x,\cdot)$ is integrable for P^∞ for each $x \in X$ and $n \in \mathbb{N}$, then the function $\alpha : X \to \mathbb{R}$ defined by*

$$\alpha(x) = \int_\Omega \varphi\big(f(x,\omega)\big) P(d\omega) \tag{1}$$

is Borel and

$$\mathbb{E}\big(\varphi \circ f^{n+1}(x,\cdot)|\mathscr{A}_n\big) = \alpha \circ f^n(x,\cdot) \quad \textit{for } x \in X \textit{ and } n \in \mathbb{N}.$$

Proof. Since $\varphi \circ f$ is $\mathscr{B} \otimes \mathscr{A}$-measurable, by Fubini's theorem α is Borel. Consequently, for every $x \in X$ and $n \in \mathbb{N}$ the function $\alpha \circ f^n(x,\cdot)$ is $\mathscr{A}_n$-measurable and for each $A \in \mathscr{A}^n$ we have

$$\begin{aligned}&\int_{\{\omega\in\Omega^\infty:\ (\omega_1,\ldots,\omega_n)\in A\}} \varphi\big(f^{n+1}(x,\omega)\big) P^\infty(d\omega)\\ &\quad= \int_{\{\omega\in\Omega^\infty:\ (\omega_1,\ldots,\omega_n)\in A\}} \varphi\left(f\big(f^n(x,\omega),\omega_{n+1}\big)\right) P^\infty(d\omega)\end{aligned}$$

$$= \int_{\{\omega\in\Omega^\infty:\ (\omega_1,\dots,\omega_n)\in A\}} \left(\int_\Omega \varphi\left(f\big(f^n(x,\omega),\omega_{n+1}\big)\right) P(d\omega_{n+1})\right) P^\infty(d\omega)$$
$$= \int_{\{\omega\in\Omega^\infty:\ (\omega_1,\dots,\omega_n)\in A\}} \alpha\big(f^n(x,\omega)\big) P^\infty(d\omega).$$

□

The following theorem is in fact a scheme of proving a strong law of large numbers for iterates of random-valued functions.

Proposition 1. *Let $\psi : X \to \mathbb{R}$ and assume that there exists a Borel and bounded $\varphi : X \to \mathbb{R}$ such that*

$$\varphi(x) = \int_\Omega \varphi\big(f(x,\omega)\big)P(d\omega) + \psi(x) \quad \textit{for } x \in X. \tag{2}$$

If $(a_n)_{n\in\mathbb{N}}$ is an increasing and unbounded sequence of positive reals such that

$$\sum_{n=1}^{\infty} \frac{1}{a_n^2} < \infty,$$

then, for every $x \in X$,

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{k=1}^{n} \psi \circ f^k(x,\cdot) = 0 \quad \textit{a.e. for } P^\infty. \tag{3}$$

Proof. Define $\alpha : X \to \mathbb{R}$ by (1). Since φ is bounded, $|\varphi(x)| \le M$ for every $x \in X$ with an $M \in (0,\infty)$. Obviously also $|\alpha(x)| \le M$ for every $x \in X$. Fix $x \in X$ and put

$$\xi_n = \varphi \circ f^n(x,\cdot) - \alpha \circ f^{n-1}(x,\cdot) \quad \text{for } n \in \mathbb{N}. \tag{4}$$

Then $|\xi_n| \le 2M$ and by Lemma 1, $\mathbb{E}(\xi_{n+1}|\mathscr{A}_n) = 0$ for each $n \in \mathbb{N}$. It now follows from Brunk-Prokhorov-type theorem (C) that

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{k=1}^{n} \big(\varphi \circ f^k(x,\cdot) - \alpha \circ f^{k-1}(x,\cdot)\big) = 0 \quad \text{a.e. for } P^\infty. \tag{5}$$

Since $\psi = \varphi - \alpha$, for every $n \in \mathbb{N}$ we have

$$\sum_{k=1}^{n} \psi \circ f^k(x,\cdot) = \sum_{k=1}^{n} \big(\varphi \circ f^k(x,\cdot) - \alpha \circ f^{k-1}(x,\cdot)\big) + \sum_{k=1}^{n} \big(\alpha \circ f^{k-1}(x,\cdot) - \alpha \circ f^k(x,\cdot)\big),$$

i.e.,

$$\sum_{k=1}^{n} \psi \circ f^k(x,\cdot) = \sum_{k=1}^{n} \big(\varphi \circ f^k(x,\cdot) - \alpha \circ f^{k-1}(x,\cdot)\big) + \alpha(x) - \alpha \circ f^n(x,\cdot) \tag{6}$$

for every $n \in \mathbb{N}$. Moreover, $|\alpha \circ f^n(x,\cdot)| \le M$. Consequently (3) holds. □

3. The Weak Limit

Assume now the following hypothesis (H).

(H) (X, ρ) is a complete and separable metric space and $f : X \times \Omega \to X$ is an rv-function such that

$$\int_\Omega \rho\big(f(x,\omega), f(z,\omega)\big) P(d\omega) \le \lambda \rho(x,z) \quad \text{for } x, z \in X \tag{7}$$

with a $\lambda \in (0,1)$, and

$$\int_\Omega \rho\big(f(x,\omega), x\big) P(d\omega) < \infty \quad \text{for } x \in X. \tag{8}$$

Then (see [1, Theorem 3.1]) there exists a probability Borel measure π^f on X such that for every $x \in X$ the sequence of distributions of $f^n(x,\cdot)$, $n \in \mathbb{N}$, converges weakly to π^f. See also [3, Lemma 2.2] and [9, Corollary 5.6 and Lemma 3.1].

This limit distribution π^f plays an important role in solving functional equations, in particular in the class of Hölder continuous functions. We call a function $\psi : X \to \mathbb{R}$ *Hölder continuous with exponent* $\delta \in (0,1]$ if there is a constant $L \in [0,\infty)$ such that

$$|\psi(x) - \psi(z)| \le L\rho(x,z)^\delta \quad \text{for } x, z \in X.$$

Moreover we call a function *Hölder continuous* if it is Hölder continuous with an exponent $\delta \in (0,1]$. The following theorem (see [3, Theorem 2.1] and [4, Corollary 2.6]) will be useful to us.

(B) Assume (H). If $\psi : X \to \mathbb{R}$ is Hölder continuous with exponent $\delta \in (0,1]$, then it is integrable for π^f and if additionally

$$\int_X \psi(x)\pi^f(dx) = 0, \tag{9}$$

then there exists a Hölder continuous with exponent δ function $\varphi : X \to \mathbb{R}$ such that (2) holds.

4. Main Results

In what follows (X, ρ) is a metric space and $f : X \times \Omega \to X$ is an rv-function.

We start with a simple consequence of Proposition 1 and (B). It is a special case of Theorem 2 given below, but shows our approach without technical details.

Theorem 1. *If (X, ρ) is complete and separable with finite diameter and (7) holds with a $\lambda \in (0,1)$, then for every Hölder continuous $\psi : X \to \mathbb{R}$ and for each $x \in X$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \psi \circ f^k(x,\cdot) = \int_X \psi d\pi^f \quad a.e.\ for\ P^\infty. \tag{10}$$

Proof. Fix a Hölder continuous $\psi : X \to \mathbb{R}$. Replacing ψ by $\psi - \int_X \psi d\pi^f$ we may assume that (9) holds. By (B) there is a Hölder continuous $\varphi : X \to \mathbb{R}$ satisfying (2). Since X is bounded, so is φ. Applying now Proposition 1 with $a_n = n$ for $n \in \mathbb{N}$ we obtain (3) which ends the proof. □

Since continuous real functions defined on a compact metric space can be uniformly approximated by Lipschitz functions (see [5, 11.2.4]), Theorem 1 implies the following corollary.

Corollary 1. *If (X, ρ) is compact and (7) holds with a $\lambda \in (0,1)$, then we have (10) for every continuous $\psi : X \to \mathbb{R}$ and for each $x \in X$.*

Theorem 2. *Assume (H). Let $x \in X$ and*

$$\sum_{n=1}^{\infty} \frac{\int_{\Omega^\infty} \rho\big(f^n(x,\omega), x\big)^{2\delta} P^\infty(d\omega)}{a_n^2} < \infty$$

with a $\delta \in (0,1]$ and an increasing and unbounded sequence $(a_n)_{n\in\mathbb{N}}$ of positive reals. If $\psi : X \to \mathbb{R}$ is Hölder continuous with exponent δ, then

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{k=1}^{n} \Big(\psi \circ f^k(x,\cdot) - \int_X \psi d\pi^f\Big) = 0 \quad \text{a.e. for } P^\infty. \tag{11}$$

The proof will be based on three lemmas.

Assume that (X, ρ) is separable, (7) holds with a $\lambda \in (0,1)$, (8) is satisfied and $\varphi : X \to \mathbb{R}$ is Hölder continuous with exponent $\delta \in (0,1]$, i.e.,

$$|\varphi(x) - \varphi(z)| \le L\rho(x,z)^\delta \quad \text{for } x, z \in X \tag{12}$$

with an $L \in [0,\infty)$.

Lemma 2. *For every $x \in X$ and $n \in \mathbb{N}$ we have*

$$\int_{\Omega^\infty} \rho\big(f^n(x,\omega), x\big) P^\infty(d\omega) \le \frac{1}{1-\lambda} \int_\Omega \varrho\big(f(x,\omega), x\big) P(d\omega),$$

$$\int_{\Omega^\infty} |\varphi\big(f^n(x,\omega)\big)| P^\infty(d\omega) \le L\left(\int_{\Omega^\infty} \rho\big(f^n(x,\omega), x\big) P^\infty(d\omega)\right)^\delta + |\varphi(x)|.$$

Proof. Fix $x \in X$, $n \in \mathbb{N}$ and assume for the inductive proof that

$$\int_{\Omega^\infty} \rho\big(f^n(x,\omega), x\big) P^\infty(d\omega) \le \sum_{k=0}^{n-1} \lambda^k \int_\Omega \varrho\big(f(x,\omega), x\big) P(d\omega).$$

Then, applying Fubini's theorem, (7) and the above inequality, we obtain

$$\begin{aligned}
&\int_{\Omega^\infty} \rho\big(f^{n+1}(x,\omega),x\big)P^\infty(d\omega)\\
&\quad\le \int_{\Omega^\infty} \rho\big(f\big(f^n(x,\omega_1,\omega_2,\ldots),\omega_{n+1}\big),f(x,\omega_{n+1})\big)P^\infty\big(d(\omega_1,\omega_2,\ldots)\big)\\
&\qquad + \int_\Omega \rho(f(x,\omega_{n+1}),x)P(d\omega_{n+1})\\
&\quad\le \lambda\int_{\Omega^\infty} \rho\big(f^n(x,\omega),x\big)P^\infty(d\omega) + \int_\Omega \varrho\big(f(x,\omega),x\big)P(d\omega)\\
&\quad\le \sum_{k=0}^{n}\lambda^k \int_\Omega \varrho\big(f(x,\omega),x\big)P(d\omega)
\end{aligned}$$

which ends the proof of the first part. To get the second one observe that by (12) and Jensen's inequality for every $x\in X$ and $n\in\mathbb{N}$ we have

$$\begin{aligned}
\int_{\Omega^\infty} |\varphi\big(f^n(x,\omega)\big)|P^\infty(d\omega) &\le L\int_{\Omega^\infty} \rho\big(f^n(x,\omega),x\big)^\delta P^\infty(d\omega) + |\varphi(x)|\\
&\le L\left(\int_{\Omega^\infty} \rho\big(f^n(x,\omega),x\big)P^\infty(d\omega)\right)^\delta + |\varphi(x)|.
\end{aligned}$$

□

Lemma 2 makes sense to define a Borel function $\alpha: X\to\mathbb{R}$ by (1).

Lemma 3. *For every $x\in X$ and $n\in\mathbb{N}$ we have*

$$\begin{aligned}
&\int_{\Omega^\infty} |\varphi\big(f^n(x,\omega)\big) - \alpha\big(f^{n-1}(x,\omega)\big)|^2 P^\infty(d\omega)\\
&\qquad\le 8L^2\int_{\Omega^\infty} \rho\big(f^n(x,\omega),x\big)^{2\delta}P^\infty(d\omega).
\end{aligned}$$

Proof. Since, for every $\omega\in\Omega^\infty$ and $\omega'\in\Omega$,

$$\begin{aligned}
|\varphi\big(f^n(x,\omega)\big)-\varphi\big(f\big(f^{n-1}(x,\omega),\omega'\big)\big)| &\le L\rho\big(f^n(x,\omega),f\big(f^{n-1}(x,\omega),\omega'\big)\big)^\delta\\
&\le L\left(\rho\big(f^n(x,\omega),x\big)^\delta + \rho\big(f\big(f^{n-1}(x,\omega),\omega'\big),x\big)^\delta\right),
\end{aligned}$$

for every $\omega\in\Omega$ we have

$$\begin{aligned}
&|\varphi\big(f^n(x,\omega)\big) - \alpha\big(f^{n-1}(x,\omega)\big)|^2\\
&\quad= \left|\int_\Omega \left(\varphi\big(f^n(x,\omega)\big) - \varphi\big(f\big(f^{n-1}(x,\omega),\omega'\big)\big)\right)P(d\omega')\right|^2\\
&\quad\le L^2\left(\rho\big(f^n(x,\omega),x\big)^\delta + \int_\Omega \rho\big(f\big(f^{n-1}(x,\omega),\omega'\big),x\big)^\delta P(d\omega')\right)^2\\
&\quad\le 4L^2\left(\rho\big(f^n(x,\omega),x\big)^{2\delta} + \left(\int_\Omega \rho\big(f\big(f^{n-1}(x,\omega),\omega'\big),x\big)^\delta P(d\omega')\right)^2\right).
\end{aligned}$$

Hence, applying Jensen's inequality and Fubini's theorem,

$$\begin{aligned}\int_{\Omega^\infty} |\varphi\big(f^n(x,\omega)\big) - \alpha\big(f^{n-1}(x,\omega)\big)|^2 P^\infty(d\omega) \\ \leq 4L^2\Big(\int_{\Omega^\infty} \rho\big(f^n(x,\omega),x\big)^{2\delta} P^\infty(d\omega) \\ + \int_{\Omega^\infty}\left(\int_\Omega \rho\big(f\big(f^{n-1}(x,\omega),\omega'\big),x\big)^{2\delta} P(d\omega')\right) P^\infty(d\omega)\Big) \\ = 8L^2 \int_{\Omega^\infty} \rho\big(f^n(x,\omega),x\big)^{2\delta} P^\infty(d\omega).\end{aligned}$$

□

Lemma 4. *Let* $(b_n)_{n\in\mathbb{N}}$ *be a converging to zero sequence of positive reals. If* $x \in X$ *and there is a* $p \in (0,\infty)$ *such that*

$$\sum_{n=1}^{\infty} b_n^p \int_{\Omega^\infty} \rho\big(f^n(x,\omega),x\big)^{p\delta} P^\infty(d\omega) < \infty,$$

then

$$\lim_{n\to\infty} b_n\, \alpha \circ f^n(x,\cdot) = 0 \quad a.e.\ for\ P^\infty.$$

Proof. If $n \in \mathbb{N}$ and $\omega \in \Omega$, then by (1), (12), Jensen's inequality and (7) we have

$$\begin{aligned}|\alpha\big(f^n(x,\omega)\big)| &\leq \int_\Omega |\varphi\left(f\big(f^n(x,\omega),\omega'\big)\right)| P(d\omega') \\ &\leq L\int_\Omega \rho\left(f\big(f^n(x,\omega),\omega'\big), f(x,\omega')\right)^\delta P(d\omega') \\ &\quad + L\int_\Omega \rho\big(f(x,\omega'),x\big)^\delta P(d\omega') + |\varphi(x)| \\ &\leq L\lambda^\delta \rho\big(f^n(x,\omega),x\big)^\delta + L\left(\int_\Omega \rho\big(f(x,\omega),x\big)P(d\omega)\right)^\delta + |\varphi(x)|.\end{aligned}$$

Now to finish the proof it is enough to show that $\lim_{n\to\infty} b_n\xi_n = 0$ a.e. for P^∞, where $\xi_n = \rho\big(f^n(x,\cdot),x\big)^\delta$ for $n \in \mathbb{N}$. To this end observe that by Markov's inequality for every $n \in \mathbb{N}$ and $\varepsilon > 0$ we have

$$P^\infty(b_n\xi_n \geq \varepsilon) \leq \frac{\mathbb{E}(\xi_n^p)}{(\frac{\varepsilon}{b_n})^p} = \frac{1}{\varepsilon^p} b_n^p \mathbb{E}(\xi_n^p).$$

Hence it follows from the assumption of the lemma that for every $\varepsilon > 0$ the series $\sum_{n=1}^\infty P^\infty(b_n\xi_n \geq \varepsilon)$ converges. Consequently, $\lim_{n\to\infty} b_n\xi_n = 0$ a.e. for P^∞. □

Proof of Theorem 2. Fix a Hölder continuous with exponent δ function $\psi : X \to \mathbb{R}$. Replacing ψ by $\psi - \int_X \psi d\pi^f$ we may assume that (9) holds. By (B) there is a Hölder continuous with exponent δ function $\varphi : X \to \mathbb{R}$ satisfying

(2). Now using Lemma 2 define a Borel function $\alpha : X \to \mathbb{R}$ by (1). Since $\psi = \varphi - \alpha$, (6) follows. Applying Lemmas 1 and 3, and the Brunk-Prokhorov-type theorem (C) to the sequence of random variables $(\xi_n)_{n\in\mathbb{N}}$ defined by (4), we have (5). Finally, by Lemma 4 with $b_n = \frac{1}{a_n}$, $n \in \mathbb{N}$, and $p = 2$,

$$\lim_{n\to\infty} \frac{1}{a_n} \alpha \circ f^n(x, \cdot) = 0 \quad \text{a.e. for } P^\infty.$$

This, (5), (6) and (9) give (11). □

Corollary 2. *Assume (H). If $\psi : X \to \mathbb{R}$ is Hölder continuous with an exponent $\delta \le \frac{1}{2}$, then we have (10) for each $x \in X$.*

Proof. It is enough to observe that by Jensen's inequality and Lemma 2 for every $x \in X$ we have

$$\begin{aligned}\int_{\Omega^\infty} \rho\big(f^n(x,\omega), x\big)^{2\delta} P^\infty(d\omega) &\le \left(\int_{\Omega^\infty} \rho\big(f^n(x,\omega), x\big) P^\infty(d\omega)\right)^{2\delta} \\ &\le \left(\frac{1}{1-\lambda}\int_\Omega \varrho\big(f(x,\omega), x\big) P(d\omega)\right)^{2\delta},\end{aligned}$$

and then to apply Theorem 2 with $a_n = n$, $n \in \mathbb{N}$. □

To get a result for exponents $\delta > \frac{1}{2}$ we accept the following hypothesis (H_δ) with parameter $\delta \in (0, \infty)$.

(H_δ) (X, ρ) is a complete and separable metric space, $f : X \times \Omega \to X$ is an rv-function such that

$$\rho\big(f(x,\omega), f(z,\omega)\big) \le \xi(\omega)\rho(x, z) \quad \text{for } \omega \in \Omega \text{ and } x, z \in X, \tag{13}$$

where $\xi : \Omega \to [0, \infty)$ is a random variable for which $\mathbb{E}(\xi^{2\delta}) < 1$, and

$$\int_\Omega \rho\big(f(x_0,\omega), x_0\big)^{2\delta} P(d\omega) < \infty$$

with an $x_0 \in X$.

Remark 1. If $\delta \ge \frac{1}{2}$, then (H_δ) implies (H).

Proof. Assume (H_δ) with a $\delta \ge \frac{1}{2}$. By Jensen's inequality

$$\mathbb{E}\xi = \mathbb{E}\big((\xi^{2\delta})^{\frac{1}{2\delta}}\big) \le \big(\mathbb{E}(\xi^{2\delta})\big)^{\frac{1}{2\delta}} < 1$$

and

$$\int_\Omega \rho\big(f(x_0,\omega), x_0\big) P(d\omega) \le \left(\int_\Omega \rho\big(f(x_0,\omega), x_0\big)^{2\delta} P(d\omega)\right)^{\frac{1}{2\delta}}.$$

Moreover, for every $x \in X$,

$$\int_\Omega \rho\big(f(x,\omega),x\big)P(d\omega) \le \int_\Omega \rho\big(f(x,\omega),f(x_0,\omega)\big)P(d\omega) + \int_\Omega \rho\big(f(x_0,\omega),x_0)\big)P(d\omega) + \rho(x_0,x) \le \big(\mathbb{E}\xi+1\big)\rho(x,x_0) + \int_\Omega \rho\big(f(x_0,\omega),x_0)\big)P(d\omega).$$

□

Theorem 3. *Assume* (H_δ) *with a* $\delta \in [\frac{1}{2},1]$*. If* $\psi : X \to \mathbb{R}$ *is Hölder continuous with exponent* δ*, then we have* (10) *for each* $x \in X$*.*

Proof. By Remark 1 we have (H), and it follows from Theorem 2 that to finish the proof it is enough to show that for every $x \in X$ the sequence

$$\Big(\int_{\Omega^\infty} \rho\big(f^n(x,\omega),x\big)^{2\delta} P^\infty(d\omega)\Big)_{n\in\mathbb{N}}$$

is bounded. This follows from the lemma that is stated below. □

Let

$$\beta_p(x) = \int_\Omega \rho\big(f(x,\omega),x\big)^p P(d\omega) \quad \text{for } p \in (0,\infty) \text{ and } x \in X.$$

Lemma 5. *Assume (13) holds with a random variable* $\xi : \Omega \to [0,\infty)$ *and let* p *be a positive real. If* $\mathbb{E}(\xi^p) < 1$ *and* $\beta_p(x_0) < \infty$ *for an* $x_0 \in X$*, then* $\beta_p(x) < \infty$ *for every* $x \in X$ *and there exists a constant* $c_p \in (0,\infty)$ *such that*

$$\int_{\Omega^\infty} \rho\big(f^n(x,\omega),x\big)^p P^\infty(d\omega) \le c_p\beta_p(x) \quad \textit{for } x \in X \textit{ and } n \in \mathbb{N}.$$

Proof. Fix $x \in X$. By (13) for every $\omega \in \Omega$ we have

$$\rho\big(f(x,\omega),x\big)^p \le 3^p\left(\xi(\omega)^p\rho(x,x_0)^p + \rho\big(f(x_0,\omega),x_0\big)^p + \rho(x_0,x)^p\right),$$

whence

$$\int_\Omega \rho\big(f(x,\omega),x\big)^p P(d\omega) \le 3^p\left(\big(\mathbb{E}(\xi^p)+1\big)\rho(x,x_0)^p + \int_\Omega \rho\big(f(x_0,\omega)x_0\big)^p P(d\omega)\right) < \infty.$$

Put now

$$\eta(\omega) = \rho\big(f(x,\omega),x\big) \quad \text{for } \omega \in \Omega,$$

and

$$\xi_n(\omega_1,\omega_2,\dots) = \xi(\omega_n), \quad \eta_n(\omega_1,\omega_2,\dots) = \eta(\omega_n)$$

for $n \in \mathbb{N}$ and $(\omega_1, \omega_2, \ldots) \in \Omega^\infty$. Then, by induction and (13),

$$\rho\big(f^n(x,\omega), x\big) \le \sum_{k=1}^{n} \eta_k(\omega)\xi_{k+1}(\omega) \cdot \ldots \cdot \xi_n(\omega) \quad \text{for } \omega \in \Omega^\infty \text{ and } n \in \mathbb{N},$$

where $\prod_{j=n+1}^{n} \xi_j(\omega) := 1$. Consequently,

$$\int_{\Omega^\infty} \rho\big(f^n(x,\omega), x\big)^p P^\infty(d\omega) \le \mathbb{E}\Big(\big(\sum_{k=1}^{n} \eta_k \prod_{j=k+1}^{n} \xi_j\big)^p\Big) \quad \text{for } n \in \mathbb{N}.$$

Moreover, for every integer $n \ge 2$ and $k \in \{1, \ldots, n-1\}$ the random variables $\eta_k, \xi_{k+1}, \ldots, \xi_n$ are independent. Hence, if $p \in (0,1)$, then for every $n \in \mathbb{N}$ we have

$$\begin{aligned}\int_{\Omega^\infty} \rho\big(f^n(x,\omega), x\big)^p P^\infty(d\omega) &\le \mathbb{E}\big(\sum_{k=1}^{n} \eta_k^p \prod_{j=k+1}^{n} \xi_j^p\big) = \sum_{k=1}^{n} \mathbb{E}(\eta_k^p) \prod_{j=k+1}^{n} \mathbb{E}(\xi_j^p)\\ &= \sum_{k=1}^{n} \mathbb{E}(\eta^p)\big(\mathbb{E}(\xi^p)\big)^{n-k} = \mathbb{E}(\eta^p)\frac{1-\big(\mathbb{E}(\xi^p)\big)^n}{1-\mathbb{E}(\xi^p)}\\ &\le \mathbb{E}(\eta^p)\frac{1}{1-\mathbb{E}(\xi^p)} = \frac{1}{1-\mathbb{E}(\xi^p)}\beta_p(x).\end{aligned}$$

If $p \in [1, \infty)$, then by Minkowski's inequality for every $n \in \mathbb{N}$ we have

$$\begin{aligned}\Big(\int_{\Omega^\infty} \rho\big(f^n(x,\omega), x\big)^p P^\infty(d\omega)\Big)^{1/p} &\le \sum_{k=1}^{n} \Big(\mathbb{E}\big(\eta_k \prod_{j=k+1}^{n} \xi_j\big)^p\Big)^{1/p}\\ = \sum_{k=1}^{n} \Big(\mathbb{E}(\eta_k^p) \prod_{j=k+1}^{n} \mathbb{E}(\xi_j^p)\Big)^{1/p} &\le \frac{1}{1-\big(\mathbb{E}(\xi^p)\big)^{1/p}}\beta_p(x)^{1/p}.\end{aligned}$$

□

Corollary 3. *Assume that either*

(i) (H_δ) holds with a $\delta \in [\frac{1}{2}, 1]$ and $\psi : X \to \mathbb{R}$ is Hölder continuous with exponent δ,

or

(ii) $(H_{\frac{1}{2}})$ is satisfied and $\psi : X \to \mathbb{R}$ is Hölder continuous with an exponent $\delta \le \frac{1}{2}$.

Then for every bounded and nonempty $A \subset X$ and for almost all $\omega \in \Omega^\infty$ with respect to P^∞,

$$\lim_{n\to\infty} \sup\Big\{\Big|\frac{1}{n}\sum_{k=1}^{n} \psi\big(f^k(x,\omega)\big) - \int_X \psi d\pi^f\Big| : x \in A\Big\} = 0.$$

Proof. It concerns both, (i) and (ii).

By induction,

$$\rho\big(f^n(x,\omega), f^n(z,\omega)\big) \le \left(\prod_{k=1}^{n} \xi_k(\omega)\right) \rho(x,z)$$

for $x, z \in X,\ \omega \in \Omega^\infty$ and $n \in \mathbb{N}$, with

$$\xi_n(\omega_1, \omega_2, \ldots) = \xi(\omega_n) \quad \text{for } (\omega_1, \omega_2, \ldots) \in \Omega^\infty \text{ and } n \in \mathbb{N}.$$

Hence

$$|\psi\big(f^n(x,\omega)\big) - \psi\big(f^n(z,\omega)\big)| \le L \left(\prod_{k=1}^{n} \xi_k(\omega)^\delta\right) \rho(x,z)^\delta$$

for $x, z \in X,\ \omega \in \Omega^\infty$ and $n \in \mathbb{N}$, with an $L \in (0,\infty)$.

Fix $z \in X$. Since, for every $x \in X,\ \omega \in \Omega^\infty$ and $n \in \mathbb{N}$,

$$\begin{aligned}
\left|\frac{1}{n}\sum_{k=1}^{n} \psi\big(f^k(x,\omega)\big) - \int_X \psi d\pi^f\right| &\le \frac{1}{n}\sum_{k=1}^{n} \left|\psi\big(f^k(x,\omega)\big) - \psi\big(f^k(z,\omega)\big)\right| \\
&\quad + \left|\frac{1}{n}\sum_{k=1}^{n} \psi\big(f^k(z,\omega)\big) - \int_X \psi d\pi^f\right| \\
&\le L\frac{1}{n}\sum_{k=1}^{n} \big(\prod_{j=1}^{k} \xi_j(\omega)^\delta\big)\rho(x,z)^\delta + \left|\frac{1}{n}\sum_{k=1}^{n} \psi\big(f^k(z,\omega)\big) - \int_X \psi d\pi^f\right|,
\end{aligned}$$

for every $r \in (0,\infty)$ and for every nonempty subset A of the ball with center at z and radius r, for every $\omega \in \Omega^\infty$ and $n \in \mathbb{N}$ we have

$$\begin{aligned}
\sup\Big\{\Big|\frac{1}{n}\sum_{k=1}^{n}\psi \circ f^k(x,\omega) - \int_X \psi d\pi^f\Big| : x \in A\Big\} \\
\le Lr^\delta \frac{1}{n}\sum_{k=1}^{n}\prod_{j=1}^{k} \xi_j(\omega)^\delta + \left|\frac{1}{n}\sum_{k=1}^{n} \psi\big(f^k(z,\omega)\big) - \int_X \psi d\pi^f\right|.
\end{aligned}$$

In view of Theorem 3 and Corollary 2, to finish the proof it is enough to show that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\prod_{j=1}^{k} \xi_j^\delta = 0 \quad \text{a.e. for } P^\infty. \tag{14}$$

To this end observe that, by Jensen's inequality, in the first case (i) we have

$$\mathbb{E}(\xi^\delta) = \mathbb{E}\big((\xi^{2\delta})^{\frac{1}{2}}\big) \le \big(\mathbb{E}(\xi^{2\delta})\big)^{\frac{1}{2}} < 1,$$

and in the second one

$$\mathbb{E}(\xi^\delta) \le (\mathbb{E}\xi)^\delta < 1.$$

Therefore, applying the monotone convergence theorem and independence of $\xi_n,\ n \in \mathbb{N}$, we get

$$\mathbb{E}\left(\sum_{n=1}^{\infty}\prod_{k=1}^{n}\xi_k^\delta\right) = \sum_{n=1}^{\infty}\mathbb{E}\left(\prod_{k=1}^{n}\xi_k^\delta\right) = \sum_{n=1}^{\infty}\prod_{k=1}^{n}\mathbb{E}\left(\xi_k^\delta\right) = \sum_{n=1}^{\infty}\left(\mathbb{E}(\xi^\delta)\right)^n < \infty.$$

Consequently, the series $\sum_{n=1}^{\infty}\prod_{k=1}^{n}\xi_k^\delta$ converges a.e. for P^∞ and (14) follows.

□

5. An Application to Random Affine Maps

Corollary 4. *Assume X is a closed subset of a separable Banach space containing the origin, $\xi : \Omega \to \mathbb{R}$ and $\eta : \Omega \to X$ are random variables such that $\xi(\omega)X + \eta(\omega) \subset X$ for $\omega \in \Omega$, and*

$$\zeta_n(\omega_1, \omega_2, \ldots) = \sum_{k=1}^{n}\left(\prod_{j=k+1}^{n}\xi(\omega_j)\right)\eta(\omega_k) \quad \textit{for } (\omega_1, \omega_2, \ldots) \in \Omega^\infty,\ n \in \mathbb{N}.$$

If either $\delta \in (0, \frac{1}{2}]$ and

$$\mathbb{E}|\xi| < 1, \quad \mathbb{E}\|\eta\| < \infty,$$

or $\delta \in [\frac{1}{2}, 1]$ and

$$\mathbb{E}(|\xi|^{2\delta}) < 1, \quad \mathbb{E}(\|\eta\|^{2\delta}) < \infty,$$

then there exists a probability Borel measure μ on X such that

$$\int_X \|x\|\mu(dx) < \infty$$

and for every Hölder continuous with exponent δ function $\psi : X \to \mathbb{R}$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\psi\circ\zeta_k = \int_X \psi d\mu \quad \textit{a.e. for } P^\infty.$$

Proof. The function $f : X \times \Omega \to X$ defined by

$$f(x, \omega) = \xi(\omega)x + \eta(\omega)$$

is an rv-function. It satisfies (H) in the first case, and (H_δ) in the second one. By induction,

$$f^n(x, \omega_1, \omega_2, \ldots) = \left(\prod_{k=1}^{n}\xi(\omega_k)\right)x + \sum_{k=1}^{n}\left(\prod_{j=k+1}^{n}\xi(\omega_j)\right)\eta(\omega_k)$$

for $x \in X,\ (\omega_1, \omega_2, \ldots) \in \Omega^\infty$ and $n \in \mathbb{N}$. Hence, $\zeta_n = f^n(0, \cdot)$ for $n \in \mathbb{N}$, so an application of Corollary 2 and Theorem 3 finishes the proof. □

Remark 2. Let $\lambda \in (0,1)$ and let $\eta : \Omega \to [0, 1-\lambda]$ be a random variable. Put

$$\zeta_n(\omega_1, \omega_2, \ldots) = \sum_{k=1}^{n} \lambda^{n-k} \eta(\omega_k)$$

for $(\omega_1, \omega_2, \ldots) \in \Omega^\infty$ and $n \in \mathbb{N}$. By Corollary 4 there exists a probability Borel measure μ on $[0,1]$ such that for every Hölder continuous $\psi : [0,1] \to \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \psi \circ \zeta_k = \int_{[0,1]} \psi d\mu \quad \text{a.e. for } P^\infty.$$

But, as observed in [2, Remark 4.3], if $(\psi \circ \zeta_n)_{n\in\mathbb{N}}$ converges in probability for a Borel $\psi : [0,1] \to \mathbb{R}$ such that

$$c|x - z| \le |\psi(x) - \psi(z)| \quad \text{for } x, z \in [0,1]$$

with a constant $c \in (0, \infty)$, then η is a.s. for P constant.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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