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Regularity, Asymptotic Solutions and Travelling Waves Analysis in a Porous Medium System to Model the Interaction between Invasive and Invaded Species

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Abstract: This work provides an analytical approach to characterize and determine solutions to a porous medium system of equations with views in applications to invasive-invaded biological dynamics. Firstly, the existence and uniqueness of solutions are proved. Afterwards, profiles of solutions are obtained making use of the self-similar structure that permits showing the existence of a diffusive front. The solutions are then studied within the Travelling Waves (TW) domain showing the existence of potential and exponential profiles in the stable connection that converges to the stationary solutions in which the invasive species predominates. The TW profiles are shown to exist based on the geometry perturbation theory together with an analytical-topological argument in the phase plane. The finding of an exponential decaying rate (related with the advection and diffusion parameters) in the invaded species TW is not trivial in the nonlinear diffusion case and reflects the existence of a TW trajectory governed by the invaded species runaway (in the direction of the advection) and the diffusion (acting in a finite speed front or support).

Keywords: porous medium equation; travelling waves; geometric perturbation; nonlinear diffusion; advection

MSC: 35K55; 35K57; 35K59; 35K65



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1. Introduction

According to the Convention of Biological Diversity (p. 1, Ch. 1 [1]), the biological invasion is defined as “. . . those alien species which threaten ecosystems, habitats or species”. The problem analyzed can be understood within the perspective of an invasive species that invades a region or space previously inhabited by the invaded species.

The interaction between invaded-invasive species has been analyzed based on a nonlinear diffusion previously in [2] to analyze biological cell dynamics related to a haptotactic process in cancer. Furthermore, stability analysis for an haptotaxis dynamics in melanoma invasion is provided in [3].

The reaction–diffusion problems with advection have been a source of research to model different physical scenarios. In [4], the author analyzes the impact of an advection coefficient to understand the hydrodynamical processes in combustion. In this case, it is particularly relevant to accurately model the advection coefficient to couple different involved mechanisms: chemical kinetic, thermodynamics, and fluid dynamics. A general nonlinear time-dependent advection does not necessarily lead to purely monotonic solutions; nonetheless, in [5], the authors analyze monotone properties of bounded Travelling Waves (TW) solutions for a density dependent diffusion and a nonlinear convection.

There are some analogies between predator–prey and invasive-invaded species models. Indeed, when an invasive species reaches into a region (a living organ for example), the invaded one may extinguish. Consequently, the invasive species may vanish as there is not invaded species to interact with. This described dynamics can be regarded as a connection with predator–prey processes as the predator vanishes whenever the prey disappears. Note that predator–prey models have been analyzed recently with analytical and stability methods. In [6], different forms of functional responses are provided for modelling the predator–prey dynamic. The cited work considers the harvesting in the predator and the density-dependent mortality in the prey to assess the Hopf bifurcation in the proximity of the equilibrium point. In addition, the Hopf bifurcation method has been followed in [7] to study a delayed density-dependent predator–prey system with Beddington–DeAngelis functional response. The periodic dynamic in the solutions has been shown to exist as a consequence of stochastic disturbance for a Holling II functional response [8]. Similar stability and bifurcation methods have been used in [9] for a predator–prey model subject to the Allee effect with a discrete-time Holling type-IV functional response. The mentioned methods have been used as well in [10] to model an infected predator that consumes the prey according to a Holling type-II response.

There exist some commonalities among Evan functions, Hopf bifurcation and Geometric Perturbation Theory. Indeed, the theories mentioned focus on the searching of solutions stability and to qualitatively describe their behaviour. Solutions may be expressed as TW profiles to this end. Note that the geometric perturbation theory, as a baseline support in the search of TW profiles, is considered in this work.

The nonlinear diffusion considered in this work for modelling purposes is particularly relevant, as it drives the mathematical approaches considered. Somehow, our work is influenced by the seminal model proposed by Keller and Segel [11] to study the chemotaxis walks in cells:

$$\begin{aligned} u_t &= \nabla \cdot (d(u)u - \chi(v)u\nabla u) & x \in \Omega, t > 0 \\ v_t &= d_v \Delta v - uv & x \in \Omega, t > 0, \end{aligned} \quad (1)$$

where v is the chemical agent concentration and u is the cell density. The term $d(u)$ represents the diffusivity and $\chi(v)$ is the distribution of chemical agents sensitive to cells. In several novel articles, the Keller and Segel model was extended to include different forms of reaction and absorption terms (refer to [12–15]). The problem discussed in [16] was proposed as an integro-differential system of equations with pressure effects to predict the behaviour of cancerous cells, as an invasive species, spreading with a nonlinear diffusion over healthy cells in a closed organ. In [17], the authors provide a complete analysis of a chemotaxis process with a nonlocal reaction source where a uniform bound of solutions are explored for different conditions in the involved coefficients.

The use of nonlinear diffusion is an oblique topic and has been used in other interesting applications where numerical and purely analytical approaches have been followed. Such nonlinear diffusion allows for accurately modelling physical phenomena in which porosity is a governing parameter. The Darcy law involving nanofluids has been considered to derive numerical solutions by the Successive Local Linearization Method [18]. The authors show the different solution profiles obtained after the numerical exercise where the exponential decay can be perceived. Such an exponential decay is immediate in the linear diffusion case; nonetheless, for the nonlinear diffusion, further analytical assessments shall be done. The effect of porosity in a partial slip for a peristaltic transport in a Jeffrey fluid has been investigated in [19]. In addition, Ref. [20] employs a nonlinear diffusion to improve the accuracy in simulating potential coagulation in an electromagnetic blood flow in annular vessel geometries. In all the cited cases, the finite speed of propagation defines a diffusive front. This is a common behaviour to all porous medium equations and shall be characterized for the problem discussed in the present work.

Other models related with perturbation under nonlocal sources and attractive and repulsion actions can be mentioned as precluding studies. In [21], the authors provide an analysis about chemotactic phenomena driven by logistic and repulsive actions. Fur-

thermore, in [22], the authors study a porous medium equation with different reaction–absorption terms, depending in some cases on the gradient of the solution over a bounded domain in \mathbb{R}^N , $N \geq 1$.

2. Model Description and Methods

The problem P analyzed is as follows:

$$\begin{aligned} u_t &= \Delta u^m + c \cdot \nabla u + v(1 - u), \\ v_t &= \Delta v^m + c \cdot \nabla v - uv, \end{aligned} \tag{2}$$

$$m > 1, \quad u_0(x), v_0(x) > 0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d),$$

where $d > 1$ or $d = 1$ for convenience and is later specified. The solving methods exposed in this work aim to obtain explicit analytical solutions in the form of maximal, minimal solutions and Travelling Waves. The spatial operator involves a nonlinear diffusion with advection and is considered together with nonlinear coupled reaction–absorption terms.

The problem P justification is based on studying the invasive-invaded dynamic and the mathematical properties introduced by the spatial operator. For this purpose:

Consider that the invasive species acts in accordance with the presence of the invaded species. If the quantity of invaded is high, the temporal increasing rate in the invaded (u_t) shall be high; otherwise, the invasion will not succeed. As the invasive species proliferates over time, there shall be a limit in its concentration; then, the temporal ratio (considered positive) decreases because the invasive species progressively reaches its maximum stationary level in the medium. In addition, the invasive temporal rate depends on the invaded concentration (v). This behaviour is intended to be modelled as a reaction term of the form:

$$u_t = v(\kappa - u), \tag{3}$$

where κ represents the maximum invasive species concentration in the given medium. Without loss of generality, assume $\kappa = 1$.

The invaded species time vanishing rate is considered as:

$$v_t = -vu, \tag{4}$$

and aims to express that the decreasing rate is absolutely higher for increasing values of the invasive species u and is dependent on its own invaded species existence in the medium.

An advection term is proposed to account for possible forced movement in the media. The interaction of the advection, through the vector c , has the intention of modelling the desertion behaviour in the invaded species, so that the runaway is oriented in the direction of c and toward spatial areas not previously populated (represented by $c \cdot \nabla v$). In response to the invaded species desertion, the invasive species will follow an advection process given by the orientation c towards the same non-populated areas (represented by $c \cdot \nabla u$).

In addition, a nonlinear diffusion (Porous Medium Equation) is considered to model the random movement of both species. As an alternative to the classical linear order two diffusion, the Porous Medium Equation introduces concepts such as the finite speed of propagation in the invaded species, which is admitted to account for a further reliable model.

The study of solutions for a Porous Medium Equation with advection and no reaction has been widely analyzed in [23]. Nonetheless, the present work considers the effect of the coupled reaction and absorption for which the mixed monotone behaviour induces a different analytical treatment compared to that in [23].

This work begins by showing some regularity results together with uniqueness of solutions. The degeneracy of the diffusivity leads to considering weak solutions defined upon a test function $\phi \in C^\infty(\mathbb{R}^d)$. Afterwards, profiles of maximal and minimal solutions are shown to exist and analytically obtained based on the mixed monotone properties in the reaction–absorption terms. In addition, the problem P is studied within the geometric

perturbation theory to support the construction of analytical TW profiles. Finally, the finite propagation and the exponential decay for v are shown in the proximity of the null solution, i.e., when $v \rightarrow \epsilon \rightarrow 0^+$. The finite propagation is proved with a maximal solution so that any other lower solution exhibits finite propagation as well. In addition, the finite propagation is shown to provide a compact support that is mainly dependent on the advection (invaded runaway) c . This means that the homogenizing effect of the diffusive front is lost and the advection term drives the dynamic. This behaviour can be interpreted as the invasive species concentrating efforts in the runaway direction c so that the invaded species decreases exponentially even when the diffusion is nonlinear.

3. Existence and Uniqueness of Solutions

Let us consider a test function $\phi \in C^\infty(\mathbb{R}^d)$ such that, for $0 < \tau < t < T$:

$$\int_{\mathbb{R}^d} u(t) \phi(t) = \int_{\mathbb{R}^d} u(\tau) \phi(\tau) + \int_{\tau}^t \int_{\mathbb{R}^d} [u \phi_t + u^m \Delta \phi - c \cdot \nabla \phi u + v(1 - u) \phi] ds, \tag{5}$$

$$\int_{\mathbb{R}^d} v(t) \phi(t) = \int_{\mathbb{R}^d} v(\tau) \phi(\tau) + \int_{\tau}^t \int_{\mathbb{R}^d} [v \phi_t + v^m \Delta \phi - c \cdot \nabla \phi v - uv \phi] ds. \tag{6}$$

Considering $r \gg r_0 > 0$, the following uniformly parabolic quasilinear set of equations is defined (named P^ϕ):

$$\begin{aligned} u \phi_t + u^m \Delta \phi - c \cdot \nabla \phi u + v(1 - u) \phi &= 0, \\ v \phi_t + v^m \Delta \phi - c \cdot \nabla \phi v - uv \phi &= 0, \end{aligned} \tag{7}$$

in $B_r \times [0, T]$, with the following set of boundary and initial conditions:

$$\begin{aligned} (\nabla \phi - uc) \cdot \nu &= 0, \\ (\nabla \phi - vc) \cdot \nu &= 0, \end{aligned} \tag{8}$$

over $\partial B_r \times [0, T]$, where ν is the outer unitary normal vector in ∂B_r and

$$\begin{aligned} u(x, 0) &= u_0(x), \\ v(x, 0) &= v_0(x), \end{aligned} \tag{9}$$

initially on B_r .

The problem P^ϕ has existence and uniqueness of solutions based on the mixed monotone properties of the forcing terms [24]. For this purpose, note that the forcing part ϕ is a monotone increasing (with ϕ) function and $-\phi$ is a monotone decreasing function.

Theorem 1. *Given $u_0(x), v_0(x) \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, the set of solutions $u(x, t), v(x, t)$ are bounded for all $(x, t) \in B_r \times [0, T]$ with $r \gg 1$.*

Proof. Consider $\eta \in \mathbb{R}^+$ and sufficiently small, the following cut off function is defined [25]:

$$\begin{aligned} \psi_\eta &\in C_0^\infty(\mathbb{R}^d), & 0 \leq \psi_\eta \leq 1, \\ \psi_\eta &= 1 \text{ in } B_{r-\eta}, & \psi_\eta = 0 \text{ in } \mathbb{R}^d - B_{r-\eta}, \end{aligned} \tag{10}$$

so that

$$\begin{aligned} |\nabla \psi_\eta| &\leq \frac{c_a}{\eta}, \\ |\Delta \psi_\eta| &\leq \frac{c_a}{\eta^2}. \end{aligned} \tag{11}$$

After multiplication of (7) by ψ_η and integrating in $B_r \times [\tau, T]$:

$$\begin{aligned} \int_{\tau}^t \int_{B_r} u \phi_t \psi_\eta + \int_{\tau}^t \int_{B_r} u^m \Delta \phi \psi_\eta - \int_{\tau}^t \int_{B_r} c \cdot \nabla \phi u \psi_\eta + \int_{\tau}^t \int_{B_r} v(1 - u) \phi \psi_\eta &= 0, \\ \int_{\tau}^t \int_{B_r} v \phi_t \psi_\eta + \int_{\tau}^t \int_{B_r} v^m \Delta \phi \psi_\eta - \int_{\tau}^t \int_{B_r} c \cdot \nabla \phi v \psi_\eta - \int_{\tau}^t \int_{B_r} uv \psi_\eta \phi &= 0. \end{aligned} \tag{12}$$

The integrals for the advection term read:

$$\int_{\tau}^t \int_{B_r} c \cdot \nabla \phi u \psi_{\eta} = \int_{\tau}^t \int_{B_r} c \cdot \nabla \phi \nabla u \psi_{\eta} + \int_{\tau}^t \int_{B_r} c \cdot \nabla \phi u \nabla \psi_{\eta}, \tag{13}$$

For some large $r \gg r_0 > 1$ [25]:

$$\int_{\tau}^t u^m \leq c_1(\tau) r^{\frac{2m}{m-1}}, \tag{14}$$

and

$$\int_{\tau}^t u \leq c_2(\tau) r^{\frac{2}{m-1}}. \tag{15}$$

Making the integration over fixed B_r domains for $r \gg r_0 \gg 1$ and admitting $\tau \leftarrow t$, the functions ϕ, ψ_{η} are considered stationary locally in time. Then, the integral term in the diffusion reads:

$$\int_{\tau}^t \int_{B_r} u^m \Delta \phi \psi_{\eta} \leq \int_{B_r} c_1(\tau) r^{\frac{2m}{m-1}} \Delta \phi \psi_{\eta}. \tag{16}$$

Given a fixed r , the following holds:

$$\int_{B_r} c_1(\tau) r^{\frac{2m}{m-1}} \Delta \phi \psi_{\eta} = c_1(\tau) r^{\frac{2m}{m-1}} \left((\nabla \phi \psi_{\eta})_{\partial B_r} - \int_{B_r} \nabla \phi \cdot \nabla \psi_{\eta} \right). \tag{17}$$

Note that $(\nabla \phi \psi_{\eta})_{\partial B_r} \ll 1$ in (17) for $r \gg 1$. Indeed, ψ_{η} is a bounded non-increasing function and $\nabla \phi \rightarrow 0$ for $r \gg 1$ as it will be shown afterwards for a particular choice of the test function. Now:

$$\int_{B_r} c_1(\tau) r^{\frac{2m}{m-1}} \nabla \phi \cdot \nabla \psi_{\eta} \leq \int_{B_r} c_1(\tau) r^{\frac{2m}{m-1}} |\nabla \phi| \frac{c_a}{\eta} \leq c_1(\tau) c_a \int_{B_r} r^{\frac{2m}{m-1}-1} |\nabla \phi|. \tag{18}$$

Similarly for the the integrals in (13) and after arranging the spatial leading terms:

$$\int_{\tau}^t \int_{B_r} c \cdot \nabla \phi \nabla u \psi_{\eta} + \int_{\tau}^t \int_{B_r} c \cdot \nabla \phi u \nabla \psi_{\eta} \leq c c_2(\tau) \int_{\tau}^t r^{\frac{2}{m-1}-1} |\nabla \phi| + c c_2(\tau) c_a \int_{B_r} r^{\frac{2}{m-1}-1} \phi. \tag{19}$$

Next, consider a test function ϕ of the form:

$$\phi(x, s) = e^{g(s)} (1 + r^2)^{-\beta}, \tag{20}$$

where $g(s) > 0$ for $0 < s < t$, $g(t) = 0$ and β shall be chosen to ensure the convergence of the integrals in (18) and (19) as $r \rightarrow \infty$. To this end, it suffices to consider:

$$\beta = \frac{1}{m-1}. \tag{21}$$

Returning to (12):

$$\int_{\tau}^t \int_{B_r} u \phi_t \psi_{\eta} + \int_{\tau}^t \int_{B_r} v(1-u) \phi \psi_{\eta} = \int_{\tau}^t \int_{B_r} c \cdot \nabla \phi \nabla u \psi_{\eta} + \int_{\tau}^t \int_{B_r} c \cdot \nabla \phi u \nabla \psi_{\eta} + \int_{\tau}^t \int_{B_r} u^m \nabla \phi \cdot \nabla \psi_{\eta}, \tag{22}$$

$$\int_{\tau}^t \int_{B_r} v \phi_t \psi_{\eta} - \int_{\tau}^t \int_{B_r} uv \psi_{\eta} \phi = \int_{\tau}^t \int_{B_r} c \cdot \nabla \phi \nabla v \psi_{\eta} + \int_{\tau}^t \int_{B_r} c \cdot \nabla \phi v \nabla \psi_{\eta} + \int_{\tau}^t \int_{B_r} v^m \nabla \phi \cdot \nabla \psi_{\eta}.$$

The intention is to show that the left-hand integrals in (22) are finite, which is equivalent to searching for the bound on the right-hand side terms. Indeed, and for the particular expression of β in (21), the right-hand integrals are indeed bounded for $r \gg r_0 > 1$ over B_r domains. Then,

$$\int_{\tau}^t \int_{B_r} u \phi_t \psi_{\eta} + \int_{\tau}^t \int_{B_r} v(1-u) \phi \psi_{\eta} \leq A(\tau, t), \tag{23}$$

$$\int_{\tau}^t \int_{B_r} v \phi_t \psi_{\eta} - \int_{\tau}^t \int_{B_r} uv \psi_{\eta} \phi \leq B(\tau, t),$$

for some finite locally in time defined A, B .

As both integrals are finite in $\tau < s < t < T$, it is possible to conclude on the theorem postulation about the boundness of solutions in $B_r \times [0, \infty)$. \square

The next intention is to show the uniqueness of solutions:

Theorem 2. *Assuming that $(u, v) > (0, 0)$ is a minimal solution for P (2) in $\mathbb{R}^d \times (0, T)$, then (u, v) coincides with the maximal solution, i.e., the solution is unique.*

Proof. Assume (\hat{u}, \hat{v}) to be the maximal solution to P in $\mathbb{R}^d \times (0, T)$, such that:

$$(\hat{u}(x, 0), \hat{v}(x, 0)) = (u_0(x) + \nu, v_0(x) + \nu), \tag{24}$$

with $\nu > 0$ being arbitrarily small. In addition, define the minimal solution:

$$(u(x, 0), v(x, 0)) = (u_0(x), v_0(x)), \tag{25}$$

such that the maximal and minimal solutions verify respectively:

$$\begin{aligned} \hat{u}_t &= \Delta \hat{u}^m + c \cdot \nabla \hat{u} + \hat{v}(1 - u), \\ \hat{v}_t &= \Delta \hat{v}^m + c \cdot \nabla \hat{v} - u\hat{v}. \end{aligned} \tag{26}$$

$$\begin{aligned} u_t &= \Delta u^m + c \cdot \nabla u + v(1 - \hat{u}), \\ v_t &= \Delta v^m + c \cdot \nabla v - \hat{u}\hat{v}. \end{aligned} \tag{27}$$

Then, for every test function $\phi \in C^\infty(\mathbb{R}^d)$, the following expressions hold:

$$0 \leq \int_{\mathbb{R}^d} (\hat{u} - u)(t) \phi(t) = \int_0^t \int_{\mathbb{R}^d} [(\hat{u} - u) \phi_t + (\hat{u}^m - u^m) \Delta \phi + c \cdot \nabla \phi (\hat{u} - u) + (\hat{v}(1 - u) - v(1 - \hat{u})) \phi] ds, \tag{28}$$

$$0 \leq \int_{\mathbb{R}^d} (\hat{v} - v)(t) \phi(t) = \int_0^t \int_{\mathbb{R}^d} [(\hat{v} - v) \phi_t + (\hat{v}^m - v^m) \Delta \phi + c \cdot \nabla \phi (\hat{v} - v) + (\hat{u}\hat{v} - uv) \phi] ds. \tag{29}$$

Let us define:

$$a_1(x, s) = \begin{cases} \frac{\hat{u}^m - u^m}{\hat{u} - u} & \text{for } \hat{u} \neq u \\ m u^{m-1} & \text{otherwise} \end{cases}, \tag{30}$$

$$a_2(x, s) = \begin{cases} \frac{\hat{v}^m - v^m}{\hat{v} - v} & \text{for } \hat{v} \neq v \\ m v^{m-1} & \text{otherwise} \end{cases}. \tag{31}$$

Given two fixed values for x and $s = t \leq T$:

$$0 \leq a_1(x, s) \leq c_1(m, \|u_0\|_\infty, T), \tag{32}$$

$$0 \leq a_2(x, s) \leq c_2(m, \|v_0\|_\infty, T). \tag{33}$$

Consider the test function:

$$\phi(|x|, s) = e^{k(T-s)}(1 + |x|^2)^{-\gamma}, \tag{34}$$

for some constants k and γ .

The test function verifies:

$$\phi_t = -k\phi(x, s),$$

$$|\nabla_{|x|}\phi| \leq c_3(\gamma, d)\phi(x, s), \tag{35}$$

$$\Delta_{|x|}\phi \leq c_4(\gamma, d)\phi(x, s),$$

such that:

$$\begin{aligned}
 (\hat{u} - u) \phi_t + (\hat{u}^m - u^m) \Delta \phi + c \cdot \nabla \phi (\hat{u} - u) + (\hat{v}(1 - u) - v(1 - \hat{u})) \phi \leq -(\hat{u} - u) k \phi \\
 + a_1(\hat{u} - u) c_4 \phi + c c_3 \phi (\hat{u} - u) + L_1(\hat{v} - v) \phi,
 \end{aligned}
 \tag{36}$$

$$\begin{aligned}
 (\hat{v} - v) \phi_t + (\hat{v}^m - v^m) \Delta \phi + c \cdot \nabla \phi (\hat{v} - v) + (\hat{u}\hat{v} - uv) \phi \leq -(\hat{v} - v) k \phi \\
 + a_2(\hat{v} - v) c_4 \phi + c c_3 \phi (\hat{v} - v) + L_2(\hat{u} - u) \phi,
 \end{aligned}
 \tag{37}$$

where $L_1 = \max_{\mathbb{R}^d} \{1 - u\}$ and $L_2 = \max_{\mathbb{R}^d} \{\hat{v}\}$ are bounded as per Theorem 1. The constant k shall be selected such that:

$$-k + a_1 c_4 + c_3 c \leq 0, \quad -k + a_2 c_4 + c_3 c \leq 0.
 \tag{38}$$

It suffices to consider:

$$k \geq \max(a_1 c_4 + c_3 c, a_2 c_4 + c_3 c),
 \tag{39}$$

so that:

$$\begin{aligned}
 (-k + a_1 c_4 + c_3 c) \phi(\hat{u} - u) \leq 0, \\
 (-k + a_1 c_4 + c_3 c) \phi(\hat{v} - v) \leq 0.
 \end{aligned}
 \tag{40}$$

The expressions (28) and (29) read:

$$0 \leq \int_{\mathbb{R}^d} (\hat{u} - u)(t) \phi(t) \leq \int_0^t \int_{\mathbb{R}^d} L_1(\hat{v} - v) \phi ds,
 \tag{41}$$

$$0 \leq \int_{\mathbb{R}^d} (\hat{v} - v)(t) \phi(t) \leq \int_0^t \int_{\mathbb{R}^d} L_2(\hat{u} - u) \phi ds.
 \tag{42}$$

Make the d/dt in the previous expressions:

$$0 \leq \frac{d}{dt} \int_{\mathbb{R}^d} (\hat{u} - u)(t) \phi(t) \leq \int_{\mathbb{R}^d} L_1(\hat{v} - v)(t) \phi(t),
 \tag{43}$$

$$0 \leq \frac{d}{dt} \int_{\mathbb{R}^d} (\hat{v} - v)(t) \phi(t) \leq \int_{\mathbb{R}^d} L_2(\hat{u} - u)(t) \phi(t).
 \tag{44}$$

Letting us operate in the last two expressions with the equality, then:

$$\int_{\mathbb{R}^d} (\hat{u} - u)(t) \phi(t) = \frac{1}{L_2} \frac{d}{dt} \int_{\mathbb{R}^d} (\hat{v} - v)(t) \phi(t),
 \tag{45}$$

which can be replaced in (43):

$$\frac{d^2}{dt^2} \int_{\mathbb{R}^d} (\hat{v} - v)(t) \phi(t) \leq L_1 L_2 \int_{\mathbb{R}^d} (\hat{v} - v)(t) \phi(t).
 \tag{46}$$

Define:

$$g(t) = \int_{\mathbb{R}^d} (\hat{v} - v)(t) \phi(t);
 \tag{47}$$

then, Equation (46) reads as a linear standard second order equation:

$$\frac{d^2 g(t)}{dt^2} = L_1 L_2 g(t),
 \tag{48}$$

with

$$\begin{aligned}
 g(0) = v \rightarrow 0, \\
 g'(0) = v \rightarrow 0.
 \end{aligned}
 \tag{49}$$

Then, Equation (48) converges to the null solution. For the sake of simplicity, consider $g(t) = 0$, concluding that:

$$\hat{v} = v. \tag{50}$$

Now, returning to (41):

$$0 \leq \int_{\mathbb{R}^d} (\hat{u} - u)(t) \phi(t) \leq 0, \tag{51}$$

to conclude that:

$$\hat{u} = u, \tag{52}$$

showing, then, the uniqueness of any positive solution to P . \square

4. Profiles of Solution

This section is devoted to the searching of upper and lower solutions considering that the invasive species $u(x, t)$ shall be non-decreasing (i.e., $u_t \geq 0$) while the invaded one $v(x, t)$ shall be non-increasing (i.e., $v_t \leq 0$). Note that in this chapter the advection coefficient may be considered as a 1D parameter when required by the context. This is an important hypothesis and reflects a constant invasive species behaviour when entering into a domain (and at the same time a constant behaviour in the invaded one). The constant advection hypothesis may be discussed further; for example, in [26], the advection coefficient is determined in a porous medium as the relationship between the water flow velocity and the range of volumetric water content. Probably a similar process can be followed to model the invasive-invaded dynamics involved, where water flow velocity can be regarded as the invasive species velocity of invasion and the volumetric water content as the range of total concentration.

Theorem 3. *A maximal solution $\hat{v}(x, t)$ for the invaded species is:*

$$\hat{v}(x, t) = v_0(x) - (At)^{\frac{1}{m-1}}, \tag{53}$$

where

$$A \geq \frac{\delta^{m-1}}{\frac{|x|}{c}}, \tag{54}$$

being $\delta = \min\{v\} > 0$, $|x|$ representing the spatial integration domain and $0 < t < \frac{|x|}{c}$.

In addition, \hat{v} propagates through the support in (x, t) defined by the minimal approximation:

$$|x|_{\min} = \left(\frac{\delta^{m-1} c 2m}{m-1} \right)^{1/3} t^{2/3}. \tag{55}$$

Proof. Let us start with the equation in v . Considering self-similar solutions:

$$\hat{v}(x, t) = t^{-\alpha} f(\xi), \quad \xi = |x|t^\beta, \tag{56}$$

to the upper profile equation:

$$\hat{v}_t = \Delta \hat{v}^m + c \cdot \nabla \hat{v}. \tag{57}$$

Then, the following holds:

$$-\alpha t^{-\alpha-1} f + \beta \xi t^{-\alpha-1} f' = G(f'', f', f, m) t^{-\alpha m + 2\beta} + c t^{-\alpha + \beta} f'. \tag{58}$$

Making the corresponding equalities in the time exponents:

$$\alpha = \frac{-1}{m-1}; \quad \beta = -1. \tag{59}$$

The self-similar profile shall, then, be obtained as a solution to the elliptic equation:

$$-\alpha f + \beta \bar{\xi} f' = (f^m)'' + \frac{d-1}{\bar{\xi}} (f^m)' + c f'. \tag{60}$$

Note that in the approximation:

$$\beta \bar{\xi} \gg c, \quad |x|t^{-1} \gg |-c|, \tag{61}$$

and the following elliptic equation holds for $t \ll \frac{|x|}{c}$:

$$\beta \bar{\xi} f' = (f^m)'' + \frac{d-1}{\bar{\xi}} (f^m)' + \alpha f, \tag{62}$$

for which a solution is available [27,28]:

$$f(\bar{\xi}) = (A - B\bar{\xi}^2)^{\frac{1}{m-1}}, \tag{63}$$

where $A > 0$ and $B = \frac{(m-1)}{2m}$. A is obtained by making $\bar{\xi} = 0$ in (63), so that:

$$\hat{v}(x, t) = A^{\frac{1}{m-1}} t^{-\alpha}. \tag{64}$$

Note that (58) provides supersolutions under the condition that each species' concentration is positive, hence $\min\{uv\} > 0$. As $u_t > 0$ and assuming $u_0 > 0$, then it is required that $\min\{v\} > 0$. Considering $\delta = \min\{v\} > 0$, then:

$$\hat{v}(x, t) = A^{\frac{1}{m-1}} t^{-\alpha} > \delta, \tag{65}$$

so that:

$$A = \frac{\delta^{m-1}}{t} \geq \frac{\delta^{m-1}}{\frac{|x|}{c}}, \tag{66}$$

in the inner region $t \ll \frac{|x|}{c}$ where $|x|$ shall be understood as the maximum spatial variable representing the integration domain.

The maximal time evolution \hat{v} is then given by:

$$\hat{v}(x, t) = v_0(x) - (At)^{\frac{1}{m-1}}. \tag{67}$$

Finally, \hat{v} is indeed an upper evolution. To show this, consider a solution v to the original equation given by the absorption term $-uv$ and define a t_0 :

$$0 < t_0 < \frac{|x|}{c}, \tag{68}$$

such that, for any τ with $t_0 < \tau < \frac{|x|}{c}$:

$$v(x, \tau) \leq \hat{v}(x, t_0). \tag{69}$$

Now, for any t with $\tau < t < \frac{|x|}{c}$:

$$v(x, \tau + t) \leq \hat{v}(x, t_0 + t). \tag{70}$$

In the limit, $\tau \rightarrow 0$:

$$v(x, t) \leq \hat{v}(x, t), \tag{71}$$

showing the upper behaviour of \hat{v} compared to v .

The propagating support is obtained making:

$$f(\zeta) = 0, \quad A = B\zeta^2, \tag{72}$$

so that A and B can be replaced by their corresponding values. Then, the support propagates as:

$$|x| \geq \left(\frac{\delta^{m-1} c 2m}{m-1} \right)^{1/3} t^{2/3} = |x|_{\min}. \tag{73}$$

□

Note that solutions are of the form (53) in $\mathbb{R}^d \times [0, \infty]$, so that, in the proximity of $\delta \rightarrow 0^+$, the following time assessment holds:

$$t_v = \frac{(\|v_0\|_\infty - \delta)^{m-1}}{A}. \tag{74}$$

The expression (74) provides an estimation in time to consider a vanishing condition for v , so that, for $t > t_v$, the equation for $\tilde{u} \leq u$ reads:

$$\tilde{u}_t = \Delta \tilde{u}^m + c \cdot \nabla \tilde{u}, \tag{75}$$

and the following theorem holds in the search for a lower solution \tilde{u} :

Theorem 4. Consider

$$A_1 = \max_{x \in \mathbb{R}^d} \{u_0(x)\}. \tag{76}$$

The lower solution $\tilde{u}(x, t)$ for the invasive species reads:

$$\tilde{u}(x, t) = u_0(x) + A_1 t^{\frac{1}{m-1}}, \tag{77}$$

for

$$t_v < t < \frac{|x|}{c}. \tag{78}$$

Proof. Again, consider self-similar solutions of the form:

$$\tilde{u}(x, t) = t^{-\alpha} F(\zeta), \quad \zeta = |x|t^\beta, \tag{79}$$

to the lower profile Equation (75). Upon substitution of the self-similar solution (79) and operating analogously as in (58), the following self-similar exponents are deterred:

$$\alpha = \frac{-1}{m-1}; \quad \beta = -1. \tag{80}$$

Assuming $t < \frac{|x|}{c}$, the self-similar profile F is obtained as a solution to the elliptic equation:

$$\beta \zeta F' = (F^m)'' + \frac{d-1}{\zeta} (F^m)' + \alpha F, \tag{81}$$

for which the solution is, again, provided in [27,28]:

$$F(\zeta) = (A_1 - B_1 \zeta^2)^{\frac{1}{m-1}}, \tag{82}$$

for $t_v < t < \frac{|x|}{c}$ and $A_1 > 0, B = \frac{(m-1)}{2m}$.

Making $\zeta = 0$, the solution for \tilde{u} is:

$$\tilde{u}(x, t) = u_0(x) + A_1 t^{\frac{1}{m-1}}, \tag{83}$$

where

$$A_1 = \max_{x \in \mathbb{R}^d} \{u_0(x)\}. \tag{84}$$

Finally, the expression in (83) is shown to be a lower solution. For such purpose, let us define t_0 , such that $t_v < t_0 < \frac{|x|}{c}$, and τ such that $t_0 < \tau < \frac{|x|}{c}$, then:

$$\tilde{u}(x, \tau) \leq u(x, t_0). \tag{85}$$

This last expression is valid provided $|t_0 - \tau| \ll 1$. Considering t' with $\tau < t' < \frac{|x|}{c}$, then:

$$\tilde{u}(x, \tau + t') \leq u(x, t_0 + t'). \tag{86}$$

After a time translation $t = t_0 + t'$:

$$\tilde{u}(x, t) \leq u(x, t), \tag{87}$$

with $t_v < \frac{t}{2} < \frac{|x|}{c}$. \square

The next objective is to define a maximal solution for u and a minimal solution for v . To this end, the following theorem holds:

Theorem 5. *There exists a minimal solution $\tilde{v}(x, t)$ and a maximal solution $\hat{u}(x, t)$ in $\mathbb{R}^d \times [0, T]$. Furthermore, consider:*

$$\tilde{v} = \gamma = \min_{x \in \mathbb{R}^d} \{v_0(x)\}, \tag{88}$$

Then, a maximal solution for u is:

$$\hat{u}(x, t) = 1 - (1 - u_0(x))e^{-\gamma t}. \tag{89}$$

Proof. For building the maximal solution $\hat{u}(x, t)$, consider the following problem in u :

$$\begin{aligned} u_t^\epsilon &= \Delta(u^\epsilon)^m + c \cdot \nabla u^\epsilon + v(1 - u^\epsilon), \\ u^\epsilon(x, 0) &= u_0(x) + \epsilon, \quad v(x, 0) = v_0(x), \\ u_0(x) > 0, \quad v_0(x) > 0 &\in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d). \end{aligned} \tag{90}$$

Solutions to problem (90) do exist under the Lipschitz condition in the reaction term and the positive initial conditions (refer to Theorem 1).

Let us consider $\epsilon_2 < \epsilon$, so that:

$$\begin{aligned} u_t^\epsilon &= \Delta(u^\epsilon)^m + c \cdot \nabla u^\epsilon + v(1 - u^\epsilon), \\ u_t^{\epsilon_2} &= \Delta(u^{\epsilon_2})^m + c \cdot \nabla u^{\epsilon_2} + v(1 - u^{\epsilon_2}), \\ u^\epsilon(x, 0) = u_0(x) + \epsilon > u^{\epsilon_2}(x, 0) = u_0(x) + \epsilon_2 > 0. \end{aligned} \tag{91}$$

The iteration process starts with ϵ , and, based on the monotony behaviour in the reaction terms of (91), then $u^{\epsilon_2} < u^\epsilon$ in accordance with the initial data for u^ϵ . The same argument can be repeated for $\epsilon_3 < \epsilon_2$, so that $u^{\epsilon_3} < u^{\epsilon_2}$. The sequence defined as $\{u^\epsilon\}$ is non-negative and non-increasing. Consequently, the following maximal solution in the limit is defined:

$$\hat{u} = \lim_{\epsilon \rightarrow 0} u^\epsilon. \tag{92}$$

To construct the minimal solution for v , consider the problem:

$$\begin{aligned} v_t^\epsilon &= \Delta(v^\epsilon)^m + c \cdot \nabla v^\epsilon - (\lim_{\epsilon \rightarrow 0} u^\epsilon) v^\epsilon \\ v^\epsilon(x, 0) &= v(x, 0) = v_0(x) > 0. \end{aligned} \tag{93}$$

Solutions to (93) do exist (refer to Theorem 1).
 Considering $\epsilon_2 < \epsilon$, then:

$$u^{\epsilon_2} < u^\epsilon \Rightarrow v^{\epsilon_2} > v^\epsilon, \tag{94}$$

based on the monotony properties in the reaction term of (93). The sequence $\{v^\epsilon\}$ is non-decreasing as $\epsilon \rightarrow 0$. The minimal solutions for v are defined as per the monotone limit:

$$\tilde{v} = \lim_{\epsilon \rightarrow 0} v^\epsilon. \tag{95}$$

Once the maximal and minimal solutions have been constructed and shown to exist, the next intention is to search for a family of flat solutions u^ϵ, v^ϵ via the resolution of:

$$\begin{aligned} u_t^\epsilon &= \gamma(1 - u^\epsilon), \\ v^\epsilon &= \gamma = \min_{x \in \mathbb{R}^d} \{v_0(x)\}, \\ u^\epsilon(x, 0) &= u_0(x) + \epsilon, \quad v^\epsilon(x, 0) = v_0(x), \\ u_0(x) > 0, \quad v_0(x) > 0 &\in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d). \end{aligned} \tag{96}$$

This last system can be solved by separation of variables, so that:

$$u^\epsilon(x, t) = 1 - (1 - u_0(x) + \epsilon)e^{-\gamma t}. \tag{97}$$

Then, for $\epsilon \rightarrow 0$,

$$\hat{u}(x, t) = 1 - (1 - u_0(x))e^{-\gamma t}. \tag{98}$$

□

On a physical basis, a value for γ shall be determined experimentally and can represent a constant homogeneous initial distribution of the invaded species in the domain. Depending on the particular application, either the maximal or minimal solution may be considered. For instance, if the intention is to model the minimal level of invaded species that can face an invasive one, then the solution \tilde{v} in (88) shall be taken into account together with the solution \hat{u} in (89). Such maximal solution would make sense in this case (minimal invaded population) as the resistance to the invasion is relatively low, leading the invasive species to prosper quickly via a maximal solution \hat{u} . In addition, note that the maximal solution for the invaded species (and hence any other solution below) propagates through the support given in (55) which elucidates the finite propagation feature. Note that in the inner region (inner compared to the diffusive front moving with finite propagation), the desertion will guide the invaded species to move along the preferred direction c . If c is high, then the propagating support (55), induced by the diffusion, will expand further.

5. Travelling Waves Analysis, Existence and Regularity

The TW profiles are provided by the following structured changes: $u(x, t) = f(\zeta)$, such that $\zeta = x \cdot n_d - at \in \mathbb{R}$. Note that n_d is a unitary vector in \mathbb{R}^d defining the propagating direction. In addition, a is the TW velocity and the profile $f : \mathbb{R} \rightarrow (0, \infty)$ is such that $f \in L^\infty(\mathbb{R}^d)$. For the sake of simplicity, the vector $n_d = (1, 0, \dots, 0)$, then $u(x, t) = f(\zeta)$, $\zeta = x - at \in \mathbb{R}$.

Previous to the formal structure of TW solutions, let us consider some remarks on TW symmetries. Note that two TW are equivalent under translation $\zeta \rightarrow \zeta + \zeta_0$ and symmetry $\zeta \rightarrow -\zeta$ as the D’Alambert waves coordinate $x - at$ is invariant under the translation group. It shall be noted that such invariant symmetry does not necessarily hold when the advection coefficient, c , is time dependent or when time-periodic boundaries are given. In addition, and under these last two cases, monotonic TW solutions may not hold (the reader can refer to [29] where periodic TW-solutions are analyzed in detail for a time-dependent advection coefficient).

Considering $u(x, t) = f(\xi)$ and $v(x, t) = g(\xi)$, then the problem P (2) is transformed to the TW domain:

$$\begin{aligned}
 -af' &= (f^m)'' + cf' + g(1 - f), \\
 -ag' &= (g^m)'' + cg' - fg, \\
 f, g &\in L^\infty(\mathbb{R}), \\
 f'(\xi) &> 0, \quad g'(\xi) < 0, \\
 f(\infty) &= 1, \quad g(\infty) = 0.
 \end{aligned}
 \tag{99}$$

Working with the density and flux variables

$$X_1 = f, \quad Y_1 = (f^m)', \quad X_2 = g, \quad Y_2 = (g^m)',
 \tag{100}$$

the following system holds:

$$\begin{aligned}
 X_1' &= \frac{1}{m} X_1^{1-m} Y_1, \\
 Y_1' &= -(a + c) \frac{1}{m} X_1^{1-m} Y_1 - X_2(1 - X_1), \\
 X_2' &= \frac{1}{m} X_2^{1-m} Y_2, \\
 Y_2' &= X_1 X_2 - (a + c) \frac{1}{m} X_2^{1-m} Y_2,
 \end{aligned}
 \tag{101}$$

with the critical point $(1, 0, 0, 0)$ that represents a situation in which the invasive species wins over the invaded.

Geometric Perturbation Theory

The singular geometric perturbation theory is employed in this section to show the asymptotic behaviour of a two-dimensional manifold defined to make simpler the assessment of a TW analytical profile.

For this purpose, define the two-dimensional manifold as:

$$M_0 = \{X_1, Y_1, X_2, Y_2 \mid X_1' = \frac{1}{m} X_1^{1-m} Y_1; Y_1' = -(a + c) \frac{1}{m} X_1^{1-m} Y_1\},
 \tag{102}$$

with critical point $(1, 0, 0, 0)$.

The perturbed manifold M_ϵ close to M_0 is defined as:

$$M_\epsilon = \{X_1, Y_1, X_2, Y_2 \mid X_2 = \epsilon; Y_2' = 0; X_1' = \frac{1}{m} X_1^{1-m} Y_1; Y_1' = -(a + c) \frac{1}{m} X_1^{1-m} Y_1 - \epsilon(1 - X_1)\}.
 \tag{103}$$

The intention is to use the Fenichel invariant manifold theorem [30] as formulated in [31,32]. For this purpose, it is required to show that M_0 is a normally hyperbolic manifold, i.e., the eigenvalues of M_0 in the linearized frame close to the critical point, and transversal to the tangent space, have a non-zero real part. This is shown based on the following two-dimensional equivalent flow associated with M_0 :

$$\begin{pmatrix} X_1' \\ Y_1' \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{m} \\ 0 & -\frac{a+c}{m} \end{pmatrix} \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}.
 \tag{104}$$

where the eigenvalues are $(0, -(a + c)/m)$. For $\lambda = 0$, the eigenvector is $[1, 0]$ which is tangent to M_0 to conclude that M_0 is a hyperbolic manifold. The next objective is to prove that M_ϵ is locally invariant under the equations (flows) in (101). To this end, and according

to [32]: for all $R > 0$, for all open interval J with $(a + c) \in J$ and for any value of $i \in \mathbb{N}$, there exists a δ such that, for $\epsilon \in (0, \delta)$, the manifold M_ϵ is invariant. Hence, consider $i \geq 1$ and the functions:

$$\begin{aligned} \phi_1 &= \epsilon, \\ \phi_2 &= 0, \\ \phi_3 &= \frac{1}{m} X_1^{1-m} Y_1, \\ \phi_4 &= -(a + c) \frac{1}{m} X_1^{1-m} Y_1 - \epsilon(1 - X_1), \end{aligned} \tag{105}$$

which are $C^i(\overline{B_R(0)} \times \bar{I} \times [0, \delta])$ close to the critical condition $(1, 0, 0, 0)$.

A value for $R > 0$ can be selected so as to assume that $M_0 \cap B_R(0)$ is large enough to study the whole TW evolution. Note that δ is determined based on the computation of the distance between the flows in M_ϵ and M_0 . To this end, assume that the functions in each of the flows are measurable a.e. in $B_R(0)$:

$$\|\phi_4^{M_0} - \phi_4^{M_\epsilon}\| \leq \delta \|1 - X_1\|. \tag{106}$$

The distance between the manifolds keeps the normal hyperbolic condition for $\delta \in (0, \infty)$. For simplicity, assume $\delta = 1$.

In the same way, given the two-dimensional manifold M_1 :

$$M_1 = \{X_1, Y_1, X_2 \sim \epsilon, Y_2 / X_2' = \frac{1}{m} X_2^{1-m} Y_2; Y_2' = X_2 - (a + c) \frac{1}{m} X_2^{1-m} Y_2\}, \tag{107}$$

with the same critical point $(1, 0, 0, 0)$, and the perturbed M'_ϵ close to M_1 :

$$M'_\epsilon = \{X_1, Y_1, X_2, Y_2 / X_1 = 1; Y_1' = 0; X_2' = \frac{1}{m} X_2^{1-m} Y_2; Y_2' = X_1 X_2 - (a + c) \frac{1}{m} X_2^{1-m} Y_2\}. \tag{108}$$

The Fenichel invariant manifold theorem can be applied in the same manner as for M_0 . Note that the two-dimensional equivalent flow associated with M_1 is:

$$\begin{pmatrix} X_2' \\ Y_2' \end{pmatrix} = \begin{pmatrix} 0 & \frac{\epsilon^{1-m}}{m} \\ 1 & -\frac{(a+c)\epsilon^{1-m}}{m} \end{pmatrix} \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}. \tag{109}$$

The associated eigenvalues are both real $\left(-\frac{(a+c)\epsilon^{1-m}}{2m} \pm \frac{1}{2} \sqrt{\frac{((a+c)\epsilon^{1-m})^2}{m^2} + \frac{4\epsilon^{1-m}}{m}}\right)$. Hence, M_1 is a hyperbolic manifold.

In the same manner, the next intention is to show that the manifold M'_ϵ is locally invariant under the flow (101) so that the manifold M_1 can be represented as an asymptotic approach to M'_ϵ . For this purpose, consider the functions:

$$\begin{aligned} \phi'_1 &= 1, \\ \phi'_2 &= 0, \\ \phi'_3 &= \frac{1}{m} X_2^{1-m} Y_2, \\ \phi'_4 &= Y_2' = X_1 X_2 - (a + c) \frac{1}{m} X_2^{1-m} Y_2, \end{aligned} \tag{110}$$

which are $C^i(\overline{B_R(0)} \times \bar{I} \times [0, \delta])$ in the proximity of the critical point $(1, 0, 0, 0)$.

In this case, δ is determined based on the following flows that are considered to be measurable a.e. in $B_R(0)$:

$$\|\phi_4'^{M_1} - \phi_4'^{M'_\epsilon}\| \leq \delta\epsilon\|1 - X_1\| = \delta'\|1 - X_1\|. \tag{111}$$

The normal hyperbolic condition for both manifolds M_1 and M'_ϵ applies for $\delta' \in (0, \infty)$. In the sake of simplicity, consider $\delta' = 1$.

Once we have shown that M_0 and M_1 , defined as two-dimensional manifolds, remain invariant with regard to the four-dimensional manifolds M_ϵ and M'_ϵ respectively under the flow (101), the TW profiles can be obtained operating in M_0 and M_1 .

6. Travelling Waves Profiles and Finite Propagation

Based on the normal hyperbolic condition in the manifold M_0 under the flow (101), asymptotic TW profiles can be obtained. For this purpose, consider firstly:

$$X_1' = \frac{1}{m}X_1^{1-m}Y_1; \quad Y_1' = -(a + c)\frac{1}{m}X_1^{1-m}Y_1 - X_2(1 - X_1), \tag{112}$$

such that the following family of trajectories in the phase plane (X_1, Y_1) holds:

$$\frac{dY_1}{dX_1} = -(a + c) - mX_2X_1^{m-1}\frac{1 - X_1}{Y_1} = H(a, c, X_1, X_2, Y_1), \tag{113}$$

where $0 < X_1 < 1; 0 < X_2 < 1; Y_1 > 0$.

The existence of a convergent trajectory is shown based on a comparison with subsolutions for $(a + c)$ sufficiently small and supersolutions for $(a + c)$ sufficiently large together with topological assessment and the continuity of H .

For this purpose, note that, when $(a + c) \rightarrow 0$, then $dY_1/dX_1 < 0$, while, when $(a + c) < 0$ with $|a + c| \gg 1$, $dY_1/dX_1 > 0$. Given the continuity of H , it is thus possible to make a conclusion about the existence of a critical trajectory of the form:

$$-(a + c) - mX_2X_1^{m-1}\frac{1 - X_1}{Y_1} = H(a, c, X_1, X_2, Y_1) = 0. \tag{114}$$

Assuming now that the selected TW speed satisfies $a \ll -c$ with $|a + c| \gg 1$, then the same solution applies with minor effects due to the advection term, which permits assessing a TW speed only making assumptions on the bound of solutions; indeed:

$$a \sim mX_2X_1^{m-1}\frac{1 - X_1}{Y_1}, \tag{115}$$

for which a maximal bound is obtained as:

$$a < \frac{m}{A}, \tag{116}$$

being $A = \min_{B_R(0)}\{Y_1\}$. Now, coming to (114) and under the infinitesimal asymptotic approach $X_2 \sim \epsilon$, the following holds:

$$Y_1 = \frac{m}{|a + c|}\epsilon X_1^{m-1}(1 - X_1), \tag{117}$$

in the approach $X_1 \nearrow 1$:

$$Y_1 \sim B(\epsilon)(1 - X_1), \tag{118}$$

for a sufficiently small $B(\epsilon)$. Equivalently, in the asymptotic approximation:

$$(f^m)' + Bf = 0 \rightarrow f(mf^{m-2}f' + B) = 0, \tag{119}$$

where $f' = \frac{d}{d\zeta}$. This last equation has the solution:

$$f(\zeta) = C(\alpha - B\zeta)^{\frac{1}{m-1}}, \tag{120}$$

where $C = \left(\frac{m-1}{m}\right)^{\frac{1}{m-1}}$ and $\alpha > 0$. Note that the growing TW is obtained replacing $(-\zeta)$ with the symmetric (ζ) . This is a meaningful result that expresses the behaviour of the invaded species ($\sim \zeta^{\frac{1}{m-1}}$) in the proximity of the critical point.

The same process shall be repeated for the manifold M_1 under the flow (101):

$$X_2' = \frac{1}{m} X_2^{1-m} Y_2; \quad Y_2' = X_2 - (a+c) \frac{1}{m} X_2^{1-m} Y_2, \tag{121}$$

such that in the phase plane (X_2, Y_2) :

$$\frac{dY_2}{dX_2} = -\frac{a+c}{m} + mX_2^m Y_2^{-1} = G(a, c, X_2, Y_2), \tag{122}$$

where $0 < X_2 < 1; Y_2 < 0$.

In the same manner, the existence of a convergent trajectory is shown based on a topological argument and the continuity of G . For $(a+c) > 0$, $dY_2/dX_2 < 0$, while, when $(a+c) < 0$ with $|a+c| \gg 1$, $dY_2/dX_2 > 0$. Given the continuity of G , a critical trajectory is given:

$$-\frac{a+c}{m} + mX_2^m Y_2^{-1} = G(a, c, X_2, Y_2) = 0, \tag{123}$$

such that

$$Y_2 = -\frac{m^2}{|a+c|} X_2^m \rightarrow (g^m)' = -\frac{m^2}{|a+c|} g^m. \tag{124}$$

The last equation can be solved with standard means:

$$g(\zeta) = D e^{-\frac{m}{|a+c|} \zeta}, \tag{125}$$

for $D > 0$.

The positivity condition for f (120) permits conclusions about some regularity results in the quasilinear parabolic operator (see Section 3). In addition, the species g verifies

$$g \rightarrow \epsilon \rightarrow 0^+ \quad \text{in} \quad B_R^T = B_R(x_0, R) \times [T - \epsilon, T + \epsilon], \tag{126}$$

for $T \gg 1$, which means the existence of a convergent tail in B_R^T towards the null condition in v . The next objective is to show the existence of finite propagation in the invaded species v .

Theorem 6. *Given $m > 2$, a finite propagating support exists whenever:*

$$f \rightarrow \epsilon \rightarrow 0^+ \quad \text{in} \quad B_R^T = B_R(x_0, R) \times [T - \epsilon, T + \epsilon], \tag{127}$$

$T > 0$.

Proof. Firstly, assume the pressure variable w :

$$w = \frac{m}{m-1} v^{m-1}, \tag{128}$$

so that the equation for v in (2) reads:

$$w_t = |\nabla w|^2 + (m - 1)w\Delta w + c \cdot \nabla w + \frac{m - 1}{m}w - \left(\frac{m - 1}{m}\right)^{\frac{m-1}{m}} w^{\frac{m-1}{m}}, \tag{129}$$

where $w \rightarrow 0^+$, then:

$$w_t \geq |\nabla w|^2 + c \cdot \nabla w. \tag{130}$$

A solution to a similar equation has been provided in [27]. Consider the following function in the search of a maximal solution:

$$W(x, t) = a \left(bt + r - \frac{1}{n} \right)_+, \quad r = |x|, \quad n \in \mathbb{N}. \tag{131}$$

Both a and $b > 0$ are constants to be determined. In particular, for $0 \leq \tau \leq 1$, impose:

$$b\tau = \frac{1}{2n}. \tag{132}$$

Under this condition:

$$W(x, t) \equiv 0 \quad \text{for } r < \frac{1}{2n} \quad \text{and } 0 \leq t \leq \tau. \tag{133}$$

Any solution to Equation (130) is bounded as per Theorem 1, then:

$$v(x, t) \leq K_1 \quad \text{for } x \in \mathbb{R}, \quad 0 \leq t \leq \tau \quad \text{and } K_1(p, \|u_0\|_\infty). \tag{134}$$

The intention is to make $W(x, t)$ as a maximal solution:

$$W(x, t) \geq v(x, t), \tag{135}$$

so that

$$a \left(bt + r - \frac{1}{n} \right)_+ \geq K_1. \tag{136}$$

Select any $r > \frac{1}{n}$, for example $r = \frac{2}{n}$. Thus, for $t = 0$:

$$a \left(\frac{2}{n} - \frac{1}{n} \right)_+ \geq K_1, \quad a \geq nK_1. \tag{137}$$

Note that:

$$W(x, t) \geq v(x, t), \tag{138}$$

in $r = \frac{2}{n}$ and $0 \leq t \leq \tau$. The value of b shall be chosen in such a way that $W(x, t)$ is a supersolution in $0 < r < \frac{2}{n}$, $0 \leq t \leq \tau$:

$$W_t \geq |\nabla W|^2 + c \cdot \nabla W, \tag{139}$$

and considering that:

$$W_t = ab, \quad W_r = a, \tag{140}$$

the following value for b is obtained:

$$b \geq a + c. \tag{141}$$

For the values of a and b in expressions (137) and (141), respectively, the function $W(x, t)$ is a supersolution locally:

$$W(x, t) \geq w(x, t), \quad 0 < |x| < \frac{2}{n}, \quad 0 \leq t \leq \tau. \tag{142}$$

The inequality (142) permits concluding that any local supersolution satisfies the null criteria in B_R^T ; then, any minimal solution $w(x, t)$ satisfies such null criteria in B_R^T . \square

The expression (125) provides the characteristic profile in relation to the diffusion front (note the parameter m) and the advection. Once the invaded species starts the desertion in the direction of c , it follows a curve in (x, t) given by (125). The invasive species movement concentrates in the same trajectory to reduce the invaded population that follows an exponential decay. Furthermore, the existence of a finite propagation suggests that, during the desertion, the diffusion is still relevant in the proximity of the null solution (i.e., the invaded tail). This can be interpreted as the existence of a random movement of invaded species in the tail where the invasive species influence is negligible.

7. Conclusions

The proposed problem P (2) has been discussed stressing aspects related with existence, uniqueness, behaviour of minimal and maximal solutions and Travelling Waves supported by the geometric perturbation theory. In addition, the finite speed of propagation, induced by the porous medium diffusion, has been shown and a characterization of such property has been explored. The propagation features of the species v when approaching the null solution have been shown. To this end, an exponential decreasing tail in the TW domain has been proved to exist. The invaded species trajectory, to escape from the invasive one, is given by the TW solution mentioned. Even when the invaded desertion is mainly governed by the advection, the nonlinear diffusion still acts in the tail leading to a finite speed propagating front.

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