Symplectic geometry of Cartan–Hartogs domains

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ORIGINAL RESEARCH

Symplectic geometry of Cartan–Hartogs domains

Roberto Mossa[1] · Michela Zedda[2]

Abstract
This paper studies the geometry of Cartan–Hartogs domains from the symplectic point of view. Inspired by duality between compact and noncompact Hermitian symmetric spaces, we construct a dual counterpart of Cartan–Hartogs domains and give explicit expression of global Darboux coordinates for both Cartan–Hartogs domains and their dual. Further, we compute their symplectic capacity and show that a Cartan–Hartogs domain admits a symplectic duality if and only if it reduces to be a complex hyperbolic space.

Keywords Cartan–Hartogs domains · Darboux coordinates · Symplectic duality · Symplectic capacity

Mathematics Subject Classification 53D05 · 32M15

1 Introduction and statement of the results

Studying the symplectic geometry of a domain $X \subset \mathbb{C}^k$ equipped with a real analytic Kähler metric $\omega = \frac{i}{2}\partial\bar{\partial}\phi$, the following questions naturally arise:

Q 1 There exist global symplectic coordinates for X?

Q 2 Is the dual domain X^* of X a well-defined Kähler manifold?

When Q 2 has a positive answer we also have:

✉ Michela Zedda
   michela.zedda@unipr.it

   Roberto Mossa
   robertom@ime.usp.br

1   Departamento de Matemática, Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, Brazil

2   Dipartimento di Scienze Matematiche, Fisiche e Informatiche, Università di Parma, Parma, Italy

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Q 3 What can we say about the symplectic capacity of X and X^*?

Q 4 Is there a symplectic duality between X and X^*?

Recall that the existence of local symplectic coordinates is guaranteed by the celebrated Darboux Theorem, but in general the answer to Q 1 is negative, as shown by Gromov's exotic symplectic structures on $\mathbb{R}^{2k}$ [1] (see also [2] for an explicit example of a symplectic manifold diffeomorphic but not symplectomorphic to $\mathbb{R}^4$).

The concept of duality in Q 2 and Q 4 is inspired by the natural duality between compact and noncompact Hermitian symmetric spaces and can be expressed as follows. The potential ϕ can be expanded as a power series of the variables $z = (z_1, \ldots, z_k)$ and $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_k)$, denoted by $\phi(z, \bar{z})$, where z is the restriction to X of the Euclidean coordinates of $\mathbb{C}^k$. By the change of variables $\bar{z} \mapsto -\bar{z}$ in this power series, one gets a new power series denoted by $\phi(z, -\bar{z})$. We say (according to [3]) that a symplectic manifold (X^*, ω^*) is the *symplectic dual* of (X, ω) if ω^* has a Kähler potential ϕ^* such that the power series $\phi^*(z, \bar{z})$ associated with ϕ^* formally satisfies

$$\phi^*(z, \bar{z}) = -\phi(z, -\bar{z}).$$

We say that a smooth map $\psi : X \to X^*$ is a *symplectic duality* if it satisfies

$$\psi^* \omega_0 = \omega \quad \text{and} \quad \psi^* \omega^* = \omega_0,$$

where we denote by $\omega_0 = \frac{i}{2} \sum_{j=1}^{k} \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j$ the restriction of the flat form of $\mathbb{C}^k$ to X and X^*.

Symplectic capacities are a class of symplectic invariants, which generalize the concept of Gromov width, defined by Ekeland and Hofer [4, 5] for domains in $\mathbb{R}^{2n}$ and generalized by Hofer and Zehnder to symplectic manifolds [6] (we refer the reader to Sect. 4 for definitions and to [7] and references therein for further details). Symplectic capacities naturally represent an obstruction for the existence of a symplectic embedding, as they generalize the concept of Gromov width introduced in [8], which gives a measure of the largest ball that can be symplectically embedded inside a symplectic manifold. Their importance arises in the celebrated Gromov's nonsqueezing Theorem, according to which a symplectic embedding of a ball into a cylinder is possible if only if the ray of the ball is less or equal the cylinder's one. Computations and estimates of the Gromov width and the Hofer–Zehnder capacity can be found, e.g., in [8–18].

All the four questions find a positive answer when X is a Hermitian symmetric space of noncompact type Ω. In particular in [19], global symplectic coordinates that realize a symplectic duality are given, while in [20] is computed the symplectic capacity of Ω, the Gromov width of the dual Ω^* and also sharp estimations of the Hofer-Zehnder capacity of Ω^*, extending G. Lu's results in [15] valid for Grassmannians. It is then natural to investigate if these results can be extended to a class of bounded domains on $\mathbb{C}^{n+1}$ called *Cartan–Hartogs domains*, which are Hartogs domains based on Hermitian symmetric spaces of noncompact type. The geometry of these domains turned out to be interesting from several points of view, see, e.g., [21–24] and references therein (see also [25] for Hartogs domains constructed over bounded homogeneous domains). More precisely, Cartan–Hartogs domains are a 1-parameter family of noncompact nonhomogeneous domains of $\mathbb{C}^{n+1}$, given by:

$$M_{\Omega,\mu} := \{(z,w) \in \Omega \times \mathbb{C} \mid |w|^2 < N_\Omega^\mu(z,\bar{z})\}, \tag{1.1}$$

where $\Omega \subset \mathbb{C}^n$ is a bounded symmetric domain, $N_\Omega(z,\bar{z})$ is its generic norm and $\mu > 0$ is a positive real parameter. We endow $M_{\Omega,\mu}$ with its Kobayashi Kähler form $\omega_{\Omega,\mu}$ whose global Kähler potential reads

$$\varphi_{\Omega,\mu}(z,w;\bar{z},\bar{w}) = -\log(N_\Omega^\mu(z,\bar{z}) - |w|^2). \tag{1.2}$$

Our first result answer positively to Q 1 when X is a Cartan–Hartogs domain:

Theorem 1 *Let $M_{\Omega,\mu}$ be an $(n+1)$-dimensional Cartan–Hartogs domain. Then, there exists a global symplectomorphism $\Psi_{\Omega,\mu} : M_{\Omega,\mu} \to \mathbb{C}^{n+1}$.*

The global coordinates we exhibit generalize those of the Hermitian symmetric space of noncompact type Ω the Cartan–Hartogs domain is based on. Indeed, if we restrict the map $\Psi_{\Omega,\mu}$ to the base Ω, we obtain the symplectic coordinates for Ω constructed by A. Di Scala and A. Loi in [19] (see also [26]). Further, we prove that as well as Di Scala and Loi's map, $\Psi_{\Omega,\mu}$ well-behaves with respect to the action of the automorphism group of Ω and enjoys a nice hereditary property (see Remarks 1 and 2).

We construct (see Lemma 4 below) a symplectic dual of Cartan–Hartogs domains, that is, $M_{\Omega,\mu}^* = \mathbb{C}^{n+1}$ equipped with the dual Kähler form $\omega_{\Omega,\mu}^*$, that we show to be strictly plurisubharmonic on $\mathbb{C}^{n+1}$. In a natural way, the dual counterpart of $\Psi_{\Omega,\mu}$ defines global Darboux coordinates for $M_{\Omega,\mu}^*$, and we have the following:

Theorem 2 *The dual Cartan–Hartogs domain $(M_{\Omega,\mu}^*, \omega_{\Omega,\mu}^*)$ is a well-defined Kähler manifold which admits global Darboux coordinates $\Phi_{\Omega,\mu} : M_{\Omega,\mu}^* \to \mathbb{C}^{n+1}$.*

We prove also that $\Phi_{\Omega,\mu}$ is compatible with the action of the automorphism group of Ω and enjoys an hereditary property analogously to $\Psi_{\Omega,\mu}$ (see Remark 5).

As third result, we compute the symplectic capacity of $(M_{\Omega,\mu}, \omega_0)$ and $(M_{\Omega,\mu}^*, \omega_{\Omega,\mu}^*)$, answering for these domains to Q 3:

Theorem 3 *Let c be a symplectic capacity. Then, for a Cartan–Hartogs domain $M_{\Omega,\mu}$ equipped with the flat form ω_0, and for its dual $M_{\Omega,\mu}^*$ endowed with the dual form $\omega_{\Omega,\mu}^*$, one has*:

$$c(M_{\Omega,\mu}, \omega_0) = \pi, \quad \text{if } \mu \leq 1;$$

$$c(M_{\Omega,\mu}^*, \omega_{\Omega,\mu}^*) = \begin{cases} \mu^2 \pi & \text{if } \mu < 1, \\ \pi & \text{if } \mu \geq 1. \end{cases}$$

The proof of the first part of Theorem 3 is based on the results in [20], on the symplectic capacity of Hermitian symmetric spaces of noncompact type. To prove the second part, we apply Theorem 2.

Unfortunately, it can be proven that $\Psi_{\Omega,\mu}$ of Theorem 1 is not a symplectic duality unless the Cartan–Hartogs reduces to be a complex hyperbolic space, i.e., when $\Omega = \mathbb{C}H^n$ and $\mu = 1$. With Theorem 4 below, we show that this is not a peculiarity of our map, giving

a negative answer to Q [4] that characterizes the complex hyperbolic space among Cartan–Hartogs domains:

Theorem 4 *There exists a symplectic duality between a Cartan–Hartogs domain* $\left(M_{\Omega,\mu}, \omega_{\Omega,\mu}\right)$ *and its dual* $\left(\mathbb{C}^{n+1}, \omega_{\Omega,\mu}^*\right)$ *if and only if* $\left(M_{\Omega,\mu}, \omega_{\Omega,\mu}\right) = \left(\mathbb{C}H^{n+1}, \omega_{hyp}\right)$. *This is equivalent to* $\Psi_{\Omega,\mu} = \Phi_{\Omega,\mu}^{-1}$, *and in this case* $\Psi_{\Omega,\mu}$ *realizes a symplectic duality.*

The proof is obtained when $\mu < 1$ as direct consequence of Theorem [3] while for $\mu \geq 1$ it is a consequence of a volume comparison.

The paper is organized as follows. In the next section, we describe the geometry of Cartan–Hartogs domains, proving Theorem [1]. Section [3] is devoted to the construction of dual Cartan–Hartogs and the proof of Theorem [2]. Finally in Sect. [4], we prove Theorems [3] and [4].

2 Cartan–Hartogs domains and the proof of Theorem [1]

Throughout this section, we use the Jordan triple system theory, referring the reader to [19, 20, 26–31] for details and further applications.

2.1 Definition and geometric properties

Consider a Hermitian symmetric space of noncompact type (from now on HSSNT) Ω and the associated Hermitian positive Jordan triple system (from now on HPJTS) $(V, \{,,\})$ (see, e.g., [19, Section 2.2]). Recall that there is a natural identification between V equipped with the flat form $\omega_0 := \frac{i}{2}\partial\bar{\partial}m_1(x,\bar{x})$, where m_1 is the generic trace of V, and $\mathbb{C}^n$ equipped with the standard flat form $\omega_0 = \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j$. By mean of this identification, from now on we will always consider Ω as a bounded symmetric domain of $\mathbb{C}^n$ in its (unique up to linear isomorphism) circled realization, which is usually called a Cartan domain when Ω is irreducible. Analogously, we will consider the Bergman operator B_Ω as operator on $\mathbb{C}^n$, and its generic norm $N_\Omega(z,\bar{z})$ as a polynomial of $\mathbb{C}^n$ (see, e.g., [19, Section 2.1]).

To any HSSNT Ω, we can associate the *Cartan–Hartogs domain* $M_{\Omega,\mu}$, defined in (1.1), equipped with its Kobayashi metric

$$\omega_{\Omega,\mu} = \frac{i}{2}\partial\bar{\partial}\varphi_{\Omega,\mu}, \tag{2.1}$$

where $\varphi_{\Omega,\mu}$ is the Kähler potential defined in (1.2). Notice that if we restrict the Kobayashi metric $\omega_{\Omega,\mu}$ to $\left\{(z,0) \in M_\Omega\right\} \cong \Omega$ we get a multiple of the hyperbolic form

$$\omega_{hyp} := -\frac{i}{2}\partial\bar{\partial}\log N_{\Omega}(z,\bar{z}), \tag{2.2}$$

of Ω (see also [19, (11)]), i.e., $\omega_{\Omega,\mu_{|\Omega}} = \mu\,\omega_{hyp}$.

Define:

$$F(s) = \frac{1}{2^r r!} \prod_{j=1}^{r} \frac{\Gamma\left(b+1+(j-1)\frac{a}{2}\right)\Gamma\left(s+1+(j-1)\frac{a}{2}\right)\Gamma\left(j\frac{a}{2}+1\right)}{\Gamma\left(s+b+2+(r+j-2)\frac{a}{2}\right)\Gamma\left(\frac{a}{2}+1\right)}, \tag{2.3}$$

for a, b the numerical invariants of Ω, r its rank and $s \in \mathbb{R}^+$. In [32, Prop. 2.1], W. Yin, K. Lu and G. Roos prove that

$$\int_{\Omega} N(z,\bar{z})^s \omega_0^n = \pi^n F(s) \int_{\mathcal{F}} \Theta,$$

where Θ is the induced volume form on Fürstenberg–Satake boundary $\mathcal{F}$ of Ω (see, e.g., [33, (1.28)] for its definition). Thus,

$$\mathrm{Vol}(M_{\Omega,\mu},\omega_0) = \int_{M_{\Omega,\mu}} \frac{\omega_0^{n+1}}{(n+1)!} = \pi \int_{\Omega} \int_0^{N^\mu} \mathrm{d}r_w \wedge \frac{\omega_0^n}{n!} = \frac{\pi}{n!} \int_{\Omega} N^\mu \omega_0^n = \frac{\pi^{n+1}}{n!} F(\mu) \int_{\mathcal{F}} \Theta,$$

, i.e., the volume of $M_{\Omega,\mu}$ with respect to the flat form induced by $\mathbb{C}^{n+1}$ is given by:

$$\mathrm{Vol}(M_{\Omega,\mu},\omega_0) = \frac{\pi^{n+1}}{n!} F(\mu) \int_{\mathcal{F}} \Theta. \tag{2.4}$$

The following is a key example for our analysis:

Example 1 (Hartogs–polydisc) Let Δ^n be the n-dimensional polydisc

$$\Delta^n = \left\{ z = (z_1,\dots,z_n) \in \mathbb{C}^n \mid |z_j|^2 < 1, j = 1,\dots,n \right\},$$

the generic norm is given by (Hua [34])

$$N_{\Delta^n}(z,\bar{z}) = \prod_{j=1}^{n}(1-|z_j|^2).$$

It follows that its hyperbolic metric reads

$$\omega_{hyp} = -\frac{i}{2}\partial\bar{\partial}\log\left(\prod_{j=1}^{n}(1-|z_j|^2)\right),$$

and that the associated Cartan–Hartogs domain, which we call *Hartogs–polydisc*, is given by

$$M_{\Delta^n,\mu} = \left\{ (z,w) \mid z \in \Delta^n,\ |w|^2 < \prod_{j=1}^{n}(1-|z_j|^2)^\mu \right\},$$

whose Kobayashi metric is $\omega_{\Delta^n,\mu} = \frac{i}{2}\partial\bar{\partial}\varphi_{\Delta^n,\mu}$, with

$$\varphi_{\Delta^n,\mu}(z,w) = -\log\left(\prod_{j=1}^{n}(1-|z_j|^2)^{\mu} - |w|^2\right). \tag{2.5}$$

2.2 Holomorphic isometries between Cartan–Hartogs domains

A totally geodesic complex immersion $f : \left(\Omega', \omega'_{hyp}\right) \to \left(\Omega, \omega_{hyp}\right)$ between two HSSNT equipped with their hyperbolic metrics preserves the triple products $\{,,\}'$ and $\{,,\}$ of the associated HPJTS V' and V (see, e.g., [19, Proposition 2.1]), i.e.,

$$f\{u,v,w\}' = \{fu, fv, fw\}. \tag{2.6}$$

Hence, also the generic norm is preserved, that is, $N_{\Omega}^{\mu}(f(z), \overline{f(z)}) = N_{\Omega'}^{\mu}(z, \bar{z})$. Thus, the natural lift

$$\widetilde{f} : M_{\Omega',\mu} \to M_{\Omega,\mu}, \qquad \widetilde{f}(z,w) = (f(z), w), \tag{2.7}$$

is a holomorphic isometric embedding with respect to the Kobayashi metrics defined by (2.1), i.e.:

$$\begin{aligned}\widetilde{f}^*\omega_{\Omega,\mu} &= -\frac{i}{2}\partial\bar{\partial}\log\left(N_{\Omega}^{\mu}(f(z), \overline{f(z)}) - |w|^2\right) \\ &= -\frac{i}{2}\partial\bar{\partial}\log\left(N_{\Omega'}^{\mu}(z, \bar{z}) - |w|^2\right) = \omega_{\Omega',\mu}.\end{aligned}$$

Thus, we get:

Proposition 1 *Let Ω, Ω' be HSSNT. Then, any totally geodesic complex immersion $f : \Omega' \to \Omega$ extends to a Kähler embedding $\widetilde{f} : M_{\Omega',\mu} \to M_{\Omega,\mu}$ to the corresponding Cartan–Hartogs domains, defined by (2.7).*

Consider now the isotropy group $K \subset \mathrm{Aut}(\Omega)$ of the automorphism's group of Ω and recall that $K = \mathrm{Aut}(V, \{,,\})$. In fact, by [35, Prop. III.2.7], the action of K preserves the triple product $\{,,\}$ of the associated HPJTS, that is $K \subseteq \mathrm{Aut}(V, \{,,\})$, where $\mathrm{Aut}(V, \{,,\})$ is the group of complex linear transformations of V preserving $\{,,\}$. Vice versa, as a transformation $f \in \mathrm{Aut}(V, \{,,\})$ preserves the triple product, it preserves also the generic norm $N(x, \bar{y})$. Hence,

$$f^*\omega_{hyp} = -\frac{i}{2}\partial\bar{\partial}\log\left(N(f(z), \overline{f(z)})\right) = -\frac{i}{2}\partial\bar{\partial}\log\left(N(z, \bar{z})\right) = \omega_{hyp},$$

that is $K \supseteq \mathrm{Aut}(V, \{,,\})$. Then, by the argument above, the holomorphic isometric action of K on Ω induces in a natural way a holomorphic isometric action of K on M_{Ω}, by

$$\tau \cdot (z,w) = (\tau(z), w), \quad \tau \in \mathrm{Aut}(\Omega). \tag{2.8}$$

Moreover, as a consequence of Proposition 1 and of the Polydisc Theorem for HSSNT (see [36]), a Cartan–Hartogs domain can be realized as a union of Kähler embedded Hartogs–Polydiscs $M_{\Delta^r,\mu}$:

$$M_{\Omega,\mu} = \cup_{\tau \in K} \tau\left(M_{\Delta^r,\mu}\right),$$

where r is the rank of Ω and $\Delta^r \subset \Omega$ is a r-dimensional complex polydisc totally geodesically embedded in Ω.

2.3 Proof of Theorem 1

Let $M_{\Omega,\mu}$ be an $(n+1)$-dimensional Cartan–Hartogs domain and $\left(\mathbb{C}^n, \{,,\}_\Omega\right)$ the HJPTS associated with Ω. Define the map $\Psi_{\Omega,\mu} : M_{\Omega,\mu} \to \mathbb{C}^{n+1}$ by

$$\Psi_{\Omega,\mu}(z, w) = \frac{1}{\sqrt{N_\Omega^\mu(z,\bar{z}) - |w|^2}} \left(\sqrt{\mu N_\Omega^\mu(z,\bar{z})}\, \mathrm{B}_\Omega(z,\bar{z})^{-\frac{1}{4}} z, w \right), \tag{2.9}$$

where B_Ω and N_Ω are, respectively, the Bergman operator and the generic norm associated with $\{,,\}_\Omega$.

In order to prove Theorem 1, we will show that $\Psi_{\Omega,\mu}$ satisfies the following properties:

(A) $\Psi_{\Omega,\mu}^* \omega_0 = \omega_{\Omega,\mu}$, where $\omega_0 = \frac{i}{2} \sum_{j=1}^{n+1} \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j$;
(B) $\Psi_{\Omega,\mu}$ is a diffeomorphism.

Let us start with the following two lemmata.

Lemma 1 *Let $f_\Omega : M_{\Omega,\mu} \to \mathbb{C}^{n+1}$ be a smooth map of the form*:

$$f_\Omega(z_1, \ldots, z_n, w) := \frac{1}{\sqrt{N_\Omega^\mu\left(z,\bar{z}\right) - |w|^2}} (h_1(z), \ldots, h_n(z), w)$$

where $h := (h_1, \ldots, h_n)$ satisfies:

$$-\partial\bar{\partial}N_\Omega^\mu = \sum_{j=1}^n \mathrm{d}h_j \wedge \mathrm{d}\bar{h}_j, \tag{2.10}$$

$$\sum_{j=1}^n (h_j \mathrm{d}\bar{h}_j - \bar{h}_j \mathrm{d}h_j) = (\partial - \bar{\partial})N^\mu. \tag{2.11}$$

Then,

$$\omega_{\Omega,\mu} = \frac{i}{2} f_\Omega^* \left(\sum_{j=1}^n \mathrm{d}z \wedge \mathrm{d}\bar{z} + \mathrm{d}w \wedge \bar{\mathrm{d}}w \right). \tag{2.12}$$

Proof Observe first that:

$$\omega_{\Omega,\mu} = -\frac{i}{2}\partial\bar{\partial}\log\left(N^{\mu} - |w|^2\right)$$

$$= -\frac{i}{2}\partial\frac{\bar{\partial}(N^{\mu} - |w|^2)}{N^{\mu} - |w|^2}$$

$$= \frac{i}{2}\left[-\frac{\partial\bar{\partial}(N^{\mu} - |w|^2)}{N^{\mu} - |w|^2} + \frac{\partial(N^{\mu} - |w|^2)\wedge\bar{\partial}(N^{\mu} - |w|^2)}{(N^{\mu} - |w|^2)^2}\right], \qquad (2.13)$$

$$= \frac{i}{2}\left[\frac{\mathrm{d}w\wedge\mathrm{d}\bar{w}}{N^{\mu} - |w|^2} - \frac{\partial\bar{\partial}N^{\mu}}{N^{\mu} - |w|^2} + \frac{(\partial N^{\mu} - \bar{w}\mathrm{d}w)\wedge\bar{\partial}(N^{\mu} - |w|^2)}{2(N^{\mu} - |w|^2)^2}\right.$$

$$\left. + \frac{\partial(N^{\mu} - |w|^2)\wedge(\bar{\partial}N^{\mu} - w\mathrm{d}\bar{w})}{2(N^{\mu} - |w|^2)^2}\right]$$

Now, let us compute $\frac{i}{2}f_{\Omega}^*\left(\sum_{j=1}^{n}\mathrm{d}z\wedge\mathrm{d}\bar{z} + \mathrm{d}w\wedge\mathrm{d}\bar{w}\right)$. To simplify the indices notation, let us write $h_0 = z_0 = w$. We have:

$$\frac{i}{2}\sum_{j=0}^{n}\mathrm{d}(f_{\Omega})_j\wedge\overline{\mathrm{d}(f_{\Omega})_j} = \frac{i}{2}\sum_{j=0}^{n}\mathrm{d}\left[\frac{h_j}{\sqrt{N^{\mu} - |w|^2}}\right]\wedge\mathrm{d}\left[\frac{\bar{h}_j}{\sqrt{N^{\mu} - |w|^2}}\right]$$

$$= \frac{i}{2}\sum_{j=0}^{n}\left[\frac{\mathrm{d}h_j\wedge\mathrm{d}\bar{h}_j}{N^{\mu} - |w|^2} + \frac{h_j\mathrm{d}\bar{h}_j - \bar{h}_j\mathrm{d}h_j}{2(N^{\mu} - |w|^2)^2}\wedge\mathrm{d}(N^{\mu} - |w|^2)\right]$$

$$= \frac{i}{2}\left[\frac{\mathrm{d}w\wedge\mathrm{d}\bar{w}}{N^{\mu} - |w|^2} + \frac{\sum_{j=1}^{n}\mathrm{d}h_j\wedge\mathrm{d}\bar{h}_j}{N^{\mu} - |w|^2} + \right.$$

$$\left. + \frac{w\mathrm{d}\bar{w} - \bar{w}\mathrm{d}w}{2(N^{\mu} - |w|^2)^2}\wedge\mathrm{d}(N^{\mu} - |w|^2) + \frac{\sum_{j=1}^{n}\left(h_j\mathrm{d}\bar{h}_j - \bar{h}_j\mathrm{d}h_j\right)}{2(N^{\mu} - |w|^2)^2}\wedge\mathrm{d}(N^{\mu} - |w|^2)\right]. \qquad (2.14)$$

Assume that (2.10) and thus (2.11) hold. Then, from (2.13) and (2.14) we get:

$$f_{\Omega}^*\omega_0 - \omega_{\mu} = \frac{i}{4(N^{\mu} - |w|^2)^2}\left[(w\mathrm{d}\bar{w} - \bar{w}\mathrm{d}w)\wedge\mathrm{d}(N^{\mu} - |w|^2) + (\partial - \bar{\partial})N^{\mu}\wedge\mathrm{d}(N^{\mu} - |w|^2)\right.$$

$$\left. - (\partial N^{\mu} - \bar{w}\mathrm{d}w)\wedge\bar{\partial}(N^{\mu} - |w|^2) - \partial(N^{\mu} - |w|^2)\wedge(\bar{\partial}N^{\mu} - w\mathrm{d}\bar{w})\right]$$

$$= \frac{i}{4(N^{\mu} - |w|^2)^2}\left[-\bar{\partial}(N^{\mu} - |w|^2)\wedge\bar{\partial}(N^{\mu} - |w|^2) + \partial(N^{\mu} - |w|^2)\wedge\partial(N^{\mu} - |w|^2)\right]$$

$$= 0,$$

$$\qquad (2.15)$$

and we are done. □

Lemma 2 *Let $F : (\Omega, \omega_{hyp}) \to (\mathbb{C}^n, w_0)$ be a holomorphic map satisfying $F^*\omega_0 = \omega_{hyp}$, and*:

$$\sum_{j=1}^{n}\left(F_j\mathrm{d}\bar{F}_j - \bar{F}_j\mathrm{d}F_j\right) = \partial\log N - \bar{\partial}\log N. \qquad (2.16)$$

Then:

$$-\partial\bar{\partial}N^{\mu} = \mu\sum_{j=1}^{n}\mathrm{d}(N^{\mu/2}F_j)\wedge\mathrm{d}(N^{\mu/2}\bar{F}_j), \qquad (2.17)$$

and

$$\mu N^{\mu/2} \sum_{j=1}^{n} \left(F_j \mathrm{d}(N^{\mu/2}\bar{F}_j) - \bar{F}_j \mathrm{d}(N^{\mu/2}F_j) \right) = (\partial - \bar{\partial})N^{\mu}. \tag{2.18}$$

Proof We start by proving (2.17). Observe first that from $F^*\omega_0 = -\frac{i}{2}\partial\bar{\partial}\log N$ one gets that $\sqrt{\mu}F$ satisfies $\sqrt{\mu}F^*\omega_0 = -\frac{i}{2}\partial\bar{\partial}\log N^{\mu}$, as it follows by

$$\mu \sum_{j=1}^{n} \mathrm{d}F_j \wedge \mathrm{d}\bar{F}_j = -\partial\bar{\partial}\log N^{\mu}.$$

Then, expanding the right hand side of (2.17) we get:

$$\mu \sum_{j=1}^{n} \mathrm{d}(N^{\mu/2}F_j) \wedge \mathrm{d}(N^{\mu/2}\bar{F}_j) = \mu \sum_{j=1}^{n} (F_j \mathrm{d}N^{\mu/2} + N^{\mu/2}\mathrm{d}F_j) \wedge (\bar{F}_j \mathrm{d}N^{\mu/2} + N^{\mu/2}\mathrm{d}\bar{F}_j)$$

$$= \mu \sum_{j=1}^{n} \left[F_j N^{\mu/2}\mathrm{d}N^{\mu/2} \wedge \mathrm{d}\bar{F}_j + \bar{F}_j N^{\mu/2}\mathrm{d}F_j \wedge \mathrm{d}N^{\mu/2} + N^{\mu}\mathrm{d}F_j \wedge \mathrm{d}\bar{F}_j \right]$$

$$= \mu N^{\mu/2} \sum_{j=1}^{n} \left[\mathrm{d}N^{\mu/2} \wedge F_j \mathrm{d}\bar{F}_j + \bar{F}_j \mathrm{d}F_j \wedge \mathrm{d}N^{\mu/2} \right] - N^{\mu}\partial\bar{\partial}\log N^{\mu}$$

$$= \mu N^{\mu/2} \sum_{j=1}^{n} \left[\mathrm{d}N^{\mu/2} \wedge F_j \mathrm{d}\bar{F}_j + \bar{F}_j \mathrm{d}F_j \wedge \mathrm{d}N^{\mu/2} \right]$$

$$\quad - \frac{N^{\mu}\partial\bar{\partial}N^{\mu} - \partial N^{\mu} \wedge \bar{\partial}N^{\mu}}{N^{\mu}}.$$

At this point, (2.17) is satisfied if and only if

$$\mu N^{\mu/2} \sum_{j=1}^{n} \left[\mathrm{d}N^{\mu/2} \wedge F_j \mathrm{d}\bar{F}_j + \bar{F}_j \mathrm{d}F_j \wedge \mathrm{d}N^{\mu/2} \right] = -4\partial N^{\mu/2} \wedge \bar{\partial}N^{\mu/2}$$

$$= -2\mathrm{d}N^{\mu/2} \wedge \bar{\partial}N^{\mu/2} - 2\partial N^{\mu/2} \wedge \mathrm{d}N^{\mu/2},$$

where we used that $\partial N^{\mu} = 2N^{\mu/2}\partial N^{\mu/2}$. This last equivalence can be rewritten as

$$\sum_{j=1}^{n} \left[\mathrm{d}N^{\mu/2} \wedge F_j \mathrm{d}\bar{F}_j + \bar{F}_j \mathrm{d}F_j \wedge \mathrm{d}N^{\mu/2} \right] = -\mathrm{d}N^{\mu/2} \wedge \bar{\partial}\log N - \partial\log N \wedge \mathrm{d}N^{\mu/2},$$

that is,

$$\mathrm{d}N^{\mu/2} \wedge \sum_{j=1}^{n} \left[F_j \mathrm{d}\bar{F}_j - \bar{F}_j \mathrm{d}F_j \right] = -\mathrm{d}N^{\mu/2} \wedge \left[\bar{\partial}\log N - \partial\log N \right],$$

which holds true once (2.16) does.

Finally, the following computation proves (2.18)

$$\mu N^{\mu/2} \sum_{j=1}^{n} \left(F_j \mathrm{d}(N^{\mu/2}\bar{F}_j) - \bar{F}_j \mathrm{d}(N^{\mu/2}F_j) \right) = \mu N^{\mu/2} \sum_{j=1}^{n} \left(F_j(N^{\mu/2}\mathrm{d}\bar{F}_j + \bar{F}_j \mathrm{d}N^{\mu/2}) + \right.$$

$$\left. -\bar{F}_j(N^{\mu/2}\mathrm{d}F_j + F_j \mathrm{d}N^{\mu/2}) \right)$$

$$= \mu N^{\mu} \sum_{j=1}^{n} \left(F_j \mathrm{d}\bar{F}_j - \bar{F}_j \mathrm{d}F_j \right) = \mu N^{\mu} \left(\partial \log N - \bar{\partial} \log N \right)$$

$$= \mu N^{\mu-1} \left(\partial N - \bar{\partial} N \right) = \partial N^{\mu} - \bar{\partial} N^{\mu}.$$

$\square$

Now, we can proceed with the proof of (A). In [19, Theorem 1.1], A. Loi and A. Di Scala show that the map $F : (\Omega, \omega_{hyp}) \to (\mathbb{C}^n, w_0)$ defined by

$$F(z) = \mathrm{B}_\Omega(z, \bar{z})^{-\frac{1}{4}} z,$$

is a global symplectomorphism, thus Lemma 2 applies once checked that F satisfies (2.16), and (A) will follow by Lemma 1.

Denote by $\mathrm{D}(x, y)$ the operator on $(V, \{,, \})$ defined by $\mathrm{D}(x, y)z = \{x, y, z\}$ and denote by $z = \sum_j \lambda_j c_j$ the spectral decomposition of z. We have (see [19, (28)]) that

$$F(z) = \left(\mathrm{id} - \frac{1}{2}\mathrm{D}(z, z) \right)^{-1/2} z = (\mathrm{id} - z \square z)^{-1/2} z$$

where we use the operator $z \square z := \frac{1}{2}\mathrm{D}(z, z)$. Therefore,

$$\sum_{j=1}^{n} \left[F_j \mathrm{d}\overline{F}_j - \overline{F}_j \mathrm{d}F_j \right] = m_1(F, dF) - m_1(dF, F)$$

$$= m_1 \left((\mathrm{id} - z \square z)^{-1/2} z, \mathrm{d}\left((\mathrm{id} - z \square z)^{-1/2} z \right) \right) - m_1 \left(\mathrm{d}\left((\mathrm{id} - z \square z)^{-1/2} z \right), (\mathrm{id} - z \square z)^{-1/2} z \right)$$

$$= m_1 \left((\mathrm{id} - z \square z)^{-1/2} z, (\mathrm{id} - z \square z)^{-1/2} \mathrm{d}z \right) - m_1 \left((\mathrm{id} - z \square z)^{-1/2} \mathrm{d}z, (\mathrm{id} - z \square z)^{-1/2} z \right),$$

where we used the identity

$$\mathrm{d}F(z) = \left(\mathrm{d}(\mathrm{id} - z \square z)^{-1/2} \right) z + (\mathrm{id} - z \square z)^{-1/2} \mathrm{d}z,$$

and the fact that $z \square z$ is self-adjoint with respect to the Hermitian metric m_1. Using [19, (34)], we get

$$m_1 \left(F(z), \left((\mathrm{id} - z \square z)^{-1/2} \right) \mathrm{d}z \right) = -\frac{\bar{\partial} N(z, \bar{z})}{N(z, \bar{z})},$$

and thus:

$$\partial \log N(z, \bar{z}) - \bar{\partial} \log N(z, \bar{z}) = \frac{\partial N(z, \bar{z})}{N(z, \bar{z})} - \frac{\bar{\partial} N(z, \bar{z})}{N(z, \bar{z})}$$

$$= m_1 \left(F(z), \left((\mathrm{id} - z \square z)^{-1/2} \right) \mathrm{d}z \right) - m_1 \left(\left((\mathrm{id} - z \square z)^{-1/2} \right) \mathrm{d}z, F(z) \right),$$

as wished.

In order to prove (B), we need the following lemma:

Lemma 3 *Property* (B) *holds for Hartogs–Polydiscs.*

Proof Let $M_{\Delta^n,\mu}$ be an n-dimensional Hartogs–polydisc, as described in Example 1. From $N_{\Delta^n}(z,\bar{z}) = \prod_{j=1}^{n}(1 - |z_j|^2)$ and $B_{\Delta^n}(z,z) = \text{diag}\big((1 - |z_1|^2)^2, \ldots, (1 - |z_n|^2)^2\big)$ (see, e.g., [19, Sec. 3]), (2.9) reads:

$$\Psi_{\Delta^n,\mu}(z,w) = \frac{1}{\sqrt{\prod_{j=1}^{n}(1 - |z_j|^2)^\mu - |w|^2}}\left(\sqrt{\mu\prod_{j=1}^{n}(1 - |z_j|^2)^\mu}\left(\frac{z_1}{\sqrt{1 - |z_1|^2}}, \ldots, \frac{z_n}{\sqrt{1 - |z_n|^2}}\right), w\right). \tag{2.19}$$

Let

$$\widetilde{M} = \left\{(x_1, \ldots, x_n, y) \in \mathbb{R}^{n+1} \,\big|\, x_j = |z_j|^2, \, y = |w|^2, \, z = (z_1, \ldots z_n, w) \in M_{\Delta^n}\right\},$$

and consider the smooth map $\widetilde{\varphi}_{\Delta^n,\mu} : \widetilde{M} \to \mathbb{R}$, defined by:

$$\tilde{\varphi}_{\Delta^n,\mu}(x_1, \ldots, x_n, y) := \varphi_{\Delta^n,\mu}(|z_1|^2, \ldots, |z_n|^2, |w|^2) = -\log\left(\prod_{j=1}^{n}(1 - x_j)^\mu - y\right).$$

By [37, Th. 1.1], $\Psi_{\Delta^n,\mu}(z,w)$ is a global symplectomorphism (in particular a diffeomorphism) if $\frac{\partial\widetilde{\varphi}_{\Delta^n,\mu}}{\partial x_k} > 0$, $\frac{\partial\widetilde{\varphi}_{\Delta^n,\mu}}{\partial y} > 0$ and $\lim_{(x,y)\to\partial M}\sum_{j=1}^{n}\frac{\partial\widetilde{\varphi}_{\Delta^n,\mu}}{\partial x_j}x_j + \frac{\partial\widetilde{\varphi}_{\Delta^n,\mu}}{\partial y}y = +\infty$. The first two conditions are easily checked:

$$\frac{\partial\widetilde{\varphi}_{\Delta^n,\mu}}{\partial x_j} = \frac{\mu\prod_{j=1}^{n}(1 - x_j)^\mu}{(1 - x_j)\left(\prod_{j=1}^{n}(1 - x_j)^\mu - y\right)} > 0,$$

$$\frac{\partial\tilde{\varphi}_{\Delta^n,\mu}}{\partial y} = \frac{1}{\prod_{j=1}^{n}(1 - x_j)^\mu - y} > 0.$$

It remains to verify the third condition:

$$\lim_{(x,y)\to\partial M_{\Delta^n,\mu}}\left(\frac{\mu\prod_{k=1}^{n}(1 - x_k)^\mu}{\left(\prod_{k=1}^{n}(1 - x_k)^\mu - y\right)}\sum_{j=1}^{n}\frac{x_j}{1 - x_j} + \frac{y}{\prod_{j=1}^{n}(1 - x_j)^\mu - y}\right) = +\infty. \tag{2.20}$$

Observe that $\partial M_{\Delta^n,\mu} = \partial\Delta^n \cup \{y = \prod_{k=1}^{n}(1 - x_k)^\mu\}$, thus (2.20) is satisfied since for any $l = 1, \ldots, n$,

$$\lim_{x_l\to 1}\left(\frac{\mu\prod_{k=1}^{n}(1 - x_k)^\mu}{\left(\prod_{k=1}^{n}(1 - x_k)^\mu - y\right)}\sum_{j=1}^{n}\frac{x_j}{1 - x_j} + \frac{y}{\prod_{j=1}^{n}(1 - x_j)^\mu - y}\right) = +\infty,$$

and

$$\lim_{y\to\prod_{k=1}^{n}(1-x_k)^\mu}\left(\frac{\mu\prod_{k=1}^{n}(1 - x_k)^\mu}{\left(\prod_{k=1}^{n}(1 - x_k)^\mu - y\right)}\sum_{j=1}^{n}\frac{x_j}{1 - x_j} + \frac{y}{\prod_{j=1}^{n}(1 - x_j)^\mu - y}\right) = +\infty,$$

concluding the proof. □

We can proceed with the proof of (B). Using the spectral decomposition, it is possible (see [28, Section 3.18]) to associate with a smooth odd function $f : (-1, 1) \to \mathbb{C}$ (resp. $f : \mathbb{R} \to \mathbb{C}$) a smooth map $\widetilde{f} : \Omega \to \mathbb{C}^n$ (resp. $\widetilde{f} : \mathbb{C}^n \to \mathbb{C}^n$) in the following way. Let $(\mathbb{C}^n, \{,\})$ be the HPJTS associated with Ω and for $z \in \mathbb{C}^n$ let:

$$z = \lambda_1 e_1 + \cdots + \lambda_k e_k, \quad \lambda_1 > \cdots > \lambda_k > 0,$$

be its spectral decomposition. Define the map $\widetilde{f}$ associated with f by

$$\widetilde{f}(z) = f(\lambda_1)e_1 + \cdots + f(\lambda_k)e_k.$$

Thus, recalling that (see [31], and also [19]):

$$B_\Omega(z, \bar{z})c_j = \left(1 - \lambda_j^2\right)^2 c_j,$$

and

$$N_\Omega(z, \bar{z}) = \prod_{j=1}^{r}(1 - \lambda_j^2),$$

we can write $\Psi_{\Omega,\mu}$ as follows:

$$\Psi_{\Omega,\mu}(z, w) = \frac{1}{\sqrt{\prod_{j=1}^{r}\left(1 - \lambda_j^2\right)^\mu - |w|^2}}\left(\sqrt{\mu \prod_{j=1}^{r}\left(1 - \lambda_j^2\right)^\mu} \sum_{j=1}^{r} \frac{\lambda_j}{\left(1 - \lambda_j^2\right)^{1/2}}c_j, w\right).$$
(2.21)

Comparing (2.21) with (2.19) and using Lemma 3, we deduce that $\Psi_{\Omega,\mu}$ is a diffeomorphism (we apply [33, Section 1.6]).

Remark 1 (Hereditary property) Observe that the map $\Psi_{\Omega,\mu}$ given in (2.9) enjoys the following hereditary property: for any bounded symmetric domain $\Omega' \subset \mathbb{C}^m$ complex and totally geodesic embedded $\Omega' \overset{f}{\hookrightarrow} \Omega$, such that $f(0) = 0$, one has

$$\Psi_{\Omega',\mu}(z, w) = \Psi_{\Omega,\mu}(f(z), w).$$

Indeed, consider a complex and totally geodesic embedded submanifold $f : \Omega' \hookrightarrow \Omega$, satisfying $f(0) = 0$. By Prop. 1, f lifts to a Kähler embedding $\widetilde{f} : M_{\Omega',\mu} \hookrightarrow M_{\Omega,\mu}$, defined by $\widetilde{f}(z, w) = (f(z), w)$. By [19, Prop. 2.2], f preserves the triple products; thus, it follows that it preserves also the Bergman operator B_Ω and the generic norm N_Ω. Hence, $\Psi_{\Omega',\mu}(z, w) = \Psi_{\Omega,\mu}(f(z), w)$.

Remark 2 The map $\Psi_{\Omega,\mu}$ commutes with the holomorphic and isometric action (2.8) of the isotropy group $K \subset \text{Aut}(\Omega)$ at the origin, i.e., for every $\tau \in K$, $\Psi_{\Omega,\mu} \circ \tau = \tau \circ \Psi_{\Omega,\mu}$. This follows since $K = \text{Aut}(\mathbb{C}^n, \{,\}_\Omega)$, and therefore,

$$\Psi_{\Omega,\mu} \circ \tau(z,w) = \frac{1}{\sqrt{N_\Omega^\mu(\tau(z),\overline{\tau(z)}) - |w|^2}} \left(\sqrt{\mu N_\Omega^\mu(\tau(z),\overline{\tau(z)})} \, \mathrm{B}_\Omega(\tau(z),\overline{\tau(z)})^{-\frac{1}{4}} \tau(z), w \right)$$

$$= \frac{1}{\sqrt{N_\Omega^\mu(z,\bar{z}) - |w|^2}} \left(\sqrt{\mu N_\Omega^\mu(z,\bar{z})} \, \mathrm{B}_\Omega(z,\bar{z})^{-\frac{1}{4}} \tau(z), w \right)$$

$$= \tau \circ \Psi_{\Omega,\mu}(z,w).$$

Remark 3 (Alternative proof of Theorem 1 for classical Cartan–Hartogs) It is possible to give a more geometric proof of (A) for Cartan–Hartogs domains based on Cartan domains of classical type, without using the Jordan triple system theory. A direct computation proves (2.16) for the Cartan–Hartogs $M_{D_I,\mu}$ based on the first classical domain D_I (see, e.g., [34]). It is known that a HSSNT Ω admits a complex and totally geodesic embedding f in $D_I[m]$, for m sufficiently large. (This is obviously true for the domains D_I, D_{II} and D_{III}, while for the domain D_{IV}—associated with the so-called Spin-factor—the explicit embedding can be found in [38].) By Proposition 1, f lifts to a complex and totally geodesic embedding $\widetilde{f}$ of M_Ω in $M_{D_I[m],\mu}$. We can assume that this embedding takes the origin $0 \in M$ to the origin $0 \in D_I[m]$. Hence, property (A) for $M_{\Omega,\mu}$ is a consequence of the Hereditary property given in Remark 1 and the fact that (A) holds true for $D_I[m]$.

3 Dual Cartan–Hartogs domains

3.1 Definition and geometric properties

We define the *dual Cartan–Hartogs domain* $M_{\Omega,\mu}^*$ as $\mathbb{C}^{n+1}$, equipped with the dual Kähler form (see Lemma 4 below)

$$\omega_{\Omega,\mu}^* = \frac{i}{2} \partial\bar{\partial}\varphi_{\Omega,\mu}^*, \tag{3.1}$$

where $\varphi_{\Omega,\mu}^* := \log(N_\Omega^\mu(z,-\bar{z}) + |w|^2)$ is the dual Kähler potential (see the introduction for the definition of symplectic dual). If we restrict $\omega_{\Omega,\mu}^*$ to $\mathbb{C}^n = \{(z,0) \in \mathbb{C}^{n+1}\}$, we get a multiple of the Kähler form dual to the hyperbolic form (4.6), i.e.:

$$\omega_{\Omega,\mu|_{\mathbb{C}^n}}^* = \mu \, \omega_{hyp}^*. \tag{3.2}$$

Example 2 (Dual Hartogs–polydisc) Consider the Hartogs–polydisc $M_{\Delta^n,\mu}$ of Example 1 at page 5. Then, by (2.5) and (3.1), the dual form on $\mathbb{C}^{n+1}$ is given by $\omega_{\Delta^n,\mu}^* = \frac{i}{2}\partial\bar{\partial}\varphi_{\Delta^n,\mu}^*$, where

$$\varphi_{\Delta^n,\mu}^*(z,w) = \log\left(\prod_{j=1}^n (1 + |z_j|^2)^\mu + |w|^2 \right).$$

In general, the dual of a Kähler form is not defined (see [3, Example 1.3]), the following lemma assures us that $\omega_{\Omega,\mu}^*$ is a Kähler metric on $\mathbb{C}^{n+1}$.

Lemma 4 *The function* $\varphi_{\Omega,\mu}^* : \mathbb{C}^{n+1} \to \mathbb{R}$ *is strictly plurisubharmonic.*

Proof To shorten the notation, let us write $\tilde{N}$ for $N_\Omega(z, -\bar{z})$. The hessian of $\varphi^*_{\Omega,\mu}$ is given by

$$H := \frac{1}{(\tilde{N}^\mu + |w|^2)^2} \left(\begin{array}{c|c} (\tilde{N}^\mu + |w|^2)\partial_j\bar{\partial}_k\tilde{N}^\mu - \partial_j\tilde{N}^\mu\bar{\partial}_k\tilde{N}^\mu & -w\partial_j\tilde{N}^\mu \\ \hline -\bar{w}\bar{\partial}_k\tilde{N}^\mu & \tilde{N}^\mu \end{array} \right).$$

Observe that it is enough to show that H is positive definite when $\Omega = \Delta^n$. Indeed, let $(\mathbb{C}^r, \omega^*_{\Delta^r})$ be the symplectic dual of (Δ^r, ω_{hyp}), i.e., $\omega^*_{\Delta^r} = \frac{i}{2}\partial\bar{\partial} \log\left(\prod_{j=1}^r \left(1 + |z_j|^2\right)\right)$, and let $(z_0, w_0) \in \mathbb{C}^{n+1}$, $(u, \xi) \in T_{(z_0,w_0)}\mathbb{C}^{n+1}$. By the dual Polydisk Theorem, there exists a totally geodesic holomorphic immersion $f : (\mathbb{C}^r, \omega^*_\Delta) \to \left(\mathbb{C}^n, \omega^*_{hyp}\right)$ such that $f(\tilde{z}_0) = z_0$ and $f_{*,\tilde{z}_0}(\tilde{u}) = u$, for suitable $\tilde{z}_0 \in \mathbb{C}^n$ and $\tilde{u} \in T_{\tilde{z}_0}\mathbb{C}^r$. Then, the map $\tilde{f} : \mathbb{C}^{r+1} \to \mathbb{C}^{n+1}$ defined by $\tilde{f}(z, w) = (f(z), w)$ satisfies:

$$\tilde{f}^*\omega^*_{\Omega,\mu} = \frac{i}{2}\partial\bar{\partial} \log\left(N^\mu_\Omega(f(z), -\overline{f(z)}) + |w|^2\right)$$

$$= \frac{i}{2}\partial\bar{\partial} \log\left(N^\mu_{\Omega'}(z, -\bar{z}) + |w|^2\right) = \omega^*_{\Omega',\mu},$$

which implies:

$$g^*_{\Delta^r,\mu}\left((\tilde{u}, \xi), (\tilde{u}, \xi)\right) = g^*_{\Omega,\mu}\left(\tilde{f}_{*,(z_0,w_0)}(\tilde{u}, \xi), \tilde{f}_{*,(z_0,w_0)}(\tilde{u}, \xi)\right) = g^*_{\Omega,\mu}((u, \xi), (u, \xi))$$

where $\tilde{f}(z, w) = (f(z), w)$. Thus, consider a dual Hartogs–polydisk of dimension $n + 1$. Then, $\tilde{N}^\mu = \prod_{h=1}^n(1 + |z_h|^2)^\mu$ and thus, for $j, k = 1, \ldots, n$:

$$\bar{\partial}_k \prod_{h=1}^n(1 + |z_h|^2)^\mu = \frac{\mu\, z_k\, \prod_{h=1}^n(1 + |z_h|^2)^\mu}{1 + |z_k|^2},$$

$$\partial_j\bar{\partial}_k \prod_{h=1}^n(1 + |z_h|^2)^\mu = \frac{\mu\, \prod_{h=1}^n(1 + |z_h|^2)^\mu}{(1 + |z_k|^2)(1 + |z_j|^2)}\left(\delta_{jk} + \mu\, z_k\bar{z}_j\right).$$

Thus:

$$(\tilde{N}^\mu + |w|^2)\partial_j\bar{\partial}_k\tilde{N}^\mu - \partial_j\tilde{N}^\mu\bar{\partial}_k\tilde{N}^\mu = \left(\prod_{h=1}^n(1 + |z_h|^2)^\mu + |w|^2\right)\frac{\mu\, \prod_{h=1}^n(1 + |z_h|^2)^\mu}{(1 + |z_k|^2)(1 + |z_j|^2)}\delta_{jk}$$

$$+ \frac{\mu^2\, z_k\bar{z}_j\, |w|^2\, \prod_{h=1}^n(1 + |z_h|^2)^\mu}{(1 + |z_k|^2)(1 + |z_j|^2)}$$

Setting:

$$A := \left(\prod_{h=1}^n(1 + |z_h|^2)^\mu + |w|^2\right)\mu \prod_{h=1}^n(1 + |z_h|^2)^\mu \begin{pmatrix} \frac{1}{(1+|z_1|^2)^2} & & \\ & \ddots & \\ & & \frac{1}{(1+|z_n|^2)^2} \end{pmatrix}$$

and:

$$B := \mu^2 \prod_{h=1}^n(1 + |z_h|^2)^\mu VV^*,$$

where V is the column vector with k-th entry $\frac{z_k}{1+|z_k|^2}$, the Hessian H reads

$$H := \frac{1}{\left(\prod_{h=1}^{n}(1+|z_h|^2)^\mu + |w|^2\right)^2}\left(\begin{array}{c|c} A+B & -w\frac{\mu\bar{z}_j\prod_{h=1}^{n}(1+|z_h|^2)^\mu}{1+|z_j|^2} \\ \hline -\bar{w}\frac{\mu z_k\prod_{h=1}^{n}(1+|z_h|^2)^\mu}{1+|z_k|^2} & \prod_{h=1}^{n}(1+|z_h|^2)^\mu \end{array}\right),$$

and since $A+B$ is positive definite (being the sum of a positive definite matrix A and a semipositive one B), H is positive definite iff its determinant is. A long but straightforward computation gives

$$\det(H) = \mu^n \frac{\prod_{h=1}^{n}(1+|z_h|^2)^{\mu(n+1)-2}}{\left(\prod_{h=1}^{n}(1+|z_h|^2)^\mu + |w|^2\right)^{n+2}},$$

and we are done. $\qquad\square$

Remark 4 Observe that it turns out (see [19, Sect. 2.4]) that the Hermitian symmetric space of compact type $\left(\Omega^*, \omega_{FS}\right)$ dual to $\left(\Omega, \omega_{hyp}\right)$ is a compactification of $\left(\mathbb{C}^n, \omega_{hyp}^*\right)$. Further, $\left\{(z,w) \in M_{\Omega,\mu}^* \mid z = 0\right\}$ is totally geodesic in $M_{\Omega,\mu}^*$ and has $\mathbb{C}P^1$ equipped with the Fubini–Study metric as compactification; therefore, $M_{\Omega,\mu}^*$ is not complete for any μ. The authors believe that $\left(\mathbb{C}^{n+1}, \omega_{\Omega,\mu}^*\right)$ admits a completion only when $M_{\Omega,\mu}$ is itself a Hermitian symmetric space of noncompact type, which actually happens only when it reduces to be the $(n+1)$-dimensional complex hyperbolic space, i.e., when $\mu = 1$ and $\text{rank}(\Omega) = 1$.

Let a and b be the two numerical invariants of Ω and denote by r its rank. Using (2.3) and the following result by Selberg [39]:

$$\begin{aligned} F(s) &= \int\limits_{1 > \lambda_1 > \cdots > \lambda_r > 0} \cdots \int \prod_{j=1}^{r}\left(1-\lambda_j^2\right)^s \prod_{j=1}^{r}\lambda_j^{2b+1} \prod_{1 \le j < k \le r}(\lambda_j^2 - \lambda_k^2)^a \, d\lambda_1 \wedge \cdots \wedge d\lambda_r \\ &= \frac{1}{2^r}\int\limits_{1 > t_1 > \cdots > t_r > 0} \cdots \int \prod_{j=1}^{r}\left(1-t_j\right)^s \prod_{j=1}^{r}t_j^{b} \prod_{1 \le j < k \le r}(t_j - t_k)^a \, dt_1 \wedge \cdots \wedge dt_r. \end{aligned} \tag{3.3}$$

we have the following lemma.

Lemma 5 *The volume of a $(n+1)$-dimensional dual Cartan–Hartogs domain $\left(\mathbb{C}^{n+1}, \omega_{\Omega,\mu}^*\right)$ is given by*

$$\text{Vol}\left(\mathbb{C}^{n+1}, \omega_{\Omega,\mu}^*\right) = \frac{\pi^{n+1}\mu^n}{(n+1)!}F(0)\int_{\mathcal{F}}\Theta,$$

where Θ is the induced volume form on Fürstenberg-Satake boundary $\mathcal{F}$ of Ω.

Proof Observe first that since (see, e.g., [40]) $\det(\mu\omega_{hyp}^*) = \mu^n(N^*)^{-\gamma}$, by (3.2) and after a long but straightforward computation we get

$$\det(\omega_{\Omega,\mu}^*) = \mu^n \frac{(N^*)^{\mu(n+1)-\gamma}}{\left((N^*)^\mu + |w|^2\right)^{n+2}};$$

second, using the polar coordinates of HSSNT, we can write (see [41, (5.1.1)] and also [29]):

$$\int_{\mathbb{C}^n} \frac{\omega_0^n}{n!} = \int_{\mathcal{F}} \Theta \int \cdots \int_{+\infty > \lambda_1 > \cdots > \lambda_r > 0} \prod_{j=1}^r \lambda_j^{2b+1} \prod_{1 \le j < k \le r} \left(\lambda_j^2 - \lambda_k^2\right)^a d\lambda_1 \wedge \cdots \wedge d\lambda_r$$

where $\mathcal{F}$ is the so-called Fürstenberg-Satake boundary of Ω. Thus,

$$\begin{aligned}
\mathrm{Vol}(\mathbb{C}^{n+1}, \omega_{\Omega,\mu}^*) &= \int_{\mathbb{C}^{n+1}} \frac{(\omega_{\Omega,\mu}^*)^{n+1}}{(n+1)!} = \mu^n \pi \int_{\mathbb{C}^n} \int_0^{+\infty} \frac{(N^*)^{\mu(n+1)-\gamma}}{\left((N^*)^\mu + r_w\right)^{n+2}} dr_w \wedge \frac{\omega_0^n}{n!} \\
&= \frac{\pi \mu^n}{(n+1)!} \int_{\mathbb{C}^n} \frac{(N^*)^{\mu(n+1)-\gamma}}{(N^*)^{\mu(n+1)}} \omega_0^n = \frac{\pi \mu^n}{(n+1)!} \int_{\mathbb{C}^n} (N^*)^{-\gamma} \omega_0^n \\
&= \frac{\pi^{n+1} \mu^n}{(n+1)!} \int_{\mathcal{F}} \Theta \int \cdots \int_{+\infty > \lambda_1 > \cdots > \lambda_r > 0} \prod_{j=1}^r \left(1 + \lambda_j^2\right)^{-\gamma} \prod_{j=1}^r \lambda_j^{2b+1} \prod_{1 \le j < k \le r} (\lambda_j^2 - \lambda_k^2)^a \, d\lambda_1 \wedge \cdots \wedge d\lambda_r \\
&= \frac{\pi^{n+1} \mu^n}{(n+1)!} F(0) \int_{\mathcal{F}} \Theta,
\end{aligned}$$

$$(3.4)$$

where $\gamma = b + 2 + (r-1)a$ is the genus of Ω, $F(s)$ is given in (2.3), and last equality follows by:

$$\begin{aligned}
&\int \cdots \int_{+\infty > \lambda_1 > \cdots > \lambda_r > 0} \prod_{j=1}^r \left(1 + \lambda_j^2\right)^{-\gamma} \prod_{j=1}^r \lambda_j^{2b+1} \prod_{1 \le j < k \le r} (\lambda_j^2 - \lambda_k^2)^a \, d\lambda_1 \wedge \cdots \wedge d\lambda_r \\
&= \frac{1}{2^r} \int \cdots \int_{+\infty > t_1 > \cdots > t_r > 0} \prod_{j=1}^r \left(1 + t_j\right)^{-\gamma} \prod_{j=1}^r t_j^b \prod_{1 \le j < k \le r} (t_j - t_k)^a \, dt_1 \wedge \cdots \wedge dt_r \\
&= \frac{1}{2^r} \int \cdots \int_{1 > s_1 > \cdots > s_r > 0} \prod_{j=1}^r \left(1 - s_j\right)^{\gamma - 2 - b} \prod_{j=1}^r s_j^b \prod_{1 \le j < k \le r} \left(\frac{s_j}{1 - s_j} - \frac{s_k}{1 - s_k}\right)^a \, ds_1 \wedge \cdots \wedge ds_r \\
&= \frac{1}{2^r} \int \cdots \int_{1 > s_1 > \cdots > s_r > 0} \prod_{j=1}^r \left(1 - s_j\right)^{\gamma - 2 - b - (r-1)a} \prod_{j=1}^r s_j^b \prod_{1 \le j < k \le r} (s_j - s_k)^a \, ds_1 \wedge \cdots \wedge ds_r \\
&= \frac{1}{2^r} \int \cdots \int_{1 > s_1 > \cdots > s_r > 0} \prod_{j=1}^r s_j^b \prod_{1 \le j < k \le r} (s_j - s_k)^a \, ds_1 \wedge \cdots \wedge ds_r = F(0),
\end{aligned}$$

where we performed in turn the change of variables $\lambda_j^2 = t_j$ and $t_j = s_j/(1 - s_j)$, and the last equality follows by (3.3). $\qquad\square$

3.2 Holomorphic isometries between dual Cartan–Hartogs domains

Consider a totally geodesic complex immersion $f : \Omega' \to \Omega$ between HSSNT. Identify Ω' with its image $f(\Omega') \subset \Omega$ and observe that f trivially extends to an injective morphism $f : V' \to V$ of the associated HPJTS V' and V (see [19, Prop. 2.1]). Hence, the map:

$$\widetilde{f} : V' \times \mathbb{C} \to V \times \mathbb{C}, \qquad \widetilde{f}(z, w) = (f(z), w), \tag{3.5}$$

satisfies:

$$\begin{aligned} \widetilde{f}^* \omega_{\Omega,\mu}^* &= \frac{i}{2} \partial\bar{\partial} \log \left(N_\Omega^\mu(f(z), -\overline{f(z)}) + |w|^2 \right) \\ &= \frac{i}{2} \partial\bar{\partial} \log \left(N_{\Omega'}^\mu(z, -\bar{z}) + |w|^2 \right) = \omega_{\Omega',\mu}^*. \end{aligned} \tag{3.6}$$

Let us identify $V \cong \mathbb{C}^n$ and $V' \cong \mathbb{C}^m$, as in the beginning of this section, we just proved the following result.

Proposition 2 *Let Ω be an HSSNT. Then, any totally geodesic complex immersion $f : \Omega' \to \Omega$ extends to the Kähler embedding $\widetilde{f} : \left(\mathbb{C}^{m+1}, \omega_{\Omega',\mu}^* \right) \to \left(\mathbb{C}^{n+1}, \omega_{\Omega,\mu}^* \right)$ to the corresponding duals Cartan–Hartogs domains, given by* (3.5).

As in Sect. 2.2, the isotropy group $K = \mathrm{Aut}(V \cong \mathbb{C}^n, \{,,\})$ of $\mathrm{Aut}(\Omega)$, by (3.6) induces a natural action by isometries of K on $\left(\mathbb{C}^{n+1}, \omega_{\Omega,\mu}^* \right)$, given by

$$\tau \cdot (z, w) = (\tau(z), w), \quad \tau \in \mathrm{Aut}(\Omega). \tag{3.7}$$

Moreover, as a consequence of Proposition 2 and of the Polydisc Theorem for HSSNT (see [36]), we can see a dual Cartan–Hartogs domain $\left(\mathbb{C}^{n+1}, \omega_{\Omega,\mu}^* \right)$ as a union of Kähler embedded dual Hartogs–Polydisc $M_{\Delta^r,\mu}^* = \left(\mathbb{C}^{r+1}, \omega_{\Delta^r,\mu}^* \right)$ (see Example 2)

$$\mathbb{C}^{n+1} = \cup_{\tau \in K} \tau \left(M_{\Delta^r,\mu}^* \right)$$

where r is the rank of Ω and $\Delta^r \subset \Omega$ is an r-dimensional complex polydisc totally geodesically embedded in Ω.

3.3 Proof of Theorem 2

Let $M_{\Omega,\mu}^* = \left(\mathbb{C}^{n+1}, \omega_{\Omega,\mu}^* \right)$ be an n-dimensional dual Cartan–Hartogs domain and $\left(\mathbb{C}^n, \{,,\}_\Omega \right)$ the HJPTS associated with Ω. By Lemma 4, $M_{\Omega,\mu}^*$ is a well-defined Kähler manifold. In order to prove the existence of global Darboux coordinates, consider the map $\Phi_{\Omega,\mu} : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ given by

$$\Phi_{\Omega,\mu}(z, w) = \frac{1}{\sqrt{N_\Omega^\mu(z, -\bar{z}) + |w|^2}} \left(\sqrt{\mu N_\Omega^\mu(z, -\bar{z})}\, \mathrm{B}_\Omega(z, -\bar{z})^{-\frac{1}{4}} z, w \right) \tag{3.8}$$

where B_Ω and N_Ω are, respectively, the Bergman operator and the generic norm associated with $\{,,\}_\Omega$. We show that $\Phi_{\Omega,\mu}$ satisfies:

(A′) $\Phi_{\Omega,\mu}^* \omega_0 = \omega_{\Omega,\mu}^*$;

(B′) $\Phi_{\Omega,\mu}$ is a diffeomorphism with its image $\mathrm{Im}\big(\Phi_{\Omega,\mu}\big)$.

As in the proof of Theorem 1, we start with the following two lemmata, where to shorten the notation we set $N_{\Omega^*}^\mu(z,\overline{z}) := N_\Omega^\mu(z,-\overline{z})$.

Lemma 6 *Let* $f_\Omega : M_{\Omega,\mu} \to \mathbb{C}^{n+1}$ *be a smooth map of the form*

$$f_\Omega(z_1,\dots,z_n,w) := \frac{1}{\sqrt{N_{\Omega^*}^\mu(z,\overline{z}) + |w|^2}}(h_1(z),\dots,h_n(z),w)$$

where $h := (h_1,\dots,h_n)$ *satisfies*

$$\partial\overline{\partial}N_{\Omega^*}^\mu(z,\overline{z}) = \sum_{j=1}^n \mathrm{d}h_j(z) \wedge \overline{\mathrm{d}h_j(z)}, \tag{3.9}$$

and

$$\sum_{j=1}^n (h_j \mathrm{d}\overline{h}_j - \overline{h}_j \mathrm{d}h_j) = (\overline{\partial} - \partial)N_{\Omega^*}^\mu. \tag{3.10}$$

Then,

$$\omega_{\Omega,\mu}^* = \frac{i}{2}f_\Omega^*\left(\sum_{j=1}^n \mathrm{d}z \wedge \mathrm{d}\overline{z} + \mathrm{d}w \wedge \mathrm{d}\overline{w}\right). \tag{3.11}$$

Proof The proof is totally similar to that of Lemma 1, taking into account that $N_{\Omega^*}^\mu(z,\overline{z}) = N_\Omega^\mu(z,-\overline{z})$. □

Lemma 7 *If* $G : (\Omega, \omega_{hyp}) \to (\mathbb{C}^n, w_0)$ *is a holomorphic map satisfying* $G^* \omega_0 = \frac{i}{2}\partial\overline{\partial}\log N_{\Omega^*}^\mu$, *and*

$$\sum_{j=1}^n \left(G_j \mathrm{d}\overline{G}_j - \overline{G}_j \mathrm{d}G_j\right) = \overline{\partial}\log N_{\Omega^*} - \partial\log N_{\Omega^*}, \tag{3.12}$$

then

$$\partial\overline{\partial}N_{\Omega^*}^\mu = \mu \sum_{j=1}^n \mathrm{d}(N_{\Omega^*}^{\mu/2}G_j) \wedge \mathrm{d}(N_{\Omega^*}^{\mu/2}\overline{G}_j), \tag{3.13}$$

and

$$\mu N_{\Omega^*}^{\mu/2} \sum_{j=1}^n \left(G_j \mathrm{d}(N_{\Omega^*}^{\mu/2}\overline{G}_j) - \overline{G}_j \mathrm{d}(N_{\Omega^*}^{\mu/2}G_j)\right) = (\overline{\partial} - \partial)N_{\Omega^*}^\mu. \tag{3.14}$$

Proof The proof is totally similar to that of Lemma 2. □

In [19, Theorem 1.1], A. Loi and A. Di Scala show that $G : \left(\mathbb{C}^n, \frac{i}{2}\partial\overline{\partial}\log N_{\Omega^*}^\mu\right) \to (\mathbb{C}^n, w_0)$ defined by

$$G(z) = B_\Omega(z, -\overline{z})^{-\frac{1}{4}} z,$$

is a global symplectomorphism. Thus, by Lemmas 6 and 7, in order to prove (A′) we need only to check that such G satisfies (3.12). Also, here the proof is very similar to that of (A), once substituting F with G and D(z, z) with $-$D$(z, -z)$.

Following the same approach as in the proof of (B), we prove first that property (B′) holds for the Hartogs-polydisc case.

Lemma 8 *Property* (B′) *holds for the dual Hartogs–polydisc.*

Proof We apply [37, Th. 1.1]. Following the notation of Example 2, we can write the Kähler potential $\varphi^*_{\Delta^n,\mu}(z, w)$ for the dual Hartogs-polydisc $M^*_{\Delta^n,\mu}$ as $\varphi^*_{\Delta^n,\mu}(z, w) = \widetilde{\varphi}^*_{\Delta^n,\mu}(|z_1|^2, \ldots, |z_n|^2, |w|^2)$, where $\widetilde{\varphi}^*_{\Delta^n,\mu} : \mathbb{C}^{n+1} \to \mathbb{R}$ is given by

$$\widetilde{\varphi}^*_{\Delta^n,\mu}(x_1, \ldots, x_n, y) := \log\left(\prod_{j=1}^n (1+x_j)^\mu + y\right). \tag{3.15}$$

Then, by [37, Th. 1.1], the map

$$\Phi_{\Delta^n,\mu}(z, w) = \frac{1}{\sqrt{\prod_{j=1}^n (1+|z_j|^2)^\mu + |w|^2}}\left(\sqrt{\mu \prod_{j=1}^n (1+|z_j|^2)^\mu}\left(\frac{z_1}{\sqrt{1+|z_1|^2}}, \ldots, \frac{z_n}{\sqrt{1+|z_n|^2}}\right), w\right), \tag{3.16}$$

is a diffeomorphism with its image if $\frac{\partial \widetilde{\varphi}_{(\Delta^n)^*,\mu}}{\partial x_k} > 0$, $\frac{\partial \widetilde{\varphi}_{(\Delta^n)^*,\mu}}{\partial y} > 0$. The two conditions are easily checked

$$\frac{\partial \widetilde{\varphi}_{(\Delta^n)^*,\mu}}{\partial x_j} = \frac{\mu \prod_{j=1}^n (1+x_j)^\mu}{(1+x_j)\left(\prod_{j=1}^n (1+x_j)^\mu + y\right)} > 0,$$

$$\frac{\partial \widetilde{\varphi}_{(\Delta^n)^*,\mu}}{\partial y} = \frac{1}{\prod_{j=1}^n (1+x_j)^\mu + y} > 0,$$

and property (B′) is verified for $\Omega = \Delta^n$. $\qquad\square$

Proceeding now as in the proof of (B), the spectral decomposition of $\Phi_{\Omega,\mu}$ reads

$$\Phi_{\Omega,\mu}(z, w) = \frac{1}{\sqrt{\prod_{j=1}^r \left(1+\lambda_j^2\right)^\mu + |w|^2}}\left(\sqrt{\mu \prod_{j=1}^r \left(1+\lambda_j^2\right)^\mu} \sum_{j=1}^r \frac{\lambda_j}{\left(1+\lambda_j^2\right)^{1/2}} c_j, w\right). \tag{3.17}$$

Comparing (3.17) with (3.16) and using Lemma 8, we deduce that also $\Phi_{\Omega,\mu}$ is a diffeomorphism (we apply [33, Section 1.6]), concluding the proof.

Remark 5 The map $\Phi_{\Omega,\mu}$ enjoys the same properties as $\Psi_{\Omega,\mu}$. In particular, it is hereditary in the sense that for any bounded symmetric domain $\Omega' \subset \mathbb{C}^m$ complex and totally geodesic embedded $\Omega' \overset{f}{\hookrightarrow} \Omega$, such that $f(0) = 0$, one has

$$\Phi_{\Omega',\mu}(z,w) = \Phi_{\Omega,\mu}\Big(\widetilde{f}(z,w)\Big),$$

where $\widetilde{f} : M^*_{\Omega',\mu} \to M^*_{\Omega,\mu}$ is the Kähler embedding given in (3.5). This can be proven as in Remark 1, using Prop. 2 instead of Prop. 1. Further, $\Phi_{\Omega,\mu}$ commutes with the holomorphic isometric action (3.7) of the isotropy group $K \subset \mathrm{Aut}(\Omega)$ at the origin, i.e., $\Phi_{\Omega,\mu} \circ \tau = \tau \circ \Phi_{\Omega,\mu}$, as it follows by

$$
\begin{aligned}
\Phi_{\Omega,\mu} \circ \tau(z,w) &= \frac{1}{\sqrt{N^\mu_\Omega(\tau(z), -\overline{\tau(z)}) + |w|^2}} \left(\sqrt{\mu N^\mu_\Omega(\tau(z), -\overline{\tau(z)})} \, \mathrm{B}_\Omega(\tau(z), -\tau(z))^{-\frac{1}{4}} \tau(z), w \right) \\
&= \frac{1}{\sqrt{N^\mu_\Omega(z, -\overline{z}) + |w|^2}} \left(\sqrt{\mu N^\mu_\Omega(z, -\overline{z})} \, \mathrm{B}_\Omega(z, -z)^{-\frac{1}{4}} \tau(z), w \right) \\
&= \tau \circ \Phi_{\Omega,\mu}(z,w).
\end{aligned}
$$

4 Proof of Theorems 3 and 4

A map c from the class $\mathcal{C}(2n)$ of all symplectic manifolds of dimension $2n$ to $[0, +\infty]$ is called a *symplectic capacity* if it satisfies the following conditions (see, e.g., [7]):

- (*monotonicity*) if there exists a symplectic embedding $(M_1, \omega_1) \to (M_2, \omega_2)$, then $c(M_1, \omega_1) \leq c(M_2, \omega_2)$;
- (*conformality*) $c(M, \lambda\omega) = |\lambda| c(M, \omega)$, for every $\lambda \in \mathbb{R} \setminus \{0\}$;
- (*nontriviality*) $c(B^{2n}(1), \omega_0) = \pi = c(Z^{2n}(1), \omega_0)$.

Here, $B^{2n}(1)$ and $Z^{2n}(1)$ are the open unit ball and the open cylinder in the standard $(\mathbb{R}^{2n}, \omega_0)$, i.e.:

$$
\begin{aligned}
B^{2n}(r) &= \left\{ (x,y) \in \mathbb{R}^{2n} \,\Big|\, \sum_{j=1}^n x_j^2 + y_j^2 < r^2 \right\}, \\
Z^{2n}(r) &= \{ (x,y) \in \mathbb{R}^{2n} \mid x_1^2 + y_1^2 < r^2 \}.
\end{aligned}
\tag{4.1}
$$

We begin computing the symplectic capacity for $(M_{\Omega,\mu}, \omega_0)$. The proof relies on the facts, pointed out in [20], that the unitary ball $(B^{2n}(1), \omega_0)$ can be embedded into (Ω, ω_0) and the domain (Ω, ω_0) can be embedded into $(Z^{2n}(1), \omega_0)$.

Proof of Theorem 3 Let Ω be an HSSNT and let $(\mathbb{C}^n, \{,,\}_\Omega)$ be its associated HJPTS. We first prove that the unitary ball $(B^{2n+2}(1), \omega_0)$ can be embedded into $(M_{\Omega,\mu}, \omega_0)$ if $\mu \in (0,1]$. Let $z = \lambda_1 c_1 + \cdots + \lambda_r c_r$ be the spectral decomposition of a regular point $z \in \Omega \subset \mathbb{C}^n$, then the distance $\mathrm{d}_0(0, v)$ from the origin $0 \in \mathcal{M}$ to z is given by

$$\mathrm{d}_0(0, z) = (z \mid z)^{\frac{1}{2}} = \sqrt{\sum_{j=1}^r \lambda_j^2},\tag{4.2}$$

(see [31, Proposition VI.3.6] for a proof). Since $1 \leq \sum_{j=1}^{r} \lambda_j^2 + \prod_{j=1}^{r} \left(1 - \lambda_j^2\right)$, and for $(z, w) \in M_{\Omega,\mu}$ we have $|w|^2 < N(z,z)^\mu = \prod_{j=1}^{r} \left(1 - \lambda_j^2\right)^\mu$, it follows that

$$(B^{2n+2}(1), \omega_0) \cap \mathbb{C}_{reg}^n \times \mathbb{C} \subset (M_{\Omega,\mu}, \omega_0) \cap \mathbb{C}_{reg}^n \times \mathbb{C}, \qquad \mu \in (0, 1].$$

Since the set of regular points $\mathbb{C}_{reg}^n$ of $\mathbb{C}^n$ is dense ([31, Proposition IV.3.1]) and $\Omega = \{z \mid \|z\|_{\max} < 1\}$ (see [28, Corollary 3.15]), we get:

$$\left(B^{2n+2}(1), \omega_0\right) \subset \left(M_{\Omega,\mu}, \omega_0\right), \qquad \mu \in (0, 1], \tag{4.3}$$

as wished.

Let now $Z^{2n}(1) = \{(x, y) \mid x_1^2 + y_1^2 < 1\}$ be the unitary cylinder in $\mathbb{R}^{2n}$. In [20, Section 5], it is proved that the domain (Ω, ω_0) can be embedded into $(Z^{2n}(1), \omega_0)$. It follows immediately that $(M_{\Omega,\mu}, \omega_0)$ can be embedded into $(Z^{2n+2}(1), \omega_0)$ for every $\mu > 0$.

Thus, the first equality of the statement of Theorem 3 follows by the monotonicity and by the nontriviality of a symplectic capacity.

Let us now compute the symplectic capacity of $\left(M_{\Omega,\mu}^*, \omega_{\Omega,\mu}^*\right)$. By Theorem 2, it follows

$$c\left(\mathbb{C}^{n+1}, \omega_{\Omega,\mu}^*\right) = c\left(\text{Im}\left(\Phi_{\Omega,\mu}\right), \omega_0\right).$$

Therefore, it is enough to show that

$$\begin{cases} B^{2n+2}(\mu) \subset \text{Im}\left(\Phi_{\Omega,\mu}\right) \subset Z^{2n+2}(\mu) \text{ if } \mu < 1 \\ B^{2n+2}(1) \subset \text{Im}\left(\Phi_{\Omega,\mu}\right) \subset Z^{2n+2}(1) \text{ if } \mu \geq 1 \end{cases}. \tag{4.4}$$

Consider the expression of the symplectomorphism $\Phi_{\Omega,\mu}$ given in (3.17) in terms of spectral decomposition

$$\Phi_{\Omega,\mu}(z, w) = \left(\sum_{j=1}^{r} \xi_j c_j, \ \xi_0 w\right),$$

for

$$\xi_j = \sqrt{\frac{\mu \prod_{k=1}^{r} \left(1 + \lambda_k^2\right)^\mu}{\prod_{k=1}^{r} \left(1 + \lambda_k^2\right)^\mu + |w|^2}} \frac{\lambda_j}{\sqrt{1 + \lambda_j^2}}, \quad j = 1, \ldots, r,$$

and

$$\xi_0 = \frac{1}{\sqrt{\prod_{k=1}^{r} \left(1 + \lambda_k^2\right)^\mu + |w|^2}},$$

where $z = \sum_{j=1}^{r} \lambda_j c_j$ is the spectral decomposition of $z \in \mathbb{C}^n$. Notice that

$$\xi_j^2 < \mu \quad j = 1, \ldots, n, \qquad \xi_0^2 |w|^2 < 1.$$

Thus:

$$\begin{cases} \mathrm{Im}\big(\Phi_{\Omega,\mu}\big) \subset Z^{2n+2}(\mu) \text{ if } \mu < 1 \\ \mathrm{Im}\big(\Phi_{\Omega,\mu}\big) \subset Z^{2n+2}(1) \text{ if } \mu \geq 1 \end{cases}.$$

It remains to show that

$$\begin{cases} B^{2n+2}(\mu) \subset \mathrm{Im}\big(\Phi_{\Omega,\mu}\big) \text{ if } \mu < 1 \\ B^{2n+2}(1) \subset \mathrm{Im}\big(\Phi_{\Omega,\mu}\big) \text{ if } \mu \geq 1 \end{cases}.$$

Notice that $\xi_0^2 |w|^2$ can assume every value in the interval $[0,1)$. Assume that δ and c are real positive constant such that

$$\xi_0^2 |w|^2 = \delta^2 \leq c^2 < \min\{1, \mu\}.$$

In order to prove that the sphere $S^{2n+2}(c)$ of $\mathbb{C}^{n+1}$ of radius c is contained in $\mathrm{Im}\big(\Phi_{\Omega,\mu}\big)$, we need to show that the following system

$$\begin{cases} \xi_0^2 |w|^2 = \delta^2, \\ \xi_j^2 = x_j^2, \text{ for } j = 1, \dots, n, \end{cases} \tag{4.5}$$

has a solution in $\lambda_1, \dots, \lambda_r, |w|$, for any $\delta^2 < c^2$ and any n-uple $x_1, \dots, x_n \in \mathbb{R}$ such that $x_1^2 + \cdots + x_n^2 = c^2 - \delta^2$. We have

$$\xi_0^2 |w|^2 = \delta^2 \iff |w|^2 = \prod_{j=1}^{r} \left(1 + \lambda_j^2\right)^\mu \frac{\delta^2}{1 - \delta^2}.$$

Substituting the last term of the previous equality in $\xi_j^2 = x_j^2$, we get

$$\mu(1 - \delta^2) \frac{\lambda_j^2}{1 + \lambda_j^2} = x_j^2.$$

Notice that the left hand side of the previous equation assume any value in the interval $[0, \mu(1 - \delta^2))$. Hence, if $c^2 - \delta^2 < \mu(1 - \delta^2)$, the system (4.5) has a solution. By hypothesis $c^2 < \min\{1, \mu\}$, hence if $\mu < 1$ we have

$$c^2 - \delta^2 \leq \mu - \delta^2 < \mu(1 - \delta^2),$$

while if $\mu \geq 1$

$$c^2 - \delta^2 < 1 - \delta^2 \leq \mu(1 - \delta^2).$$

Thus, we get (4.4) and conclusions follow by the monotonicity, conformality and nontriviality of a symplectic capacity. $\qquad\square$

Recall that given a HSSNT Ω, with associated Bergman operator B_Ω, the map $\Xi_\Omega : \Omega \to \mathbb{C}^n$,

$$\Xi_\Omega(z) = B_\Omega(z, z)^{-\frac{1}{4}} z,$$

satisfies the analogous of properties as the map $\Psi_{\Omega,\mu}$ of Theorem 1 (see Remarks 1 and 2) and in addition it is a symplectic duality between $\big(\Omega, \omega_{hyp}\big)$ and its dual $\big(\mathbb{C}^n, \omega_{hyp}^*\big)$ (see

[19]). Notice that, according with the definition of symplectic dual given in the introduction and (2.2), we have (see also [19, (13)])

$$\omega_{hyp}^* = \frac{i}{2} \partial \bar{\partial} \log N_\Omega(z, -\bar{z}). \tag{4.6}$$

We are now in the position of proving Theorem 4.

Proof of Theorem 4 Assume $\Omega = \mathbb{CH}^n$ and $\mu = 1$, it is immediate to check that $M_{\mathbb{CH}^n,1} = \mathbb{CH}^{n+1}$. Recall that the generic norm and the Bergmann operator for $\mathbb{CH}^n$ are given by

$$N_{\mathbb{CH}^n}(z, \bar{z}) = 1 - |z|^2 \quad \text{and} \quad B_{\mathbb{CH}^n}(z, \bar{z})^{-\frac{1}{4}} z = \frac{z}{\sqrt{1 - |z|^2}},$$

where $|z|^2 = \sum_{j=1}^n |z_j|^2$. Substituting the previous expressions in (3.1), (4.6) and (2.9), we see that $\left(\mathbb{C}^{n+1}, \omega_{\mathbb{CH}^n,1}^* \right) = \left(\mathbb{C}^{n+1}, \omega_{hyp}^* \right)$ and that

$$\Psi_{\mathbb{CH}^n,1} = \Xi_{\mathbb{CH}^{n+1}},$$

we conclude by [19, Theorem 1.1] that $\Psi_{\mathbb{CH}^n,1}$ is a symplectic duality. Moreover, by substituting the previous expressions in (3.8) and by [19, Theorem 1.1] we see that

$$\Phi_{\mathbb{CH}^n,1}(z) = B_{\mathbb{CH}^{n+1}}(z, -z)^{-\frac{1}{4}} z = \Xi_{\mathbb{CH}^{n+1}}^{-1}(z).$$

Viceversa, when $\mu < 1$, a symplectic duality does not exist due to Theorem 3 while, for $\mu \geq 1$, if a symplectic duality between $(\mathbb{C}^{n+1}, \omega_{\Omega,\mu}^*)$ and $(M_{\Omega,\mu}, \omega_0)$ exists, then $\mathrm{Vol}(\mathbb{C}^{n+1}, \omega_{\Omega,\mu}^*) = \mathrm{Vol}(M_{\Omega,\mu}, \omega_0)$. In this case, by Lemma 5 and (2.4) we have

$$\frac{F(\mu)}{F(0)} = \frac{\mu^n}{n+1}. \tag{4.7}$$

Since

$$\frac{F(\mu)}{F(0)} = \prod_{j=1}^r \frac{\Gamma\left(\mu + 1 + (j-1)\frac{a}{2}\right)\Gamma\left(b + 2 + (r+j-2)\frac{a}{2}\right)}{\Gamma\left(1 + (j-1)\frac{a}{2}\right)\Gamma\left(\mu + b + 2 + (r+j-2)\frac{a}{2}\right)},$$

when Ω is the complex hyperbolic space, $\mu = 1$ is a solution to (4.7). In fact, in this case $r = 1$ and $b = n - 1$, thus

$$\frac{F(\mu)}{F(0)} = \frac{\Gamma(\mu+1)\Gamma(n+1)}{\Gamma(\mu+n+1)} = \frac{n!}{(\mu+n)\cdots(\mu+1)},$$

is equal to $\mu^n/(n+1)$ if and only if $\mu = 1$. We claim that

$$\frac{F(1)}{F(0)} = \prod_{j=1}^r \frac{\left(1 + (j-1)\frac{a}{2}\right)}{\left(b + 2 + (r+j-2)\frac{a}{2}\right)} \leq \frac{1}{n+1}, \tag{4.8}$$

and the equality holds if and only if $r = 1$.

If the claim holds, since the left hand side of (4.7) is strictly decreasing in μ while the right hand side is strictly increasing, we can see that the only positive solution to (4.7) must lie in $(0, 1)$, concluding the proof.

In order to prove the claim recall that $n = r\left(b + 1 + \frac{a}{2}(r - 1)\right)$. Thus, substituting $r = 1$ in (4.8), which happens iff $\Omega = \mathbb{C}H^n$, one readily gets that the equality is verified. To conclude, let us proceed by induction on r. Assume $r \geq 2$, by the inductive hypothesis we have

$$
\prod_{j=1}^{r} \frac{\left(1 + (j-1)\frac{a}{2}\right)}{\left(b + 2 + (r + j - 2)\frac{a}{2}\right)} \leq \frac{\left(1 + (r-1)\frac{a}{2}\right)}{\left(b + 2 + (2r - 2)\frac{a}{2}\right)} \cdot \frac{1}{r\left(b + 1 + \frac{a}{2}(r-1)\right)}
$$

$$
\leq \frac{1}{\left(b + 2 + (2r-2)\frac{a}{2}\right)} \cdot \frac{1}{r} = \frac{1}{r\left(b + 1 + \frac{a}{2}(r-1)\right) + r\left(1 + \frac{a}{2}(r-1)\right)}
$$

$$
< \frac{1}{r\left(b + 1 + \frac{a}{2}(r-1)\right) + 1} = \frac{1}{n+1},
$$

which proves (4.8) (notice that the last inequality is strict since we assumed $r \geq 2$). $\qquad\square$

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