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# Boundary regularity for manifold constrained $p(x)$-harmonic maps

Iwona Chlebicka, Cristiana De Filippis and Lukas Koch

Abstract

We prove partial and full boundary regularity for manifold constrained $p(x)$-harmonic maps.

*Contents*



## 1. *Introduction*

In this paper we complete the partial regularity theory for $p(x)$-harmonic maps studied in [**10**] providing partial and full boundary regularity for manifold constrained minima of the variable exponent energy:

$$g+\left(W^{1,p(\cdot)}(\Omega,\mathcal{M})\cap W_0^{1,p(\cdot)}(\Omega,\mathbb{R}^N)\right)\ni w\mapsto\mathcal{E}(w,\Omega):=\int_\Omega k(x)|Dw|^{p(x)}\ \mathrm{d}x \tag{1.1}$$

for a suitable boundary datum $g\colon\bar\Omega\to\mathcal{M}$. We immediately refer to Section 2.2 for the complete list of assumptions in force concerning the regularity of $\partial\Omega$, the coefficients appearing in the energies displayed in (1.1)–(1.2) and the topology of the manifold $\mathcal{M}$. Our main accomplishment is that there exists a relatively (to $\bar\Omega$) open subset $\Omega_0\subseteq\bar\Omega$ of full $n$-dimensional Lebesgue measure on which $u$ is the locally Hölder continuous and the singular set $\Sigma_0:=\bar\Omega\setminus\Omega_0$ has Hausdorff dimension at the most equal to $n-\gamma_1$; see $(2.2)_1$ for more information on this quantity. This is the content of the following theorem.

Theorem 1.1. *Under assumptions* (2.1), (2.2), (2.3) *and* (2.6), *let* $u\in W^{1,p(\cdot)}(\Omega,\mathcal{M})$ *be a solution to the Dirichlet problem* (1.1) *with boundary datum* $g\in W^{1,q}(\bar\Omega,\mathcal{M})$ *satisfying* (2.7). *Then there exists a relatively (to* $\bar\Omega$*) open subset* $\Omega_0\subseteq\bar\Omega$ *so that* $u\in C^{0,1-\frac{n}{q}}_{loc}(\Omega_0,\mathcal{M})$ *with* $q$ *as in* (2.7) *and* $\mathcal{H}^{n-\gamma_1}(\Sigma_0)=0$.

Moreover, after strengthening the hypotheses on the variable exponent $p(\cdot)$, we can prove that the singular set of solutions to problem

$$g+\left(W^{1,p(\cdot)}(\Omega,\mathcal{M})\cap W_0^{1,p(\cdot)}(\Omega,\mathbb{R}^N)\right)\ni w\mapsto\mathcal{J}(w,\Omega):=\int_\Omega |Dw|^{p(x)}\ \mathrm{d}x \tag{1.2}$$

does not intersect the boundary $\partial\Omega$. In this respect we have

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Theorem 1.2. *Under assumptions* (2.1), (2.4) *and* (2.6), *let* $u \in W^{1,p(\cdot)}(\Omega, \mathcal{M})$ *be a solution to the Dirichlet problem* (1.2) *with boundary datum* $g \colon \bar{\Omega} \to \mathcal{M}$ *satisfying* (2.7). *Then there exists a constant* $\Upsilon \equiv \Upsilon(\texttt{data}) \in (0, 1]$ *such that if*

$$[g]_{0,1-\frac{n}{q};\bar{\Omega}} < \Upsilon, \tag{1.3}$$

*then* $\Sigma_0 \Subset \Omega$ *and so* $u$ *is* $(1 - \frac{n}{q})$*-Hölder continuous in a neighborhood of* $\partial\Omega$.

The results exposed in Theorems 1.1–1.2 are new already in the case $p(\cdot) \equiv \text{const}$. In fact, we recover for the $p(x)$-Laplacian the boundary regularity theory already available for $p$-harmonic maps, under weaker assumptions on the boundary datum than those considered in [**23, 31, 52**]. Let us put our results into the context of the available literature. The regularity theory for vector-valued minimizers of functionals modeled upon the $p$-Laplacean integral, that is, variational problems, such as

$$\begin{gathered} W^{1,p}_{loc}(\Omega, \mathbb{R}^N) \ni w \mapsto \int_\Omega F(x, Dw) \ \mathrm{d}x \\ |z|^p \lesssim F(x, z) \lesssim (1 + |z|^2)^{\frac{p}{2}}, \quad 1 < p < \infty, \end{gathered} \tag{1.4}$$

started with the seminal paper [**55**] and received several contributions later on; see [**24–26, 28, 40, 43**] and references therein for an overview of the state of the art concerning $p$-Laplacean type problems. On the other hand, the regularity theory in the case when both minimizers and competitors take values into a manifold $\mathcal{M} \subset \mathbb{R}^N$ faces additional difficulties. The cornerstones of the theory were laid down by the fundamental papers [**17, 19, 51, 52**] analyzing harmonic maps, that is, constrained minimizers of the functional in (1.4) for $p = 2$; see also [**30, 53**]. We mention also the recent works [**46, 47**] for a fine analysis of the singular set of harmonic maps. The extension of the basic results to the case $p \neq 2$ has been done in the by now classical papers: [**21–23, 31, 42**]. Moreover, several of these results have been extended to more general functionals with $p$-growth, for instance the quasiconvex case has been treated in [**36**] while a purely PDE approach has been proposed in [**16**]. The matter of boundary regularity for vectorial problems is rather delicate and received lots of attention in the literature, starting from [**37, 52**], which covers the case of quadratic functionals. This theory has been extended later on to variational integrals of $p$-Laplacean type; see [**14**] for the first results in this direction and [**3, 15, 27–29, 39**] for general systems with standard $p$-growth. On the other hand, we note that energies of the type in (1.1) do not satisfy conditions as in (1.4), but rather, the more general and flexible one

$$\begin{gathered} W^{1,p}_{loc}(\Omega, \mathbb{R}^N) \ni w \mapsto \int_\Omega F(x, Dw) \ \mathrm{d}x \\ |z|^p \lesssim F(x, z) \lesssim (1 + |z|^2)^{\frac{q}{2}} \quad 1 < p \leqslant q < \infty. \end{gathered} \tag{1.5}$$

The systematic study of functionals as in (1.5) started in [**44, 45**] and, subsequently, has undergone an intensive development over the last years; see for instance [**2, 4–6, 11, 18, 20, 32, 34, 35**]. In particular, the energy in (1.1) have been introduced in the setting of Calculus of Variations and Homogenization in the seminal works [**38, 56**]. Energies as in (1.1) also occur in the modeling of electrorheological fluids, a class of non-Newtonian fluids whose viscosity properties are influenced by the presence of external electromagnetic fields [**1, 50**] or image restoration [**7**]; see also [**13**] for the basic properties of the $p(x)$-Laplacian. As for regularity, the first result in the vectorial case has been obtained in [**9**], where it is shown that local minimizers of energy (1.2) are locally $C^{1,\beta}$-regular in the unconstrained case. Subsequently, the regularity theory of functionals with variable exponent growth has been developed in a series of interesting papers [**48, 49, 54**], where the authors established partial regularity results for unconstrained minimizers that are on the other hand obviously related to the constrained case. Especially, in [**54**] is given an interesting partial regularity result and some singular set estimates for a class of functionals related to the constrained minimization problem in which minimizers are assumed to take values in a single chart. Finally, [**10**] is devoted to the study of partial inner regularity of manifold constrained $p(x)$-harmonic maps and to the analysis and dimension-reduction of their singular set.

*Organization of the paper.* This paper is organized as follows. Section 2 contains our notation, the list of the assumptions which will rule problems (1.1)–(1.2), several by now classical tools in the framework of regularity theory and some results of geometric and topological nature on Lipschitz retractions. Finally, Sections 3–4 are devoted to the proof of Theorems 1.1 and 1.2, respectively.

## 2. *Preliminaries*

In this section we display our notation, list the main assumptions in force throughout the paper and collect some useful tools for regularity theory and several well-known results in the framework of manifold-valued maps.

### 2.1. *Notation*

Following a usual custom, we denote by $c$ a general constant larger than 1. Different occurrences from line to line will be still denoted by $c$, while special occurrences will be denoted by $c_1, c_2, \tilde{c}$ or the like. Relevant dependencies on parameters will be emphasized using parentheses, that is, $c \equiv c(p, \nu, L)$ means that $c$ depends on $p, \nu, L$. Given any measurable subset $U \subset \mathbb{R}^n$, we denote by $|U|$ its $n$-dimensional Lebesgue measure and with $\mathcal{H}^k(U)$ its $k$-dimensional Hausdorff measure, for some $k \geqslant 0$. For a point $x_0 \in \mathbb{R}^n$ and a number $\varrho > 0$ we indicate with $B_\varrho(x_0) := \{x \in \mathbb{R}^n \colon |x - x_0| < \varrho\}$ the open ball centered at $x_0$ and with radius $\varrho$ and further, $B_\varrho \equiv B_\varrho(0)$. Similarly, for $x_0 \in \mathbb{R}^{n-1} \times \{0\}$ we define the half ball centered at $x_0$ as $B_\varrho^+(x_0) := \{x \in B_\varrho(x_0) \colon x^n > 0\}$. We moreover set $B_\varrho^+ \equiv B_\varrho^+(0)$. We also name $\Gamma_\varrho(x_0)$ the set $\{x \in \mathbb{R}^n \colon x^n = 0 \text{ and } |x_0 - x| < \varrho\}$ and $\partial^+ B_\varrho^+(x_0) := \partial B_\varrho^+(x_0) \setminus \Gamma_\varrho(x_0)$. As before, $\Gamma_\varrho \equiv \Gamma_\varrho(0)$. With $U \subset \mathbb{R}^n$ being a measurable subset having finite and positive $n$-dimensional Lebesgue measure, and with $h \colon U \to \mathbb{R}^k$, being a measurable map, we shall denote by

$$(h)_U \equiv \fint_U h(x)\ \mathrm{d}x := \frac{1}{|U|} \int_U h(x)\ \mathrm{d}x$$

its integral average. Similarly, with $\gamma \in (0,1]$ we denote the Hölder seminorm of $h$ as

$$[h]_{0,\gamma;U} := \sup_{x,y \in U, x \neq y} \frac{|h(x) - h(y)|}{|x - y|^\gamma}.$$

It is well known that the quantity defined above is a seminorm and when $[h]_{0,\gamma;U} < \infty$, we will say that $h$ belongs to the Hölder space $C^{0,\gamma}(U, \mathbb{R}^k)$. When clear from the context, we will omit the reference to $U$, that is: $[h]_{0,\gamma;U} \equiv [h]_{0,\gamma}$. Finally, given any set $\Gamma$ allowing for a trace operator, we denote by $\mathtt{tr}_\Gamma(h)$ the trace of $h$ on $\Gamma$.

### 2.2. *Main assumptions*

Let us turn to the main assumptions that will characterize our problem. The set $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$ is open, bounded, connected and

$$\partial\Omega \text{ is } C^2\text{-regular.} \tag{2.1}$$

When considering the functional in (1.1), the exponent $p(\cdot)$ will always satisfy

$$\begin{cases} p\in C^{0,\alpha}(\bar\Omega) \ \text{ for some } \ \alpha\in(0,1],\\ 1<\gamma_1:=\inf_{x\in\bar\Omega}p(x)\leqslant p(x)\leqslant \gamma_2:=\sup_{x\in\bar\Omega}p(x)<\infty, \end{cases} \tag{2.2}$$

while the coefficient $k(\cdot)$ is so that

$$\begin{cases} k\in C^{0,\nu}(\bar\Omega) \ \text{ for some } \ \nu\in(0,1],\\ 0<\lambda\leqslant k(x)\leqslant\Lambda<\infty \ \text{ for all } \ x\in\bar\Omega \end{cases} \tag{2.3}$$

holds true. We anticipate that in the estimates contained in Section 3.2, only $\min\{\alpha,\nu\}$ will be relevant, so for simplicity, for the proof of Theorem 1.1 we will assume that $\alpha=\nu$, that is: $p(\cdot)$, $k(\cdot)\in C^{0,\alpha}(\bar\Omega)$. When dealing with the question of full boundary regularity, we need higher regularity for $p(\cdot)$. Precisely, we shall suppose that

$$\begin{cases} p\in C^{0,1}(\bar\Omega),\\ 2\leqslant\gamma_1\leqslant p(x)\leqslant\gamma_2<\infty, \end{cases} \tag{2.4}$$

with $\gamma_1$ and $\gamma_2$ as in $(2.2)_2$. Given an half ball $B_R^+$ and a ball $B_\varrho(x_0)$ with $x_0\in B_R^+$ and $\varrho\in(0,R-|x_0|)$, we denote

$$p_1(x_0,\varrho):=\inf_{x\in B_\varrho(x_0)\cap B_R^+}p(x) \quad\text{and}\quad p_2(x_0,\varrho):=\sup_{x\in B_\varrho(x_0)\cap B_R^+}p(x). \tag{2.5}$$

Since in (2.5) we will always consider the intersection with the same ball $B_R^+$, the reference to $R$ in the symbols $p_1$, $p_2$ is omitted. When clear from the context, in (2.5) we shall not mention $x_0$ that is: $p_i(x_0,\varrho)\equiv p_i(\varrho)$ for $i\in\{1,2\}$. With a little abuse, we will adopt the notation in (2.5) also to denote the infimum (respectively, the supremum) of $p(\cdot)$ on $B_R^+$: the context will remove any ambiguity. Note that there is no loss of generality in assuming $\gamma_1<\gamma_2$, otherwise $p(\cdot)\equiv$ const on $\bar\Omega$, and in this case the problem is very well understood [**23, 31, 52**]. Furthermore, we need to impose some topological restriction on the manifold $\mathcal{M}$. Precisely, we ask that

$$\begin{cases} \mathcal{M} \text{ is a compact, } m\text{-dimensional, } C^3 \text{ Riemannian submanifold of } \mathbb{R}^N \text{ with } N\geqslant 3,\\ \mathcal{M} \text{ is } [\gamma_2]-1 \text{ connected},\\ \partial\mathcal{M}=\emptyset. \end{cases} \tag{2.6}$$

Here $[x]$ denotes the integer part of $x$ and the definition of $j$-connectedness is given in Section 2.4, Definition 4. Moreover, we assume that the boundary datum satisfies:

$$g\in W^{1,q}(\bar\Omega,\mathcal{M}) \quad \text{for some} \quad q>\max\{n,\gamma_2\}. \tag{2.7}$$

Combining (2.7) with Morrey's embedding theorem we automatically get that

$$g\in C^{0,1-(n/q)}(\bar\Omega,\mathcal{M}). \tag{2.8}$$

Finally, to shorten the notation we shall collect the main parameters of the problem in the quantities

$$\texttt{data}_{p(\cdot)}:=(n,N,\mathcal{M},\lambda,\Lambda,\gamma_1,\gamma_2,q,[p]_{0,\alpha},\alpha),$$

$$\texttt{data}:=(n,N,\mathcal{M},\lambda,\Lambda,\gamma_1,\gamma_2,q,[k]_{0,\nu},[p]_{0,\alpha},\nu,\alpha).$$

Any dependencies of the constants appearing in the forthcoming estimates on quantities depending on the characteristics of $\mathcal{M}$, such as, for instance, the $L^\infty$-norm of maps with range in $\mathcal{M}$ (which is clearly finite being $\mathcal{M}$ compact) will be simply denoted as a dependency on $\mathcal{M}$ in the form: $c\equiv c(\mathcal{M})$.

REMARK 1. Assumption (2.1) assures that there exists a positive constant $\hat{r} \equiv \hat{r}(n, \Omega)$ such that $B_\varrho(x_0) \cap \Omega$ is simply connected for all $\varrho \in (0, \hat{r}]$ and any $x_0 \in \partial\Omega$. This renders the existence of a positive constant $c \equiv c(n, \Omega)$ such that

$$\frac{\mathcal{H}^{n-1}(B_\varrho(x_0) \cap \partial\Omega)}{\mathcal{H}^{n-1}(\partial B_\varrho(x_0) \cap \Omega)} > c \quad \text{for all} \quad \varrho \in (0, \hat{r}],\ x_0 \in \partial\Omega.$$

Moreover, the Ahlfors condition yields that

$$|B_\varrho(x_0) \cap \Omega| \approx \varrho^n \quad \text{for all} \quad x_0 \in \bar{\Omega},\ \varrho \in (0, \hat{r}],$$

with constants implicit in '$\approx$' depending on $n, \Omega$. We shall refer to such constants with the term 'Ahlfors constants'; see [**14**, Section 2].

As to fully clarify the framework we are going to adopt, we need to introduce some basic terminology on the so-called Musielak–Orlicz–Sobolev spaces. Essentially, these are Sobolev spaces defined by the fact that the distributional derivatives lie in a suitable Musielak– Orlicz space, rather than in a Lebesgue space as usual. Classical Sobolev spaces are then a particular case. Such spaces and related variational problems are discussed for instance in [**8, 13, 33, 38**], to which we refer for more details. Here we will consider spaces related to the variable exponent case in both unconstrained and manifold-constrained settings.

DEFINITION 1. Given an open set $\Omega \subset \mathbb{R}^n$, the Musielak–Orlicz space $L^{p(\cdot)}(\Omega, \mathbb{R}^k)$, $k \geqslant 1$, with $p(\cdot)$ satisfying (2.2), is defined as

$$L^{p(\cdot)}(\Omega, \mathbb{R}^k) := \left\{ w\colon \Omega \to \mathbb{R}^k \text{ measurable and } \int_\Omega |w|^{p(x)}\ \mathrm{d}x < \infty \right\}$$

endowed with the Luxemburg norm $\|w\|_{L^{p(\cdot)}(\Omega,\mathbb{R}^k)} = \inf\{\lambda > 0 :\ \int_\Omega |w/\lambda|^{p(x)}\ \mathrm{d}x < 1\}$. Consequently,

$$W^{1,p(\cdot)}(\Omega, \mathbb{R}^k) := \left\{ w \in W^{1,1}(\Omega, \mathbb{R}^k) \cap L^{p(\cdot)}(\Omega, \mathbb{R}^k) \ \text{ such that } |Dw| \in L^{p(\cdot)}(\Omega, \mathbb{R}^{k\times n}) \right\}$$

with the norm $\|w\|_{W^{1,p(\cdot)}(\Omega,\mathbb{R}^k)} = \|w\|_{L^{p(\cdot)}(\Omega,\mathbb{R}^k)} + \|\,|Dw|\,\|_{L^{p(\cdot)}(\Omega,\mathbb{R}^k)}$. The variant $W^{1,p(\cdot)}_{loc}(\Omega, \mathbb{R}^k)$ is defined as in the classical case, whereas $W^{1,p(\cdot)}_0(\Omega, \mathbb{R}^k)$ is a closure of smooth and compactly supported functions in the norm $\|\cdot\|_{W^{1,p(\cdot)}(\Omega,\mathbb{R}^k)}$.

It is well known that, under assumptions (2.2), the set of smooth maps is dense in $W^{1,p(\cdot)}(\Omega, \mathbb{R}^k)$; see, for example, [**18, 38**]. Following [**10**] we also recall the analogous definition of such spaces when mappings take values into $\mathcal{M}$.

DEFINITION 2. Let $\mathcal{M}$ be a compact submanifold of $\mathbb{R}^k$, $k \geqslant 2$, without boundary and $\Omega \subset \mathbb{R}^n$ an open set. For $p(\cdot)$ satisfying (2.2), the Musielak–Orlicz–Sobolev space $W^{1,p(\cdot)}(\Omega, \mathcal{M})$ of functions into $\mathcal{M}$ can be defined as

$$W^{1,p(\cdot)}(\Omega, \mathcal{M}) := \left\{ w \in W^{1,p(\cdot)}(\Omega, \mathbb{R}^k):\ w(x) \in \mathcal{M} \text{ for a.e. } x \in \Omega \right\}.$$

The local space $W^{1,p(\cdot)}_{loc}(\Omega, \mathcal{M})$ consists of maps belonging to $W^{1,p(\cdot)}(B, \mathcal{M})$ for all open sets $B \Subset \Omega$.

Of course, when $p(\cdot) \equiv \text{const}$, Definitions 1 and 2 reduce to the classical Sobolev spaces $W^{1,p}(\Omega, \mathbb{R}^k)$ and $W^{1,p}(\Omega, \mathcal{M})$, respectively. Since the regularity question in $\Omega$ is local in nature, we can choose coordinates $\{x^i\}_{i=1}^n$ centered at $x_0 \in \partial\Omega$ such that locally $\Omega$ is the upper half space $\mathbb{R}^n \cap \{x^n > 0\}$, therefore, to avoid unnecessary complications, from now on we will assume that $\Omega \equiv B_1^+$; see [14, 15, 31, 37, 39, 52] for a more detailed discussion on this matter. Let us display the definition of constrained $W^{1,p(\cdot)}$-minimizer of (1.1) in $B_1^+$.

Definition 3. Let assumptions (2.1)–(2.6) and (2.7) be in force and consider the Dirichlet class $\mathcal{C}_g^{p(\cdot)}(B_1^+,\mathcal{M}) := \{w \in W^{1,p(\cdot)}(B_1^+,\mathcal{M})\colon \mathtt{tr}_{\Gamma_1}(w) = \mathtt{tr}_{\Gamma_1}(g)\}$. A map $u \in W^{1,p(\cdot)}(B_1^+,\mathcal{M})$ with $\mathtt{tr}_{\Gamma_1}(u) = \mathtt{tr}_{\Gamma_1}(g)$, is a constrained minimizer of the functional in (1.1) in the Dirichlet class $\mathcal{C}_g^{p(\cdot)}(B_1^+,\mathcal{M})$ provided that $\mathcal{E}(u,B_1^+) \leqslant \mathcal{E}(w,B_1^+)$ holds for all maps $w \in \mathcal{C}_g^{p(\cdot)}(B_1^+,\mathcal{M})$ so that $(u-w) \in W_0^{1,p}(B_1^+,\mathbb{R}^N)$.

To shorten the notation, for $\varrho \in (0,1]$, $x_0 \in \mathbb{R}^n \cap \{x^n \geqslant 0\}$, $f \in W^{1,p(\cdot)}(\bar{B}_\varrho^+(x_0),\mathcal{X})$ and a subset $\mathcal{X} \subseteq \mathbb{R}^N$, we also introduce the general Dirichlet class

$$\hat{\mathcal{C}}_f^{p(\cdot)}(B_\varrho^+(x_0),\mathcal{X}) := f + \Big(W^{1,p(\cdot)}(B_\varrho^+(x_0),\mathcal{X}) \cap W_0^{1,p(\cdot)}(B_\varrho^+(x_0),\mathbb{R}^N)\Big).$$

Clearly, the previous position makes sense also when $p(\cdot) \equiv \text{const}$.

### 2.3. *Well-known results*

When dealing with $p$-Laplacean type problems, we shall often use the auxiliary vector fields $V_{s,t}\colon \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$, defined by

$$V_{s,t}(z) := (s^2 + |z|^2)^{(t-2)/4} z, \qquad t \in (1,\infty) \ \text{ and } \ s \in [0,1] \tag{2.9}$$

whenever $z \in \mathbb{R}^{N\times n}$. If $s = 0$ we shall simply write $V_{s,t} \equiv V_t$. A useful related inequality is contained in the following:

$$|V_{s,t}(z_1) - V_{s,t}(z_2)| \approx (s^2 + |z_1|^2 + |z_2|^2)^{(t-2)/4} |z_1 - z_2|, \tag{2.10}$$

where the equivalence holds up to constants depending only on $n, N, t$. An important property which is usually related to such field is recorded in the following lemma.

Lemma 2.1. *Let $t > -1$, $s \in [0,1]$ and $z_1, z_2 \in \mathbb{R}^{N\times n}$ be so that $s + |z_1| + |z|_2 > 0$. Then*

$$\int_0^1 \left[s^2 + |z_1 + \lambda(z_2 - z_1)|^2\right]^{\frac{t}{2}} \, \mathrm{d}\lambda \approx (s^2 + |z_1|^2 + |z_2|^2)^{\frac{t}{2}},$$

*with constants implicit in '$\approx$' depending only on $n, N, t$.*

The next are a couple of simple inequalities which will be used several times throughout the paper. They are elementary; see, for example, [9, 10, 48, 54].

Lemma 2.2. *The following inequalities hold true.*

(i) *For any $\varepsilon_0 > 0$, there exists a constant $c \equiv c(\varepsilon_0)$ such that for all $t \geqslant 0$, $l \geqslant m \geqslant 1$ there holds $|t^l - t^m| \leqslant c(l-m)(1 + t^{(1+\varepsilon_0)l})$.*

(ii) *For $t \in (0,1]$, consider the function $g_1(t) := t^{\tilde{c}t^\gamma}$, where $\tilde{c}$ is an absolute real constant and $\gamma \in (0,1]$. Then $\lim_{t\to 0} g_1(t) = 1$ and $\sup_{t\in(0,1]} g_1(t) \leqslant c(\tilde{c},\gamma)$. Via the substitution $t \mapsto t^{-1}$, we have an analogous property for the function $[1,\infty) \ni t \mapsto g_2(t) := t^{\tilde{c}t^{-\gamma}}$, for $\tilde{c}$ and $\gamma$ as before. Precisely there holds that $\lim_{t\to\infty} g_2(t) = 1$ and $\sup_{t\in[1,\infty)} g_2(t) \leqslant c(\tilde{c},\gamma)$.*

We conclude this section by recalling the celebrated iteration lemma [26].

LEMMA 2.3. *Let* $h\colon [\varrho, R_0] \to \mathbb{R}$ *be a non-negative, bounded function and* $0 < \theta < 1$, $0 \leqslant A$, $0 < \beta$. *Assume that* $h(r) \leqslant A(d-r)^{-\beta} + \theta h(d)$, *for* $\varrho \leqslant r < d \leqslant R_0$. *Then* $h(\varrho) \leqslant cA/(R_0 - \varrho)^{-\beta}$ *holds, where* $c \equiv c(\theta, \beta) > 0$.

## 2.4. *Extensions*

In this section we shall borrow from [10] some useful lemmas concerning locally Lipschitz retractions. Such results were first introduced in [31] and intensively used in the literature for dealing with possibly non-homogeneous variational problems whose structure is *a priori* non-compatible with any kind of monotonicity formulae [10, 12, 36]. We refer to Remark 2 for a quick discussion on this matter. We start with clarifying a key assumption in our paper, which is the concept of $j$-connectedness.

DEFINITION 4. Given an integer $j \geqslant 0$, a manifold $\mathcal{M}$ is said to be $j$-connected if its first $j$ homotopy groups vanish identically, that is $\pi_0(\mathcal{M}) = \pi_1(\mathcal{M}) = \cdots = \pi_{j-1}(\mathcal{M}) = \pi_j(\mathcal{M}) = 0$.

It is well known that a compact manifold $\mathcal{M} \subset \mathbb{R}^N$ without boundary admits a tubular neighborhood $\mathcal{M} \subset \omega \subset \mathbb{R}^N$. Identifying $\mathcal{M}$ with its image in $\mathbb{R}^N$, we say that a neighborhood $\omega$ of $\mathcal{M}$ has the nearest point property if for every $x \in \omega$ there is a unique point $\Pi_{\mathcal{M}}(x) \in \mathcal{M}$ such that $\operatorname{dist}(x, \mathcal{M}) = |x - \Pi_{\mathcal{M}}(x)|$. The map $\Pi_{\mathcal{M}}\colon \omega \to \mathcal{M}$ is called the retraction onto $\mathcal{M}$, we shall refer to it also as 'projector'. Moreover, the regularity of $\mathcal{M}$ influences the regularity of $\Pi_{\mathcal{M}}$ in the following way:

$$\mathcal{M} \text{ is } C^k\text{-regular for } k \geqslant 2 \implies \Pi_{\mathcal{M}} \in C^{k-1}(\omega, \mathcal{M}), \tag{2.11}$$

see [36] for a deeper discussion on this matter. It is important to stress that manifolds endowed with the relatively simple topology described by Definition 4 enjoy good properties in terms of retractions; cf. [31, 36].

LEMMA 2.4. *Let* $\mathcal{M} \subset \mathbb{R}^N$ *be a compact,* $j$*-connected submanifold for some integer* $j \in \{0, \dots, N-2\}$ *contained in an* $N$*-dimensional cube* $Q$. *Then there exists a closed* $(N-j-2)$*-dimensional Lipschitz polyhedron* $X \subset Q \setminus \mathcal{M}$ *and a locally Lipschitz retraction* $\psi\colon Q \setminus X \to \mathcal{M}$ *such that for any* $x \in Q \setminus X$, $|D\psi(x)| \leqslant c/\operatorname{dist}(x, X)$ *holds, for some positive* $c \equiv c(N, j, \mathcal{M})$.

The next lemma allows modifying the image of a map while keeping under control boundary values and $p(\cdot)$-energy; see also [10, Lemma 5].

LEMMA 2.5. *Let* $\mathcal{M}$ *be as in* (2.6) *and* $U \subseteq B_1^+$ *a subset with positive measure and Lipschitz boundary. If* $w \in W^{1,p(\cdot)}(U, \mathbb{R}^N) \cap L^\infty(U, \mathbb{R}^N)$ *is so that* $w(\partial U) \subset \mathcal{M}$, *then there exists* $\tilde{w} \in \hat{\mathcal{C}}_w^{p(\cdot)}(U, \mathcal{M})$ *satisfying*

$$\int_U |D\tilde{w}|^{p(x)} \, \mathrm{d}x \leqslant c \int_U |Dw|^{p(x)} \, \mathrm{d}x,$$

*where* $c \equiv c(N, \mathcal{M}, \gamma_2)$.

REMARK 2. When dealing with manifold constrained minima of the $p$-Laplacean energy it is customary to recover the fundamental Caccioppoli inequality by exploiting the so-called monotonicity formula; see [21–23, 42, 51–53]. This way cannot be used in our case. Even though it is possible to show a monotonicity formula for the $p(x)$-energy, that is, Lemma 4.2, see also [10, Lemma 12; 54, Lemma 4.1], its proof crucially requires some corollaries of Gehring Lemma, which, in turn, is implied by Caccioppoli inequality, whose proof requires the monotonicity formula. Lemma 2.5 breaks this vicious circle giving the chance of deriving Caccioppoli inequality directly by minimality, as we will see in Section 3.1.

## 3. *Partial boundary regularity*

As mentioned in Section 2.2, to avoid unnecessary complications, we shall take $\Omega \equiv B_1^+$. In fact, since $\partial\Omega$ is $C^2$-regular, given any $x_0 \in \partial\Omega$, there exists an open neighborhood $B_{x_0}$ of $x_0$ and a change of variable $\Psi_0 \in C^2(\bar{B}_{x_0}, \mathbb{R}^n)$ so that in the new coordinates $y^i := \Psi_0^i(x)$ it holds that

$$\Psi_0(x_0) = 0, \qquad \Psi_0(\bar{B}_{x_0} \cap \bar{\Omega}) = \bar{B}_1^+, \qquad \Psi_0(\bar{B}_{x_0} \cap \partial\Omega) = \Gamma_1.$$

Moreover, there exists a positive constant $c_0 \equiv c_0(n, \partial\Omega)$ such that

$$0 < c_0^{-1} \leqslant \|D\Psi_0\|_{L^\infty(\bar{B}_{x_0} \cap \bar{\Omega})} \leqslant c_0 < \infty.$$

We stress that, being $\partial\Omega$ compact, the constant $c_0$ does not depend on $x_0$. A straightforward computation shows that, if $u \in W^{1,p(\cdot)}(\Omega, \mathcal{M})$ solves (1.1), then the map $\tilde{u} := u \circ \Psi_0^{-1}$ solves an analogous problem still satisfying (2.2) and (2.3). Assumption (2.7) on the boundary condition is preserved as well: if $g \in W^{1,q}(\bar{\Omega}, \mathcal{M})$ then $\tilde{g} := g \circ \Psi_0^{-1} \in W^{1,q}(\bar{B}_1^+, \mathcal{M})$. We refer to [**14, 31, 37**] for more details on this matter. Therefore, keeping Definition 3 in mind, we shall study problem

$$\mathcal{C}_g^{p(\cdot)}(B_1^+, \mathcal{M}) \ni w \mapsto \min \int_{B_1^+} k(x)|Dw|^{p(x)} \, \mathrm{d}x, \tag{3.1}$$

with $k(\cdot)$ and $p(\cdot)$ as in (2.3)–(2.2), respectively, and $g$ as in (2.7).

### 3.1. *Basic regularity results*

We first fix a threshold radius $R_* \in (0, 1]$ so that

$$0 < R_* \leqslant \min\left\{1, \left(\frac{\gamma_1^2}{4n[p]_{0,\alpha}}\right)^{\frac{1}{\alpha}}, \left(\frac{\gamma_1 q\left(1 - \frac{n}{q}\right)}{4n[p]_{0,\alpha}}\right)^{\frac{1}{\alpha}}\right\} \tag{3.2}$$

and choose $R \in (0, R_*]$. Further restrictions on the size of $R_*$ will be imposed in Section 3.2. An immediate consequence of (3.2) is that, given any half-ball $B_R^+$ and all balls $B_\varrho(x_0)$ with $x_0 \in B_R^+$ and $\varrho \in (0, R - |x_0|)$, there holds

$$\begin{cases} p_1^*(x_0, \varrho) > p_2(x_0, \varrho) \\ \frac{np_2(x_0,\varrho)}{q} \leqslant p_1(x_0, \varrho) \end{cases} \quad \text{for all} \quad R \in (0, R_*], \quad \varrho \in (0, R - |x_0|), \tag{3.3}$$

which is, on the other hand, automatic when $p_1(x_0, \varrho) \geqslant n$. Obviously, in (3.3) we adopted the usual terminology

$$p^* := \begin{cases} \frac{np}{n-p} & \text{if } \ 1 < p < n, \\ \text{any finite number larger than } p & \text{if } \ p \geqslant n. \end{cases}$$

Recall now that, if $B_\varrho(x_0) \Subset B_R^+$ and $w \in W^{1,p}(B_\varrho(x_0), \mathbb{R}^N)$ is such that $w \equiv 0$ on $U \subset B_\varrho(x_0)$ with $|U| > \hat{c}|B_\varrho(x_0)|$ for some positive, absolute $\hat{c}$, then Sobolev–Poincaré's inequality gives

$$\int_{B_\varrho(x_0)} |w/\varrho|^p \, \mathrm{d}x \leqslant c\varrho^{-n(p/p_*-1)} \left(\int_{B_\varrho(x_0)} |Dw|^{p_*} \, \mathrm{d}x\right)^{\frac{p}{p_*}}, \tag{3.4}$$

for $c \equiv c(n, N, p, \hat{c})$. Here $p_* := \max\{1, \frac{np}{n+p}\}$. We consider now an intrinsic version of [14, Theorem 2.4].

PROPOSITION 3.1. *Let $U \subset \mathbb{R}^n$ be an open, bounded domain with Lipschitz boundary and finite Ahlfors constants depending only on $n$. Let also $A \subset \bar{U}$ be a closed subset. Consider two non-negative functions $f_1 \in L^1(U)$ and $f_2 \in L^{1+\hat{\sigma}}(U)$ for some $\hat{\sigma} > 0$. With $\theta \in (0,1)$, assume that there holds*

$$\fint_{B_{\varrho/2}(x_0)\cap U} f_1 \, \mathrm{d}x \leqslant b\left\{\left(\fint_{B_\varrho(x_0)\cap U} f_1^\theta \, \mathrm{d}x\right)^{\frac{1}{\theta}} + \fint_{B_\varrho(x_0)\cap U} f_2 \, \mathrm{d}x\right\} \tag{3.5}$$

*for almost all $x_0 \in U \setminus A$ with $B_\varrho(x_0) \cap A = \emptyset$ and a positive constant $b$. Set*

$$d(x) := \frac{|B_{\operatorname{dist}(x,A)}(x) \cap U|}{|U|} \quad \text{and} \quad \tilde{f}_1(x) := d(x) f_1(x).$$

*Then there exists a positive threshold $\sigma_g \equiv \sigma_g(b, \theta, \hat{\sigma}) \in (0, \hat{\sigma})$ such that*

$$\left(\fint_U \tilde{f}_1^{1+\sigma} \, \mathrm{d}x\right)^{\frac{1}{1+\sigma}} \leqslant c(n, \theta, b, \hat{\sigma})\left\{\left(\fint_U f_1 \, \mathrm{d}x\right) + \left(\fint_U f_2^{1+\sigma} \, \mathrm{d}x\right)^{\frac{1}{1+\sigma}}\right\}$$

*for all $\sigma \in [0, \sigma_g)$.*

*Proof.* The proof is essentially the same as the one in [14] with minor changes due to the fact that in our case (3.5) involves the whole integrand; see also [26, Lemma 6.2]. □

As a consequence of Proposition 3.1, we derive some higher integrability results for solutions to problem (1.1).

LEMMA 3.2. *Under assumptions (2.2), (2.3), (2.6) and (2.7), let $u \in W^{1,p(\cdot)}(B_R^+, \mathcal{M})$ be a solution of problem (3.1). Then, for $x_0 \in \bar{B}_R^+$, with $R \in (0, R_*]$, $R_*$ as in (3.2) and $0 < \varrho < R - |x_0|$, there exists a positive threshold $\sigma_g \equiv \sigma_g(\mathrm{data}_{p(\cdot)}, q) \in (0, \frac{q}{\gamma_2} - 1)$ such that for all $\sigma \in (0, \sigma_g)$ there holds that*

$$\left(\fint_{B_{\varrho/2}(x_0)\cap B_R^+} (1 + |Du|^2)^{\frac{p(x)(1+\sigma)}{2}} \, \mathrm{d}x\right)^{\frac{1}{1+\sigma}}$$
$$\leqslant c\left[\fint_{B_\varrho(x_0)\cap B_R^+} (1 + |Du|^2)^{\frac{p(x)}{2}} \, \mathrm{d}x + \left(\fint_{B_\varrho(x_0)\cap B_R^+} |Dg|^{p(x)(1+\sigma)} \, \mathrm{d}x\right)^{\frac{1}{1+\sigma}}\right], \tag{3.6}$$

*for $c \equiv c(\mathtt{data}_{p(\cdot)}, q)$. If $B_\varrho(x_0) \Subset B_R^+$ then, there exists a positive threshold $\sigma'_g \equiv \sigma'_g(\mathtt{data}_{p(\cdot)}) > 0$ so that*

$$\left(\fint_{B_{\varrho/2}(x_0)} (1 + |Du|^2)^{\frac{p(x)(1+\sigma)}{2}} \, \mathrm{d}x\right)^{\frac{1}{1+\sigma}} \leqslant c \fint_{B_\varrho(x_0)} (1 + |Du|^2)^{\frac{p(x)}{2}} \, \mathrm{d}x, \tag{3.7}$$

*for all $\sigma \in (0, \sigma'_g)$ with $c \equiv c(\mathtt{data}_{p(\cdot)})$. In particular,*

$$|Du|^{p(\cdot)(1+\sigma)} \in L^1(B_R^+) \quad \text{for all} \quad \sigma \in \big[0, \min\{\sigma_g, \sigma'_g\}\big). \tag{3.8}$$

*Proof.* We take $x_0 \in \bar{B}_R^+$, $0 < \varrho < R - |x_0|$ and distinguish two cases: $x_0^n \leqslant \frac{3\varrho}{4}$ and $x_0^n > \frac{3\varrho}{4}$.
*Case 1*: $x_0^n \leqslant \frac{3\varrho}{4}$

We fix parameters $\frac{\varrho}{2} < \tau_1 < \tau_2 \leqslant \varrho$ and a cutoff function $\eta \in C^1_c(B_{\tau_2}(x_0))$ with the following specifics:

$$\mathbb{1}_{B_{\tau_1}(x_0)} \leqslant \eta \leqslant \mathbb{1}_{B_{\tau_2}(x_0)} \qquad \text{and} \qquad |D\eta| \leqslant \frac{4}{\tau_2 - \tau_1}. \tag{3.9}$$

Note that in this case the intersection $B_{\tau_2}(x_0) \cap \Gamma_R$ can be non-empty and the map $w := u - \eta(u-g)$ agrees with $u$ in the sense of traces on $\partial(B_{\tau_2}(x_0) \cap B_R^+)$. This means that we can use Lemma 2.5 to recover a map $\tilde{w} \in \hat{\mathcal{C}}^{p(\cdot)}_u(B_{\tau_2}(x_0) \cap B_R^+, \mathcal{M})$ satisfying the energy inequality (3.9) and so that

$$\begin{aligned}\int_{B_{\tau_2}(x_0)\cap B_R^+} |Du|^{p(x)}\ \mathrm{d}x &\leqslant \lambda^{-1}\int_{B_{\tau_2}(x_0)\cap B_R^+} k(x)|Du|^{p(x)}\ \mathrm{d}x \\ &\leqslant \lambda^{-1}\int_{B_{\tau_2}(x_0)\cap B_R^+} k(x)|D\tilde{w}|^{p(x)}\ \mathrm{d}x \leqslant \frac{\Lambda}{\lambda}\int_{B_{\tau_2}(x_0)\cap B_R^+} |D\tilde{w}|^{p(x)}\ \mathrm{d}x \\ &\leqslant c\int_{(B_{\tau_2}(x_0)\setminus B_{\tau_1}(x_0))\cap B_R^+} |Du|^{p(x)}\ \mathrm{d}x + c\int_{B_{\tau_2}(x_0)\cap B_R^+}\left[|Dg|^{p(x)} + \left|\frac{u-g}{\tau_2-\tau_1}\right|^{p(x)}\right]\ \mathrm{d}x,\end{aligned}$$

with $c \equiv c(N, \lambda, \Lambda, \gamma_2, \mathcal{M})$. Once the inequality of the previous display is available, we can use Widmann's hole-filling technique; Lemmas 2.3 and Lemma 2.2 (ii) to end up with

$$\begin{aligned}\int_{B_{\varrho/2}(x_0)\cap B_R^+} |Du|^{p(x)}\ \mathrm{d}x &\leqslant c\int_{B_\varrho(x_0)\cap B_R^+} |Dg|^{p(x)}\ \mathrm{d}x + c\varrho^{-p_2(\varrho)}\int_{B_\varrho(x_0)\cap B_R^+} |u-g|^{p(x)}\ \mathrm{d}x \\ &\leqslant c\int_{B_\varrho(x_0)\cap B_R^+} |Dg|^{p(x)}\ \mathrm{d}x + c\int_{B_\varrho(x_0)\cap B_R^+}\left|\frac{u-g}{\varrho}\right|^{p(x)}\ \mathrm{d}x \\ &\leqslant c\int_{B_\varrho(x_0)\cap B_R^+} |Dg|^{p(x)}\ \mathrm{d}x + c\int_{B_\varrho(x_0)\cap B_R^+}\left|\frac{u-g}{\varrho}\right|^{p_1(\varrho)}\ \mathrm{d}x,\end{aligned} \tag{3.10}$$

where $c \equiv c(N, \lambda, \Lambda, \gamma_2, \mathcal{M})$. Now we extend $u = g$ in $B_\varrho(x_0) \setminus B_R^+$, note that condition $x_0^n \leqslant 3\varrho/4$ implies that $|B_\varrho(x_0) \setminus B_R^+| \geqslant c(n)|B_\varrho(x_0)|$ and use (3.4) to bound

$$\begin{aligned}\fint_{B_\varrho(x_0)\cap B_R^+}\left|\frac{u-g}{\varrho}\right|^{p_1(\varrho)}\ \mathrm{d}x \leqslant & c\left(\fint_{B_\varrho(x_0)\cap B_R^+} |Du-Dg|^{(p_1(\varrho))_*}\ \mathrm{d}x\right)^{\frac{p_1(\varrho)}{(p_1(\varrho))_*}} \\ \leqslant & c\left(\fint_{B_\varrho(x_0)\cap B_R^+} (1+|Du|^2)^{\frac{p(x)(p_1(\varrho))_*}{2p_1(\varrho)}}\ \mathrm{d}x\right)^{\frac{p_1(\varrho)}{(p_1(\varrho))_*}} \\ & + c\left(\fint_{B_\varrho(x_0)\cap B_R^+} |Dg|^{\frac{p(x)(p_1(\varrho))_*}{p_1(\varrho)}}\ \mathrm{d}x\right)^{\frac{p_1(\varrho)}{(p_1(\varrho))_*}},\end{aligned}$$

for $c \equiv c(n, \gamma_1, \gamma_2)$. Merging the content of the two previous displays we obtain

$$\begin{aligned}\fint_{B_{\varrho/2}(x_0)\cap B_R^+} (1+|Du|^2)^{\frac{p(x)}{2}}\ \mathrm{d}x \leqslant & c\fint_{B_\varrho(x_0)\cap B_R^+} |Dg|^{p(x)}\ \mathrm{d}x \\ & + c\left(\fint_{B_\varrho(x_0)\cap B_R^+} (1+|Du|^2)^{\frac{p(x)}{2}\cdot\frac{(p_1(\varrho))_*}{p_1(\varrho)}}\ \mathrm{d}x\right)^{\frac{p_1(\varrho)}{(p_1(\varrho))_*}},\end{aligned} \tag{3.11}$$

where $c \equiv c(n, N, \lambda, \Lambda, \gamma_1, \gamma_2, \mathcal{M})$.

*Case 2:* $x_0^n > \frac{3\varrho}{4}$

In this case, we see that $B_{\frac{3\varrho}{4}} \Subset B_R^+$, so as in [**10**, Lemma 9] we recover

$$\begin{aligned}\fint_{B_{\varrho/2}(x_0)} (1+|Du|^2)^{\frac{p(x)}{2}} \ \mathrm{d}x \leqslant & c\left(\fint_{B_{3\varrho/4}(x_0)} (1+|Du|^2)^{\frac{p(x)}{2}\cdot\frac{(p_1(\varrho))_*}{p_1(\varrho)}} \ \mathrm{d}x\right)^{\frac{p_1(\varrho)}{(p_1(\varrho))_*}} \\ \leqslant & c\left(\fint_{B_{\varrho}(x_0)} (1+|Du|^2)^{\frac{p(x)}{2}\cdot\frac{(p_1(\varrho))_*}{p_1(\varrho)}} \ \mathrm{d}x\right)^{\frac{p_1(\varrho)}{(p_1(\varrho))_*}}, \end{aligned} \tag{3.12}$$

for $c \equiv c(n, N, \lambda, \Lambda, \gamma_1, \gamma_2, \mathcal{M})$. Once (3.11)-(3.12) are available, we can apply Proposition 3.1 with $U \equiv B_\varrho(x_0) \cap B_R^+$ and $A \equiv \partial B_\varrho(x_0) \cap B_R^+$ to conclude with (3.6)–(3.7).

Combining (3.6), (3.7) and a standard covering argument, we obtain (3.8) and the proof is complete. □

Remark 3. Since $Dg \in L^q(B_1^+, \mathbb{R}^{N\times n})$ with (2.7) in force, by the Hölder inequality we can rearrange (3.6) as follows:

$$\begin{aligned}&\left(\fint_{B_{\varrho/2}(x_0)\cap B_R^+} (1+|Du|^2)^{\frac{p(x)(1+\sigma)}{2}} \ \mathrm{d}x\right)^{\frac{1}{1+\sigma}} \\ &\qquad \leqslant c\left[\fint_{B_{\varrho}(x_0)\cap B_R^+} (1+|Du|^2)^{\frac{p(x)}{2}} \ \mathrm{d}x + \left(\fint_{B_{\varrho}(x_0)\cap B_R^+} |Dg|^q \ \mathrm{d}x\right)^{\frac{p_2(\varrho)}{q}}\right], \end{aligned} \tag{3.13}$$

for $c \equiv c(\mathtt{data}_{p(\cdot)}, q)$.

Let us point out a particularly helpful inequality contained in the proof of Lemma 3.2.

Corollary 3.3. *Under assumptions (2.2), (2.3), (2.6) and (2.7), let $u \in W^{1,p(\cdot)}(B_1^+, \mathcal{M})$ be a solution of problem (3.1). Then for any half-ball $B_R \subset \bar{B}_1^+$ and all balls $B_\varrho(x_0)$ with $x_0 \in B_R^+$, $\varrho \in (0, R-|x_0|)$, $R \in (0, R_*]$ and $R_*$ as in (3.2), there holds that*

$$\fint_{B_{\varrho/2}(x_0)\cap B_R^+} |Du|^{p(x)} \ \mathrm{d}x \leqslant c \fint_{B_{\varrho}(x_0)\cap B_R^+} \left[\left|\frac{u-g}{\varrho}\right|^{p(x)} + |Dg|^{p(x)}\right] \ \mathrm{d}x \tag{3.14}$$

*with $c \equiv c(\mathtt{data}_{p(\cdot)})$. In case $B_\varrho(x_0) \Subset B_1^+$, the inequality*

$$\fint_{B_{\varrho/2}(x_0)\cap B_R^+} |Du|^{p(x)} \ \mathrm{d}x \leqslant c \fint_{B_{\varrho}(x_0)\cap B_R^+} \left|\frac{u-(u)_\varrho}{\varrho}\right|^{p(x)} \ \mathrm{d}x, \tag{3.15}$$

*for $c \equiv c(\mathtt{data}_{p(\cdot)})$. Moreover, the following inequalities are satisfied:*

$$\fint_{B_{\varrho/4}(x_0)\cap B_R^+} |Du|^{p(x)} \ \mathrm{d}x \leqslant c\varrho^{-p_2(\varrho)} \quad \text{and} \quad \fint_{B_{\varrho/4}(x_0)\cap B_R^+} |Du|^{p(x)(1+\sigma)} \ \mathrm{d}x \leqslant c\varrho^{-p_2(\varrho)(1+\sigma)}, \tag{3.16}$$

*with $c \equiv c(n, N, \mathcal{M}, \gamma_1, \gamma_2, q, \|Dg\|_{L^q(B_1^+)})$ and for all $\sigma \in [0, \min\{\sigma_g, \frac{q}{n}-1\}]$, where $\sigma_g$ is the same higher integrability threshold appearing in Lemma 3.2.*

*Proof.* Inequality (3.14) is similar to (3.10) in the proof of Lemma 3.2, while the proof of (3.15) is contained in [**10**, Lemma 8]. To prove (3.16) we only need to note that by $(2.7)_2$ it immediately follows that

$$\varrho^{p_2(\varrho)} \fint_{B_\varrho(x_0)\cap B_R^+} |Dg|^{p(x)(1+\sigma)}\,\mathrm{d}x \leqslant c\left[\varrho^{p_2(\varrho)} + \varrho^{p_2(\varrho)}\left(\fint_{B_\varrho(x_0)\cap B_R^+} |Dg|^{p_2(\varrho)(1+\sigma)}\,\mathrm{d}x\right)\right]$$
$$\leqslant c\left[\varrho^{p_2(\varrho)} + \varrho^{p_2(\varrho)\left(1-\frac{n(1+\sigma)}{q}\right)}\left(1+\|Dg\|^{2\gamma_2}_{L^q(B_1^+)}\right)\right] \leqslant c(n,\gamma_2,\|Dg\|_{L^q(B_1^+)}). \tag{3.17}$$

Using this information together with (3.14) and $(2.6)_1$, we obtain $(3.16)_1$. Combining (3.6)–(3.7) with $(3.16)_1$ and (3.17), we get $(3.16)_2$ and the proof is complete. □

By Proposition 3.1 with $A \equiv \emptyset$, we can prove a globally higher integrability result for $p$-harmonic functions; see, for example, [**10**, Lemma 10; **14**, Lemma 3.3].

LEMMA 3.4. *Let $R \in (0,1]$, $x_0 \in \Gamma_R$ and $\varrho \in (0, R-|x_0|)$. Assume $(2.2)_2$, $(2.3)_2$ and (2.6) take $p \in [\gamma_1,\gamma_2]$ and $f \in W^{1,p}(\bar{B}_\varrho(x_0)\cap \bar{B}_R^+,\mathcal{M})$ so that $|Df|^p \in L^{1+\hat{\delta}}(\bar{B}_\varrho(x_0)\cap\bar{B}_R^+)$. If $v \in W^{1,p}(B_\varrho(x_0)\cap B_R^+,\mathcal{M})$ is a solution of the Dirichlet problem*

$$\hat{\mathcal{C}}^p_f(B_\varrho(x_0)\cap B_R^+,\mathcal{M}) \ni w \mapsto \min \int_{B_\varrho(x_0)\cap B_R^+} k(x)|Dw|^p\,\mathrm{d}x, \tag{3.18}$$

*then there exists a positive threshold $\delta_g \equiv \delta_g(n,N,\mathcal{M},\gamma_1,\gamma_2,\lambda,\Lambda) \in (0,\hat{\delta})$ so that*

$$\left(\fint_{B_\varrho(x_0)\cap B_R^+} |Dv|^{p(1+\delta)}\,\mathrm{d}x\right)^{\frac{1}{1+\delta}}$$
$$\leqslant c\left\{\fint_{B_\varrho(x_0)\cap B_R^+} |Dv|^p\,\mathrm{d}x + \left(\fint_{B_\varrho(x_0)\cap B_R^+} |Df|^{p(1+\delta)}\,\mathrm{d}x\right)^{\frac{1}{1+\delta}}\right\} \tag{3.19}$$

*for all $\delta \in [0,\delta_g)$. In (3.19), $c \equiv c(n,N,\mathcal{M},\gamma_1,\gamma_2,\lambda,\Lambda)$.*

### 3.2. *Proof of Theorem 1.1*

The proof of Theorem 1.1 relies on the following result.

PROPOSITION 3.5. *Under assumptions (2.1), (2.2), (2.3) and (2.6), let $u \in W^{1,p(\cdot)}(B_1^+,\mathcal{M})$ be a solution of problem (3.1) with boundary datum $g\colon \bar{B}_1^+ \to \mathcal{M}$ satisfying (2.7). Then, there exist a threshold radius $R_* \equiv R_*(\mathtt{data}) \in (0,1]$ and a smallness parameter $\varepsilon \equiv \varepsilon(\mathtt{data}) \in (0,1]$ such that if*

$$\left(\varrho^{p_2(x_0,\varrho)-n}\int_{B_\varrho(x_0)\cap B_R^+} |Du|^{p_2(x_0,\varrho)}\,\mathrm{d}x\right)^{\frac{1}{p_2(x_0,\varrho)}} + \left(\varrho^{q-n}\int_{B_\varrho(x_0)\cap B_R^+} |Dg|^q\,\mathrm{d}x\right)^{\frac{1}{q}} < \varepsilon, \tag{3.20}$$

*for some $R \in (0,R_*]$, $x_0 \in B_R^+$ and $\varrho \in (0,R-|x_0|)$, then*

$$u \in C^{0,1-\frac{n}{q}}_{loc}\big((B_\varrho(x_0)\cap\bar{B}_R^+)\setminus\Sigma_0(u,B_\varrho(x_0)\cap\bar{B}_R^+),\mathcal{M}\big),$$

*where $\Sigma_0(u,B_\varrho(x_0)\cap\bar{B}_R^+) \subset \bar{B}_R^+$ is a closed subset with $\dim_{\mathcal{H}}(\Sigma_0(u,B_\varrho(x_0)\cap\bar{B}_R^+)) < n-\gamma_1$.*

*Proof.* For the sake of simplicity, we split the proof into six steps.
*Step 1: Setting a threshold radius.* As mentioned in Section 3.1, there is no loss of generality in reducing the size of the half ball we are working on. Precisely, in addition to (3.2), we choose

a radius $R \in (0, R_*]$, where now it is

$$0 < R_* < \min\left\{1, \left[\frac{\gamma_1^2}{4n[p]_{0,\alpha}}\right]^{\frac{1}{\alpha}}, \left(\frac{\gamma_1 q\left(1-\frac{n}{q}\right)}{4n[p]_{0,\alpha}}\right)^{\frac{1}{\alpha}}, \left(\frac{\sigma_0\gamma_1}{2[p]_{0,\alpha}(2+\sigma_0)}\right)^{\frac{1}{\alpha}}\right\}, \tag{3.21}$$

for $\sigma_0 \in (0,1)$ defined as

$$\sigma_0 := \min\left\{\frac{1}{4}, \frac{\sigma_g'}{2}, \frac{\sigma_g}{2}, \frac{2}{\gamma_2 - 1}, \frac{\alpha}{\gamma_2}, \frac{q-\gamma_2}{\gamma_2}\right\}. \tag{3.22}$$

In (3.22), $\sigma_g$ and $\sigma_g'$ are the higher integrability thresholds appearing Lemma 3.2, therefore, given an half-ball $B_R^+ \subset B_1^+$, by (3.8) there holds that

$$|Du|^{p(\cdot)(1+\sigma)} \in L^1(B_R^+) \quad \text{for all} \quad \sigma \in [0, \sigma_0]. \tag{3.23}$$

Moreover, in addition to (3.3), another straightforward consequence of the restriction imposed in (3.21) yields that

$$p_2(x_0, \varrho) < p_2(x_0, \varrho)\left(1 + \frac{\sigma_0}{2}\right) < (1+\sigma_0)p_1(x_0, \varrho), \tag{3.24}$$

whenever $x_0 \in B_R^+$ and $\varrho \in (0, R - |x_0|)$. Hence, combining (3.23) and (3.24) we can conclude that

$$|Du|^{p_2(x_0,\varrho)} \in L^1(B_\varrho(x_0) \cap B_R^+). \tag{3.25}$$

Let us stress that by continuity, for any point $\bar{x} \in \bar{B}_R^+$ for which $p(\bar{x}) \geqslant n$, we can find a small ball $B_{\varrho_{\bar{x}}}(\bar{x}) \subset \bar{B}_R^+$ so that $p(x) > n - \frac{\sigma_0}{2}$ for all $x \in B_{\varrho_{\bar{x}}}(\bar{x})$. Combining this information with (3.6)–(3.7), the fact that by (3.22) we have

$$\left(n - \frac{\sigma_0}{2}\right)(1+\sigma_0) > n + \frac{\sigma_0}{4},$$

and with Morrey's embedding theorem we obtain that $u \in C^{0,\frac{\sigma_0}{4n+\sigma_0}}(B_{\varrho_{\bar{x}}/2}(\bar{x}) \cap B_1^+, \mathcal{M})$. Therefore, for the rest of the paper, we shall assume that $\gamma_2 < n$. Moreover, since from now on we work on sets of the type $B_\varrho(x_0) \cap B_R^+$ with $x_0 \in B_R^+$ and $\varrho \in (0, R - |x_0|)$, we shall simplify the notation in (2.5) as follows: $p_1(x_0, \varrho) \equiv p_1(\varrho)$ and $p_2(x_0; \varrho) \equiv p_2(\varrho)$.

*Step 2: Comparison, first time.* Let $u \in W^{1,p(\cdot)}(B_1^+, \mathcal{M})$ be a solution to the minimization problem (3.1) with (2.7) in force. We introduce the extensions

$$\tilde{u}(x) := \begin{cases} u(x', x^n) - g(x', x^n) & \text{if} \quad x^n \geqslant 0, \\ -(u(x', -x^n) - g(x', -x^n)) & \text{if} \quad x^n < 0. \end{cases} \tag{3.26}$$

Since $\mathtt{tr}_{\Gamma_1}(u) = \mathtt{tr}_{\Gamma_1}(g)$, it easily follows that $\tilde{u} \in W^{1,p(\cdot)}(B_1, \mathbb{R}^N)$ and, by (3.23), for all $B_\varrho(x_0) \subseteq B_R \subset B_{R_*}$ with $R_*$ as in (3.21) there holds that

$$\int_{B_\varrho(x_0)} |D\tilde{u}|^{p_2(\varrho)} \, \mathrm{d}x \leqslant c \int_{B_\varrho(x_0)\cap B_R^+} \left[|Du|^{p_2(\varrho)} + |Dg|^{p_2(\varrho)}\right] \, \mathrm{d}x \tag{3.27}$$

with $c \equiv c(\gamma_1, \gamma_2)$. Before going on we define the following quantities:

$$\phi(x_0, \varrho, p) := \left(\varrho^p \fint_{B_\varrho(x_0)} (1 + |D\tilde{u}|^2)^{p/2} \, \mathrm{d}x\right)^{\frac{1}{p}};$$

$$\phi^+(x_0, \varrho, p) := \left(\varrho^p \fint_{B_\varrho(x_0)\cap B_R^+} (1 + |Du|^2)^{p/2} \, \mathrm{d}x\right)^{\frac{1}{p}};$$

$$\psi(x_0,\varrho) := \phi(x_0,\varrho,p_2(\varrho)), \qquad \psi^+(x_0,\varrho) := \phi^+(x_0,\varrho,p_2(\varrho));$$

$$\chi^+(x_0,\varrho) := \psi^+(x_0,\varrho) + \left(\varrho^{q-n}\int_{B_\varrho(x_0)\cap B_R^+} |Dg|^q \,\mathrm{d}x\right)^{\frac{1}{q}},$$

where $x_0$, $\varrho$ and $R$ satisfy the usual relation $R \in (0, R_*]$, $x_0 \in B_R^+$ and $\varrho \in (0, R - |x_0|)$. In the definition of $\psi(x_0,\varrho)$, $p_2(\varrho)$ is as in (2.5). Clearly, if $B_\varrho(x_0) \Subset B_R^+$, both $\phi^+(\cdot)$ and $\psi^+(\cdot)$ denote the average on the full ball $B_\varrho(x_0)$. We shall start our analysis by considering a point $x_0 \in \Gamma_R$ and imposing (3.20) on $B_\varrho(x_0) \cap B_R^+ \equiv B_\varrho^+(x_0)$, which, with the terminology introduced above reads as

$$\chi^+(x_0,\varrho) < \varepsilon, \tag{3.28}$$

where $\varepsilon \in (0,1)$ is a small parameter whose size will be suitably reduced along the proof. Note that, as done in the case of (3.27), for all balls $B_\varrho(x_0) \subset B_R$, by the Hölder inequality we have

$$\begin{aligned}\chi^+(x_0,\varrho) &\leqslant c'\left[\psi(x_0,\varrho) + \left(\varrho^{p_2(\varrho)-n}\int_{B_\varrho(x_0)\cap B_R^+} |Dg|^{p_2(\varrho)} \,\mathrm{d}x\right)^{\frac{1}{p_2(\varrho)}}\right] \\ &\quad + c'\left(\varrho^{q-n}\int_{B_\varrho(x_0)\cap B_R^+} |Dg|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \\ &\leqslant c'\left[\psi(x_0,\varrho) + \left(\varrho^{q-n}\int_{B_\varrho(x_0)\cap B_R^+} |Dg|^q \,\mathrm{d}x\right)^{\frac{1}{q}}\right],\end{aligned} \tag{3.29}$$

for $c' \equiv c'(n,\gamma_1,\gamma_2,q)$. Now we compare $u$ to a solution $v \in W^{1,p_2(\varrho)}(B_{\varrho/2}^+(x_0),\mathcal{M})$ of the Dirichlet problem

$$\hat{\mathcal{C}}_u^{p_2(\varrho)}(B_{\varrho/2}^+(x_0),\mathcal{M}) \ni w \mapsto \min \int_{B_{\varrho/2}^+(x_0)} k(x)|Dw|^{p_2(\varrho)} \,\mathrm{d}x. \tag{3.30}$$

Such a solution exists, given that by (3.25), class $\hat{\mathcal{C}}_u^{p_2(\varrho)}(B_{\varrho/2}^+(x_0),\mathcal{M})$ is non-empty. The minimality of $v$ in class $\hat{\mathcal{C}}_u^{p_2(\varrho)}(B_{\varrho/2}^+(x_0),\mathcal{M})$ yields that it satisfies the Euler–Lagrange equation

$$0 = \int_{B_{\varrho/2}^+(x_0)} k(x)p_2(\varrho)|Dv|^{p_2(\varrho)-2}[Dv\cdot D\varphi - A_v(Dv,Dv)\varphi] \,\mathrm{d}x, \tag{3.31}$$

for any $\varphi \in W_0^{1,p_2(\varrho)}(B_{\varrho/2}^+(x_0),\mathbb{R}^N) \cap L^\infty(B_{\varrho/2}^+(x_0),\mathbb{R}^N)$, where, for $y \in \mathcal{M}$, $A_y\colon T_y\mathcal{M} \times T_y\mathcal{M} \to (T_y\mathcal{M})^\perp$ denotes the second fundamental form of $\mathcal{M}$. In particular, by tangentiality, we have

$$\nabla^2\Pi(v)(Dv,Dv) = -A_v(Dv,Dv) \quad \text{and} \quad |A_v(Dv,Dv)| \leqslant c_{\mathcal{M}}|Dv|^2, \tag{3.32}$$

where $c_{\mathcal{M}}$ depends only on the geometry of $\mathcal{M}$; see [**53**, Appendix to Chapter 2]. Let us quantify the $L^{p_2(\varrho)}$-distance between $Du$ and $Dv$. We first note that, by (3.25), the map $\varphi := u - v$ is admissible as a test in (3.31), thus exploiting the monotonicity properties of the integrand in (3.30), (2.10) and Lemma 2.1 we obtain

$$\int_{B^+_{\varrho/2}(x_0)} |V_{p_2(\varrho)}(Du)-V_{p_2(\varrho)}(Dv)|^2 \, \mathrm{d}x$$
$$\leqslant c\int_{B^+_{\varrho/2}(x_0)} k(x)\left[|Du|^{p_2(\varrho)}-|Dv|^{p_2(\varrho)}\right] \, \mathrm{d}x + c\int_{B^+_{\varrho/2}(x_0)} |Dv|^{p_2(\varrho)}|u-v| \, \mathrm{d}x, \quad (3.33)$$

where $c \equiv c(n, N, \gamma_1, \gamma_2, \lambda, \Lambda, \mathcal{M})$. Let us estimate the two quantities appearing on the right-hand side of (3.33). Note that, being $v$ a solution of (3.30), it satisfies the assumptions of Lemma 3.4 with $p = p_2(\varrho)$, $f = u$ and $\hat{\delta} = \frac{\sigma_0}{2}$, therefore, choosing any $\sigma' \in (0, \min\{\delta_g, \widehat{\delta}, \frac{1}{\gamma_2-1}\})$, by the Hölder inequality we control:

$$\int_{B^+_{\varrho/2}(x_0)} |Dv|^{p_2(\varrho)}|u-v| \, \mathrm{d}x \leqslant c\varrho^n \left(\fint_{B^+_{\varrho/2}(x_0)} |Dv|^{p_2(\varrho)(1+\sigma')} \, \mathrm{d}x\right)^{\frac{1}{1+\sigma'}}$$
$$\cdot\left(\fint_{B^+_{\varrho/2}(x_0)} |u-v|^{\frac{1+\sigma'}{\sigma'}} \, \mathrm{d}x\right)^{\frac{\sigma'}{1+\sigma'}} =: c(n)\varrho^n[(\mathrm{I})\cdot(\mathrm{II})].$$

By (3.6), (3.19), (3.24), the minimality of $v$ in class $\hat{\mathcal{C}}^{p_2(\varrho)}_u(B^+_{\varrho/2}(x_0), \mathcal{M})$, Hölder inequality, (3.28) and Lemma 2.2 (ii) we bound

$$\begin{aligned}
(\mathrm{I}) &\leqslant c\left(\fint_{B^+_{\varrho/2}(x_0)} |Du|^{p_2(\varrho)(1+\sigma')} \, \mathrm{d}x\right)^{\frac{1}{1+\sigma'}} \\
&\leqslant c\left(\fint_{B^+_{\varrho/2}(x_0)} |Du|^{p_1(\varrho)(1+\sigma_0)} \, \mathrm{d}x\right)^{\frac{p_2(\varrho)}{p_1(\varrho)(1+\sigma_0)}} \\
&\leqslant c\left[\fint_{B^+_{\varrho}(x_0)} (1+|Du|^2)^{\frac{p(x)}{2}} \, \mathrm{d}x + \left(\fint_{B^+_{\varrho}(x_0)} |Dg|^{p(x)(1+\sigma_0)} \, \mathrm{d}x\right)^{\frac{1}{1+\sigma_0}}\right]^{\frac{p_2(\varrho)}{p_1(\varrho)}} \\
&\leqslant c\left[\fint_{B^+_{\varrho}(x_0)} (1+|Du|^2)^{\frac{p(x)}{2}} \, \mathrm{d}x + \left(\fint_{B^+_{\varrho}(x_0)} |Dg|^{q} \, \mathrm{d}x\right)^{\frac{p_2(\varrho)}{q}}\right]^{\frac{p_2(\varrho)}{p_1(\varrho)}} \\
&\leqslant c\varepsilon^{\frac{p_2(\varrho)(p_2(\varrho)-p_1(\varrho))}{p_1(\varrho)}}\left[\fint_{B^+_{\varrho}(x_0)} (1+|Du|^2)^{\frac{p_2(\varrho)}{2}} \, \mathrm{d}x + \left(\fint_{B^+_{\varrho}(x_0)} |Dg|^{q} \, \mathrm{d}x\right)^{\frac{p_2(\varrho)}{q}}\right],
\end{aligned}$$

with $c \equiv c(\texttt{data}_{p(\cdot)})$. With the Poincaré inequality, (3.22), the minimality of $v$ in class $\hat{\mathcal{C}}^{p_2(\varrho)}_u(B^+_{\varrho/2}(x_0), \mathcal{M})$ and (3.28) we get

$$\begin{aligned}
(\mathrm{II}) \leqslant& c\left(\varrho^{p_2(\varrho)}\fint_{B^+_{\varrho/2}(x_0)} |Du-Dv|^{p_2(\varrho)} \, \mathrm{d}x\right)^{\frac{\sigma'}{1+\sigma'}} \\
\leqslant& c\left(\varrho^{p_2(\varrho)-n}\int_{B^+_{\varrho}(x_0)} (1+|Du|^2)^{\frac{p_2(\varrho)}{2}} \, \mathrm{d}x\right)^{\frac{\sigma'}{1+\sigma'}} \leqslant c\varepsilon^{\frac{\gamma_1\sigma'}{1+\sigma'}},
\end{aligned}$$

where $c \equiv c(n, \mathcal{M}, \gamma_1, \gamma_2, \lambda, \Lambda)$. Finally, by (3.13), (3.19), Lemma 2.2(i) with $\varepsilon_0 = \sigma'$, the minimality of $v$ in class $\hat{\mathcal{C}}_u^{p_2(\varrho)}(B^+_{\varrho/2}(x_0), \mathcal{M})$, (3.22) and (3.28) we have

$$
\begin{aligned}
\int_{B^+_{\varrho/2}(x_0)} k(x)\Big[|Du|^{p_2(\varrho)} - |Dv|^{p_2(\varrho)}\Big] \, dx &\leqslant \int_{B^+_{\varrho/2}(x_0)} k(x)\Big| \, |Du|^{p_2(\varrho)} - |Du|^{p(x)} \, \Big| \, dx \\
&\quad + \int_{B^+_{\varrho/2}(x_0)} k(x)\Big| \, |Dv|^{p(x)} - |Dv|^{p_2(\varrho)} \, \Big| \, dx \\
&\leqslant c\varrho^{n+\alpha}\left[ \fint_{B^+_{\varrho/2}(x_0)} (1+|Du|^2)^{\frac{p_2(\varrho)}{2}(1+\sigma')} \, dx + \fint_{B^+_{\varrho/2}(x_0)} (1+|Dv|^2)^{\frac{p_2(\varrho)}{2}(1+\sigma')} \, dx \right] \\
&\leqslant c\varrho^{n+\alpha}\left( \fint_{B^+_{\varrho/2}(x_0)} (1+|Du|^2)^{\frac{p_1(\varrho)(1+\sigma_0)}{2}} \, dx \right) \\
&\leqslant c\varrho^{n+\alpha}\left[ \left( \fint_{B^+_{\varrho}(x_0)} (1+|Du|^2)^{\frac{p(x)}{2}} \, dx \right) + \left( \fint_{B^+_{\varrho}(x_0)} |Dg|^q \, dx \right)^{\frac{p_2(\varrho)}{q}} \right]^{1+\sigma_0} \\
&\leqslant c\varrho^{n+\alpha-\sigma_0 p_2(\varrho)}\left[ \varrho^{p_2(\varrho)-n} \int_{B^+_{\varrho}(x_0)} (1+|Du|^2)^{\frac{p_2(\varrho)}{2}} \, dx + \left( \varrho^{q-n} \int_{B^+_{\varrho}(x_0)} |Dg|^q \, dx \right)^{\frac{p_2(\varrho)}{q}} \right]^{\sigma_0} \\
&\quad \cdot \left[ \fint_{B^+_{\varrho}(x_0)} (1+|Du|^2)^{\frac{p_2(\varrho)}{2}} \, dx + \left( \fint_{B^+_{\varrho}(x_0)} |Dg|^q \, dx \right)^{\frac{p_2(\varrho)}{q}} \right] \\
&\leqslant c\varepsilon^{\sigma_0\gamma_1}\left[ \int_{B^+_{\varrho}(x_0)} (1+|Du|^2)^{\frac{p_2(\varrho)}{2}} \, dx + \varrho^{n\left(1-\frac{p_2(\varrho)}{q}\right)} \left( \int_{B^+_{\varrho}(x_0)} |Dg|^q \, dx \right)^{\frac{p_2(\varrho)}{q}} \right],
\end{aligned}
$$

where $c \equiv c(\mathtt{data}_{p(\cdot)})$. Merging the content of all the previous displays we end up with

$$
\begin{aligned}
&\int_{B^+_{\varrho/2}(x_0)} |V_{p_2(\varrho)}(Du) - V_{p_2(\varrho)}(Dv)|^2 \, dx \\
&\quad \leqslant c\varepsilon^{\frac{\sigma'\gamma_1}{1+\sigma'}}\left[ \int_{B^+_{\varrho}(x_0)} (1+|Du|^2)^{\frac{p_2(\varrho)}{2}} \, dx + \varrho^{n\left(1-\frac{p_2(\varrho)}{q}\right)} \left( \int_{B^+_{\varrho/2}(x_0)} |Dg|^q \, dx \right)^{\frac{p_2(\varrho)}{q}} \right],
\end{aligned} \tag{3.34}
$$

where $c \equiv c(\mathtt{data}_{p(\cdot)})$. If $p_2(\varrho) \geqslant 2$, by (2.10) and (3.34) we directly obtain that

$$
\begin{aligned}
\int_{B^+_{\varrho/2}(x_0)} |Du - Dv|^{p_2(\varrho)} \, dx &\leqslant \int_{B^+_{\varrho/2}(x_0)} |V_{p_2(\varrho)}(Du) - V_{p_2(\varrho)}(Dv)|^2 \, dx \\
&\leqslant c\varepsilon^{\frac{\sigma'\gamma_1}{1+\sigma'}}\left[ \int_{B^+_{\varrho}(x_0)} (1+|Du|^2)^{\frac{p_2(\varrho)}{2}} \, dx + \varrho^{n\left(1-\frac{p_2(\varrho)}{q}\right)} \left( \int_{B^+_{\varrho}(x_0)} |Dg|^q \, dx \right)^{\frac{p_2(\varrho)}{q}} \right],
\end{aligned}
$$

while, when $1 < p_2(\varrho) < 2$, via Hölder inequality, (3.34), the minimality of $v$ in class $\hat{\mathcal{C}}_u^{p_2(\varrho)}(B^+_{\varrho/2}(x_0),\mathcal{M})$ and (2.10) we can conclude that

$$\begin{aligned}\int_{B^+_{\varrho/2}(x_0)} & |Du-Dv|^{p_2(\varrho)}\ \mathrm{d}x\\ &\leqslant \left(\int_{B^+_{\varrho/2}(x_0)} |Du-Dv|^2(|Du|^2+|Dv|^2)^{\frac{p_2(\varrho)-2}{2}}\ \mathrm{d}x\right)^{\frac{p_2(\varrho)}{2}}\\ &\quad\cdot\left(\int_{B^+_{\varrho/2}(x_0)} (|Du|^2+|Dv|^2)^{\frac{p_2(\varrho)}{2}}\ \mathrm{d}x\right)^{\frac{2-p_2(\varrho)}{2}}\\ &\leqslant c\varepsilon^{\frac{\sigma'\gamma_1^2}{2(1+\sigma')}}\left[\int_{B^+_\varrho(x_0)} (1+|Du|^2)^{\frac{p_2(\varrho)}{2}}\ \mathrm{d}x+\varrho^{n\left(1-\frac{p_2(\varrho)}{q}\right)}\left(\int_{B^+_\varrho(x_0)}|Dg|^q\ \mathrm{d}x\right)^{\frac{p_2(\varrho)}{q}}\right].\end{aligned}$$

All in all, setting $\kappa := \frac{\gamma_1\sigma'}{1+\sigma'}\min\{1,\frac{\gamma_1}{2}\}$, we get

$$\begin{aligned}\int_{B^+_{\varrho/2}(x_0)} |Du-Dv|^{p_2(\varrho)}\ \mathrm{d}x &\leqslant c\varepsilon^\kappa\int_{B^+_\varrho(x_0)}(1+|Du|^2)^{\frac{p_2(\varrho)}{2}}\ \mathrm{d}x\\ &\quad+c\varepsilon^\kappa\varrho^{n\left(1-\frac{p_2(\varrho)}{q}\right)}\left(\int_{B^+_\varrho(x_0)}|Dg|^q\ \mathrm{d}x\right)^{\frac{p_2(\varrho)}{q}},\end{aligned}\tag{3.35}$$

for $c\equiv c(\mathtt{data}_{p(\cdot)})$.

*Step 3: Comparison, second time.* Set $k_0 := k(x_0)$. We confront $v$ with the solution $h \in W^{1,p_2(\varrho)}(B^+_{\varrho/4}(x_0),\mathbb{R}^N)$ of the Dirichlet problem

$$\hat{\mathcal{C}}_v^{p_2(\varrho)}(B^+_{\varrho/4}(x_0),\mathbb{R}^N)\ni w\mapsto\int_{B^+_{\varrho/4}(x_0)} k_0|Dw|^{p_2(\varrho)}\ \mathrm{d}x.\tag{3.36}$$

Furthermore, $h$ solves the Euler–Lagrange equation

$$0=\int_{B^+_{\varrho/4}(x_0)} k_0p_2(\varrho)|Dh|^{p_2(\varrho)}Dh\cdot D\varphi\ \mathrm{d}x,\tag{3.37}$$

for all $\varphi\in W_0^{1,p_2(\varrho)}(B^+_{\varrho/4}(x_0),\mathbb{R}^N)$. Note that, by the results in [**41**] there holds that

$$\|h\|_{L^\infty(B^+_{\varrho/4}(x_0))}\leqslant c(N)\|v\|_{L^\infty(B^+_{\varrho/4}(x_0))}\leqslant c(N,\mathcal{M}).\tag{3.38}$$

Recalling [**14**, Lemma 3.4] there holds that

$$\begin{aligned}\int_{B^+_\varsigma(x_0)}|Dh|^{p_2(\varrho)}\ \mathrm{d}x \leqslant & c\left(\frac{\varsigma}{\varrho}\right)^\vartheta\int_{B^+_{\varrho/2}(x_0)}|Du|^{p_2(\varrho)}\ \mathrm{d}x\\ &+c\varsigma^{n\left(1-\frac{p_2(\varrho)}{q}\right)}\left(\int_{B^+_{\varrho/2}(x_0)}|Dg|^q\ \mathrm{d}x\right)^{\frac{p_2(\varrho)}{q}},\end{aligned}\tag{3.39}$$

for all $\varsigma\in(0,\frac{\varrho}{4})$ and any $\vartheta\in(n(1-\frac{p_2(\varrho)}{q}),n)$, with $c\equiv c(n,N,\gamma_1,\gamma_2,\lambda,\Lambda,q)$. For (3.39) we also used that, by (3.36) and (3.30) it is $\mathtt{tr}_{\Gamma_{\varrho/4}}(h)=\mathtt{tr}_{\Gamma_{\varrho/4}}(v)=\mathtt{tr}_{\Gamma_{\varrho/4}}(u)=\mathtt{tr}_{\Gamma_{\varrho/4}}(g)$, the minimality of $h$ in class $\hat{\mathcal{C}}_v^{p_2(\varrho)}(B^+_{\varrho/4}(x_0),\mathbb{R}^N)$ and the one of $v$ in class $\hat{\mathcal{C}}_u^{p_2(\varrho)}(B^+_{\varrho/2}(x_0),\mathcal{M})$. Exploiting now the monotonicity properties of the integrand in (3.36), Lemma 2.1, (3.31), (3.37), Hölder inequality and the minimality of $h$ in class $\hat{\mathcal{C}}_v^{p_2(\varrho)}(B^+_{\varrho/4}(x_0),\mathbb{R}^N)$. We estimate

$$
\begin{aligned}
c\int_{B^+_{\varrho/4}(x_0)} &|V_{p_2(\varrho)}(Dv)-V_{p_2(\varrho)}(Dh)|^2 \, \mathrm{d}x \\
&\leqslant \int_{B^+_{\varrho/4}(x_0)} k_0 p_2(\varrho)\Big(|Dv|^{p_2(\varrho)-2}Dv-|Dh|^{p_2(\varrho)-2}Dh\Big)\cdot(Dv-Dh) \, \mathrm{d}x \\
&= \int_{B^+_{\varrho/4}(x_0)} (k_0-k(x))p_2(\varrho)|Dv|^{p_2(\varrho)-2}Dv\cdot(Dv-Dh) \, \mathrm{d}x \\
&\quad + \int_{B^+_{\varrho/4}(x_0)} k(x)p_2(\varrho)|Dv|^{p_2(\varrho)-2}Dv\cdot(Dv-Dh) \, \mathrm{d}x \\
&\leqslant c\varrho^\alpha \int_{B^+_{\varrho/4}(x_0)} |Dv|^{p_2(\varrho)-1}|Dv-Dh| \, \mathrm{d}x + c\int_{B^+_{\varrho/4}(x_0)} |Dv|^{p_2(\varrho)}|v-h| \, \mathrm{d}x \\
&\leqslant c\varrho^\alpha \int_{B^+_{\varrho/4}(x_0)} |Dv|^{p_2(\varrho)} \, \mathrm{d}x + c\int_{B^+_{\varrho/4}(x_0)} |Dv|^{p_2(\varrho)}|v-h| \, \mathrm{d}x =: c[\varrho^\alpha(\mathrm{I})+(\mathrm{II})],
\end{aligned}
$$

with $c\equiv c(n,N,\lambda,\Lambda,\gamma_1,\gamma_2,[k]_{0,\alpha},\alpha)$. The minimality of $v$ in class $\hat{\mathcal{C}}_u^{p_2(\varrho)}(B^+_{\varrho/2}(x_0),\mathcal{M})$ yields that

$$
(\mathrm{I}) \leqslant \lambda^{-1}\int_{B^+_{\varrho}(x_0)} |Du|^{p_2(\varrho)} \, \mathrm{d}x
$$

and, recalling also (3.28), we see that

$$
\begin{aligned}
\Big(\frac{\varrho}{2}\Big)^{p_2(\varrho)-n} \int_{B^+_{\varrho/2}(x_0)} |Dv|^{p_2(\varrho)} \, \mathrm{d}x &\leqslant 2^{n-\gamma_1}\varrho^{p_2(\varrho)-n}\int_{B^+_{\varrho/2}(x_0)} |Du|^{p_2(\varrho)} \, \mathrm{d}x \\
&\leqslant 2^{n-\gamma_1}\left[\varrho^{p_2(\varrho)-n}\int_{B^+_{\varrho}(x_0)} (1+|Du|^2)^{\frac{p_2(\varrho)}{2}} \, \mathrm{d}x + \left(\varrho^{q-n}\int_{B^+_{\varrho}(x_0)} |Dg|^q \, \mathrm{d}x\right)^{\frac{p_2(\varrho)}{q}}\right] \\
&< 2^{n-\gamma_1}\varepsilon^{p_2(\varrho)}. \qquad (3.40)
\end{aligned}
$$

By the Hölder inequality, the minimality of $h$ in class $\hat{\mathcal{C}}_v^{p_2(\varrho)}(B^+_{\varrho/4}(x_0),\mathbb{R}^N)$ and the one of $v$ in class $\hat{\mathcal{C}}_u^{p_2(\varrho)}(B^+_{\varrho/2}(x_0),\mathcal{M})$, Lemma 3.4, (3.7), (3.38) and (3.40) we bound

$$
\begin{aligned}
(\mathrm{II}) \leqslant& c\varrho^n\left(\fint_{B^+_{\varrho/4}(x_0)} |Dv|^{p_2(\varrho)(1+\sigma')} \, \mathrm{d}x\right)^{\frac{1}{1+\sigma'}}\left(\fint_{B^+_{\varrho/4}(x_0)} |v-h|^{p_2(\varrho)} \, \mathrm{d}x\right)^{\frac{\sigma'}{1+\sigma'}} \\
\leqslant& c\varrho^n\left[\fint_{B^+_{\varrho/2}(x_0)} |Du|^{p_2(\varrho)} \, \mathrm{d}x + \left(\fint_{B^+_{\varrho/2}(x_0)} |Dg|^q \, \mathrm{d}x\right)^{\frac{p_2(\varrho)}{q}}\right] \\
&\cdot\left(\varrho^{p_2(\varrho)-n}\int_{B^+_{\varrho/4}(x_0)} |Dv-Dh|^{p_2(\varrho)} \, \mathrm{d}x\right)^{\frac{\sigma'}{1+\sigma'}}
\end{aligned}
$$

$$\leqslant c\varepsilon^{\frac{\gamma_1\sigma'}{1+\sigma'}}\left[\int_{B^+_\varrho(x_0)}(1+|Du|^2)^{\frac{p_2(\varrho)}{2}}\,\mathrm{d}x+\varrho^{n\left(1-\frac{p_2(\varrho)}{q}\right)}\left(\int_{B^+_\varrho(x_0)}|Dg|^q\,\mathrm{d}x\right)^{\frac{p_2(\varrho)}{q}}\right],$$

for $c\equiv c(\mathtt{data}_{p(\cdot)})$. Merging the content of the two previous displays and proceeding as in the last part of *Step* 2 we end up with

$$\int_{B^+_{\varrho/4}(x_0)}|Dv-Dh|^{p_2(\varrho)}\,\mathrm{d}x$$
$$\leqslant c[\varepsilon^\kappa+\varrho^\alpha]\left[\int_{B^+_\varrho(x_0)}(1+|Du|^2)^{\frac{p_2(\varrho)}{2}}\,\mathrm{d}x+\varrho^{n\left(1-\frac{p_2(\varrho)}{q}\right)}\left(\int_{B^+_\varrho(x_0)}|Dg|^q\,\mathrm{d}x\right)^{\frac{p_2(\varrho)}{q}}\right], \tag{3.41}$$

with $c\equiv c(\mathtt{data})$. Collecting inequalities (3.35) and (3.41) we obtain

$$\int_{B^+_{\varrho/4}(x_0)}|Du-Dh|^{p_2(\varrho)}\,\mathrm{d}x$$
$$\leqslant c[\varepsilon^\kappa+\varrho^\alpha]\left[\int_{B^+_\varrho(x_0)}(1+|Du|^2)^{\frac{p_2(\varrho)}{2}}\,\mathrm{d}x+\varrho^{n\left(1-\frac{p_2(\varrho)}{q}\right)}\left(\int_{B^+_\varrho(x_0)}|Dg|^q\,\mathrm{d}x\right)^{\frac{p_2(\varrho)}{q}}\right], \tag{3.42}$$

where $c\equiv c(\mathtt{data})$.

*Step 4: Morrey decay estimates at the boundary.* Let $\varsigma\in(0,\frac{\varrho}{4})$ and estimate, via (3.27), (3.39) and (3.42),

$$\int_{B_\varsigma(x_0)}(1+|D\tilde{u}|^2)^{\frac{p_2(\varrho)}{2}}\,\mathrm{d}x\leqslant c\left[\int_{B^+_\varsigma(x_0)}|Du|^{p_2(\varrho)}\,\mathrm{d}x+\int_{B^+_\varsigma(x_0)}|Dg|^{p_2(\varrho)}\,\mathrm{d}x\right]+c\varsigma^n$$
$$\leqslant c\left[\int_{B^+_\varsigma(x_0)}|Du-Dh|^{p_2(\varrho)}\,\mathrm{d}x+\int_{B^+_\varsigma(x_0)}|Dh|^{p_2(\varrho)}\,\mathrm{d}x+\int_{B^+_\varsigma(x_0)}|Dg|^{p_2(\varrho)}\,\mathrm{d}x\right]+c\varsigma^n$$
$$\leqslant c\left[\left(\frac{\varsigma}{\varrho}\right)^n+\varepsilon^\kappa+\varrho^\alpha+\left(\frac{\varsigma}{\varrho}\right)^\vartheta\right]$$
$$\cdot\left[\int_{B^+_\varrho(x_0)}(1+|Du|^2)^{\frac{p_2(\varrho)}{2}}\,\mathrm{d}x+\varrho^{n\left(1-\frac{p_2(\varrho)}{q}\right)}\left(\int_{B^+_\varrho(x_0)}|Dg|^q\,\mathrm{d}x\right)^{\frac{p_2(\varrho)}{q}}\right]$$
$$+c\left(\frac{\varsigma}{\varrho}\right)^{n\left(1-\frac{p_2(\varrho)}{q}\right)}\left(\varrho^{q-n}\int_{B^+_\varrho(x_0)}|Dg|^q\,\mathrm{d}x\right)^{\frac{p_2(\varrho)}{q}},$$

where $c\equiv c(\mathtt{data})$. Now recall that $n>\vartheta>n(1-\frac{p_2(\varrho)}{q})$, so we can always find $\beta'\in(n(1-\frac{p_2(\varrho)}{q}),\vartheta)$. Moreover set $\tilde{p}_2(\varrho):=p_2(\varrho)-n+\beta'$ and choose $\varsigma=\tau\varrho$ for some $\tau\in(0,\frac{1}{4})$. Multiplying both sides of the previous inequality by $(\tau\varrho)^{p_2(\varrho)-n}$ we obtain

$$(\tau\varrho)^{p_2(\varrho)-n}\int_{B_{\tau\varrho}(x_0)}(1+|D\tilde{u}|^2)^{\frac{p_2(\varrho)}{2}}\,\mathrm{d}x$$
$$\leqslant\tau^{\tilde{p}_2(\varrho)}\left[c\tau^{n-\beta'}+c\tau^{-\beta'}(\varepsilon^\kappa+R_*^\alpha)+c\tau^{\vartheta-\beta'}\right]$$

$$
\cdot \left[ \varrho^{p_2(\varrho)-n} \int_{B^+_\varrho(x_0)} (1+|Du|^2)^{\frac{p_2(\varrho)}{2}} \ \mathrm{d}x + \left( \varrho^{q-n} \int_{B^+_\varrho(x_0)} |Dg|^q \ \mathrm{d}x \right)^{\frac{p_2(\varrho)}{q}} \right]
$$

$$
+ c\tau^{p_2(\varrho)(1-\frac{n}{q})} \left( \varrho^{q-n} \int_{B^+_\varrho(x_0)} |Dg|^q \ \mathrm{d}x \right)^{\frac{p_2(\varrho)}{q}}, \tag{3.43}
$$

for $c \equiv c(\texttt{data})$. With the notation introduced in *Step* 2, the inequality in (3.43) reads as

$$
\phi(x_0, \tau\varrho, p_2(\varrho)) \leqslant \tau^{\frac{\tilde{p}_2(\varrho)}{p_2(\varrho)}} \left[ c\tau^{\frac{(n-\beta')}{p_2(\varrho)}} + c\tau^{-\frac{\beta'}{p_2(\varrho)}} \left( \varepsilon^{\frac{\kappa}{p_2(\varrho)}} + R_*^{\frac{\alpha}{p_2(\varrho)}} \right) + c\tau^{\frac{\vartheta-\beta'}{p_2(\varrho)}} \right]
$$

$$
\cdot \left[ \psi^+(x_0,\varrho) + \left( \varrho^{q-n} \int_{B^+_\varrho(x_0)} |Dg|^q \ \mathrm{d}x \right)^{\frac{1}{q}} \right] + c\tau^{1-\frac{n}{q}} \left( \varrho^{q-n} \int_{B^+_\varrho(x_0)} |Dg|^q \ \mathrm{d}x \right)^{\frac{1}{q}}, \tag{3.44}
$$

therefore since $\phi(x_0, r, t_1) \leqslant \phi(x_0, r, t_2)$ for $1 \leqslant t_1 \leqslant t_2$, we obtain from (3.44):

$$
\begin{aligned}
\psi(x_0, \tau\varrho) \leqslant & \tau^{\frac{\tilde{p}_2(\varrho)}{p_2(\varrho)}} \left[ c\tau^{\frac{(n-\beta')}{p_2(\varrho)}} + c\tau^{-\frac{\beta'}{p_2(\varrho)}} \left( \varepsilon^{\frac{\kappa}{p_2(\varrho)}} + R_*^{\frac{\alpha}{p_2(\varrho)}} \right) + c\tau^{\frac{\vartheta-\beta'}{p_2(\varrho)}} \right] \\
& \cdot \left[ \psi^+(x_0,\varrho) + \left( \varrho^{q-n} \int_{B^+_\varrho(x_0)} |Dg|^q \ \mathrm{d}x \right)^{\frac{1}{q}} \right] \\
& + c\tau^{1-\frac{n}{q}} \left( \varrho^{q-n} \int_{B^+_\varrho(x_0)} |Dg|^q \ \mathrm{d}x \right)^{\frac{1}{q}}
\end{aligned}
$$

with $c \equiv c(\texttt{data})$. Recalling that $\tau \in (0, \frac{1}{4})$, it is easy to see that

$$
\begin{aligned}
& \tau^{\frac{\tilde{p}_2(\varrho)}{p_2(\varrho)}} \left[ c\tau^{\frac{(n-\beta')}{p_2(\varrho)}} + c\tau^{-\frac{\beta'}{p_2(\varrho)}} \left( \varepsilon^{\frac{\kappa}{p_2(\varrho)}} + R_*^{\frac{\alpha}{p_2(\varrho)}} \right) + c\tau^{\frac{\vartheta-\beta'}{p_2(\varrho)}} \right] \\
& \qquad \leqslant \tau^{\frac{\tilde{p}_2(\varrho)}{p_2(\varrho)}} \left[ c\tau^{\frac{n-\vartheta}{\gamma_2}} + c\tau^{-\frac{\vartheta}{\gamma_1}} \left( \varepsilon^{\frac{\kappa}{\gamma_2}} + R_*^{\frac{\alpha}{\gamma_2}} \right) + c\tau^{\frac{\vartheta-\beta'}{\gamma_2}} \right],
\end{aligned}
$$

therefore, merging the content of the two above displays we obtain

$$
\begin{aligned}
\psi(x_0, \tau\varrho) \leqslant & \tau^{\frac{\tilde{p}_2(\varrho)}{p_2(\varrho)}} \left[ c\tau^{\frac{n-\vartheta}{\gamma_2}} + c\tau^{-\frac{\vartheta}{\gamma_1}} \left( \varepsilon^{\frac{\kappa}{\gamma_2}} + R_*^{\frac{\alpha}{\gamma_2}} \right) + c\tau^{\frac{\vartheta-\beta'}{\gamma_2}} \right] \\
& \cdot \left[ \psi^+(x_0,\varrho) + \left( \varrho^{q-n} \int_{B^+_\varrho(x_0)} |Dg|^q \ \mathrm{d}x \right)^{\frac{1}{q}} \right] \\
& + c\tau^{1-\frac{n}{q}} \left( \varrho^{q-n} \int_{B^+_\varrho(x_0)} |Dg|^q \ \mathrm{d}x \right)^{\frac{1}{q}} \\
\leqslant & \tau^{\frac{\tilde{p}_2(\varrho)}{p_2(\varrho)}} \left[ c\tau^{\frac{n-\vartheta}{\gamma_2}} + c\tau^{-\frac{\vartheta}{\gamma_1}} \left( \varepsilon^{\frac{\kappa}{\gamma_2}} + R_*^{\frac{\alpha}{\gamma_2}} \right) + c\tau^{\frac{\vartheta-\beta'}{\gamma_2}} \right]
\end{aligned}
$$

$$
\cdot \left[ \psi(x_0,\varrho) + \left( \varrho^{q-n} \int_{B^+_\varrho(x_0)} |Dg|^q \ \mathrm{d}x \right)^{\frac{1}{q}} \right]
$$

$$
+ c\tau^{1-\frac{n}{q}} \left( \varrho^{q-n} \int_{B^+_\varrho(x_0)} |Dg|^q \ \mathrm{d}x \right)^{\frac{1}{q}}, \tag{3.45}
$$

with $c \equiv c(\texttt{data})$. Select $\tau$, $\varepsilon$ and $R_*$ so small that

$$
\begin{cases} \tau^{\frac{\tilde{p}_2(\varrho)}{p_2(\varrho)}} \leqslant \frac{1}{8}, \qquad c'c\tau^{\frac{n-\vartheta}{\gamma_2}} \leqslant \frac{1}{3}, \qquad c'c\tau^{-\frac{\vartheta}{\gamma_1}} \left( \varepsilon^{\frac{\kappa}{\gamma_2}} + R_*^{\frac{\alpha}{\gamma_2}} \right) \leqslant \frac{1}{3} \\ c'c\tau^{\frac{\vartheta-\beta'}{\gamma_2}} \leqslant \frac{1}{3}, \qquad (c'+c)\tau^{1-\frac{n}{q}} \leqslant \frac{1}{8}, \end{cases} \tag{3.46}
$$

where $c'$ is the same constant appearing in (3.29). By (3.29) and (3.28), with the choice made above we can conclude that

$$
\chi^+(x_0,\tau\varrho) \leqslant \frac{1}{2} \left[ \psi^+(x_0,\varrho) + \left( \varrho^{q-n} \int_{B^+_\varrho(x_0)} |Dg|^q \ \mathrm{d}x \right)^{\frac{1}{q}} \right]
$$

$$
+ \frac{1}{2} \left( \varrho^{q-n} \int_{B^+_\varrho(x_0)} |Dg|^q \ \mathrm{d}x \right)^{\frac{1}{q}} < \varepsilon,
$$

so iterations are legal. Moreover, combining (3.45) and (3.46) we have

$$
\psi(x_0,\tau\varrho) \leqslant \tau^{\frac{\tilde{p}_2(\varrho)}{p_2(\varrho)}} \psi(x_0,\varrho) + c\varrho^{1-\frac{n}{q}} \left( \int_{B^+_\varrho(x_0)} |Dg|^q \ \mathrm{d}x \right)^{\frac{1}{q}}, \tag{3.47}
$$

for $c \equiv (\texttt{data}, q)$. Iterating (3.47) on integers $k \geqslant 1$ we end up with

$$
\psi(x_0,\tau^k\varrho) \leqslant \tau^{k\frac{\tilde{p}_2(\varrho)}{p_2(\varrho)}} \psi(x_0,\varrho)
+ c\left( \int_{B^+_\varrho(x_0)} |Dg|^q \ \mathrm{d}x \right)^{\frac{1}{q}} \varrho^{1-\frac{n}{q}} \tau^{(k-1)\left(1-\frac{n}{q}\right)} \sum_{j=0}^{k-1} \tau^{j\left(\frac{\tilde{p}_2(\varrho)}{p_2(\varrho)}-1+\frac{n}{q}\right)}. \tag{3.48}
$$

Since $\frac{\tilde{p}_2(\varrho)}{p_2(\varrho)} - 1 + \frac{n}{q} > 0$, the series on the right-hand side of (3.47) converges, so we have

$$
\psi(x_0,\tau^k\varrho) \leqslant \tau^{k\frac{\tilde{p}_2(\varrho)}{p_2(\varrho)}} \psi(x_0,\varrho) + c\left( \int_{B^+_\varrho(x_0)} |Dg|^q \ \mathrm{d}x \right)^{\frac{1}{q}} \varrho^{1-\frac{n}{q}} \tau^{(k-1)\left(1-\frac{n}{q}\right)}, \tag{3.49}
$$

for $c \equiv c(\texttt{data})$. Whenever $0 < \varsigma < \varrho$ we can find $k \in \mathbb{N}$ so that $\tau^{k+1}\varrho \leqslant \varsigma < \tau^k\varrho$, so using (3.49) and the very definition of $\tilde{p}_2(\varrho)$ we obtain

$$
\begin{aligned}
\psi(x_0,\varsigma) \leqslant& \tau^{-\frac{n}{p_2(\varrho)}} \psi(x_0,\tau^k\varrho) \\
\leqslant& \tau^{-\frac{n}{p_2(\varrho)}} \left[ \tau^{k\frac{\tilde{p}_2(\varrho)}{p_2(\varrho)}} \psi(x_0,\varrho) + c\varrho^{1-\frac{n}{q}} \tau^{(k-1)\left(1-\frac{n}{q}\right)} \left( \int_{B^+_\varrho(x_0)} |Dg|^q \ \mathrm{d}x \right)^{\frac{1}{q}} \right] \\
\leqslant& c\tau^{-2\left(1+\frac{n}{\gamma_1}\right)} \left[ \left( \frac{\varsigma}{\varrho} \right)^{\frac{\tilde{p}_2(\varrho)}{p_2(\varrho)}} \psi(x_0,\varrho) + \varrho^{1-\frac{n}{q}} \left( \int_{B^+_\varrho(x_0)} |Dg|^q \ \mathrm{d}x \right)^{\frac{1}{q}} \right]
\end{aligned}
$$

$$\leqslant c\left[\left(\frac{\varsigma}{\varrho}\right)^{1-\frac{n}{q}}\psi(x_0,\varrho)+\varsigma^{1-\frac{n}{q}}\left(\int_{B_\varrho^+(x_0)}|Dg|^q\,\mathrm{d}x\right)^{\frac{1}{q}}\right], \tag{3.50}$$

for $c\equiv c(\texttt{data})$. To summarize, we just got that, if $x_0\in\Gamma_R$ is any point satisfying (3.28) on $B_\varrho^+(x_0)$ for some $\varrho\in(0,R-|x_0|)$ then

$$\begin{aligned}\psi(x_0,\varsigma)\leqslant& c\left[\left(\frac{\varsigma}{\varrho}\right)^{1-\frac{n}{q}}\psi(x_0,\varrho)+\varsigma^{1-\frac{n}{q}}\left(\int_{B_\varrho^+(x_0)}|Dg|^q\,\mathrm{d}x\right)^{\frac{1}{q}}\right]\\ \leqslant& c\left(\frac{\varsigma}{\varrho}\right)^{1-\frac{n}{q}}\left(\varrho^{p_2(\varrho)}\fint_{B_\varrho^+(x_0)}(1+|Du|^2)^{\frac{p_2(\varrho)}{2}}\,\mathrm{d}x\right)^{\frac{1}{p_2(\varrho)}}\\ &+\left(\frac{\varsigma}{\varrho}\right)^{1-\frac{n}{q}}\left(\varrho^{q-n}\int_{B_\varrho^+(x_0)}|Dg|^q\,\mathrm{d}x\right)^{\frac{1}{q}}\\ \leqslant& c\left(\frac{\varsigma}{\varrho}\right)^{1-\frac{n}{q}}\chi^+(x_0,\varrho)\leqslant c\left(\frac{\varsigma}{\varrho}\right)^{1-\frac{n}{q}}\end{aligned} \tag{3.51}$$

for $c\equiv c(\texttt{data})$. In (3.51) we also used (3.28) to control $\chi^+(x_0,\varrho)$ with $\varepsilon\in(0,1]$.

*Step 5: Partial Hölder continuity.*

Now we aim to prove an estimate analogous to (3.51) valid also for points $x_0\in\bar{B}_R^+$ not necessarily belonging to $\Gamma_R$. As in [**37**, Proof of Lemma 2], we shall determine a threshold $\iota\equiv\iota(\texttt{data})\in(0,10\,000^{-1})$ and $x_0\in B_R^+$ satisfying (3.28) for some $\varrho\in(0,R-|x_0|)$. For $0<\varsigma<\varrho$ we distinguish two main cases: $\varsigma<2\iota\varrho$ or $\varsigma\geqslant 2\iota\varrho$.

*Case* 1: $\varsigma<2\iota\varrho$. We take $\hat{x}\in\Gamma_R$ so that $d:=\operatorname{dist}(x_0,\Gamma_R)=|x_0-\hat{x}|$. Now, if $2\iota\varrho\geqslant d$ we note that $B_d(x_0)\subset B_{2d}(\hat{x})\subset B_{\varrho/2}(\hat{x})\subset B_\varrho(x_0)$, therefore according to (3.28) it is

$$\chi^+\left(\hat{x},\frac{\varrho}{2}\right)\leqslant c\chi^+(x_0,\varrho)\leqslant c(n,\gamma_1,\gamma_2,q)\varepsilon,$$

so reducing the size of $\varepsilon\equiv\varepsilon(\texttt{data})$ determined in (3.46) to $\varepsilon':=\frac{\varepsilon}{2c}$ we end up with

$$\chi^+\left(\hat{x},\frac{\varrho}{2}\right)<\varepsilon'. \tag{3.52}$$

If $2\iota\varrho>\varsigma\geqslant d$ we immediately note that

$$B_\varsigma(x_0)\subset B_{4\varsigma}(\hat{x})\subset B_{\varrho/4}(\hat{x})\subset B_\varrho(x_0), \tag{3.53}$$

and, since (3.52) legalizes (3.51) with $x_0$ replaced by $\hat{x}$, we obtain

$$\psi(x_0,\varsigma)\leqslant c\psi(\hat{x},4\varsigma)\leqslant c\left(\frac{\varsigma}{\varrho}\right)^{1-\frac{n}{q}}, \tag{3.54}$$

for $c\equiv c(\texttt{data})$. If $2\iota\varrho\geqslant d>\varsigma$, we separately look at two possible occurrences: $2\iota\varrho\geqslant d\geqslant 4\varsigma$ and $2\iota\varrho\geqslant d$ with $4\varsigma>d$. In the first case, we note that $B_\varsigma(x_0)\subseteq B_{d/4}(x_0)\subset B_{d/2}(x_0)\Subset B_R^+$ and, since $B_{d/2}(x_0)\subset B_{2d}^+(\hat{x})$ and (3.52) is in force, by (3.51) it is

$$\psi^+\left(x_0,\frac{d}{2}\right)\leqslant c\psi^+(\hat{x},2d)\leqslant c\left(\frac{d}{\varrho}\right)^{1-\frac{n}{q}}\chi^+\left(\hat{x},\frac{\varrho}{2}\right)<c\iota^{1-\frac{n}{q}}\varepsilon',$$

with $c\equiv c(\texttt{data})$. After reducing the size of $\iota>0$ in such a way that $c\iota^{1-\frac{n}{q}}\varepsilon'<\varepsilon_0$, where $\varepsilon_0$ is the smallness threshold appearing in [**10**, (3.16)] we get that $\psi(x_0,d/2)<\varepsilon_0$ so [**10**, estimates

(3.40)–(3.43)] and (3.51) apply and render for arbitrary $\beta\in(0,1)$,

$$\psi^{+}(x_0,\varsigma)\leqslant c\left(\frac{\varsigma}{d}\right)^{\beta}\psi^{+}\left(x_0,\frac{d}{2}\right)+c\varsigma^{\beta}\leqslant c\left(\frac{\varsigma}{d}\right)^{\beta}\left(\frac{d}{\varrho}\right)^{1-\frac{n}{q}}\chi^{+}\left(\hat{x},\frac{\varrho}{2}\right)+c\varsigma^{\beta},$$

for $c\equiv c(\texttt{data})$. This in particular determines the dependency $\iota\equiv\iota(\texttt{data})$. Using (3.52) and choosing $\beta=1-n/q$ in the above display we can conclude with (3.54). On the other hand, when $4\varsigma>d$ we see that inclusion (3.53) holds with $B_{4\varsigma}(\hat{x})$ replaced by $B_{8\varsigma}(\hat{x})$ (keep in mind that $\iota<10\,000^{-1}$) and this yields (3.54). Now we consider the occurrence $\varsigma<2\iota\varrho<d$. It follows that $B_{2\iota\varrho}(x_0)\Subset B^{+}_{R}$ and

$$\psi^{+}(x_0,2\iota\varrho)\leqslant c\iota^{1-\frac{n}{\gamma_1}}\chi^{+}(x_0,\varrho)<c(n,\gamma_1,\gamma_2,q)\varepsilon.$$

Restricting further the size of $\varepsilon$ in such a way that $c\varepsilon\leqslant\varepsilon_0$, where $\varepsilon_0$ is the smallness threshold appearing in [**10**, (3.16)] we obtain $\psi^{+}(x_0,2\iota\varrho)<\varepsilon_0$ and again estimates [**10**, (3.40)–(3.43)] apply, thus getting

$$\begin{aligned}\psi(x_0,\varsigma)\leqslant&c\psi^{+}(x_0,\varsigma)+c\left(\frac{\varsigma}{\varrho}\right)^{1-\frac{n}{q}}\left(\varrho^{q-n}\int_{B_{\varrho}(x_0)\cap B^{+}_{R}}|Dg|^{q}\ \mathrm{d}x\right)^{1/q}\\ \leqslant&\iota^{-\beta_0}c(\texttt{data},\beta_0)\left(\frac{\varsigma}{\varrho}\right)^{\beta_0}+c\left(\frac{\varsigma}{\varrho}\right)^{1-\frac{n}{q}}\chi^{+}(x_0,\varrho),\end{aligned}$$

for all $\beta_0\in(0,1)$, so we can conclude using (3.28) and fixing $\beta_0=1-n/q$ above.

*Case* 2: $\varsigma\geqslant 2\iota\varrho$. Estimate (3.51) trivially holds with a constant $c\equiv c(\texttt{data})$.

All in all, we have just proved that if $x_0\in\bar{B}^{+}_{R}$ satisfies (3.28) on $B_{\varrho}(x_0)\cap B^{+}_{R}$ for some $\varrho\in(0,R-|x_0|)$, then

$$\psi(x_0,\varsigma)\leqslant c(\texttt{data})\left(\frac{\varsigma}{\varrho}\right)^{1-\frac{n}{q}}. \tag{3.55}$$

Now, by the continuity of Lebesgue's integral and of the mapping $x_0\mapsto p_2(x_0,\varrho)$, we can conclude that if (3.28) holds for $x_0$ on $B_{\varrho}(x_0)\cap B^{+}_{R}$ then it holds also on $B_{\varrho}(y)\cap B^{+}_{R}$ for all $y\in\bar{B}^{+}_{1}$ belonging to a sufficiently small, relatively open neighborhood of $x_0$, say, $B_{x_0}\subset\bar{B}^{+}_{R}$. Then the set

$$D_0:=\left\{y\in B_{x_0}\colon\chi^{+}(y,\varrho)<\varepsilon\text{ on }B_{\varrho}(y)\cap B^{+}_{R},\ R\in(0,R_*],\ \varrho\in(0,R-|y|)\right\}$$

is relatively open, so via (3.55) we can conclude that

$$\left(\varsigma^{-n\left(1-\frac{\gamma_1}{q}\right)}\int_{B_{\varsigma}(x_0)}|D\tilde{u}|^{\gamma_1}\ \mathrm{d}x\right)^{\frac{1}{\gamma_1}}\leqslant c\varrho^{\frac{n}{q}-1}, \tag{3.56}$$

where $c\equiv c(\texttt{data})$. By (3.56) and Morrey's embedding theorem we can conclude that $\tilde{u}$ is $(1-\frac{n}{q})$-Hölder continuous in a neighborhood of $D_0$, which in turn implies that $u\in C^{0,1-\frac{n}{q}}_{loc}(D_0,\mathcal{M})$.

*Step* 6: *Hausdorff dimension of the singular set* Given the characterization of $D_0$, we easily see that the singular set $\Sigma_0(u,B_{\varrho}(x_0)\cap B^{+}_{R})$ can be defined as

$$\Sigma_0(u,B_{\varrho}(x_0)\cap B^{+}_{R}):=\left(\bar{B}^{+}_{R}\cap B_{\varrho}(x_0)\right)\setminus D_0.$$

Moreover, for $y\in B_{\varrho}(x_0)\cap B^{+}_{R}$, via (2.7) we see that

$$\limsup_{\varsigma\to0}\chi^{+}(y,\varsigma)\leqslant\limsup_{\varsigma\to0}\left(\varsigma^{p_2(y,\varsigma)-n}\int_{B_{\varsigma}(y)\cap B^{+}_{R}}(1+|Du|^2)^{\frac{p_2(y,\varsigma)}{2}}\ \mathrm{d}x\right)^{\frac{1}{p_2(y,\varsigma)}}$$

$$+\limsup_{\varsigma\to 0}\left(\varsigma^{q-n}\int_{B_\varsigma(y)\cap B_R^+}|Dg|^q\,\mathrm{d}x\right)^{\frac{1}{q}}$$

$$\leqslant \limsup_{\varsigma\to 0}\left(\varsigma^{p_2(y,\varsigma)-n}\int_{B_\varsigma(y)\cap B_R^+}(1+|Du|^2)^{\frac{p_2(y,\varsigma)}{2}}\,\mathrm{d}x\right)^{\frac{1}{p_2(y,\varsigma)}},$$

therefore

$$\Sigma_0(u,B_\varrho(x_0)\cap B_R^+)\subset\left\{y\in\bar{B}_\varrho(x_0)\cap\bar{B}_R^+\colon \limsup_{\varsigma\to 0}\psi^+(y,\varsigma)>0\right\}.$$

Now, note that, as in (3.24),

$$p_2(y,\varsigma){<}(1+\sigma_0)p_1(x_0,R_*)\ \text{ for all }\ 0<\varsigma\leqslant R_*,\quad B_\varsigma(y)\cap B_R^+\subset B_\varrho(x_0)\cap B_R^+, \tag{3.57}$$

so we obtain,

$$\left(\varsigma^{p_2(y,\varsigma)}\fint_{B_\varsigma(y)\cap B_R^+}(1+|Du|^2)^{\frac{p_2(y,\varsigma)}{2}}\,\mathrm{d}x\right)^{\frac{1}{p_2(y,\varsigma)}}$$

$$\leqslant\left(\varsigma^{p_1(x_0,R_*)(1+\sigma_0)}\fint_{B_\varsigma(y)\cap B_R^+}(1+|Du|^2)^{\frac{p_1(x_0,R_*)(1+\sigma_0)}{2}}\,\mathrm{d}x\right)^{\frac{1}{p_1(x_0,R_*)(1+\sigma_0)}},$$

which by (3.6) is finite. This allows concluding that $\Sigma_0(u,B_\varrho(x_0)\cap B_R^+)$ is contained into the set

$$D_1:=\left\{y\in\bar{B}_\varrho(x_0)\cap\bar{B}_R^+\colon \limsup_{\varsigma\to 0}\phi^+(y,\varsigma,p_1(x_0,R_*)(1+\sigma_0))^{p_1(x_0,R_*)(1+\sigma_0)}>0\right\}.$$

By [**26**, Proposition 2.7] it follows that $\dim_{\mathcal{H}}(D_1)\leqslant n-p_1(x_0,R_*)(1+\sigma_0)$, so by $(2.2)_2$ we easily have that $\dim_{\mathcal{H}}(D_1)<n-\gamma_1$ and so $\dim_{\mathcal{H}}(\Sigma_0(u,B_\varrho(x_0)\cap B_R^+))<n-\gamma_1$. The proof is complete. □

Once Proposition 3.5 is available, we can cover $B_1^+$ with balls having the same features of $B_\varrho(x_0)\cap B_R^+$ and remembering that, by $(2.2)_2$, $p_1(x_0,R_*)\geqslant\gamma_1$, we obtain that $\dim_{\mathcal{H}}(\Sigma_0(u))\leqslant n-\gamma_1(1+\sigma_0)<n-\gamma_1$, and so $\dim_{\mathcal{H}}(\Sigma_0(u))<n-\gamma_1$. Via a standard covering argument, we can conclude that $u\in C^{0,1-\frac{n}{q}}_{loc}(\bar{B}_1^+\setminus\Sigma_0(u),\mathcal{M})$ and the proof of Theorem 1.1 is complete.

Remark 4. The result in Theorem 1.1 essentially shows that solutions of problem (1.1) are as regular as the boundary datum allows, in particular, if instead of (2.7) we assume $g\in W^{1,\infty}(\bar{\Omega},\mathcal{M})$, we can prove that $u\in C^{0,\beta}_{loc}(\Omega_0,\mathcal{M})$ for all $\beta\in(0,1)$, as done in the $p$-Laplacean case in [**31, 52**].

## 4. *Full boundary regularity*

In this section we recover a regularity criterion based on the result in Theorem 1.1. The main preliminary step consists in proving compactness of sequences of minimizers of (3.1) under uniform assumptions; see [**10, 14, 48**].

Remark 5. We will always assume that $\gamma_2<n$, otherwise, as stressed in *Step* 1 of the proof of Theorem 1.1, by Morrey's embedding theorem we would have $u$ Hölder continuous in a small neighborhood of any point $\bar{x}\in\bar{B}_1^+$ so that $p(\bar{x})\geqslant n$.

LEMMA 4.1. *Let $\{k_j\}, \{p_j\}$ be two sequences of Hölder continuous functions satisfying*

$$\begin{cases} \sup_{j\in\mathbb{N}}[k_j]_{0,\nu} < c_k \quad \text{for some } \ \nu \in (0,1] \\ \lambda \leqslant k_j(x) \leqslant \Lambda \text{ for all } x \in \bar{B}_1^+ \\ \|k_j - k_0\|_{L^\infty(\bar{B}_1^+)} \to 0, \ k_0(\cdot) \in C^{0,\nu}(\bar{B}_1^+) \end{cases} \tag{4.1}$$

*and*

$$\begin{cases} \sup_{j\in\mathbb{N}}[p_j]_{0,\alpha} < c_p \quad \text{for some } \ \alpha \in (0,1] \\ p_j(x) \geqslant \gamma_1 > 1 \text{ for all } x \in \bar{B}_1^+, j \in \mathbb{N} \\ \|p_j - p_0\|_{L^\infty(\bar{B}_1^+)} \to 0, \ p_0 \geqslant \gamma_1 > 1 \text{ constant}, \end{cases} \tag{4.2}$$

*respectively. For each $j \in \mathbb{N}$, let $u_j \in W^{1,p_j(\cdot)}(B_1^+, \mathcal{M})$ be a constrained minimizer of*

$$\mathcal{E}_j(w, B_1^+) := \int_{B_1^+} k_j(x)|Dw|^{p_j(x)} \ \mathrm{d}x,$$

*in class $\mathcal{C}^{p_j(\cdot)}_{g_j}(B_1^+, \mathcal{M})$, where the manifold $\mathcal{M}$ is as in (2.6) and the sequence $\{g_j\} \subset W^{1,q}(\bar{B}_1^+, \mathcal{M})$, uniformly satisfying (2.7), is weakly convergent to some $g_0 \in W^{1,q}(\bar{B}_1^+, \mathcal{M})$. Then, there exists a subsequence, still denoted by $\{u_j\}$, such that*

$$u_j \rightharpoonup u_0 \quad \text{weakly in } W^{1,(1+\tilde{\sigma})p_0}(B_R^+, \mathcal{M}) \tag{4.3}$$

*for some $\tilde{\sigma} > 0$ and any $R \in (0,1)$. In particular, $u_0$ is a constrained minimizer of the functional*

$$\mathcal{E}_0(w, B_R^+) := \int_{B_R^+} k_0(x)|Dw|^{p_0} \ \mathrm{d}x$$

*in class $\mathcal{C}^{p_0}_{g_0}(B_R^+, \mathcal{M})$. Moreover,*

$$\mathcal{E}_j(u_j, B_R^+) \to \mathcal{E}_0(u_0, B_R^+) \quad \textit{for all} \ \ R \in (0,1).$$

*Finally, if $x_j$ is a singular point of $u_j$ and $x_j \to x_0$, then $x_0$ is a singular point for $u_0$.*

*Proof.* For the reader's convenience, we split the proof into three steps.

*Step 1: Weak convergence.* By assumption, the sequence $\{g_j\}$ is weakly convergent in $W^{1,q}(\bar{B}_1, \mathcal{M})$, so we can find a positive, finite constant $M \equiv M(n, \mathcal{M}, q)$ so that

$$\sup_{j\in\mathbb{N}} \|g_j\|_{W^{1,q}(B_1^+)} \leqslant M. \tag{4.4}$$

Since the whole sequence $\{u_j\}$ has image contained in $\mathcal{M}$, which, by $(2.6)_1$ is compact, we immediately have that $\sup_{j\in\mathbb{N}} \|u_j\|_{L^\infty(B_1^+)} \leqslant c(\mathcal{M}) < \infty$, thus, up to extracting a non-relabeled subsequence,

$$u_j \rightharpoonup u_0 \quad \text{weakly in } \ L^t(B_1^+, \mathcal{M}) \quad \text{for all } \ t \in (1, \infty). \tag{4.5}$$

Moreover, being the assumptions in (4.1)–(4.2) uniform in $j \in \mathbb{N}$, we deduce that Lemmas 3.2 and 3.4 for the associated frozen problem hold with constants independent of $j$. In particular, recalling the uniform features of the functions $g_j$ and combining (3.8) with a standard covering argument we can conclude that $\{u_j\} \subset W^{1,p(\cdot)(1+\sigma)}_{loc}(B_1^+, \mathcal{M})$ for all $\sigma \in [0, \min\{\sigma_g, \sigma_g'\})$. Now we take any ball $B_\varrho(x_0) \subset B_1$ with $\varrho \in (0, \frac{1}{4}\min\{1 - |x_0|, R_*\}]$ and $R_*$ as in (3.21), so we can apply $(3.16)_2$ with any $\sigma \in (0, \min\{\sigma_g, \frac{n-\gamma_2}{\gamma_2}, \frac{q}{n} - 1\})$ to deduce that

$$\int_{B_\varrho(x_0)\cap B_1^+} |Du_j|^{p_j(x)(1+\sigma)} \ \mathrm{d}x \leqslant c\varrho^{n-p_2(x_0,\varrho)(1+\sigma)} \leqslant c(n, N, \mathcal{M}, \gamma_1, \gamma_2, q). \tag{4.6}$$

In (4.6) we also used (4.4) to incorporate the dependency on the constant from $\|Dg_j\|_{L^q(B_1^+)}$ into the one from $(n,\mathcal{M},q)$. Now set

$$\hat{\sigma}_g := \frac{1}{4}\min\left\{\sigma_g,\sigma_g',\delta_g,\frac{n-\gamma_2}{\gamma_2},\frac{q}{n}-1\right\},$$

where $\sigma_g$, $\sigma_g'$ and $\delta_g$ are the same higher integrability threshold determined in Lemmas 3.2 and 3.4, respectively, and choose any $\sigma\in(0,\hat{\sigma}_g)$. Because of the uniform convergence of the sequence $\{p_j\}$ to the constant $p_0$, taking $j\in\mathbb{N}$ sufficiently large we can find positive constants $\gamma_1\leqslant q_1\leqslant q_2\leqslant\gamma_2$ such that

$$1<q_1\leqslant p_j(\cdot)\leqslant q_2<\infty \text{ on } \bar{B}_1^+,\quad q_2\Big(1+\frac{\sigma}{2}\Big)<q_1(1+\sigma),\quad q_2<p_0\Big(1+\frac{\sigma}{2}\Big) \tag{4.7}$$

and

$$0\leqslant q_2-q_1<\frac{\delta_g\gamma_1}{16}\quad\text{and}\quad 1\leqslant\frac{q_2}{q_1}<2. \tag{4.8}$$

Combining (4.6), (4.7) and the choice of $\sigma>0$ we made, we can conclude that

$$\int_{B_\varrho(x_0)\cap B_1^+}|Du_j|^{q_2\left(1+\frac{\sigma}{2}\right)}\,\mathrm{d}x\leqslant c(n,\mathcal{M},\gamma_1,\gamma_2,q). \tag{4.9}$$

By (4.5) and (4.9) we derive the uniform boundedness of the functions $u_j$ in $W^{1,(1+\sigma/2)q_2}(B_\varrho(x_0)\cap B_1^+,\mathcal{M})$, so, up to extract a (non-relabeled) subsequence, we obtain that $u_j\rightharpoonup\bar{u}_0$ weakly in $W^{1,(1+\sigma/2)q_2}(B_\varrho(x_0)\cap B_1^+,\mathcal{M})$, for some $\bar{u}_0\in W^{1,(1+\sigma/2)q_2}(B_\varrho(x_0)\cap B_1^+,\mathcal{M})$. Anyway, by (4.5), $\bar{u}_0(x)=u_0(x)$, $u_0(x)\in\mathcal{M}$ for a.e. $x\in B_\varrho(x_0)\cap B_1^+$ and, by the Rellich–Kondrachov theorem,

$$u_j\to u_0\quad\text{strongly in}\ \ L^{(1+\sigma/2)q_2}(B_\varrho(x_0)\cap B_1^+,\mathcal{M}), \tag{4.10}$$

$$Du_j\to Du_0\quad\text{weakly in}\ \ L^{(1+\sigma/2)q_2}(B_\varrho(x_0)\cap B_1^+,\mathbb{R}^{N\times n}). \tag{4.11}$$

From $(4.7)_1$ and $(4.2)_3$, we see that $q_2\geqslant p_0$, therefore (4.3) is proved for instance with

$$\tilde{\sigma}=\frac{\hat{\sigma}_g}{4}. \tag{4.12}$$

Using the lower semicontinuity of the norm, we also have that

$$\int_{B_\varrho(x_0)\cap B_1^+}|Du_0|^{q_2\left(1+\frac{\tilde{\sigma}}{2}\right)}\,\mathrm{d}x\leqslant c(\mathtt{data}_{p(\cdot)}). \tag{4.13}$$

Inequality (4.13) and the convergence in (4.10)–(4.11) hold on $B_\varrho(x_0)\cap B_1^+$, but we will show that they actually hold on half balls having any radius $R\in(0,1)$. In fact, being $\bar{B}_1^+$ compact, we can find $m\equiv m(n)$ and a finite family of balls $\{B_{\varrho_k}(x_k)\}_{k=1}^m$ so that $\{\varrho_k\}\subset(0,\frac{R_*}{4})$ and $B_1^+\subseteq\bigcup_{k=1}^m B_{\varrho_k}(x_k)$. Then, given any measurable subset $U\subseteq B_R^+$ with $R\in(0,1)$, we trivially have that $U\subseteq\bigcup_{k=1}^m(B_{\varrho_k}(x_k)\cap B_1^+)$ and, recalling (4.10), (4.11) and (4.13):

$$\int_U|Du_0|^{q_2\left(1+\frac{\tilde{\sigma}}{2}\right)}\,\mathrm{d}x\leqslant\sum_{k=1}^m\int_{B_{\varrho_k}(x_k)\cap B_1^+}|Du_0|^{q_2\left(1+\frac{\tilde{\sigma}}{2}\right)}\,\mathrm{d}x\leqslant mc\leqslant c(\mathtt{data}_{p(\cdot)}) \tag{4.14}$$

$$\|Du_j\|_{L^{q_2\left(1+\frac{\tilde{\sigma}}{2}\right)}(U)}\leqslant\sum_{k=1}^m\|Du_j\|_{L^{q_2\left(1+\frac{\tilde{\sigma}}{2}\right)}(B_{\varrho_k}(x_k)\cap B_1^+)}\leqslant c(\mathtt{data}_{p(\cdot)}) \tag{4.15}$$

$$\|u_j-u_0\|_{L^{q_2\left(1+\frac{\tilde{\sigma}}{2}\right)}(U)}\leqslant\sum_{k=1}^m\|u_j-u_0\|_{L^{q_2\left(1+\frac{\tilde{\sigma}}{2}\right)}(B_{\varrho_k}(x_k)\cup B_1^+)}\to 0, \tag{4.16}$$

so (4.3) is completely proved. Note that (4.3) and the weak continuity of the trace operator yield in particular that

$$\mathtt{tr}_{\Gamma_R}(u_0) = \mathtt{tr}_{\Gamma_R}(g_0) \quad \text{for all} \quad R \in (0,1). \tag{4.17}$$

*Step 2: Compactness.* We fix $R \in (0,1)$ and, as a first step toward the proof of the minimality of $\mathcal{E}_0(u_0, B_R^+)$ in class $\mathcal{C}^{p_0}_{g_0}(B_R^+, \mathcal{M})$ we show that

$$\mathcal{E}_0(u_0, B_R^+) \leqslant \liminf_{j\to\infty} \mathcal{E}_j(u_j, B_R^+). \tag{4.18}$$

Since $\mathcal{E}_j(u_j, B_R^+) = (\mathcal{E}_j(u_j, B_R^+) - \mathcal{E}_0(u_j, B_R^+)) + \mathcal{E}_0(u_j, B_R^+)$ and, by weak lower semicontinuity and (4.5) it is

$$\mathcal{E}_0(u_0, B_R^+) \leqslant \liminf_{j\to\infty} \mathcal{E}_0(u_j, B_R^+), \tag{4.19}$$

we only need to show that

$$|\mathcal{E}_j(u_j, B_R^+) - \mathcal{E}_0(u_j, B_R^+)| \to 0, \tag{4.20}$$

which is a consequence of $(4.1)_3$, $(4.2)_3$, Lemma 2.2 (i) with $\varepsilon_0 = \frac{\sigma}{2}$ and (4.9). In fact,

$$\begin{aligned}
|\mathcal{E}_j(u_j, B_R^+) - \mathcal{E}_0(u_j, B_R^+)| &\leqslant \left| \int_{B_R^+} (k_j(x) - k_0(x))|Du_j|^{p_j(x)} \, \mathrm{d}x \right| \\
&+ \left| \int_{B_R^+} k_0(x)\Big[|Du|^{p_j(x)} - |Du_j|^{p_0}\Big] \, \mathrm{d}x \right| \leqslant \|k_j - k_0\|_{L^\infty(B_R^+)} \int_{B_R^+} |Du|^{p_j(x)} \, \mathrm{d}x \\
&+ c\|p_j - p_0\|_{L^\infty(B_R^+)} \int_{B_R^+} (1 + |Du_j|^2)^{\frac{q_2}{2}(1+\frac{\sigma}{2})} \, \mathrm{d}x \\
&\leqslant c\Big[\|k_j - k_0\|_{L^\infty(B_R^+)} + \|p_j - p_0\|_{L^\infty(B_R^+)}\Big] \to 0.
\end{aligned}$$

The constant $c$ appearing in the previous display depends only on $\mathtt{data}_{p(\cdot)}$. Combining (4.20) and (4.19) we end up with (4.18). Now, let $\tilde{u}_0 \in W^{1,p_0}(B_R^+, \mathcal{M})$ be a solution of the Dirichlet problem

$$\hat{\mathcal{C}}^{p_0}_{u_0}(B_R^+, \mathcal{M}) \ni w \mapsto \min \mathcal{E}_0(w, B_R^+). \tag{4.21}$$

As in [**10, 14, 30**] we fix any $\theta \in (0,1)$, a cut-off function $\eta \in C^1_c(B_R)$ satisfying

$$\mathbb{1}_{B_{(1-\theta)R}} \leqslant \eta \leqslant \mathbb{1}_{B_R} \quad \text{and} \quad |D\eta| \lesssim \frac{1}{R\theta}, \tag{4.22}$$

and consider a bi-Lipschitz transformation $\Phi\colon \bar{B}_R^+ \to \bar{B}_R$ so that

$$\Phi|_{\partial^+ B_R^+} = \mathbb{I}_{\partial^+ B_R^+} \quad \text{and} \quad \Phi(\Gamma_R) = \{x \in \partial B_R \colon x^n < 0\}. \tag{4.23}$$

Being $\Phi$ bi-Lipschitz, if $\mathcal{J}_\Phi$ is its jacobian, we have that

$$0 < c(n)^{-1} \leqslant |\mathcal{J}_\Phi(x)| \leqslant c(n) < \infty. \tag{4.24}$$

Let us look at the function

$$\tilde{u}_j(x) := \tilde{u}_0(x) + (1 - \eta(\Phi(x)))(u_j(x) - u_0(x)) \quad \text{for} \quad x \in B_R^+.$$

By $(4.22)_1$ and (4.23) we see that

$$B_R^+ \cap \{0 \leqslant \eta(\Phi(x)) < 1\} = B_R^+ \cap \Phi^{-1}(\bar{B}_R \setminus \bar{B}_{(1-\theta)R}). \tag{4.25}$$

Since

$$\partial\big(B_R^+\cap\{0\leqslant\eta(\Phi(x))<1\}\big)=\partial B_R^+\cup\partial\{\eta(\Phi(x))=1\},$$

by (4.17) we infer also that

$$\text{in a neighborhood of }\ \partial\big(B_R^+\cap\{0\leqslant\eta(\Phi(x))<1\}\big),\ \tilde u_j \text{ takes values in } \mathcal{M}. \tag{4.26}$$

In particular, according to (4.17) and to the definition given in (4.21), we have

$$\begin{cases} \mathtt{tr}_{\Gamma_R}(\tilde u_j)=\mathtt{tr}_{\Gamma_R}(g_j)\\ \mathtt{tr}_{\partial^+B_R^+}(\tilde u_j)=\mathtt{tr}_{\partial^+B_R^+}(u_j)\\ \mathtt{tr}_{\partial\{\eta(\Phi(x))=1\}}(\tilde u_j)=\mathtt{tr}_{\partial\{\eta(\Phi(x))=1\}}(\tilde u_0).\end{cases} \tag{4.27}$$

Conditions (4.26)–(4.27) justify the application of Lemma 2.5 on the set $B_R^+\cap\{0\leqslant\eta(\Phi(x))<1\}$ to end up with a function $\bar w_j\in W^{1,p_j(\cdot)}_{loc}(B_1^+,\mathcal{M})$ satisfying

$$\begin{cases} \bar w_j\big(\partial\big(B_R^+\cap\{0\leqslant\eta(\Phi(x))<1\}\big)\big)\subset\mathcal{M}\\ \mathtt{tr}_{\Gamma_R}(\bar w_j)=\mathtt{tr}_{\Gamma_R}(g_j)\\ \mathtt{tr}_{\partial^+B_R^+}(\bar w_j)=\mathtt{tr}_{\partial^+B_R^+}(u_j)\\ \mathtt{tr}_{\partial\{\eta(\Phi(x))=1\}}(\bar w_j)=\mathtt{tr}_{\partial\{\eta(\Phi(x))=1\}}(g_0),\\ \int_{B_R^+\cap\{0\leqslant\eta(\Phi(x))<1\}}|D\bar w_j|^{p_j(x)}\ \mathrm{d}x\lesssim\int_{B_R^+\cap\{0\leqslant\eta(\Phi(x))<1\}}|D\tilde u_j|^{p_j(x)}\ \mathrm{d}x\end{cases} \tag{4.28}$$

with constants implicit in '$\lesssim$' depending on $(N,\mathcal{M},\gamma_2)$. Finally, define

$$\tilde w_j(x):=\begin{cases}\tilde u_0(x) & \text{if } \ x\in B_R^+\cap\{\eta(\Phi(x))=1\}\\ \bar w_j(x) & \text{if } \ x\in B_R^+\cap\{0\leqslant\eta(\Phi(x))<1\}.\end{cases}$$

Now, note that the choices we made in (4.8) and (4.12) imply that

$$\frac{q_2}{p_0}\left(1+\frac{\tilde\sigma}{2}\right)=1+\left[\frac{q_2-p_0}{p_0}+\frac{q_2\tilde\sigma}{2p_0}\right]<1+\frac{\delta_g}{8}, \tag{4.29}$$

so by Lemma 3.4, (4.29), (4.14) and the minimality of $\tilde u_0$ in class $\hat{\mathcal{C}}^{p_0}_{u_0}(B_R^+,\mathcal{M})$, we get

$$\begin{aligned}\int_{B_R^+\cap\{0\leqslant\eta(\Phi(x))<1\}}|D\tilde u_0|^{p_j(x)}\ \mathrm{d}x&\leqslant c|B_R^+\cap\{0\leqslant\eta(\Phi(x))<1\}|+c\int_{B_R^+}|D\tilde u_0|^{q_2\left(1+\frac{\tilde\sigma}{2}\right)}\ \mathrm{d}x\\ &\leqslant c|B_R^+\cap\{0\leqslant\eta(\Phi(x))<1\}|+c\int_{B_R^+}|Du_0|^{q_2\left(1+\frac{\tilde\sigma}{2}\right)}\ \mathrm{d}x<\infty,\end{aligned} \tag{4.30}$$

for $c\equiv c(\mathtt{data}_{p(\cdot)})$. In (4.30) we used, in particular, that

$$\int_{B_R^+}|D\tilde u_0|^{q_2\left(1+\frac{\tilde\sigma}{2}\right)}\ \mathrm{d}x\leqslant c\int_{B_R^+}|Du_0|^{q_2\left(1+\frac{\tilde\sigma}{2}\right)}\ \mathrm{d}x, \tag{4.31}$$

with $c\equiv c(n,N,\mathcal{M},\gamma_1,\gamma_2,\lambda,\Lambda)$, which follows by the minimality of $\tilde u_0$ in class $\hat{\mathcal{C}}^{p_0}_{u_0}(B_R^+,\mathcal{M})$ and Lemma 3.4. Via (4.24), (4.25) and a straightforward change of variables we have

$$\begin{aligned}\big|\ B_R^+\cap\{0\leqslant\eta(\Phi(x))<1\}\ \big|&=\int_{B_R^+}\mathbb{1}_{\{0\leqslant\eta(\Phi(x))<1\}}\ \mathrm{d}x\\ &\leqslant\int_{B_R^+\cap\{\Phi^{-1}(\bar B_R\setminus B_{(1-\theta)R})\}}\mathrm{d}x\leqslant\int_{B_R\setminus\bar B_{(1-\theta)R}}|\mathcal{J}_\Phi(x)|^{-1}\ \mathrm{d}x\\ &\leqslant c(n)|\bar B_R\setminus\bar B_{(1-\theta)R}|\to0\quad\text{as}\quad\theta\to0.\end{aligned} \tag{4.32}$$

We then estimate

$$\begin{aligned}\mathcal{E}_j(u_j,B_R^+) \leqslant & \mathcal{E}_j(\tilde{w}_j,B_R^+)\\ \leqslant & \mathcal{E}_j(\tilde{w}_j,B_R^+\cap\{0\leqslant\eta(\Phi(x))<1\})+\mathcal{E}_j(\tilde{u}_0,B_R^+\cap\{\eta(\Phi(x))=1\})\\ =: & (\mathrm{I})_j+(\mathrm{II})_j.\end{aligned} \tag{4.33}$$

In the previous display, we used that, in view of $(4.28)_{2,3}$, $\tilde{w}_j$ is a legitimate comparison map to $u_j$. The bounds in (4.30), $(4.28)_4$ and (4.32) then legalize the following estimate:

$$\begin{aligned}(\mathrm{I})_j \leqslant & c\int_{B_R^+\cap\{0\leqslant\eta(\Phi(x))<1\}}\left[|D\tilde{u}_0|^{p_j(x)}+|Du_j-Du_0|^{p_j(x)}+\left|\frac{u_j-u_0}{R\theta}\right|^{p_j(x)}\right]\,\mathrm{d}x\\ \leqslant & c\int_{B_R^+\cap\{0\leqslant\eta(\Phi(x))<1\}}|D\tilde{u}_0|^{p_j(x)}\,\mathrm{d}x+c\int_{B_R^+\cap\{0\leqslant\eta(\Phi(x))<1\}}\left[|Du_j|^{p_j(x)}+|Du_0|^{p_j(x)}\right]\,\mathrm{d}x\\ &+c\int_{B_R^+\cap\{0\leqslant\eta(\Phi(x))<1\}}\left|\frac{u_j-u_0}{R\theta}\right|^{p_j(x)}\,\mathrm{d}x=:c\left[(\mathrm{I})_j^1+(\mathrm{I})_j^2+(\mathrm{I})_j^3\right]\end{aligned}$$

where $c\equiv c(N,\mathcal{M},\gamma_1,\gamma_2)$. Let us bound the three terms appearing on the right-hand side of the above inequality. By Lemma 2.2 (i) with $\varepsilon_0=\frac{\tilde{\sigma}}{2}$, (4.31), (4.7), (3.19), (4.32), $(4.2)_3$ and the absolute continuity of Lebesgue's integral we have

$$\begin{aligned}(\mathrm{I})_j^1 \leqslant & c\int_{B_R^+\cap\{0\leqslant\eta(\Phi(x))<1\}}\left[|D\tilde{u}_0|^{p_j(x)}-|D\tilde{u}_0|^{p_0}\right]\,\mathrm{d}x+c\int_{B_R^+\cap\{0\leqslant\eta(\Phi(x))<1\}}|D\tilde{u}_0|^{p_0}\,\mathrm{d}x\\ \leqslant & c\|p_j-p_0\|_{L^\infty(B_1^+)}\int_{B_R^+\cap\{0\leqslant\eta(\Phi(x))<1\}}|D\tilde{u}_0|^{q_2\left(1+\frac{\tilde{\sigma}}{2}\right)}\,\mathrm{d}x+o(\theta)\\ \leqslant & c\|p_j-p_0\|_{L^\infty(B_1^+)}\int_{B_R^+}|Du_0|^{q_2\left(1+\frac{\tilde{\sigma}}{2}\right)}\,\mathrm{d}x+o(\theta)=o(j)+o(\theta),\end{aligned}$$

with $c\equiv c(\mathtt{data}_{p(\cdot)})$. By (4.7), (4.14), (4.15), (4.32) we get that

$$\begin{aligned}(\mathrm{I})_j^2 &\leqslant o(\theta)+\int_{B_R^+\cap\{0\leqslant\eta(\Phi(x))<1\}}\left[|Du_j|^{q_2}+|Du_0|^{q_2}\right]\,\mathrm{d}x\\ &\leqslant o(\theta)+c(\mathtt{data}_{p(\cdot)})|B_R^+\cap\{0\leqslant\eta(\Phi(x))<1\}|^{\frac{\tilde{\sigma}}{1+\tilde{\sigma}}}\leqslant o(\theta).\end{aligned}$$

Moreover, using (4.16), Hölder inequality and (4.32) we have

$$\begin{aligned}(\mathrm{I})_j^3 \leqslant & |B_R^+\cap\{0\leqslant\eta(\Phi(x))<1\}|+\int_{B_R^+\cap\{0\leqslant\eta(\Phi(x))<1\}}\left|\frac{u_j-u_0}{r\theta}\right|^{q_2}\,\mathrm{d}x\\ \leqslant & o(\theta)+(R\theta)^{-q_2}|B_R^+\cap\{0\leqslant\eta(\Phi(x))<1\}|^{\frac{\tilde{\sigma}}{1+\tilde{\sigma}}}\|u_j-u_0\|^{q_2}_{L^{q_2\left(1+\frac{\tilde{\sigma}}{2}\right)}(B_R^+)}\\ \leqslant & o(\theta)+(R\theta)^{-q_2}o(j),\end{aligned}$$

and, trivially,

$$(\mathrm{II})_j\leqslant\mathcal{E}_j(\tilde{u}_0,B_R^+).$$

Finally, by $(4.1)_3$, $(4.2)_3$, (4.30) and (4.31) we get

$$\begin{aligned}|\mathcal{E}_j(\tilde u_0, B_R^+) - \mathcal{E}_0(\tilde u_0, B_R^+)| \\ &\leqslant \left[\|k_j - k_0\|_{L^\infty(B_1^+)} + \|p_j - p_0\|_{L^\infty(B_1^+)}\right]\left(1 + \int_{B_R^+} |Du_0|^{q_2\left(1+\frac{\tilde\sigma}{2}\right)}\,\mathrm{d}x\right) \\ &\leqslant c(\mathtt{data}_{p(\cdot)})\left[\|k_j - k_0\|_{L^\infty(B_1^+)} + \|p_j - p_0\|_{L^\infty(B_1^+)}\right] = o(j).\end{aligned}$$

Plugging the content of all the previous estimates in (4.33) we end up with

$$\mathcal{E}_j(u_j, B_R^+) \leqslant \mathcal{E}_0(\tilde u_0, B_R^+) + o(j) + o(\theta) + (R\theta)^{-q_2} o(j).$$

By (4.18) we can take the liminf as $j \to \infty$ in the above display to obtain

$$\begin{aligned}\mathcal{E}_0(u_0, B_R^+) &\leqslant \liminf_{j\to\infty} \mathcal{E}_j(u_j, B_R^+) \\ &\leqslant \limsup_{j\to\infty} \left[\mathcal{E}_0(\tilde u_0, B_R^+) + o(j) + o(\theta) + (R\theta)^{-q_2} o(j)\right] \\ &\leqslant \mathcal{E}_0(\tilde u_0, B_R^+) + o(\theta). \end{aligned} \tag{4.34}$$

Sending $\theta \to 0$ in (4.34) and using the minimality of $\tilde u_0$ in class $\hat{\mathcal{C}}^{p_0}_{u_0}(B_R^+, \mathcal{M})$, we end up with

$$\mathcal{E}_0(u_0, B_R^+) \leqslant \mathcal{E}_0(\tilde u_0, B_R^+) \leqslant \mathcal{E}_0(w, B_R^+)$$

for all $w \in \hat{\mathcal{C}}^{p_0}_{u_0}(B_R^+, \mathcal{M})$. Therefore, by Definition 3 and (4.17), the minimality of $u_0$ in class $\mathcal{C}^{p_0}_{g_0}(B_R^+, \mathcal{M})$ is proved. Finally, combining (4.34) with the minimality of $\tilde u_0$ in class $\hat{\mathcal{C}}^{p_0}_{u_0}(B_R^+, \mathcal{M})$, we can conclude that $\mathcal{E}_j(u_j, B_R^+) \to \mathcal{E}_0(u_0, B_R^+)$.

*Step* 3. *Singular points.* Let $\{x_j\} \subset \bar B_1^+$ be the sequence of singular points in the statement. The interior case $x_0 \in B_1^+$ has already been analyzed in [**10**, Section 4.1], so we can assume that $x_0 \in \Gamma_1$. Up to choose $j \in \mathbb{N}$ sufficiently large and then relabel, we can also suppose that $\{x_j\} \subset B_R^+$ for some $R \in (0, \frac{R_*}{4})$, $x_0 \in \Gamma_R$ and (4.7)–(4.8) are in force. By Theorem 1.1, (2.7) and (4.4), we can find a radius $\tilde R > 0$ and a positive constant $\tilde\varepsilon$, both independent of $j \in \mathbb{N}$ so that if $x_j$ is a singular point of $u_j$, then

$$\left(\varrho^{p_{2,j}(\varrho)-n} \int_{B_\varrho^+(x_j)} (1 + |Du_j|^2)^{\frac{p_{2,j}(\varrho)}{2}}\,\mathrm{d}x\right)^{\frac{1}{p_{2,j}(\varrho)}} > \tilde\varepsilon > 0 \tag{4.35}$$

for all $\varrho \in (0, \frac14 \min\{\tilde R, R_* - R\})$, with $R_*$ as in (3.21). In the above display, we denoted $p_{2,j}(\varrho) := \sup_{x \in B_\varrho(x_j) \cap B_R^+} p_j(x)$. Set $\sigma' := \min\{\tilde\sigma, \frac{\alpha}{\gamma_2}\}$. By Lemma 2.2 (i) with $\varepsilon_0 = \frac{\sigma'}{2}$ and $(3.16)_2$, we estimate

$$\begin{aligned}&\left|\varrho^{p_{2,j}(\varrho)-n} \int_{B_\varrho^+(x_j)} \left[(1 + |Du_j|^2)^{\frac{p_{2,j}(\varrho)}{2}} - (1 + |Du_j|^2)^{\frac{p_j(x)}{2}}\right]\,\mathrm{d}x\right|^{\frac{1}{p_{2,j}(\varrho)}} \\ &\qquad \leqslant c\varrho^{1+\frac{\alpha}{\gamma_1}} \left(\fint_{B_\varrho^+(x_j)} (1 + |Du_j|^2)^{\frac{p_{2,j}(\varrho)}{2}\left(1+\frac{\sigma'}{2}\right)}\,\mathrm{d}x\right)^{\frac{1}{p_{2,j}(\varrho)}} \leqslant c\varrho^{-\frac{\sigma'}{2}+\frac{\alpha}{\gamma_2}} \to 0, \end{aligned}\tag{4.36}$$

for $c \equiv c(n, N, \mathcal{M}, \gamma_1, \gamma_2, q)$. By (4.35), (4.36), (4.4) and (3.14) we then get

$$\tilde\varepsilon < c\varrho^{-\frac{\sigma'}{2}+\frac{\alpha}{\gamma_2}} + c\left(\varrho^{p_{2,j}(\varrho)-n} \int_{B_\varrho^+(x_j)} (1 + |Du_j|^2)^{\frac{p_j(x)}{2}}\,\mathrm{d}x\right)^{\frac{1}{p_{2,j}(\varrho)}}$$

$$
\begin{aligned}
&\leqslant c\varrho^{-\frac{\sigma'}{2}+\frac{\alpha}{\gamma_2}}+c\varrho+c\varrho^{1-\frac{n}{p_{2,j}(\varrho)}}\left[\int_{B^+_{2\varrho}(x_j)}\left|\frac{u_j-g_j}{\varrho}\right|^{p_j(x)}\,\mathrm{d}x+\int_{B^+_{2\varrho}(x_j)}|Dg_j|^{p_j(x)}\,\mathrm{d}x\right]^{\frac{1}{p_{2,j}(\varrho)}}\\
&\leqslant c\varrho^{-\frac{\sigma'}{2}+\frac{\alpha}{\gamma_2}}+c\varrho+c\varrho^{1-\frac{n}{q}}\left(\int_{B^+_{\varrho}(x_j)}|Dg_j|^{q}\,\mathrm{d}x\right)^{\frac{1}{q}}\\
&\quad+c\varrho^{1-\frac{n}{p_{2,j}(\varrho)}}\left[\int_{B^+_{2\varrho}(x_j)}\left|\frac{u_j-u_0}{\varrho}\right|^{q_2\left(1+\frac{\tilde\sigma}{2}\right)}\,\mathrm{d}x+\int_{B^+_{2\varrho}(x_j)}\left|\frac{g_j-u_0}{\varrho}\right|^{p_j(x)}\,\mathrm{d}x\right]^{\frac{1}{p_{2,j}(\varrho)}}\\
&\leqslant c\varrho^{\sigma''}+c\left[\fint_{B^+_{2\varrho}(x_j)}|u_j-u_0|^{q_2\left(1+\frac{\tilde\sigma}{2}\right)}\,\mathrm{d}x+\fint_{B^+_{2\varrho}(x_j)}|g_j-u_0|^{p_j(x)}\,\mathrm{d}x\right]^{\frac{1}{p_{2,j}(\varrho)}},
\end{aligned}
\tag{4.37}
$$

where we set $\sigma'':=\min\{1-\frac{n}{q},\frac{\alpha}{\gamma_2}-\frac{\sigma'}{2}\}>0$ and $c\equiv c(n,N,\mathcal{M},\gamma_1,\gamma_2,q)$. By (4.16) we get

$$
\fint_{B^+_{2\varrho}(x_j)}|u_j-u_0|^{q_2\left(1+\frac{\tilde\sigma}{2}\right)}\,\mathrm{d}x\to 0\quad\text{as}\quad j\to\infty. \tag{4.38}
$$

Since $g_j\rightharpoonup g_0$ weakly in $W^{1,q}(\bar B^+_1,\mathcal{M})$, then by the Rellich–Kondrachov theorem there holds that, up to subsequences, $g_j\to g_0$ strongly in $L^q(\bar B^+_1,\mathcal{M})$ and pointwise a.e., therefore, keeping also $(4.2)_3$ in mind, we can apply dominated convergence theorem to end up with

$$
\fint_{B^+_{2\varrho}(x_j)}|g_j-u_0|^{p_j(x)}\,\mathrm{d}x\to\fint_{B^+_{2\varrho}(x_j)}|g_0-u_0|^{p_0}\,\mathrm{d}x\quad\text{as}\quad j\to\infty. \tag{4.39}
$$

By $(4.2)_3$, (4.38) and (4.39) we can take the limit superior with respect to $j\in\mathbb{N}$ on both sides of the inequality in (4.37) to obtain

$$
\tilde\varepsilon\leqslant c\varrho^{\sigma''}+c\left(\fint_{B^+_{2\varrho}(x_j)}|g_0-u_0|^{p_0}\,\mathrm{d}x\right)^{\frac{1}{p_0}}. \tag{4.40}
$$

We finally pass to the limit superior for $\varrho\to\infty$ in (4.40) and have

$$
0<\bar\varepsilon^{p_0}\leqslant\limsup_{\varrho\to 0}\fint_{B^+_{2\varrho}(x_j)}|u_0-g_0|^{p_0}\,\mathrm{d}x,
$$

meaning that $x_0$ is a singular point for $u_0$. In the previous display, we set $\bar\varepsilon:=\tilde\varepsilon/c$. □

The next lemma is a monotonicity formula in the spirit of [**10, 23, 52, 54**].

Lemma 4.2. *Under assumptions* (2.3), (2.4), (2.6) *and* (2.7), *let* $u\in W^{1,p(\cdot)}(B^+_1,\mathcal{M})$ *be a solution of problem* (3.1). *Suppose also that*

$$
k(0)=1. \tag{4.41}
$$

*Then, for all* $\kappa\in(0,1-\frac{n}{q})$, *there exist* $\Upsilon\equiv\Upsilon(n,N,\mathcal{M},\gamma_1,\gamma_2,q)\in(0,1]$ *and a threshold* $T\equiv T(\mathtt{data},\kappa)\in(0,1]$ *such that if*

$$
[g]_{0,1-\frac{n}{q};\bar B^+_1}<\Upsilon, \tag{4.42}
$$

*then the map* $\Phi\colon(0,\frac{T}{4})\to[0,\infty)$ *defined as*

$$
\Phi(\tau):=\exp\left(\frac{\tilde c}{\beta''}\tau^{\beta''}\right)\left[\tau^{p_2(\tau)-n}\int_{B^+_\tau}k(x)|D\tilde u|^{p_2(\tau)}\,\mathrm{d}x+c\frac{\tau^\kappa}{\kappa}\right], \tag{4.43}
$$

*with* $\tilde{u}$ *as in* (3.26), $\beta'' \equiv \beta''(n,q,\nu)$ *and* $c, \tilde{c} \equiv c, \tilde{c}(\texttt{data}, \|Dg\|_{L^q(B_1^+)}, \kappa)$, *is monotone non-decreasing. Moreover, the following inequality holds true:*

$$\begin{aligned}&\int_{\partial B_1^+} |u(Rx) - u(\varrho x)|^{p_2(\varrho)} \ \mathrm{d}\mathcal{H}^{n-1}(x)\\ &\qquad \leqslant c \log(R/\varrho)^{p_2(\varrho)-1} \left[\varrho^{p_2(\varrho)-p_2(R)}(\Phi(R) - \Phi(\varrho))\right] + c(R-\varrho)^{\gamma_1\left(1-\frac{n}{q}\right)}, \end{aligned} \tag{4.44}$$

*for* $c \equiv c(\texttt{data}, \|Dg\|_{L^q(B_1^+)}, \kappa)$.

*Proof.* Let $u \in W^{1,p(\cdot)}(B_1^+, \mathcal{M})$ be a solution of problem (3.1), $\kappa \in (0,1)$ be a fixed constant and select $T \in (0,1]$ so that

$$0 < T \leqslant \min\left\{R_*, \frac{1-\kappa}{32[p]_{0,1}}, \left(\frac{\lambda}{4[k]_{0,\nu}}\right)^{\frac{2}{\nu}}\right\},$$

where $R_*$ is as in (3.21). Such a position assures that, whenever $\tau \in (0, \frac{T}{4}]$, (3.23)-(3.25) hold with $R$ replaced by $\tau$, moreover,

$$p_2(4\tau) - p_1(\tau) \leqslant \frac{1-\kappa}{2} \quad \text{and} \quad 4[k]_{0,\nu}\tau^{\frac{\nu}{2}} \leqslant \lambda, \tag{4.45}$$

with $\nu$ as in $(2.3)_1$. For $\tau \in (0, \frac{T}{4}]$, we introduce the functional

$$W^{1,p_2(\tau)}(B_\tau^+, \mathcal{M}) \ni w \mapsto \mathcal{E}_\tau(w, B_\tau^+) := \int_{B_\tau^+} k(x)|Dw|^{p_2(\tau)} \ \mathrm{d}x$$

and let $v \in W^{1,p_2(\tau)}(B_\tau^+, \mathcal{M})$ be a solution of problem

$$\hat{\mathcal{C}}_u^{p_2(\tau)}(B_\tau^+, \mathcal{M}) \ni w \mapsto \min \mathcal{E}_\tau(w, B_\tau^+). \tag{4.46}$$

By the minimality of $v$ in class $\hat{\mathcal{C}}_u^{p_2(\tau)}(B_\tau^+, \mathcal{M})$ and that of $u$ in class $\mathcal{C}_g^{p(\cdot)}(B_1^+, \mathcal{M})$ we bound

$$\begin{aligned}|\mathcal{E}_\tau(u, B_\tau^+) - \mathcal{E}_\tau(v, B_\tau^+)| &= \mathcal{E}_\tau(u, B_\tau^+) - \mathcal{E}_\tau(v, B_\tau^+)\\ &\leqslant |\mathcal{E}_\tau(u, B_\tau^+) - \mathcal{E}(u, B_\tau^+)| + |\mathcal{E}_\tau(v, B_\tau^+) - \mathcal{E}(v, B_\tau^+)| =: (\mathrm{I}) + (\mathrm{II}).\end{aligned}$$

Let

$$\sigma'' := \frac{1}{4}\min\left\{\sigma_g, \delta_g, \frac{n-\gamma_2}{\gamma_2}, \frac{1-\kappa}{2\gamma_2}\right\}, \tag{4.47}$$

where $\sigma_g$ and $\delta_g$ are the higher integrability threshold from Lemmas 3.2–3.4, respectively. Combining $(2.2)_1$, $(2.3)_1$, Lemma 3.2, Lemma 2.2 (i) with $\varepsilon_0 = \sigma''$ and $(3.16)_2$ we end up with

$$(\mathrm{I}) \leqslant c\tau \int_{B_\tau^+} (1 + |Du|^2)^{\frac{p_2(\tau)(1+\sigma'')}{2}} \ \mathrm{d}x \leqslant c\tau^{1+n-p_2(4\tau)(1+\sigma'')},$$

for $c \equiv c(\texttt{data}_{p(\cdot)}, \|Dg\|_{L^q(B_1^+)})$. In a totally similar way, using this time Lemma 3.4, $(3.16)_2$ and Lemma 2.2 (ii) with $\varepsilon_0 = \sigma''$ we get

$$\begin{aligned}(\mathrm{II}) &\leqslant c\tau \int_{B_\tau^+} (1 + |Dv|^2)^{\frac{p_2(\tau)(1+\sigma'')}{2}} \ \mathrm{d}x\\ &\leqslant c\tau \int_{B_\tau^+} (1 + |Dv|^2)^{\frac{p_2(\tau)(1+\sigma'')}{2}} \ \mathrm{d}x \leqslant c\tau^{1+n-p_2(4\tau)(1+\sigma'')},\end{aligned}$$

with $c \equiv c(\texttt{data}_{p(\cdot)}, \|Dg\|_{L^q(B_1^+)})$. Merging the content of the previous displays we obtain

$$\mathcal{E}_\tau(u, B_\tau^+) \leqslant \mathcal{E}_\tau(v, B_\tau^+) + c\tau^{1+n-p_2(4\tau)(1+\sigma'')}. \tag{4.48}$$

Now, for $\tau$ as above, define $x_\tau := \tau \frac{x}{|x|}$. As in [**52**, Lemma 1.3] we consider the following comparison map:

$$w_\tau(x) := \begin{cases} u(x) & \text{if } \ x \in B_1^+ \setminus B_\tau^+ \\ \tilde{u}(x_\tau) + g(x) & \text{if } \ x \in B_\tau^+, \end{cases}$$

where $\tilde{u}$ is defined in (3.26). Note that, by (3.24)–(3.25) there holds that

$$w_\tau \in u + W_0^{1,p(\cdot)}(B_\tau^+, \mathbb{R}^N) \quad \text{and} \quad w_\tau \in W^{1,p_2(\tau)}(B_\tau^+, \mathbb{R}^N). \tag{4.49}$$

Moreover, since (2.11) and (4.42) are in force, we see that

$$\operatorname{dist}(w_\tau, \mathcal{M}) \leqslant c(n, q, \beta_0)\Upsilon\tau^{\beta_0} \quad \text{with } \ \beta_0 := 1 - \frac{n}{q},$$

therefore, choosing $\Upsilon$ small enough, and thus determining the dependency $\Upsilon \equiv \Upsilon(n, N, \mathcal{M}, \gamma_1, \gamma_2, q)$, we can project $w_\tau$ onto $\mathcal{M}$ thus obtaining a map $\bar{w}_\tau := \Pi_{\mathcal{M}}(w_\tau)$ satisfying

$$\bar{w}_\tau \in \hat{\mathcal{C}}_u^{p_2(\tau)}(B_\tau^+, \mathcal{M}) \quad \text{and} \quad \int_{B_\tau^+} |D\bar{w}_\tau|^{p_2(\tau)} \ \mathrm{d}x \leqslant (1 + c\Upsilon\tau^{\beta_0}) \int_{B_\tau^+} |Dw_\tau|^{p_2(\tau)} \ \mathrm{d}x, \tag{4.50}$$

for $c \equiv c(n, N, \mathcal{M}, \gamma_1, \gamma_2, q)$. Note that by the mean value theorem applied to the function $[0, \infty) \ni s \mapsto (t + s)^{p_2(\tau)}$ there holds that

$$(|D\tilde{u}| + |Dg|)^{p_2(\tau)} \leqslant |D\tilde{u}|^{p_2(\tau)} + p_2(\tau)(|D\tilde{u}| + |Dg|)^{p_2(\tau)-1}|Dg|, \tag{4.51}$$

so by Hölder inequality with conjugate exponents $(\frac{p_2(\tau)}{p_2(\tau)-1}, p_2(\tau))$, (4.42) and (4.50) we get

$$\begin{aligned} \int_{B_\tau^+} |Dw_\tau|^{p_2(\tau)} \ \mathrm{d}x \leqslant & \int_{B_\tau^+} (|D\tilde{u}(x_\tau)| + |Dg|)^{p_2(\tau)} \ \mathrm{d}x \\ \leqslant & (1 + c\tau^{\beta_0}) \int_{B_\tau^+} |D\tilde{u}(x_\tau)|^{p_2(\tau)} \ \mathrm{d}x \\ & + c\left[\tau^{-\beta_0(p_2(\tau)-1)} \int_{B_\tau^+} |Dg|^{p_2(\tau)} \ \mathrm{d}x + \int_{B_\tau^+} |Dg|^{p_2(\tau)} \ \mathrm{d}x\right] \\ \leqslant & (1 + c\tau^{\beta_0}) \int_{B_\tau^+} |D\tilde{u}(x_\tau)|^{p_2(\tau)} \ \mathrm{d}x \\ & + c\left[\tau^{-\beta_0(p_2(\tau)-1)+n\left(1-\frac{p_2(\tau)}{q}\right)} + \tau^{n\left(1-\frac{p_2(\tau)}{q}\right)}\right] \|Dg\|_{L^q(B_1^+)} \\ \leqslant & (1 + c\tau^{\beta_0}) \int_{B_\tau^+} |D\tilde{u}(x_\tau)|^{p_2(\tau)} \ \mathrm{d}x + c\tau^{n\left(1-\frac{1}{q}\right)+1-p_2(\tau)} \end{aligned} \tag{4.52}$$

for $c \equiv c(n, \gamma_1, \gamma_2, q, \|Dg\|_{L^q(B_1^+)})$. In the previous expression, we also used the original value of $\beta_0$. By (2.3), (4.41) and (4.50) we can refine (4.52) as

$$\begin{aligned} \int_{B_\tau^+} k(x)|D\bar{w}_\tau|^{p_2(\tau)} \ \mathrm{d}x \leqslant & (1 + 4[k]_{0,\nu}\tau^\nu) \int_{B_\tau^+} |D\bar{w}_\tau|^{p_2(\tau)} \ \mathrm{d}x \\ \leqslant & (1 + c\tau^{\beta'}) \int_{B_\tau^+} |D\tilde{u}(x_\tau)|^{p_2(\tau)} \ \mathrm{d}x + c\tau^{n\left(1-\frac{1}{q}\right)+1-p_2(\tau)}, \end{aligned} \tag{4.53}$$

where $\beta' := \min\{\beta_0, \nu\}$ and $c \equiv c(n, N, \mathcal{M}, \gamma_1, \gamma_2, q, [k]_{0,\nu}, \|Dg\|_{L^q(B_1^+)})$. Let us evaluate the $p_2(\tau)$-energy of $\tilde{u}$. First, recall that if $\frac{\partial \tilde{u}}{\partial r} := D\tilde{u} \cdot \frac{x}{|x|}$ denotes the radial derivative of $\tilde{u}$, then

$$\left| \frac{\partial \tilde{u}}{\partial r} \right| \leqslant |D\tilde{u}|. \tag{4.54}$$

Moreover, if $p_2(\tau) \geqslant 2$ and $t \geqslant s \geqslant 0$ there holds that

$$(t-s)^{p_2(\tau)} \leqslant t^{p_2(\tau)} - s^{p_2(\tau)}. \tag{4.55}$$

A straightforward computation renders, for $x \in B_\tau^+$ that

$$|D\tilde{u}(x_\tau)|^2 = \frac{\tau^2}{|x|^2}\left[|D\tilde{u}(x_\tau)|^2 - \left| D\tilde{u}(x_\tau) \cdot \frac{x_\tau}{|x_\tau|} \right|^2\right],$$

so by (4.54), (4.55), area formula, (2.3), (4.41) and $(4.45)_2$

$$\begin{aligned}
\int_{B_\tau^+} |D\tilde{u}(x_\tau)|^{p_2(\tau)} \, \mathrm{d}x &= \frac{\tau}{n-p_2(\tau)} \int_{\partial B_\tau^+} \left[|D\tilde{u}(x)|^2 - \left| \frac{\partial \tilde{u}}{\partial r} \right|^2\right]^{\frac{p_2(\tau)}{2}} \mathrm{d}\mathcal{H}^{n-1}(x) \\
&\leqslant \frac{\tau}{n-p_2(\tau)} \left[\int_{\partial B_\tau^+} |D\tilde{u}(x)|^{p_2(\tau)} - \left| \frac{\partial \tilde{u}}{\partial r} \right|^{p_2(\tau)}\right] \mathrm{d}\mathcal{H}^{n-1}(x) \\
&\leqslant \frac{\tau}{n-p_2(\tau)} \left[(1+\tau^{\frac{\nu}{2}}) \int_{\partial B_\tau^+} k(x)|D\tilde{u}(x)|^{p_2(\tau)} \, \mathrm{d}\mathcal{H}^{n-1}(x) - \int_{\partial B_\tau^+} \left| \frac{\partial \tilde{u}}{\partial r} \right|^{p_2(\tau)} \mathrm{d}\mathcal{H}^{n-1}(x)\right].
\end{aligned} \tag{4.56}$$

Recalling the position made in (3.26), proceeding as in (4.51) and using Young inequality with conjugate exponents $(\frac{p_2(\tau)}{p_2(\tau)-1}, p_2(\tau))$, (4.42), (2.3), (4.48) and the minimality of $v$ in class $\hat{\mathcal{C}}_u^{p_2(\tau)}(B_\tau^+, \mathcal{M})$ with $(4.50)_1$ we have

$$\begin{aligned}
\mathcal{E}_\tau(\tilde{u}, B_\tau^+) &\leqslant (1+c\tau^{\beta_0})\mathcal{E}_\tau(u, B_\tau^+) + c\tau^{-\beta_0(p_2(\tau)-1)} \int_{B_\tau^+} |Dg|^{p_2(\tau)} \, \mathrm{d}x \\
&\leqslant (1+c\tau^{\beta_0})\mathcal{E}_\tau(v, B_\tau^+) + c\left[\tau^{n\left(1-\frac{1}{q}\right)+1-p_2(\tau)} + \tau^{n+1-p_2(4\tau)(1+\sigma'')}\right] \\
&\leqslant (1+c\tau^{\beta_0})\mathcal{E}_\tau(\bar{w}_\tau, B_\tau^+) + c\left[\tau^{n\left(1-\frac{1}{q}\right)+1-p_2(\tau)} + \tau^{n+1-p_2(4\tau)(1+\sigma'')}\right] \\
&\leqslant (1+c\tau^{\beta'}) \int_{B_\tau^+} |D\tilde{u}(x_\tau)|^{p_2(\tau)} \, \mathrm{d}x + c\left[\tau^{n\left(1-\frac{1}{q}\right)+1-p_2(\tau)} + \tau^{n+1-p_2(4\tau)(1+\sigma'')}\right].
\end{aligned} \tag{4.57}$$

with $c(\mathtt{data}_{p(\cdot)}, \|Dg\|_{L^q(B_1^+)})$. Merging (4.57) with (4.56) and using (4.54), (2.3) and $(4.45)_2$ we obtain

$$\begin{aligned}
\mathcal{E}_\tau(\tilde{u}, B_\tau^+) \leqslant &\frac{\tau}{n-p_2(\tau)} \left[(1+c\tau^{\beta''}) \int_{\partial B_\tau^+} k(x)|D\tilde{u}|^{p_2(\tau)} \, \mathrm{d}\mathcal{H}^{n-1}(x)\right. \\
&\left. - \int_{\partial B_\tau^+} \left| \frac{\partial \tilde{u}}{\partial r} \right|^{p_2(\tau)} \mathrm{d}\mathcal{H}^{n-1}(x) + c\tau^{\beta'}(\tau^{\frac{\nu}{2}}+1) \int_{\partial B_\tau^+} k(x)|D\tilde{u}|^{p_2(\tau)} \, \mathrm{d}\mathcal{H}^{n-1}(x)\right] \\
&+ c\left[\tau^{n\left(1-\frac{1}{q}\right)+1-p_2(\tau)} + \tau^{n+1-p_2(4\tau)(1+\sigma'')}\right]
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{\tau(1+c\tau^{\beta''})}{n-p_2(\tau)} \int_{\partial B_\tau^+} k(x)|D\tilde{u}|^{p_2(\tau)} \ \mathrm{d}\mathcal{H}^{n-1}(x) \\
&\quad - \frac{\tau}{n-p_2(\tau)} \int_{\partial B_\tau^+} \left| \frac{\partial \tilde{u}}{\partial r} \right|^{p_2(\tau)} \ \mathrm{d}\mathcal{H}^{n-1}(x) \\
&\quad + c\left[\tau^{n\left(1-\frac{1}{q}\right)+1-p_2(\tau)} + \tau^{n+1-p_2(4\tau)(1+\sigma'')}\right],
\end{aligned} \tag{4.58}
$$

with $\beta'' := \min\{\frac{\nu}{2}, \beta'\}$ and $c \equiv c(\mathtt{data}, \|Dg\|_{L^q(B_1^+)})$. To summarize, we got

$$
\begin{aligned}
\tau \int_{\partial B_\tau^+} k(x)|D\tilde{u}|^{p_2(\tau)} \ \mathrm{d}\mathcal{H}^{n-1}(x) &\geqslant \frac{n-p_2(\tau)}{1+c\tau^{\beta''}} \int_{B_\tau^+} k(x)|D\tilde{u}|^{p_2(\tau)} \ \mathrm{d}x \\
&\quad + \frac{\tau}{1+c\tau^{\beta''}} \int_{\partial B_\tau^+} \left| \frac{\partial \tilde{u}}{\partial r} \right|^{p_2(\tau)} \ \mathrm{d}\mathcal{H}^{n-1}(x) \\
&\quad - \frac{c(n-p_2(\tau))}{1+c\tau^{\beta''}} \left[\tau^{n\left(1-\frac{1}{q}\right)+1-p_2(\tau)} + \tau^{n+1-p_2(4\tau)(1+\sigma'')}\right]
\end{aligned} \tag{4.59}
$$

for $c \equiv c(\mathtt{data}, \|Dg\|_{L^q(B_1^+)}, \beta_0)$. Now, set

$$
\left(0, \frac{T}{4}\right) \ni \tau \mapsto \mathfrak{f}(\tau) := \tau^{p_2(\tau)-n} \int_{B_\tau^+} k(x)|D\tilde{u}|^{p_2(\tau)} \ \mathrm{d}x. \tag{4.60}
$$

Multiplying both sides of (4.59) by $\tau^{p_2(\tau)-n-1}$ and using $(4.45)_1$ and (4.47) we obtain

$$
\begin{aligned}
\tau^{p_2(\tau)-n} \int_{\partial B_\tau^+} k(x)|D\tilde{u}|^{p_2(\tau)} \ \mathrm{d}\mathcal{H}^{n-1}(x) \geqslant& \frac{n-p_2(\tau)}{1+c\tau^{\beta''}} \left[\tau^{-1}\mathfrak{f}(\tau) - c\left(\tau^{\kappa-1} + \tau^{-\frac{n}{q}}\right)\right] \\
&+ \frac{\tau^{p_2(\tau)-n}}{1+c\tau^{\beta''}} \int_{\partial B_\tau^+} \left| \frac{\partial \tilde{u}}{\partial r} \right|^{p_2(\tau)} \ \mathrm{d}\mathcal{H}^{n-1}(x)
\end{aligned} \tag{4.61}
$$

for $c \equiv c(\mathtt{data}, \|Dg\|_{L^q(B_1^+)}, \kappa)$. From (2.4) follows that $(0, \frac{T}{4}) \ni \tau \mapsto p_2(\tau)$ is differentiable with bounded, non-negative first derivative $0 \leqslant p'(\tau) \leqslant c(n, [p]_{0,1})$. We compute:

$$
\begin{aligned}
\mathfrak{f}'(\tau) =& (p_2(\tau)-n)\tau^{p_2(\tau)-n-1} \int_{B_\tau^+} k(x)|D\tilde{u}|^{p_2(\tau)} \ \mathrm{d}x \\
&+ \tau^{p_2(\tau)-n} \int_{\partial B_\tau^+} k(x)|D\tilde{u}|^{p_2(\tau)} \ \mathrm{d}\mathcal{H}^{n-1}(x) \\
&+ p_2'(\tau)\log(\tau)\tau^{p_2(\tau)-n} \int_{B_\tau^+} k(x)|D\tilde{u}|^{p_2(\tau)} \ \mathrm{d}x \\
&+ p_2'(\tau)\tau^{p_2(\tau)-n} \int_{B_\tau^+} k(x)\log(|D\tilde{u}|)|D\tilde{u}|^{p_2(\tau)} \ \mathrm{d}x.
\end{aligned}
$$

We record that, for all $\varepsilon_0 \in (0,1)$ there holds that

$$
|\log(t)| \leqslant c(\varepsilon_0)(1+t)t^{-\varepsilon_0} \quad \text{for any} \quad t > 0. \tag{4.62}
$$

Let us estimate the last two terms appearing in the expansion of $\mathfrak{f}'(\tau)$. Using (4.62) with $\varepsilon_0 = 1-\beta''$ we bound

$$
p_2'(\tau)\log(\tau)\tau^{p_2(\tau)-n} \int_{B_\tau^+} k(x)|D\tilde{u}|^{p_2(\tau)} \ \mathrm{d}x \leqslant cp_2'(\tau)\tau^{\beta''-1+p_2(\tau)-n} \int_{B_\tau^+} k(x)|D\tilde{u}|^{p_2(\tau)} \ \mathrm{d}x.
$$

By (4.62) with $\varepsilon_0 = 1 - p_2(\tau)\min\{\frac{\sigma_0}{2}, \frac{1-\kappa}{2\gamma_2}\}$, (keep (3.22)–(3.24) in mind) and $(3.16)_2$ we obtain

$$p_2'(\tau)\tau^{p_2(\tau)-n}\int_{B_\tau^+} k(x)\log(|D\tilde{u}|)|D\tilde{u}|^{p_2(\tau)}\ \mathrm{d}x \leqslant c\tau^{p_2(\tau)} \fint_{B_\tau^+}(1+|D\tilde{u}|)^{p_2(\tau)(1+\varepsilon_0)}\ \mathrm{d}x$$
$$\leqslant c\tau^{p_2(\tau)-p_2(4\tau)(1+\varepsilon_0)} \leqslant c\tau^{\kappa-1},$$

for $c \equiv c(\mathtt{data}_{p(\cdot)}, \|Dg\|_{L^1(B_1^+)}, \kappa)$. All in all, we got the following lower bound for $\mathfrak{f}'(\tau)$:

$$\begin{aligned}\mathfrak{f}'(\tau) \geqslant&(p_2(\tau)-n)\tau^{p_2(\tau)-n-1}\int_{B_\tau^+} k(x)|D\tilde{u}|^{p_2(\tau)}\ \mathrm{d}x\\ &+\tau^{p_2(\tau)-n}\int_{\partial B_\tau^+} k(x)|D\tilde{u}|^{p_2(\tau)}\ \mathrm{d}\mathcal{H}^{n-1}(x)\\ &-cp_2'(\tau)\tau^{\beta''-1+p_2(\tau)-n}\int_{B_\tau^+} k(x)|D\tilde{u}|^{p_2(\tau)}\ \mathrm{d}x - c\tau^{\kappa-1}\\ =&\tau^{p_2(\tau)-n}\int_{\partial B_\tau^+} k(x)|D\tilde{u}|^{p_2(\tau)}\ \mathrm{d}\mathcal{H}^{n-1}(x)\\ &+\left(p_2(\tau)-n-cp_2'(\tau)\tau^{\beta''}\right)\frac{\mathfrak{f}(\tau)}{\tau} - c\tau^{\kappa-1},\end{aligned} \tag{4.63}$$

with $c \equiv c(\mathtt{data}, \|Dg\|_{L^q(B_1^+)}, \kappa)$. Set

$$\varphi(\tau) := n - p_2(\tau) + cp_2'(\tau)\tau^{\beta''}.$$

Merging (4.61) and (4.63) we obtain

$$\begin{aligned}&\mathfrak{f}'(\tau) + \left(\varphi(\tau) - \frac{n-p_2(\tau)}{1+c\tau^{\beta''}}\right)\frac{\mathfrak{f}(\tau)}{\tau}\\ &\geqslant \frac{\tau^{p_2(\tau)-n}}{1+c\tau^{\beta''}}\int_{\partial B_\tau^+}\left|\frac{\partial\tilde{u}}{\partial r}\right|^{p_2(\tau)}\ \mathrm{d}\mathcal{H}^{n-1}(x) - c\tau^{\kappa-1}\left[\frac{2(n-p_2(\tau))}{1+c\tau^{\beta''}}+1\right],\end{aligned}$$

where $c \equiv c(\mathtt{data}, \|Dg\|_{L^q(B_1^+)}, \kappa)$. In the previous display we also used that $\kappa \leqslant \beta_0$. It is easy to see that

$$\left|\varphi(\tau) - \frac{n-p_2(\tau)}{1+c\tau^{\beta''}}\right| \leqslant \tilde{c}(\mathtt{data}, \|Dg\|_{L^q(B_1^+)}, \kappa)\tau^{\beta''},$$

therefore we get

$$\mathfrak{f}'(\tau) + \tilde{c}\tau^{\beta''-1}\mathfrak{f}(\tau) + c\tau^{\kappa-1} \geqslant \frac{\tau^{p_2(\tau)-n}}{1+c\tau^{\beta''}}\int_{\partial B_\tau^+}\left|\frac{\partial\tilde{u}}{\partial r}\right|^{p_2(\tau)}\ \mathrm{d}\mathcal{H}^{n-1}(x). \tag{4.64}$$

Let $\Phi(\cdot)$ be the function defined in (4.43). Combining (4.64) and the fact that $\tau \in (0,1]$, we immediately see that

$$\begin{aligned}\Phi'(\tau) \geqslant& \exp\left\{\frac{\tilde{c}\tau^{\beta''}}{\beta''}\right\}\left[\tilde{c}\tau^{\beta''-1}\mathfrak{f}(\tau) + \mathfrak{f}'(\tau) + c\tau^{\kappa-1}\right]\\ \geqslant&\frac{\tau^{p_2(\tau)-n}}{1+c}\int_{\partial B_\tau^+}\left|\frac{\partial\tilde{u}}{\partial r}\right|^{p_2(\tau)}\ \mathrm{d}\mathcal{H}^{n-1}(x),\end{aligned}$$

with $c \equiv c(\mathtt{data}, \|Dg\|_{L^q(B_1^+)}, \kappa)$. At this stage, we integrate the inequality in the previous display over $\tau \in (\varrho, R)$ with $0 < \varrho < R \leqslant T \leqslant 1$ to get

$$\begin{aligned}\Phi(R) - \Phi(\varrho) \geqslant & \frac{1}{1+c} \int_{\varrho}^{R} \tau^{p_2(\tau)-n} \left( \int_{\partial B_\tau^+} \left| \frac{\partial \tilde{u}}{\partial r} \right|^{p_2(\tau)} \, \mathrm{d}\mathcal{H}^{n-1}(x) \right) \, \mathrm{d}\tau \\ \geqslant & \frac{\varrho^{p_2(R)-p_2(\varrho)}}{1+c} \int_{\varrho}^{R} \tau^{p_2(\varrho)-n} \left( \int_{\partial B_\tau^+} \left| \frac{\partial \tilde{u}}{\partial r} \right|^{p_2(\tau)} \, \mathrm{d}\mathcal{H}^{n-1}(x) \right) \, \mathrm{d}\tau. \end{aligned} \tag{4.65}$$

Once (4.65) is available, we can proceed exactly as in [**54**, Lemma 4.1] to end up with

$$\begin{aligned} & \int_{\partial B_1^+} |\tilde{u}(Rx) - \tilde{u}(\varrho x)|^{p_2(\varrho)} \, \mathrm{d}\mathcal{H}^{n-1}(x) \\ & \quad \leqslant \log(R/\varrho)^{p_2(\varrho)-1} \int_{\varrho}^{R} \tau^{p_2(\varrho)-n} \left( \int_{\partial B_\tau^+} \left| \frac{\partial \tilde{u}}{\partial r} \right|^{p_2(\varrho)} \, \mathrm{d}\mathcal{H}^{n-1}(x) \right) \, \mathrm{d}\tau \\ & \quad \leqslant \frac{1+c}{\varrho^{p_2(R)-p_2(\varrho)}} \log(R/\varrho)^{p_2(\varrho)-1} [\Phi(R) - \Phi(\varrho)], \end{aligned} \tag{4.66}$$

for $c \equiv c(\mathtt{data}, \|Dg\|_{L^q(B_1^+)}, \kappa)$. Finally, keeping in mind (2.7) and position (3.26) we bound via (4.66):

$$\begin{aligned} \int_{\partial B_1^+} |u(Rx) - u(\varrho x)|^{p_2(\varrho)} \, \mathrm{d}\mathcal{H}^{n-1}(x) \leqslant & \, c \int_{\partial B_1^+} |\tilde{u}(Rx) - \tilde{u}(\varrho x)|^{p_2(\varrho)} \, \mathrm{d}\mathcal{H}^{n-1}(x) \\ & + c \int_{\partial B_1^+} |g(Rx) - g(\varrho x)|^{p_2(\varrho)} \, \mathrm{d}\mathcal{H}^{n-1}(x) \\ \leqslant & c \log(R/\varrho)^{p_2(\varrho)-1} \left[ \varrho^{p_2(\varrho)-p_2(R)} (\Phi(R) - \Phi(\varrho)) \right] + c(R-\varrho)^{\gamma_1\left(1-\frac{n}{q}\right)}, \end{aligned}$$

with $c \equiv c(\mathtt{data}, \|Dg\|_{L^q(B_1^+)}, \kappa)$ and the proof is complete. □

Before going on, let us stress that, as in Section 3, we can reduce problem (1.2) to an equivalent one defined on the half-ball $B_1^+$. In fact, in the proof of Theorem 1.2 we shall consider $u \in W^{1,p(\cdot)}(B_1^+, \mathcal{M})$ solution to

$$\mathcal{C}_g^{p(\cdot)}(B_1^+, \mathcal{M}) \ni w \mapsto \int_{B_1^+} |Dw|^{p(x)} \, \mathrm{d}x, \tag{4.67}$$

with boundary datum $g(\cdot)$ as in (2.7) (of course $\bar{\Omega}$ is replaced by $\bar{B}_1^+$). Now we are ready to prove Theorem 1.2.

### 4.1. *Proof of Theorem* 1.2

As a consequence of Theorem 1.1, we know that $u \in C^{0,\beta_0}_{loc}(\bar{B}_1^+ \setminus \Sigma_0(u), \mathcal{M})$, for a closed, negligible set $\Sigma_0 \subset \bar{B}_1^+$. Let us prove that $\Sigma_0 \cap \partial B_1^+ = \emptyset$. By contradiction, assume that $x_0 \in \Gamma_1$ is a singular point for $u \in W^{1,p(\cdot)}(B_1^+, \mathcal{M})$, solution to (4.67). Up to translations, there is no loss of generality in assuming $x_0 = 0$. Now, for $j \in \mathbb{N}$, define the rescaled maps

$$u_j(x) := u(x/j), \qquad p_j(x) := p(x/j), \qquad k_j(x) := j^{p_j(x)-p(0)}, \qquad g_j(x) := g(x/j).$$

Since $u \in W^{1,p(\cdot)}(B_1^+,\mathcal{M})$ solves (4.67), we deduce that each $u_j \in W^{1,p_j(\cdot)}(B_j^+,\mathcal{M})$ solves problem

$$\mathcal{C}^{p_j(\cdot)}_{g_j}(B_j^+,\mathcal{M}) \ni w \mapsto \min \int_{B_j^+} k_j(x)|Dw|^{p_j(x)} \ \mathrm{d}x, \tag{4.68}$$

therefore it is easy to see that it also solves

$$\mathcal{C}^{p_j(\cdot)}_{g_j}(B_1^+,\mathcal{M}) \ni w \mapsto \min \int_{B_1^+} k_j(x)|Dw|^{p_j(x)} \ \mathrm{d}x. \tag{4.69}$$

Note that, whenever $x \in \bar{B}_1^+$, a straightforward computation shows that

$$\{p_j\} \text{ and } \{k_j\} \text{ are Lipschitz continuous uniformly on } j \in \mathbb{N} \text{ in } \bar{B}_1^+. \tag{4.70}$$

Again for $x \in \bar{B}_1^+$, recalling Morrey's embedding theorem we see that

$$\sup_{x\in\bar{B}_1^+} |p_j(x)-p(0)| \leqslant 4[p]_{0,1}|x/j| \leqslant 4[p]_{0,1}(1/j) \to 0 \tag{4.71}$$

$$\sup_{x\in\bar{B}_1^+} |k_j(x)-1| \leqslant \max\left\{\exp\left(\frac{4[p]_{0,1}\log(j)}{j}\right)-1, 1-\exp\left(\frac{-4[p]_{0,1}\log(j)}{j}\right)\right\} \to 0 \tag{4.72}$$

$$\sup_{x\in\bar{B}_1^+} |g_j(x)-g(0)| \leqslant 4[g]_{0,1-\frac{n}{q}}|x/j|^{1-\frac{n}{q}} \leqslant 4[g]_{0,1-\frac{n}{q}}(1/j)^{1-\frac{n}{q}} \to 0. \tag{4.73}$$

Furthermore, recalling (2.7) and (4.73) we see that

$$\int_{B_1^+} |Dg_j|^q \ \mathrm{d}x \leqslant j^{q-n}\|Dg\|^q_{L^q(B_1^+)} \ \mathrm{d}x \to 0,$$

so

$$g_j \to g(0) \quad \text{in} \ \ W^{1,q}(\bar{B}_1^+,\mathcal{M}). \tag{4.74}$$

Collecting (4.68) and (4.70)–(4.74) we see that the assumptions of Lemma 4.1 are satisfied in $B_1^+$, so in particular $u_j \rightharpoonup u_0$ weakly in $W^{1,(1+\tilde{\sigma})p(0)}_{loc}(B_1^+,\mathcal{M})$, $u_0$ is a solution of problem

$$\mathcal{C}^{p(0)}_{g(0)}(B_R^+,\mathcal{M}) \ni w \mapsto \min \int_{B_R^+} |Dw|^{p(0)} \ \mathrm{d}x \tag{4.75}$$

for any $R \in (0,1)$ and, since $x_0 = 0$ is a singular point of all the functions $u_j$, then it is also a singular point for $u_0$. We fix $0<\mu_1<\mu_2<1$ and let $j \in \mathbb{N}$ be so large that $j^{-1} < \frac{T}{4}$ with $T$ as in Lemma 4.2. Recalling also (1.3) (on $\bar{B}_1^+$ of course), we see that the assumptions of Lemma 4.2 are satisfied, we can apply (4.44) with $\varrho = \mu_1/j$ and $R = \mu_2/j$ to get

$$\begin{aligned}
&\int_{\partial B_1^+} |u_j(\mu_1 x)-u_j(\mu_2 x)|^{p_2(\mu_1/j)} \ \mathrm{d}\mathcal{H}^{n-1}(x) \\
&= \int_{\partial B_1^+} |u(j^{-1}\mu_1 x)-u(j^{-1}\mu_2 x)|^{p_2(\mu_1/j)} \ \mathrm{d}\mathcal{H}^{n-1}(x) \\
&\leqslant c\log(\mu_2/\mu_1)^{p_2(\mu_1/j)}(\Phi(\mu_2/j)-\Phi(\mu_1/j)) + cj^{-\gamma_1\left(1-\frac{n}{q}\right)}(\mu_2-\mu_1)^{\gamma_1\left(1-\frac{n}{q}\right)},
\end{aligned} \tag{4.76}$$

with $\Phi(\cdot)$ defined as in (4.43) with $k(\cdot) \equiv 1$. By Lemma 4.2, we deduce that

$$\lim_{j\to\infty} \Phi(\mu_1/j) = \lim_{j\to\infty} \Phi(\mu_2/j) = L \ \text{ for some finite } \ L \geqslant 0,$$

thus

$$c\log(\mu_2/\mu_1)^{p_2(\mu_1/j)}(\Phi(\mu_2/j)-\Phi(\mu_1/j))+cj^{-\gamma_1\left(1-\frac{n}{q}\right)}(\mu_2-\mu_1)^{\gamma_1\left(1-\frac{n}{q}\right)}\to 0. \tag{4.77}$$

Furthermore, in light of (4.3) we have that $u_j \to u_0$ almost everywhere in $B_1^+$, so recalling also (4.71) we get

$$|u_j(\mu_2 x)-u_j(\mu_1 x)|^{p_2(\mu_1/j)}\to |u_0(\mu_2 x)-u_0(\mu_1 x)|^{p(0)} \quad \text{for a.e. } \ x\in B_1^+. \tag{4.78}$$

Combining (4.78), $(2.6)_1$ and the dominated convergence theorem, we obtain

$$\begin{aligned}\lim_{j\to\infty}\int_{\partial B_1^+}&|u_j(\mu_2 x)-u_j(\mu_1 x)|^{p_2(\mu_1/j)}\ \mathrm{d}\mathcal{H}^{n-1}(x)\\ &=\int_{\partial B_1^+}|u_0(\mu_2 x)-u_0(\mu_1 x)|^{p(0)}\ \mathrm{d}\mathcal{H}^{n-1}(x).\end{aligned} \tag{4.79}$$

Inserting (4.79) and (4.77) in (4.76), we end up with

$$\int_{\partial B_1^+}|u_0(\mu_2 x)-u_0(\mu_1 x)|^{p(0)}\ \mathrm{d}\mathcal{H}^{n-1}(x)=0,$$

which in turn implies that $u_0$ is homogeneous of degree zero. Recalling that $u_0$ is a solution of (4.75), by [**31**, Theorem 5.7] we can conclude that $u_0$ is constant, so $x_0 = 0$ cannot be a singular point. This means that $\Sigma_0 \Subset B_1^+$ and the proof is complete.

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*Iwona Chlebicka*
*Faculty of Mathematics*
*Informatics and Mechanics University of Warsaw*
*ul. Banacha 2*
*Warsaw 02-097*
*Poland*

i.chlebicka@mimuw.edu.pl

*Cristiana De Filippis*
*Dipartimento di Matematica 'Giuseppe Peano'*
*Università di Torino*
*Via Carlo Alberto 10*
*Torino 10123*
*Italy*

cristiana.defilippis@unito.it

*Lukas Koch*
*Mathematical Institute*
*University of Oxford*
*Andrew Wiles Building, Radcliffe*
*Observatory Quarter, Woodstock Road*
*Oxford OX26GG*
*United Kingdom*

Lukas.Koch@maths.ox.ac.uk