Mathematical modeling of oxygen control in biocell composting plants

# Mathematical modeling of oxygen control in biocell composting plants

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**Abstract**

We propose an optimal control problem to determine the best aeration strategy for the aerobic biodegradation in a composting cell. The goal is to minimize the deviation of the oxygen level from its reference value for the entire duration of the biodegradation process. The mathematical model includes several chemical phenomena, like the aerobic biodegradation of the soluble substrate by means of a bacterial biomass, the hydrolysis of insoluble substrate and the biomass decay. The oxygen and the optimal mechanical aeration time profiles are obtained and discussed. Finally, the plant performance is evaluated in absence and presence of external aeration by means of several specific indices.

*Keywords:* Waste, composting, bioreactor, chemostat, optimal control

## 1. Introduction

Waste treatment and disposal processes are a very important and urgent issue, especially for local authorities that have to handle this problem in

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their area of responsibility [12, 13, 15]. In order to face such a problem some procedures and technologies have been proposed, overcoming the classical storage approach.

Presently, there are several techniques for the waste management [10, 15]. The traditional one is the storage in a containment vessel, where one tries to isolate the polluting agents as much as possible [20]. This is a good approach in the short term but it is not successful in the long term. As a matter of fact, the risk of soil and aquifers contamination increases over time due to several reasons like, for example, the formation and diffusion of leachate. Since it is a long time treatment method, additional vessels are often required in order to stock other waste [8].

In order to overcome these issues, in recent years a different approach has been proposed where the containment vessels are conceived as bioreactors, i.e. biologically active environments where degradation occurs by means of suitable chemical reactions [34]. Starting from the organic matter, i.e. a mixture of green, food and agricultural waste, biodegradable paper and plastic, bioreactors may produce fertilizer for agriculture activities, natural gases like methane and reduce the total amount of solid waste (and consequently increase the capacity of the vessel) [38].

It is known that degradation is a natural and spontaneous process but it can be improved by using the mechanical aeration or the internal moisture [14, 17, 27]. For example, to this purpose leachate can be used, since it is rich of bacteria that are able to force the biodegradation process [6, 44].

In order to validate the known methods and to get preliminary insights about the effectiveness of new procedures, it may be useful to have a mathematical modeling approach. In fact, theoretical analysis of models representing the main processes involved in waste treatment and disposals may produce realistic predictions and suggest the adoption of efficient procedures [21].

A well established research field concerns the formulation of mathematical models for the treatment of the organic matter. This kind of treatment involves both anaerobic and aerobic bacteria digestion [12, 13]. Anaerobic processes are usually involved in biogas production as well as in the treatment of polluting leachate [18, 26]. The aerobic digestion occurs in several composting techniques like aerated static pile composting, in-vessel composting and windrow composting [20]. The latter is a very recently used method which have a low energetic cost because it does not require a frequent mechanical aeration, although it does require a long time treatment [24].

As pointed out in [1, 43], a pioneering mathematical approach to biodegra-

dation processes is the paper by Jacques Monod in 1942 [33]. There, a basic model for batch microbial culture is proposed, where batch culture usually is used to describe a close system for the bacterial activity where the growth occurs in a fixed volume of nutrient and under specific environmental conditions.
Let us indicate by $X(t)$ and $S(t)$ the bacterial and substrate concentration, respectively, at time $t$. A very simple description of the growth [33] is given by

$$\mu(S) = \frac{S}{K_S + S}$$

where $K_S$ is the so called semi–saturation (positive) constant. Then, the basic growth equation for the bacterial growth is given by:

$$\dot{X} = \mu_m \, X \, \mu(S)$$

where the upper dot denotes the time derivative and $\mu_m$ is a positive constant representing the maximal growth rate. It is possible to show that, in presence of a single organic substrate, the two variables are related by the following equation

$$\dot{X} = -Y \, \dot{S}$$

where $Y$ is a positive constant yield rate.
Now, when modeling a continuous culture, i.e. an open system where additional nutrient is introduced constantly and a part of the biological material is expelled out of the culture, one has to consider these flows and the usual growth processes [43]. Therefore, the previous system of equations may be modified as follows

$$\begin{aligned} \dot{X} &= \mu_m X \, \mu(S) - \mathcal{D} X \\ \dot{S} &= \mathcal{D} S_{in} - \mathcal{D} S - \frac{\mu_m}{Y} X \, \mu(S), \end{aligned}$$

where the positive constants $\mathcal{D}$ and $S_{in}$ represent the input/output flow rate and the inlet substrate concentration, respectively.
A growth vessel where nutrient is continuously added while matter containing nutrients, products and microorganisms is removed at the same rate is usually called chemostat; the mathematical models for this kind of bioreactor have been extensively studied in mathematical literature [1, 28, 43].
These basic models can be considered as a starting point to create more

complex and realistic models of bacterial growth where some effects, like e.g. the production of biogas generated by suitable chemical reactions, may be incorporated.

Another important feature concerns the individuation of suitable decision policies to improve the performance of a waste deposit. From this point of view, the optimal control theory provides useful tools to find proper control strategies, which satisfy a given optimality criterion.

The use of optimal control theory has given useful insights in waste management problems [4, 5, 39], although to our knowledge applications to biodegradation in composting biocells has not been investigated, despite of its current interest and utility in the integrated systems [32].

In this paper, we introduce a mathematical model for composting in biocell and formulate an optimal control problem to determine the best aeration strategy for the degradation. As matter of fact, in absence of any external operation the oxygen in a closed system is consumed and part of the digestion process becomes anaerobic. Therefore, in order to guarantee the aerobic feature of the process, additional oxygen is required and its injection in the cell can be regulated to reach a satisfactory plant performance. In our control problem we seek to minimize the deviation of the oxygen level from its reference value for the entire duration of the biodegradation process.

The paper is organized as follows: after a presentation of the management techniques of the organic waste in Section 2, we will focus on the composting process in Section 3 and present a model for aerobic degradation in biocells. In Section 4 the optimal control problem will be formulated and analyzed. Some numerical simulations and the evaluation of the plant performance will be presented in Section 5.

## 2. Organic waste management

As mentioned before, in composting processes a part of the solid waste is made by the organic matter, that is a mixture of green, food and agricultural waste, biodegradable paper and plastic, and it can be treated separately in order to reduce the total amount of waste and to give some useful products, like biogas and compost.

The organic matter may undergo an anaerobic or aerobic digestion, that degrades it respectively in absence or in presence of oxygen by means of microorganisms.

In this section we will present both approaches and discuss some relevant papers concerning the mathematical modeling approach.

*2.1. Anaerobic digestion*

The anaerobic digestion process is the biodegradation of the organic matter induced by the action of microorganisms in absence of oxygen. The oxygen molecules are not required to activate the chemical process and sometimes they can be even toxic for specific microorganisms that can not survive to oxidation phenomena [30].
This chemical process occurs in the production of renewable energy in the form of burnable biogas, similar to methane but with a low calorific value [25]. An additional advantage of the anaerobic degradation concerns the possibility to treat all types of organic waste and no specific levels of temperature or moisture are required. On the other hand, the remaining material, called digestate, is a low quality product and it can not be used as fertilizer, since it requires an additional stage of treatment. Another disadvantage regards the quality of the leachate that can contain some pollutants and the relative high cost for its treatment [9].
From a mathematical point of view, in [31] a basic model describing the consumption of organic waste and the production of methane (methanogenesis) is proposed. By denoting by $S$ the substrate concentration, $X$ the biomass and $P$ the produced methane, the evolution of the system is governed by the dynamical system

$$\dot{S} = -\mu(S,X) \; , \; \dot{X} = -Y\dot{S} \, , \; \dot{P} = -\alpha\dot{S}$$

where $\mu$ is the Contois growth function [11] and the variations of biomass and methane are weighted respectively by the yield coefficient $Y$ and the positive methane production coefficient $\alpha$.
In [22, 23] a two substrates and two populations model is proposed; more precisely, biodegradable waste is used by bacteria forming acids, while substrate given by volatile fatty acids is consumed by bacteria forming methane. The bacterial growth is described by Monod-like functions.
An important phenomenon to be taken into account is the hydrolysis, that splits the substrate in simple molecules. In the previous models the substrate is immediately available for the biomass microorganisms, and this assumption is reasonable only if the hydrolytic process occurs in a very short time. Otherwise, it is necessary to distinguish the substrate according to the availability for short or long term biodegradation as shown in [46].

In [7] a biodegradation model is proposed for a single substrate, two different populations and product. In addition to Monod functions, the authors consider also the Haldane growth function, including inhibition due to high concentrations of the substrate.
In [47] the authors propose a space-dependent model for a batch reactor that describes several processes like hydrolysis, acidogenesis and acetogenesis/methanogenesis. Their model takes also into account the recirculation of leachate inside the bioreactor.

*2.2. Aerobic digestion*

In aerobic biodegradation processes the oxygen plays a fundamental role in activating and speeding up the biodegradation of waste. Moreover the aeration can be useful also in controlling the temperature. In fact, the chemical reactions increase the system temperature and consequently the substrate can overly dry. In presence of dry matter, the degradation can be decelerated or even inhibited and the bacterial activity stops [19, 45].
Unlike the anaerobic process, the aerobic digestion requires a sufficient level of moisture in order to be activated; such level can be guaranteed by the recirculation of leachate that does not contain pollutant materials. Moreover, the remaining product of the aerobic degradation is of high quality and it can be used as fertilizer in agriculture. The main disadvantage of the aerobic process is to have a good level of oxygen in the entire process; usually this requires some external operations, like mechanical aeration, and consequently a consumption of energy, and its relative cost, has to be considered [19, 45].
The mathematical approaches to aerobic biodegradation processes are far to be deeply explored in the literature. In particular, only few deterministic mathematical models have been proposed and the recent attention of scholars to this topic (see e.g [29, 41, 42]) clearly shows the need of further progresses in this direction.
Although the equations describing the evolution of aerobic processes are very similar to those of the anaerobic ones, an accurate description of the oxygen concentration variation has a crucial role, since the aerobic feature of the degradation process must be guaranteed. Moreover, the influence of the other transformation processes (like hydrolysis and bacteria death) on the oxygen consumption has to be analyzed, also to prevent low levels of oxygen and the inhibition of the aerobic phenomenon.
In [29] a mathematical model for an open vessel is introduced. Several processes are considered: the aeration in the bacterial growth process, the decay

rate to transform the biomass in insoluble substrate and the production of soluble substrate from the insoluble one. The state variables, given as concentrations, are: oxygen ($\Omega$), soluble substrate ($S$), insoluble substrate ($I$), aerobic biomass ($X$) and the liquid part ($L$).
The biomass growth function is given by

$$\mathcal{R}_G\left(\Omega, S, X\right) \; = \; \mu_m \frac{\Omega}{\Omega + K_\Omega \varepsilon_w} \frac{S}{S + K_S \varepsilon_w} X$$

where $\mu_m$ is the maximal growth rate, $\varepsilon_w$ is the water fraction, $K_S$ e $K_\Omega$ are the saturation constants. The evolution is governed by the following system of differential equations:

$$\begin{cases} \dot{\Omega} & = \dfrac{1}{Y_\Omega} \mathcal{R}_G\left(\Omega, S, X\right) \, f_1\left(T\right) \\ \dot{S} & = -\dfrac{1}{Y_S} \mathcal{R}_G\left(\Omega, S, X\right) \, f_1\left(T\right) + k_h I f_3\left(T\right) \\ \dot{X} & = \mathcal{R}_G\left(\Omega, S, X\right) \, f_1\left(T\right) - bX f_2\left(T\right) \\ \dot{I} & = -k_h I f_3\left(T\right) + \dfrac{1}{Y_I} bX f_2\left(T\right) \\ \dot{L} & = \left(-\dfrac{\psi}{Y_\Omega} + \dfrac{1}{Y_L}\right) \mathcal{R}_G\left(\Omega, S, X\right) \, f_1\left(T\right) - r_{H_2O} \end{cases} \tag{1}$$

where the quantities $Y$ are the yield coefficients, $k_h$ the transformation rate of the insoluble substrate, $b$ the biomass decay rate, $\psi$ is the biomass water content and $r_{H_2O}$ represents the rate of water loss from substrate. The last equation takes account of the presence of water in the biomass (first term) and its production due to oxidation (second term). The functions $f_i(T)$, $i = 1, 2, 3$ are some correction terms that describes the contribution of the temperature $T$ in the conversion processes.
In order to compute the amount of inert mass, i.e the stabilized matter that does not undergo the degradation process, the conservation of the total mass is assumed under the hypothesis of homogenous temperature.
In [29] it is shown that the dynamics is qualitatively consistent with the experimental results.
Similar approaches have been presented in [41, 42] for composting processes in windrow, while, to our knowledge, there is lack of applications to the degradation in biocells.

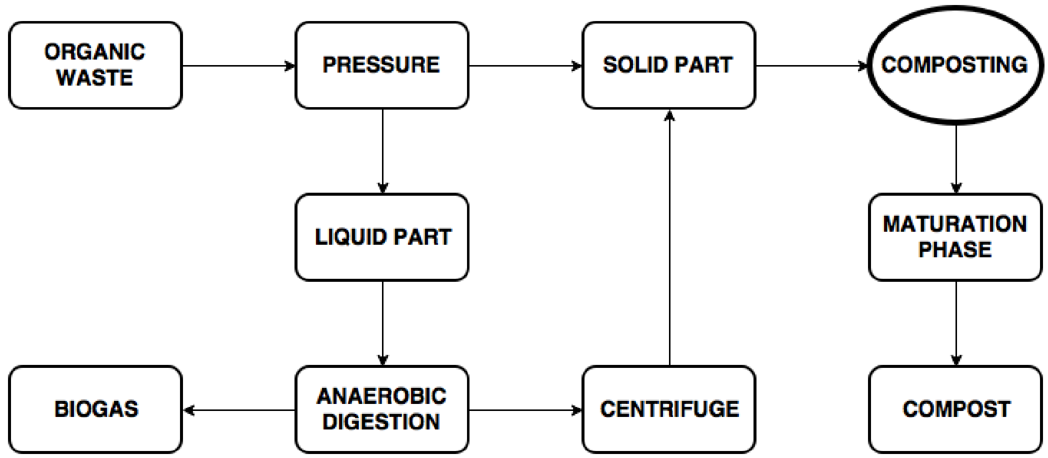


Figure 1: Scheme of the organic waste treatment in an integrated aerobic/anaerobic system. The liquid part is reduced by means of the anaerobic digestion giving biogas as product; the solid part is treated by an aerobic process giving the compost, after an additional maturation phase.

### *2.3. Integrated systems*

An alternative approach to waste treatment combines the anaerobic and aerobic digestion processes in order to catch the positive effects of both methods. In particular, such approach allows to use the waste as a source of energy, by producing the biogas, and as raw material for the composting process that gives a high quality product for agriculture [32].

As shown in Figure 1, the transformation process of the organic waste consists in several stages. More precisely, the waste is first pressed and mashed and the liquid part is separated by the solid one. The liquid is treated by an anaerobic digestion process while the solid part undergoes an aerobic process; the liquid part passes in a biodigester, i.e. a structure where the digestion of organic waste matter by bacteria takes place with the production of a burnable biogas. The solid part obtained at the end of the anaerobic process is added to the previous pressed solid part; the solid contributions are treated by means of an aerobic digestion and they are used mainly to produce a high quality compost. The product of this process is not still usable as fertilizer and it has to undergo an additional maturation phase out of the bioreactor to complete the transformation process [20].

In the next section we will concentrate on the composting phase in biocells and propose a possible mathematical formulation of the dynamics.

## 3. Composting in biocells

A mathematical idealization of the aerobic processes in biocells may be obtained by adapting the approach proposed in [29]. In particular, unlike static piles and windrows, a biocell is a closed system with no interaction with the atmosphere; consequently, the oxygen in the cell is set to be consumed and an external aeration operation is required. Such operation can be controlled and hence the oxygen level can be monitored, in order to guarantee the aerobic feature of the digestion process.

Let us denote by $S$ the concentration of soluble substrate, i.e. the waste available for the digestion process, by $I$ the concentration of insoluble substrate, i. e. the waste not yet available for the digestion process, by $X$ the concentration of biomass, i.e. the aerobic bacteria, by $L$ the concentration of the liquid part, by $M$ the concentration of inert material, i.e. the precompost, and finally by $\Omega$ the concentration of oxygen.

We take into account some chemical phenomena; in particular, we consider: (i) the aerobic biodegradation, where the soluble substrate is digested by aerobic bacteria using oxygen, the concentration of biomass increases and water and inert matter are produced; (ii) the hydrolysis, that transforms the insoluble substrate in the soluble one; (iii) the biomass decay, where the death of bacteria generates a part of insoluble substrate and a part of inert material.

The total (liquid and solid) mass is assumed to be conserved since the mass variation due to different levels of oxygen can be considered negligible.

The dynamics is governed by the following set of ordinary differential equations

$$
\begin{aligned}
\dot{S} &= -\frac{1}{Y_S}\,\mu\,\frac{S}{S+K_S}\,\frac{\Omega}{\Omega+K_\Omega}\,X \;+\; K_h I \\
\dot{I} &= -K_h I \;+\; \frac{1}{Y_I}\,bX \\
\dot{X} &= \mu\,\frac{S}{S+K_S}\,\frac{\Omega}{\Omega+K_\Omega}\,X \;-\; bX \\
\dot{L} &= \frac{1}{Y_L}\,\mu\,\frac{S}{S+K_S}\,\frac{\Omega}{\Omega+K_\Omega}\,X \\
\dot{M} &= -\left(1-\frac{1}{Y_S}+\frac{1}{Y_L}\right)\mu\,\frac{S}{S+K_S}\,\frac{\Omega}{\Omega+K_\Omega}\,X + \left(1-\frac{1}{Y_I}\right)bX
\end{aligned}
$$

and

$$\dot{\Omega} = -\frac{1}{Y_\Omega} \mu \frac{S}{S+K_S} \frac{\Omega}{\Omega+K_\Omega} X$$

where upper dots denote the derivative with respect to the time variable $\tilde{t}$, $\mu$ is the maximum growth rate, $K_S$ and $K_\Omega$ are respectively the half saturation constant of soluble substrate and oxygen, $b$ is the biomass decay rate constant, $K_h$ is the hydrolysis rate constant and $Y_S$, $Y_I$, $Y_L$ and $Y_\Omega$ are the yield coefficients. Let us notice that the conversion terms

$$\mu \frac{S}{S+K_S} \frac{\Omega}{\Omega+K_\Omega} X \,, \; K_h I \,, \; bX \,,$$

express respectively the degradation of the soluble substrate by means of aerobic bacteria, the hydrolysis conversion of insoluble substrate and the biomass decay.

Typical values of the physical parameters and of the initial configuration

| Parameter | value |
|---|---|
| $\mu$ | 2e-4 $s^-1$ |
| $K_S$ | 0.2573 $mol/m^3$ |
| $K_\Omega$ | 2.822$mol/m^3$ |
| $K_h$ | 4.9e-7 $s^-1$ |
| $b$ | 3.8e-5 $s^-1$ |
| $Y_S$ | 0.53 |
| $Y_I$ | 1.02 |
| $Y_L$ | 1.34 |
| $Y_\Omega$ | 1.12 |
| $S_0$ | 1420 $mol/m^3$ |
| $I_0$ | 4750 $mol/m^3$ |
| $X_0$ | 1.5 $mol/m^3$ |
| $L_0$ | 12297 $mol/m^3$ |
| $M_0$ | 450 $mol/m^3$ |
| $T_0$ | 18918,5 $mol/m^3$ |

| Parameter | value |
|---|---|
| $c_s$ | 1.36e-5 |
| $c_\omega$ | 0.74 |
| $c_h$ | 2.45e-3 |
| $\beta$ | 1.9e-1 |
| $\gamma$ | 3.7e3 |
| $s_0$ | 0.07506 |
| $i_0$ | 0.25108 |
| $x_0$ | 0.00008 |
| $\ell_0$ | 0.64999 |
| $m_0$ | 0.02379 |

Table 1: Values of the physical parameters and the relative nondimensional ones.

are given in Table 1 (left) and are inspired by the paper [19].
We observe again that the equation for oxygen time variation just takes account of a loss term due to the degradation. Unlike the open system described

by (1), no gain term due to the interaction with the atmosphere is taken into account, and oxygen is set to be consumed.
The time variation of the inert matter $M$ is obtained by means of the conservation of the total mass, that is,

$$\dot{M} = -\left(\dot{S} + \dot{I} + \dot{X} + \dot{L}\right) .$$

Denoting by $\tilde{\mathcal{M}}$ the total mass (both solid and liquid), $\tilde{\mathcal{M}} = S+I+X+L+M$, it follows for all $t \geq 0$, $\tilde{\mathcal{M}}(t) = \tilde{\mathcal{M}}(0) = S(0) + I(0) + X(0) + L(0) + M(0) =: \tilde{\mathcal{M}}_0$. The system can be rewritten in a non-dimensional form by introducing the following scaling

$$t = \mu\tilde{t}, \; s = \frac{S}{\tilde{\mathcal{M}}_0}, \; i = \frac{I}{\tilde{\mathcal{M}}_0}, \; x = \frac{X}{\tilde{\mathcal{M}}_0}, \; \ell = \frac{L}{\tilde{\mathcal{M}}_0}, \; m = \frac{M}{\tilde{\mathcal{M}}_0}, \; \omega = \frac{\Omega}{\Omega_0}$$

where $\Omega_0$ is a reference operational value of the oxygen concentration for the biodigestion process. The aerobic process occurs in presence of a suitable oxygen level. The aerobic organisms can survive with as little as 5% oxygen concentration in the system atmosphere but if the oxygen level goes under 10% part of the process can become anaerobic. In addition, an oxygen level around this value guarantees a very performing degradation of the organic matter [48]. Such reasons suggest to consider 10% as a reference value for oxygen concentration, corresponding to the reference value 1 for the nondimensional variable $\omega$. In order to give a value to the parameter $\Omega_0$, we remind that the concentration of wet air is $1220\, g/m^3$ and that oxygen percentage is approximately 21%; therefore the oxygen concentration is around $256.2\, g/m^3$ corresponding to $8\, mol/m^3$. By proportion we can set $\Omega_0 = 3.81\, mol/m^3$.
The evolution of the nondimensional variables is described by the following system of ordinary differential equations

$$\begin{aligned}
\dot{s} &= -\frac{1}{Y_S}\frac{s}{s+c_s}\,\frac{\omega}{\omega+c_\omega}\,x + c_h i \\
\dot{i} &= -c_h i + \frac{1}{Y_I}\beta x \\
\dot{x} &= \frac{s}{s+c_s}\frac{\omega}{\omega+c_\omega}x - \beta x \\
\dot{\ell} &= \frac{1}{Y_L}\frac{s}{s+c_s}\frac{\omega}{\omega+c_\omega}x \\
\dot{m} &= -\left(1 - \frac{1}{Y_S} + \frac{1}{Y_L}\right)\frac{s}{s+c_s}\,\frac{\omega}{\omega+c_\omega}\,x + \left(1 - \frac{1}{Y_I}\right)\beta x
\end{aligned} \tag{2}$$

and

$$\dot{\omega} \,=\, -\gamma \,\frac{s}{s+c_s}\frac{\omega}{\omega+c_\omega}x. \tag{3}$$

where now the upper dots denote the derivative with respect to the nondimensional variable $t$, $c_s = K_S/T_0$, $c_\omega = K_\Omega/\Omega_0$, $c_h = K_h/\mu$, $\beta = b/\mu$ and $\gamma = T_0/\left(Y_\Omega \Omega_0\right)$ (values of the parameters and of the initial configuration in the nondimensional setting are given in Table 1 (right)). We can observe that

$$0 \,\leq\, s\,,\, i\,,\, x\,,\, \ell\,,\, m \,\leq\, 1$$

while $\omega > 0$ and its reference value is equal to 1.
As concerns the long time behavior of the solution, it is easy to notice that equilibria are characterized by

$$x \,=\, 0\,,\, i \,=\, 0$$

and hence infinitely many steady states are possible; the reached equilibrium strictly depends on the initial configuration and any small perturbation of the initial datum leads to a different asymptotic state.
Moreover, as can be seen in equation (3), $\omega$ decreases and the oxygen will be (totally or partially) consumed. This means that a sufficient level of oxygen can not be long guaranteed and the aerobic process can be compromised.
In order to ensure the bacterial growth and the substrate consumption, it is necessary to inject additional oxygen in the system. Such operation will be considered in Section 4 where we will try to individuate the best way to maintain a good level of oxygen in the bioreactor.
From a mathematical point of view, the difficulty to analyze the steady states of the system is due also to the closed system structure, where nutrient or medium is not added. This kind of analysis is more treatable in open systems, where biological matter incomes and/or outgoes; such models are typically used in the wastewater management descriptions and a simple system will be presented in the Appendix A.

## 4. An optimal control problem for composting

As mentioned in Section 3 the level of oxygen as described by system (2-3) rapidly decays and the aerobic biodegradation process cannot be maintained for long.
In order to guarantee the transformation of the soluble substrate into pre-compost, an external operation, like mechanical aeration, is required.

Let us introduce the control function $u = u(t)$ that describes the contribution of the additional aeration in the oxygen balance. The model (2)-(3) is modified by substituting equation (3) by

$$\dot{\omega} \, = \, -\gamma \, \frac{s}{s+c_s} \frac{\omega}{\omega + c_\omega} x \, + \, u\,(t) \tag{4}$$

where $0 \le u(t) \le u_{\max}$ $\forall t \ge 0$. The positive value $u_{\max}$ is an upper bound for the control variable $u$ corresponding to the maximal value of oxygen that can be introduced in the biological system.
The optimal time profile is in the admissible control set

$$\mathcal{U} \, = \, \{u \text{ Lebesgue measurable on } (0, t_f) \,|\, 0 \le u(t) \le u_{\max}, \, \forall t \ge 0\}$$

and the objective functional is given by

$$J\,(u) \, = \, \int_0^{t_f} \left(\omega\,(t) - 1\right)^2 dt \tag{5}$$

where $t_f$ is the final time in our observation. The functional expresses the deviation of the oxygen level from the reference one in the time range $(0, t_f)$. The aim is to determine $(s^\star, i^\star, x^\star, \ell^\star, m^\star, \omega^\star)$ associated to an admissible control $u^\star \in \mathcal{U}$ satisfying (2)-(4) and minimizing the objective functional (5), i.e.

$$J\,(u^\star) \, = \, \min_{u \in \mathcal{U}} \, J\,(u) \; . \tag{6}$$

A possible approach to optimal control problems is strictly connected to Pontryagin's work that gives necessary conditions that an optimal solution has to satisfy [16, 37].
This principle converts the problem (6) into the problem of minimization of the following Hamiltonian function

$$\begin{aligned} \mathcal{H}\,(X, u, t) \, &= \, (\omega\,(t) - 1)^2 \\ &\quad - \frac{s}{s+c_s} \frac{\omega}{\omega + c_\omega} x \left[ \frac{\lambda_s}{Y_S} - \lambda_x - \frac{\lambda_\ell}{Y_L} + \left(1 - \frac{1}{Y_S} + \frac{1}{Y_L}\right) \lambda_m + \gamma \lambda_\omega \right] \\ &\qquad + c_h i\,(\lambda_s - \lambda_i) + \beta x \left[ \frac{\lambda_i}{Y_I} - \lambda_x + \left(1 - \frac{1}{Y_I}\right) \lambda_m \right] + u\,(t)\,\lambda_\omega \end{aligned}$$

where $X = (s, i, x, \ell, m, \omega)$ and $\Lambda = (\lambda_s, \lambda_i, \lambda_x, \lambda_\ell, \lambda_m, \lambda_\omega)$ are, respectively, the sets of state variables and of adjoint variables.

The adjoint variables solve a set of ordinary differential equations given by

$$\begin{aligned}\dot{\Lambda} &= A\Lambda + b \\ \Lambda\left(t_f\right) &= (0,0,0,0,0,0)^T\end{aligned}$$

where $b = (0,0,0,0,0,2\left(1-\omega\left(t\right)\right))^T$, $A$ has the following form

$$\begin{bmatrix} \frac{\alpha_s}{Y_S} & 0 & -\alpha_s & -\frac{\alpha_s}{Y_L} & \left(1-\frac{1}{Y_S}+\frac{1}{Y_L}\right)\alpha_s & \gamma\alpha_s \\ -c_h & c_h & 0 & 0 & 0 & 0 \\ \frac{\alpha_x}{Y_s} & -\frac{\beta}{Y_I} & -\alpha_x+\beta & -\frac{\alpha_x}{Y_L} & \left(1-\frac{1}{Y_S}+\frac{1}{Y_L}\right)\alpha_x - \left(1-\frac{1}{Y_I}\right)\beta & \gamma\alpha_x \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\alpha_\omega}{Y_S} & 0 & -\alpha_\omega & -\frac{\alpha_\omega}{Y_L} & \left(1-\frac{1}{Y_S}+\frac{1}{Y_L}\right)\alpha_\omega & \gamma\alpha_\omega \end{bmatrix}$$

and

$$\alpha_s = \frac{c_s}{\left(s+c_s\right)^2}\frac{\omega}{\omega+c_\omega}x\,,\ \alpha_x = \frac{s}{s+c_s}\frac{\omega}{\omega+c_\omega}\,,\ \alpha_\omega = \frac{s}{s+c_s}\frac{c_\omega}{\left(\omega+c_\omega\right)^2}x\,.$$

It is possible to give a characterization of the control in terms of the so-called switching function

$$\sigma(t) = \frac{\partial H}{\partial u}$$

when the Hamiltonian depends linearly on the control variable $u$ [16]. Let us recall some definitions.

Let the Hamiltonian $H$ be linear in the control variable $u$. The optimal control $u^\star$ is called a singular control on $[\underline{t},\bar{t}]$ if

$$\frac{\partial H}{\partial u}\left(X^\star, u^\star, \Lambda, t\right) = 0$$

for every $t \in [\underline{t},\bar{t}]$. The corresponding solution $(X^\star, u^\star)$ is called singular arc. If $u^*$ is a singular control, the problem order is the smallest number $q^\star$ such that the $2q$-th derivative

$$\frac{d^{2q}}{dt^{2q}} \frac{\partial H}{\partial u} \left(X^\star, u^\star, \Lambda, t\right)$$

explicitly contains the control variable $u$ (if no derivative satisfies this condition then $q = \infty$).
In our case, the switching function is given as

$$\sigma(t) \;=\; \lambda_\omega$$

and we can give the following characterization to the control function

$$u\left(t\right) \;=\; \begin{pmatrix} 0 \\ \text{singular} \\ u_{\max} \end{pmatrix} \qquad \text{if} \qquad \lambda_\omega \begin{pmatrix} > \\ = \\ < \end{pmatrix} 0\,. \tag{7}$$

It is also easy to show that

$$\begin{aligned} \frac{d^2}{dt^2}\frac{\partial H}{\partial u} \;=\; & -2u\left(t\right) + 2\gamma \, \frac{s}{s+c_s}\frac{\omega}{\omega + c_\omega} x \\ & + \dot{\alpha}_\omega\left(X\right)\left[\frac{\lambda_s}{Y_S} - \lambda_x - \frac{\lambda_\ell}{Y_L} + \left(1 - \frac{1}{Y_S} + \frac{1}{Y_L}\right)\lambda_m + \gamma\lambda_\omega\right] \\ & + \alpha_\omega\left(X\right)\left[\frac{\dot{\lambda}_s}{Y_S} - \dot{\lambda}_x - \frac{\dot{\lambda}_\ell}{Y_L} + \left(1 - \frac{1}{Y_S} + \frac{1}{Y_L}\right)\dot{\lambda}_m + \gamma\dot{\lambda}_\omega\right] \end{aligned}$$

and hence our problem is of order 1. We observe also that such derivative depends on the control variable $u$ also by means of the term $\dot{\alpha}_\omega$.
Let $(X^\star, u^\star)$ be an optimal solution of the linear problem of order 1 in $(\underline{t}, \bar{t})$; the following property holds

$$\frac{\partial}{\partial u}\frac{d^2}{dt^2}\frac{\partial H}{\partial u}\left(X^\star, u^\star, \Lambda, t\right) \;\leq\; 0 \tag{8}$$

for any $t \in (\underline{t}, \bar{t})$ [16, 40]. Such necessary condition, called *Legendre-Clebsch condition*,will be checked numerically. When the inequality (8) holds true the occurrence of singular arcs cannot be ruled out.

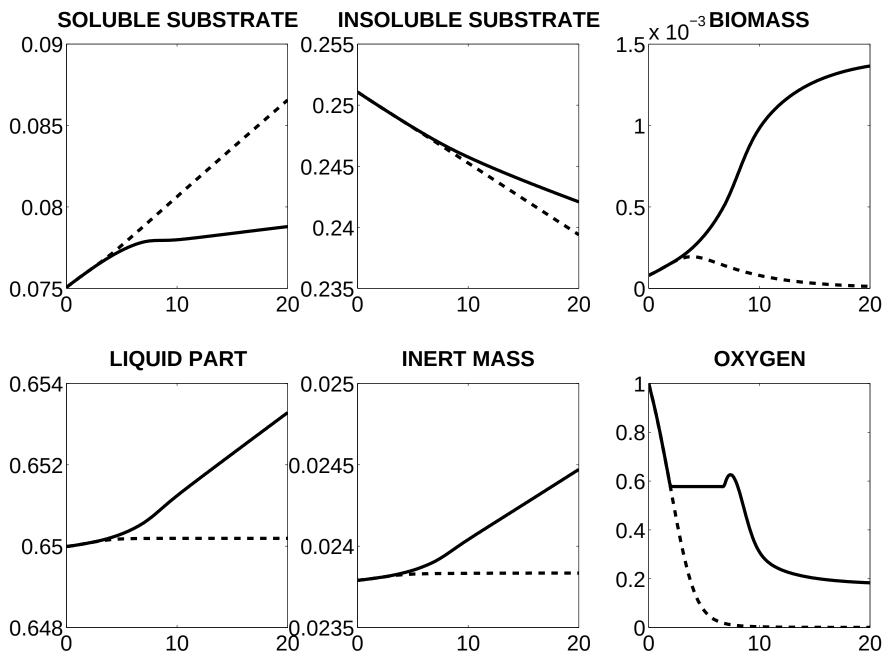


Figure 2: Trends of the state variables versus time in presence (continuous line) or in absence (dashed line) of the control. The nondimensional final time is set equal to 20 (corresponding to approximately 28 hours).

## 5. Numerical simulations, plant performance

In order to show the optimal control solutions, we perform numerical simulations and use a gradient method described in [3].
In Figure 2 it is shown that in absence of aeration ($u \equiv 0$), degradation occurs until the oxygen is exhausted. As soon as the oxygen is exhausted, the evolution of the system is governed by the decay process of the biomass and by the hydrolytic process that converts the insoluble substrate in the soluble one. The soluble substrate concentration increases and no consumption occurs because oxygen is not present in the system. Long time behavior shows that the system reaches a configuration where the insoluble substrate and the biomass are zero ($i = 0$, $x = 0$).
The presence of aeration ($u \not\equiv 0$) guarantees the survival of the biomass population, the degradation process is not completed in a short time and consequently the consumption of soluble substrate increases. In addition,

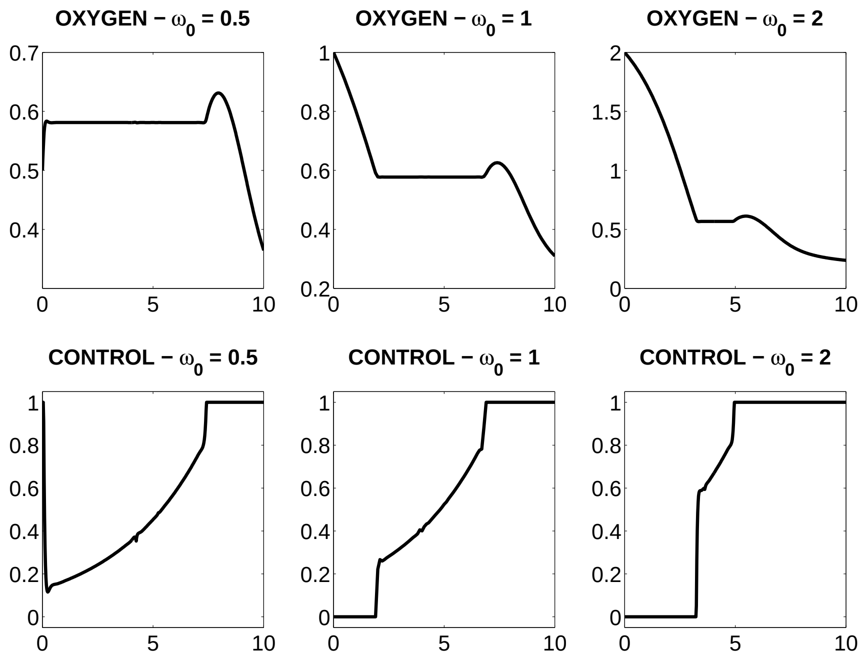


Figure 3: Comparison of oxygen and optimal mechanical aeration time profiles for different values of the initial condition $\omega_0$. The maximal value $u_{\max}$ for $u$ is 1.

it implies a higher level of the products: in particular it is easy to notice a higher concentration of the liquid part $\ell$ and of the pre-compost $m$ with respect to the process without additional aeration. By means of the decay phenomenon, a higher concentration of biomass gives a contribution to insoluble substrate that decreases slowly.

In order to evaluate the plant performance, we can introduce some quantitative indices that describe the gain of pre-compost and the consumption of substrate with respect to the initial configuration (respectively $\mathbf{I}_1 = m(t_f) - m(0)$ and $\mathbf{I}_3 = (s+i)(t_f) - (s+i)(0)$), the effect of the aeration with respect to the case in absence of external operation in terms of pre-compost ($\mathbf{I}_2 = (m(t_f)_{u\not\equiv 0} - m(t_f)_{u\equiv 0})/m(t_f)_{u\equiv 0}$) and substrate ($\mathbf{I}_4 = ((s+i)(t_f)_{u\not\equiv 0} - (s+i)(t_f)_{u\equiv 0})(s+i)(t_f)_{u\equiv 0})$.
In the realistic case of a biocell capacity of 115 tonnes [36], the performance improvements can be computed in terms of kilos of produced precompost and reduced substrate in a time range of about 28 hours. As shown in Table 2,

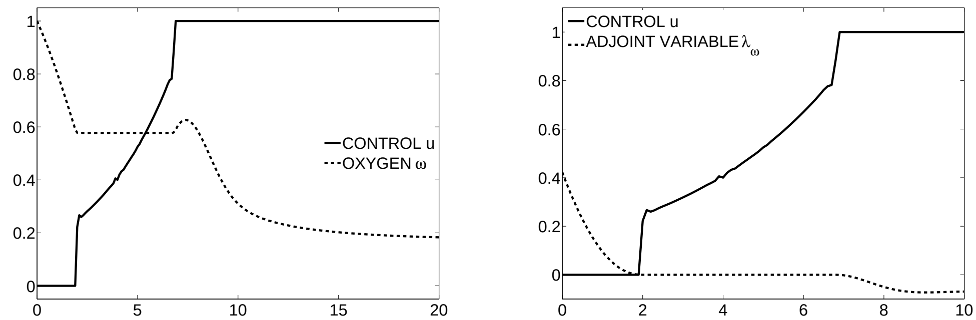


Figure 4: The figure compares the evolution of the control $u$ with the oxygen concentration $\omega$ (left) and with the switching function $\lambda_\omega$ (right).

| | Aeration | $\omega_0 = 0.5$ | $\omega_0 = 1$ | $\omega_0 = 2$ |
|---|---|---|---|---|
| $\mathbf{I}_1 = m(t_f) - m(0)$ | no | +3Kg (+0.096% ) | +5Kg (+0.187%) | +10Kg (+0.369%) |
| $\mathbf{I}_1 = m(t_f) - m(0)$ | yes | +76Kg (+2.766% ) | +78Kg (+2.863%) | +87Kg (+3.173%) |
| $\mathbf{I}_2 = \dfrac{m(t_f)_{u\not\equiv 0} - m(t_f)_{u\equiv 0}}{m(t_f)_{u\equiv 0}}$ | – | +2.667% | +2.670% | +2.794% |
| $\mathbf{I}_3 = (s+i)(t_f) - (s+i)(0)$ | no | -6Kg (-0.016% ) | -20Kg (-0.055%) | -50Kg (-0.133%) |
| $\mathbf{I}_3 = (s+i)(t_f) - (s+i)(0)$ | yes | -589Kg (-1.570% ) | -604Kg (-1.612%) | -655Kg (-1,746%) |
| $\mathbf{I}_4 = \dfrac{(s+i)(t_f)_{u\not\equiv 0} - (s+i)(t_f)_{u\equiv 0}}{(s+i)(t_f)_{u\equiv 0}}$ | – | -1.555% | -1.558% | -1.616% |

Table 2: Performance indices in absence and in presence of external aeration ($\omega_0 = 0.5, 1, 2$).

the presence of an external operation implies a product gain of about 80Kg and a reduction of the total (soluble and insoluble) substrate of about 600kg while these variations are of order of few kilos in absence of aeration.
As shown in Figure 3, the trend of the mechanical aeration $u$ strongly depends on the initial value of oxygen concentration. In particular, initial values around or higher than the reference one clearly do not require additional aeration at the beginning; aeration occurs just afterwards when the oxygen concentration is sufficiently less than the reference one.
Of course, when the initial oxygen concentration is below the reference value, additional aeration is required immediately in order to bridge the gap with respect to the reference concentration. The aeration slows down when the oxygen level is close to the reference value and subsequently it rapidly in-

creases when the consumption increases.
We can notice also that the control variable $u$ assumes the maximal value definitively. Unfortunately it does not guarantee a high level of oxygen because of a high concentration of biomass that stimulate the degradation process.
Although not reported here, the plot of the left hand side term in (8) is negative, when the control does not assume the extremal values 0 and $u_{\max}$, due to the optimality of the singular arc.
The time profiles of oxygen concentration in Figure 4 (left) and Figure 3 show a pick in correspondence of a jump in the control variable $u$ where an oxygen pulse occurs.
Figure 4 (right) finally shows that the characterization (7) holds. In particular $u$ admits the minimal/maximal value where the switching function is respectively positive/negative; when the switching function is zero the singular arc is correctly reproduced.

## 6. Concluding remarks

We have formulated a mathematical model for aerobic degradation of organic waste in a composting biocell. Then, we have set an optimal control problem, whose aim is to minimize the deviation of the oxygen level from its reference value by controlling the injection of additional oxygen in the cell atmosphere.
We have obtained the optimal time profiles of the state variables and the optimal control in presence of mechanical aeration. We found that the presence of aeration gives satisfactory levels of oxygen and guarantees the survival of the bacterial population. The results are compared with the corresponding profiles in absence of oxygen injection: the aeration leads to an higher production of pre-compost and to an higher reduction of total substrate. The plant performance has been evaluated by using several specific indices.
The objective functional takes into account of the deviation of oxygen from its reference value. Clearly, other choices of the optimality criterion are possible. In particular, the maximization of the product and/or the minimization of the substrate can be investigated by using a similar approach.
Moreover, time optimal control problems can be also formulated in order to find the minimal time to reach a given target on the state variables, like, for example, the reduction of the global substrate below a given threshold value.
We have focused our attention on the composting process in the integrated

system. We plan also to investigate the anaerobic process involved in the production of biogas as well as the treatment of the wastewater sludge which may represent a risk of contaminating the soil and ground water.


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## Appendix A. Continuous stirred tank bioreactor

As already pointed out in Section 3, the limited qualitative analysis of the steady states is partially due to the closed system structure. We want to show how different may be the analysis for an open system with ingoing/outgoing flow of material.
As example of open system [1, 2, 35], let us consider a simple biological well aerated system composed by the soluble substrate $S$, the insoluble one $I$ and the biomass $X$. The dynamics is governed by several phenomena: the ingoing/outgoing flow with the same flow rate $F$ and inflow values $S^0$, $I^0$ and $X^0$; the degradation process that implies the biomass growth and the consumption of soluble substrate; the hydrolysis of the insoluble substrate that increases the amount of the soluble one.
Unlike the dynamics of the previous bioreactor, in this case we consider a well-aerated system, where the oxygen level can be maintained constant. Moreover, the biomass decay process is not taken into account.
Mathematically, the balances can be expressed by

$$\begin{aligned} V\dot{S} &= F\left(S^0 - S\right) - V\frac{1}{Y}\mu\frac{S}{S+K_S}X + VK_hI \\ V\dot{I} &= F\left(I^0 - I\right) - VK_hI \\ V\dot{X} &= F\left(X^0 - X\right) + V\mu\frac{S}{S+K_S}X \end{aligned}$$

where upper dots denote the time derivative with respect to $\tilde{t}$, $V$ is the volume of the bioreactor, $Y$ is the yield coefficient, $\mu$ is the maximal growth rate, $K_S$ is the half saturation constant and $K_h$ is the hydrolitic coefficient.

Let us define the residence time $\tau = V/F$ and introduce the nondimensional variables

$$s = \frac{S}{K_S}\,,\ i = \frac{I}{K_S}\,,\ x = \frac{X}{K_S}\,,\ t = \mu\tilde{t}\,.$$

The differential equations can be rewritten as

$$\begin{aligned} \dot{s} &= \alpha\left(s^0 - s\right) - \frac{1}{Y}\frac{s}{s+1}x + \kappa_h i \\ \dot{i} &= \alpha\left(i^0 - i\right) - \kappa_h i \\ \dot{x} &= \alpha\left(x^0 - x\right) + \frac{s}{s+1}x \end{aligned} \tag{A.1}$$

where upper dots denote the derivative with respect to the nondimensional variable $t$, $\alpha = 1/(\tau\mu)$ and $\kappa_h = K_h/\mu$.
We can notice that the second equation can be solved analytically

$$i(t) = \frac{\alpha I^0}{\alpha + \kappa_h} + \left(i_0 - \frac{\alpha I^0}{\alpha + \kappa_h}\right)\exp\left[-\left(\alpha + \kappa_h\right)t\right]$$

where $i_0$ is the initial value: $i(0) = i_0$. At the asymptotic limit, when $t \to +\infty$, $i(t) \to \dfrac{\alpha I^0}{\alpha + \kappa_h} = \tilde{i}$, that is the equilibrium value for $i$.
In order to determine the equilibria of the system, we can consider the reduced system in $(s, x)$. We have to solve

$$\begin{aligned} \alpha\left(s^0 - s\right) - \frac{1}{Y}\frac{s}{s+1}x + \kappa_h\tilde{i} &= 0 \\ \alpha\left(x^0 - x\right) + \frac{s}{s+1}x &= 0 \end{aligned} \tag{A.2}$$

and the following result holds.

**Proposition 1.** *System (A.2) admits a unique equilibrium* $\left(-\dfrac{\tilde{x}}{Y} + \beta, \tilde{x}\right)$ *where* $\beta = s^0 + \dfrac{x^0}{Y} + \dfrac{\kappa_h}{\alpha}\tilde{i}$ *and* $\tilde{x}$ *is the positive solution of*

$$\frac{1}{Y}\left(1-\alpha\right)x^2 + \left(\alpha y x^0 + \alpha\beta + \alpha - \beta\right)x - \alpha\left(1+\beta\right)x^0 = 0\,. \tag{A.3}$$

*Proof.* (Sketch) Equation (A.3) may be obtained from (A.2) by simple manipulations. The discriminant of the equation is positive and consequently

the equation admits two real roots. In addition, by using the Descartes' rule, one can conclude that there is a unique positive solution, that is indicated by $\tilde{x}$.
Let us denote by $\tilde{s} = -\frac{1}{Y}\tilde{x} + \beta$ the corresponding value for the variable $s$; we have to prove that $\tilde{s}$ is an admissible value, i.e. $\tilde{s} > 0$ or, equivalently, $\tilde{x} < \beta Y$.
This can be easily proved by showing that the evaluation of the function

$$r(x) \;=\; \frac{1}{Y}(1-\alpha)\,x^2 + \big(\alpha y x^0 + \alpha\beta + \alpha - \beta\big)\,x - \alpha\,(1+\beta)\,x^0$$

in $\beta Y$ is positive. □

As regards the stability, the following result (illustrated in Figure A.5) is obtained.

**Proposition 2.** *The unique equilibrium* $(\tilde{s}, \tilde{x})$ *of system (A.2) is linearly asymptotically stable.*

*Proof.* (Sketch) The eigenvalues of the jacobian matrix evaluated at the equilibrium are given by

$$\lambda_1 \;=\; -\alpha \;<\; 0\,,\; \lambda_2 \;=\; -\alpha - \frac{1}{Y}\frac{1}{(\tilde{s}+1)^2}\tilde{x} + \frac{\tilde{s}}{\tilde{s}+1}\,.$$

The linear stability of the equilibrium follows from $\lambda_2 < 0$, i.e.

$$(1-\alpha)\,(\tilde{s}+1)^2 - (1+\beta) \;<\; 0\,.$$

If $\alpha > 1$, the inequality trivially holds. If $\alpha < 1$, then

$$\lambda_2 \;<\; 0 \iff \tilde{s} \;<\; \sqrt{\frac{1+\beta}{1-\alpha}} - 1 \iff \tilde{x} \;<\; \hat{x} \;=:\; Y\left[(1+\beta) - \sqrt{\frac{1+\beta}{1-\alpha}}\right]\,.$$

The last inequality is easy proven by showing that $r(\hat{x}) < 0$. □

Unlike the case of closed system, the open one admits a unique stable equilibrium and any perturbation of the initial configuration does not affect the reached asymptotic state.

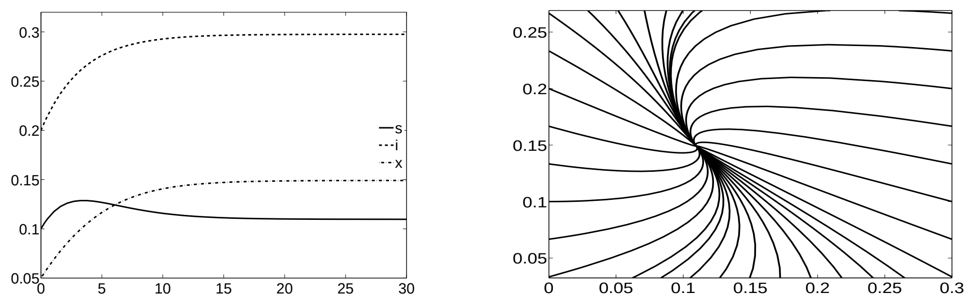


Figure A.5: The figure shows the evolution of the state variables (left) and the projection of the phase diagram on the $(s, x)$-plane (right) for the continuous stirred tank bioreactor model (A.1). The parameters values are: $\alpha = 0.3$, $Y = 0.53$, $c_h = 2.45 \times 10^{-3}$, $s^0 = 0.2$, $i^0 = 0.3$ and $x^0 = 0.1$. The initial conditions in the left figure are: $s(0) = 0.1$, $i(0) = 0.2$ and $x(0) = 0.05$.

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