Optimal control of leachate recirculation for anaerobic processes in landfills

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OPTIMAL CONTROL OF LEACHATE RECIRCULATION FOR ANAEROBIC PROCESSES IN LANDFILLS

MARZIA BISI, MARIA GROPPI, GIORGIO MARTALÒ* AND ROMINA TRAVAGLINI

Department of Mathematical, Physical and Computer Sciences, University of Parma
Parco Area delle Scienze 53/A, 43124, Parma, Italy

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ABSTRACT. A mathematical model for the degradation of the organic fraction of solid waste in landfills, by means of an anaerobic bacterial population, is proposed. Additional phenomena, like hydrolysis of insoluble substrate and biomass decay, are taken into account. The evolution of the system is monitored by controlling the effects of leachate recirculation on the hydrolytic process. We investigate the optimal strategies to minimize substrate concentration and recirculation operation costs. Analytical and numerical results are presented and discussed for linear and quadratic cost functionals.

1. **Introduction.** Waste management is nowadays a problem of urgency and interest for national and local authorities that have to implement strategies and decision policies in their intervention area.

Traditionally, landfills were conceived as containment vessels, where waste was simply stored; such approach has already revealed some criticisms, like the requirement of new stocking sites and the formation and diffusion of contaminated leachate that can pollute soils and aquifers [7, 11]. These raised findings can be overcome by a different recent approach, according to which landfills are controlled sites (bioreactors), where the solid waste is treated and stabilized. In particular, the organic fraction can be reduced and used to produce some byproducts [20, 23], like biogas and compost.

As regards agricultural fertilizers (compost), they can be produced from organic waste by means of an aerobic phenomenon of degradation [11]; the digestion process due to a bacterial population can be controlled, for example, by manipulating the levels of oxygen concentration in the composting system atmosphere. Some optimal control problems have been formulated recently [16, 17] to give some indications about the optimal aeration strategies to improve biocell performance.

Under anaerobic conditions, the transformation of soluble component of the organic fraction implies the production of biogas, similar to methane but with a heating value lower than the methane one [9, 13]; in this context several optimization problems have been formulated [4, 5, 6] to individuate the best feeding strategies that maximize the biogas production and the soluble substrate reduction. Some factors can influence the process performance; for example, the leachate recirculation can

be manipulated and controlled in order to improve the system mixing [29, 31], and consequently to stimulate the degradation.

A recent paper [25] presents a simple model of landfill describing the degradation of the soluble substrate by means of a bacterial population, acting under anaerobic conditions in a batch bioreactor. More precisely, a two components substrate and a bacterial population (biomass) are taken into account and their evolution is governed by two different phenomena: (i) the soluble substrate is degraded by the anaerobic biomass; (ii) the insoluble substrate is solubilized by means of a hydrolytic process. The leachate recirculation is the process to be controlled and it can stimulate or inhibit solubilization of the insoluble component. Such scenario can be modeled by a system of ordinary differential equations, that constitute an appropriate description of the evolution of main variables in the case of perfect mixing. In [25] the authors have individuated and analyzed the optimal control to reach a given target configuration in minimum time.

In this paper, we propose a control problem for anaerobic degradation in a batch bioreactor, under perfect mixing conditions. The natural phenomenon of bacteria death, not taken into account in [25], is now introduced in the model. Therefore, the organic matter is modeled as a two components substrate - soluble and insoluble - and the following transformation phenomena are considered: (i) anaerobic digestion: the bacterial population grows by consuming soluble organic matter; (ii) hydrolysis: i.e. solubilization of the insoluble substrate; (iii) biomass decay: bacteria death produces new insoluble substrate.

The control variable represents the effects of leachate recirculation on the solubilization of the insoluble substrate. The leachate recirculation can be achieved by means of a proper device spraying the leachate in the system. The traditional approach based on the injection of leachate directly on the organic waste surface [15] has been overcome by current strategies of horizontal trenches [24], vertical wells [8] and permeable blankets [12], preventing the increase of odors and gas emissions. The control is carried out by monitoring the leachate flow through the system. Unlike [25], we deal here with a finite horizon control problem, whose main goal is to find the best strategies optimizing an objective functional, that balances the minimization of leachate recirculation cost in a given time range and the minimization of substrate components at the final time.

As concerns this last task, it is known [25] that anaerobic processes in biogas production show in the first phase a negligible amount of methane [10]. In this paper we focus on such phase, which is mainly aimed at improving the mixing in the system, by reaching at the end a better configuration with low concentrations of substrate components as effect of a proper leachate recirculation.

As regards the recirculation cost, we remind here that an economic contribution is required in terms of technology, electricity and working hours; because of the several cost sources, we will analyze two different objective functionals depending linearly and quadratically on the control variable. The consequent optimal profiles will be computed and discussed.

The paper is organized as follows: we introduce the mathematical model and some basic properties in Section 2; the optimal control problem is formulated and analyzed by means of Pontryagin's theory in Section 3; some numerical results, for both linear and quadratic cost functionals, are presented and discussed in Section 4; some concluding remarks are given in Section 5.

2. **The model.** We propose a model describing the action of an anaerobic bacterial population, that degrades the soluble fraction of the substrate in a batch bioreactor. The insoluble part undergoes a hydrolytic process before being digested; in addition, a biomass decay phenomenon is taken into account.

Under perfect mixing conditions, the time evolution of the involved quantities is governed by a system of ordinary differential equations

$$\begin{aligned}
\frac{dS_1}{d\tau} &= -\tilde{\gamma}(Q)\tilde{c}_h S_1 + \tilde{b}X \\
\frac{dS_2}{d\tau} &= -\mu\tilde{g}(S_2)X + \tilde{\gamma}(Q)\tilde{c}_h S_1 \\
\frac{dX}{d\tau} &= \mu\tilde{g}(S_2)X - \tilde{b}X\,,
\end{aligned} \tag{1}$$

where τ is the time variable, $S_1(\tau)$, $S_2(\tau)$ and $X(\tau)$ denote the insoluble substrate, the soluble component and the biomass at time τ, respectively. The term $\mu\tilde{g}(S_2)X$ represents the degradation and $\tilde{\gamma}(Q)$ describes the effects of the leachate recirculation on the hydrolytic phenomenon; the solubilization of insoluble substrate depends on the flow rate $Q \in [0, Q_{\max}]$. The positive constants μ, $\tilde{b}$ and $\tilde{c}_h$ represent the maximum growth rate, the biomass decay parameter and the hydrolysis coefficient, respectively.

The total mass conservation

$$S_1(\tau) + S_2(\tau) + X(\tau) = S_1(0) + S_2(0) + X(0) =: m \tag{2}$$

is used to introduce the following scaled quantities

$$t = \mu\tau\,, \qquad s_1 = \frac{S_1}{m}\,, \qquad s_2 = \frac{S_2}{m}\,, \qquad x = \frac{X}{m}\,, \qquad q = \frac{Q}{Q_{\max}}\,,$$

where $0 \le s_1, s_2, x, q \le 1$ and the flow rate Q has been scaled with its maximum admissible value $Q_{\max}$.

From (1) the nondimensional system of equations is deduced

$$\begin{aligned}
\dot{s}_1 &= -\gamma(q)c_h s_1 + b(1 - s_1 - s_2) \\
\dot{s}_2 &= -g(s_2)(1 - s_1 - s_2) + \gamma(q)c_h s_1\,,
\end{aligned} \tag{3}$$

where the overdots denote the derivative with respect to t; the conservation (2) is used to replace x by $1 - s_1 - s_2$; functions γ, g and parameters b and c_h are the nondimensional version of the corresponding physical quantities (in particular $c_h = \tilde{c}_h/\mu$ and $b = \tilde{b}/\mu$).

We assume that

H0 - $\gamma \in \mathcal{C}^1([0,1])$, $\gamma(0) = 0$, $\gamma'(q) > 0$ for any $q \in [0,1]$;
H1 - $g \in \mathcal{C}^0([0,1])$ with $g(0) = 0$ and $g(s_2) > 0$ for any $s_2 \in (0,1]$.

Under hypothesis **H0**, we may introduce the control variable

$$u := \gamma(q)\,,$$

representing the effects of leachate recirculation on the hydrolytic process. The admissible control set for u is given by

$$\mathcal{U} := \{\nu : [0, t_f] \longrightarrow [0, u_{\max}]\,, \nu \text{ Lebesgue measurable}\}\,, \tag{4}$$

and without any loss of generality we may assume $u_{\max} = 1$.

Then the model (3) can be written as

$$
\begin{aligned}
\dot{s}_1 &= -uc_h s_1 + b\left(1 - s_1 - s_2\right) \\
\dot{s}_2 &= -g\left(s_2\right)\left(1 - s_1 - s_2\right) + uc_h s_1 \,.
\end{aligned}
\tag{5}
$$

Note also that, under hypothesis **H1**, the set

$$
\mathcal{F} = \{(s_1, s_2) \in [0, 1] \times [0, 1] \text{ such that } s_1 + s_2 \leq 1\}
\tag{6}
$$

is positively invariant, i.e any solution starting from a state in $\mathcal{F}$ remains in $\mathcal{F}$ for any time.

2.1. **Basic properties in presence of a constant flow rate.** We shortly analyze some properties of model (5) when a suitable strategy guarantees the same flow rate at any time, i.e. $u(t) = \theta \in [0, 1]$ for any $t > 0$; the model can be rewritten as

$$
\begin{aligned}
\dot{s}_1 &= -\theta c_h s_1 + b\left(1 - s_1 - s_2\right) \\
\dot{s}_2 &= -g\left(s_2\right)\left(1 - s_1 - s_2\right) + \theta c_h s_1 \,.
\end{aligned}
\tag{7}
$$

We consider also the following additional hypothesis about the smoothness of bacterial growth function g

H2 - $g \in \mathcal{C}^1\left([0, 1]\right)$ and $g'\left(s_2\right) > 0$ for any $s_2 \in [0, 1]$.

We discuss separately the case $\theta = 0$, in which the hydrolytic process does not play any role and the evolution is driven only by biomass decay phenomenon and soluble substrate degradation. In this case system (7) reduces to

$$
\begin{aligned}
\dot{s}_1 &= b\left(1 - s_1 - s_2\right) \\
\dot{s}_2 &= -g\left(s_2\right)\left(1 - s_1 - s_2\right) \,.
\end{aligned}
\tag{8}
$$

We can easily observe that any configuration in

$$
\mathcal{C} = \{(s_1, s_2) \in \mathcal{F} \text{ such that } s_1 + s_2 = 1\}
$$

is a steady solution of system (8); in addition any other solution in phase space can be explicitly computed for any initial state $\sigma^0 = \left(s_1^0, s_2^0\right) \in \mathcal{F}$; more precisely, trajectory $\mathcal{T}_{\sigma^0}$ starting from σ^0 is given by

$$
\mathcal{T}_{\sigma^0} = \left\{(s_1, s_2) \in \mathcal{F} \text{ such that } b\mathcal{G}\left(s_2\right) + s_1 = b\mathcal{G}\left(s_2^0\right) + s_1^0\right\} \,,
$$

where $\mathcal{G}\left(s_2\right)$ is a primitive function of $1/g\left(s_2\right)$.

We observe that the state vector defined by (8) is such that

$$
\left(\dot{s}_1, \dot{s}_2\right) \cdot n = \frac{1}{\sqrt{2}}\left(b - g\left(s_2\right)\right)\left(1 - s_1 - s_2\right) > 0 \text{ if and only if } b - g\left(s_2\right) > 0 \,,
$$

where n is the unit outgoing vector orthogonal to line $\mathcal{C}$; by reminding that $\mathcal{F}$ is the positive invariant set (6), one can conclude about stability of equilibria on line $\mathcal{C}$. In particular, let $s_2^* \in [0, 1]$ be (if it exists) the unique solution of

$$
b - g\left(s_2\right) = 0
\tag{9}
$$

(its uniqueness follows from hypothesis **H2**); then line $\mathcal{C}$ can be rewritten as union of two subsets

$$
\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2
$$

where

$$
\begin{aligned}
\mathcal{C}_1 &= \{(s_1, s_2) \in \mathcal{C} \text{ such that } 0 \leq s_2 \leq s_2^*\} \\
\mathcal{C}_2 &= \{(s_1, s_2) \in \mathcal{C} \text{ such that } s_2^* < s_2 \leq 1\} \,;
\end{aligned}
$$

equilibria in $\mathcal{C}_1$ are stable, while those in $\mathcal{C}_2$ are unstable. We notice here that

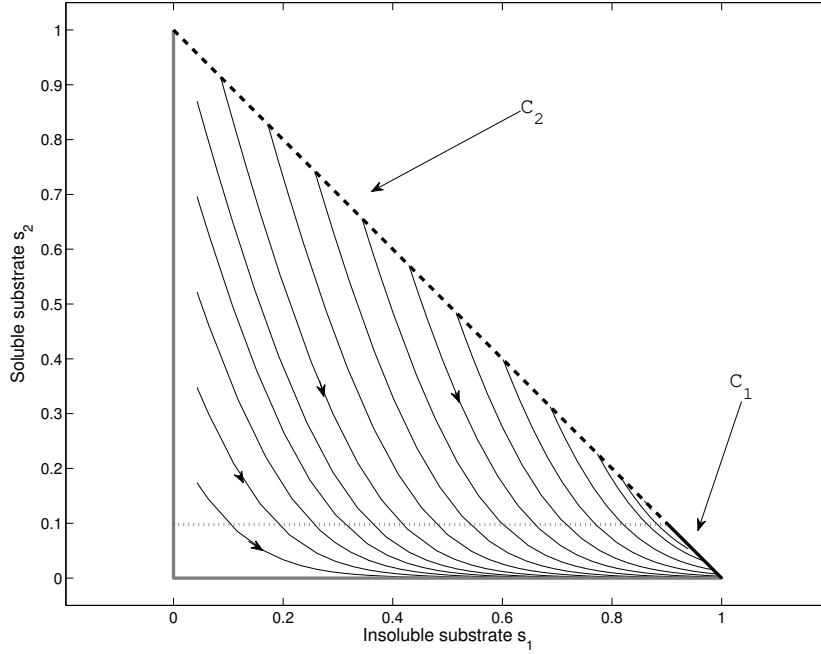FIGURE 1. Phase portrait in absence of solubilization of the insoluble substrate ($u(t) = 0$ for any $t > 0$). Bacterial growth is described by a Monod response function (10); parameters $c = 0.417$ and $b = 0.19$ are purely illustrative.

$\mathcal{C}_1 = \mathcal{C}$ and $\mathcal{C}_2 = \emptyset$ if equation (9) has no solution in $[0, 1]$. Results are summarized in Figure 1, when the bacterial growth is modeled by a Monod function [18]

$$g\left(s_2\right) = \frac{s_2}{s_2 + c}, \tag{10}$$

where c is the half saturation constant.

As regards the case $\theta \neq 0$, we notice that system (7) admits the equilibrium configuration $E_1 = (0, 1)$, representing total solubilization of insoluble substrate and absence of bacterial population; we observe also that E_1 does not depend on the parameter θ. A second equilibrium E_2 is admitted if (9) has solution s_2^*; such equilibrium is given by (s_1^*, s_2^*), where

$$s_1^* = \frac{b\left(1 - s_2^*\right)}{b + \theta c_h}. \tag{11}$$

As concerns the local stability of equilibria, the Jacobian matrix $\mathbf{J}$ associated to system (7), evaluated at the equilibrium state E_2, has two negative eigenvalues, guaranteeing local stability of E_2 when it exists. As regards equilibrium E_1, the jacobian matrix $\mathbf{J}$ evaluated in such state has eigenvalues

$$\lambda_1 = -\theta c_h < 0 \text{ and } \lambda_2 = g\left(1\right) - b$$

and the stability strictly depends on the sign of λ_2.

We notice that if $b > g(1)$ then $\lambda_2 < 0$ and E_1 is stable, while E_2 is not admitted, since equation (9) has not solution in $[0, 1]$. When $b < g(1)$, system (7) admits
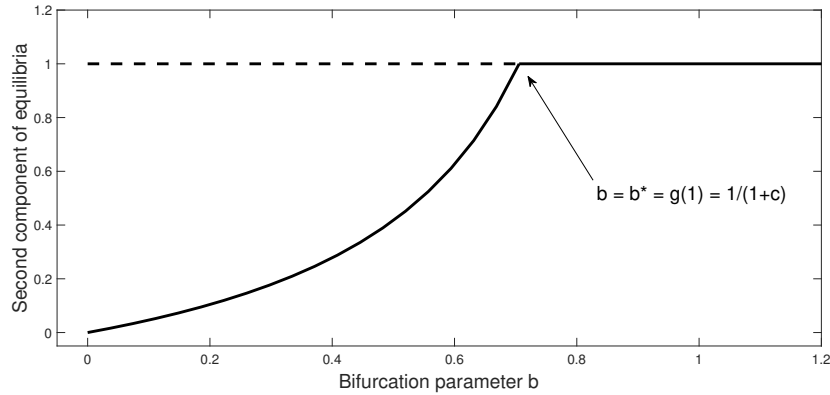
FIGURE 2. Second component of equilibria versus the bifurcation parameter b in the case of Monod response function (10). Continuous and dashed lines denote stability and instability of equilibria, respectively. The bifurcation value is $b^* \simeq 0.705$.
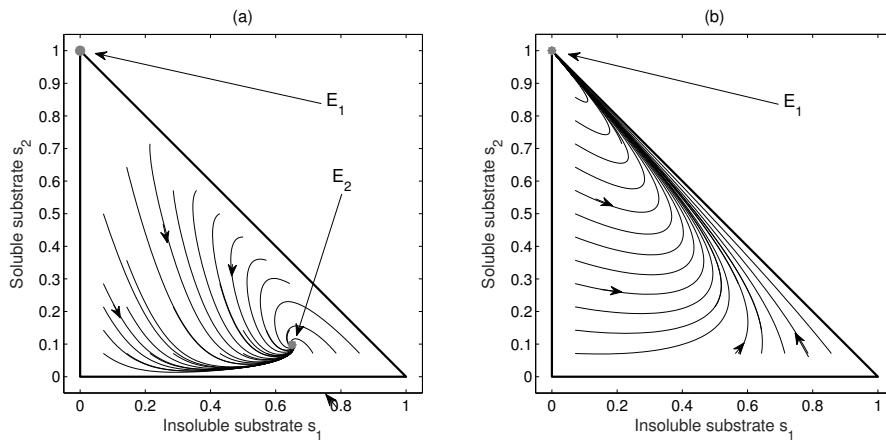

FIGURE 3. Phase portrait for system (7), when the bifurcation parameter b is less (panel (a)) or greater (panel (b)) than the bifurcation value $b^* \simeq 0.705$. Bacterial growth is modeled by Monod growth function (10); parameters $c = 0.417$, $c_h = 0.245$, $\theta = 0.3$.

equilibrium E_2, that is stable, while E_1 is unstable. Therefore, we can conclude that $b^* = g(1)$ is a bifurcation value for parameter b and a transcritical bifurcation occurs when $b = b^*$ (see Figure 2). Phase portraits are depicted in Figure 3, for $b < b^*$ (panel (a)) and $b > b^*$ (panel (b)). We observe that global stability of

equilibria follows from the Poincaré-Bendixson theorem, thanks to local stability results, together with the existence of the attractive bounded invariant region $\mathcal{F}$ in (6) and by excluding the occurrence of periodic solutions, by means of the Dulac function [21]

$$\psi(s_1, s_2) = \frac{1}{1 - s_1 - s_2}.$$

As last remark, we remind here that the above analysis has been performed under hypothesis **H2**; in absence of the monotonicity of g, other response functionals can be taken into account, like Haldane growth kinetics [1] that considers inhibition effects due to high concentrations of soluble substrate. When g is not monotonic, the expected scenario turns out to be richer than the one obtained for monotonic response functionals, since equation (9) may have several solutions.

3. **Pontryagin's formulation.** In this section, we formulate an optimal control problem whose main goal is to find the best recirculation strategy, that combines the maximum reduction of any component of the organic matter at the final time and the limitation of the costs of the recirculation operation in the entire time interval.

As concerns the reduction of the organic matter, the most desired scenario is provided by the total consumption of both components, i.e. $(s_1(t_f), s_2(t_f)) = (0, 0)$. We model the first contribution in the objective of the optimal control problem by requiring the minimization of the quadratic distance from such desired scenario. Moreover, as pointed out also in other frameworks (see [19, 26, 30]), this requirement has the effect of emphasizing/de-emphasizing the large/small deviations from the target configuration.

As concerns the costs of the recirculation operation, it is known that they can be modeled as a proper, but unknown, function of the state variables and the control [27]

$$\varphi = \varphi(s_1, s_2; u),$$

taking into account several cost sources (technology, electricity, working hours,...). In this manuscript, we focus only on fixed costs, that do not depend on the specific scenario; therefore, we assume

$$\varphi = \varphi(u).$$

Since the dependence of the cost function φ on the control variable u is unknown, in the following analysis we will discuss two particular cases commonly discussed in the literature on optimal control problems [3, 14], assuming linear and quadratic dependence on the control.

Mathematically, our goal is therefore to determine the optimal control function $u = u(t)$ in the admissible control set (4) (with $u_{\max} = 1$), which minimizes

$$\mathcal{L}_k(u) = \left(s_1^2(t_f) + s_2^2(t_f)\right) + \alpha \int_0^{t_f} u^k(t)\, dt, \tag{12}$$

along the solutions of (5), where $k = 1, 2$ and $\alpha > 0$. The functional $\mathcal{L}_k$ is a weighted average of the two contributions described above and the relative weight of each term on the strategy to be adopted is given by the coefficient α.

By means of Pontryagin's minimum principle [22], the optimal control problem for system (5) subject to the minimization of (12) can be formulated in terms of the Hamiltonian function

$$\mathcal{H}_k = \alpha u^k + u c_h s_1 (\lambda_2 - \lambda_1) + b(1 - s_1 - s_2)\lambda_1 - g(s_2)(1 - s_1 - s_2)\lambda_2.$$

The adjoint variables λ_i, $i = 1, 2$, solve the adjoint system of ordinary differential equations

$$
\begin{aligned}
\dot{\lambda}_1 &= -\frac{\partial \mathcal{H}_k}{\partial s_1} = (b + uc_h)\lambda_1 - [g(s_2) + uc_h]\lambda_2 \\
\dot{\lambda}_2 &= -\frac{\partial \mathcal{H}_k}{\partial s_2} = b\lambda_1 + [g'(s_2)(1 - s_1 - s_2) - g(s_2)]\lambda_2 ,
\end{aligned}
\tag{13}
$$

with given condition at final time t_f

$$(\lambda_1(t_f), \lambda_2(t_f)) = (2s_1(t_f), 2s_2(t_f)) .$$

For the linear case, when $k = 1$, we can give the following characterization of the optimal control

$$
\begin{cases}
u = 1 & \text{if } \phi_1 < 0 \\
u \in (0, 1) & \text{if } \phi_1 = 0 \\
u = 0 & \text{if } \phi_1 > 0 ,
\end{cases}
\tag{14}
$$

where the function ϕ_1 is given by

$$
\phi_1 = \frac{\partial \mathcal{H}_1}{\partial u} = \alpha + c_h s_1 (\lambda_2 - \lambda_1) .
\tag{15}
$$

If there exist two times $t_1, t_2 \in [0, t_f]$ such that $\phi_1(t) = 0$ for any $t \in [t_1, t_2]$, then the corresponding control $u(t) \in (0, 1)$ for any $t \in [t_1, t_2]$ is called *singular*.

If no singular control occurs in the time range $[0, t_f]$, then the optimal control must be constant or piecewise constant (assuming the minimum or the maximum value) and is said to be of *bang* or *bang-bang* type, respectively. In the latter case, times at which control passes instantaneously from minimum to maximum value, or vice versa, are called *switching times*.

Analogously, for the quadratic control, when $k = 2$, we can introduce the function $\phi_2 = \dfrac{\partial \mathcal{H}_2}{\partial u}$ and the condition $\phi_2 = 0$ provides the following explicit characterization of the optimal control in terms of state and adjoint variables

$$
u = \frac{c_h s_1 (\lambda_1 - \lambda_2)}{2\alpha} .
\tag{16}
$$

3.1. Optimality of singular control for the linear cost functional.

We are interested now in discussing the presence of minimizing singular controls in the case of linear cost functional ($k = 1$ in (12)).

Let u be a singular control in $[t_1, t_2] \subset [0, t_f]$ and $(s_1(t), s_2(t))$, $t \in [t_1, t_2]$, the corresponding solution. The *problem order* is the smallest number n such that the $2n$-th derivative

$$
\frac{d^{2n}}{dt^{2n}} \frac{\partial \mathcal{H}_1}{\partial u} (s_1, s_2, \lambda_1, \lambda_2, u, t)
\tag{17}
$$

explicitly contains the control variable u (if no derivative satisfies this condition then $n = \infty$). By assuming that

H3 - $g \in \mathcal{C}^2([0, 1])$, $g''(s_2) \leq 0$ for any $s_2 \in [0, 1]$,

the second order derivative of ϕ_1 in (15) is

$$
\begin{aligned}
\ddot{\phi}_1 &= c_h^2 (1 - s_1 - s_2) \left[b(\lambda_2 - \lambda_1) - s_1 g'(s_2)\lambda_2 + s_1^2 g''(s_2)\lambda_2 \right] u \\
&\quad + \mathcal{R}(s_1, s_2, \lambda_1, \lambda_2) ,
\end{aligned}
$$

and explicitly contains the control variable; therefore the problem is of order 1. We remark here that Monod function (10) satisfies also hypothesis **H3**.

The Legendre-Clebsch condition [27] provides a sufficient condition for the singular control to be a minimizer; it reads as

$$\mathcal{M}\left(s_1, s_2, \lambda_1, \lambda_2\right) := \frac{\partial}{\partial u} \frac{d^2}{dt^2} \frac{\partial \mathcal{H}_1}{\partial u} \leq 0 \tag{18}$$

for problems of order 1.

In our case, such derivative is given by

$$\mathcal{M}\left(s_1, s_2, \lambda_1, \lambda_2\right) = c_h^2 \left(1 - s_1 - s_2\right) \left[b\left(\lambda_2 - \lambda_1\right) - s_1 g'\left(s_2\right) \lambda_2 + s_1^2 g''\left(s_2\right) \lambda_2\right]$$

that in the interior of $\mathcal{F}$ and in correspondence of a singular control reduces to

$$\mathcal{N}\left(s_1, s_2\right) = -\frac{\alpha b c_h}{s_1} \left(1 - s_1 - s_2\right) \left[2 - \frac{g''\left(s_2\right)}{g'\left(s_2\right)} s_1\right], \tag{19}$$

since $s_1 > 0$, $g'\left(s_2\right) > 0$ and $\phi_1\left(t\right) = \dot{\phi}_1\left(t\right) = 0$ for any $t \in [t_1, t_2]$, where

$$\dot{\phi}_1\left(t\right) = c_h \left(1 - s_1 - s_2\right) \left[b\left(\lambda_2 - \lambda_1\right) + s_1 g'\left(s_2\right) \lambda_2\right].$$

For Monod response function (10), the Legendre-Clebsch condition is satisfied since

$$\mathcal{N}\left(s_1, s_2\right) = -2 \frac{\alpha b c_h}{s_1 \left(s_2 + c\right)} \left(1 - s_1 - s_2\right) \left(s_1 + s_2 + c\right) \leq 0,$$

and then the singular control can be a minimizer and the optimal control is not bang-bang in general.

4. **Optimal controls.** In this section we numerically compute the optimal controls by means of a technique based on a gradient method [2]; optimal strategies that minimize the objective functional (12), in linear and quadratic case, will be commented on.

We consider the case of Monod response function (10) and fix from now on the parameters values

$$c = 0.417, \quad b = 0.19, \quad c_h = 0.245. \tag{20}$$

We will discuss separately the cases of linear and quadratic objectives.

4.1. **Optimal profiles for the linear cost functional.** We recall that the optimal control has to minimize the objective functional

$$\mathcal{L}_1\left(u\right) = s_1^2(t_f) + s_2^2(t_f) + \alpha \int_0^{t_f} u\left(t\right) dt \tag{21}$$

along the solution of (5) for u belonging to the set $\mathcal{U}$ in (4). We consider the following parameter α and initial configuration:

$$\alpha = 0.01, \quad \left(s_1(0), s_2(0)\right) = \left(s_1^0, s_2^0\right) = (0.1, 0.5); \tag{22}$$

the initial amount of bacteria comes from conservation (2)

$$x(0) = x^0 = 1 - s_1^0 - s_2^0 = 0.4.$$

We fix in our simulations a temporal horizon $t_f = 10$.

Figure 4 shows that the optimal control is bang-bang in this case

$$u = \begin{cases} 0 & \text{for } t \in [0, t_s] \\ 1 & \text{for } t \in (t_s, t_f] \end{cases} \tag{23}$$

where $t_s \simeq 5.25$ is the switching time.

The characterization (14) for optimal controls is checked numerically, as shown in Figure 5, where the optimal control profile and the switching function ϕ_1 (given in
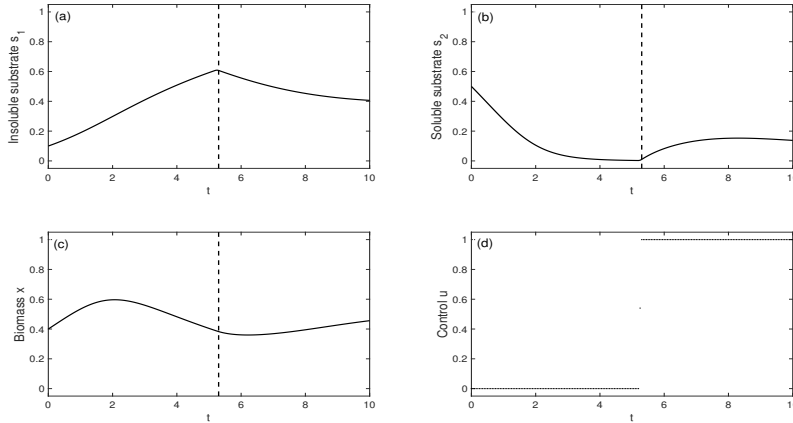
FIGURE 4. State variables profiles and optimal control for objective functional (21), Monod response function (10) and configuration (22). Parameters are given in (20). The optimal control is of bang-bang type with a unique switch from 0 to 1 for $t = t_s \simeq 5.25$ (dashed line).
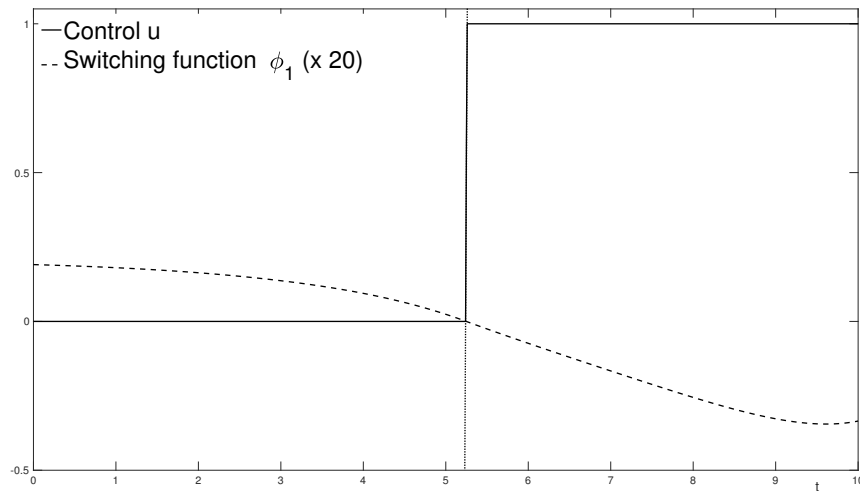

FIGURE 5. Optimal control u and scaled ($\times 20$) switching function ϕ_1 for objective functional (21), when parameters and initial configuration are given in (20) and (22), respectively.

(15)) are reported.

For $0 \leq t \leq t_s$ the control $u = 0$ corresponds to the absence of the hydrolytic process and the system evolution is driven only by degradation and biomass decay. As clearly indicated in (8), the variation of insoluble substrate concentration s_1 is due only to a gain term, coming from the biomass decay phenomenon. Bacteria
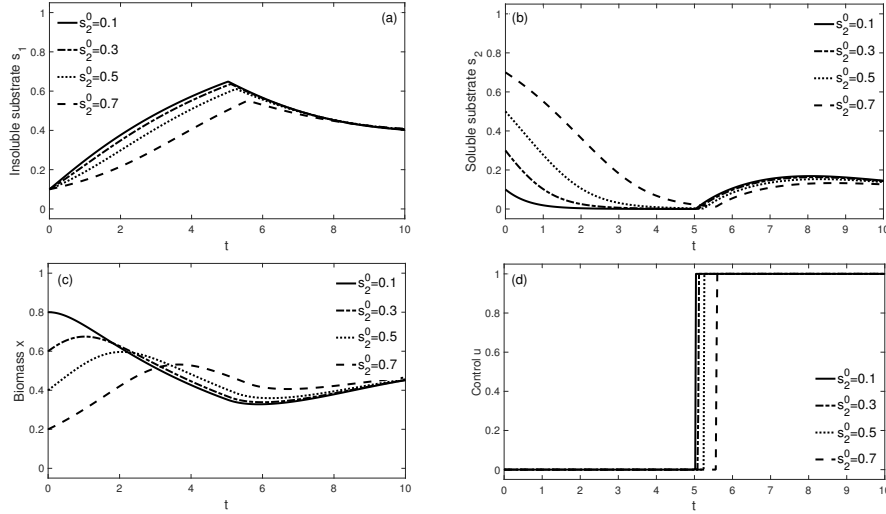
FIGURE 6. State variables and optimal control for objective functional (21); initial soluble concentrations s_2^0 varies from 0.1 to 0.7, while parameter $\alpha = 0.01$ and initial insoluble substrate $s_1^0 = 0.1$ remain fixed.

death produces new insoluble substrate, whose concentration increases (panel (a) in Figure 4). Analogously, we notice that the equation for soluble component s_2 has just a loss term due to degradation, and such component is expected to be totally consumed (panel (b) in Figure 4). As concerns biomass concentration x, its profile exhibits a peak at $t \simeq 2.05$ (panel (c) in Figure 4); from this time on the concentration of soluble substrate is not sufficient for degradation to balance the effects of bacteria death, and the biomass concentration decreases until the switching time t_s.

At the switching time, the hydrolysis starts to play a role; for $t > t_s$ the insoluble substrate decreases since the loss term now balances the gain one due to biomass decay (see first equation in system (5) and Figure 4(a)). Similarly, the variation of soluble substrate combines the loss term due to anaerobic degradation and the gain one coming from the solubilization of the insoluble component that also stimulates the bacterial growth (see second equation in system (5) and Figure 4(b-c)).

We discuss now the dependence of optimal profiles on the initial configuration.

First, we fix parameter α ($\alpha = 0.01$) and the first component of substrate s_1^0 at time $t = 0$ ($s_1^0 = 0.1$). Optimal profiles for varying s_2^0 are given in Figure 6. Figure 6 and Table 1 show that any difference in the initial value of soluble substrate does not affect significantly the solution, especially the switching time and the final configuration; in fact substrate components and biomass concentration at $t = t_f$, as well as switching times, are very close to each other. Small differences in the switching times suggest that a higher concentration of soluble substrate at $t = 0$ slightly delays the action of leachate recirculation, since initially a significant part of soluble substrate has to be consumed.

For $\alpha = 0.01$ and given initial soluble substrate $s_2^0 = 0.5$, we now investigate the

s_2^0	t_s	$s_1(t_f)$	$s_2(t_f)$
0.1	5.03	0.4031	0.1454
0.3	5.10	0.4043	0.1428
0.5	5.25	0.4063	0.1378
0.7	5.58	0.4084	0.1262

TABLE 1. Switching times and final substrate concentrations for objective functional (21), when soluble substrate s_2^0 varies ($\alpha = 0.01$, $s_1^0 = 0.1$).
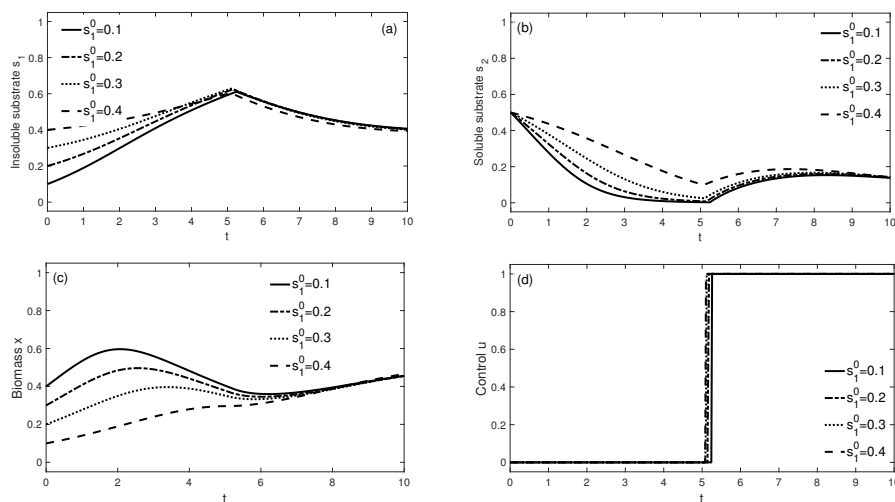

FIGURE 7. State variables and optimal control for objective functional (21); initial insoluble concentrations s_1^0 varies from 0.1 to 0.4, while parameter $\alpha = 0.01$ and initial soluble substrate $s_2^0 = 0.5$ remain fixed.

optimal solutions relevant to different values of the first component of substrate at $t = 0$. As in the previous case, we can observe (see Table 2) that the final configuration almost does not change when the initial component of the insoluble substrate is varied, since no significant difference can be noticed in substrate (and hence biomass) concentrations at $t = t_f$. As concerns the optimal controls, they are still piecewise constant with a unique switch from 0 to 1 (see Figure 7(d)) and the switching time is anticipated for increasing initial values of insoluble substrate, as shown in Table 2. This latter is confirmed also in Figure 8, where the biomass is almost absent and the total mass m is mainly given by the insoluble component of the substrate; in this case, the leachate recirculation starts to play a significant role very early and its action leads to a significant reduction of the total substrate (-42.6%).

Finally, we want to analyze the role of the parameter α in determining the optimal strategy. We remind here that the parameter α represents the relative weight of each term in the objective functional (12). Low values of α imply that the decision policy is mainly determined by the minimization of substrate concentrations at final
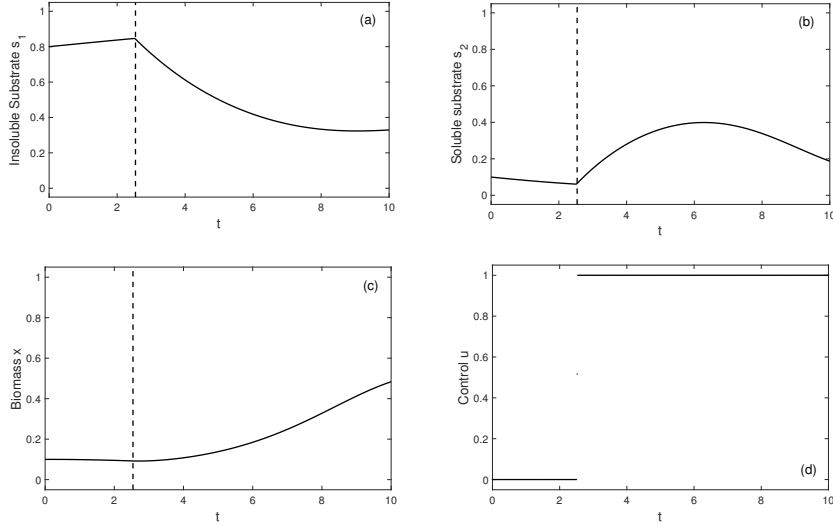
FIGURE 8. State variables and optimal control for objective functional (21), when the initial configuration is given by $(s_1^0, s_2^0) = (0.8, 0.1)$ and $\alpha = 0.01$.

s_1^0	t_s	$s_1(t_f)$	$s_2(t_f)$
0.1	5.25	0.4063	0.1378
0.2	5.16	0.4046	0.1406
0.3	5.10	0.4021	0.1422
0.4	5.09	0.3933	0.1414

TABLE 2. Switching times and final substrate concentrations for objective functional (21), when insoluble substrate s_1^0 varies ($\alpha = 0.01$, $s_2^0 = 0.5$).

time. When α is large the cost term plays the key role in the adopted strategy.

We consider the initial state $(s_1^0, s_2^0) = (0.1, 0.5)$ and let α vary from 0.001 to 1. Optimal controls (see Figure 9) are of bang-bang type in all of these considered cases. Initially the solution of the system is driven by biomass decay and degradation phenomena only; then, at the switching time t_s, the hydrolysis takes part to the process and is stimulated by leachate recirculation.

For increasing values of α, the time interval where the control is not zero is reduced and the effort spent in controlling the process is lower and lower (see index $I = \int_0^{t_f} u(t)dt$ in Table 3). In fact, for high values of α, the leachate recirculation operation is considered so expensive that the optimal strategy avoids it; in this case the total amount of substrate at final time is very large, since the absence of leachate recirculation does not allow the solubilization of the insoluble component of the substrate.

We point out that, even if in principle singular arcs have been proved to exist, for data set used in these simulations they never occurred.
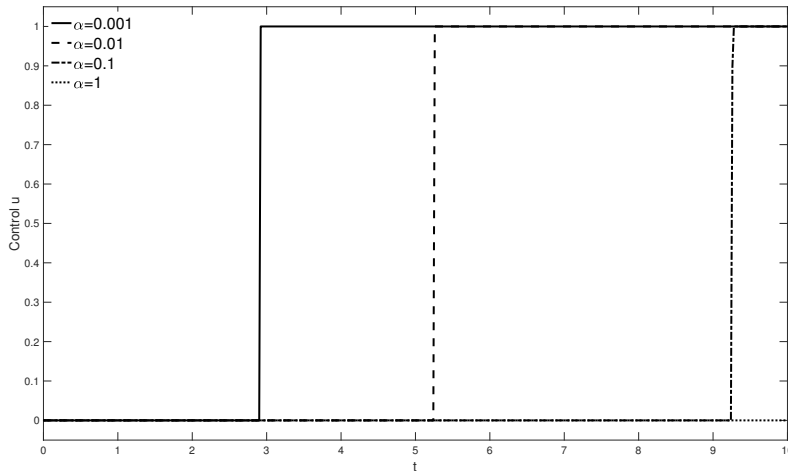
FIGURE 9. Optimal controls for objective functional (21), $\alpha = 0.001, 0.01, 0.1, 1$ and given initial configuration $(s_1^0, s_2^0) = (0.1, 0.5)$.

α	t_s	$I = \int_0^{t_f} u(t)dt$	$s_1(t_f)$	$s_2(t_f)$
0.001	2.91	7.0900	0.3943	0.0989
0.01	5.25	4.7500	0.4063	0.1378
0.1	9.26	0.7479	0.7037	0.1222
1	$-$	0	0.8420	0.0001

TABLE 3. Switching times, global effort required to control the system and final substrate concentrations for objective functional, when $\alpha = 0.001$, 0.01, 0.1, 1 and $(s_1^0, s_2^0) = (0.1, 0.5)$.

4.2. **Optimal control with the quadratic cost functional.** We now discuss the case of quadratic cost functional

$$\mathcal{L}_2(u) = s_1^2(t_f) + s_2^2(t_f) + \alpha \int_0^{t_f} u^2(t) \, dt. \tag{24}$$

We consider the same reference case of Subsection 4.1, by taking a Monod response function for bacterial growth and setting α and initial substrate concentrations as

$$\alpha = 0.01, \qquad (s_1^0, s_2^0) = (0.1, 0.5). \tag{25}$$

The optimal control in such case results

$$u = \begin{cases} \tilde{u} & \text{if } \phi_2 = 0 \\ 1 & \text{if } \phi_2 < 0 \, , \end{cases}$$

where $\tilde{u}$ is given by (16), and the optimal solution is shown in Figure 10.

The above characterization of optimal controls can be verified numerically, as shown in Figure 11, where the function ϕ_2 is plotted together with the optimal control function u versus time.
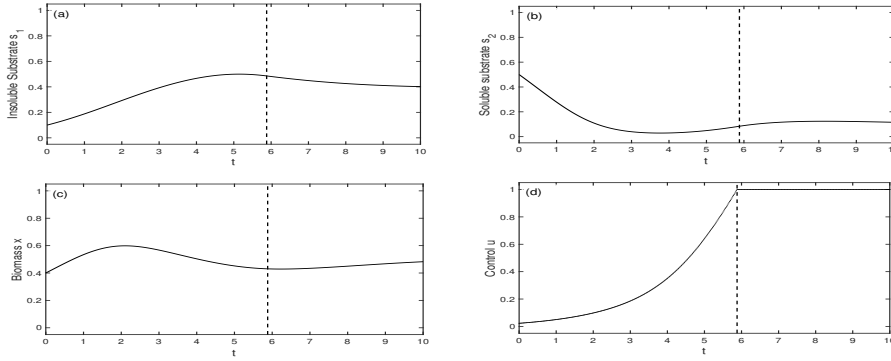
FIGURE 10. State variables profiles and optimal control objective functional (24) with Monod response function (10). Initial configuration and parameter α are given in (25); other parameters are given in (20).
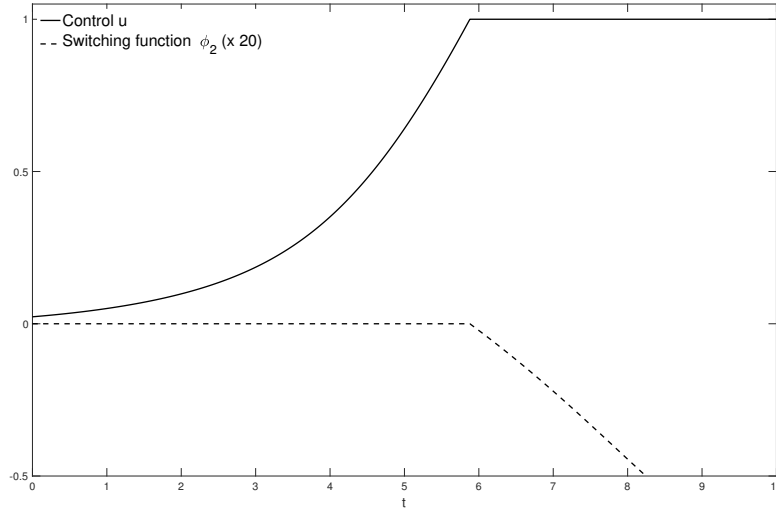

FIGURE 11. Optimal control u and scaled ($\times 20$) switching function ϕ_2 for objective functional (24), when parameter α and initial configuration are given in (25); other parameters are given in (20).

When the cost is modeled by a quadratic function of the control u, in our simulations the optimal control is strictly positive for any time $t \geq 0$, contrary to the linear case. In detail, the initial value of the control is very small, then it gradually increases and reaches its maximal admissible value $u = 1$ at $\tilde{t} \simeq 5.88$; from this time on, the control assumes constantly the value 1.

We observe that, in the first phase when the controll is small, the evolution is mainly driven by degradation and biomass decay phenomena. The soluble substrate is partially consumed, while the insoluble component increases. When the
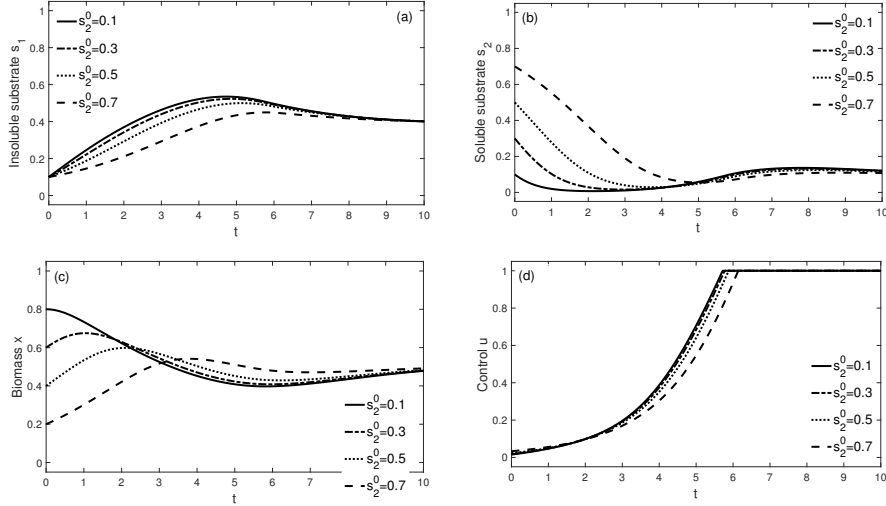
FIGURE 12. State variables and optimal control for objective functional (24), for different values of s_2^0 ($\alpha = 0.01$, $s_1^0 = 0.1$).

control increases and the effects of the leachate recirculation influence significantly the hydrolytic process, the trend of substrate components changes; the insoluble substrate strictly decreases, since the hydrolysis balances the biomass decay process. Analogously, the soluble substrate concentration increases, since the loss term due to the degradation is now balanced by the hydrolytic gain term. As concerns biomass concentration, its profile exhibits a peak (see Figure 10(c)), as in the case of linear cost functional.

As concerns the dependence of optimal profiles on the initial state, first we consider the case with given parameter $\alpha = 0.01$ and fixed initial insoluble substrate concentration $s_1^0 = 0.1$, and let s_2^0 vary from 0.1 to 0.7. We observe (see Figure 12 and Table 4) that configurations at $t = t_f$ are almost the same for any choice of s_2^0; in particular, different values of s_2^0 do not affect the final amounts of substrate and biomass.

Slight differences can be observed in optimal control profiles in Figure 12(d); in more detail, time $\tilde{t}$ at which the control assumes its constant maximal value is delayed for increasing values of s_2^0, as also shown in Table 4.

Analogous results are obtained when we fix $s_2^0 = 0.5$ and let s_1^0 vary (see Figure 13 and Table 5); also in this case, the optimal control profiles are qualitatively similar and show only negligible differences (as in the linear case).

Finally we briefly discuss the dependence of optimal controls on parameter α.

We remind that the weight of the cost term on the decision policy is negligible, when α is very small; the main goal is the reduction of substrate components at the final time. For such reason, the optimal strategy is based on the key role of the leachate recirculation on the evolution (see Figure 14). The optimal control $u(t)$ assumes lower and lower values for increasing parameter α, since any control operation is considered too expensive to be implemented when α is large; consequently, the effort I is lower and lower (see Table 6) for increasing α. Moreover, we can

s_2^0	$\tilde{t}$	$s_1(t_f)$	$s_2(t_f)$
0.1	5.72	0.4008	0.1210
0.3	5.78	0.4012	0.1192
0.5	5.88	0.4018	0.1157
0.7	6.16	0.4010	0.1075

TABLE 4. First time $\tilde{t}$ at which the control assumes constantly its maximal value and final substrate concentrations for objective functional (24) and different values of s_2^0 ($\alpha = 0.01$, $s_1^0 = 0.1$).
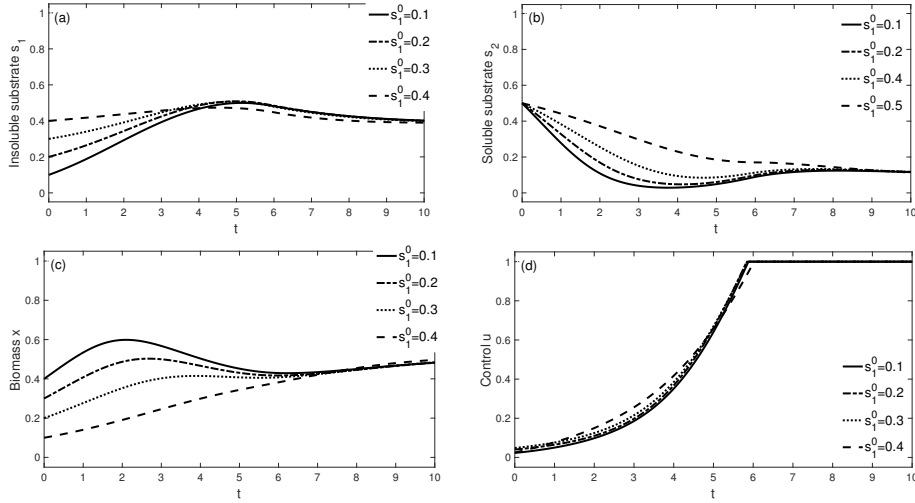

FIGURE 13. State variables and optimal control for objective functional (24), for different values of s_1^0 ($\alpha = 0.01$, $s_2^0 = 0.5$).

s_1^0	$\tilde{t}$	$s_1(t_f)$	$s_2(t_f)$
0.1	5.88	0.4018	0.1157
0.2	5.84	0.4009	0.1171
0.3	5.86	0.3990	0.1169
0.4	6.04	0.3902	0.1135

TABLE 5. First time $\tilde{t}$ at which the control assumes constantly its maximal value and final substrate concentrations for objective functional (24) and different values of s_1^0 ($\alpha = 0.01$, $s_2^0 = 0.5$).

observe that the final amount of insoluble substrate s_1 is very large, because of the negligible role played by the hydrolysis in system evolution (see Table 6).

5. **Conclusions.** We have proposed a mathematical model for anaerobic degradation under perfect mixing conditions. We have taken into account a two components substrate whose soluble component is degraded by an anaerobic bacterial population
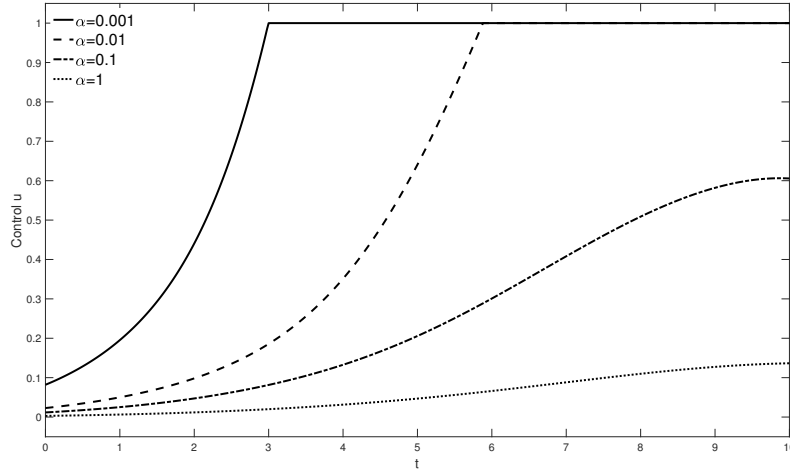
FIGURE 14. Optimal controls for objective functional (24) and varying $\alpha = 0.001, 0.01, 0.1, 1$, when $(s_1^0, s_2^0) = (0.1, 0.5)$.

α	$\tilde{t}$	$I = \int_0^{t_f} u(t)dt$	$s_1(t_f)$	$s_2(t_f)$
0.001	3.00	8.1174	0.3942	0.0961
0.01	5.88	5.8249	0.4018	0.1157
0.1	–	2.6023	0.5600	0.1186
1	–	0.5777	0.7659	0.0383

TABLE 6. First time $\tilde{t}$ at which the control assumes constantly its maximal value, global effort I required to control the process, final substrate concentrations for objective functional (24), when $\alpha = 0.001, 0.01, 0.1, 1$ and $(s_1^0, s_2^0) = (0.1, 0.5)$.

and the insoluble one undergoes a solubilization process. We have also considered the biomass decay. The control variable is assumed to model the effects of leachate recirculation on the hydrolytic process.

We have discussed the optimal strategies of leachate recirculation in order to minimize the substrate concentrations at a fixed time, which represents the end of the first phase of the anaerobic processes in biogas production, and also to minimize the operation costs for monitoring the hydrolysis process. We have found the optimal control when the cost of the recirculation operation is modeled by integral functionals depending linearly or quadratically on the control variable.

In the case of linear costs, the optimal control is of bang-bang type for the cases considered in the analysis above. The evolution is first driven by degradation and biomass decay only; the hydrolytic process occurs just in the last part of the evolution. Instead, in case of quadratic costs, the optimal solution is smooth and the control assumes intermediate values and reaches the maximum gradually. In the extensive numerical investigation, not shown here for brevity, we have found that

the time at which the control becomes effective depends crucially on the fixed final horizon t_f; the switching time (in the linear case) or the first instant in which the control is not zero (in the quadratic case) becomes larger and larger if t_f is increased. Moreover, for both linear and quadratic cases, for increasing values of α, the leachate recirculation is not considered cost-effective and the optimal control tends to the constant $u = 0$.

We have discussed also the dependence of the final state on the initial datum, pointing out that results seem not to depend deeply on the initial substrate concentrations. The adopted optimal strategies allow to reach similar final amount of soluble and insoluble substrates starting from different initial configurations.

Such preliminary results can be seen as a contribution to individuate optimal control strategies for solid waste reduction in landfills, in support of the few results available in literature [5, 25]; other distinctive features, like the maximization of the biogas production and the minimization of storage times, could also be taken into account. In the first case, the objective functional has to be modified by requiring the minimization of substrate components in the entire time range; in fact, any substrate reduction corresponds to methane production, whose conversion rate should be eventually scaled by a given coefficient. In the second case, a time optimal control problem has to be formulated, in order to move from a given initial configuration to a target scenario in minimal time. These problems will be subject of future investigations.

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E-mail address: marzia.bisi@unipr.it

E-mail address: maria.groppi@unipr.it

E-mail address: giorgio.martalo@unipr.it

E-mail address: romina.travaglini@unipr.it