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Analyticity of nonsymmetric Ornstein-Uhlenbeck semigroup with respect to a weighted Gaussian measure

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Abstract

In this paper we show that the realization in $\mathrm{L}^p(X,\nu_\infty)$ of a nonsymmetric Ornstein-Uhlenbeck operator L is sectorial for any $p\in(1,+\infty)$ and we provide an explicit sector of analyticity. Here (X,μ_∞,H_∞) is an abstract Wiener space, i.e., X is a separable Banach space, μ_∞ is a centred non degenerate Gaussian measure on X and H_∞ is the associated Cameron-Martin space. Further, ν_∞ is a weighted Gaussian measure, that is, $\nu_\infty=e^{-U}\mu_\infty$ where U is a convex function which satisfies some minimal conditions. Our results strongly rely on the theory of nonsymmetric Dirichlet forms and on the divergence form of the realization of L in $\mathrm{L}^2(X,\nu_\infty)$.

Keywords: Infinite dimensional analysis; Wiener spaces; analytic semigroups; Ornstein-Uhlenbeck operators; numerical range

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1 Introduction

In this paper we prove that the realization in in $\mathrm{L}^p(X,\nu_\infty)$ of the nonsymmetric perturbed Ornstein-Uhlenbeck operator L_p operator defined on smooth functions f by

$$L_pf(x)=\frac{1}{2}\mathrm{Tr}[D^2f(x)]_H+\langle x,A^*Df(x)\rangle_{X\times X^*}+[D_Hf(x),D_HU(x)]_H,\quad x\in X, \tag{1.1}$$

where U is a suitable function (see e.g. [6, 10, 15]), is sectorial and we provide an explicit sector of analyticity.

In finite dimension, the Ornstein-Uhlenbeck operator is the uniformly elliptic second order differential operator $\mathscr{L}$ defined on smooth functions φ by

$$\mathscr{L}\varphi(\xi)=\sum_{i,j=1}^n q_{ij}D^2_{ij}\varphi(\xi)+\sum_{i,j=1}^n a_{ij}\xi_jD_i\varphi(\xi),\quad \xi\in\mathbb{R}^n,$$

where $Q=(q_ij)_{i,j=1}^n$ is a positive definite matrix and $A=(a_{ij})_{i,j=1}^n$. It is well known (see [27, 28]) that $\mathscr{L}$ may fail to generate an analytic semigroup on $\mathrm{L}^p(\mathbb{R}^n)$. The additional assumption $\sigma(A)\subseteq\{z\in\mathbb{C}:\mathrm{Re}z<0\}$ implies that the integral

$$Q_\infty:=\int_0^{+\infty}e^{tA}Qe^{tA^*}dt,$$

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is well defined. The centred Gaussian measure μ_∞ with covariance Q_∞ is an invariant measure for $\mathscr{L}$, i.e.,

$$\int_{\mathbb{R}^n} \mathscr{L} f d\mu_\infty = 0, \quad f \in D(\mathscr{L}).$$

$\mathscr{L}$ behaves well on $\mathrm{L}^p(\mathbb{R}^n, \mu_\infty)$. Indeed, the realization L_p of $\mathscr{L}$ in $\mathrm{L}^p(\mathbb{R}^n, \mu_\infty)$ generates an analytic semigroup for any $p \in (1, +\infty)$. Further, in [8] the authors explicitly provide a sector

$$\Sigma_{\theta_p} := \{re^{i\phi} \in \mathbb{C} : \ r > 0, \ |\phi| \le \theta_p\}, \tag{1.2}$$

where $\theta_p \in (0, \pi/2)$ is an angle which depends on Q, A and p, such that L_p is sectorial in Σ_{θ_p}. This sector is optimal, in the sense that if $\theta \in (0, \pi/2)$ is an angle such that L_p is sectorial in Σ_θ, then $\theta \le \theta_p$. In [9] the same authors extend this result to nonsymmetric submarkovian semigroups.

In infinite dimension the situation is much more complicated. We consider an abstract Wiener spaces $(X, \mu_\infty, H_\infty)$, where X is a separable Banach space μ_∞ is a centred nondegenerate Gaussian measure on X and H_∞ is the associated Cameron-Martin space (see e.g. [4]). It is well known that $H_\infty \subseteq X$ is a Hilbert space with inner product $[\cdot, \cdot]_{H_\infty}$. Let us denote by $Q_\infty : X^* \to X$ the covariance operator of μ_∞. In this setting, the definition of the Ornstein-Uhlenbeck operator can be given in terms of bilinear forms: for smooth functions $f, g : X \to \mathbb{R}$ we set

$$\mathcal{E}(f, g) := \int_X [D_{H_\infty} f, D_{H_\infty} g]_{H_\infty} d\mu_\infty,$$

where $D_{H_\infty} = Q_\infty D$ is the gradient along the directions of H_∞. Following [24] it is possible to associate an operator $\mathscr{L}_2$ to $\mathcal{E}$ as follows: for any $f \in D(\mathscr{L}_2)$ and any g smooth enough we have

$$\mathcal{E}(f, g) = -\int_X \mathscr{L}_2 f g d\mu_\infty$$

The operator $\mathscr{L}_2$ is self-adjoint and generates a analytic contraction C_0-semigroup on $\mathrm{L}^2(X, \mu_\infty)$. Moreover, if $f = \varphi(x_1^*, \ldots, x_n^*)$ for some smooth function φ and $x_i^* \in X^*$, $i = 1, \ldots, n$, then the operator $\mathscr{L}_2$ reads as

$$\mathscr{L}_2 f := \sum_{i,j=1}^n q_{ij}^0 \frac{\partial^2 \varphi}{\partial \xi_i \partial \xi_j} - \sum_{,i=1}^n |Q_\infty x_i^*|_{H_\infty}^{-1} x_i^* \frac{\partial \varphi}{\partial \xi_i},$$

where $q_{ij}^0 = \langle Q_\infty x_j^*, x_i^* \rangle_{X \times X^*}$. In [19] the authors provide a generalization of $\mathscr{L}_2$, defining the Wiener space $(X, \mu_\infty, H_\infty)$ as follows. They consider two operators $Q : X^* \to X$ and $A : D(A) \subset X \to X$ such that Q is a linear, bounded positive and symmetric operator (see Hypothesis 2.1) and A is the infinitesimal generator of a strongly continuous semigroup. Further, if we denote by $(e^{tA})_{t \ge 0}$ the semigroup generated by A, they assume that the integral

$$\int_0^\infty e^{tA} Q e^{tA^*} dt,$$

with values in $\mathcal{L}(X^*; X)$, exists as a Pettis integral and the operator $Q_\infty : X^* \to X$ defined by

$$Q_\infty x^* := \int_0^\infty e^{tA} Q e^{tA^*} dt \, x^*$$

is the covariance operator of the Gaussian measure μ_∞. In such a way they can define the Reproducing Kernel Hilbert Space H associated to Q, and they prove the closability of a gradient operator $D_H = QD$. Thanks to a stochastic representation, the authors define the semigroup $P(t)$ and its infinitesimal generator $\mathbb{L}$ on $\mathrm{L}^p(X,\mu_\infty)$ which on smooth functions f (with $f=\varphi(x_1^*,\dots,x_n^*)$, for some smooth function φ and $x_i^*\in D(A^*)$, $i=1,\dots,n$) reads as

$$\mathbb{L}f:=\sum_{i,j=1}^n q_{ij}\frac{\partial^2\varphi}{\partial\xi_i\partial\xi_j}+\sum_{i=1}^n Ax_i^*\frac{\partial\varphi}{\partial\xi_i},$$

with $q_{ij}=\langle Qx_i^*,x_j^*\rangle_{X\times X^*}$. Further, from the results in [18], the authors deduce that the set

$$\mathscr{F}_0:=\{f\in\mathscr{F}:\langle\cdot,A^*Df\rangle_{X\times X^*}\in C_b(X)\},$$

is a core for $\mathbb{L}$. Here $\mathscr{F}$ is the set of functions $f\in C_b^2(X)$ such that there exists $\varphi\in C_b^2(\mathbb{R}^n)$ and $x_1^*,\dots,x_n^*\in D(A^*)$ such that $f(x)=\varphi(\langle x,x_1^*\rangle_{X\times X^*},\dots,\langle x,x_1^*\rangle_{X\times X^*})$ for any $x\in X$. Finally, arguing as in [16], the authors show different characterizations of the analyticity of $P(t)$. In particular, they prove that $P(t)$ is analytic in $\mathrm{L}^2(X,\mu_\infty)$ if and only if $Q_\infty A^*x^*\in H$ for any $x^*\in D(A^*)$ and there exists a positive constant c such that

$$|Q_\infty A^*x^*|_H\le c|Qx^*|,\quad x^*\in D(A^*).$$

This characterization is the starting point of [25], where the authors generalize the results in [8] to the infinite dimensional case without any assumption on the nondegeneracy of Q. To begin with, they prove that the operator $B\in\mathcal{L}(H)$, which is the extension of $Q_\infty A^*$ to the whole H, satisfies $B+B^*=-Id_H$. Further, setting

$$\mathcal{E}_B(u,v):=-\int_X[BD_Hu,D_Hv]_Hd\mu_\infty,$$

on smooth functions u,v, the authors show that $\mathbb{L}$ is indeed the operator associated in $\mathrm{L}^2(X,\mu_\infty)$ to the nonsymmetric bilinear form $\mathcal{E}_B$ in the sense of [24, Chapter 1], i.e., for any u,v smooth enough,

$$\mathcal{E}_B(u,v)=-\int_X\mathbb{L}uvd\mu_\infty.$$

This implies that, if we denote by D_H^* the adjoint operator of D_H in $\mathrm{L}^2(X,\mu_\infty)$, then $\mathbb{L}=D_HBD_H$, and by means of the divergence form of $\mathbb{L}$ the authors avoid the nondegeneracy assumption on Q. Finally, by applying well known results on the numerical range (see [3, 22]) the authors prove that for any $p\in(1,+\infty)$ the semigroup $P(t)$ is analytic in $\mathrm{L}^p(X,\mu_\infty)$ with sector of analiticity Σ_{θ_p} defined in (1.2). Also in this case, this sector is optimal. We remark that, differently from $\mathscr{L}_2$, in general the operator $\mathbb{L}$ is not self-adjoint and therefore it is not possible to use the theory of self-adjoint operators to prove the analyticity of $\mathbb{L}$.

We prove that (1.1) is the operator associated in $\mathrm{L}^2(X,\nu_\infty)$ to the nonsymmetric bilinear form in

$$\mathcal{E}_B^\nu(u,v):=-\int_X[BD_Hu,D_Hv]_Hd\nu_\infty,$$

in the sense of [24], where

$$\nu_\infty:=e^{-U}\mu_\infty.$$

Further, $L_2=D_H^*BD_H$, where D_H^* denotes the adjoint operator of D_H in $\mathrm{L}^2(X,\nu_\infty)$. By taking advantage of the divergence form of L_2, we use analytic techniques to extend L_2 and the associated semigroup to $\mathrm{L}^p(X,\nu_\infty)$, $p\in(1,+\infty)$. Finally, we prove that the semigroup associated to L_p is analytic in $\mathrm{L}^p(X,\nu_\infty)$.

We stress that, at the best of our knowledge, in the case of perturbed Ornstein-Uhlenbeck operator no explicit core of L_p is known. However, the explicit representation (1.1) of L_p on smooth functions allows us to find a suitable sets of smooth functions which will play the role of $\mathscr{F}_0$.

The paper is organized as follows. In Section 2 we uniform the notations used in the symmetric and in nonsymmetric case, which are different and sometimes may give rise to confusion and misunderstandings. Then, we prove that D_H is closable on smooth functions in $\mathrm{L}^p(X,\nu_\infty)$ for any $p\in(1,+\infty)$ and define the Sobolev spaces as the domain of the closure of D_H. Section 3 is devoted to define the nonsymmetric Ornstein-Uhlenbeck operator and semigroup in $\mathrm{L}^p(X,\nu_\infty)$. At first, thanks to the theory of nonsymmetric Dirichlet forms, we provide the definition of the Ornstein-Uhlenbeck operator and semigroup in $\mathrm{L}^2(X,\nu_\infty)$. Later, we extend both the operator L_2 and the semigroup to any $\mathrm{L}^p(X,\nu_\infty)$, $p\in(1,\infty)$, and we conclude the section by showing an explicit formula for L_p on smooth functions when $p\in(1,\infty)$, and the inclusion $D(L_p)\subset D(L_2)$ for any $p\in[2,+\infty)$. These results allow us to overcome the fact that we don't know a core for L_p. In Section 4 we use the numerical range to show that L_p generates an analytic semigroup in $\mathrm{L}^p(X,\nu_\infty)$ with sector Σ_{θ_p} for any $p\in(1+\infty)$. We are not able to show the optimality of this sector since the techniques applied both in [8] and in [25] don't work in infinite dimension with a weighted Gaussian measure. Finally, in Section 5 we provide a explicit example of operators Q and A and of function U which satisfy our assumptions.

1.1 Notations

Let X be a separable Banach space. We denote by $\langle\cdot,\cdot\rangle_{X\times X^*}$ the duality, by $\|\cdot\|_X$ its norm and by $\|\cdot\|_{X^*}$ the norm of its dual. Further, for a general Banach space V we denote by $\mathcal{L}(V)$ the space of linear operators from V onto V endowed with the operator norm. For any $k\in\mathbb{N}\cup\{\infty\}$ and any $n\in\mathbb{N}$ we denote by $C_b^k(\mathbb{R}^n)$ the continuous and bounded functions on $\mathbb{R}^n$ whose derivatives up to the order k are continuous and bounded.

2 Preliminaries and Sobolev spaces

We state the following assumptions on the operators Q and A.

Hypothesis 2.1. (i) $Q: X^*\to X$ is a linear and bounded operator which is symmetric and nonnegative, i.e.,

$$\langle Qx^*,y^*\rangle_{X\times X^*}=\langle Qy^*,x^*\rangle_{X\times X^*},\quad \langle Qx^*,x^*\rangle_{X\times X^*}\geq 0,\quad \forall x^*,y^*\in X^*.$$

(ii) $A: D(A)\subseteq X\to X$ is the infinitesimal generator of a strongly continuous contraction semigroup $\left(e^{tA}\right)_{t\geq 0}$ on X.

We recall that for any positive and symmetric operator we can define the associated Reproducing Kernel.

Definition 2.2. Let $F: X^*\to X$ be a linear, bounded, positive and symmetric operator. On FX^* we define the inner product $[Fx^*,Fy^*]_K:=\langle Fx^*,y^*\rangle_{X\times X^*}$ for any $x^*,y^*\in X^*$. We denote by $|Kx^*|_K^2:=\langle Fx^*,x^*\rangle_{X\times X^*}$ the associated norm. We set $K:=\overline{FX^*}^{|\cdot|_K}$ and we call it the Reproducing Kernel Hilbert Space (RKHS) associated with F.

From [31, Proposition 1.2] the function $s\mapsto e^{sA}Qe^{sA^*}$ is strongly measurable and we may define, for any $t>0$, the positive symmetric operator $Q_t\in\mathcal{L}(X^*;X)$ by

$$Q_t:=\int_0^t e^{sA}Qe^{sA^*}ds.$$

Further, we denote by H_t the Reproducing Kernel Hilbert Space associated to Q_t. We assume that the family of operators $(Q_t)_{t\geq 0}$ satisfies the following hypotheses (see e.g. [19, Sections 2 & 6]).

Hypothesis 2.3. 1. The operator Q_t is the covariance operator of a centred Gaussian measure μ_t on X for any $t>0$.

2. For any $x^* \in X^*$, there exists weak$-\lim_{t\to+\infty} Q_t x^* =: Q_\infty x^*$ and Q_∞ is the covariance operator of a centred nondegenerate Gaussian measure μ_∞.

Hypothesis 2.3(2) implies that

$$\widehat{\mu_\infty}(f) = \exp\left(-\frac{1}{2}\langle Q_\infty f, f\rangle_{X\times X^*}\right), \quad f \in X^*.$$

We follow [4, Chapter 2] to construct the Cameron-Martin space H_∞ associated to μ_∞, which gives the abstract Wiener space $(X, \mu_\infty, H_\infty)$. In particular, we focus on a characterization of H_∞ which allows us to associate a Hilbert space $H \subset X$ to the operator Q.

From [4, Fernique Theorem 2.8.5] it follows that $X^* \subset \mathrm{L}^2(X, \mu_\infty)$, and we denote by $j : X^* \to \mathrm{L}^2(X, \mu_\infty)$ the injection of X^* in $\mathrm{L}^2(X, \mu_\infty)$. Further, from [4, Theorem 2.2.4] we have

$$\langle Q_\infty f, g\rangle_{X\times X^*} = \int_X fg d\mu_\infty, \quad f, g \in X^*. \tag{2.1}$$

We denote by $X^*_{\mu_\infty}$ the closure of $j(X^*)$ in $\mathrm{L}^2(X, \mu_\infty)$ and we define $R : X^*_{\mu_\infty} \to (X^*)'$ by

$$R(f)(g) := \int_X fg d\mu_\infty, \quad f \in X^*_{\mu_\infty},\ g \in X^*. \tag{2.2}$$

It is possible to prove that $R(X^*_{\mu_\infty})f$ is weakly*-continuous for any $f \in X^*$, and therefore $R(X^*_{\mu_\infty}) \subset X$. For any $f \in X^*_{\mu_\infty}$ we still denote by $R(f)$ the unique element $y \in X$ such that $R(f)(g) = \langle y, g\rangle_{X\times X^*}$ for any $g \in X^*$. Further, the injection j is the adjoint operator of R. The Cameron-Martin space H_∞ associated to μ_∞ is defined as follows (see e.g. [4, Chapter 2, Section 2]):

$$\begin{aligned} |h|_{H_\infty} &:= \sup\left\{\langle h, \ell\rangle_{X\times X^*} : \ell \in X^*,\ R(\ell)(\ell) = \|R^*\ell\|^2_{\mathrm{L}^2(X,\mu_\infty)} \le 1\right\}, \\ H_\infty &:= \{h \in X : |h|_{H_\infty} < +\infty\}. \end{aligned}$$

From [4, Lemma 2.4.1] it follows that $h \in H_\infty$ if and only if there exists $\widehat{h} \in X^*_{\mu_\infty}$ such that $R(\widehat{h}) = h$. Further, H_∞ is a Hilbert space if endowed with inner product

$$[h, k]_{H_\infty} = \langle \widehat{h}, \widehat{k}\rangle_{\mathrm{L}^2(X,\mu_\infty)}, \quad h, k \in H_\infty. \tag{2.3}$$

We stress that for any $f \in X^*$, from (2.1) and (2.2) we have $Q_\infty f \in H_\infty$ and that $R(R^* f) = Q_\infty f$, i.e., $\widehat{Q_\infty f} = R^* f$. Further, from (2.3) we deduce that

$$\langle Q_\infty f, g\rangle_{X\times X^*} = [Q_\infty f, Q_\infty g]_{H_\infty}, \quad f, g \in X^*. \tag{2.4}$$

We get the following characterization of H_∞.

Lemma 2.4. $H_\infty = \overline{Q_\infty X^*}^{|\cdot|_{H_\infty}}$*, that is, the Cameron-Martin space* H_∞ *is the closure in* $|\cdot|_{H_\infty}$ *of* $Q_\infty X^* \subset X$.

Proof. The proof is quite simple but we provide it for reader's convenience. Let $h \in H_\infty$. Then, there exists $\widehat{h} \in X^*_{\mu_\infty}$ such that $R_{\mu_\infty}(\widehat{h}) = h$. In particular, there exists $(R^* f_n) \subset X^*$ such that $R^* f_n \to \widehat{h}$ in $\mathrm{L}^2(X, \mu_\infty)$. We claim that $Q_\infty f_n \to h$ in H_∞. Indeed, from (2.3) and recalling that $\widehat{Q_\infty f_n} = R^* f_n$ for any $n \in \mathbb{N}$, it follows that

$$|Q_\infty f_n - h|^2_{H_\infty} = [Q_\infty f_n - h, Q_\infty f_n - h]_{H_\infty} = \int_X |R^* f_n - \widehat{h}|^2 d\mu_\infty \to 0, \quad n \to +\infty.$$

This means that $H_\infty \subseteq \overline{Q_\infty X^*}^{|\cdot|_{H_\infty}}$. The converse inclusion follows from analogous arguments. $\square$

Let us consider the continuous injection of $Q_\infty X^*$ into X which can be continuously extend to H_∞. We denote by i_∞ the extension of the injection. If we denote by $i_\infty^* : X^* \to (H_\infty)'$ the adjoint operator and we identify $(H_\infty)'$ with H_∞ by means of the Riesz Representation Theorem, then $Q_\infty = i_\infty \circ i_\infty^*$. Further, for any $f, g \in X^*$ we have

$$\langle i_\infty \circ i_\infty^* f, g\rangle_{X\times X^*} = [i_\infty^* f, i_\infty^* g]_{H_\infty} = \langle R^* f, R^* g\rangle_{\mathrm{L}^2(X,\mu_\infty)} = \langle Q_\infty f, g\rangle_{X\times X^*}, \tag{2.5}$$

which gives $Q_\infty = i_\infty \circ i_\infty^*$.

Lemma 2.5. *H_∞ admits an orthonormal basis $\Theta := \{e_n : n \in \mathbb{N}\}$ such that $e_n = i_\infty^* x_n^*$ with $x_n^* \in D(A^*)$ for any $n \in \mathbb{N}$.*

Proof. It is well known (see e.g. [20, Theorem 2.2]) that the weak*-closure of $D(A^*)$ coincides with X^*. Then, for any $x^* \in X^*$ there exists a sequence $(x_n^*) \subset D(A^*)$ such that $x_n^* \to x^*$ in the weak*-topology, that is, $\langle x, x_n^*\rangle_{X\times X^*} \to \langle x, x^*\rangle_{X\times X^*}$ for any $x \in X$. Therefore, for any $x \in X$ there exists a positive constant c_x such that $\sup_{n\in\mathbb{N}} |\langle x, x_n^*\rangle_{X\times X^*}| \le c_x$. The uniform boundedness principle gives $\sup_{n\in\mathbb{N}} \|x_n^*\|_{X^*} \le c$ for some positive constant c. By the dominated convergence theorem and the Fernique Theorem it follows that $R^* x_n^* \to R^* x^*$ in $\mathrm{L}^2(X, \mu_\infty)$. Combining this fact and (2.5) gives

$$|i_\infty^* x_n^* - i_\infty^* x^*|_{H_\infty}^2 = \int_X |\langle x, x_n^* - x^*\rangle_{X\times X^*}|^2 \mu_\infty(dx) \to 0,$$

as $n \to +\infty$. Therefore, $Q_\infty(D(A^*))$ is dense in $Q_\infty X$ with respect to $|\cdot|_{H_\infty}$. Since from [4, Corollary 3.2.8] $Q_\infty X$ is dense in H_∞, we conclude that $Q_\infty(D(A^*))$ is dense in H_∞. In particular, this implies that there exists an orthonormal basis of H_∞ of elements of $Q_\infty(D(A^*))$. □

We fix an orthonormal basis $\Theta := \{e_n : n \in \mathbb{N}\}$ of H_∞ such that $e_n = i_\infty^* x_n^*$ and $x_n^* \in D(A^*)$ for any $n \in \mathbb{N}$. We denote by $P_n : X \to H_\infty$ the projection on $\mathrm{span}\{e_1, \ldots, e_n\}$ defined by

$$P_n x := \sum_{k=1}^{n} \widehat{e}_n(x) e_n, \quad x \in X,\ n \in \mathbb{N},$$

where $\widehat{e}_j := R^* x_j^*$ for any $j \in \mathbb{N}$.

Definition 2.6. For any $k \in \mathbb{N} \cup \{\infty\}$ we denote by $\mathscr{F}\mathscr{C}^k_{b,\Theta}(X)$ the space of cylindrical functions $f \in C_b^k(X)$ such that there exists $n \in \mathbb{N}$ and $\varphi \in C_b^k(\mathbb{R}^n)$ which satisfies $f(x) = \varphi(\widehat{e}_1(x), \ldots, \widehat{e}_n(x))$ for any $x \in X$.

Remark 2.7. We stress that the space $\mathscr{F}\mathscr{C}^k_{b,\Theta}(X)$ is different from those considered in [1, 6, 10, 17, 19, 25, 26]. Indeed, in these papers the spaces $\mathscr{F}\mathscr{C}^k_b(X)$ or $\mathscr{F}\mathscr{C}^{k,\ell}_b(X)$, with $k, \ell \in \mathbb{N}$, are considered. The former is the space of cylindrical functions f such that there exists $\varphi \in C_b^k(\mathbb{R}^n)$ and $y_1, \ldots, y_n \in X^*$ such that $f(x) = \varphi(\langle x, y_1^*\rangle_{X\times X^*}, \ldots, \langle x, y_n^*\rangle_{X\times X^*})$ for any $x \in X$, the latter is the space of cylindrical functions f such that there exists $\varphi \in C_b^k(\mathbb{R}^n)$ and $z_1, \ldots, z_n \in D((A^*)^\ell)$ such that $f(x) = \varphi(\langle x, z_1^*\rangle_{X\times X^*}, \ldots, \langle x, z_n^*\rangle_{X\times X^*})$ for any $x \in X$. Even if the space $\mathscr{F}\mathscr{C}^k_{b,\Theta}(X)$ is smaller than $\mathscr{F}\mathscr{C}^k_b(X)$ and of $\mathscr{F}\mathscr{C}^{k,1}_b(X)$, it is "good" in the sense that it is big enough, since $\{x_n^* : n \in \mathbb{N}\}$ is an orthonormal basis of H_∞. Further, it is well known that $\mathscr{F}\mathscr{C}^k_{b,\Theta}(X)$ is dense in $\mathrm{L}^p(X, \nu_\infty)$ for any $p \in [1, +\infty)$ and any $k \in \mathbb{N}$ (see [4, Corollary 3.5.2]).

2.1 Reproducing Kernel associated to Q and Sobolev Spaces

Starting from (2.4) we can define the Reproducing Kernel Hilbert Space associated to Q (see also [31]).

We recall that Q is positive and symmetric. Then, following Definition 2.2 we can define a scalar product on QX^* and then, inspired by Lemma 2.4, the Reproducing Kernel Hilbert Space H associated to Q. H is a Hilbert space if endowed with the scalar product $[\cdot,\cdot]_H$. The inclusion $QX^* \hookrightarrow X$ can be extended to the injection $i : H \to X$ and we consider the adjoint operator $i^* : X^* \to H$, where again we have identify H' and H. Arguing as for i_∞ and i_∞^* we infer that $Q = i \circ i^*$.

The following hypothesis is very important since [19, Theorem 8.3] states that it is equivalent to the analyticity in $\mathrm{L}^p(X,\mu_\infty)$ of the Ornstein-Uhlenbeck semigroup $P(t)$ defined by

$$(P(t)f)(x) := \int_X f(e^{tA}x + y)\mu_t(dy), \quad f \in C_b(X),$$

and extended to $\mathrm{L}^p(X,\mu_\infty)$ for any $p \in (1,+\infty)$.

Hypothesis 2.8. For any $x^* \in D(A^*)$ we have $i_\infty^* A^* x^* \in H$ and there exists a positive constant c such that

$$|i_\infty^* A^* x^*|_H \le c|i^* x^*|_H, \qquad x \in D(A^*). \tag{2.6}$$

Since i^* is continuous with respect the weak* topology on X^* and the weak topology on H and $D(A^*)$ is weak* dense in X^*, itfollows that i^* maps $D(A^*)$ onto a dense subspace of H. Then, there exists an operator $B \in \mathcal{L}(H)$ such that $Bi^*x^* = i_\infty^* A^* x^*$ for any $x^* \in D(A^*)$ and $\|B\|_{\mathcal{L}(H)} \le c$. The operator B enjoys the following properties.

Lemma 2.9. *[25, Lemma 2.2] $B + B^* = -I_H$ and $[Bh,h]_H = -\frac{1}{2}|h|_H^2$ for any $h \in H$.*

We now introduce two operators which are crucial for the definition of Sobolev spaces in our context. The first one is the gradient along the directions of the Reproducing Kernel H, while the second allows to prove an integration by parts formula with respect to suitable directions in H (see e.g. [17, Section 3]).

Definition 2.10. Let $\Theta := \{e_n : n \in \mathbb{N}\}$ be the orthonormal basis of H_∞ introduced in Lemma 2.5. For any $p \in [1,+\infty)$ we define the operator $D_H : \mathscr{F}\mathscr{C}^1_{b,\Theta}(X) \to \mathrm{L}^p(X,\mu_\infty;H)$ by

$$D_H f(x) := i^* Df(x) = \sum_{j=1}^n \frac{\partial \varphi}{\partial \xi_j}(\langle x_1, x\rangle_{X\times X^*}, \ldots, \langle x_n, x\rangle_{X\times X^*}) i^* x_j^*, \quad x \in X,$$

where $f \in \mathscr{F}\mathscr{C}^1_{b,\Theta}(X)$ and $f(x) = \varphi(\langle x_1, x\rangle_{X\times X^*}, \ldots, \langle x_n, x\rangle_{X\times X^*})$ for some $n \in \mathbb{N}$, $\varphi \in C_b(\mathbb{R}^n)$ and any $x \in X$.

Definition 2.11. We define the operator $V : D(V) \subseteq H_\infty \to H$ as follows:

$$D(V) := \{i_\infty^* x^* : x^* \in X^*\}, \quad V(i_\infty^* x^*) = i^* x^*, \quad x^* \in X^*. \tag{2.7}$$

Since V is densely defined on H_∞ it is possible to consider the adjoint operator $V^* : D(V^*) \subset H \to H_\infty$. Thanks to Hypothesis 2.8 and [19, Theorems 8.1, 8.3 & Proposition 8.7] it follows that D_H is closable in $\mathrm{L}^p(X,\mu_\infty)$ and [17, Theorem 3.5] gives that the operator V is closable. We still denote by D_H the closure of D_H and by $W_H^{1,p}(X,\mu_\infty)$ the domain of the closure.

Lemma 2.12. *For any $x^* \in D(A^*)$, we have $Bi^*x^* \in D(V^*)$ and $V^*(Bi^*x^*) = i_\infty^* A^* x^*$.*

Proof. The statement is contained in the proof of [25, Theorem 2.3], but for reader's convenience we provide the simple proof. Let $x^* \in D(A^*)$. Then, for any $y^* \in X^*$, from the definition of $[\cdot,\cdot]_H$, of $[\cdot,\cdot]_{H_\infty}$ and of V we have

$$[Bi^*x^*, V(i_\infty^* y^*)]_H = [Bi^*x^*, i^*y^*]_H = [i_\infty^* A^* x^*, i^* y^*]_H = \langle i_\infty^* A^* x^*, y^* \rangle_{X\times X^*} = [i_\infty^* A^* x^*, i_\infty^* y^*]_{H_\infty},$$

which means that $Bi^*x^* \in D(V^*)$ and $V^*(Bi^*x^*) = i_\infty^* A^* x^*$. □

Remark 2.13. If $Q = Q_\infty$, i.e., the Malliavin setting, D_H is the Malliavin derivative and V is the identity operator. Finally, for any $p \in [1, +\infty)$ the space $W^{1,p}_H(X, \mu_\infty)$ is the Sobolev space considered in [4, Chapter 5].

Remark 2.14. Since $(X, \mu_\infty, H_\infty)$ is a Wiener space, we can always consider the Malliavin derivative D_{H_∞} and the Sobolev spaces $W^{1,p}(X, \mu_\infty)$ (see e.g. [4, Chapter 5]).

Remark 2.15. It is not hard to see that, even if we consider a space of test functions which is smaller with respect to those considered in [25, 26], we obtain the same Sobolev space $W^{1,p}_H(X, \mu_\infty)$ for any $p \in [1, +\infty)$.

We are now ready to state the hypotheses on the weighted function U.

Hypothesis 2.16. U is a proper $\|\cdot\|_X$-lower semi-continuous convex function which belongs to $W^{1,p}_H(X, \mu_\infty)$ for any $p \in [1, +\infty)$.

It is useful to notice that Hypothesis 2.16 and [2, Lemma 7.5] imply that $e^{-U} \in W^{1,p}_H(X, \mu_\infty)$ for any $p \in [1, +\infty)$. This allows us to introduce the weighted measure

$$\nu_\infty := e^{-U} d\mu_\infty. \tag{2.8}$$

We want to prove that $D_H : \mathscr{F}\mathscr{C}^1_{b,\Theta}(X) \to \mathrm{L}^p(X, \nu_\infty; H)$ is closable in $\mathrm{L}^p(X, \nu_\infty)$. To this aim we prove an intermediate result, which is the extension of [17, Lemma 3.3] for the weighted measure ν_∞.

Lemma 2.17. *Let $f \in \mathscr{F}\mathscr{C}^1_{b,\Theta}(X)$ and let $h \in D(V^*)$. Then,*

$$\int_X [D_H f, h]_H d\nu_\infty = \int_X f\widehat{V^* h} d\nu_\infty + \int_X f[D_H U, h]_H d\nu_\infty. \tag{2.9}$$

Proof. From [17, Lemma 3.3] we already know that

$$\int_X [D_H g, h]_H d\mu_\infty = \int_X g\widehat{V^* h} d\mu_\infty,$$

for any $g \in \mathscr{F}\mathscr{C}^1_{b,\Theta}(X)$ and any $h \in D(V^*)$. By density, it holds for any $g \in W^{1,p}_H(X, \mu_\infty)$ and any $p \in [1, +\infty)$. Since $e^{-U} \in W^{1,p}_H(X, \mu_\infty)$, it follows that $fe^{-U} \in W^{1,p}_H(X, \mu_\infty)$ for any $p \in (1, +\infty)$. Finally, [26, Lemma 3.3] gives $D_H(fe^{-U}) = (D_H f)e^{-U} - (D_H U)fe^{-U}$. Then,

$$\begin{aligned}
\int_X [D_H f, h]_H d\nu_\infty &= \int_X [D_H f, h]_H e^{-U} d\mu_\infty = \int_X [D_H(fe^{-U}), h]_H d\mu_\infty + \int_X f[D_H U, h]_H e^{-U} d\mu_\infty \\
&= \int_X fe^{-U}\widehat{V^* h} d\mu_\infty + \int_X f[D_H U, h]_H d\nu_\infty \\
&= \int_X f\widehat{V^* h} d\nu_\infty + \int_X f[D_H U, h]_H d\nu_\infty.
\end{aligned}$$

□

Integration by parts (2.9) is the key tool to prove the closability of D_H.

Proposition 2.18. *$D_H : \mathscr{F}\mathscr{C}^1_{b,\Theta}(X) \to \mathrm{L}^p(X, \nu_\infty; H)$ is closable in $\mathrm{L}^p(X, \nu_\infty)$ for any $p \in (1, +\infty)$. We still denote by D_H the closure of D_H and we denote by $W^{1,p}_H(X, \nu_\infty)$ the domain of its closure. Finally, for any $p \in (1, +\infty)$ the space $W^{1,p}_H(X, \nu_\infty)$ endowed with the norm*

$$\|f\|_{1,p,H} := \|f\|_{\mathrm{L}^p(X,\nu_\infty)} + \|D_H f\|_{\mathrm{L}^p(X,\nu_\infty;H)}, \quad f \in W^{1,p}_H(X, \nu_\infty),$$

is a Banach space, and for $p = 2$ it is a Hilbert space with inner product

$$\langle f, g\rangle_{W^{1,2}_H(X,\nu_\infty)} := \int_X fg d\nu_\infty + \int_X [D_H f, D_H g]_H d\nu_\infty, \quad f, g \in W^{1,2}_H(X, \nu_\infty).$$

Proof. Let us fix $p \in (1, +\infty)$. Since $(V, D(V))$ is closable from H_∞ onto H, from [17, Theorem 3.4] it follows that $D(V^*)$ is dense in H, and therefore there exists an orthonormal basis $\{v_n : n \in \mathbb{N}\} \subset D(V^*)$ of H. To show that D_H is closable, let us consider a sequence $(f_n) \subset \mathscr{FC}^1_{b,\Theta}(X)$ such that $f_n \to 0$ and $D_H f_n \to F$ in $\mathrm{L}^p(X, \nu_\infty)$ and in $\mathrm{L}^p(X, \nu_\infty; H)$, respectively. If we show that $F = 0$ we infer the closability of D_H. To prove that $F = 0$ let us consider $g \in \mathscr{FC}^1_{b,\Theta}(X)$. From (2.9) applied to the function $\widetilde{f}_n := f_n g \in \mathscr{FC}^1_{b,\Theta}(X)$ we have

$$\begin{aligned}\int_X [D_H f_n, v_j]_H g d\nu_\infty &= \int_X [D_H(\widetilde{f}_n), v_j]_H d\nu_\infty - \int_X [D_H g, v_j]_H f_n d\nu_\infty \\ &= \int_X f_n g \widehat{V^* v_j} d\nu_\infty + \int_X [D_H U, v_j]_H f_n g d\nu_\infty - \int_X [D_H g, v_j]_H f_n d\nu_\infty, \end{aligned} \tag{2.10}$$

for any $j \in \mathbb{N}$. Letting $n \to +\infty$ in the right-hand side of (2.10) we infer that

$$\int_X [F, v_j]_H g d\nu_\infty = \lim_{n \to +\infty} \int_X [D_H f_n, v_j]_H g d\nu_\infty = 0,$$

for any $j \in \mathbb{N}$ and any $g \in \mathscr{FC}^1_{b,\Theta}(X)$. Since $\mathscr{FC}^1_{b,\Theta}(X)$ is dense in $\mathrm{L}^q(X, \nu_\infty)$ for any $q \in (1, +\infty)$ we obtain that $[F(x), v_j]_H = 0$ for ν_∞-a.e. $x \in X$ for any $j \in \mathbb{N}$, which gives $F(x) = 0$ for ν_∞-a.e. $x \in X$. The second part of the statement follows from standard arguments. □

Remark 2.19. As one expects, for any $k \in \mathbb{N} \cup \{\infty\}$ the operator $D_H : \mathscr{FC}^k_{b,\Theta}(X) \to \mathrm{L}^p(X, \nu_\infty; H)$ is closable in $\mathrm{L}^p(X, \nu_\infty)$ for any $p \in (1, +\infty)$, and the domain of its closure coincides with $W^{1,p}_H(X, \nu_\infty)$.

3 The perturbed nonsymmetric Ornstein-Uhlenbeck operator

3.1 The perturbed nonsymmetric Ornstein-Uhlenbeck operator in $\mathrm{L}^2(X, \nu_\infty)$

We introduce the nonsymmetric Ornstein-Uhlenbeck operator by means of the theory of bilinear Dirichlet forms. We introduce the nonsymmetric bilinear form

$$\mathcal{E}(u, v) = -\int_X [B D_H u, D_H v]_H d\nu_\infty, \tag{3.1}$$

with domain $\mathcal{D} = W^{1,2}_H(X, \nu_\infty)$. From Lemma 2.9 we get

$$\mathcal{E}(u, u) = -\int_X [B D_H u, D_H u]_H d\nu_\infty = \frac{1}{2} \int_X [D_H u, D_H u]_H d\nu_\infty = \frac{1}{2} \|D_H u\|^2_{\mathrm{L}^2(X, \nu_\infty; H)}, \tag{3.2}$$

which implies that $\mathcal{E}$ is positive definite. Further, if we consider the symmetric part $\overline{\mathcal{E}}(u, v) := \frac{1}{2}(\mathcal{E}(u, v) + \mathcal{E}(v, u))$ of $\mathcal{E}$, with $u, v \in \mathcal{D}$, we have

$$\begin{aligned}\overline{\mathcal{E}}(u, v) =& \frac{1}{2} \int_X ([B D_H u, D_H v]_H + [B D_H v, D_H u]_H) d\nu_\infty \\ =& \frac{1}{2} \int_X ([B D_H u, D_H v]_H + [B^* D_H u, D_H v]_H) d\nu_\infty = \frac{1}{2} \int_X [D_H u, D_H v] d\nu_\infty.\end{aligned}$$

Hence, Proposition 2.18 implies that $(\overline{\mathcal{E}}, \mathcal{D})$ is a symmetric closed form on $\mathrm{L}^2(X, \nu_\infty)$. Finally, for any $u, v \in \mathcal{D}$, from Hypothesis 2.8 we have

$$\begin{aligned}|\mathcal{E}(u, v)| \leq& \int_X |[B D_H u, D_H v]_H| d\nu_\infty = \|B\|_{\mathcal{L}(H)} \int_X |D_H u|_H |D_H v|_H d\nu_\infty \\ \leq& c \, \|D_H u\|_{\mathrm{L}^2(X, \nu_\infty; H)} \|D_H v\|_{\mathrm{L}^2(X, \nu_\infty; H)} = 4c \, \mathcal{E}(u, u)^{1/2} \mathcal{E}(v, v)^{1/2}.\end{aligned}$$

This implies that $(\mathcal{E},\mathcal{D})$ satisfies the *strong* (and hence the *weak*) *sector condition* (see [24, Chapter 1, Section 2 and Exercise 2.1]) and therefore $(\mathcal{E},\mathcal{D})$ is a coercive closed form on $\mathrm{L}^2(X,\nu_\infty)$. According to [24, Chapter 1] we define a densely defined operator L as follows:

$$\begin{cases} D(L) := \Big\{u \in W^{1,2}_H(X,\nu_\infty) : \text{ there exists } g \in \mathrm{L}^2(X,\nu_\infty) \text{ such that} \\ \qquad\qquad\qquad\qquad\qquad \mathcal{E}(u,v) = -\displaystyle\int_X gv d\nu_\infty,\ \forall v\, \mathscr{F}\mathscr{C}^1_{b,\Theta}(X)\Big\}, \\ Lu := g. \end{cases} \tag{3.3}$$

Remark 3.1. From [24, Chapter 1, Sections 1 and 2] it follows that L generates a strongly continuous contraction semigroup on $\mathrm{L}^2(X,\nu_\infty)$ which we denote by $(T(t))_{t\geq 0}$. In particular, $1 \in \rho(L)$. The operator L is called *perturbed Ornstein-Uhlenbeck operator in* $\mathrm{L}^2(X,\nu_\infty)$ and the associated semigroup $(T(t))_{t\geq 0}$ is called *perturbed Ornstein-Uhlenbeck semigroup in* $\mathrm{L}^2(X,\nu_\infty)$.

In the following we will need of the adjoint operator L^* of L. We recall that formally L^* is defined as follows:

$$\begin{cases} D(L^*) := \Big\{v \in \mathrm{L}^2(X,\nu_\infty) : \exists g \in \mathrm{L}^2(X,\nu_\infty) \text{ such that} \\ \qquad\qquad\qquad\qquad \displaystyle\int_X gud\nu_\infty = \int_X vLud\nu_\infty, \quad u \in D(L)\Big\}, \\ L^*v := g. \end{cases}$$

Moreover, let us consider the adjoint semigroup $(T^*(t))_{t\geq 0}$ of $(T(t))_{t\geq 0}$. Even if in general it is not a strongly continuous semigroup, [24, Chapter 1, Theorem 2.8] ensures that $(T^*(t))_{t\geq 0}$ is strongly continuous and L^* is its generator. Further, [24, Chapter 1, Corollary 2.10] implies that $D(L^*) \subset \mathcal{D} = W^{1,2}_H(X,\nu_\infty)$.

We give a characterization of L^* in terms of bilinear form on $\mathrm{L}^2(X,\nu_\infty)$. Let us introduce the nonsymmetric bilinear form

$$\widetilde{\mathcal{E}}(u,v) := -\int_X [B^* D_H u.D_H v]_H d\nu_\infty, \tag{3.4}$$

with domain $\mathcal{D} := W^{1,2}_H(X,\nu_\infty)$. Arguing as for $\mathcal{E}$ it is possible to prove that $\widetilde{\mathcal{E}}$ is a coercive closed form on $\mathrm{L}^2(X,\nu_\infty)$ and therefore the operator $\widetilde{L}$ defined as

$$\begin{cases} D(\widetilde{L}) := \Big\{u \in W^{1,2}_H(X,\nu_\infty) : \text{ there exists } g \in \mathrm{L}^2(X,\nu_\infty) \text{ such that} \\ \qquad\qquad\qquad\qquad\qquad \widetilde{\mathcal{E}}(u,v) = -\displaystyle\int_X gv d\nu_\infty,\ \forall v\, \mathscr{F}\mathscr{C}^1_{b,\Theta}(X)\Big\}, \\ \widetilde{L}u := g, \end{cases} \tag{3.5}$$

generates a strongly continuous semigroup $(\widetilde{T}(t))_{t\geq 0}$ on $\mathrm{L}^2(X,\nu_\infty)$. The next result shows that $\widetilde{L}$ is indeed the adjoint operator of L and $(\widetilde{T}(t))_{t\geq 0}$ is the adjoint semigroup of $(T(t))_{t\geq 0}$.

Proposition 3.2. *$D(\widetilde{L}) = D(L^*)$ and $\widetilde{L}u = L^*u$ for any $u \in D(L^*)$. Therefore, $\widetilde{T}(t) = T^*(t)$ for any $t \geq 0$.*

Proof. Let $u \in D(\widetilde{L})$. Then, for any $v \in D(L)$ we have

$$\int_X \widetilde{L}uv d\nu_\infty = -\int_X [B^* D_H u, D_H v]_H d\nu_\infty = \int_X [BD_H v, D_H u] d\nu_\infty = \int_X Lvu d\nu_\infty.$$

Therefore, from the definition of L^* it follows that $u \in D(L^*)$ and $L^*u = \widetilde{L}u$. To prove the converse inclusion, let $u \in D(L^*)$. We recall that, in particular, $u \in W^{1,2}_H(X,\nu_\infty)$. Hence, for any $v \in D(L)$ we have

$$\int_X L^* uv d\nu_\infty = \int_X uLv d\nu_\infty = -\int_X [BD_H v, D_H u]_H d\nu_\infty = -\int_X [B^* D_H u, D_H v]_H d\nu_\infty = -\widetilde{\mathcal{E}}(u,v). \tag{3.6}$$

From [24, Chapter 1, Theorem 2.13(ii)] it follows that $D(L)$ is dense in $\mathcal{D} = W_H^{1,2}(X,\nu_\infty)$. Therefore, (3.6) gives $u \in D(\widetilde{L})$ and $\widetilde{L}u = L^* u$. □

3.2 The nonsymmetric Ornstein-Uhlenbeck operator in $\mathrm{L}^p(X,\nu_\infty)$

In this subsection we consider the realization of the semigroup $(T(t))_{t\geq 0}$ in $\mathrm{L}^p(X,\nu_\infty)$ with $p \in (1,+\infty)$, showing some important properties of the perturbed Ornstein-Uhlenbeck semigroup in $\mathrm{L}^p(X,\nu_\infty)$. We need of a technical lemma, which is the analogous of [10, Lemma 2.7] in our setting, about the differentiability of the positive and negative part of a function $u \in W_H^{1,2}(X,\nu_\infty)$.

Lemma 3.3. *Let $u \in W_H^{1,2}(X,\nu_\infty)$. Then, $|u|, u^+, u^- \in W_H^{1,2}(X,\nu_\infty)$ and $D_H|u| = \mathrm{sign}(u) D_H u$. Further, $D_H u$ vanishes on $u^{-1}(0)$ ν_∞-a.e.; $D_H(u^+) = \mathbb{1}_{\{u>0\}} D_H u$ and $D_H(u^-) = -\mathbb{1}_{\{u<0\}} D_H u$.*

Proof. The proof is analogous to the one of [10, Lemma 2.7] and we omit it. We simply remark that, to prove that second part, as in the proof of Proposition 2.18 we consider the basis $\{v_n : n \in \mathbb{N}\}$ of H of elements of $D(V^*)$ and we show that

$$\int_{\{u=0\}} [D_H u, v_i]_H \varphi d\nu_\infty = 0,$$

for any $u \in W_H^{1,2}(X,\nu_\infty)$ and any $\varphi \in \mathscr{F}\mathscr{C}_b^{*,1}(X)$. □

Thanks to Lemma 3.3 we can prove that both L and L^* are Dirichlet operators and therefore that $(T(t))_{t\geq 0}$ and $(T^*(t))_{t\geq 0}$ are sub-Markovian operators. For reader's convenience, we recall the definitions of Dirichlet and sub-Markovian operators and their main properties (see e.g. [24, Chapter 1, Definition 4.1 & Proposition 4.3]).

Definition 3.4. Let $\mathscr{H} := \mathrm{L}^2(E,\mu)$ be a measure space.

(i) A semigroup $(S(t))_{t\geq 0}$ on $\mathscr{H}$ is called sub-Markovian if for any $t \geq 0$ and any $f \in \mathscr{H}$ with $0 \leq f \leq 1$ μ-a.e., we have $0 \leq S(t)f \leq 1$ μ-a.e.

(ii) A closed linear densely defined operator A on $\mathscr{H}$ is called Dirichlet operator if

$$\int_E Au(u-1)^+ d\mu \leq 0, \quad u \in D(A).$$

Proposition 3.5. *Let $(S(t))_{t\geq 0}$ be a strongly continuous contraction semigroup on $\mathrm{L}^2(E,\mu)$ with generator $\mathcal{A}$. Then, the following are equivalent:*

(i) $(S(t))_{t\geq 0}$ is a sub-Markovian semigroup on $\mathrm{L}^2(E,\mu)$.

(ii) $\mathcal{A}$ is a Dirichlet operator on $\mathrm{L}^2(E,\mu)$.

We prove that it is possible to extend the semigroup $(T(t))_{t\geq 0}$ to a strongly continuous contraction semigroup on $\mathrm{L}^p(X,\nu_\infty)$ for any $p \in [1,+\infty)$. We follow the proof of [12, Theorem 1.4.1].

Proposition 3.6. *The semigroup $(T(t))_{t\geq 0}$ can be uniquely extended to a positive contraction semigroup $(T_p(t))_{t\geq 0}$ on $\mathrm{L}^p(X,\nu_\infty)$ for any $p \in [1,+\infty)$. These semigroups are strongly continuous if $p \in [1,+\infty)$ and are consistent in the sense that $T_p(t)f = T_q(t)f$ if $f \in \mathrm{L}^p(X,\nu_\infty) \cap \mathrm{L}^q(X,\nu_\infty)$.*

Proof. For reader's convenience, we split the proof into different steps.

Step 1. At first, we prove that both L and L^* are Dirichlet operators on $\mathrm{L}^2(X,\nu_\infty)$. Let $u\in D(L)$. Then, $u\in W^{1,2}_H(X,\nu_\infty)$ and from Lemma 3.3 we infer that $(u-1)^+\in W^{1,2}_H(X,\nu_\infty)$ and $D_H(u-1)^+=\mathbb{1}_{u\geq 1}D_Hu$. Therefore,

$$\int_X Lu(u-1)^+d\nu_\infty=\int_X[BD_Hu,D_H(u-1)^+]_Hd\nu_\infty=\int_{\{u>1\}}[BD_Hu,D_Hu]_Hd\nu_\infty\leq 0,$$

thanks to Lemma 2.9. The computations for L^* are analogous. Hence, both L and L^* are Dirichlet operators on $\mathrm{L}^2(X,\nu_\infty)$, which means that $(T(t))_{t\geq0}$ and $(T^*(t))_{t\geq0}$ are sub-Markovian semigroups on $\mathrm{L}^2(X,\nu_\infty)$.

Step 2. Here, we prove that $\mathrm{L}^1(X,\nu_\infty)\cap\mathrm{L}^\infty(X,\nu_\infty)$ is invariant for $T(t)$, for any $t\geq0$. From Step 1 we know that for any $f\in\mathrm{L}^2(X,\nu_\infty)$ such that $0\leq f\leq 1$ ν_∞-a.e.we have $0\leq T(t)f\leq 1$ ν_∞-a.e. Then, it follows that $\mathrm{L}^\infty(X,\nu_\infty)$ is invariant under $(T(t))_{t\geq0}$. Hence, for any $f\in\mathrm{L}^1(X,\nu_\infty)\cap\mathrm{L}^\infty(X,\nu_\infty)$, which is a subspace of $\mathrm{L}^2(X,\nu_\infty)\cap\mathrm{L}^\infty(X,\nu_\infty)$, we have

$$\|T(t)f\|_{\mathrm{L}^\infty(X,\nu_\infty)}\leq\|f\|_{\mathrm{L}^\infty(X,\nu_\infty)},\quad t\geq0.$$

Further, if also $g\in\mathrm{L}^1(X,\nu_\infty)\cap\mathrm{L}^\infty(X,\nu_\infty)$, then

$$\left|\int_X T(t)fgd\nu_\infty\right|=\left|\int_X fT^*(t)gd\nu_\infty\right|\leq\|f\|_{\mathrm{L}^1(X,\nu_\infty)}\|g\|_{\mathrm{L}^\infty(X,\nu_\infty)},\quad t\geq0,$$

since also $T^*(t)$ is a contraction on $\mathrm{L}^\infty(X,\nu_\infty)$. This implies that

$$\|T(t)f\|_{\mathrm{L}^1(X,\nu_\infty)}\leq\|f\|_{\mathrm{L}^1(X,\nu_\infty)},\quad t\geq0,$$

and therefore $\mathrm{L}^1(X,\nu_\infty)\cap\mathrm{L}^\infty(X,\nu_\infty)$ is invariant under $(T(t))_{t\geq0}$. By applying the Riesz-Thorin Interpolation Theorem [29, Section 1.18.7, Theorem 1] we conclude that $(T(t))_{t\geq0}$ extends to a positive contraction semigroup $(T_p(t))_{t\geq0}$ on $\mathrm{L}^p(X,\nu_\infty)$ for any $p\in[1,+\infty)$. Uniqueness follows by density.

Step 3. Now we show that $(T_p(t))_{t\geq0}$ is strongly continuous if $p\in[1,+\infty)$. Let $f\geq0$ be a bounded function which vanishes outside a set E of bounded measure. Then,

$$\lim_{t\to0}\int_X\mathbb{1}_ET_1(t)fd\nu_\infty=\lim_{t\to0}\int_X\mathbb{1}_ET(t)fd\nu_\infty=\int_E fd\nu_\infty=\|f\|_{\mathrm{L}^1(X,\nu_\infty)},$$

since $(T(t))_{t\geq0}$ is strongly continuous. We recall that $(T(t))_{t\geq0}$ is the Ornstein-Uhlenbeck semigroup on $\mathrm{L}^2(X,\nu_\infty)$. But $\|T_1(t)f\|_{\mathrm{L}^1(X,\nu_\infty)}\leq\|f\|_{\mathrm{L}^1(X,\nu_\infty)}$, and therefore

$$\lim_{t\to0}\|T_1(t)f-f\|_{\mathrm{L}^1(X,\nu_\infty)}=\lim_{t\to0}\int_X|T_1(t)f-f|\mathbb{1}_Ed\nu_\infty\leq\lim_{t\to0}\nu_\infty(E)^{1/2}\|T(t)f-f\|_{\mathrm{L}^2(X,\nu_\infty)}=0.$$

By density, we deduce that $(T_1(t))_{t\geq0}$ is strongly continuous on $\mathrm{L}^1(X,\nu_\infty)$. By interpolation, we infer the strong continuity of $(T_p(t))_{t\geq0}$ on $\mathrm{L}^p(X,\nu_\infty)$ for any $p\in(1,2)$. Finally, the riflexivity of $\mathrm{L}^p(X,\nu_\infty)$ (see e.g. [13, Section 4, Theorem 1]) for any $p\in(1,+\infty)$ and [11, Theorem 1.34] allow us to conclude that $(T_p(t))_{t\geq0}$ is strongly continuous on $\mathrm{L}^p(X,\nu_\infty)$ for any $p\in(2,+\infty)$. □

For any $p\in[1,+\infty)$ let us denote by L_p the infinitesimal generator of $(T_p(t))_{t\geq0}$. Since $(T_p(t))_{t\geq0}$ is a positive strongly continuous semigroup for any $p\in[1,+\infty)$, we get $1\in\rho(L_p)$ for any $p\in[1,+\infty)$.

Following [25, Theorem 2.3], we show that $\mathscr{F}\mathscr{C}^2_{b,\Theta}(X)\subset D(L)$ and for any $u\in\mathscr{F}\mathscr{C}^2_{b,\Theta}(X)$ an explicit formula for Lu is available. To this aim, we recall the definition of Trace class operator on $\mathcal{L}(H)$: given a nonnegative operator $\Phi\in\mathcal{L}(H)$, we say that Φ is a trace class operator if

$$\sum_{n=1}^\infty[\Phi h_n,h_n]_H<+\infty,$$

where $\{h_n : n \in \mathbb{N}\}$ is any orthonormal basis of H. We define the Trace $\mathrm{Tr}[\Phi]$ of Φ as

$$\mathrm{Tr}[\Phi]_H := \sum_{n=1}^{\infty} [\Phi h_n, h_n]_H.$$

We observe that for any $f \in \mathscr{F}\mathscr{C}^2_{b,\Theta}(X)$ such that $f(x) = \varphi(\widehat{e}_1(x), \dots, \widehat{e}_n(x))$ for some $\varphi \in C^2_b(\mathbb{R}^n)$, we define the second order derivative along H as

$$D^2_H f(x) := \sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial \xi_j \xi_k}(\widehat{e}_1(x), \dots, \widehat{e}_n(x)) Q x_j^* \otimes Q x_k^*.$$

$D^2_H f(x)$ is a trace class operator for any $x \in X$ and

$$\mathrm{Tr}[D^2_H f(x)]_H = \sum_{j,k=1}^{n} \langle Q x_j^*, x_k^* \rangle_{X \times X^*} \frac{\partial^2 \varphi}{\partial \xi_j \partial \xi_k}(\widehat{e}_1(x), \dots, \widehat{e}_n(x)), \quad x \in X.$$

Proposition 3.7. *$\mathscr{F}\mathscr{C}^2_{b,\Theta}(X) \subset D(L)$ and for any $u \in \mathscr{F}\mathscr{C}^2_{b,\Theta}(X)$ we have*

$$Lu(x) = \frac{1}{2}\mathrm{Tr}[D^2_H u(x)]_H + \langle x, A^* Du(x) \rangle_{X \times X^*} + [BD_H u(x), D_H U(x)]_H, \quad \nu_\infty\text{–a.e. } x \in X. \tag{3.7}$$

Proof. Let $u \in \mathscr{F}\mathscr{C}^2_{b,\Theta}(X)$ be such that $u(x) = \varphi(\widehat{e}_1(x), \dots, \widehat{e}_m(x))$, with $\varphi \in C^2_b(\mathbb{R}^m)$ and let $v \in \mathscr{F}\mathscr{C}^1_{b,\Theta}(X)$. From Lemma 2.12 for any $x^* \in D(A^*)$ we have $Bi^* x^* \in D(V^*)$ and $V^*(Bi^* x^*) = i^*_\infty A^* x^*$. The form of u, integration by parts formula (2.10) and the computations in the proof of [25, Theorem 2.3] give

$$\begin{aligned}
\mathcal{E}(u,v) &= -\int_X [BD_H u(x), D_H v(x)]_H \nu_\infty(dx) \\
&= -\sum_{n=1}^{m} \int_X [D_H v(x), Bi^* x_n^*]_H \frac{\partial \varphi}{\partial \xi_n}(\widehat{e}_1(x), \dots, \widehat{e}_n(x)) \nu_\infty(dx) \\
&= \sum_{n=1}^{m} \int_X v(x) \Big(\sum_{j=1}^{m} \frac{\partial^2 \varphi}{\partial \xi_n \partial \xi_j} [i^* x_j^*, Bi^* x_n^*]_H - v(x) \frac{\partial \varphi}{\partial \xi_n}(\widehat{e}_1(x), \dots, \widehat{e}_n(x)) \widehat{V^* B i^* x_n^*}(x) \\
&\qquad - [D_H U(x), BD_H u(x)]_H \Big) \nu_\infty(dx) \\
&= -\int_X v(x) \Big(\frac{1}{2}\mathrm{Tr}[D^2_H u(x)]_H + \langle x, A^* Du(x) \rangle_{X \times X^*} + [BD_H u(x), D_H U(x)]_H \Big) \nu_\infty(dx).
\end{aligned}$$

Since

$$x \mapsto \frac{1}{2}\mathrm{Tr}[D^2_H u(x)]_H + \langle x, A^* Du(x) \rangle_{X \times X^*} + [BD_H u(x), D_H U(x)]_H \in \mathrm{L}^2(X, \nu_\infty),$$

it follows that $u \in D(L)$ and

$$Lu(x) = \frac{1}{2}\mathrm{Tr}[D^2_H u(x)]_H + \langle x, A^* Du(x) \rangle_{X \times X^*} + [BD_H u(x), D_H U(x)]_H,$$

for ν_∞-a.e. $x \in X$. □

Now we show that $\mathscr{F}\mathscr{C}^2_{b,\Theta}(X)$ is contained in $D(L_p)$ for any $p \in (1, +\infty)$.

Proposition 3.8. *$\mathscr{F}\mathscr{C}^2_{b,\Theta}(X) \subset D(L_p)$ for any $p \in (1, +\infty)$. Further, $L_p u = Lu$ for any $u \in \mathscr{F}\mathscr{C}^2_{b,\Theta}(X)$ and any $p \in (1, +\infty)$.*

Proof. At first we stress that $Lu \in \mathrm{L}^p(X,\nu_\infty)$ for any $p \in (1,+\infty)$. We study separately two cases. In the former we take $p \in (1,2)$, in the latter we consider $p \in (2,+\infty)$.

Let $p \in (1,2)$ and let $u \in \mathscr{F}\mathscr{C}^2_{b,\Theta}(X)$. Then,

$$\|t^{-1}(T_p(t)u-u)-Lu\|_{\mathrm{L}^p(X,\nu_\infty)} \le (\nu_\infty(X))^{1/p'}\,\|t^{-1}(T(t)u-u)-Lu\|_{\mathrm{L}^2(X,\nu_\infty)} \to 0, \quad t\to 0,$$

where p' is the conjugate exponent of p. Hence, $u \in D(L_p)$ and $L_pu = Lu$.

Let us consider $p \in (2,+\infty)$ and let $u \in \mathscr{F}\mathscr{C}^2_{b,\Theta}(X)$. Since $T_p(t)u = T(t)u$, from Proposition 3.7 we deduce that for any sequence of positive numbers (t_m) decreasing to 0 there exists a subsequence $(t_{m_n}) \subset (t_m)$ such that $t_{m_n}^{-1}(T_p(t_{m_n})u-u) \to Lu$ for ν_∞-a.e. $x \in X$. Let us consider $q > p$. For any $v \in \mathscr{F}\mathscr{C}^1_{b,\Theta}(X)$ we have

$$\lim_{n\to\infty}\int_X \frac{T_p(t_{m_n})u-u}{t_{m_n}}v\,d\nu_\infty = \lim_{n\to\infty}\int_X \frac{T(t_{m_n})u-u}{t_{m_n}}v\,d\nu_\infty = \int_X Luv\,d\nu_\infty,$$

and from the density of $\mathscr{F}\mathscr{C}^2_{b,\Theta}(X)$ in $\mathrm{L}^{q'}(X,\nu_\infty)$ we infer that $t_{m_n}^{-1}(T_p(t_{m_n})u-u) \to Lu$ weakly in $\mathrm{L}^q(X,\nu_\infty)$ as $n\to\infty$, which implies that $(\Delta_n u := t_{m_n}^{-1}(T_p(t_{m_n})u-u)-Lu)_{n\in\mathbb{N}}$ is uniformly bounded in $\mathrm{L}^q(X,\nu_\infty)$. We claim that $(|\Delta_n u|^p)_{n\in\mathbb{N}}$ is uniformly integrable. To this aim, we introduce the function $\varphi : [0,+\infty) \to [0,+\infty)$ defined by $\varphi(t) := t^{q/p}$. Since $q > p$ we have

$$\lim_{t\to+\infty}\frac{\varphi(t)}{t} = +\infty,$$

and

$$\sup_{n\in\mathbb{N}}\int_X \varphi(|\Delta_n u|^p)\,d\nu_\infty = \sup_{n\in\mathbb{N}}\int_X |\Delta_n u|^q\,d\nu_\infty < +\infty.$$

Then, from [5, Theorem 4.5.9] the claim follows. We are almost done. Further, from the Egoroff Theorem (see e.g. [5, Theorem 2.2.1]) we know that for any $\delta > 0$ there exists a Borel set $X_\delta \subset X$ such that $\nu_\infty(X\setminus X_\delta) \le \delta$ and $\Delta_n u \to 0$ as $n\to\infty$ uniformly on X_δ. Let us fix $\varepsilon > 0$. Since $(|\Delta_n u|^p)_{n\in\mathbb{N}}$ is uniformly integrable, there exists $\delta > 0$ such that

$$\int_E |\Delta_n u|^p\,d\nu_\infty \le \varepsilon, \quad n\in\mathbb{N}, \tag{3.8}$$

for any Borel set $E \subset X$ such that $\nu_\infty(E) \le \delta$. Then,

$$\int_X |\Delta_n u|^p\,d\nu_\infty = \int_{X\setminus X_\delta} |\Delta_n u|^p\,d\nu_\infty + \int_{X_\delta} |\Delta_n u|^p\,d\nu_\infty. \tag{3.9}$$

By taking the lim sup as $n\to\infty$ in both the sides of (3.9), by (3.8) and dominated convergence theorem we deduce that

$$\limsup_{n\to\infty}\int_X |\Delta_n u|^p\,d\nu_\infty \le \varepsilon.$$

The arbitrariness of $\varepsilon > 0$ implies that

$$\lim_{n\to\infty}\int_X |\Delta_n u|^p\,d\nu_\infty = 0.$$

Therefore, we have shown that for any sequence (t_m) of positive numbers decreasing to 0 there exists a subsequence $(t_{m_n}) \subset (t_m)$ such that $t_{m_n}^{-1}(T_p(t_{m_n})u-u)-Lu \to 0$ in $\mathrm{L}^p(X,\nu_\infty)$ as $n\to\infty$. This gives $t^{-1}(T(t)u-u) \to 0$ in $\mathrm{L}^p(X,\nu_\infty)$ as $t\to 0$, which implies that $u \in D(L_p)$ and $L_pu = Lu$ for any $p > 2$. □

Remark 3.9. For any $p \in [2,+\infty)$ we have $D(L_p) \subset D(L)$ and for any $u \in D(L_p)$ it follows that $L_p u = Lu$. Indeed, for any $u \in D(L_p)$ we have

$$\begin{aligned}\|t^{-1}(T(t)u-u)-L_pu\|_{\mathrm{L}^2(X,\nu_\infty)} =&\|t^{-1}(T_p(t)u-u)-L_pu\|_{\mathrm{L}^2(X,\nu_\infty)}\\ \leq&(\nu_\infty(X))^{(p-2)/p}\|t^{-1}(T_p(t)u-u)-L_pu\|_{\mathrm{L}^p(X,\nu_\infty)}\to 0,\end{aligned}$$

as $t \to 0$. Hence, $u \in D(L)$ and $Lu = L_p u$.

4 Analyticity of the semigroup associated to L_p

We want to show that L is sectorial in $\mathrm{L}^p(X,\nu_\infty)$ for any $p \in (1,+\infty)$, i.e., $(T_p(t))_{t\geq 0}$ is an analytic semigroup on the sector $\Sigma_{\theta_p} := \left\{re^{i\phi} : r > 0, |\phi| < \theta_p\right\}$, where

$$\mathrm{cotg}(\theta_p) = \frac{\sqrt{(p-2)^2+p^2\gamma^2}}{2\sqrt{p-1}}, \quad \gamma := \|B-B^*\|_{\mathcal{L}(H)}. \tag{4.1}$$

To this aim we follow the approach of [25, Section 3]. We introduce the following spaces of functions.

Definition 4.1. For any $p \in (1,+\infty)$ we set $\mathrm{L}^p_{\mathbb{C}}(X,\nu_\infty) := \mathrm{L}^p(X,\nu_\infty) + i\,\mathrm{L}^p(X,\nu_\infty)$ with dual product $(f,g) := \int_X f\overline{g}d\nu_\infty$ for any $f \in \mathrm{L}^p(X,\nu_\infty)$ and $g \in \mathrm{L}^{p'}(X,\nu_\infty)$. For any $k \in \mathbb{N}\cup\{\infty\}$ we denote by $\mathscr{F}\mathscr{C}^k_{b,\Theta}(X;\mathbb{C})$ the functions $f = u + iv$ such that $u, v \in \mathscr{F}\mathscr{C}^k_{b,\Theta}(X)$.

We consider the operator $L^{\mathbb{C}}_p$, on $D(L^{\mathbb{C}}_p) := D(L_p) + iD(L_p)$ endowed with the complexified norm of $D(L_p)$, defined by $L^{\mathbb{C}}_p f := L_p u + iL_p v$, where $f := u + iv \in D(L^{\mathbb{C}}_p)$.

Remark 4.2. It is not hard to prove that all the results in Section 2 and Section 3 can be extended by complexification to the complex case.

Remark 4.3. For any $p \in (1,+\infty)$ and any $f \in \mathrm{L}^p_{\mathbb{C}}(X,\nu_\infty)$, with respect to the duality pairing $\langle f, g\rangle := \int_X fg d\nu_\infty$, we have $\partial f = \{\|f\|^{2-p}_p f^*\}$ with $f^* := \overline{f}|f|^{p-2}$, with $f^* = 0$ at those point where $f = 0$, where ∂f is the duality set of f in $\mathrm{L}^p_{\mathbb{C}}(X,\nu_\infty)$.

For any $\theta \in [0,\pi/2)$ we set $C_\theta := \mathrm{cotg}(\theta)$. We want to apply the following proposition, which is an adaptation of [25, Proposition 3.2] to our situation.

Proposition 4.4. *Let $\mathscr{A}$ be a densely defined operator on $\mathrm{L}^p(X,\nu_\infty)$ and assume that $1 \in \rho(\mathscr{A})$. Then, the following are equivalent:*

(i) $\mathscr{A}$ generates an analytic C_0-semigroup on $\mathrm{L}^p(X,\nu_\infty)$ which is contractive on Σ_θ;

(ii) for any $f \in D(\mathscr{A})$ we have

$$\left|\mathrm{Im}\left(\int_X \mathscr{A}ff^*d\nu_\infty\right)\right| \leq -C_\theta \mathrm{Re}\left(\int_X \mathscr{A}ff^*d\nu_\infty\right). \tag{4.2}$$

Remark 4.5. For any $f \in \mathscr{F}\mathscr{C}^1_{b,\Theta}(X;\mathbb{C})$ and $p \geq 2$ we have

$$D_H f^* = D_H(\overline{f}|f|^{p-2}) = |f|^{p-2}D_H\overline{f} + (p-2)|f|^{p-4}f\overline{f}(D_H u + D_H v),$$

where $f = u + iv$. Hence, $D_H f^*$ is well defined and bounded.

Finally, we recall [25, Lemma 3.3], which is obtained by repeating the computations of [8, Lemma 5].

Lemma 4.6. *For any $f \in \mathscr{F}\mathscr{C}^1_{b,\Theta}(X;\mathbb{C})$ and any $p \in [2,+\infty)$ we have*

$$\begin{aligned}-\mathrm{Re}[BD_Hf, D_Hf^*]_H =& - \mathrm{Re}[B^*D_Hf, D_Hf^*]_H \\ =&\frac{1}{2}|f|^{p-4}\left((p-1)|\mathrm{Re}(\overline{f}D_Hf)|_H^2 + |\mathrm{Im}(\overline{f}D_Hf)|_H^2\right), \end{aligned} \tag{4.3}$$

and

$$\mathrm{Im}[BD_Hf, D_Hf^*]_H = p|f|^{p-4}\left[\left(B + \frac{1}{2}I_H\right)\mathrm{Im}(\overline{f}D_Hf), \mathrm{Re}(\overline{f}D_Hf)\right], \tag{4.4}$$

$$\mathrm{Im}[B^*D_Hf, D_Hf^*]_H = p|f|^{p-4}\left[\left(B^* + \frac{1}{2}I_H\right)\mathrm{Im}(\overline{f}D_Hf), \mathrm{Re}(\overline{f}D_Hf)\right]. \tag{4.5}$$

Following the arguments of [25, Theorem 3.4] we obtain the analyticity of the semigroup $(T_p(t))_{t\geq 0}$ for any $p \in (1,+\infty)$.

Proposition 4.7. *$(T_p(t))_{t\geq 0}$ is analytic in $\mathrm{L}^p(X,\nu_\infty)$ on the sector Σ_{θ_p}.*

Proof. We show that Proposition 4.4(ii) is satisfied with $\mathscr{A} = L_p$ and $\theta = \theta_p$. To begin with, the positivity of $(T_p(t))_{t\geq 0}$ implies that $1 \in \rho(L_p)$ for any $p \in (1,+\infty)$. At first we consider $p \in [2,+\infty)$ and then we deal with the case $p \in (1,2)$.

Step 1. Let $p \in [2,+\infty)$ and let $f \in \mathscr{F}\mathscr{C}^2_{b,\Theta}(X;\mathbb{C})$. From Proposition 3.8 and Remark 4.2 it follows that $f \in D(L_p^{\mathbb{C}})$. We set

$$a := |\mathrm{Re}(\overline{f}D_Hf)|_H, \quad b := |\mathrm{Im}(\overline{f}D_Hf)|_H.$$

From (4.3) we infer that

$$-\mathrm{Re}[BD_Hf, D_Hf^*]_H = \frac{1}{2}|f|^{p-4}\left((p-1)a^2 + b^2\right). \tag{4.6}$$

Since $B + B^* = -I_H$ we easily get

$$\left|B + \frac{1}{2}I_H\right|_{\mathcal{L}(H)} = \left|\frac{1}{2}B - \frac{1}{2}B^*\right|_{\mathcal{L}(H)} = \frac{1}{4}\gamma^2 + \left(\frac{1}{2} - \frac{1}{p}\right)^2, \tag{4.7}$$

where γ has been introduced in (4.1). The Cauchy-Schwarz inequality and (4.4) give

$$|\mathrm{Im}[BD_Hf, D_Hf^*]_H| \leq |f|^{p-4}C_{\theta_p}ab\sqrt{p-1}. \tag{4.8}$$

Thanks to the Young's inequality $2ab\sqrt{p-1} \leq (p-1)a^2 + b^2$ we deduce that

$$|\mathrm{Im}[BD_Hf, D_Hf^*]_H| \leq \frac{1}{2}|f|^{p-4}C_{\theta_p}\left((p-1)a^2 + b^2\right) = -\mathrm{Re}[BD_Hf, D_Hf^*]_H. \tag{4.9}$$

Then, from Remark 3.9 and (4.9) we infer

$$\begin{aligned}\left|\mathrm{Im}\left(\int_X L_pff^*d\nu_\infty\right)\right| =& \left|\mathrm{Im}\left(\int_X Lff^*d\nu_\infty\right)\right| = \left|\mathrm{Im}\left(\int_X [BD_Hf, D_Hf^*]_Hd\nu_\infty\right)\right| \\ \leq& - C_{\theta_p}\int_X \mathrm{Re}[BD_Hf, D_Hf^*]_Hd\nu_\infty = -C_{\theta_p}\mathrm{Re}\left(\int_X Lff^*d\nu_\infty\right) \\ =& - C_{\theta_p}\mathrm{Re}\left(\int_X L_pff^*d\nu_\infty\right).\end{aligned}$$

Hence, Proposition 4.4(ii) holds true for any $f \in \mathscr{F}\mathscr{C}^2_{b,\Theta}(X;\mathbb{C})$. For a generic $f = u + iv \in D(L_p^{\mathbb{C}})$ let us consider a sequence $(f_n := u_n + iv_n) \subset \mathscr{F}\mathscr{C}^2_{b,\Theta}(X;\mathbb{C})$ such that $u_n \to$ and $v_n \to v$ in $W^{1,2}_H(X,\nu_\infty)$.

This sequence exists thanks to Remark 2.19, to Proposition 3.9 (since $D(L_p) \subset D(L) \subset W^{1,2}_H(X,\nu_\infty)$) and thanks to Remark 4.2. Further, from Remark 4.5 it follows that $f^*_m \in W^{1,2}_H(X,\nu_\infty)$ for any $m \in \mathbb{N}$. In particular, for any $m \in \mathbb{N}$ we have

$$\lim_{n\to+\infty} f_n = f, \quad \lim_{n\to+\infty} f^*_n = f^*, \quad \text{in } \mathrm{L}^2(X,\nu_\infty), \tag{4.10}$$

$$\lim_{n\to+\infty} \mathrm{Re}[BD_Hf_n, D_Hf^*_m]_H = \mathrm{Re}[BD_Hf, D_Hf^*_m]_H, \quad \text{in } \mathrm{L}^2(X,\nu_\infty), \tag{4.11}$$

$$\lim_{n\to+\infty} \mathrm{Im}[BD_Hf_n, D_Hf^*_m]_H = \mathrm{Im}[BD_Hf, D_Hf^*_m]_H, \quad \text{in } \mathrm{L}^2(X,\nu_\infty). \tag{4.12}$$

Therefore, from Proposition 3.9, (4.9), (4.10), (4.11) and (4.12) we get

$$\begin{aligned}
\left|\mathrm{Im}\left(\int_X L_pff^*d\nu_\infty\right)\right| &= \lim_{m\to+\infty}\left|\mathrm{Im}\left(\int_X Lff^*_md\nu_\infty\right)\right| = \lim_{m\to+\infty}\left|\mathrm{Im}\left(\int_X [BD_Hf, D_Hf^*_m]_Hd\nu_\infty\right)\right| \\
&= \lim_{m\to+\infty}\lim_{n\to+\infty}\left|\mathrm{Im}\left(\int_X [BD_Hf_n, D_Hf^*_m]_Hd\nu_\infty\right)\right| \\
&\leq - C_{\theta_p}\lim_{m\to+\infty}\lim_{n\to+\infty}\int_X \mathrm{Re}[BD_Hf_n, D_Hf^*_m]_Hd\nu_\infty \\
&= - C_{\theta_p}\lim_{m\to+\infty}\int_X \mathrm{Re}[BD_Hf, D_Hf^*_m]_Hd\nu_\infty \\
&= - C_{\theta_p}\lim_{m\to+\infty}\mathrm{Re}\left(\int_X Lff^*_md\nu_\infty\right) = -C_{\theta_p}\mathrm{Re}\left(\int_X Lff^*d\nu_\infty\right) \\
&= - C_{\theta_p}\mathrm{Re}\left(\int_X L_pff^*d\nu_\infty\right).
\end{aligned}$$

This shows that Proposition 4.4(ii) holds true for any $f \in D(L_p^{\mathbb{C}})$ for any $p \in [2,+\infty)$.

Step 2. Let $p \in (1,2)$ and let $f \in \mathscr{F}\mathscr{C}^2_{b,\Theta}(X)$. Then, if we set $g = f^*$, we have $g \in \mathrm{L}^{p'}(X,\nu_\infty)$ with $p' \in (2,+\infty)$, $g^* = f$ and therefore

$$\int_X L_pff^*d\nu_\infty = \int_X Lff^*d\nu_\infty = \int_X [BD_Hf, D_Hf^*]_Hd\nu_\infty = \int_X [B^*D_Hg, D_Hg^*]_Hd\nu_\infty.$$

Arguing as in the first part of Step 1 and by applying (4.5) with f replaced by g we infer that

$$\left|\mathrm{Im}\left(\int_X L_pff^*d\nu_\infty\right)\right| \leq= - C_{\theta_p}\mathrm{Re}\left(\int_X L_pff^*d\nu_\infty\right).$$

Let $f \in D(L_p^{\mathbb{C}})$ and let us set again $g := f^*$. Approximating g with a sequence $(g_n) \subset \mathscr{F}\mathscr{C}^2_{b,\Theta}(X;\mathbb{C})$ we can repeat the argument of the second part of Step 1, and therefore we get

$$\left|\mathrm{Im}\left(\int_X L_pff^*d\nu_\infty\right)\right| \leq - C_{\theta_p}\mathrm{Re}\left(\int_X L_pff^*d\nu_\infty\right), \quad f \in D(L_p^{\mathbb{C}}).$$

This concludes the proof. □

5 Example

In this subsection we provide an example of operators A and Q which satisfy Hypotheses 2.1, 2.3 and 2.8. Let $X := \mathrm{L}^2(0,1)$, let A be the realization of the Laplace operator in $\mathrm{L}^2(0,1)$ with domain $W^{2,2}((0,1),d\xi)\cap W^{1,2}_0((0,1),d\xi)$ and let $Q : W \to X$ be the covariance operator of the Wiener measure on X, i.e.,

$$Qf(x) := \int_0^1 \min\{x,y\}f(y)dy, \quad x \in (0,1),$$

for any $f \in \mathrm{L}^2(0,1)$ (see e.g. [30]). It is well known that A is self-adjoint and that $e_k = \sqrt{2}\sin(k\pi\cdot)$, $k \in \mathbb{N}$, is an orthonormal basis of $\mathrm{L}^2((0,1), d\xi)$ of eigenvectors of A with corresponding eigenvalues $\lambda_k = -k^2\pi^2$. We denote by $(e^{tA})_{t\geq 0}$ the semigroup generated by A. $(e^{tA})_{t\geq 0}$ is analytic on $\mathrm{L}^2((0,1), d\xi)$ and $e^{tA}e_k = e^{-k^2\pi^2 t}e_k$ for any $k \in \mathbb{N}$. Then, it is not hard to see that for any smooth function f we have

$$(Qe^{sA}f)(x) = \sqrt{2}\sum_{k=1}^{\infty} e^{-k^2\pi^2 s}\langle f, \sqrt{2}\sin(k\pi\cdot)\rangle_{\mathrm{L}^2}\left(\frac{1}{k^2\pi^2}\sin(k\pi x) + \frac{(-1)^{k+1}}{k\pi}x\right).$$

Moreover,

$$\begin{aligned}(e^{sA}Qe^{sA}f)(x) =& \sqrt{2}\sum_{k=1}^{\infty} e^{-2k^2\pi^2 s}\langle f, \sqrt{2}\sin(k\pi\cdot)\rangle_{\mathrm{L}^2}\frac{1}{k^2\pi^2}\sin(k\pi x)\\ &+ 2\sum_{k,j=1}^{\infty} e^{-(k^2+j^2)\pi^2 s}\langle f, \sqrt{2}\sin(k\pi\cdot)\rangle_{\mathrm{L}^2}\frac{(-1)^{k+1}}{k\pi}\langle x, \sqrt{2}\sin(j\pi\cdot)\rangle_{\mathrm{L}^2}\sin(j\pi x)\\ =& \sqrt{2}\sum_{k=1}^{\infty} e^{-2k^2\pi^2 s}\langle f, \sqrt{2}\sin(k\pi\cdot)\rangle_{\mathrm{L}^2}\frac{1}{k^2\pi^2}\sin(k\pi x)\\ &+ 2\sqrt{2}\sum_{k,j=1}^{\infty} e^{-(k^2+j^2)\pi^2 s}\langle f, \sqrt{2}\sin(k\pi\cdot)\rangle_{\mathrm{L}^2}\frac{(-1)^{k+j+2}}{kj\pi^2}\sin(j\pi x).\end{aligned}$$

Integrating between 0 and t we get

$$\begin{aligned}(Q_t)f(x) =& \sqrt{2}\sum_{k=1}^{\infty}\langle f, e_k\rangle_{\mathrm{L}^2}\frac{1-e^{-2k^2\pi^2 t}}{2k^4\pi^4}\sin(k\pi x)\\ &+ 2\sqrt{2}\sum_{k,j=1}^{\infty}\langle f, e_k\rangle_{\mathrm{L}^2}\frac{(-1)^{k+j+2}(1-e^{-(k^2+j^2)\pi^2 t})}{kj(k^2+j^2)\pi^4}\sin(j\pi x).\end{aligned}$$

Proposition 5.1. *Q_t is a trace class operator for any $t > 0$, $Q_t \to Q_\infty$ in the operator norm and Q_∞ is a trace class operator, where*

$$\begin{aligned}Q_\infty f(x) =& \sqrt{2}\sum_{k=1}^{\infty}\langle f, e_k\rangle_{\mathrm{L}^2}\frac{1}{2k^4\pi^4}\sin(k\pi x) + 2\sqrt{2}\sum_{k,j=1}^{\infty}\langle f, e_k\rangle_{\mathrm{L}^2}\frac{(-1)^{k+j+2}}{kj(k^2+j^2)\pi^4}\sin(j\pi x)\\ =& \frac{3\sqrt{2}}{2}\sum_{k=1}^{\infty}\langle f, e_k\rangle_{\mathrm{L}^2}\frac{1}{2k^4\pi^4}\sin(k\pi x) + 2\sqrt{2}\sum_{j\neq k}^{\infty}\langle f, e_k\rangle_{\mathrm{L}^2}\frac{(-1)^{k+j+2}}{kj(k^2+j^2)\pi^4}\sin(j\pi x).\end{aligned}$$

Proof. We have

$$\sum_{k=1}^{\infty}\langle Q_t e_k, e_k\rangle_{\mathrm{L}^2} = \frac{3\sqrt{2}}{2}\sum_{k=1}^{\infty}\frac{1-e^{-k^2\pi^2 t}}{k^4\pi^4} < +\infty,$$

and

$$\sum_{k=1}^{\infty}\langle Q_\infty e_k, e_k\rangle_{\mathrm{L}^2} = \frac{3\sqrt{2}}{2}\sum_{k=1}^{\infty}\frac{1}{k^4\pi^4} < +\infty.$$

□

Finally, let us take $U : X \to \mathbb{R}$ defined by

$$U(f) := \int_0^1 f(\xi)^2 d\xi, \quad f \in X.$$

Further, from [6, Subection 7.1] we infer that $U \in W^{1,p}_H(X, \mu_\infty)$ for any $p \in (1, +\infty)$. Hence, the Ornstein-Uhlenbeck operator L_p is sectorial in $\mathrm{L}^p(\mathrm{L}^2(0,1), e^{-U}\mu_\infty)$ for any $p \in (1, +\infty)$.