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Abstract

To help to achieve high performances in the regulation of linear scalar (SISO) nonminimum-phase systems, an inversion-based (feedforward) control method is proposed. The aim is designing an inverse input to smoothly switch from a current, arbitrary, steady-state regime to a new, future, desired steady-state output. A new-found polynomial basis solves the related interpolation problem to join the current output to the future one while ensuring the necessary or desired smoothness. The (interpolation) transition time can be minimized in order to optimally reduce the delay with which the desired output occurs. By applying a behavioral stable inversion formula to the overall smoothed output, detailed expressions of the inverse input are finally derived. A simulation of a flexible arm rotating in the horizontal plane exemplifies the presented method.

Keywords: Feedforward control, inversion-based control, behavioral approach, steady-state, nonminimum-phase linear systems, polynomial interpolation

1. Introduction

Feedforward control helps to improve the performances of control systems [2, 3]. Among the various feedforward methods (bang-bang control, input shaping techniques, etc.) inversion-based control methods have found their

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way in mid 90's and subsequent years [4, 5, 6, 7]. These methods share a common idea. First, an output signal is designed according to the pertinent application. Then, by system inversion, the corresponding (inverse) input that causes the desired output is determined. In this approach, a difficulty was found in the application to nonminimum-phase systems, i.e. systems whose zero dynamics [8] is unstable. Indeed, for these systems the standard inversion procedure fails to provide an acceptable solution insofar the inverse input is unbounded even in presence of a bounded desired output. This theoretical obstruction was overcome by the works in [9, 10, 4, 5]. The idea that led to the breakthrough was to search for solutions among noncausal signals. In such a way, it emerged a line of research devoted to *noncausal stable inversion*. In this line, one of the first addressed problems was that of feedforward regulation, i.e. the problem to make an output transition from a current constant value to a future one [7, 11, 12]. In particular, in [12] *transition polynomials* were used to smoothly shape a monotonically increasing output signal between the current and future output values.

In this paper, still in the context of scalar (i.e. single-input single-output or SISO) linear nonminimum-phase systems, we extend the results of [12] by addressing and solving a *generalized feedforward regulation* problem. This is about designing a control input to smoothly switch from a current, arbitrary, steady-state regime (forming an input-output pair) to a new, future, desired steady-state output. The way to achieve this smooth transition is to join the current output to the future one with a polynomial, solution of an interpolation problem over a time interval of duration τ (cf. Problem 2). In such a way, an overall output having the necessary or desired *smoothness degree* (cf. Definition 3) is obtained. A closed-form expression of the interpolating polynomial is then provided by a *polynomial basis* (cf. Proposition 4) which is deduced by means of the Spitzbart's generalized interpolation formula [13]. This polynomial is parameterized by the transition time τ which is a free parameter that can be minimized in order to reduce the delay with which the desired output occurs (cf. Problem 3). By applying the stable inversion formula (7) to the overall output (17), detailed expressions of the inverse input to be used as a feedforward control are then determined (cf. (28)-(30)). The problem formulation and the solution provided require a behavioral approach to inversion-based control (cf. [14] and [15]) and a new general definition of steady-state solutions. In particular, the concept of (input-output) *steady-state pair* is introduced (cf. Definition 7) and its connection with stable inversion is established by a converse theorem (Theorem 2).

In a multivariable state-space setting, the problem of tracking-transition switching, which is similar to the addressed generalized feedforward regulation problem, was solved in [16]. This work uses inversion-based control with preview and the minimization of an integral quadratic index to design the output during the transition periods. However, in [16] the transition times between the output tracking sections cannot be minimized because all the time instances defining the tracking/transition sections are required to be fixed. Moreover, it is not possible to arbitrarily choose the smoothness degree of the output. In [17] for a scalar nonlinear system, an interpolating polynomial is designed as output signal to solve the feedforward regulation problem in the simpler case of an output transition between two constant values. This technique that uses a numerical routine to solve a two-point boundary value problem for the system's zero dynamics has the advantage to obtain a causal feedforward input, but at the price of an output transition that can exhibits large overshooting and/or undershooting. Here, as in [16], the smoothness degree of the output (or the input) cannot be arbitrarily chosen.

Polynomials or polynomial B-splines are also used in the approximate stable inversion methods to feedforward control in the works of [18, 19, 20]. Polynomials and other basis functions are used in [21]. It proposes a general pseudo-inversion method that addresses the smoothness issue. The required continuity of the input can be achieved by suitably reducing the solution searching space. All the inversion methods in [18, 19, 20, 21] require that the system to be inverted is asymptotically stable and the desired output to be approximated is only defined for positive times. These assumptions are overcome by the presented approach because herein the system to be inverted is allowed to be unstable (cf. Remark 1) and the overall desired output (cf. (17)) is, in general, a noncausal signal (i.e. a signal that is not identically zero for negative times). Just a few parts of this article are taken from [1]. Indeed, the present paper solves a more general feedforward regulation problem. (In [1] the current steady-state output is identically zero.) Moreover, new results on steady-state solutions and stable inversion have been added (cf. Subsection 3.1).

By summarizing, the main novelties herein presented are:

- A new problem and an inversion-based solution for the generalized feedforward regulation of scalar (SISO) linear nonminimum-phase systems

- A behavioral presentation of steady-state solutions and its connection with stable input-output inversion

- A new-found polynomial basis for the closed-form expression of an interpolating polynomial to smoothly join two distinct steady-state outputs
- Minimization of the transition time to reduce the delay of the future desired output.

Paper organization: Section 2 provides the preliminaries and summarizes the main results of the behavioral approach to inversion-based control [15]. Section 3 has two subsections. Subsection 3.1 reports the behavioral definition of steady-state and the converse result linking steady-state pairs to stable inversion (Theorem 2). The generalized feeedfoward regulation problem is introduced in Subsection 3.2 along with the associated interpolation problem (Problem 2). Solution to this problem is given by the parameterized interpolating polynomial presented in Section 4. The inverse input that is a solution of the generalized feedforward regulation problem is presented in Section 5 along with the pertinent analysis on the *preaction* and *postaction control* phenomena (cf. Propositions 5 and 6). Then the minimization of the transition time is addressed by Problem 3. Section 6 presents a simulation example of feedforward regulation for a flexible arm rotating in the horizontal plane. Finally, a summary and a perspective on the paper's contribution are reported in Section 7.

Notation: The set of natural numbers comprising zero is denoted by $\mathbb{N}$. We say that a real function $f : \mathbb{R} \to \mathbb{R}$ has continuity order n if it belongs to C^n, the set of continuous functions with continuous derivatives up to the nth-order. The nth-order derivative of a real function f is denoted by $f^{(n)}$. The nth-order derivative operator is denoted by D^n so that $D^n f \equiv f^{(n)}$. Given a real function f and $n \in \mathbb{N}$, the following shorthand notation stands for the left and right limits: $f^{(n)}(t^-) := \lim_{v \to t^-} f^{(n)}(v)$, $f^{(n)}(t^+) := \lim_{v \to t^+} f^{(n)}(v)$. The analytical extension over $\mathbb{R}$ of the inverse Laplace transform is denoted by $\mathcal{L}_{\mathtt{ae}}^{-1}[\cdot]$. The set of polynomial with real (complex) coefficients is denoted by $\mathcal{P}$. The degree of $p \in \mathcal{P}$ is $\deg p$. If p is the null polynomial then $\deg p = -1$ conventionally.

2. Preliminaries and stable input-output inversion

2.1. $C_{\mathtt{p}}^{\infty}$, the set of piecewise C^{∞}-functions

A set $S \subset \mathbb{R}$ is said to be *sparse* if for any real finite interval $[a, b]$, the intersection $S \cap [a, b]$ has finite cardinality or it is the empty set. The space of signals used herein is $C_{\mathtt{p}}^{\infty}$ according to this definition [15].

Definition 1 ($C_{\mathtt{p}}^{\infty}$, set of piecewise C^{∞}-functions). *A function f belongs to $C_{\mathtt{p}}^{\infty}$, called the* set of piecewise C^{∞}-functions*, if there exists a sparse set S for which $f \in C^{\infty}(\mathbb{R}\backslash S, \mathbb{R})$ and for any $n \in \mathbb{N}$ and $t \in S$ the limits $f^{(n)}(t^{-})$ and $f^{(n)}(t^{+})$ exist and are finite.*
When f is defined in $t \in S$, conventionally $f(t) := f(t^{+})$; in particular $C^{-1} := C_{\mathtt{p}}^{\infty}(\mathbb{R})$ denotes the set of piecewise C^{∞}-functions defined over the whole set of reals.

The integral/differential operator acting on $C_{\mathtt{p}}^{\infty}$ can be introduced as follows. Let $f \in C_{\mathtt{p}}^{\infty}$ and define $\int f(t) \equiv \int^{1} f(t) \equiv (\int f)(t) := \int_0^t f(\xi)d\xi$, $\int^0 f := f$. Given $k \in \mathbb{Z}$, $\int^k f$ is defined by the recursion $\int^k f := \int(\int^{k-1} f)$ if $k \geq 1$ whereas $\int^k f := D^{-k} f$ if $k \leq -1$.

In the signal space $C_{\mathtt{p}}^{\infty}$, useful definitions are the following.

Definition 2 (Polynomial order [15]). *A signal $f \in C_{\mathtt{p}}^{\infty}$ has polynomial order $l \in \mathbb{N}$ if there exist constants $M > 0$ and $N > 0$ such that*

$$|f(t^{+})| < M\,|t|^{l} + N, \forall t \in \mathbb{R}. \tag{1}$$

Definition 3 (Smoothness degree [15]). *A signal $f \in C_{\mathtt{p}}^{\infty}(\mathbb{R})$ is said to have smoothness degree $k \geq -1$ if $f \in C^{k}$ and $f \notin C^{k+1}$. Signal f has* infinite *smoothness, i.e. $k = \infty$, when $f \in C^{\infty}$.*

Note that a smoothness degree k of $f \in C_{\mathtt{p}}^{\infty}(\mathbb{R})$ means that k is the maximal continuity order of f. Straightforward useful lemmas are the following (for brevity their proofs are omitted).

Lemma 1. *Let $f \in C_{\mathtt{p}}^{\infty}$ have finite polynomial order. Then the integral $\int_0^t f(v)dv$ has finite polynomial order too.*

Lemma 2. *Let $f, g \in C_{\mathtt{p}}^{\infty}$ have finite polynomial orders. Then their convolution $\int_0^t f(t-v)g(v)dv$ has finite polynomial order too.*

2.2. Stable input-output inversion

Let us consider a linear time-invariant system Σ whose transfer function is

$$H(s) = \frac{b(s)}{a(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0}.$$

Input and output are $u \in C_{\mathtt{p}}^{\infty}(\mathbb{R})$ and $y \in C_{\mathtt{p}}^{\infty}(\mathbb{R})$ respectively. Polynomials $a(s)$ and $b(s)$ have real coefficients; they are coprime with $a_n \neq 0$, $b_m \neq 0$, and $m \leq n$. The order of Σ is n and its relative degree is $r := n - m$. Moreover, we assume that the zero dynamics of Σ is hyperbolic, i.e. any zero of Σ has a positive or negative real part (the zeros of Σ are the roots of $b(s)$).

The behavior of Σ, i.e. the set of all pairs of input and output signals, can be introduced as the set of *weak solutions* of the differential equation associated to Σ:

$$\sum_{i=0}^{n} a_i D^i y(t) = \sum_{i=0}^{m} b_i D^i u(t). \tag{2}$$

Definition 4 (Weak solution [15]). *A pair $(u, y) \in C_{\mathtt{p}}^{\infty}(\mathbb{R})^2$ is a weak solution of differential equation (2) if there exists a polynomial $g \in \mathcal{P}$ with $\deg g \leq n-1$ such that the integral equation*

$$\sum_{i=0}^{n} a_i \int^{n-i} y(t) = \sum_{i=0}^{m} b_i \int^{n-i} u(t) + g(t) \tag{3}$$

is satisfied for all $t \in \mathbb{R}$.

The behavior of Σ can be then formally introduced as follows.

Definition 5 (Behaviour of Σ).

$$\mathcal{B} := \{\, (u, y) \in C_{\mathtt{p}}^{\infty}(\mathbb{R})^2 : (u, y) \text{ is a weak solution of (2)} \,\}.$$

A property on the continuity order of the output functions is the following.

Proposition 1 ([15]). *Let $(u, y) \in \mathcal{B}$, then $y \in C^{r-1}$.*

A simple relation between smoothness degrees of input and output is given by the following result.

Proposition 2 ([15]). *Consider a pair $(u, y) \in \mathcal{B}$. Then, input u has smoothness degree k if and only if output y has smoothness degree $k + r$.*

In the inversion-based control a relevant concept is that of *zero modes* of Σ [15].

Definition 6 (Zero modes of Σ). *Given a real (complex) zero of Σ $z \in \mathbb{R}$ ($z = \rho \pm j\psi \in \mathbb{C}$) with multiplicity ν, the associated modes are $e^{zt}, te^{zt}, \ldots,$ $t^{\nu-1}e^{zt}$ ($e^{\rho t}\cos(\psi t), e^{\rho t}\sin(\psi t), \ldots, t^{\nu-1}e^{\rho t}\cos(\psi t), t^{\nu-1}e^{\psi t}\sin(\psi t)$). All the zero modes of Σ are denoted by $m_i(t)$, $i = 1\ldots, m$. These modes can be split into stable and unstable ones according to: $m_i^-(t)$, $i = 1, \ldots, m^-$ denote the stable zero modes ($\lim_{t\to+\infty} m_i^-(t) = 0$) whereas $m_i^+(t)$, $i = 1, \ldots, m^+$ denote the unstable ones ($\lim_{t\to-\infty} m_i^+(t) = 0$). By our assumption $m^+ + m^- = m$.*

The stable input-output inversion problem can be introduced as follows [15].

Problem 1 (Stable inversion problem). *Let be given a desired output signal $y_{\mathtt{d}} \in C_{\mathtt{p}}^{\infty}(\mathbb{R})$ with smoothness degree k. Assume that $y_{\mathtt{d}}$ and its derivatives $Dy_{\mathtt{d}}, \ldots, D^r y_{\mathtt{d}}$ have all polynomial order l. Find an (inverse) input $u_{\mathtt{d}} \in C_{\mathtt{p}}^{\infty}(\mathbb{R})$ with polynomial order l such that $(u_{\mathtt{d}}, y_{\mathtt{d}}) \in \mathcal{B}$.*

The stable inversion procedure can be summarized as follows. By Euclidean division we express $a(s) = q(s)b(s) + d(s)$, with $q(s) = q_r s^r + q_{r-1}s^{r-1} + \cdots + q_0$, $q_r = \frac{a_n}{b_m} \neq 0$ and $\deg d(s) < m$. Let $H_0(s) := \frac{d(s)}{b(s)} = \frac{d(s)}{b_m\, b^-(s)\, b^+(s)}$ be the transfer function of the zero dynamics of Σ with $b^-(s)$ and $b^+(s)$ being monic polynomials having all the roots with negative and positive real parts respectively. By partial fraction expansion $H_0(s) = H_0^-(s) + H_0^+(s)$ where $H_0^-(s) := \frac{d^-(s)}{b^-(s)}$ and $H_0^+(s) := \frac{d^+(s)}{b^+(s)}$ with $d^-(s)$ and $d^+(s)$ being suitable polynomials. Let $h_0(t) := \mathcal{L}_{\mathtt{ae}}^{-1}[H_0(s)])$, $h_0^-(t) := \mathcal{L}_{\mathtt{ae}}^{-1}[H_0^-(s)])$ and $h_0^+(t) := \mathcal{L}_{\mathtt{ae}}^{-1}[H_0^+(s)])$, so that

$$h_0(t) = h_0^-(t) + h_0^+(t),\ t \in \mathbb{R}. \tag{4}$$

By taking into account Definition 6, there exist real coefficients α_i and β_i such that

$$h_0^-(t) = \sum_{i=1}^{m^-} \alpha_i m_i^-(t), \quad h_0^+(t) = \sum_{i=1}^{m^+} \beta_i m_i^+(t),\ t \in \mathbb{R}. \tag{5}$$

Also define $q(D)$ as the differential operator associated to polynomial $q(s)$. The solution to the stable inversion problem can be then introduced as follows [15].

Theorem 1. *The stable inversion problem (Problem 1) has a solution if and only if*

$$k \geq r - 1, \tag{6}$$

(i.e. the smoothness degree of $y_{\mathtt{d}}$ is greater than or equal to the relative degree of Σ minus one). When condition (6) is satisfied the solution is unique and can be expressed as

$$u_{\mathtt{d}}(t) = q(D)y_{\mathtt{d}}(t^+) + \int_{-\infty}^{t} h_0^-(t-v)y_{\mathtt{d}}(v)dv$$
$$- \int_{t}^{+\infty} h_0^+(t-v)y_{\mathtt{d}}(v)dv,\ t \in \mathbb{R}. \tag{7}$$

Remark 1. Theorem 1 can be applied to systems that can be either stable or unstable (no assumptions are made on the poles of Σ). When a system, typically a plant to be controlled, is (asymptotically) stable the inverse input $u_{\mathtt{d}}$ can be injected to the system as a purely feedforward (open-loop) control (cf. the example in Section 6). However, in the presence of significant model uncertainties or perturbations on the system, it is advisable to add feedback. By using output feedback, this can be done by the following feedforward-feedback schemes: i) the *plant inversion* architecture [22, 23] and ii) the *closed-loop inversion* architecture [7, 24]. In the first scheme the stable inversion is performed on the nominal plant. Then, the feedback controller adds a correcting input to the plant's inverse input to reduce the tracking error between the desired output and the actual one. The second scheme uses a unity feedback controller to reduce the sensitivity of the closed-loop system to disturbances and plant perturbations. Then, stable inversion is applied to the nominal closed-loop system to determine the actual input to inject.

When the plant is unstable the inverse input $u_{\mathtt{d}}$ cannot be directly injected to the system because the pair $(u_{\mathtt{d}}, y_{\mathtt{d}})$ is an unstable trajectory. In this case the implementation necessarily requires a feedforward-feedback scheme, specifically the plant inversion architecture in which closed-loop stability is ensured by the feedback controller. However, also the closed-loop inversion architecture can be adopted (to obtain on the plant the desired output $y_{\mathtt{d}}$) but in this case the stable inversion is applied to the closed-loop (stabilized) system. An example of set-point regulation of an unstable nonminimum-phase plant by the the closed-loop inversion architecture is reported in [25]. Comparisons between the plant and the closed-loop inversion architectures are presented in [26, 27].

Remark 2. If the zero dynamics of Σ is nonhyperbolic, i.e. there are zeros on the imaginary axis of the complex plane, the solution provided by the inversion formula (7) to the stable inversion problem (Problem 1) is no longer valid. However, there is the possibility to solve an approximate stable inversion problem by suitably perturbing the system to obtain a near nonhyperbolic zero dynamics. Then, formula (7) can be applied but at the price to accept a large *preaction time* (cf. (33) and Proposition 5). A reference on this kind of approximation can be found in [28].

3. Steady-state pairs and problem motivation

3.1. Steady-state solutions and stable inversion

Steady-state solutions of system Σ can be introduced in a coherent and simple way within the behavioral framework herein adopted.

Definition 7. *A pair* $(u_{\mathsf{ss}}, y_{\mathsf{ss}}) \in \mathcal{B}$ *is said to be a steady-state pair if both* u_{ss} *and* y_{ss} *have finite polynomial orders.*

Definition 8. *An input* u_{ss} *(output* y_{ss}*) is said to be steady-state if there exists an output* y_{ss} *(input* u_{ss}*) such that* $(u_{\mathsf{ss}}, y_{\mathsf{ss}}) \in \mathcal{B}$ *is steady-state.*

The present definition of steady-state can be regarded as a generalization of some common definitions currently used (cf. e.g. [29, 30, 31]). Indeed, this new concept of steady-state:

1. is defined over the entire time axis (i.e. for both negative and positive times) whereas the usual definitions focus on the positive times only.

2. is not restricted to polynomials and sinusoids only but it embraces more general functions within the limit of a finite polynomial order (cf. Definition 2 and Remark 3).

3. involves not only the output as the common definitions do (*steady-state response* by using the standard terminology) but also the input forming in such away the more complete concept of steady-state solutions or pairs.

In addition, we can remark that the introduced steady-state concept is still meaningful for unstable systems. First, note that a steady-state solution is a particular (weak) solution of the differential equation (2) regardless of whether or not the system is stable. (In [31] the steady-state is just introduced as a particular integral solution of the system's equation.) Secondly, note that pair $(u_{\mathtt{d}}, y_{\mathtt{d}})$ as constructed by stable inversion (cf. Theorem 1) is actually a steady-state solution. So, when the system is unstable this solution as well as any other steady-state pair can be implemented by adopting a feedforward-feedback architecture (cf. Remark 1).

The following converse result establishes a close connection between steady-state pairs and the stable input-output inversion.

Theorem 2. *Let be given a steady-state pair* $(u_{\mathtt{ss}}, y_{\mathtt{ss}}) \in \mathcal{B}$. *Then* $q(D)y_{\mathtt{ss}}(t^+)$ *has finite polynomial order and*

$$u_{\mathtt{ss}}(t) = q(D)y_{\mathtt{ss}}(t^+) + \int_{-\infty}^{t} h_0^-(t-v)y_{\mathtt{ss}}(v)dv - \int_{t}^{+\infty} h_0^+(t-v)y_{\mathtt{ss}}(v)dv,\ t \in \mathbb{R}. \tag{8}$$

Proof. The output $y_{\mathtt{ss}}$ has continuity order equal to $r-1$, i.e. $y_{\mathtt{ss}} \in C^{r-1}$ (cf. Proposition 1), so that by the *output-input representation* of the behavior $\mathcal{B}$ (cf. Theorem 5 in [15]) there must exist real coefficients g_i such that

$$u_{\mathtt{ss}}(t) = q(D)y_{\mathtt{ss}}(t^+) + \int_0^t h_0(t-v)y_{\mathtt{ss}}(v)dv + \sum_{i=1}^{m} g_i m_i(t),\ t \in \mathbb{R}. \tag{9}$$

Define $I(t) := \int_{-\infty}^{t} h_0^-(t-v)y_{\mathtt{ss}}(v)dv - \int_t^{+\infty} h_0^+(t-v)y_{\mathtt{ss}}(v)\ dv$, $t \in \mathbb{R}$ which is a function with finite polynomial order (cf. the proof of Theorem 6 in [15]) and note that by (4):

$$I(t) = \int_0^t h_0(t-v)y_{\mathtt{ss}}(v)dv + \int_{-\infty}^{0} h_0^-(t-v)y_{\mathtt{ss}}(v)dv - \int_0^{+\infty} h_0^+(t-v)y_{\mathtt{ss}}(v)dv,\ t \in \mathbb{R}.$$

On the other hand, $\int_{-\infty}^{0} h_0^-(t-v)y_{\mathtt{ss}}(v)dv$ and $\int_0^{+\infty} h_0^+(t-v)y_{\mathtt{ss}}(v)dv$ are linear combinations of the stable and unstable zero modes respectively (cf. the proof of Theorem 6 in [15]), i.e. there exist real coefficients δ_i, γ_i for which $\int_{-\infty}^{0} h_0^-(t-v)y_{\mathtt{ss}}(v)dv = \sum_{i=1}^{m^-} \delta_i m_i^-(t)$ and $\int_0^{+\infty} h_0^+(t-v)y_{\mathtt{ss}}(v)dv = \sum_{i=1}^{m^+} \gamma_i m_i^+(t)$, $t \in \mathbb{R}$. Hence, input $u_{\mathtt{ss}}$ can be rewritten as $u_{\mathtt{ss}}(t) = q(D)y_{\mathtt{ss}}(t^+) + I(t) - \sum_{i=1}^{m^-} \delta_i m_i^-(t) + \sum_{i=1}^{m^+} \gamma_i m_i^+(t) + \sum_{i=1}^{m} g_i m_i(t),\ t \in \mathbb{R}$ or in a more compact way (cf. Definition 6) $u_{\mathtt{ss}}(t) = q(D)y_{\mathtt{ss}}(t^+) + I(t) -$

$\sum_{i=1}^{m}\tilde{g}_i m_i(t),\ t\in\mathbb{R}$ having suitably defined the coefficients $\tilde{g}_i$. Define $f(t) := u_{\mathtt{ss}}(t) - I(t)$ which is a function of finite polynomial order as it is the difference of functions having finite polynomial orders and then

$$q(D)y_{\mathtt{ss}}(t^+) = f(t) + \sum_{i=1}^{m}\tilde{g}_i m_i(t),\ t\in\mathbb{R}. \tag{10}$$

Now, we prove that $q(D)y_{\mathtt{ss}}(t^+)$ has finite polynomial order which implies $\tilde{g}_i = 0,\ i = 1,\ldots,m$ and consequently relation (8) holds. By contradiction, assume that $q(D)y_{\mathtt{ss}}(t^+)$ hasn't finite polynomial order so that there must exist some $\tilde{g}_i$ that are not zeros. Without loss of generality, consider $\tilde{g}_1 \neq 0$ and all the other $\tilde{g}_i$ to be zeros. For simplicity, set $m_1(t) = e^{\rho t}$ with $\rho\in\mathbb{R}$. Relation (10) can be then interpreted as the following differential equation ($y_{\mathtt{ss}} \in C^{r-1}$ by Proposition 1):

$$q_r D^r y_{\mathtt{ss}}(t^+) + \sum_{i=0}^{r-1} q_i D^i y_{\mathtt{ss}}(t) = f(t) + \tilde{g}_1 e^{\rho t},\ t\in\mathbb{R}. \tag{11}$$

By introducing the *discontinuity set* $S^{(r)}_{y_{\mathtt{ss}}} := \{t\in\mathbb{R} : y^{(r)}_{\mathtt{ss}}$does not exist in $t\}$ (cf. [15]) it follows that

$$q_r D^r y_{\mathtt{ss}}(t) + \sum_{i=0}^{r-1} q_i D^i y_{\mathtt{ss}}(t) = f(t) + \tilde{g}_1 e^{\rho t},\ t\in\mathbb{R}\setminus S^{(r)}_{y_{\mathtt{ss}}}. \tag{12}$$

By virtue of the differential-integral characterization of weak solutions of differential equations (cf. Theorem 3 in [15]) the pair $(f(t) + \tilde{g}_1 e^{\rho t}, y_{\mathtt{ss}}(t))$ is a weak solution of $\sum_{i=0}^{r} q_i D^i y = u$ associated to the system transfer function $H_q(s) = 1/q(s)$. By the *input-output representation* of the behavior of this system (cf. Theorem 4 in [15]) there must exist real coefficients f_i such that

$$y_{\mathtt{ss}}(t) = \int_0^t h_q(t-v)(f(v) + \tilde{g}_1 e^{\rho v})dv + \sum_{i=1}^{r} f_i m_i^{\mathtt{p}}(t),\ t\in\mathbb{R} \tag{13}$$

where $h_q(t) := \mathcal{L}^{-1}_{\mathtt{ae}}[1/q(s)]$ and the $m_i^{\mathtt{p}}(t)$ are the *pole modes* associated to the roots of $q(s)$ (defined in analogy to the zero modes of Definition 6, also cf. [15]). Similarly to the splitting of $h_0(t)$ in (4), $h_q(t)$ can be obtained as the sum of three functions, i.e. $h_q(t) = h_q^-(t) + h_q^0(t) + h_q^+(t)$ where $h_q^-(t) := \mathcal{L}^{-1}_{\mathtt{ae}}[H_q^-(s)]$, $h_q^0(t) := \mathcal{L}^{-1}_{\mathtt{ae}}[H_q^0(s)]$ and $h_q^+(t) := \mathcal{L}^{-1}_{\mathtt{ae}}[H_q^+(s)]$ are associated to the roots of $q(s)$ with negative, zero and positive real parts ($H_q(s) = H_q^-(s) + H_q^0(s) + H_q^+(s)$). Hence, the integral $\int_0^t h_q(t-v)f(v)dv = I_1(t) + \sum_{i=1}^{r}\tilde{f}_i m_i^{\mathtt{p}}(t)$ where $I_1(t) := \int_{-\infty}^{t} h_q^-(t-v)f(v)dv + \int_0^t h_q^0(t-v)f(v)dv - \int_t^{+\infty} h_q^+(t-v)f(v)dv$ is a function with finite polynomial order because all the integral addends have finite polynomial orders too and the $\tilde{f}_i$ are real values. Indeed note, in particular, that $h_q^0(t)$ has finite polynomial order and so is the convolution $\int_0^t h_q^0(t-v)f(v)dv$ by Lemma 2. Expression (13) is then rewritten as

$$y_{\mathtt{ss}}(t) = I_1(t) + \tilde{g}_1 \int_0^t h_q(t-v)e^{\rho v}dv + \sum_{i=1}^{r}(\tilde{f}_i + f_i)m_i^{\mathtt{P}}(t),\ t \in \mathbb{R}. \tag{14}$$

In computing the integral appearing in (14) two possible cases arise: ρ does not coincide with any of the poles of $H_q(s)$ or it does coincide with one of them (it's the *resonant* case and let's say ρ has multiplicity ν as a pole of $H_q(s)$). In the former case

$$\int_0^t h_q(t-v)e^{\rho v}dv = H_q(\rho)e^{\rho t} + \sum_{i=1}^{r}\mu_i m_i^{\mathtt{P}}(t),\ t \in \mathbb{R}, \tag{15}$$

$\mu_i \in \mathbb{R}$ and $H_q(\rho) \neq 0$ (because $H_q(s) = 1/q(s)$ does not have any zeros) whereas in the latter

$$\int_0^t h_q(t-v)e^{\rho v}dv = \xi_0 t^{\nu} e^{\rho t} + \sum_{i=1}^{r}\xi_i m_i^{\mathtt{P}}(t),\ t \in \mathbb{R}, \tag{16}$$

$\xi_i \in \mathbb{R}$, $i = 0, 1, \ldots, r$ and $\xi_0 \neq 0$. To keep computations simple, we verify the above relations under the assumption that the poles of $H_q(s)$ are all real and simple, i.e. $q(s) = q_r \prod_{i=1}^{r}(s - p_i)$, $p_i \in \mathbb{R}$. By partial fraction expansion $H_q(s) = \sum_{i=1}^{r} k_i/(s - p_i)$ so that $h_q(t) = \sum_{i=1}^{r} k_i e^{p_i t}$ ($m_i^{\mathtt{P}}(t) \equiv e^{p_i t}$). After some passages, relation (15) is verified with $\mu_i = k_i/(p_i - \rho)$, $i = 1, \ldots, r$ and relation (16) with (let $p_1 = \rho$ and note that under the current assumption $\nu = 1$) $\xi_0 = 1/(q_r \prod_{i=2}^{r}(\rho - p_i))$, $\xi_1 = \sum_{i=2}^{r} k_i/(\rho - p_i)$, and $\xi_i = k_i/(p_i - \rho)$, $i = 2, \ldots, r$. Hence, in both cases, the right hand-side of (14) has an addend, $\tilde{g}_1 H_q(\rho)e^{\rho t}$ or $\tilde{g}_1 \xi_0 t^{\nu} e^{\rho t}$ that cannot be canceled by any combination of the modes $m_i^{\mathtt{P}}(t)$. Therefore, $y_{\mathtt{ss}}(t)$ cannot have finite polynomial order and this contradiction completes the proof. $\square$

In setting the stable inversion problem (cf. Problem 1), the assumption is to consider the desired output to be of finite polynomial order along with its derivatives up to the r-th order. This assumption is highlighted by the following result (whose proof is reported in the Appendix).

Proposition 3. *Let $y \in C_{\mathtt{P}}^{\infty}(\mathbb{R}) \cap C^{r-1}$ with $r \geq 1$ and assume y has finite polynomial order. Then the following statements are equivalent:*

(a) The derivatives $y^{(1)}, y^{(2)}, \ldots, y^{(r)}$ have all finite polynomial orders.

(b) *The derivative* $y^{(r)}$ *has finite polynomial order.*

(c) *The function* $q(D)y$ *has finite polynomial order where* $q(s)$ *is any polynomial with* $\deg q = r$.

Remark 3. It is worth stressing that the introduced new concept of steady-state (cf. Definitions 7 and 8) is actually a broader one (cf. [29, 30, 31]). Indeed, any signal belonging to $C_{\mathtt{p}}^{\infty} \cap C^{r-1}$ and having, along with its r-th order derivative, finite polynomial order is a *steady-state output* because by Proposition 3 and Theorem 1 there exists the (inverse) input for which they form a steady-state pair.

On the other hand, any signal (belonging to $C_{\mathtt{p}}^{\infty}(\mathbb{R})$) that has finite polynomial order can be a *steady-state input* $u_{\mathtt{ss}}$ because always there exists a corresponding output $y_{\mathtt{ss}}$ that has finite polynomial order. Indeed, for example, if Σ is hyperbolic (i.e., no poles on the imaginary axis) the unique corresponding output having finite polynomial order is $y_{\mathtt{ss}}(t) = \int_{-\infty}^{t} h^{-}(t - v)u_{\mathtt{ss}}(v)dv - \int_{t}^{+\infty} h^{+}(t-v)u_{\mathtt{ss}}(v)dv$, $t \in \mathbb{R}$ where $h^{-}(t) := \mathcal{L}_{\mathtt{ae}}^{-1}[H^{-}(s)]$, $h^{+}(t) := \mathcal{L}_{\mathtt{ae}}^{-1}[H^{+}(s)]$ (similarly to (4) $H(s) = H^{-}(s) + H^{+}(s)$ with $H^{-}(s)$ and $H^{+}(-s)$ being both asymptotically stable). Proof of this statement is omitted for brevity.

3.2. The generalized feedforward regulation problem

The generalized feedforward regulation problem is about designing a control input to switch from a current, arbitrary, steady-state pair $(u_0, y_0) \in \mathcal{B}$ for $t < 0$ to a new desired steady-state output $y_1 \in C_{\mathtt{p}}^{\infty}(\mathbb{R})$ when $t \geq 0$. This switching cannot be instantaneous. Indeed, the composite output resulting by joining $y_0(t)$, $t < 0$ with $y_1(t)$, $t \geq 0$ cannot be reproduced by any input because, in general, $y_0^{(i)}(0^-) \neq y_1^{(i)}(0^+)$, $i = 0, 1, \ldots, r-1$ and this clashes with the necessary requirement of the composite output having a smoothness degree greater or equal to $r-1$ (cf. Theorem 1). Moreover, depending on the control application, a sufficiently high smoothness degree of the input or the output may be required (cf. Proposition 2 and the example in Section 6). Hence, the necessity to adequately smooth the switching from y_0 to y_1 emerges.

A solution to this feedforward regulation problem is proposed. It uses inversion-based control coupled with the following interpolation scheme. To achieve the required smoothness a period of duration τ is inserted between y_0 and y_1 to allow a suitable interpolation of the current output with the desired future output. Hence, an interpolating function $p(t)$ may be designed to form the following overall (smoothed) output:

$$\tilde{y}(t) := \begin{cases} y_0(t) & t < 0 \\ p(t) & t \in [0, \tau] \\ y_1(t-\tau) & t > \tau \end{cases} . \tag{17}$$

In (17), the transition time τ is also the delay time that allows the insertion of the desired output. Therefore we consider the following interpolation problem.

Problem 2 (The interpolation problem). *Consider a current steady-state pair $(u_0, y_0) \in \mathcal{B}$ with $y_0 \in C^{k_0}((-\infty, 0])$ ($k_0 \geq r-1$, cf. Proposition 1). Also consider a desired output $y_1 \in C^{\infty}_{\mathtt{p}}(\mathbb{R}) \cap C^{k_1}([0, +\infty))$ with $k_1 \geq r-1$. Find a sufficiently smooth function $p(t)$ defined over $[0, \tau]$ such that the following interpolation conditions are satisfied at the endpoints of $[0, \tau]$:*

$$p^{(i)}(0) = y_0^{(i)}(0),\ i = 0, \ldots, k_0, \tag{18}$$

$$p^{(i)}(\tau) = y_1^{(i)}(0),\ i = 0, \ldots, k_1. \tag{19}$$

In the next section, a polynomial solution to the above interpolating problem is provided.

Remark 4. Since y_1 is a steady-state output there exists a corresponding input u_1 for which (u_1, y_1) is a steady-state pair (cf. Definition 8). (Moreover, by virtue of Theorem 2 input u_1 is the unique input such that (u_1, y_1) is steady-state.) Hence, the addressed feedforward regulation can be also seen as the *controllability problem* to smoothly steer the system from a given steady-state regime (u_0, y_0) to any desired steady-state (u_1, y_1) (cf. [14]).

Remark 5. To apply the stable inversion formula (7) of Theorem 1 requires that condition (6) be verified and the derivatives up to the r-th order of the output have all finite polynomial orders. Remarkably, if we know that a signal is a steady-state output (cf. Definition 8) then the inverse input can be determined by expression (8) of Theorem 2 — which is equal to the inversion formula (7) — without the need to ascertain the polynomial order finiteness of the output derivatives. Indeed, Theorem 2 along with Proposition 3 ensures that all the output derivatives up to the r-th order actually have finite polynomial orders. Also note that, still by virtue of Theorem 2 and Proposition 3, the introduced generalized feedforward regulation problem is set out without any assumption on the derivatives of y_0 and y_1.

4. The parameterized interpolating polynomial

To satisfy the $k_0 + k_1 + 2$ interpolating conditions given by (18) and (19) of Problem 2 we consider a $(k_0 + k_1 + 1)$-order polynomial $p(t)$. This polynomial can be deduced as a closed-form expression parameterized by the transition time τ (cf. the next Proposition 4). This deduction relies on Spitzbart's generalized interpolation formula [13]. When restricted to a two-nodes problem this formula can be introduced as follows.

Theorem 3. *[13] Let there be given t_j, k_j, $p_j^{(l)}$, $j = 0, 1$ and $l = 0, 1, \ldots, k_j$. Let $f_j(t)$ and $g_j(t)$ be defined by $(j = 0, 1)$*

$$f_0(t) = (t - t_1)^{k_1+1}, \; g_0(t) = [f_0(t)]^{-1}, \tag{20}$$

$$f_1(t) = (t - t_0)^{k_0+1}, \; g_1(t) = [f_1(t)]^{-1}. \tag{21}$$

Then the polynomial $p(t)$ of degree $k_0 + k_1 + 1$, such that

$$p^{(l)}(t_j) = p_j^{(l)}, \quad j = 0, 1, \quad l = 0, 1, \ldots, k_j,$$

is given by

$$p(t) = \sum_{j=0}^{1} \sum_{l=0}^{k_j} A_{jl}(t) p_j^{(l)} \tag{22}$$

where

$$A_{jl}(t) = f_j(t) \frac{(t - t_j)^l}{l!} \sum_{i=0}^{k_j - l} \frac{1}{i!} g_j^{(i)}(t_j)(t - t_j)^i. \tag{23}$$

The parameterized interpolating polynomial is provided as follows.

Proposition 4. *A solution to Problem 2 is given by the following parameterized interpolating polynomial*

$$p(t; \tau) = \sum_{l=0}^{k_0} q^0_{k_0 k_1 l}\,(t/\tau)\,\tau^l y_0^{(l)}(0) + \sum_{l=0}^{k_1} q^1_{k_0 k_1 l}\,(t/\tau)\,\tau^l y_1^{(l)}(0) \tag{24}$$

where

$$q^0_{k_0 k_1 l}(v) := \frac{k_1 + 1}{l!} \sum_{j=0}^{k_1+1} \sum_{i=0}^{k_0 - l} \frac{(-1)^{-k_1-1+j}(k_1 + i)!}{i! j! (k_1 + 1 - j)!} v^{k_1+1+l+i-j}, \tag{25}$$

$$q^1_{k_0 k_1 l}(v) := \frac{1}{l! k_0!} \sum_{i=0}^{k_1 - l} \sum_{j=0}^{l+i} \frac{(-1)^{i+j}(k_0 + i)!(l + i)!}{i! j! (l + i - j)!} v^{k_0+1+l+i-j}. \tag{26}$$

Proof. Apply Theorem 3 by setting $p_j^{(l)} = y_j^{(l)}(0)$, $j = 0, 1$, $l = 0, 1, \ldots, k_j$, $t_0 = 0$, $t_1 = \tau$ and rewrite (22) as

$$p(t;\tau) = \sum_{l=0}^{k_0} A_{0l}(t) y_0^{(l)}(0) + \sum_{l=0}^{k_1} A_{1l}(t) y_1^{(l)}(0). \tag{27}$$

By setting $j = 0$ in relation (23) we obtain $A_{0l}(t) = f_0(t)\frac{t^l}{l!}\sum_{i=0}^{k_0-l}\frac{1}{i!}g_0^{(i)}(0)t^i$. Definitions (20) imply that $g_0^{(i)}(0) = (-1)^{-k_1-1}\frac{(k_1+i)!}{k_1!}\tau^{-k_1-1-i}$ and then

$$A_{0l}(t) = (t-\tau)^{k_1+1}\frac{t^l}{l!}\sum_{i=0}^{k_0-l}\frac{1}{i!}(-1)^{-k_1-1}\frac{(k_1+i)!}{k_1!}\tau^{-k_1-1-i}t^i.$$

By expansion of the binomial power appearing above, $A_{0l}(t)$ can be expressed and manipulated as follows:

$$\begin{aligned}
&\frac{t^l}{l!}\sum_{j=0}^{k_1+1}\binom{k_1+1}{j}t^{k_1+1-j}(-\tau)^j\sum_{i=0}^{k_0-l}\frac{(-1)^{-k_1-1}(k_1+i)!}{i!k_1!}\tau^{-k_1-1-i}t^i\\
&=\frac{t^l}{l!}\sum_{j=0}^{k_1+1}\frac{(k_1+1)!}{j!(k_1+1-j)!}\sum_{i=0}^{k_0-l}\frac{(-1)^{-k_1-1+j}(k_1+i)!}{i!k_1!}(t/\tau)^{k_1+1-j+i}\\
&=\frac{t^l}{l!}\sum_{j=0}^{k_1+1}\sum_{i=0}^{k_0-l}\frac{(-1)^{-k_1-1+j}(k_1+1)!(k_1+i)!}{i!j!k_1!(k_1+1-j)!}(t/\tau)^{k_1+1-j+i}\\
&=\tau^l\frac{k_1+1}{l!}\sum_{j=0}^{k_1+1}\sum_{i=0}^{k_0-l}\frac{(-1)^{-k_1-1+j}(k_1+i)!}{i!j!(k_1+1-j)!}(t/\tau)^{k_1+1+l+i-j}.
\end{aligned}$$

Eventually, by definition (25) $A_{0l}(t) = q^0_{k_0k_1l}(t/\tau)\tau^l$.

Similarly, by setting $j = 1$ in relation (23) we obtain

$$A_{1l}(t) = f_1(t)\frac{(t-\tau)^l}{l!}\sum_{i=0}^{k_1-l}\frac{1}{i!}g_1^{(i)}(\tau)(t-\tau)^i.$$

Definitions (21) imply $g_1^{(i)}(\tau) = (-1)^i \frac{(k_0+i)!}{k_0!}\tau^{-k_0-i-1}$ and then

$$
\begin{aligned}
A_{1l}(t) &= t^{k_0+1}\frac{(t-\tau)^l}{l!}\sum_{i=0}^{k_1-l}\frac{1}{i!}(-1)^i\frac{(k_0+i)!}{k_0!}\tau^{-k_0-i-1}(t-\tau)^i \\
&=\frac{t^{k_0+1}}{l!}\sum_{i=0}^{k_1-l}\frac{(-1)^i(k_0+i)!}{i!k_0!}\tau^{-k_0-i-1}(t-\tau)^{l+i} \\
&=\frac{t^{k_0+1}}{l!}\sum_{i=0}^{k_1-l}\frac{(-1)^i(k_0+i)!}{i!k_0!}\tau^{-k_0-i-1}\sum_{j=0}^{l+i}\binom{l+i}{j}t^{l+i-j}(-\tau)^j \\
&=\frac{t^{k_0+1}}{l!}\sum_{i=0}^{k_1-l}\frac{(-1)^i(k_0+i)!}{i!k_0!}\tau^{-k_0-i-1}\sum_{j=0}^{l+i}\frac{(l+i)!}{j!(l+i-j)!}t^{l+i-j}(-\tau)^j \\
&=\frac{\tau^l}{l!k_0!}\sum_{i=0}^{k_1-l}\sum_{j=0}^{l+i}\frac{(-1)^{i+j}(k_0+i)!(l+i)!}{i!j!(l+i-j)!}(t/\tau)^{k_0+1+l+i-j}.
\end{aligned}
$$

By definition (26), $A_{1l}(t) = q^1_{k_0k_1l}(t/\tau)\tau^l$. Hence, expression (27) coincides with the parameterized interpolating polynomial given by (24). $\square$

Expressions appearing in (25) and (26) form a *polynomial basis* that leads to a straightforward writing of the interpolating polynomial (24). This polynomial basis does not depend on the interpolating data, nor does it depend on the transition time τ, but it just depends on k_0 and k_1, i.e. the imposed continuity orders at the endpoints of interval $[0, \tau]$ (cf. the example in Section 6).

Remark 6. For the special case of $k_0 = k_1 = k$ and $y_0^{(i)}(0) = 0$, $i = 0, 1, \dots, k$, $y_1(0) = 1$, $y_1^{(i)}(0) = 0$, $i = 1, \dots, k$, polynomial (24) becomes the *transition polynomial* introduced in [32, 12] for the inversion-based feedforward regulation of linear scalar systems.

5. Solution to the generalized feedforward regulation problem

By using the parameterized interpolating polynomial provided by the closed-form expression (24), the resulting output signal $\tilde{y}(t)$ given by (17) has an overall continuity order equal to $\min\{k_0, k_1\} \geq r - 1$. Hence, the smoothness degree of $\tilde{y}$ is greater or equal to $r - 1$ (cf. condition (6) of Theorem 1). On the other hand, $\tilde{y}$ and its derivatives $\tilde{y}^{(1)}, \dots, \tilde{y}^{(r)}$ have all finite polynomial orders because both y_0 and $y_1(t-\tau)$ with their derivatives $y_0^{(1)}, \ldots, y_0^{(r)}$ and $Dy_1(t-\tau), \ldots, D^r y_1(t-\tau)$ have all finite polynomial orders. Indeed, both (u_0, y_0) and (u_1, y_1) are steady-state (cf. Remark 4), hence by Theorem 2 and Proposition 3 all the derivatives $y_0^{(i)}$, $y_1^{(i)}$, $i = 1, \ldots, r$ have all finite polynomial orders. In turn, the delayed signals $D^i y_1(t-\tau)$, $i = 1, \ldots, r$, have finite polynomial orders too.

Therefore, by Theorem 1, the inversion formula (7) can be applied to $\tilde{y}(t)$ to obtain the inverse input $\tilde{u}(t)$ which is a solution of the generalized feedforward regulation problem. Detailed expressions of $\tilde{u}(t)$ on the relevant time intervals are the following. When $t < 0$

$$\begin{aligned}\tilde{u}(t) = q(D)y_0(t^+) &+ \int_{-\infty}^{t} h_0^-(t-v)y_0(v)dv \\ &- \int_t^0 h_0^+(t-v)y_0(v)dv - \int_0^\tau h_0^+(t-v)p(v;\tau)dv \\ &- \int_\tau^{+\infty} h_0^+(t-v)y_1(v-\tau)dv, \end{aligned} \tag{28}$$

if $t \in [0, \tau]$

$$\begin{aligned}\tilde{u}(t) &= q(D)p(t;\tau) \\ &+ \int_{-\infty}^{0} h_0^-(t-v)y_0(v)dv + \int_0^t h_0^-(t-v)p(v;\tau)dv \\ &- \int_t^\tau h_0^+(t-v)p(v;\tau)dv - \int_\tau^{+\infty} h_0^+(t-v)y_1(v-\tau)dv, \end{aligned} \tag{29}$$

and finally, with $t > \tau$

$$\begin{aligned}\tilde{u}(t) = q(D)y_1(t^+ - \tau) &+ \int_{-\infty}^{0} h_0^-(t-v)y_0(v)dv \\ &+ \int_0^\tau h_0^-(t-v)p(v;\tau)dv + \int_\tau^t h_0^-(t-v)y_1(v-\tau)dv \\ &- \int_t^{+\infty} h_0^+(t-v)y_1(v-\tau)dv. \end{aligned} \tag{30}$$

In nonminimum-phase systems the inverse feedforward control exhibits the so-called *preaction* (or *preactuation*) *control* (cf. [4, 33]) introduced by the following result.

Proposition 5 (Preaction Control). *The inverse input $\tilde{u}(t)$, for $t < 0$, is given by the steady-state input u_0 plus a linear combination of the unstable zero modes, i.e. there exist real coefficients γ_i such that*

$$\tilde{u}(t) = u_0(t) + \sum_{i=1}^{m^+} \gamma_i m_i^+(t), \ t < 0. \tag{31}$$

PROOF. When $t < 0$ the inverse input $\tilde{u}(t)$ is given by expression (28). Then add and subtract $\int_0^{+\infty} h_0^+(t-v) y_0(v) dv$ to this expression. Since pair (u_0, y_0) is steady-state, by Theorem 2 $u_0(t) = q(D) y_0(t^+) + \int_{-\infty}^{t} h_0^-(t-v) y_0(v) dv - \int_t^{+\infty} h_0^+(t-v)\, y_0(v) dv$. Hence, by taking into account definition (17) it follows that

$$\tilde{u}(t) = u_0(t) + \int_0^{+\infty} h_0^+(t-v)[y_0(v) - \tilde{y}(v)] dv. \tag{32}$$

Since $h_0^+(t) = \sum_{i=1}^{m_i^+} \beta_i m_i^+(t), t \in \mathbb{R}$, (cf. (5)) it follows that $\int_0^{+\infty} h_0^+(t-v)[y_0(v) - \tilde{y}(v)] dv = \sum_{i=1}^{m_i^+} \beta_i \int_0^{+\infty} m_i^+(t-v)[y_0(v) - \tilde{y}(v)] dv$. Note that function $y_0(t) - \tilde{y}(t)$ has finite polynomial order so that the integral $\int_0^{+\infty} m_i^+(t-v)[y_0(v) - \tilde{y}(v)] dv$ is, in general, a linear combination of a subset of the unstable zeros modes of Σ (cf. [15]). For example if $m_i^+(t) = e^{z^+ t}$ we simply obtain $\int_0^{+\infty} m_i^+(t-v)[y_0(v) - \tilde{y}(v)] dv = \gamma e^{z^+ t}$ with $\gamma := \int_0^{+\infty} e^{-z^+ v}[y_0(v) - \tilde{y}(v)] dv$. Hence, there exist real coefficients γ_i such that $\int_0^{+\infty} h_0^+(t-v)[y_0(v) - \tilde{y}(v)] dv = \sum_{i=1}^{m^+} \gamma_i m_i^+(t), t < 0$ and by relation (32) we obtain (31). □

The linear combination of the unstable zero modes in (31) is the preaction control that exponentially decays to zero as $t \to -\infty$. In practice, it is negligible term when $t < -t_{\mathtt{pre}}$ where $t_{\mathtt{pre}}$ is *preaction time*, i.e. the time span in which the preaction control is significantly different from 0. Preaction time can be estimated by

$$t_{\mathtt{pre}} := \frac{f_{\mathtt{pre}}}{d_{\mathtt{rhp}}} \tag{33}$$

where $f_{\mathtt{pre}}$ is a factor that may be selected in the interval $[5, 10]$ according to the desired accuracy (cf. [11]) and $d_{\mathtt{rhp}}$ is the minimum distance of the right half-plane zeros from the imaginary axis $j\mathbb{R}$.

Having defined the desired output y_1, the inverse input u_1 for which the pair (u_1, y_1) is steady-state can be determined by the inversion formula (8) or (7). Hence, the delayed pair $(u_1(t-\tau), y_1(t-\tau))$ is still a steady-state pair of system Σ. In our context, the so-called *postaction* (or *postactuation*) *control* (cf. [34, 35]) can be then presented as follows.

Proposition 6 (Postaction Control). *The inverse input $\tilde{u}(t)$, for $t > \tau$, is given by the steady-state input $u_1(t-\tau)$ plus a linear combination of the stable zero modes, i.e. there exist real coefficients δ_i such that*

$$\tilde{u}(t) = u_1(t-\tau) + \sum_{i=1}^{m^-} \delta_i m_i^-(t), \;\; t > \tau. \tag{34}$$

Proof. It is similar to that of Proposition 5. When $t > \tau$ the inverse input $\tilde{u}(t)$ is given by expression (30). Then add and subtract $\int_{-\infty}^{\tau} h_0^-(t-v)y_1(v-\tau)dv$ to this expression. Since pair $(u_1(t-\tau), y_1(t-\tau))$ is steady-state, by Theorem 2 $u_1(t-\tau) = q(D)y_1(t^+-\tau) + \int_{-\infty}^{t} h_0^-(t-v)y_1(v-\tau)dv - \int_{t}^{+\infty} h_0^+(t-v)y_1(t-\tau)dv$. Hence, by taking into account definition (17) it follows that

$$\tilde{u}(t) = u_1(t-\tau) + \int_{-\infty}^{\tau} h_0^-(t-v)[\tilde{y}(v) - y_1(v-\tau)]dv. \tag{35}$$

Since $h_0^-(t) = \sum_{i=1}^{m^-} \alpha_i m_i^-(t)$ (cf. (5)) it follows that $\int_{-\infty}^{\tau} h_0^-(t-v)[\tilde{y}(v) - y_1(v-\tau)]dv = \sum_{i=1}^{m^-} \alpha_i \int_{-\infty}^{\tau} m_i^-(t-v)[\tilde{y}(v) - y_1(v-\tau)]dv$. The integral $\int_{-\infty}^{\tau} m_i^-(t-v)[\tilde{y}(v) - y_1(v-\tau)]dv$ is, in general, a linear combination of a subset of the stable zero modes because function $\tilde{y}(t) - y_1(t-\tau)$ has finite polynomial order. Hence, there exist real coefficients δ_i such that $\int_{-\infty}^{\tau} h_0^-(t-v)[\tilde{y}(v) - y_1(v-\tau)]dv = \sum_{i=1}^{m^-} \delta_i m_i^-(t)$, $t > \tau$ and by relation (35), the expression (34) follows. $\square$

Remark 7. The presented preaction and postaction control properties (Propositions 5 and 6) improve over the analogous propositions reported in [15]. Specifically, Proposition 5 extends preaction control property to the case of a noncausal desired output (an output that is not identically zero on the negative time axis). In Proposition 6 the postaction control statement is simplified and clarified in relation to the role of the delayed steady-state pair $(u_1(t-\tau), y_1(t-\tau))$. Both propositions benefit of the new concept the steady-state (cf. Definition 7) and its connection with stable inversion (cf. Theorem 2). Due to this connection the proofs of Propositions 5 and 6 are direct and straightforward.

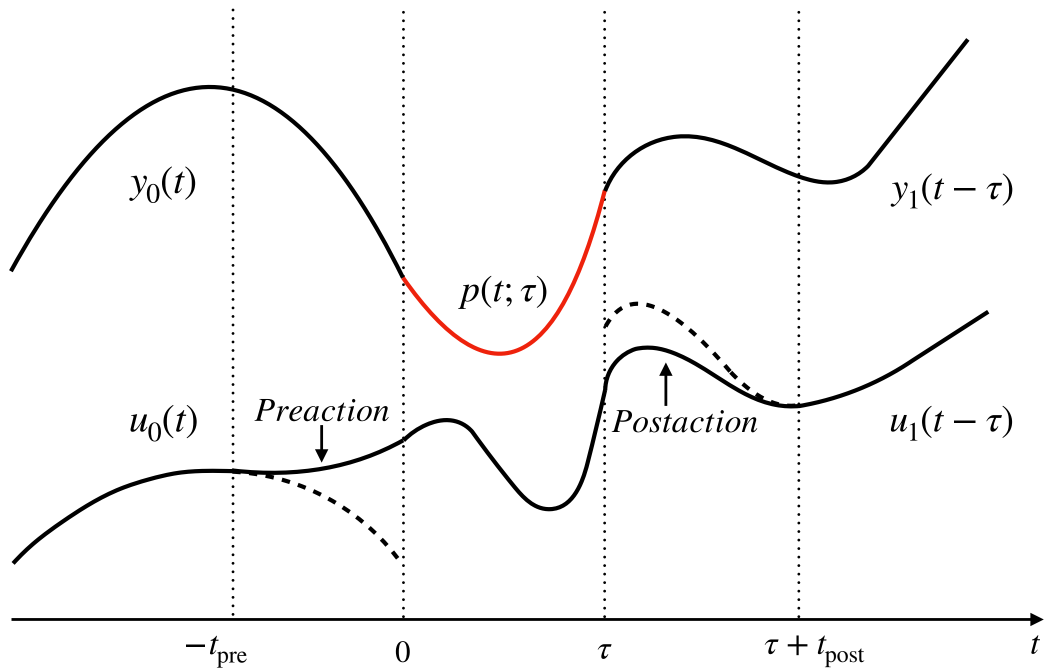

Figure 1: The pair $(\tilde{u}(t), \tilde{y}(t))$: the upper (lower) side displays the smoothed output $\tilde{y}(t)$ (inverse input $\tilde{u}(t)$). The red line highlights the interpolating polynomial. The dashed lines plot $u_0(t)$ in $[-t_{\mathtt{pre}}, 0]$ and $u_1(t-\tau^*)$ in $[\tau, \tau^* + t_{\mathtt{post}}]$. In these intervals $\tilde{u}(t)$ differs form $u_0(t)$ and $u_1(t-\tau^*)$ due to the presence of the preaction and postaction control respectively.

The linear combination of the stable zero modes in (34) is the postaction control that appears when there is a nontrivial stable zero dynamics (there exists at least one stable zero). The postaction control is negligible for $t > \tau + t_{\mathtt{post}}$ where $t_{\mathtt{post}}$ is the *postaction time*, i.e. the time span in which the postaction control significantly differs from zero. Analogously to (33), postaction time can be computed by

$$t_{\mathtt{post}} := \frac{f_{\mathtt{post}}}{d_{\mathtt{lhp}}} \tag{36}$$

where $f_{\mathtt{post}} \in [5, 10]$ (cf. [11]) and $d_{\mathtt{lhp}}$ is the minimum distance of the left half-plane zeros from the imaginary axis $j\mathbb{R}$.

Figure 1 illustrates the pair $(\tilde{u}, \tilde{y})$ highlighting the transition from pair (u_0, y_0) to the delayed pair $(u_1(t-\tau), y_1(t-\tau))$. Actually, the transition time τ delays the occurrence on the output of the desired y_1 and therefore it is sensible to minimize it. This can be done by solving the following optimization problem.

Problem 3 (Minimization of the transition time). *Define* $k_{\mathtt{max}} := \max\{k_0, k_1\}$ *and set*

$$\tau^* = \min\{\tau > 0 : |p^{(i)}(t;\tau)| \leq p_{ub}^{(i)}, \forall t \in [0,\tau], i = 0,1,\ldots,k_{\mathtt{max}}+1\} \quad (37)$$

where $p_{ub}^{(i)}$ *are selectable bounds to be chosen according to*

$$p_{ub}^{(i)} \geq \max\{|y_0^{(i)}(0)|, |y_1^{(i)}(0)|\},\ i = 0,1,\ldots,\min\{k_0,k_1\}, \quad (38)$$

$$p_{ub}^{(i)} \geq \begin{cases} |y_0^{(i)}(0)| & \text{if } k_0 > k_1,\ i = k_1+1,\ldots,k_0 \\ |y_1^{(i)}(0)| & \text{if } k_0 < k_1,\ i = k_0+1,\ldots,k_1 \end{cases} \quad (39)$$

and

$$p_{\mathtt{ub}}^{(k_{\mathtt{max}}+1)} > 0. \quad (40)$$

Obviously inequalities (38)-(40) are necessary conditions in order Problem 3 has a solution. In general, however, these inequalities do not make a sufficient condition. Hence, some care must be paid in choosing the bounds $p_{\mathtt{ub}}^{(i)}$. Problem 3 is evidently equivalent to the following one:

$$\min\left\{\tau > 0 : \max_{0\leq t\leq\tau} |p^{(i)}(t;\tau)| \leq p_{\mathtt{ub}}^{(i)}, i = 0,1,\ldots,k_{\mathtt{max}}+1\right\}. \quad (41)$$

Taking into account that $p(t;\tau)$ is a polynomial, for a given τ the maximum appearing in (41) can be easily determined. Hence, a standard local optimization routine to compute the solution τ^* of Problem 3 can be used. On the other hand, to obtain a guaranteed global solution, global optimization methods such as, e.g. those based on interval analysis, could be adopted (cf. [36]). Remarkably, for the special case of $k_0 = k_1 = k_{\mathtt{max}}$, $y_0^{(i)}(0) = 0$, $i = 0,1,\ldots,k_{\mathtt{max}}$, $y_1(0) \neq 0$, $y_1^{(i)}(0) = 0$, $i = 1\ldots,k_{\mathtt{max}}$ and $p_{\mathtt{ub}}^{(k_{\mathtt{max}}+1)} = +\infty$ (i.e. there is no bound on the derivative of order $k_{\mathtt{max}}+1$) a closed-form expression that gives the global solution τ^* is available (cf. [32]).

Remark 8. The rationale of Problem 3 is to search for a smoothing (or delay) time τ as small as possible while limiting the possible winding and oscillations of the interpolating polynomial $p(t;\tau)$. This limitation is achieved by imposing constraints on the absolute values of $p(t;\tau)$ and its derivatives $p^{(i)}(t;\tau)$, $i = 1\ldots,k_{\mathtt{max}}+1$, (cf. (37)).

6. An example

A flexible arm is rotated by a hub motor in the horizontal plane (see Figure 2). The (control) input is the hub angle u (measured in radians [rad]). The output is the tip position y of the flexible arm (measured in meters [m] along the tip's arc path). Considering the dominant dynamics only with data taken from [37] the resulting second order transfer function is

$$H(s) = -0.1913\frac{(s-9.31)(s+6.93)}{[(s+1.16)^2+2.99^2]}.$$

It is a nonminimum-phase system with stable and unstable zero modes given by $m_1^-(t) = e^{-6.93t}$ and $m_1^+(t) = e^{9.31t}$ respectively.

The following generalized feedforward regulation problem is addressed. A smooth transition from the harmonic steady-state regime given by the input-output pair (u_0, y_0) with $u_0(t) = 0.5\sin(t-1.6817)$, $y_0(t) = 0.6552\sin(t-1.8902)$, $t<0$ to a new desired output is sought. This is defined by $y_1(t) = 1+t$, $t \geq 0$ which is a ramp function with the velocity of 1 m/s for the arm's end-point. Solution to this problem is achieved by the proposed inversion-based control.

The control implementation requires that the input have a smoothness degree 2, i.e. the hub angle position, velocity, and acceleration be all continuous signals. The system relative degree is $r=0$ so that by Proposition 2 the overall output $\tilde{y}(t)$ (17) must have degree 2 of smoothness. Hence, in the design of the interpolating polynomial (24) we set $k_0 = k_1 = 2$. The data for

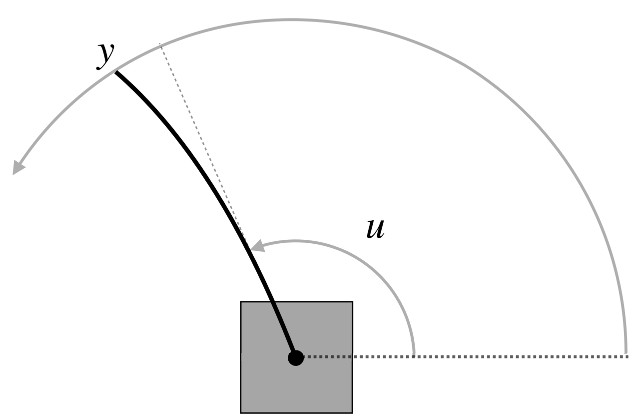

Figure 2: A sketch of the rotary flexible arm.

the interpolating conditions (18) and (19) are then the following:

$$y_0^{(0)}(0) = -0.6221,\ y_0^{(1)}(0) = -0.2057,\ y_0^{(2)}(0) = 0.6221;$$
$$y_1^{(0)}(0) = 1,\ y_1^{(1)}(0) = 1,\ y_0^{(2)}(0) = 0.$$

By Proposition 4, the parameterized interpolating polynomial is expressed by

$$p(t;\tau) = \sum_{l=0}^{2} q^0_{22l}(t/\tau)\tau^l y_0^{(l)}(0) + \sum_{l=0}^{2} q^1_{22l}(t/\tau)\tau^l y_1^{(l)}(0)$$

with associated polynomial basis given by (cf. (25), (26))

$$\begin{aligned}
q^0_{220}(v) &= -6v^5 + 15v^4 - 10v^3 + 1,\\
q^0_{221}(v) &= -3v^5 + 8v^4 - 6v^3 + v,\\
q^0_{222}(v) &= -\frac{1}{2}v^5 + \frac{3}{2}v^4 - \frac{3}{2}v^3 + \frac{1}{2}v^2,\\
q^1_{220}(v) &= 6\,v^5 - 15\,v^4 + 10\,v^3,\\
q^1_{221}(v) &= -3\,v^5 + 7\,v^4 - 4\,v^3,\\
q^1_{222}(v) &= \frac{1}{2}v^5 - v^4 + \frac{1}{2}v^3.
\end{aligned}$$

The minimization of the transition time (cf. Problem 3) is posed with the constraints defined by (cf. (37))

$$p^{(0)}_{\mathtt{ub}} = 2,\ p^{(1)}_{\mathtt{ub}} = 20,\ p^{(2)}_{\mathtt{ub}} = 20,\ p^{(3)}_{\mathtt{ub}} = 200,$$

and the obtained solution is $\tau^* = 0.7438$ s. The corresponding smoothed output $\tilde{y}(t)$ is plotted in Figure 3. By the stable inversion procedure (cf. Section 2.2) $q(D) = -5.22739$, $h_0^-(t) = 13.59411e^{-6.93t}$, $h_0^+(t) = -38.16286e^{9.31t}$ so that by applying formulae (28)-(30) the inverse input $\tilde{u}(t)$ is obtained. If $t < 0$ (cf. (31))

$$\begin{aligned}
\tilde{u}(t) &= u_0(t) + u_{\mathtt{pre}}(t)\\
&= 0.5\sin(t - 1.6817) + 0.4003\mathrm{e}^{9.31\,t}
\end{aligned}$$

where $u_{\mathtt{pre}}(t) = 0.4003\,\mathrm{e}^{9.31\,t}$ is the preaction control. If $t \in [0, \tau^*]$

$$\begin{aligned}
\tilde{u}(t) = {}& 28.5189t^5 - 27.1113t^4 + 47.2062t^3 - 54.3180t^2\\
&+ 24.4690t - 3.0931 - 0.001928e^{9.31t} + 2.99884e^{-6.93t}.
\end{aligned}$$

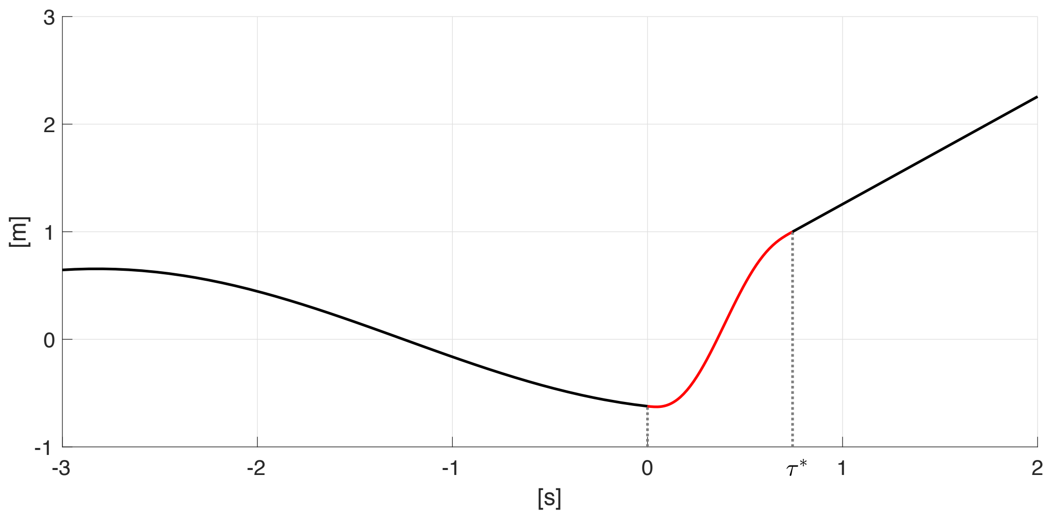

Figure 3: The smoothed output $\tilde{y}(t)$ of the example: the red line plots the interpolating polynomial $p(t;\tau^*)$ which joins the previous output $y_0(t)$, $t < 0$ with delayed desired output $y_1(t-\tau^*)$, $t > \tau^*$.

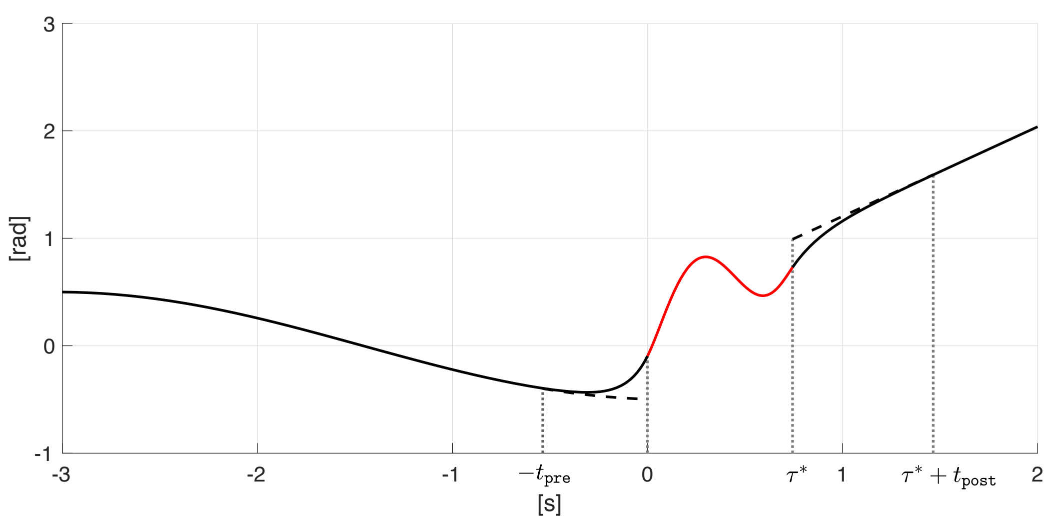

Figure 4: The inverse input $\tilde{u}$ of the example: the red line and the black ones plot the input inside and outside the time interval $[0, \tau^*]$ respectively. The dashed lines plot $u_0(t)$ in $[-t_{\mathtt{pre}}, 0)$ and $u_1(t-\tau^*)$ in $[\tau, \tau^* + t_{\mathtt{post}}]$. In these intervals $\tilde{u}(t)$ differs form $u_0(t)$ and $u_1(t-\tau^*)$ due to the presence of the preaction and postaction control respectively.

Finally, for $t > \tau^*$,

$$\begin{aligned}\tilde{u}(t) &= u_1(t-\tau^*) + u_{\mathtt{post}}(t)\\ &= 0.8334t + 0.3707 - 45.1136e^{-6.93t}\end{aligned}$$

where $u_{\mathtt{post}}(t) = -45.1136e^{-6.93t}$ is the decaying postaction control (cf. 34) and $u_1(t-\tau^*) = 0.8334t + 0.3707$ is the delayed inverse input for which $(u_1(t-\tau^*), y_1(t-\tau^*))$ is a steady-state pair (cf. Theorem 2).

Preaction and postaction times (cf. (33) and (36)) can be determined, for example, by choosing $f_{\mathtt{pre}} = f_{\mathtt{post}} = 5.3$ with a negligible error on input $\tilde{u}$ (less than $2 \cdot 10^{-3}$ rad in both cases) to respectively obtain $t_{\mathtt{pre}} = 0.5693$ s and $t_{\mathtt{post}} = 0.7648$ s. Figure 4 displays the plotting of the inverse input $\tilde{u}$.

7. Conclusions

In behavioral terms, the classic feedforward regulation is about designing a control input that makes a transition from $(0, 0)$ to $(y_{1\mathtt{c}}/H(0), y_{1\mathtt{c}})$ with $y_{1\mathtt{c}}$ being the desired constant output. The design of a smooth transition between two arbitrary steady-state pairs, specifically from (u_0, y_0) to (u_1, y_1), has been the topic of this paper. This generalized feedforward regulation has been solved by inversion-based control. To this aim, by inserting a delay or transition time τ, an interpolating polynomial that smoothly joins the current output with the future one has been devised. This polynomial, parameterized by τ, is given in closed-form by means of a polynomial basis only depending on the boundary continuity orders k_0 and k_1. Remarkably, the polynomial basis can be easily computed offline and this speeds up the real-time implementation of the method. Moreover, the time parameter τ can be minimized in order to reduce the delay of the desired y_1.

The proposed method can be iteratively applied. Once the transition from (u_0, y_0) to (u_1, y_1) is completed, i.e. after the time interval $[-t_{\mathtt{pre}}, \tau + t_{\mathtt{post}}]$ is elapsed, with sufficient preview time (cf. [6]) another transition can start to reach a new steady-state pair (u_2, y_2) and so on for reaching a next pair. Actually, by considering that pair $(\tilde{u}, \tilde{y})$ is itself steady-state (cf. (17) and Figure 1) a new transition to (u_2, y_2) can start even before the current transition is completed, i.e. starting at any time in the interval $[-t_{\mathtt{pre}}, \tau + t_{\mathtt{post}}]$ (always allowing enough preview time). This feature of the method makes it interesting for consideration for event-based control applications [38, 39, 40].

Appendix A. Proof of Proposition 3

The proof is obvious when $r = 1$ so that in the following consider $r > 1$. First we prove that (a) $\Leftrightarrow$ (b). Evidently (a) $\Rightarrow$ (b), so we show (b) $\Rightarrow$ (a). By Lemma 1, the polynomial order finiteness of $y^{(r)}$ implies the same finiteness of $\int_0^t y^{(r)}(v)dv$. On the other hand $\int_0^t y^{(r)}(v)dv = \int_0^t Dy^{(r-1)}(v)dv$ and $y^{(r-1)} \in C^0$ so that $\int_0^t y^{(r)}(v)dv = y^{(r-1)}(t) - y^{(r-1)}(0)$ (cf. Lemma 3 in [15]). Hence, $y^{(r-1)}(t) = y^{(r-1)}(0) + \int_0^t y^{(r)}(v)dv$ therefore $y^{(r-1)}$ has finite polynomial order. The reasoning applied to $y^{(r)}$ can be in turn applied to $y^{(r-1)}$ so that $y^{(r-2)}$ is proved to have finite polynomial order. Hence, by repeating the reasoning iteratively we conclude that all the derivatives $y^{(r-1)}, y^{(r-2)}, \ldots, y^{(1)}$ have finite polynomial orders.

To complete the proof we show that (a) $\Leftrightarrow$ (c) where the deduction of (a) $\Rightarrow$ (c) is evidently immediate. To prove (c) $\Rightarrow$ (a) consider the identity

$$q_r D^r y(t^+) + q_{r-1} D^{r-1} y(t) + \cdots + q_0 y(t) \equiv q(D)y(t^+). \tag{A.1}$$

This identity implies

$$q_r D^{r-1} y_1(t) + q_{r-1} D^{r-2} y_1(t) + \cdots + q_1 y_1(t) = u_1(t),\ t \in \mathbb{R} \setminus S_{y_1}^{(r-1)} \tag{A.2}$$

where $u_1(t) := q(D)y(t^+) - q_0 y(t)$, $t \in \mathbb{R}$ is a function with finite polynomial order, $y_1(t) := Dy(t)$, $t \in \mathbb{R}$ and $S_{y_1}^{(r-1)} := \{t \in \mathbb{R} : y_1^{(r-1)} \text{ does not exist in } t\}$ is the discontinuity set of order $r-1$ of y_1 (cf. [15]). Hence, by the differential-integral characterization of weak solutions (cf. Theorem 3 in [15]), the pair (u_1, y_1) belongs to the behavior associated to the transfer function $H_1 := 1/(q_r s^{r-1} + \cdots + q_1)$. By the input-output representation of this behavior (cf. Theorem 4 in [15]) there exist real coefficients f_i^-, f_i^0 and f_i^+ such that $(t \in \mathbb{R})$

$$y_1(t) = \int_0^t h_1(t-v)u_1(v)dv + \sum_{i=1}^{r_1^-} f_i^- m_i^{\mathtt{p}-}(t) + \sum_{i=1}^{r_1^0} f_i^0 m_i^{\mathtt{p}0}(t) + \sum_{i=1}^{r_1^+} f_i^+ m_i^{\mathtt{p}+}(t) \tag{A.3}$$

where $h_1(t) := \mathcal{L}_{\mathtt{ae}}^{-1}[H_1(s)]$ and the $m_i^{\mathtt{p}-}(t)$, $i = 1, \ldots, r_1^-$, $m_i^{\mathtt{p}0}(t)$, $i = 1, \ldots, r_1^0$, $m_i^{\mathtt{p}+}(t)$, $i = 1, \ldots, r_1^+$ ($r_1^- + r_1^0 + r_1^+ = r - 1$) are the pole modes of H_1 associated to the poles with negative, zero and positive real parts respectively (cf. [15]). By partial fraction expansion $H_1(s) = H_1^-(s) + H_1^0(s) + H_1^+(s)$ where H_1^-, H_1^0 and H_1^+ are associated to the poles of H_1 with negative, zero and positive real parts respectively. Hence, by inverse Laplace transform and analytical extension over $\mathbb{R}$ of H_1^+, H_1^0 and H_1^-, $h_1(t)$ can be decomposed as $h_1(t) = h_1^-(t) + h_1^0(t) + h_1^+(t)$. The integral appearing in (A.3) can be written as $\int_0^t h_1(t-v)u_1(v)dv = \int_{-\infty}^t h_1^-(t-v)u_1(v)dv + \int_0^t h_1^0(t-v)u_1(v)dv -$

$\int_t^{+\infty} h_1^+(t-v)u_1(v)dv + \sum_{i=1}^{r_1^-} g_i^- m_i^{\mathtt{p}-}(t) + \sum_{i=1}^{r_1^+} g_i^+ m_i^{\mathtt{p}+}(t)$ (with suitable g_i^- and g_i^+) and by setting $I_1(t) := \int_{-\infty}^t h_1^-(t-v)u_1(v)dv + \int_0^t h_1^0(t-v)u_1(v)dv - \int_t^{+\infty} h_1^+(t-v)u_1(v)dv + \sum_{i=1}^{r_1^0} f_i^0 m_i^{\mathtt{p}0}(t)$ it follows that

$$y_1(t) = I_1(t) + \sum_{i=1}^{r_1^-}(f_i^- + g_i^-)m_i^{\mathtt{p}-}(t) + \sum_{i=1}^{r_1^+}(f_i^+ + g_i^+)m_i^{\mathtt{p}+}(t),\ t \in \mathbb{R}. \tag{A.4}$$

Note that $I_1(t)$ has finite polynomial order because all its addends have polynomial order finiteness (in particular cf. Lemma 2). Then apply the integral operator $\int$ to the above relation (A.4) and by noting $\int y_1(t) = \int Dy(t) = y(t) - y(0)$ it follows that ($t \in \mathbb{R}$)

$$y(t) = y(0) + \int I_1(t) + \sum_{i=1}^{r_1^-}(f_i^- + g_i^-)\int m_i^{\mathtt{p}-}(t) + \sum_{i=1}^{r_1^+}(f_i^+ + g_i^+)\int m_i^{\mathtt{p}+}(t). \tag{A.5}$$

In (A.5) $\int I_1(t)$ has finite polynomial order (by Lemma 1) and the integrals $\int m_i^{\mathtt{p}-}(t)$, $\int m_i^{\mathtt{p}+}(t)$ are exponential functions that cannot vanish over $\mathbb{R}$. By taking into account the polynomial order finiteness of $y(t)$, relation (A.5) can only be valid if all the coefficients satisfy $f_i^- = -g_i^-$ and $f_i^+ = -g_i^+$, i.e. mathematical cancellations between exponential addends occur. Therefore, relation (A.4) becomes $Dy(t) = I_1(t)$ and this proves that $y^{(1)}$ has finite polynomial order.

Now, $y^{(2)}$ can be proved to have finite polynomial order by rearranging the identity (A.1) as

$$q_r D^r y(t^+) + \cdots + q_2 D^2 y(t) = -q_1 Dy(t) - q_0 y(t) + q(D)y(t^+)\ .$$

Then set $y_2(t) := Dy_1(t)$ ($= D^2 y(t)$) and $u_2(t) := -q_1 Dy(t) - q_0 y(t) + q(D)y(t^+)$, $t \in \mathbb{R}$ and reapply the same reasoning on pair (u_2, y_2) as previously done on (u_1, y_1). By iterating this argument we prove that $y^{(3)}, \ldots, y^{(r-1)}$ have all finite polynomial orders. Eventually, $y^{(r)}$ is proved to have finite polynomial order by just noting that $D^r y(t^+) = \frac{1}{q_r}(q(D)y(t^+) - q_{r-1}D^{r-1}y(t) - \cdots - q_0 y(t))$. $\square$

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