



UNIVERSITÀ DI PARMA

ARCHIVIO DELLA RICERCA

University of Parma Research Repository

Domains of elliptic operators on sets in Wiener space

This is the peer reviewed version of the following article:

Original

Domains of elliptic operators on sets in Wiener space / Addona, D.; Cappa, G.; Ferrari, S.. - In: INFINITE DIMENSIONAL ANALYSIS QUANTUM PROBABILITY AND RELATED TOPICS. - ISSN 0219-0257. - 23:1(2020), p. 2050004. [10.1142/S0219025720500046]

Availability:

This version is available at: 11381/2887637 since: 2021-12-29T13:23:46Z

Publisher:

World Scientific Publishing Co. Pte Ltd

Published

DOI:10.1142/S0219025720500046

Terms of use:

openAccess

Anyone can freely access the full text of works made available as "Open Access". Works made available

Publisher copyright

(Article begins on next page)

Infinite Dimensional Analysis, Quantum Probability and Related Topics
© World Scientific Publishing Company

Domains of elliptic operators on sets in Wiener space

ADDONA Davide

*Dipartimento di Matematica e Applicazioni, Università degli studi di Milano Bicocca,
Via Cozzi 55, Milano, 20126, Italy
_davide.addona@unimib.it*

CAPPA Gianluca

*Dipartimento di Economia e Finanza, Università LUISS - Guido Carli,
Viale Romania 32, Roma, 00197, Italy
_gcappa@luiss.it*

FERRARI Simone

*Dipartimento di Scienze Matematiche, Fisiche e Informatiche, Università di Parma,
Parco Area delle Scienze 53/a (Campus), Parma, 43124, Italy
_simone.ferrari1@unipr.it*

Received (Day Month Year)

Revised (Day Month Year)

Published (Day Month Year)

Communicated by (xxxxxxxxxx)

Let X be a separable Banach space endowed with a non-degenerate centered Gaussian measure μ . The associated Cameron–Martin space is denoted by H . Consider two sufficiently regular convex functions $U : X \rightarrow \mathbb{R}$ and $G : X \rightarrow \mathbb{R}$. We let $\nu = e^{-U}\mu$ and $\Omega = G^{-1}(-\infty, 0]$. In this paper we study the domain of the self-adjoint operator associated with the quadratic form

$$(\psi, \varphi) \mapsto \int_{\Omega} \langle \nabla_H \psi, \nabla_H \varphi \rangle_H d\nu \quad \psi, \varphi \in W^{1,2}(\Omega, \nu), \quad (0.1)$$

and we give sharp embedding results for it. In particular we obtain a characterization of the domain of the Ornstein–Uhlenbeck operator in Hilbert space with $\Omega = X$ and on half-spaces, namely if $U \equiv 0$ and G is an affine function, then the domain of the operator defined via (0.1) is the space

$$\{u \in W^{2,2}(\Omega, \mu) \mid \langle \nabla_H u(x), \nabla_H G(x) \rangle_H = 0 \text{ for } \rho\text{-a.e. } x \in G^{-1}(0)\},$$

where ρ is the Feyel–de La Pradelle Hausdorff–Gauss surface measure.

Keywords: Domain of operator, elliptic operator, Wiener space, weighted Gaussian measure, maximal regularity, divergence operator.

2010 AMS Subject Classification 2010: 28C20, 35J15, 46G12, 47A07, 47A30

1. Introduction

Let X be a separable Banach space with norm $\|\cdot\|_X$, endowed with a non-degenerate centered Gaussian measure μ . The associated Cameron–Martin space is denoted by

2 ADDONA Davide, CAPPA Gianluca, FERRARI Simone

H , its inner product by $\langle \cdot, \cdot \rangle_H$ and its norm by $|\cdot|_H$. The spaces $W^{k,p}(X, \mu)$ for $p \geq 1$ and $k \in \mathbb{N}$ are the classical Sobolev spaces of the Malliavin calculus (see⁸ Chapter 5).

The aim of this paper is to study the domain of the self-adjoint operator $L_{\nu, \Omega}$ associated with the quadratic form

$$(\psi, \varphi) \mapsto \int_{\Omega} \langle \nabla_H \psi, \nabla_H \varphi \rangle_H d\nu \quad \psi, \varphi \in W^{1,2}(\Omega, \nu),$$

where Ω is a convex subset of X , $\nu := e^{-U} \mu$ and $U : X \rightarrow \mathbb{R}$ is a convex function, $\nabla_H \psi$ is the gradient along H of ψ and $W^{1,2}(\Omega, \nu)$ is the Sobolev space on Ω associated to the measure ν (see Section 2). These operators arise in Kolmogorov equations in Hilbert spaces corresponding to stochastic variational inequalities with reflection, such as

$$\begin{cases} dY(t, x) - Y(t, x)dt - \nabla_H U(Y(t, x))dt + N_{\Omega}(Y(t, x))dt \ni dW(t, x); \\ Y(0, x) = x, \end{cases}$$

where N_{Ω} is the normal cone to Ω and $W(t, \cdot)$ is a X -valued cylindrical Wiener process (here X is a Hilbert space). This is because, at least formally, the transition semigroup $T(t)f(x) := \mathbb{E}[f(Y(t, x))]$ is generated by $L_{\nu, \Omega}$.

In the case of the standard Gaussian measure in a convex subset $\Omega \subseteq \mathbb{R}^n$ with sufficiently regular boundary, the operator $L_{\nu, \Omega}$ reads as

$$L_{\nu, \Omega} u(\xi) = \Delta u(\xi) - \langle \nabla U(\xi) + \xi, \nabla u(\xi) \rangle \quad u \in \mathcal{C}_b^2(\Omega),$$

so that, if U is sufficiently regular, $L_{\nu, \Omega}$ is an elliptic operator with possibly unbounded coefficients, and its domain in $L^2(\Omega, \nu)$ is

$$D(L_{\nu, \Omega}) = \begin{cases} \{u \in W^{2,2}(\mathbb{R}^n, \nu) \mid \langle \nabla U + \xi, \nabla u \rangle \in L^2(\mathbb{R}^n, \nu)\}, & \Omega = \mathbb{R}^n; \\ \{u \in W^{2,2}(\Omega, \nu) \mid \langle \nabla U + \xi, \nabla u \rangle \in L^2(\Omega, \nu), \partial u / \partial n = 0 \text{ at } \partial \Omega\}, & \Omega \neq \mathbb{R}^n, \end{cases}$$

where $\partial / \partial n$ is the exterior normal derivative at the boundary of $\partial \Omega$ (see¹⁶ and³²). In the infinite dimensional case there is a characterization for the Ornstein–Uhlenbeck operator, when Ω is the whole space and $U \equiv 0$ (see⁸ Section 5.6). In this case the operator $L_{\mu, X}$ is the infinitesimal generator of the Ornstein–Uhlenbeck semigroup

$$T_t f(x) = \int_X f\left(e^{-t}x + \sqrt{1 - e^{-2t}}y\right) d\mu(y),$$

in $L^2(X, \mu)$ and its domain is $D(L_{\mu, X}) = W^{2,2}(X, \mu)$. Further results were obtained in,¹¹ assuming U has H -Lipschitz gradient, and Ω is the whole space. In this case too the domain is $D(L_{\nu, X}) = W^{2,2}(X, \nu)$. We want to point out that in³⁴ the authors study in detail the case of non-symmetric Ornstein–Uhlenbeck operators on the whole space.

This paper is a first attempt to give a characterization of the domain of $L_{\nu, \Omega}$ in a more general setting. In order to state the main results of this paper we need some hypotheses on the set Ω and on the weighted measure ν .

Throughout the paper we take $\Omega = G^{-1}(-\infty, 0]$, where G satisfies the following assumptions.

Hypothesis 1.1. Let $G : X \rightarrow \mathbb{R}$ be a version of a function belonging to $W^{2,q}(X, \mu) \cap C^2(X)$ for every $q > 1$ such that

- (1) G is convex;
- (2) $|\nabla_H G|_H^{-1} \in L^q(G^{-1}(-\infty, 0], \mu)$ for every $q > 1$;
- (3) $|\nabla_H G(x)|_H \neq 0$ for ρ -a.e $x \in G^{-1}(0)$.

Hypotheses 1.1(1)-(2) are taken from¹³ and¹² in order to define traces of Sobolev functions on level sets of G and to get maximal Sobolev regularity estimates for elliptic equations associated to the operator $L_{\nu, \Omega}$. Hypotheses 1.1(3) allow us to prove Lemma 4.2 which is generalization of a classical result in differential geometry (see,³⁰⁶ and¹⁰).

Remark 1.1. Our results also hold true if we consider weaker assumptions on G . However, these weaker assumptions are quite technical, and since our examples satisfy Hypothesis 1.1, we decided to provide these set of hypotheses.

Hypothesis 1.2. $U : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is a proper, convex, lower semicontinuous and twice continuously differentiable along H function belonging to $W^{2,t}(X, \mu)$ for some $t > 3$ (see Section 2 for the definition of differentiability along H). We set

$$\nu := e^{-U} \mu.$$

The assumption $t > 3$ may sound strange, but it is helpful to define the weighted Sobolev spaces $W^{1,2}(X, \nu)$. Indeed, let us observe that, by¹ Lemma 7.5, e^{-U} belongs to $W^{1,r}(X, \mu)$ for every $r < t$. Thus if U satisfies Hypothesis 1.2, then it satisfies²² Hypothesis 1.1; namely $e^{-U} \in W^{1,s}(X, \mu)$ for some $s > 1$ and $U \in W^{1,r}(X, \mu)$ for some $r > s'$. Then following²² it is possible to define the space $W^{1,2}(X, \nu)$ as the domain of the closure of the gradient operator along H (see Section 2 for an in-depth discussion).

From here on, we will denote by Tr the trace operator acting on Sobolev functions (see Section 2.6), by ρ the Feyel–de La Pradelle Hausdorff–Gauss surface measure (see²³) and by $\mathcal{F}\mathcal{C}_b^2(\Omega)$ the space of the restriction to Ω of cylindrical twice differentiable functions on X with bounded derivatives (see Section 2.2). We remark that, by²⁵ Theorem 3.1(2), $\langle \nabla_H^2 U(x)h, h \rangle_H \geq 0$, for μ -a.e. $x \in X$ and every $h \in H$. An important space in our investigation is

$$W_U^{2,2}(\Omega, \nu) = \left\{ u \in W^{2,2}(\Omega, \nu) \mid \int_{\Omega} \langle \nabla_H^2 U \nabla_H u, \nabla_H u \rangle d\nu < +\infty \right\},$$

endowed with the norm

$$\|u\|_{W_U^{2,2}(\Omega, \nu)}^2 = \|u\|_{W^{2,2}(\Omega, \nu)}^2 + \int_{\Omega} \langle \nabla_H^2 U \nabla_H u, \nabla_H u \rangle_H d\nu. \quad (1.1)$$

4 ADDONA Davide, CAPPÀ Gianluca, FERRARI Simone

We remark that $W_U^{2,2}(\Omega, \nu)$ is a Hilbert space. We will also study the following subspace of $W_U^{2,2}(\Omega, \nu)$

$$W_{U,N}^{2,2}(\Omega, \nu) = \left\{ u \in W^{2,2}(\Omega, \nu) \mid \int_{\Omega} \langle \nabla_H^2 U \nabla_H u, \nabla_H u \rangle d\nu < +\infty, \right. \\ \left. \langle \text{Tr}(\nabla_H u), \nabla_H G \rangle_H = 0 \text{ } \rho\text{-a.e. in } G^{-1}(0) \right\}$$

endowed with the norm (1.1).

Our main results are the following characterizations of the domain of the self-adjoint operator $L_{\nu, \Omega}$ when Ω is the whole space or a half-space. We recall that by $\|\cdot\|_{D(L_{\nu, \Omega})}$ we denote the graph norm, i.e. for $u \in D(L_{\nu, \Omega})$

$$\|u\|_{D(L_{\nu, \Omega})}^2 := \|u\|_{L^2(X, \nu)}^2 + \|L_{\nu, \Omega} u\|_{L^2(X, \nu)}^2.$$

Theorem 1.3. *Assume that Hypothesis 1.2 holds and that $\mathcal{F}\mathcal{C}_b^2(X)$ is dense in $W_U^{2,2}(X, \nu)$. Then $D(L_{\nu, X}) = W_U^{2,2}(X, \nu)$. Moreover, for every $u \in D(L_{\nu, X})$, it holds*

$$\|u\|_{D(L_{\nu, X})} \leq \|u\|_{W_U^{2,2}(X, \nu)} \leq 2\sqrt{2}\|u\|_{D(L_{\nu, X})}.$$

and fixed any orthonormal basis $\{h_n \mid n \in \mathbb{N}\}$ of H

$$L_{\nu, X} u = \sum_{n=1}^{+\infty} \left(\partial_{nn} u - \partial_n u \partial_n U - \partial_n u \widehat{h}_n \right),$$

where the series converges in $L^2(X, \nu)$ (See Section 2 for the definition of the $\widehat{\cdot}$ operator).

We remark that if the weight U is such that $\nabla_H U$ is Lipschitz continuous, or more generally H -Lipschitz (see Section 2), then $\mathcal{F}\mathcal{C}_b^2(X)$ is dense in $W_U^{2,2}(X, \nu)$, so that the assumption of Theorem 1.3 is satisfied (see Corollary 6.2). When X is a Hilbert space, in Section 7 we provide sufficient conditions on U which imply that $W_U^{2,2}(X, \nu) = W^{2,2}(X, \nu)$. Therefore, the assumptions of Theorem 1.3 are satisfied and we deduce the characterization of $D_{L_{\nu, X}}$.

When $G = x^* - r$ where $x^* \in X^* \setminus \{0\}$ and $r \in \mathbb{R}$, i.e. if Ω is a half-space, we want to remark that the Neumann boundary condition: $\langle \text{Tr}(\nabla_H u), \nabla_H G \rangle_H = 0$ for ρ -a.e. $x \in G^{-1}(0)$, read

$$x^*(\text{Tr}(\nabla_H u)(x)) = \langle \text{Tr}(\nabla_H u)(x), h_{x^*} \rangle_H = 0$$

for ρ -a.e. $x \in G^{-1}(0)$, where h_{x^*} is the unique vector of H such that

$$x^*(h) = \langle h_{x^*}, h \rangle_H \text{ for every } h \in H. \quad (1.2)$$

Such an element exists since x^* is a continuous linear functional on H .

Theorem 1.4. *Assume that Hypothesis 1.2 holds and G is an affine function, namely $G = x^* - r$ where $x^* \in X^* \setminus \{0\}$ and $r \in \mathbb{R}$. If the space*

$$\mathcal{Z}(\Omega) = \{u \in \mathcal{F}\mathcal{C}_b^2(\Omega) \mid x^*(\text{Tr}(\nabla_H u)(x)) = 0 \text{ for } \rho\text{-a.e. } x \in G^{-1}(0)\},$$

is dense in the space $W_{U,N}^{2,2}(\Omega, \nu)$, then $D(L_{\nu, \Omega}) = W_{U,N}^{2,2}(\Omega, \nu)$. Moreover, for every $u \in D(L_{\nu, \Omega})$, it holds

$$\|u\|_{D(L_{\nu, \Omega})} \leq \|u\|_{W_{U,N}^{2,2}(\Omega, \nu)} \leq 2\sqrt{2}\|u\|_{D(L_{\nu, \Omega})}.$$

and fixed any orthonormal basis $\{h_n \mid n \in \mathbb{N}\}$ of H

$$L_{\nu, \Omega} u = \sum_{n=1}^{+\infty} \left(\partial_{nn} u - \partial_n u \partial_n U - \partial_n u \widehat{h}_n \right),$$

where the series converges in $L^2(\Omega, \nu)$ (See Section 2 for the definition of the $\widehat{\cdot}$ operator).

We remark that showing the density of $\mathcal{Z}(\Omega)$ in $W_{U,N}^{2,2}(\Omega)$ is not an easy task. This difficulty can be overcome if Ω belongs to the class of *Neumann extension domains*.

Definition 1.5. Let $Z_U^{2,2}(X, \nu)$ be the completion of the space $\mathcal{F}\mathcal{C}_b^2(X)$ with respect to the norm defined in (1.1). We say that Ω is a Neumann extension domain if there exists a linear operator $E^{\mathcal{N}}$ from $W_{U,N}^{2,2}(\Omega, \nu)$ into $Z_U^{2,2}(X, \nu)$ such that for every $\phi \in W_{U,N}^{2,2}(\Omega, \nu)$

- (1) $E^{\mathcal{N}}\phi(x) = \phi(x)$ for ν -a.e $x \in \Omega$;
- (2) there is $K > 0$, independent of ϕ , such that $\|E^{\mathcal{N}}\phi\|_{Z_U^{2,2}(X, \nu)} \leq K\|\phi\|_{W_{U,N}^{2,2}(\Omega, \nu)}$.

The operator $E^{\mathcal{N}}$ is called *Neumann extension operator*.

Theorem 1.6. Assume that Hypothesis 1.2 holds and that Ω is a Neumann extension domain satisfying Hypothesis 1.1. Then $D(L_{\nu, \Omega}) = W_{U,N}^{2,2}(\Omega, \nu)$. Moreover, for every $u \in D(L_{\nu, \Omega})$, it holds

$$\|u\|_{D(L_{\nu, \Omega})} \leq \|u\|_{W_{U,N}^{2,2}(\Omega, \nu)} \leq 2\sqrt{2}\|u\|_{D(L_{\nu, \Omega})}.$$

and fixed any orthonormal basis $\{h_n \mid n \in \mathbb{N}\}$ of H

$$L_{\nu, \Omega} u = \sum_{n=1}^{+\infty} \left(\partial_{nn} u - \partial_n u \partial_n U - \partial_n u \widehat{h}_n \right),$$

where the series converges in $L^2(\Omega, \nu)$ (See Section 2 for the definition of the $\widehat{\cdot}$ operator).

The characterization of Neumann extension domains is an open problem in Wiener space theory. The only known results are mainly negative (see⁷), but if Ω is a half-space and $U \equiv 0$, it is known that an extension operator can be constructed (see⁷). Since we were unable to find explicit computations in the literature, we made them in Lemma 8.1. Applying Theorems 1.4, 1.6 and Lemma 8.1 we get the following characterization of the domain of the Ornstein–Uhlenbeck operator on half-spaces, i.e. $U \equiv 0$ and G is an affine function.

6 ADDONA Davide, CAPPA Gianluca, FERRARI Simone

Theorem 1.7. *Assume that Hypothesis 1.2 holds and G is an affine function, namely $G(x) = x^*(x) - r$ with $x^* \in X^* \setminus \{0\}$ and $r \in \mathbb{R}$. Then*

$$D(L_{\mu,\Omega}) = \{u \in W^{2,2}(\Omega, \mu) \mid x^*(\text{Tr}(\nabla_H u)(x)) = 0 \text{ for } \rho\text{-a.e. } x \in G^{-1}(0)\}.$$

Moreover, for every $u \in D(L_{\mu,\Omega})$, it holds

$$\|u\|_{D(L_{\mu,\Omega})} \leq \|u\|_{W^{2,2}(\Omega, \mu)} \leq 2\sqrt{2}\|u\|_{D(L_{\mu,\Omega})}.$$

and fixed any orthonormal basis $\{h_n \mid n \in \mathbb{N}\}$ of H

$$L_{\mu,\Omega}u = \sum_{n=1}^{+\infty} (\partial_{nn}u - \partial_n u \hat{h}_n),$$

where the series converges in $L^2(\Omega, \mu)$ (See Section 2 for the definition of the $\hat{\cdot}$ operator). In addition the space

$$\mathcal{Z}(\Omega) = \{u \in \mathcal{F}\mathcal{C}_b^2(\Omega) \mid x^*(\text{Tr}(\nabla_H u)(x)) = 0 \text{ for } \rho\text{-a.e. } x \in G^{-1}(0)\}$$

is dense in $D(L_{\mu,\Omega})$ with respect to the graph norm.

The paper is organized as follows: in Section 2 we recall some basic definitions and we fix the notations. Section 3 is dedicated to the study of the second order analysis of the Moreau–Yosida approximations along H , that are used to prove Theorems 1.3. In section 4 we will introduce the divergence operator $\text{div}_{\nu,\Omega}$ as minus the formal adjoint of the gradient operator along H and investigate its properties. Namely, we consider the space

$$\mathcal{Z}(\Omega, H) := \left\{ \Phi : \Omega \rightarrow H \left| \begin{array}{l} \text{there exists } n \in \mathbb{N} \text{ and } \{h_1, \dots, h_n\} \subseteq H \\ \text{such that } \Phi = \sum_{i=1}^n \varphi_i h_i \text{ for some } n \in \mathbb{N}, \\ \text{and } \varphi_i \in \mathcal{F}\mathcal{C}_b^2(\Omega) \text{ for } i = 1, \dots, n. \\ \text{In addition } \langle \Phi, \nabla_H G \rangle_H = 0 \text{ } \rho\text{-a.e. in } G^{-1}(0). \end{array} \right. \right\}. \quad (1.3)$$

For every $\Phi \in \mathcal{Z}(\Omega, H)$ put

$$\begin{aligned} \|\Phi\|_{Z_U^{1,2}(\Omega, \nu; H)}^2 &:= \|\Phi\|_{W^{1,2}(\Omega, \nu; H)}^2 + \int_{\Omega} \langle \nabla_H^2 U \Phi, \Phi \rangle_H d\nu + \\ &+ \int_{G^{-1}(0)} \langle \nabla_H^2 G \Phi, \Phi \rangle_H \frac{e^{-\text{Tr}(U)}}{|\nabla_H G|_H} d\rho. \end{aligned} \quad (1.4)$$

Let $Z_U^{1,2}(\Omega, \nu; H)$ be the completion of the space $\mathcal{Z}(\Omega, H)$ with respect to the norm defined in (1.4). As usual the elements of $Z_U^{1,2}(\Omega, \nu; H)$ can be identified as equivalence classes of vector fields with respect to the ν -a.e. equivalence relation. It is easy to see that $Z_U^{1,2}(\Omega, \nu; H)$ is a Hilbert space. In Proposition 4.2 we will prove that the space $Z_U^{1,2}(\Omega, \nu; H)$ is contained in the domain of the divergence operator $\text{div}_{\nu,\Omega}$ in L^2 and $\text{div}_{\nu,\Omega} \Phi \in L^2(\Omega, \nu)$ for every $\Phi \in Z_U^{1,2}(\Omega, \nu; H)$. Furthermore an explicit formula for the calculation of $\text{div}_{\nu,\Omega}$ is given by (4.13).

We remark that without loss of generality we can assume that the sequence $\{h_1, \dots, h_n\}$ in (1.3) is a sequence of orthonormal elements of H (indeed, it is enough to apply the Gram-Schmidt procedure). Moreover, we stress that the boundary

integral in (1.4) in general cannot be estimated by the $W^{1,2}$ -norm of Φ . This fact depends not only from the presence of the second order derivatives of G , but also from the trace theory in infinite dimensions. Indeed, as shown in²² the trace of $f \in W^{1,p}(\Omega)$ belongs to $L^q(G^{-1}(0), e^{-U}\rho)$ for any $q \in [1, p(t-2)(t-1)^{-1}]$, where t is the number fixed in Hypothesis 1.2. In particular if $p = 2$ then we do not know if the trace operator is continuous in $L^2(G^{-1}(0), e^{-U}\rho)$.

In Section 5 we obtain maximal Sobolev regularity estimates for the weak solution of the problem

$$\lambda u(x) - L_{\nu,\Omega}u(x) = f(x) \quad \mu\text{-a.e. } x \in \Omega, \quad (1.5)$$

where $\lambda > 0$, and $f \in L^2(\Omega, \nu)$. We say that $u \in W^{1,2}(\Omega, \nu)$ is a *weak solution* of problem (1.5) if

$$\lambda \int_{\Omega} u\varphi d\nu + \int_{\Omega} \langle \nabla_H u, \nabla_H \varphi \rangle_H d\nu = \int_{\Omega} f\varphi d\nu \quad \text{for every } \varphi \in W^{1,2}(\Omega, \nu).$$

Notice that the unique weak solution u of problem (1.5) satisfies $u = R(\lambda, L_{\nu,\Omega})f$, where $R(\lambda, L_{\nu,\Omega})$ is the resolvent of $L_{\nu,\Omega}$. We recall that results about existence, uniqueness and regularity of the weak solution of problem (5.1), in domains with sufficiently regular boundary, are known in the finite dimensional case (see the classical books²⁶ and²⁹ for a bounded Ω and,^{6, 16, 33, 17} and¹⁸ for an unbounded Ω). If X is infinite dimensional maximal Sobolev regularity results are known when X is a separable Hilbert space. See for example² and³ where $U \equiv 0$ and²⁰ where U is bounded from below. When $\Omega = X$ more results are known, see for example,^{15, 35} and³¹ if X is finite dimensional,¹⁹ if X is a Hilbert space and¹¹ if X is a separable Banach space. If X is general separable Banach space and $\Omega \subsetneq X$, then the only results regarding maximal Sobolev regularity are those contained in,¹⁰ where the second named author studied problem (5.1) when $U \equiv 0$, namely when $L_{\nu,\Omega}$ is the Ornstein–Uhlenbeck operator on Ω , and in,¹² where the second and third named authors studied the general case.

In Section 6 we prove Theorems 1.3, 1.4 and 1.6 and some related corollaries. Finally, in Section 8 we provide some examples to which our results can be applied. In particular we study the case when Ω is the unit ball of a Hilbert space and we prove Theorem 1.7.

2. Notation and preliminaries

We will denote by X^* the topological dual of X . We recall that $X^* \subseteq L^2(X, \mu)$. The linear operator $R_{\mu} : X^* \rightarrow (X^*)'$

$$R_{\mu}x^*(y^*) = \int_X x^*(x)y^*(x)d\mu(x) \quad (2.1)$$

is called the covariance operator of μ . Since X is separable, then it is actually possible to prove that $R_{\mu} : X^* \rightarrow X$ (see⁸ Theorem 3.2.3). We denote by X_{μ}^* the closure of X^* in $L^2(X, \mu)$. The covariance operator R_{μ} can be extended by

8 ADDONA Davide, CAPPA Gianluca, FERRARI Simone

continuity to the space X_μ^* , still by formula (2.1). By⁸ Lemma 2.4.1, for every $h \in H$ there exists a unique $g \in X_\mu^*$ with $h = R_\mu g$, in this case we set

$$\widehat{h} := g. \quad (2.2)$$

Throughout the paper we fix an orthonormal basis $\{e_i\}_{i \in \mathbb{N}}$ of H such that \widehat{e}_i belongs to X^* , for every $i \in \mathbb{N}$. Such basis exists by⁸ Corollary 3.2.8(ii).

2.1. Differentiability along H

We say that a function $f : X \rightarrow \mathbb{R}$ is *differentiable along H* at x if there exists $v \in H$ such that

$$\lim_{t \rightarrow 0} \frac{f(x + th) - f(x)}{t} = \langle v, h \rangle_H,$$

uniformly with respect to $h \in H$, with $|h|_H = 1$. In this case, the vector $v \in H$ is unique and we set $\nabla_H f(x) := v$. Moreover, for every $k \in \mathbb{N}$ the derivative of f in the direction of e_k exists and it is given by

$$\partial_k f(x) := \lim_{t \rightarrow 0} \frac{f(x + te_k) - f(x)}{t} = \langle \nabla_H f(x), e_k \rangle_H.$$

We denote by \mathcal{H}_2 the space of the Hilbert–Schmidt operators in H , that is the space of the bounded linear operators $A : H \rightarrow H$ such that $\|A\|_{\mathcal{H}_2}^2 = \sum_i |Ae_i|_H^2$ is finite (see²¹). We say that a function $f : X \rightarrow \mathbb{R}$ is *twice differentiable along H* at x if it is differentiable along H at x and there exists $A \in \mathcal{H}_2$ such that

$$H\text{-}\lim_{t \rightarrow 0} \frac{\nabla_H f(x + th) - \nabla_H f(x)}{t} = Ah,$$

uniformly with respect to $h \in H$, with $|h|_H = 1$. In this case the operator A is unique and we set $\nabla_H^2 f(x) := A$. Moreover, for every $i, j \in \mathbb{N}$ we set

$$\partial_{ij} f(x) := \lim_{t \rightarrow 0} \frac{\partial_j f(x + te_i) - \partial_j f(x)}{t} = \langle \nabla_H^2 f(x) e_j, e_i \rangle_H.$$

2.2. Special classes of functions

For $k \in \mathbb{N} \cup \{\infty\}$, we denote by $\mathcal{FC}^k(X)$ ($\mathcal{FC}_b^k(X)$ respectively) the space of the cylindrical function of the type $f(x) = \varphi(x_1^*(x), \dots, x_n^*(x))$ where $\varphi \in \mathcal{C}^k(\mathbb{R}^n)$ ($\varphi \in \mathcal{C}_b^k(\mathbb{R}^n)$, respectively) and $x_1^*, \dots, x_n^* \in X^*$, for some $n \in \mathbb{N}$. We remark that $\mathcal{FC}_b^\infty(X)$ is dense in $L^p(X, \nu)$ for all $p \geq 1$ (see²² Proposition 3.6). We recall that if $f \in \mathcal{FC}^2(X)$, then $\partial_{ij} f(x) = \partial_{ji} f(x)$ for every $i, j \in \mathbb{N}$ and $x \in X$.

If Y is a Banach space, a function $F : X \rightarrow Y$ is said to be *H -Lipschitz* if there exists a positive constant C such that

$$\|F(x + h) - F(x)\|_Y \leq C|h|_H, \quad (2.3)$$

for every $h \in H$ and μ -a.e. $x \in X$ (see⁸ Section 4.5 and Section 5.11). We denote with $[F]_{H\text{-Lip}}$ the best constant C appearing in (2.3).

A function $F : X \rightarrow \mathbb{R}$ is said to be *H -continuous*, if $\lim_{|h|_H \rightarrow 0} F(x + h) = F(x)$, for μ -a.e. $x \in X$.

2.3. Sobolev spaces

The Gaussian Sobolev spaces $W^{1,p}(X, \mu)$ and $W^{2,p}(X, \mu)$, with $p \geq 1$, are the completions of the smooth cylindrical functions $\mathcal{FC}_b^\infty(X)$ in the norms

$$\|f\|_{W^{1,p}(X,\mu)} := \|f\|_{L^p(X,\mu)} + \left(\int_X |\nabla_H f(x)|_H^p d\mu(x) \right)^{\frac{1}{p}};$$

$$\|f\|_{W^{2,p}(X,\mu)} := \|f\|_{W^{1,p}(X,\mu)} + \left(\int_X \|\nabla_H^2 f(x)\|_{\mathcal{H}_2}^p d\mu(x) \right)^{\frac{1}{p}}.$$

Such spaces can be identified with subspaces of $L^p(X, \mu)$ and the (generalized) gradient and Hessian along H , $\nabla_H f$ and $\nabla_H^2 f$, are well defined and belong to $L^p(X, \mu; H)$ and $L^p(X, \mu; \mathcal{H}_2)$, respectively. The spaces $W^{1,p}(X, \mu; H)$ are defined in a similar way, replacing smooth cylindrical functions with H -valued smooth cylindrical functions (i.e. the linear span of the functions $x \mapsto f(x)h$, where f is a smooth cylindrical function and $h \in H$). For more details see⁸ Section 5.2.

Now we consider the operators $\nabla_H : \mathcal{FC}_b^\infty(X) \rightarrow L^p(X, \nu; H)$ and $(\nabla_H, \nabla_H^2) : \mathcal{FC}_b^\infty(X) \rightarrow L^p(X, \nu; H) \times L^p(X, \nu; \mathcal{H}_2)$. These operators are closable in $L^p(X, \nu)$ and $L^p(X, \nu; H)$, respectively, for any $p > \frac{t-1}{t-2}$ (see²² Definition 4.3 and¹¹ Proposition 2.1), where t has been defined in Hypothesis 1.2. For such p we denote by $W^{1,p}(X, \nu)$ the domain of the closure of ∇_H in $L^p(X, \nu)$, and by $W^{2,p}(X, \nu)$ the domain of the closure of (∇_H, ∇_H^2) in $L^p(X, \nu)$. The spaces $W^{1,p}(X, \nu; H)$ are defined in a similar way, replacing smooth cylindrical functions with H -valued smooth cylindrical functions.

We want to point out that if Hypothesis 1.2 holds, then $\frac{t-1}{t-2} < 2$. In particular the above arguments allow us to define the Sobolev spaces $W^{1,2}(X, \nu)$ and $W^{2,2}(X, \nu)$.

We shall use the integration by parts formula (see²² Lemma 4.1): for $\varphi \in W^{1,p}(X, \nu)$ with $p > \frac{t-1}{t-2}$:

$$\int_X \partial_k \varphi d\nu = \int_X \varphi (\partial_k U + \widehat{e}_k) d\nu \quad \text{for every } k \in \mathbb{N},$$

where \widehat{e}_k is defined in formula (2.2). Finally, we recall that if U satisfies Hypothesis 1.2 then for every $u \in \mathcal{FC}_b^2(X)$

$$L_{\nu, X} u = \sum_{i=1}^{+\infty} (\partial_{ii} u - (\partial_i U + \widehat{e}_i) \partial_i u), \quad (2.4)$$

where the series converges in $L^2(X, \nu)$ (see²² Proposition 5.3).

2.4. Capacity

Let L_p be the infinitesimal generator of the Ornstein–Uhlenbeck semigroup $T(t)$ in $L^p(X, \mu)$, where

$$T(t)f(x) := \int_X f\left(e^{-t}x + (1 - e^{-2t})^{\frac{1}{2}}y\right) d\mu(y), \quad t > 0.$$

10 ADDONA Davide, CAPPÀ Gianluca, FERRARI Simone

For $k = 1, 2, 3$, we define the $C_{k,p}$ -capacity of an open set $A \subseteq X$ as

$$C_{k,p}(A) := \inf \left\{ \|f\|_{L^p(X,\mu)} \mid (I - L_p)^{-\frac{k}{2}} f \geq 1 \text{ } \mu\text{-a.e. in } A \right\}.$$

For a general Borel set $B \subseteq X$ we let $C_{k,p}(B) = \inf \{C_{k,p}(A) \mid B \subseteq A \text{ open}\}$. By $f \in W^{k,p}(X, \mu)$ we mean an equivalence class of functions and we call every element “version”. For any $f \in W^{k,p}(X, \mu)$ there exists a version \bar{f} of f which is Borel measurable and $C_{k,p}$ -quasicontinuous, i.e. for every $\varepsilon > 0$ there exists an open set $A \subseteq X$ such that $C_{k,p}(A) \leq \varepsilon$ and $\bar{f}|_{X \setminus A}$ is continuous. Furthermore, for every $r > 0$

$$C_{k,p}(\{x \in X \mid |\bar{f}(x)| > r\}) \leq \frac{1}{r} \left\| (I - L_p)^{-\frac{k}{2}} \bar{f} \right\|_{L^p(X,\mu)}.$$

See⁸ Theorem 5.9.6. Such a version is called a (k, p) -precise version of f . Two precise versions of the same f coincide outside sets with null $C_{k,p}$ -capacity. All our results will be independent on our choice of a precise version of G in Hypothesis 1.1. With obvious modifications the same definition can be adapted to functions belonging to $W^{k,p}(X, \mu; H)$ and $W^{k,p}(X, \mu; \mathcal{H}_2)$.

2.5. Sobolev spaces on sublevel sets

The proof of the results stated in this subsection can be found in¹³ and²² Let G be a function satisfying Hypothesis 1.1. We are interested in Sobolev spaces on sublevel sets of G .

For $k \in \mathbb{N} \cup \{\infty\}$, we denote by $\mathcal{F}\mathcal{C}_b^k(\Omega)$ the space of the restriction to Ω of functions in $\mathcal{F}\mathcal{C}_b^k(X)$. For any $p \geq 1$, the spaces $W^{1,p}(\Omega, \mu)$ and $W^{2,p}(\Omega, \mu)$ are defined as the domain of the closure of the operators $\nabla_H : \mathcal{F}\mathcal{C}_b^\infty(\Omega) \rightarrow L^p(\Omega, \mu; H)$ and $(\nabla_H, \nabla_H^2) : \mathcal{F}\mathcal{C}_b^\infty(\Omega) \rightarrow L^p(\Omega, \mu; H) \times L^p(\Omega, \mu; \mathcal{H}_2)$, respectively. See¹³ Lemma 2.2, and¹⁰ Proposition 1.

We recall that $\nabla_H : \mathcal{F}\mathcal{C}_b^\infty(\Omega) \rightarrow L^p(\Omega, \nu; H)$ and $(\nabla_H, \nabla_H^2) : \mathcal{F}\mathcal{C}_b^\infty(\Omega) \rightarrow L^p(\Omega, \nu; H) \times L^p(\Omega, \nu; \mathcal{H}_2)$ are closable operators in $L^p(\Omega, \nu)$, whenever $p > \frac{t-1}{t-2}$ (see²² Proposition 6.1 and¹² Proposition 2.2). For such values of p we denote by $W^{1,p}(\Omega, \nu)$ the domain of its closure in $L^p(\Omega, \nu)$ and we will still denote by ∇_H the closure operator. The space $W^{2,p}(\Omega, \nu)$ is defined in the same way.

Finally we want to remark that if Hypotheses 1.1 and 1.2 hold, then $\frac{t-1}{t-2} < 2$. In particular the Sobolev spaces $W^{1,2}(\Omega, \nu)$ and $W^{2,2}(\Omega, \nu)$ are well defined.

2.6. Traces of Sobolev functions

By ρ we indicate the Feyel–de La Pradelle Hausdorff–Gauss surface measure. For a comprehensive treatment of surface measures in infinite dimensional Banach spaces with Gaussian measures we refer to,²⁴²³ and¹³

Traces of Sobolev functions in infinite dimensional Banach spaces have been studied in¹³ in the Gaussian case and in²² in the weighted Gaussian case. We stress that in¹³ the definition of Sobolev Spaces is different with respect to the our one,

but these two definitions coincide in the case of Gaussian measure. Assume that Hypotheses 1.1 and 1.2 hold and let $p > \frac{t-1}{t-2}$. If $\varphi \in W^{1,p}(\Omega, \nu)$ we define the trace of φ on $G^{-1}(0)$ as follows:

$$\text{Tr } \varphi = \lim_{n \rightarrow +\infty} \varphi_{n|_{G^{-1}(0)}} \quad \text{in } L^1(G^{-1}(0), e^{-U} \rho),$$

and it is possible to prove that $\text{Tr } \varphi \in L^q(G^{-1}(0), e^{-U} \rho)$ for any $q \in [1, p(t-2)(t-1)^{-1}]$, where t is the number fixed in Hypothesis 1.2. Here, $(\varphi_n)_{n \in \mathbb{N}}$ is any sequence in $\text{Lip}_b(\Omega)$, the space of bounded and Lipschitz functions on Ω , which converges in $W^{1,p}(\Omega, \nu)$ to φ . The definition does not depend on the choice of the sequence $(\varphi_n)_{n \in \mathbb{N}}$ in $\text{Lip}_b(\Omega)$ approximating φ in $W^{1,p}(\Omega, \nu)$ (see²² Proposition 7.1). In addition the following result holds.

Proposition 2.1. *Assume that Hypotheses 1.1 and 1.2 hold. Then the operator $\text{Tr} : W^{1,p}(\Omega, \nu) \rightarrow L^q(G^{-1}(0), e^{-U} \rho)$ is continuous for every $p > \frac{t-1}{t-2}$ and $q \in [1, p \frac{t-2}{t-1}]$. Moreover, if $U \equiv 0$, then the trace operator is continuous from $W^{1,p}(\Omega, \mu)$ to $L^q(G^{-1}(0), \rho)$ for every $p > 1$ and $q \in [1, p)$ (see¹³ Corollary 4.2 and²² Corollary 7.3).*

We will still denote by $\text{Tr } \Psi = \sum_{n=1}^{+\infty} (\text{Tr } \psi_n) e_n$ if $\Psi \in W^{1,p}(\Omega, \nu; H)$, for $p > \frac{t-1}{t-2}$, and $\psi_n = \langle \Psi, e_n \rangle_H$. The main result of²² is the following integration by parts formula.

Theorem 2.2. *Assume that Hypotheses 1.1 and 1.2 hold and let $p > \frac{t-1}{t-2}$. For every $\varphi \in W^{1,p}(\Omega, \nu)$ and $k \in \mathbb{N}$ we have*

$$\int_{\Omega} (\partial_k \varphi - \varphi \partial_k U - \varphi \widehat{e}_k) d\nu = \int_{G^{-1}(0)} \text{Tr}(\varphi) \frac{\partial_k G}{|\nabla_H G|_H} e^{-\text{Tr}(U)} d\rho.$$

Another important result, that we will use in this paper, is the following (see¹³ Proposition 4.8 and²² Proposition 7.5).

Proposition 2.3. *Assume that Hypotheses 1.1 and 1.2 hold and let $p > \frac{t-1}{t-2}$. Then for every $\varphi \in W^{1,p}(\Omega, \nu)$, the trace of $\text{Tr}(\varphi)$ at $G^{-1}(0)$ coincides ρ -a.e. with the restriction to $G^{-1}(0)$ of any precise version $\tilde{\varphi}$ of φ .*

2.7. The spaces $W_{U,N}^{2,2}$ and $Z_{U,N}^{2,2}$

We recall the definition of the space $W_U^{2,2}$ and $W_{U,N}^{2,2}$.

$$W_U^{2,2}(\Omega, \nu) = \left\{ u \in W^{2,2}(\Omega, \nu) \mid \int_{\Omega} \langle \nabla_H^2 U \nabla_H u, \nabla_H u \rangle d\nu < +\infty \right\},$$

endowed with the norm

$$\|u\|_{W_U^{2,2}(\Omega, \nu)}^2 = \|u\|_{W^{2,2}(\Omega, \nu)}^2 + \int_{\Omega} \langle \nabla_H^2 U \nabla_H u, \nabla_H u \rangle_H d\nu. \quad (2.5)$$

12 ADDONA Davide, CAPPÀ Gianluca, FERRARI Simone

We consider the space $W_U^{2,2}(\Omega, \nu)$

$$W_{U,N}^{2,2}(\Omega, \nu) = \left\{ u \in W^{2,2}(\Omega, \nu) \left| \int_{\Omega} \langle \nabla_H^2 U \nabla_H u, \nabla_H u \rangle d\nu < +\infty, \right. \right. \\ \left. \left. \langle \text{Tr}(\nabla_H u), \nabla_H G \rangle_H = 0 \text{ } \rho\text{-a.e. in } G^{-1}(0) \right\}$$

endowed with the norm (2.5).

We denote by $Z_U^{2,2}(X, \nu)$ the completion of the space $\mathcal{F}\mathcal{C}_b^2(X)$ with respect to the norm defined in (2.5) and by $Z_{U,N}^{2,2}(\Omega, \nu)$ the completion of the space

$$\mathcal{Z}(\Omega) := \left\{ f \in \mathcal{F}\mathcal{C}_b^2(\Omega) \mid \langle \nabla_H f(x), \nabla_H G(x) \rangle_H = 0 \text{ for } \rho\text{-a.e. } x \in G^{-1}(0) \right\},$$

with respect to the norm (2.5).

3. Second-order analysis of the Moreau–Yosida approximations along H

We start this section by recalling the definition of the subdifferential of a convex semicontinuous function. If $f : X \rightarrow \mathbb{R}$ is a proper, convex and lower semicontinuous function, we denote by $\text{dom}(f)$ the domain of f , namely $\text{dom}(f) := \{x \in X \mid f(x) < +\infty\}$, and by $\partial f(x)$ the subdifferential of f at the point x , i.e.

$$\partial f(x) := \begin{cases} \{x^* \in X^* \mid f(y) \geq f(x) + x^*(y - x) \text{ for every } y \in X\} & x \in \text{dom}(f); \\ \emptyset & x \notin \text{dom}(f). \end{cases}$$

For a classical treatment of subdifferentials of convex functions we refer to³⁸ and⁴

We recall that for $\alpha > 0$ the *Moreau–Yosida approximation along H* of a proper convex and lower semicontinuous function $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is

$$f_{\alpha}(x) := \inf \left\{ f(x+h) + \frac{1}{2\alpha} |h|_H^2 \mid h \in H \right\}. \quad (3.1)$$

See¹¹ Section 3 and¹² Section 4 for more details and⁹ and⁵ Section 12.4 for a treatment of the classical Moreau–Yosida approximations in Hilbert spaces, which are different from the ones defined in (3.1). Second-order analysis of the classical Moreau–Yosida approximations have been studied in various papers, e.g.,^{39,37} and³⁶

In the following proposition we recall some results contained in¹¹ Section 3 and in¹² Section 4.

Proposition 3.1. *Let $x \in X$, $\alpha > 0$ and let $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper convex and lower semicontinuous function. The following properties hold:*

- (1) *the function $g_{\alpha,x} : H \rightarrow \mathbb{R}$ defined as $g_{\alpha,x}(h) := f(x+h) + \frac{1}{2\alpha} |h|_H^2$, has a unique global minimum point $P(x, \alpha) \in H$. Moreover $P(x, \alpha) \rightarrow 0$ in H as α goes to zero;*
- (2) *$f_{\alpha}(x) \nearrow f(x)$ as $\alpha \rightarrow 0^+$. In particular $f_{\alpha}(x) \leq f(x)$ for every $\alpha > 0$ and $x \in X$;*

- (3) for $p \in H$, we have $p = P(x, \alpha)$ if, and only if, $f(x + p) \leq f(x + h) + \frac{1}{\alpha} \langle p, h - p \rangle_H$, for every $h \in H$;
- (4) the function $P_{x, \alpha} : H \rightarrow H$ defined as $P_{x, \alpha}(h) := P(x + h, \alpha)$ is Lipschitz continuous, with Lipschitz constant less than or equal to 1;
- (5) f_α is differentiable along H at every point $x \in X$. In addition, for every $x \in X$, we have $\nabla_H f_\alpha(x) = -\alpha^{-1} P(x, \alpha)$;
- (6) f_α belongs to $W^{2,p}(X, \mu)$, whenever $f \in L^p(X, \mu)$ for some $1 \leq p < +\infty$;
- (7) let $x \in \text{dom}(f)$ and assume that f belongs to $W^{1,p}(X, \mu)$ for some $p > 1$. If we define $F : H \rightarrow \mathbb{R}$ as $F(h) := f(x + h)$, then F is proper convex and lower semicontinuous function. Moreover, $\nabla_H f(x) \in \partial F(0)$ and $\nabla_H f_\alpha(x) \in \partial F(P(x, \alpha))$;
- (8) let $x \in \text{dom}(f)$ and assume that f belongs to $W^{1,p}(X, \mu)$ for some $p > 1$. Then $\nabla_H f_\alpha(x)$ converges to $\nabla_H f(x)$ as α goes to zero.

The last property we need is the convergence of the second-order derivative along H .

Proposition 3.2. *Let $f \in W^{2,p}(X, \mu)$ for some $p > 1$. Assume that f is twice differentiable along H at every point $x \in \text{dom} f$. Then for every $x \in \text{dom}(f)$ there exists $\nabla_H^2 f_\alpha(x)$, and $\nabla_H^2 f_\alpha(x)$ converges to $\nabla_H^2 f(x)$ as α goes to zero.*

Proof. By Proposition 3.1(7) we get $\nabla_H f_\alpha(x) = \nabla_H f(x + P(x, \alpha))$. We can differentiate along H since $P(x, \alpha)$ admits a H -gradient (it is H -Lipschitz).

$$\nabla_H^2 f_\alpha(x) = \nabla_H^2 f(x + P(x, \alpha))(I_H + \nabla_H P(x, \alpha)) = \nabla_H^2 f(x + P(x, \alpha))(I_H - \alpha \nabla_H^2 f_\alpha(x)).$$

If we let $\alpha \rightarrow 0$ then, by 3.1(8), we get $\lim_{\alpha \rightarrow 0} \nabla_H^2 f_\alpha(x) = \nabla_H^2 f(x)$. \square

4. The divergence operator

We start this section by recalling the definition of divergence, see⁸ Section 5.8 for the case $\Omega = X$. For every measurable map $\Phi : \Omega \rightarrow X$ and for every $f \in \mathcal{F}\mathcal{C}_b^\infty(\Omega)$ we define

$$\partial_\Phi f(x) = \lim_{t \rightarrow 0} \frac{f(x + t\Phi(x)) - f(x)}{t}, \quad x \in \Omega. \quad (4.1)$$

Definition 4.1. Let $\Phi \in L^1(\Omega, \nu; X)$ be a vector field. We say that Φ admits *divergence* if there exists a function $g \in L^1(\Omega, \nu)$ such that

$$\int_\Omega \partial_\Phi f d\nu = - \int_\Omega f g d\nu, \quad (4.2)$$

for every $f \in \mathcal{F}\mathcal{C}_b^\infty(\Omega)$, where $\partial_\Phi f$ has been defined in (4.1). If such a function g exists, then we set $\text{div}_{\nu, \Omega} \Phi := g$. Observe that, when $\text{div}_{\nu, \Omega} \Phi$ exists, it is unique by the density of $\mathcal{F}\mathcal{C}_b^\infty(\Omega)$ in $L^p(\Omega, \nu)$ (see²²). We denote by $D(\text{div}_{\nu, \Omega})$ the domain

14 ADDONA Davide, CAPPÀ Gianluca, FERRARI Simone

of $\operatorname{div}_{\nu,\Omega}$ in $L^1(\Omega, \nu; X)$. Lastly, we observe that if $\Phi \in L^1(\Omega, \nu; H)$, then $\partial_\Phi f(x) = \langle \nabla_H f(x), \Phi(x) \rangle_H$ for $x \in \Omega$. In this case (4.2) becomes

$$\int_{\Omega} \langle \nabla_H f, \Phi \rangle_H d\nu = - \int_{\Omega} f g d\nu, \quad \text{for every } f \in \mathcal{F}\mathcal{C}_b^\infty(\Omega). \quad (4.3)$$

We remark that in L^2 -setting, the divergence operator $\operatorname{div}_{\nu,\Omega}$ is $-\nabla_H^*$, the L^2 -adjoint of the gradient along H operator. Indeed, for any $\Phi \in L^2(\Omega, \nu; H)$ and any $f \in W^{1,2}(\Omega, \nu)$ we get

$$\int_{\Omega} \langle \nabla_H f, \Phi \rangle_H d\nu = - \int_{\Omega} f \operatorname{div}_{\nu,\Omega} \Phi d\nu.$$

The following two technical lemmata are crucial to show Theorems 1.3 and 1.4. In particular, the second one is a generalization of a well known result in differential geometry, see,³⁰⁶ and.¹⁰

Lemma 4.1. *If Hypothesis 1.2 holds, then*

$$\begin{aligned} \int_X \left(\partial_h f - f \partial_h U - f \widehat{h} \right) \left(\partial_k g - g \partial_k U - g \widehat{k} \right) d\nu &= \\ &= \int_X f g \partial_h \partial_k U d\nu + \langle h, k \rangle_H \int_X f g d\nu + \int_X \partial_k f \partial_h g d\nu. \end{aligned} \quad (4.4)$$

If $\Omega \subsetneq X$, let Hypotheses 1.1 and 1.2 hold true, and let $f, g \in \mathcal{F}\mathcal{C}_b^2(\Omega)$ and $h, k \in H$. Then

$$\begin{aligned} &\int_{\Omega} \left(\partial_h f - f \partial_h U - f \widehat{h} \right) \left(\partial_k g - g \partial_k U - g \widehat{k} \right) d\nu = \\ &= \int_{G^{-1}(0)} f \left(\partial_k g - g \operatorname{Tr}(\partial_k U) - g \widehat{k} \right) \frac{\partial_h G}{|\nabla_H G|_H} e^{-\operatorname{Tr}(U)} d\rho + \\ &- \int_{G^{-1}(0)} f \partial_h g \frac{\partial_k G}{|\nabla_H G|_H} e^{-\operatorname{Tr}(U)} d\rho + \int_{\Omega} f g \partial_h \partial_k U d\nu + \\ &+ \langle h, k \rangle_H \int_{\Omega} f g d\nu + \int_{\Omega} \partial_k f \partial_h g d\nu. \end{aligned} \quad (4.5)$$

Proof. We will only prove (4.5), since the proof of (4.4) is essentially the same. We will use Theorem 2.2 several times. We have

$$\begin{aligned} &\int_{\Omega} \left(\partial_h f - f \partial_h U - f \widehat{h} \right) \left(\partial_k g - g \partial_k U - g \widehat{k} \right) d\nu = \\ &= \int_{\Omega} \partial_h f \left(\partial_k g - g \partial_k U - g \widehat{k} \right) d\nu - \int_{\Omega} \left(f \partial_h U + f \widehat{h} \right) \left(\partial_k g - g \partial_k U - g \widehat{k} \right) d\nu = \\ &= \int_{\Omega} \partial_h \left(f \left(\partial_k g - g \partial_k U - g \widehat{k} \right) \right) d\nu - \int_{\Omega} f \partial_h \left(\partial_k g - g \partial_k U - g \widehat{k} \right) d\nu + \\ &- \int_{\Omega} \left(f \partial_h U + f \widehat{h} \right) \left(\partial_k g - g \partial_k U - g \widehat{k} \right) d\nu = \end{aligned}$$

$$\begin{aligned}
 &= \int_{\Omega} \partial_h \left(f \left(\partial_k g - g \partial_k U - g \widehat{k} \right) \right) - f \left(\partial_k g - g \partial_k U - g \widehat{k} \right) \partial_h U - f \left(\partial_k g - g \partial_k U - g \widehat{k} \right) \widehat{h} d\nu + \\
 &\quad - \int_{\Omega} f \partial_h \left(\partial_k g - g \partial_k U - g \widehat{k} \right) d\nu = \\
 &= \int_{G^{-1}(0)} f \left(\partial_k g - g \operatorname{Tr}(\partial_k U) - g \widehat{k} \right) \frac{\partial_h G}{|\nabla_H G|_H} e^{-\operatorname{Tr}(U)} d\rho - \int_{\Omega} f \partial_h \left(\partial_k g - g \partial_k U - g \widehat{k} \right) d\nu = \\
 &= \int_{G^{-1}(0)} f \left(\partial_k g - g \operatorname{Tr}(\partial_k U) - g \widehat{k} \right) \frac{\partial_h G}{|\nabla_H G|_H} e^{-\operatorname{Tr}(U)} d\rho - \int_{\Omega} f \partial_k \partial_h g d\nu + \int_{\Omega} f \partial_h g \partial_k U d\nu + \\
 &\quad + \int_{\Omega} f g \partial_h \partial_k U d\nu + \int_{\Omega} f \partial_h g \widehat{k} d\nu + \langle h, k \rangle_H \int_{\Omega} f g d\nu = \\
 &= \int_{G^{-1}(0)} f \left(\partial_k g - g \operatorname{Tr}(\partial_k U) - g \widehat{k} \right) \frac{\partial_h G}{|\nabla_H G|_H} e^{-\operatorname{Tr}(U)} d\rho + \int_{\Omega} f g \partial_h \partial_k U d\nu + \langle h, k \rangle_H \int_{\Omega} f g d\nu + \\
 &\quad + \int_{\Omega} \partial_k f \partial_h g d\nu - \int_{\Omega} \partial_k (f \partial_h g) - f \partial_h g \partial_k U - f \partial_h g \widehat{k} d\nu = \\
 &= \int_{G^{-1}(0)} f \left(\partial_k g - g \operatorname{Tr}(\partial_k U) - g \widehat{k} \right) \frac{\partial_h G}{|\nabla_H G|_H} e^{-\operatorname{Tr}(U)} d\rho - \int_{G^{-1}(0)} f \partial_h g \frac{\partial_k G}{|\nabla_H G|_H} e^{-\operatorname{Tr}(U)} d\rho + \\
 &\quad + \int_{\Omega} f g \partial_h \partial_k U d\nu + \langle h, k \rangle_H \int_{\Omega} f g d\nu + \int_{\Omega} \partial_k f \partial_h g d\nu. \quad \square
 \end{aligned}$$

Lemma 4.2. *Assume Hypothesis 1.1. Let $\Phi \in \mathcal{Z}(\Omega, H)$ the space defined in (1.3). Then for ρ -a.e. $x \in G^{-1}(0)$*

$$\langle (\nabla_H^2 G)(x) \Phi(x), \Phi(x) \rangle_H = -\langle (\nabla_H \Phi(x)) \Phi(x), (\nabla_H G)(x) \rangle_H. \quad (4.6)$$

Proof. The proof is rather long and it will be split into various steps. Let $\{h_i\}_{i \in \mathbb{N}}$ be the orthonormal basis of H associated with Φ given by the definition of the space $\mathcal{Z}(\Omega, H)$. By Hypothesis 1.1, Proposition 2.3 and the very definition of $\mathcal{Z}(\Omega, H)$ the set

$$A = \{x \in G^{-1}(0) \mid |\nabla_H G(x)|_H \neq 0, \langle \Phi(x), \nabla_H G(x) \rangle_H = 0\} \quad (4.7)$$

has full ρ measure. We will prove that (4.6) holds for every point x_0 belonging to A . By (4.7) we have $\nabla_H G(x_0) \neq 0$, so there exists $n_0 \in \mathbb{N}$ such that

$$\partial_{n_0} G(x_0) \neq 0.$$

Without loss of generality, we can assume that $n_0 = 1$. By the very definition of the space $\mathcal{Z}(\Omega, H)$ there exist $K(\Phi) > 0$, $k \in \mathbb{N}$ and $(\varphi_i)_{i=1}^k \subseteq \mathcal{F}\mathcal{C}_b^2(\Omega)$ such that for every $k_1, k_2 \in H$ it holds

$$|\Phi(x_0 + k_1) - \Phi(x_0 + k_2)|_H \leq K(\Phi) |k_1 - k_2|_H, \quad (4.8)$$

and $\Phi(x) = \sum_{i=1}^k \varphi_i(x) h_i$. For $i > k$ we set $\varphi_i(x) \equiv 0$.

Step 1: Let us consider the space

$$h_1^\perp = \{\bar{h} \in H \mid \langle \bar{h}, h_1 \rangle_H = 0\},$$

endowed with the Hilbert space norm $|\bar{h}|_{h_1^\perp} = \sum_{i=2}^{+\infty} \langle \bar{h}, h_i \rangle_H^2$. We denote its inner product by $\langle \cdot, \cdot \rangle_{h_1^\perp}$ and recall that $\{h_i\}_{i \geq 2}$ is an orthonormal basis for h_1^\perp and $H = h_1^\perp \oplus \text{span}\{h_1\}$. We want to apply the implicit function theorem to a function defined on $h_1^\perp \oplus \mathbb{R}$. Let $G_{x_0} : h_1^\perp \oplus \mathbb{R} \rightarrow \mathbb{R}$ be the function defined as

$$G_{x_0}(\bar{h}, \alpha) := G(x_0 + \bar{h} + \alpha h_1).$$

Observe that $G_{x_0}((0, 0)) = G(x_0) = 0$ and

$$D_2 G_{x_0}((0, 0)) = \partial_1 G(x_0) \neq 0,$$

where D_2 is the derivative with respect the second variable. Hypothesis 1.1 implies that G_{x_0} is Fréchet differentiable at 0, applying the implicit function theorem, see³⁰ Theorem 5.9, we get an open neighborhood $U_0 \subseteq h_1^\perp$ of the origin and a continuously Fréchet differentiable function $g_{x_0} : U_0 \rightarrow \mathbb{R}$ such that for every $\bar{h} \in U_0$ we have

$$g_{x_0}(0) = 0, \quad G_{x_0}(\bar{h}, g_{x_0}(\bar{h})) = 0. \quad (4.9)$$

Moreover, the function $g_{x_0} : U_0 \rightarrow \mathbb{R}$ satisfying (4.9) is uniquely determined. Without loss of generality we may assume that U_0 is an open ball centered at the origin of radius R . We remark that (4.9) implies that for every $\bar{h} \in U_0$

$$G(x_0 + \bar{h} + g_{x_0}(\bar{h})h_1) = G_{x_0}(\bar{h}, g_{x_0}(\bar{h})) = 0. \quad (4.10)$$

Step 2: We denote by $D_{h_1^\perp} g_{x_0}(0)$ the Fréchet derivative of g_{x_0} at the origin. For $t > 0$ sufficiently small and by (4.10), for any $i \geq 2$ we get

$$\begin{aligned} 0 &= G_{x_0}(th_i, g_{x_0}(th_i)) - G_{x_0}(0, g_{x_0}(0)) = \\ &= G(x_0 + th_i + g_{x_0}(th_i)h_1) - G(x_0 + g_{x_0}(0)h_1) = \\ &= G\left(x_0 + th_i + g_{x_0}(0)h_1 + t\left\langle D_{h_1^\perp} g_{x_0}(0), h_i \right\rangle_{h_1^\perp} h_1 + o(t)h_1\right) - G(x_0 + g_{x_0}(0)h_1) = \\ &= \left\langle \nabla_H G(x_0), th_i + \left\langle D_{h_1^\perp} g_{x_0}(0), th_i \right\rangle_{h_1^\perp} h_1 + o(t)h_1 \right\rangle_H. \end{aligned}$$

Letting t go to zero, for any $i \geq 2$ we get

$$\left\langle D_{h_1^\perp} g_{x_0}(0), h_i \right\rangle_{h_1^\perp} = -\frac{\partial_i G(x_0)}{\partial_1 G(x_0)}. \quad (4.11)$$

Step 3: The vector field $\Phi_{x_0}(\bar{h}) = \sum_{i=2}^{+\infty} \varphi_i(x_0 + \bar{h})h_i$ is defined from h_1^\perp to itself. Let δ be a positive real number which satisfies

$$\delta \leq \frac{R}{2(K(\Phi)R + |\Phi(x_0)|_H)},$$

where $K(\Phi)$ has been introduced in (4.8). We consider the complete metric space $\mathcal{C}_b([-\delta, \delta], \overline{U_0})$, i.e. the set

$$\mathcal{C}_b([-\delta, \delta], \overline{U_0}) := \{f : [-\delta, \delta] \rightarrow \overline{U_0} \mid f \text{ is continuous}\},$$

endowed with the complete metric $d(f, g) := \sup_{t \in [-\delta, \delta]} |f(t) - g(t)|_{h_1^\perp}$. Let $\Gamma : \mathcal{C}_b([-\delta, \delta], \overline{U_0}) \rightarrow \mathcal{C}_b([-\delta, \delta], h_1^\perp)$ be the function defined as follows:

$$\Gamma(\gamma)(t) = \int_0^t \Phi_{x_0}(\gamma(s)) ds, \quad (4.12)$$

for any $t \in [-\delta, \delta]$. The integral in (4.12) should be understood in the Bochner sense. We look for a fixed point of Γ in $\mathcal{C}_b([-\delta, \delta], \overline{U_0})$. We want to use Banach fixed-point theorem, so

$$\begin{aligned} d(\Gamma(\gamma_1), \Gamma(\gamma_2)) &= \sup_{t \in [-\delta, \delta]} |\Gamma(\gamma_1)(t) - \Gamma(\gamma_2)(t)|_{h_1^\perp} = \\ &= \sup_{t \in [-\delta, \delta]} \left| \int_0^t \Phi_{x_0}(\gamma_1(s)) ds - \int_0^t \Phi_{x_0}(\gamma_2(s)) ds \right|_{h_1^\perp} \leq \sup_{t \in [-\delta, \delta]} \int_0^t |\Phi_{x_0}(\gamma_1(s)) - \Phi_{x_0}(\gamma_2(s))|_{h_1^\perp} ds \leq \\ &\leq \sup_{t \in [-\delta, \delta]} \int_0^t |\Phi(x_0 + \gamma_1(s)) - \Phi(x_0 + \gamma_2(s))|_H ds \leq K(\Phi) \sup_{t \in [-\delta, \delta]} \int_0^t |\gamma_1(s) - \gamma_2(s)|_{h_1^\perp} ds \leq \\ &\leq \delta K(\Phi) \sup_{t \in [-\delta, \delta]} |\gamma_1(t) - \gamma_2(t)|_{h_1^\perp} \leq \frac{1}{2} \sup_{t \in [-\delta, \delta]} |\gamma_1(t) - \gamma_2(t)|_{h_1^\perp} = \frac{1}{2} d(\gamma_1, \gamma_2). \end{aligned}$$

Therefore Γ is a contraction in $\mathcal{C}_b([-\delta, \delta], \overline{U_0})$. We claim that Γ maps $\mathcal{C}_b([-\delta, \delta], \overline{U_0})$ into itself. The continuity of $\Gamma(\gamma)(t)$ is clear, and

$$\begin{aligned} \sup_{t \in [-\delta, \delta]} |\Gamma(\gamma)(t)|_{h_1^\perp} &\leq \sup_{t \in [-\delta, \delta]} |\Gamma(\gamma)(t) - \Gamma(0)(t)|_{h_1^\perp} + \sup_{t \in [-\delta, \delta]} |\Gamma(0)(t)|_{h_1^\perp} \leq \\ &\leq \frac{R}{2} + \sup_{t \in [-\delta, \delta]} \left| \int_0^t \Phi_{x_0}(0) ds \right|_{h_1^\perp} \leq \frac{R}{2} + \frac{R}{2} = R. \end{aligned}$$

By the Banach fixed-point theorem there exists a unique fixed point $\gamma_{x_0} \in \mathcal{C}_b([-\delta, \delta], \overline{U_0})$ of Γ . We remark that $\gamma_{x_0}(0) = 0$ and that, up to replace $\delta > 0$ with a smaller one, we can assume that $\gamma_{x_0}([-\delta, \delta]) \subseteq U_0$.

Step 4: We consider the function $\psi_{x_0} : U_0 \rightarrow H$, defined as $\psi_{x_0}(\bar{h}) = \bar{h} + g_{x_0}(\bar{h})h_1$. We now want to evaluate the function $\sigma_{x_0} : (-\delta, \delta) \rightarrow H$ defined as

$$\sigma_{x_0}(t) = \psi_{x_0}(\gamma_{x_0}(t)),$$

and its derivative at the origin. Observe that

$$\begin{aligned} \sigma_{x_0}(t) &= \psi_{x_0}(\gamma_{x_0}(t)) = \psi_{x_0} \left(\int_0^t \Phi_{x_0}(\gamma_{x_0}(s)) ds \right) = \\ &= \int_0^t \Phi_{x_0}(\gamma_{x_0}(s)) ds + g_{x_0} \left(\int_0^t \Phi_{x_0}(\gamma_{x_0}(s)) ds \right) h_1, \end{aligned}$$

so $\sigma_{x_0}(0) = 0$. Furthermore

$$\begin{aligned} \sigma'_{x_0}(0) &= \lim_{t \rightarrow 0} \frac{\sigma(t) - \sigma(0)}{t} = \lim_{t \rightarrow 0} \frac{1}{t} \left(\int_0^t \Phi_{x_0}(\gamma_{x_0}(s)) ds + g_{x_0} \left(\int_0^t \Phi_{x_0}(\gamma_{x_0}(s)) ds \right) h_1 \right) = \\ &= \Phi_{x_0}(\gamma_{x_0}(0)) + \lim_{t \rightarrow 0} \frac{1}{t} \left(g_{x_0} \left(\int_0^t \Phi_{x_0}(\gamma_{x_0}(s)) ds \right) h_1 \right) = \end{aligned}$$

18 ADDONA Davide, CAPPÀ Gianluca, FERRARI Simone

$$= \Phi_{x_0}(0) + \left\langle D_{h_1^\perp} g_{x_0}(0), \Phi_{x_0}(0) \right\rangle_{h_1^\perp} h_1 = \sum_{i=2}^{+\infty} \varphi_i(x_0) h_i + \left(\sum_{i=2}^{+\infty} \varphi_i(x_0) \left\langle D_{h_1^\perp} g_{x_0}(0), h_i \right\rangle_{h_1^\perp} \right) h_1,$$

by (4.7) and (4.11) we get

$$\sigma'_{x_0}(0) = \sum_{i=2}^{+\infty} \varphi_i(x_0) h_i - \left(\sum_{i=2}^{+\infty} \varphi_i(x_0) \frac{\partial_i G(x_0)}{\partial_1 G(x_0)} \right) h_1 = \sum_{i=1}^{+\infty} \varphi_i(x_0) h_i = \Phi(x_0).$$

We finally claim that for every $t \in (-\delta, \delta)$ we have $G(x_0 + \sigma_{x_0}(t)) = 0$. Indeed, recalling that $\Gamma(\gamma_{x_0})(t) \in U_0$ and (4.10), we get

$$G(x_0 + \sigma_{x_0}(t)) = G(x_0 + \psi_{x_0}(\gamma_{x_0}(t))) = G(x_0 + \Gamma(\gamma_{x_0})(t) + g_{x_0}(\Gamma(\gamma_{x_0})(t))h_1) = 0.$$

Step 5: Now We are able to prove (4.6). Indeed, from the smoothness of G , (4.1), $\sigma_{x_0}(0) = 0$ and $\sigma'_{x_0}(0) = \Phi(x_0)$ we deduce that

$$\begin{aligned} \frac{d}{dt} \nabla_H G(x_0 + \sigma_{x_0}(t))|_{t=0} &= \lim_{t \rightarrow 0} \frac{\nabla_H G(x_0 + \sigma_{x_0}(t)) - \nabla_H G(x_0)}{t} \\ &= \lim_{t \rightarrow 0} \frac{\nabla_H G(x_0 + \Phi(x_0)t + \Phi(x_0)o(t)) - \nabla_H G(x_0)}{t} \\ &= \partial_{\Phi(x_0)} \nabla_H^2 G(x_0) = \nabla_H^2 G(x_0)(\Phi(x_0)). \end{aligned}$$

Then, we have

$$\begin{aligned} 0 &= \frac{d}{dt} (\langle \Phi(x_0 + \sigma_{x_0}(t)), \nabla_H G(x_0 + \sigma_{x_0}(t)) \rangle_H)|_{t=0} = \\ &= \langle \nabla_H \Phi(x_0) \Phi(x_0), \nabla_H G(x_0) \rangle_H + \langle \Phi(x_0), \nabla_H^2 G(x_0) \Phi(x_0) \rangle_H. \quad \square \end{aligned}$$

In the next theorem we prove that the space $Z_U^{1,2}(\Omega, \nu; H)$ is contained in the domain of the divergence, where $Z_U^{1,2}(\Omega, \nu; H)$ is the completion of the space $\mathcal{Z}(\Omega, H)$ with respect to the norm defined in (1.4).

Theorem 4.2. *Assume that either Hypotheses 1.1 and 1.2 hold or Hypothesis 1.2 holds and Ω is the whole space. Every vector field $\Phi \in Z_U^{1,2}(\Omega, \nu; H)$ has a divergence $\operatorname{div}_{\nu, \Omega} \Phi \in L^2(\Omega, \nu)$ and for every $f \in W^{1,2}(\Omega, \nu)$, the following equality holds:*

$$\int_{\Omega} \langle \nabla_H f(x), \Phi(x) \rangle_H d\nu(x) = - \int_{\Omega} f(x) \operatorname{div}_{\nu, \Omega} \Phi(x) d\nu(x).$$

Furthermore, if $\varphi_n = \langle \Phi, h_n \rangle_H$ for every $n \in \mathbb{N}$ where $(h_n)_{n \in \mathbb{N}}$ is an orthonormal basis of H , then

$$\operatorname{div}_{\nu, \Omega} \Phi = \sum_{n=1}^{+\infty} \left(\partial_n \varphi_n - \varphi_n \partial_n U - \varphi_n \widehat{h}_n \right), \quad (4.13)$$

where the series converges in $L^2(\Omega, \nu)$. In addition $\|\operatorname{div}_{\nu} \Phi\|_{L^2(\Omega, \nu)} \leq \|\Phi\|_{Z_U^{1,2}(\Omega, \nu; H)}$.

Proof. We prove the theorem assuming Hypotheses 1.1 and 1.2 hold, since the case when Hypothesis 1.2 holds and Ω is the whole space can be proved in a similar

way. We start with a preliminary computation. Let $\Phi \in \mathcal{Z}(\Omega, H)$, so there exists an orthonormal basis $\{h_i\}_{i \in \mathbb{N}}$ of H such that $\Phi = \sum_{i=1}^n \varphi_i h_i$ for some $n \in \mathbb{N}$ and $\varphi_i \in \mathcal{F}\mathcal{C}_b^2(\Omega)$ for every $i = 1, \dots, n$. In addition $\langle \Phi(x), \nabla_H G(x) \rangle_H = 0$ for ρ -a.e $x \in G^{-1}(0)$. By the integration by parts formula if $f \in \mathcal{F}\mathcal{C}_b^\infty(\Omega)$ we have

$$\begin{aligned} \int_{\Omega} \langle \nabla_H f, \Phi \rangle_H d\nu &= \int_{\Omega} \sum_{i=1}^n \partial_i f \varphi_i d\nu = \sum_{i=1}^n \int_{\Omega} \partial_i f \varphi_i d\nu = \sum_{i=1}^n \left(\int_{\Omega} \partial_i (f \varphi_i) d\nu - \int_{\Omega} f \partial_i \varphi_i d\nu \right) = \\ &= \sum_{i=1}^n \left(\int_{G^{-1}(0)} f \varphi_i \frac{\partial_i G}{|\nabla_H G|_H} e^{-\text{Tr}(U)} d\rho - \int_{\Omega} f (\partial_i \varphi_i - \varphi_i \partial_i U - \varphi_i \widehat{h}_i) d\nu \right) = \\ &= \int_{G^{-1}(0)} \langle \Phi, \nabla_H G \rangle_H \frac{f e^{-\text{Tr}(U)}}{|\nabla_H G|_H} d\rho - \sum_{i=1}^n \left(\int_{\Omega} f (\partial_i \varphi_i - \varphi_i \partial_i U - \varphi_i \widehat{h}_i) d\nu \right) = \\ &= - \sum_{i=1}^n \left(\int_{\Omega} f (\partial_i \varphi_i - \varphi_i \partial_i U - \varphi_i \widehat{h}_i) d\nu \right). \end{aligned} \tag{4.14}$$

So we have

$$\text{div}_{\nu, \Omega} \Phi = \sum_{i=1}^n (\partial_i \varphi_i - \varphi_i \partial_i U - \varphi_i \widehat{h}_i).$$

We recall the definition of the trace operator for nuclear operators A . Let $x \in \Omega$ and let $\{h_n\}_{n \in \mathbb{N}}$ be an orthonormal basis of H ; we say that A is a trace class operator if $\sum_{n=1}^{\infty} \langle (A^* A)^{1/2} h_n, h_n \rangle_H$ is finite, and we set $\text{trace}_H(A) := \sum_{n=1}^{\infty} \langle (A^* A)^{1/2} h_n, h_n \rangle_H$. In particular, $(\nabla_H \Phi)^2$ is a trace class operator and $\text{trace}_H(\nabla_H \Phi(x)^2) \leq \|\nabla_H \Phi(x)\|_{\mathcal{H}_2}^2$ (see⁸ Appendix A.2). By Lemmata 4.1 and 4.2

$$\begin{aligned} \int_{\Omega} (\text{div}_{\nu, \Omega} \Phi)^2 d\nu &= \sum_{i=1}^n \int_{\Omega} |\varphi_i|^2 d\nu + \sum_{i=1}^n \sum_{j=1}^n \int_{\Omega} \partial_j \varphi_i \partial_i \varphi_j d\nu + \sum_{i=1}^n \sum_{j=1}^n \int_{\Omega} \varphi_i \varphi_j \partial_i \partial_j U d\nu + \\ &+ \sum_{i=1}^n \sum_{j=1}^n \int_{G^{-1}(0)} \varphi_i (\partial_j \varphi_j - \varphi_j \text{Tr}(\partial_j U) - \varphi_j \widehat{h}_j) \frac{\partial_i G}{|\nabla_H G|} e^{-\text{Tr}(U)} d\rho + \\ &\quad - \sum_{i=1}^n \sum_{j=1}^n \int_{G^{-1}(0)} \varphi_i \partial_i \varphi_j \frac{\partial_j G}{|\nabla_H G|_H} e^{-\text{Tr}(U)} d\rho = \\ &= \|\Phi\|_{L^2(\Omega, \nu)}^2 + \int_{\Omega} \langle \nabla_H^2 U \Phi, \Phi \rangle_H d\nu + \int_{\Omega} \text{trace}_H((\nabla_H \Phi)^2) d\nu + \\ &+ \sum_{j=1}^n \int_{G^{-1}(0)} (\partial_j \varphi_j - \varphi_j \text{Tr}(\partial_j U) - \varphi_j \widehat{h}_j) \frac{\langle \Phi, \nabla_H G \rangle_H}{|\nabla_H G|} e^{-\text{Tr}(U)} d\rho + \\ &\quad + \int_{G^{-1}(0)} \frac{\langle \nabla_H^2 G \Phi, \Phi \rangle_H}{|\nabla_H G|_H} e^{-\text{Tr}(U)} d\rho \leq \\ &\leq \|\Phi\|_{L^2(\Omega, \nu; H)}^2 + \int_{\Omega} \|\nabla_H \Phi\|_{\mathcal{H}_2}^2 d\nu + \int_{\Omega} \langle \nabla_H^2 U \Phi, \Phi \rangle_H d\nu + \\ &\quad + \int_{G^{-1}(0)} \frac{\langle \nabla_H^2 G \Phi, \Phi \rangle_H}{|\nabla_H G|_H} e^{-\text{Tr}(U)} d\rho = \|\Phi\|_{Z^{1,2}(\Omega, \nu; H)}^2. \end{aligned} \tag{4.15}$$

20 ADDONA Davide, CAPPÀ Gianluca, FERRARI Simone

Let $(\Phi^n)_{n \in \mathbb{N}} \subseteq \mathcal{Z}(\Omega, H)$ be a sequence of vector fields which converges to Φ in $Z_U^{1,2}(\Omega, \nu; H)$. By (4.15), $(\operatorname{div}_{\nu, \Omega} \Phi^n)$ is a Cauchy sequence in $L^2(\Omega, \nu)$ and therefore it converges to an element of $L^2(\Omega, \nu)$ which we denote by $\operatorname{div}_{\nu, \Omega} \Phi$. By formula (4.14), it is easily seen that $\operatorname{div}_{\nu, \Omega} \Phi$ satisfies (4.3). Finally, by a standard approximation argument we can conclude that $\operatorname{div}_{\nu, \Omega} \Phi$ fulfills (4.3) also for every $f \in W^{1,2}(\Omega, \nu)$. \square

We say that a subspace S of $W^{1,2}(\Omega, \nu; H)$, endowed with a Banach norm $\|\cdot\|_S$, is a *Neumann extension subspace* if any $\Phi \in S$ satisfies $\langle \Phi, \nabla_H G \rangle_H = 0$ ρ -a.e. on $G^{-1}(0)$ and it admits a continuous linear extension operator, i.e. if there exists a linear operator $E_S : S \rightarrow Z_U^{1,2}(X, \nu, H)$ such that for every $\Phi \in S$

- (1) $E_S \Phi(x) = \Phi(x)$ for μ -a.e $x \in \Omega$;
- (2) there is $K_S > 0$, independent of Φ , such that $\|E_S \Phi\|_{Z_U^{1,2}(X, \mu; H)} \leq K_S \|\Phi\|_S$.

As a corollary of Theorem 4.2 we get the following result.

Corollary 4.3. *Assume that Hypotheses 1.1 and 1.2 hold and let S be a Neumann extension subspace with norm $\|\cdot\|_S$. Every field $\Phi \in S$ has a divergence $\operatorname{div}_{\nu, \Omega} \Phi \in L^2(\Omega, \nu)$ and for every $f \in W^{1,2}(\Omega, \nu)$, the following equality holds:*

$$\int_{\Omega} \langle \nabla_H f(x), \Phi(x) \rangle_H d\nu(x) = - \int_{\Omega} f(x) \operatorname{div}_{\nu, \Omega} \Phi(x) d\nu(x).$$

Furthermore, if $\varphi_n = \langle \Phi, h_n \rangle_H$ for every $n \in \mathbb{N}$, where $(h_n)_{n \in \mathbb{N}}$ is an orthonormal basis of H , then

$$\operatorname{div}_{\nu, \Omega} \Phi = \sum_{n=1}^{+\infty} \left(\partial_n \varphi_n - \varphi_n \partial_n U - \varphi_n \widehat{h}_n \right),$$

where the series converges in $L^2(\Omega, \nu)$. In addition, $\|\operatorname{div}_{\nu} \Phi\|_{L^2(\Omega, \nu)} \leq K_S \|\Phi\|_S$.

Proof. Let us consider the divergence $\operatorname{div}_{\nu, X} E_S \Phi$ (Theorem 4.2). For ν -a.e. $x \in \Omega$ let

$$D_k(x) := \sum_{n=1}^k \left(\partial_n \varphi_n(x) - \varphi_n(x) \partial_n U(x) - \varphi_n(x) \widehat{h}_n(x) \right).$$

We have that

$$\int_{\Omega} |D_k - D_m|^2 d\nu \leq \int_X \left| \sum_{n=k+1}^m \partial_n E_S \varphi_n - E_S \varphi_n \partial_n U - E_S \varphi_n \widehat{h}_n \right|^2 d\nu, \quad (4.16)$$

where $E_S \varphi_n := \langle E_S \Phi, h_n \rangle_H$. Since the right hand side of (4.16) converges to zero (the series converges to $\operatorname{div}_{\nu, X} E_S \Phi$) we get that $(D_k)_{k \in \mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega, \nu)$. We denote by $D_{\infty} \Phi$ the limit of D_n in $L^2(\Omega, \nu)$ and we observe that for

every $f \in W^{1,2}(\Omega, \nu)$

$$\begin{aligned} \int_{\Omega} \langle \nabla_H f, \Phi \rangle_H d\nu &= \lim_{n \rightarrow +\infty} \sum_{i=1}^n \int_{\Omega} \partial_i f \varphi_i d\nu = \\ &= \lim_{n \rightarrow +\infty} \sum_{i=1}^n \left(\int_{\Omega} f (\varphi_i (\partial_i U + \hat{h}_i) - \partial_i \varphi_i) d\nu + \int_{G^{-1}(0)} f \varphi_i \frac{\partial_i G}{|\nabla_H G|_H} e^{-U} d\rho \right). \end{aligned}$$

We remark that ρ -a.e we have

$$\sum_{i=1}^n f \varphi_i \frac{\partial_i G}{|\nabla_H G|_H} e^{-U} \rightarrow f \langle \Phi, \nabla_H G \rangle_H \frac{e^{-U}}{|\nabla_H G|_H} = 0,$$

and

$$\left| \sum_{i=1}^n f \varphi_i \frac{\partial_i G}{|\nabla_H G|_H} e^{-U} \right| \leq |f| e^{-U} |\Phi|_H \in L^1(G^{-1}(0), \rho).$$

Therefore, by the Lebesgue's dominated convergence theorem and the continuity of the trace operator (Proposition 2.1) we get $\int_{\Omega} \langle \nabla_H f, \Phi \rangle_H d\nu = - \int_{\Omega} f D_{\infty} \Phi d\nu$ for any $f \in W^{1,2}(\Omega, \nu)$. This means that $\text{div}_{\nu, \Omega} \Phi$ exists and $\text{div}_{\nu, \Omega} \Phi = D_{\infty} \Phi$. Moreover

$$\begin{aligned} \|\text{div}_{\nu, \Omega} \Phi\|_{L^2(\Omega, \nu)} &\leq \liminf_{k \rightarrow +\infty} \|D_k\|_{L^2(\Omega, \nu)} \leq \liminf_{k \rightarrow +\infty} \|D_k\|_{L^2(X, \nu)} = \\ &= \|\text{div}_{\nu, X} E_S \Phi\|_{L^2(X, \nu)} \leq \|E_S \Phi\|_{Z_U^{1,2}(X, \nu; H)} \leq K_S \|\Phi\|_S. \quad \square \end{aligned}$$

We remark that the subspace of the vector fields $\Phi \in Z_U^{1,2}(\Omega, \nu; H)$ such that the extension

$$\tilde{\Phi}(x) := \begin{cases} \Phi(x) & x \in \Omega; \\ 0 & x \notin \Omega, \end{cases}$$

belongs to $Z_U^{1,2}(X, \nu; H)$ satisfies the hypotheses of Corollary 4.3.

5. Maximal Sobolev regularity

This Section is devoted to the study of maximal Sobolev regularity for the equation

$$\lambda u(x) - L_{\nu, \Omega} u(x) = f(x), \quad \mu\text{-a.e. } x \in \Omega, \quad (5.1)$$

where $\lambda > 0$, and $f \in L^2(\Omega, \nu)$, since a part of the proofs of Theorems 1.3, 1.4 and 1.6 relies on them. The results of this section are sharper than the results contained in¹¹ and¹²

Our main result is the following theorem.

Theorem 5.1. *Assume that Hypotheses 1.1 and 1.2 hold. For every $\lambda > 0$ and $f \in L^2(\Omega, \nu)$ problem (5.1) has a unique weak solution $u \in W_U^{2,2}(\Omega, \nu)$. In addition*

22 ADDONA Davide, CAPPÀ Gianluca, FERRARI Simone

the following hold

$$\langle \nabla_H u(x), \nabla_H G(x) \rangle_H = 0 \text{ for } \rho\text{-a.e. } x \in G^{-1}(0); \quad (5.2)$$

$$\|u\|_{L^2(\Omega, \nu)} \leq \frac{1}{\lambda} \|f\|_{L^2(\Omega, \nu)}; \quad \|\nabla_H u\|_{L^2(\Omega, \nu; H)} \leq \frac{1}{\sqrt{\lambda}} \|f\|_{L^2(\Omega, \nu)}; \quad (5.3)$$

$$\|\nabla_H^2 u\|_{L^2(\Omega, \nu; \mathcal{H}_2)}^2 + \int_{\Omega} \langle \nabla_H^2 U \nabla_H u, \nabla_H u \rangle d\nu \leq 2 \|f\|_{L^2(\Omega, \nu)}^2. \quad (5.4)$$

In particular $u \in W_{U, N}^{2,2}(\Omega, \nu)$.

We split the proof of Theorem 5.1 into two parts: in the Section 5.1 we study the case of $\Omega = X$ and U with H -Lipschitz gradient, in Section 5.2 we use the results of Section 5.1 to prove Theorem 5.1.

5.1. Ω is the whole space

We start this subsection assuming the following hypothesis on the weight:

Hypothesis 5.2. Let $U : X \rightarrow \mathbb{R}$ be a function satisfying Hypothesis 1.2. Assume that U is differentiable along H at every point $x \in X$, and $\nabla_H U$ is H -Lipschitz.

We remark that every convex function in $\mathcal{F}\mathcal{C}_b^2(X)$ and every continuous linear functional $x^* \in X^*$ satisfy Hypothesis 5.2.

We will recall some results about maximal Sobolev regularity contained in.¹¹ Let us consider the problem

$$\lambda u(x) - L_{\nu} u(x) = f(x) \quad \mu\text{-a.e. } x \in X, \quad (5.5)$$

where $\lambda > 0$, $f \in L^2(X, \nu)$, and $L_{\nu} := L_{\nu, X}$. A function $u \in D(L_{\nu})$ is said to be a *strong solution* of problem (5.5) if there exists a sequence $\{u_n\}_{n \in \mathbb{N}} \subseteq \mathcal{F}\mathcal{C}_b^3(X)$ such that u_n converges to u in $L^2(X, \nu)$ and

$$L^2(X, \nu)\text{-} \lim_{n \rightarrow +\infty} \lambda u_n - L_{\nu} u_n = f.$$

Moreover a sequence $\{u_n\}_{n \in \mathbb{N}} \subseteq \mathcal{F}\mathcal{C}_b^3(X)$ satisfying the above conditions is called a *strong solution sequence* for u . The following proposition is borrowed from¹¹ Proposition 5.8.

Theorem 5.3. *Assume that Hypothesis 5.2 holds. For every $\lambda > 0$ and $f \in L^2(X, \nu)$, there exists a unique strong solution of equation (5.5). Such strong solution is also a weak solution of problem (5.5). In addition, if $\{u_n\}_{n \in \mathbb{N}} \subseteq \mathcal{F}\mathcal{C}_b^3(X)$ is a strong solution sequence for u , then (u_n) converges to u in $W^{2,2}(X, \nu)$.*

When U satisfies Hypothesis 5.2 we have the following regularity result.

Theorem 5.4. *Let U be a function satisfying Hypothesis 5.2, let $\lambda > 0$, $f \in L^2(X, \nu)$, and let u be the strong solution of equation (5.5). Then $u \in W_U^{2,2}(X, \nu)$*

and

$$\|u\|_{L^2(X,\nu)} \leq \frac{1}{\lambda} \|f\|_{L^2(X,\nu)}; \quad \|\nabla_H u\|_{L^2(X,\nu;H)} \leq \frac{1}{\sqrt{\lambda}} \|f\|_{L^2(X,\nu)}; \quad (5.6)$$

$$\|\nabla_H^2 u\|_{L^2(X,\nu;\mathcal{H}_2)}^2 + \int_X \langle \nabla_H^2 U \nabla_H u, \nabla_H u \rangle_H d\nu \leq 2 \|f\|_{L^2(X,\nu)}^2. \quad (5.7)$$

The difference between Theorem 5.4 and the results of¹¹ is that estimate (5.7) is sharper, since it contains the integral $\int_X \langle \nabla_H^2 U \nabla_H u, \nabla_H u \rangle_H d\nu$. We stress that, even if $\nabla_H U$ is H -Lipschitz, which means that $\nabla_H^2 U$ is essentially bounded, we can not use the second inequality in (5.6) to estimate (5.7). Indeed, (5.7) is independent of λ , while (5.6) does not.

Proof. The proof of (5.6) can be found in¹¹ Theorem 5.10. By Proposition 5.3 there exists a sequence $\{u_n\}_{n \in \mathbb{N}} \subseteq \mathcal{FC}_b^3(X)$ and a function $u \in W^{1,2}(X,\nu)$ such that u_n converges to u in $L^2(X,\nu)$ and

$$L^2(X,\nu)\text{-}\lim_{n \rightarrow +\infty} \lambda u_n - L_\nu u_n = f.$$

Let $f_n := \lambda u_n - L_\nu u_n$. Using formula (2.4), we differentiate the equality $\lambda u_n - L_\nu u_n = f_n$ with respect to the e_j direction, multiply the result by $\partial_j u$, sum over j and finally integrate over X with respect to ν . Then we obtain

$$\begin{aligned} (1 + \lambda) \int_X |\nabla_H u_n|_H^2 d\nu + \int_X \|\nabla_H^2 u_n\|_{\mathcal{H}_2}^2 d\nu + \int_X \langle \nabla_H^2 U \nabla_H u_n, \nabla_H u_n \rangle_H d\nu &= \\ &= \int_X f_n^2 d\nu - \lambda \int_X f_n u_n d\nu. \end{aligned}$$

By Fatou's Lemma and recalling that u_n and f_n converge to u and f in $L^2(X,\nu)$, respectively, we get

$$\begin{aligned} &\|\nabla_H^2 u\|_{L^2(X,\nu;\mathcal{H}_2)}^2 + \int_X \langle \nabla_H^2 U \nabla_H u, \nabla_H u \rangle_H d\nu \leq \\ &\leq \liminf_{n \rightarrow +\infty} \left(\|\nabla_H^2 u_n\|_{L^2(X,\nu;\mathcal{H}_2)}^2 + \int_X \langle \nabla_H^2 U \nabla_H u_n, \nabla_H u_n \rangle_H d\nu \right) \leq \liminf_{n \rightarrow +\infty} \left(\int_X f_n^2 d\nu - \lambda \int_X f_n u_n d\nu \right) = \\ &= \int_X f^2 d\nu - \lambda \int_X f u d\nu. \end{aligned}$$

Using inequalities (5.6) we get

$$\|\nabla_H^2 u\|_{L^2(X,\nu;\mathcal{H}_2)}^2 + \int_X \langle \nabla_H^2 U \nabla_H u, \nabla_H u \rangle_H d\nu \leq 2 \|f\|_{L^2(X,\nu)}^2. \quad \square$$

We will not give the prove of the following theorem, since it can be easily deduced using the results of¹¹ and the arguments in the proof of Theorem 5.1.

24 ADDONA Davide, CAPPÀ Gianluca, FERRARI Simone

Theorem 5.5. *Assume Hypothesis 1.2 holds. Let $\lambda > 0$, $f \in L^2(X, \nu)$, and let u be the strong solution of equation (5.5). Then $u \in W_U^{2,2}(X, \nu)$ and*

$$\begin{aligned} \|u\|_{L^2(X, \nu)} &\leq \frac{1}{\lambda} \|f\|_{L^2(X, \nu)}; & \|\nabla_H u\|_{L^2(X, \nu; H)} &\leq \frac{1}{\sqrt{\lambda}} \|f\|_{L^2(X, \nu)}; \\ \|\nabla_H^2 u\|_{L^2(X, \nu; \mathcal{H}_2)}^2 &+ \int_X \langle \nabla_H^2 U \nabla_H u, \nabla_H u \rangle d\nu &\leq 2 \|f\|_{L^2(X, \nu)}^2. \end{aligned}$$

5.2. The general case

Assume that Hypotheses 1.1 and 1.2 hold. Let $x \in X$ and let $\mathcal{C} \subseteq X$ be a Borel set. We define

$$d_H(x, \mathcal{C}) := \begin{cases} \inf \{|h|_H \mid h \in H \cap (x - \mathcal{C})\} & \text{if } H \cap (x - \mathcal{C}) \neq \emptyset; \\ +\infty & \text{if } H \cap (x - \mathcal{C}) = \emptyset. \end{cases}$$

d_H can be seen as a distance function from \mathcal{C} along H . This function has been already considered in,^{28, 408} Example 5.4.10,²⁷ and.¹² For $\alpha \in (0, 1]$ let U_α be the Moreau–Yosida approximation along H of the weight U defined in Section 3.

We approach the problem in Ω by penalized problems in the whole space X , replacing U by

$$V_\alpha(x) := U_\alpha(x) + \frac{1}{2\alpha} d_H^2(x, \Omega).$$

for $\alpha \in (0, 1]$. Namely for $\alpha \in (0, 1]$, we consider the problem

$$\lambda u_\alpha - L_{\nu_\alpha} u_\alpha = f \tag{5.8}$$

where $\lambda > 0$, $f \in L^2(X, \nu_\alpha)$, $\nu_\alpha = e^{-V_\alpha} \mu$ and $L_{\nu_\alpha} := L_{\nu_\alpha, X}$. The first result we need to recall is¹² Proposition 5.2.

Proposition 5.6. *Assume that Hypotheses 1.1 and 1.2 hold and let $\alpha \in (0, 1]$. Then the following properties hold:*

- (1) V_α is a convex and H -continuous function;
- (2) V_α is differentiable along H for μ -a.e. $x \in X$, and $\nabla_H V_\alpha$ H -Lipschitz;
- (3) $e^{-V_\alpha} \in W^{1,p}(X, \mu)$, for every $p \geq 1$;
- (4) $V_\alpha \in W^{2,t}(X, \mu)$, where t is given by Hypothesis 1.2;
- (5) $\lim_{\alpha \rightarrow 0^+} V_\alpha(x) = \begin{cases} U(x) & x \in \Omega; \\ +\infty & x \notin \Omega. \end{cases}$

By Proposition 5.6 we can apply Theorem 5.4 to problem (5.8) and get the following maximal Sobolev regularity result (see also¹² Theorem 5.3).

Theorem 5.7. *Assume Hypotheses 1.1 and 1.2 hold and let $\alpha \in (0, 1]$, $\lambda > 0$ and $f \in L^2(X, \nu_\alpha)$. Problem (5.8) has a unique weak solution u_α . Moreover $u_\alpha \in$*

$W_{V_\alpha}^{2,2}(X, \nu_\alpha)$ and

$$\|u_\alpha\|_{L^2(X, \nu_\alpha)} \leq \frac{1}{\lambda} \|f\|_{L^2(X, \nu_\alpha)}; \quad \|\nabla_H u_\alpha\|_{L^2(X, \nu_\alpha; H)} \leq \frac{1}{\sqrt{\lambda}} \|f\|_{L^2(X, \nu_\alpha)}; \quad (5.9)$$

$$\|\nabla_H^2 u_\alpha\|_{L^2(X, \nu_\alpha; \mathcal{H}_2)}^2 + \int_X \langle \nabla_H^2 V_\alpha \nabla_H u_\alpha, \nabla_H u_\alpha \rangle d\nu_\alpha \leq 2 \|f\|_{L^2(X, \nu_\alpha)}^2. \quad (5.10)$$

In addition, for every $\alpha \in (0, 1]$, there exists a sequence $\{u_\alpha^{(n)}\}_{n \in \mathbb{N}} \subseteq \mathcal{FC}_b^3(X)$ such that $u_\alpha^{(n)}$ converges to u_α in $W^{2,2}(X, \nu_\alpha)$ and $\lambda u_\alpha^{(n)} - L_{\nu_\alpha} u_\alpha^{(n)}$ converges to f in $L^2(X, \nu_\alpha)$.

We are now ready to prove Theorem 5.1.

Proof. [Proof of Theorem 5.1] The Neumann condition (5.2) and estimates (5.3) have been proved in¹² Theorems 1.3 and 1.4. Hence, it remains to prove (5.4). Let $f \in \mathcal{FC}_b^\infty(X)$. By Theorem 5.7, for every $\alpha \in (0, 1]$ the equation (5.8) has a unique weak solution $u_\alpha \in W^{2,2}(X, \nu_\alpha)$ such that inequalities (5.9) and (5.10) hold. Moreover, for every $\varphi \in \mathcal{FC}_b^\infty(X)$ we have

$$\lambda \int_X u_\alpha \varphi d\nu_\alpha + \int_X \langle \nabla_H u_\alpha, \nabla_H \varphi \rangle_H d\nu_\alpha = \int_X f \varphi d\nu_\alpha.$$

By Proposition 5.6 and Proposition 3.1(2) we have

$$e^{-U(x)} \leq e^{-U_\alpha(x)} = e^{-V_\alpha(x)}, \quad x \in \Omega,$$

and so the inclusion $W^{2,2}(\Omega, \nu_\alpha) \subseteq W^{2,2}(\Omega, \nu)$ follows, for every $\alpha \in (0, 1]$.

Let $\{\alpha_n\}_{n \in \mathbb{N}}$ be a sequence converging to zero such that $0 < \alpha_n \leq 1$ for every $n \in \mathbb{N}$. By inequalities (5.9) and (5.10) the sequence $\{u_{\alpha_n} \mid n \in \mathbb{N}\}$ is bounded in $W^{2,2}(\Omega, \nu)$. By weak compactness there exists a subsequence, that we will still denote by $\{\alpha_n\}_{n \in \mathbb{N}}$, such that u_{α_n} weakly converges to an element $u \in W^{2,2}(\Omega, \nu)$. Without loss of generality we can assume that u_{α_n} , $\nabla_H u_{\alpha_n}$ and $\nabla_H^2 u_{\alpha_n}$ converge pointwise μ -a.e. respectively to u , $\nabla_H u$ and $\nabla_H^2 u$. By Fatou's lemma and inequality (5.10) we get

$$\begin{aligned} & \|\nabla_H^2 u\|_{L^2(\Omega, \nu; \mathcal{H}_2)}^2 + \int_\Omega \langle \nabla_H^2 U \nabla_H u, \nabla_H u \rangle d\nu \leq \\ & \leq \liminf_{n \rightarrow +\infty} \left(\|\nabla_H^2 u_{\alpha_n}\|_{L^2(\Omega, \nu_{\alpha_n}; \mathcal{H}_2)}^2 + \int_\Omega \langle \nabla_H^2 U_{\alpha_n} \nabla_H u_{\alpha_n}, \nabla_H u_{\alpha_n} \rangle d\nu_{\alpha_n} \right) \leq \\ & \leq \liminf_{n \rightarrow +\infty} \left(\|\nabla_H^2 u_{\alpha_n}\|_{L^2(X, \nu_{\alpha_n}; \mathcal{H}_2)}^2 + \int_X \langle \nabla_H^2 V_{\alpha_n} \nabla_H u_{\alpha_n}, \nabla_H u_{\alpha_n} \rangle d\nu_{\alpha_n} \right) \leq \\ & \leq 2 \|f\|_{L^2(X, \nu_{\alpha_n})}^2 \leq 2 \|f\|_{L^2(\Omega, \nu)}^2. \end{aligned}$$

Finally, if $f \in L^2(\Omega, \nu)$, a standard density argument gives us the assertions of our theorem. \square

6. Proof of the main results

Theorems 1.3, 1.4 and 1.6 are consequence of the following result.

Theorem 6.1. *Assume that either Hypotheses 1.1 and 1.2 hold or Hypothesis 1.2 holds and Ω is the whole space. Then $Z_{U,N}^{2,2}(\Omega, \nu) \subseteq D(L_{\nu,\Omega}) \subseteq W_{U,N}^{2,2}(\Omega, \nu)$. Furthermore if we denote with $\|\cdot\|_{D(L_{\nu,\Omega})}$ the graph norm in $D(L_{\nu,\Omega})$, i.e. for $u \in D(L_{\nu,\Omega})$*

$$\|u\|_{D(L_{\nu,\Omega})}^2 := \|u\|_{L^2(\Omega,\nu)}^2 + \|L_{\nu,\Omega}u\|_{L^2(\Omega,\nu)}^2,$$

then for $u \in Z_U^{2,2}(\Omega, \nu)$ and $v \in D(L_{\nu,\Omega})$ it holds that

$$\|u\|_{D(L_{\nu,\Omega})} \leq \|u\|_{Z_{U,N}^{2,2}(\Omega,\nu)} \quad \text{and} \quad \|v\|_{W_{U,N}^{2,2}(\Omega,\nu)} \leq 2\sqrt{2}\|v\|_{D(L_{\nu,\Omega})}.$$

Proof. We prove the theorem assuming Hypotheses 1.1 and 1.2 hold, since in the case when Hypothesis 1.2 holds and Ω is the whole space the proof can be obtained in a similar way using Theorem 5.5.

Let $u \in D(L_{\nu,\Omega})$. Hence, $\lambda u - L_{\nu,\Omega}u \in L^2(\Omega, \nu)$, for every $\lambda \in (0, 1)$, and by Theorem 5.1 we get $u \in W_{U,N}^{2,2}(\Omega, \nu)$. Moreover

$$\begin{aligned} \|u\|_{W_{U,N}^{2,2}(\Omega,\nu)}^2 &\leq \left(\frac{1}{\lambda^2} + \frac{1}{\lambda} + 2\right) \|\lambda u - L_{\nu,\Omega}u\|_{L^2(\Omega,\nu)}^2 \leq \\ &\leq \left(\frac{1}{\lambda^2} + \frac{1}{\lambda} + 2\right) \left(2\lambda^2 \|u\|_{L^2(\Omega,\nu)}^2 + 2\|L_{\nu,\Omega}u\|_{L^2(\Omega,\nu)}^2\right) \leq \\ &\leq 2\left(\frac{1}{\lambda^2} + \frac{1}{\lambda} + 2\right) \left(\|u\|_{L^2(\Omega,\nu)}^2 + \|L_{\nu,\Omega}u\|_{L^2(\Omega,\nu)}^2\right) = 2\left(\frac{1}{\lambda^2} + \frac{1}{\lambda} + 2\right) \|u\|_{D(L_{\nu,\Omega})}^2. \end{aligned} \tag{6.1}$$

Letting $\lambda \rightarrow 1^-$ in inequality (6.1) we get $\|u\|_{W_{U,N}^{2,2}(\Omega,\nu)} \leq 2\sqrt{2}\|u\|_{D(L_{\nu,\Omega})}$.

Assume that $u \in Z_{U,N}^{2,2}(\Omega, \nu)$. Proposition 4.2 implies that $\operatorname{div}_{\nu,\Omega} \nabla_H u \in L^2(\Omega, \nu)$ and

$$\int_{\Omega} \langle \nabla_H f, \nabla_H u \rangle_H d\nu = - \int_{\Omega} f \operatorname{div}_{\nu,\Omega} \nabla_H u d\nu,$$

for every $f \in \mathcal{FC}_b^\infty(\Omega)$. Then we have $u \in D(L_{\nu,\Omega})$ and $L_{\nu,\Omega}u = \operatorname{div}_{\nu,\Omega} \nabla_H u$. By Proposition 4.2 we have

$$\begin{aligned} \|u\|_{D(L_{\nu,\Omega})}^2 &= \|u\|_{L^2(\Omega,\nu)}^2 + \|L_{\nu,\Omega}u\|_{L^2(\Omega,\nu)}^2 = \|u\|_{L^2(\Omega,\nu)}^2 + \|\operatorname{div}_{\nu,\Omega} \nabla_H u\|_{L^2(\Omega,\nu)}^2 \leq \\ &\leq \|u\|_{L^2(\Omega,\nu)}^2 + \|\nabla_H u\|_{Z_U^{1,2}(\Omega,\nu;H)}^2 = \|u\|_{Z_{U,N}^{2,2}(\Omega,\nu)}^2, \end{aligned}$$

for every $u \in Z_{U,N}^{2,2}(\Omega, \nu)$. □

We can actually simplify the statement of Theorem 1.3 when $\nabla_H U$ is H -Lipschitz and $\Omega = X$. Indeed, let us observe that if $\nabla_H U$ is H -Lipschitz then the function $x \mapsto \|\nabla_H^2 U(x)\|_{\mathcal{H}_2}$ is essentially bounded (see⁸ Theorem 5.11.2(ii)). So $W^{2,2}(X, \nu)$ is isomorphic to $W_U^{2,2}(X, \nu)$, with

$$\|u\|_{W^{2,2}(X,\nu)} \leq \|u\|_{W_U^{2,2}(X,\nu)} \leq \max\{1, \operatorname{ess\,sup}_{x \in X} \|\nabla_H^2 U(x)\|_{\mathcal{H}_2}\} \|u\|_{W^{2,2}(X,\nu)}.$$

In particular if $\nabla_H U$ is H -Lipschitz, then $\mathcal{F}\mathcal{C}_b^2(X)$ is dense in $W_U^{2,2}(X, \nu)$.

Corollary 6.2. *Assume Hypothesis 1.2 holds and $\nabla_H U$ is H -Lipschitz. Then $D(L_{\nu, X}) = W^{2,2}(X, \nu)$. Moreover, for every $u \in D(L_{\nu, X})$, it holds $L_{\nu, X} u = \operatorname{div}_{\nu, X} \nabla_H u$ and*

$$\frac{1}{\max\{1, \operatorname{ess\,sup}_{x \in X} \|\nabla_H^2 U(x)\|_{\mathcal{H}_2}\}} \|u\|_{D(L_{\nu, X})} \leq \|u\|_{W^{2,2}(X, \nu)} \leq 2\sqrt{2} \|u\|_{D(L_{\nu, X})}.$$

The same holds true, with obvious modifications, when Ω is a Neumann extension domain.

This result has been already proved in¹¹ Theorem 6.2.

7. A characterization in the Hilbert case when $\Omega = X$

When X is a Hilbert space and $\Omega = X$, then we characterize the domain of L_{ν} . In particular, we show that, under suitable assumptions on U , it is possible to show that $D(L) = W_U^{2,2}(X, \nu) = W^{2,2}(X, \nu)$. In this section, we denote by Q the covariance operator of μ (see⁸).

Hypothesis 7.1. In addition to Hypothesis 1.2, we assume that there exists a positive constant C such that

$$-\langle \nabla_H U(x), x \rangle_X \leq C|x|_X, \quad (7.1)$$

μ -a.e. $x \in X$.

The following proposition is an improvement of¹⁴ Corollary 1. Let $\{e_k\}_{k \in \mathbb{N}}$ be an orthonormal basis of X of eigenvectors of Q such that $Qe_k = \lambda_k e_k$ for any $k \in \mathbb{N}$. At first, we recall the following integration-by-parts formula in Hilbert spaces:

$$\int_X \langle D\varphi, Qe_k \rangle_X \psi d\nu + \int_X \langle D\psi, Qe_k \rangle_X \varphi d\nu = \int_X \varphi \psi \langle \nabla_H U + x, e_k \rangle_X d\nu, \quad (7.2)$$

for any $\varphi, \psi \in C_b^1(X)$ and any $k \in \mathbb{N}$, where $D\varphi$ denotes the Fréchet derivative of φ . We recall that, if we consider the orthonormal basis $\{h_n := \sqrt{\lambda_n} e_n\}_{n \in \mathbb{N}}$ in H , then the integration-by-parts formula (7.2) reads

$$\int_X \partial_{h_k} \varphi \psi d\nu + \int_X \partial_{h_k} \psi \varphi d\nu = \int_X \varphi \psi (\widehat{h}_k + \partial_{h_k} U) d\nu.$$

Proposition 7.2. *For any $\varphi \in W^{1,2}(X, \nu)$ we get*

$$x \mapsto |x|_X \varphi(x) \in L^2(X, \nu),$$

Moreover, if $\varphi \in W^{2,2}(X, \mu)$ then

$$x \mapsto |x|_X |\nabla_H \varphi(x)|_H \in L^2(X, \nu).$$

Finally, for any $s \in \mathbb{N}$, if $\varphi \in W^{s,2}(X, \nu)$ then the function $x \mapsto |x|_X^m \|\nabla_H^r \varphi(x)\|_{\mathcal{L}_r(H)}$ belongs to $L^2(X, \nu)$ for any $m, r \in \mathbb{N}$ such that $m + r \leq s$.

28 ADDONA Davide, CAPPÀ Gianluca, FERRARI Simone

Proof. In (7.2) we consider $\varphi \in \mathcal{F}\mathcal{C}_b^1(X)$ and $\psi(x) := \varphi(x)\langle x, e_k \rangle_X$. We have

$$\begin{aligned} & 2 \int_X \langle D\varphi(x), Qe_k \rangle_X \varphi(x)\langle x, e_k \rangle_X \nu(dx) + \lambda_k \int_X \varphi(x)^2 \nu(dx) \\ &= \int_X \varphi(x)^2 \langle x, e_k \rangle_X^2 \nu(dx) + \int_X \varphi(x)^2 \langle \nabla_H U(x), e_k \rangle_X \langle x, e_k \rangle_X \nu(dx). \end{aligned}$$

Applying the Young's inequality to the first integral in the left-hand side and summing up over $k \in \mathbb{N}$, we deduce that

$$\begin{aligned} \frac{1}{2} \int_X \varphi(x)^2 |x|_X^2 \nu(dx) &\leq - \int_X \varphi(x)^2 \langle \nabla_H U(x), x \rangle_X \nu(dx) + \text{trace } Q \int_X \varphi(x)^2 \nu(dx) \\ &\quad + 2 \text{trace } Q \int_X |Q^{1/2} D\varphi(x)|_X^2 \nu(dx), \end{aligned} \quad (7.3)$$

where $\text{trace } Q := \sum_{n \in \mathbb{N}} \lambda_n$. We want to estimate the first integral in the right-hand side. From Hypothesis 7.1 we obtain

$$-\varphi(x)^2 \langle \nabla_H U(x), x \rangle_X \leq C\varphi(x)^2 |x|_X.$$

Hence,

$$- \int_X \varphi(x)^2 \langle \nabla_H U(x), x \rangle_X \nu(dx) \leq C \int_X \varphi(x)^2 |x|_X \nu(dx) \leq \tilde{C} \int_X \varphi(x)^2 \nu(dx) + \frac{1}{4} \int_X \varphi(x)^2 |x|_X^2 \nu(dx),$$

where the last inequality has been obtained applying once again the Young's inequality and \tilde{C} is a positive constant. Therefore, replacing in (7.3) it follows that

$$\frac{1}{4} \int_X \varphi(x)^2 |x|_X^2 \nu(dx) \leq (\text{trace } Q + \tilde{C}) \int_X \varphi(x)^2 \nu(dx) + 2 \text{trace } Q \int_X |Q^{1/2} D\varphi(x)|_X^2 \nu(dx).$$

Since for smooth functions we have $|\nabla_H \varphi|_H = |Q^{1/2} D\varphi|_X$ we get

$$\int_X \varphi(x)^2 |x|_X^2 \nu(dx) \leq 4(\text{trace } Q + \tilde{C}) \int_X \varphi(x)^2 \nu(dx) + 8 \text{trace } Q \int_X |\nabla_H \varphi(x)|_H^2 \nu(dx). \quad (7.4)$$

and applying Fatou's Lemma and the dominated convergence theorem, the density of $\mathcal{F}\mathcal{C}_b^1(X)$ in $W^{1,2}(X, \mu)$ gives the thesis. Further, if we consider $\varphi \in \mathcal{F}\mathcal{C}_b^2(X)$, replacing φ with $\partial_k \varphi$ in (7.4) and summing up over k , we obtain

$$\int_H |\nabla_H \varphi(x)|_H^2 |x|_X^2 \nu(dx) \leq 4(\text{trace } Q + \tilde{C}) \int_X |\nabla_H \varphi(x)|_H^2 \nu(dx) + 8 \text{trace } Q \int_X |\nabla_H^2 \varphi(x)|_H^2 \nu(dx).$$

Again, an approximation argument implies that the function $x \mapsto |x|_X |\nabla_H \varphi(x)|_H \in L^2(X, \mu)$ for any $\varphi \in W^{2,2}(X, \mu)$.

The general case $s \in \mathbb{N}$ follows taking $\vartheta \in \mathcal{F}\mathcal{C}_b^s(H)$ and setting $\varphi(x) := \partial_{h_1 \dots h_r} \vartheta(x)$, with $h_1, \dots, h_r \in \{e_k : k \in \mathbb{N}\}$, and $\psi(x) := \varphi(x)\langle x, e_k \rangle_X \prod_{i=1}^{s-r} \langle x, e_i \rangle_X^2$. Then, starting from $r = s - 1$ and $m = 1$ and concluding with $r = 0$ and $m = s$, arguing as above we conclude. \square

Theorem 7.3. *Assume that Hypothesis 7.1 holds true, and that there exists a positive constant C such that $|\nabla_H^2 U(x)|_H \leq C(1 + |x|_X^2)$, then $W_U^{2,2}(X, \nu) = W^{2,2}(X, \nu)$ and Theorem 1.3 applies.*

Proof. We recall that

$$\|u\|_{W_U^{2,2}(X, \nu)} = \|u\|_{W^{2,2}(X, \nu)} + \int_X \langle \nabla_H^2 U \nabla_H u, \nabla_H u \rangle_H d\nu.$$

From Proposition 7.2 and the assumptions on U we have

$$\int_X |\langle \nabla_H^2 U \nabla_H u, \nabla_H u \rangle_H| d\nu \leq C \int_X |\nabla_H u|_H^2 d\nu + C \int_X |x|_X^2 |\nabla_H u(x)|_H^2 d\nu \leq K \|u\|_{W^{1,2}(X, \nu)},$$

for some positive constant K . This means that $\|\cdot\|_{W^{2,2}(X, \nu)}$ and $\|\cdot\|_{W_U^{2,2}(X, \nu)}$ are equivalent norms, and therefore $W_U^{2,2}(X, \nu) = W^{2,2}(X, \nu)$ and therefore $\mathcal{F}\mathcal{C}_b^2(X)$ is dense in $W_U^{2,2}(X, \nu)$. \square

8. Examples

We conclude the paper by presenting some examples. In Subsection 8.1 we study in detail the case when Ω is the ball sphere of a Hilbert space and we show that, in this case, the spaces $\mathcal{Z}(\Omega, H)$ is non-trivial, namely it is infinite dimensional, but the space $\mathcal{Z}(\Omega)$ contains only the constant functions. In Subsection 8.2 we prove Theorem 1.7 giving a characterization of the domain of the Ornstein–Uhlenbeck operator on half-spaces. Finally, in Subsection 8.3 we provide some examples of U satisfying Hypothesis 7.1 and $\Omega = X$.

8.1. The unit sphere of a Hilbert space

Let X be a separable Hilbert space, with norm $\|\cdot\|_X$ and inner product $(\cdot, \cdot)_X$, and let μ be a centered non-degenerate Gaussian measure on X . Let $\{h_n\}_{n \in \mathbb{N}}$ be an orthonormal basis of X which consists of eigenvector of the covariance operator Q , i.e. $Qh_n = \lambda_n h_n$, it is known that an orthonormal basis of the Cameron–Martin space H is $\{\sqrt{\lambda_n} h_n\}_{n \in \mathbb{N}}$ (see⁸).

Consider $G(x) = (x, x)_X - 1$, for any $x \in X$, then

$$\Omega = \{x \in X \mid \|x\|_X \leq 1\} =: \mathcal{B}_X.$$

Clearly, $G(x) = 0$ if and only if $x \in \mathcal{S}_X$ the unit sphere of X . Moreover, easy computations show that $\partial_h G(x) = 2(x, h)_X$ for any $x \in X$ and any $h \in H$. Hence, if , we have

$$\nabla_H G(x) = 2Q^{1/2}x = 2 \sum_{n=1}^{\infty} \sqrt{\lambda_n} (x, h_n)_X (\sqrt{\lambda_n} h_n),$$

for any $x \in X$, and so $|\nabla_H G(x)|_H^2 = 4\|Q^{1/2}x\|_X^2 = 4 \sum_{n=1}^{\infty} \lambda_n (x, h_n)_X^2$. So $\nabla_H G(x) = 0$ if, and only if, $x = 0$. Finally G satisfies Hypothesis 1.1 (see¹²) and $\partial_{n,m} G(x) = 2\lambda_n \delta_{n,m}$.

30 ADDONA Davide, CAPPÀ Gianluca, FERRARI Simone

As an admissible weight we can take $U(x) := \Phi(\|x\|_X^2)$, where $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ is a C^2 convex function which satisfies

$$|\Phi'(t)|, |\Phi''(t)| \leq t^k, \quad t \in \mathbb{R},$$

for some positive integer k . It is easy to prove that U is convex and satisfies the Hypothesis 1.2.

Observe that

$$\mathcal{Z}(\mathcal{B}_X, H) := \left\{ \Phi : \mathcal{B}_X \rightarrow H \left| \begin{array}{l} \text{there exists } n \in \mathbb{N} \text{ and } \{k_1, \dots, k_n\} \subseteq H \\ \text{such that } \Phi = \sum_{i=1}^n \varphi_i k_i \text{ for some } n \in \mathbb{N}, \\ \text{and } \varphi_i \in \mathcal{F}\mathcal{C}_b^2(\Omega) \text{ for } i = 1, \dots, n. \\ \text{In addition } \langle \Phi, \nabla_H G \rangle_H = 0 \text{ } \rho\text{-a.e. in } \mathcal{S}_X. \end{array} \right. \right\}.$$

In particular all the vector fields

$$\Phi_{i,j}(x) = -\frac{(x, h_i)_X}{\sqrt{\lambda_i}} h_j + \frac{(x, h_j)_X}{\sqrt{\lambda_j}} h_i$$

belongs to $\mathcal{Z}(\mathcal{B}_X, H)$, so the space $Z_U^{1,2}(\mathcal{B}_X, \nu; H)$ is infinite dimensional and contained in the domain of the divergence operator (see Theorem 4.2).

The domain of the operator L_{ν, \mathcal{B}_X} contains the space $Z_{U,N}^{2,2}(\mathcal{B}_X, \nu)$, i.e. the completion of the space

$$\mathcal{Z}(\mathcal{B}_X) = \{u \in \mathcal{F}\mathcal{C}_b^2(\mathcal{B}_X) \mid \langle \nabla_H u, \nabla_H G \rangle_H = 0 \text{ for } \rho\text{-a.e. in } x \in \mathcal{S}_X\};$$

with respect to the norm

$$\begin{aligned} \|u\|_{Z_{U,N}^{2,2}(\mathcal{B}_X, \nu)}^2 &= \|u\|_{W^{2,2}(\mathcal{B}_X, \nu)}^2 + \int_{\mathcal{B}_X} \langle \nabla_H^2 U, \nabla_H u, \nabla_H u \rangle_H d\nu + \\ &\quad + 2 \sum_{n,m=1}^{+\infty} \lambda_n \delta_{n,m} \int_{\mathcal{S}_X} \partial_n u \partial_m u \frac{e^{-\Phi(1)}}{\|Q^{1/2}x\|} d\rho. \end{aligned}$$

We want to show that in this case the space $Z_{U,N}^{2,2}(\Omega, \nu)$ only contains the constant functions. Indeed let $u \in \mathcal{Z}(\mathcal{B}_X)$, without loss of generality assume that $u(x) = \varphi((x, h_1)_X, (x, h_2)_X)$ with $\varphi \in \mathcal{C}_b^2(\mathbb{R}^2)$. The Neumann boundary condition

$$\sum_{n=1}^{+\infty} \sqrt{\lambda_n} (x, h_n)_X \partial_n u(x) = 0 \text{ for } \rho\text{-a.e. in } x \in \mathcal{S}_X$$

implies

$$\sqrt{\lambda_1} (x, h_1)_X \partial_1 \varphi((x, h_1)_X, (x, h_2)_X) + \sqrt{\lambda_2} (x, h_2)_X \partial_2 \varphi((x, h_1)_X, (x, h_2)_X) = 0$$

for ρ -a.e $x \in \mathcal{S}_X$. So the function φ satisfies the differential equation

$$\sqrt{\lambda_1} \xi_1 \partial_1 \varphi(\xi) + \sqrt{\lambda_2} \xi_2 \partial_2 \varphi(\xi) = 0 \text{ for every } \xi \in \mathcal{B}_{\mathbb{R}^2}. \quad (8.1)$$

We want to remark that the condition $\xi \in \mathcal{B}_{\mathbb{R}^2}$ is a consequence of the fact that, if $x \in \mathcal{S}_X$, then the vector $(\xi_1, \xi_2) = ((x, h_1), (x, h_2))$ belongs to the unit ball of \mathbb{R}^2 . All the solutions of (8.1) are functions of the form

$$\varphi(\xi) = g\left(\xi_1^{\sqrt{\lambda_2}} \xi_2^{-\sqrt{\lambda_1}}\right),$$

where g is a sufficiently regular function in \mathbb{R} . It is easy to see that if φ is non-constant, then φ cannot be continuous at the origin.

So Theorem 6.1 only gives us

$$D(L_{\mathcal{B}_X, \nu}) \subseteq W_{U, N}^{2,2}(\mathcal{B}_X, \nu).$$

We want to remark that a positive answer to the question “Is \mathcal{B}_X a Neumann extension domain?” would allow us to apply Theorem 1.6 and get a characterization of the domain of $L_{\mathcal{B}_X, \nu}$.

8.2. The Ornstein–Uhlenbeck operator on half-spaces

In this section we give a characterization of the domain of the operator $L_{\mu, \Omega}$, where Ω is a half-space and μ is a centered non-degenerate Gaussian measure on a separable Banach space X . To do so we need some preliminary results, in particular a lemma about extensions of Sobolev functions and a proposition about finite dimensional approximations. We recall that $Z_0^{2,2}(X, \mu) = W^{2,2}(X, \mu)$ (see⁸).

Let $x^* \in X^* \setminus \{0\}$ and $r \in \mathbb{R}$, throughout this section we set $G(x) := x^*(x) - r$ and $\Omega := G^{-1}(-\infty, 0]$. We recall that x^* is a linear and continuous functional on H , so there exists $h_{x^*} \in H$ such that for every $h \in H$

$$x^*(h) = \langle h_{x^*}, h \rangle_H.$$

Finally we remind the reader that

$$W_{0, N}^{2,2}(\Omega, \mu) = \{u \in W^{2,2}(\Omega, \mu) \mid x^*(\text{Tr}(\nabla_H u)(x)) = 0 \text{ for } \rho\text{-a.e. } x \in G^{-1}(0)\}.$$

Lemma 8.1. *There exists a Neumann extension operator from $W_{0, N}^{2,2}(\Omega, \mu)$ to $W^{2,2}(X, \mu)$.*

Proof. We use a generalization of the reflection method, adapted to our Gaussian measure. Let $f \in \mathcal{F}\mathcal{C}_b^2(\Omega)$ and put

$$Ef(x) := \begin{cases} f(x), & G(x) \leq 0, \\ \sum_{j=1}^7 a_j f\left(x - (j+1)G(x) \frac{h_{x^*}}{|h_{x^*}|_H^2}\right) \exp\left(-\frac{c_j G(x) + b_j G^2(x)}{2|h_{x^*}|_H}\right), & G(x) > 0; \end{cases} \quad (8.2)$$

where for every $j = 1, \dots, 7$,

$$b_j = 1 - \frac{1}{j^2}, \quad c_j = \frac{2(j+1)r}{j^2} \left(2 - \frac{1}{j^2}\right); \quad (8.3)$$

and

$$\sum_{j=1}^7 a_j = 1, \quad \sum_{j=1}^7 a_j(j+1) = 0, \quad \sum_{j=1}^7 a_j(j+1)^2 = 0, \quad (8.4)$$

$$\sum_{j=1}^7 a_j c_j(j+1) = 0, \quad \sum_{j=1}^7 a_j c_j^2 = 0, \quad \sum_{j=1}^7 a_j c_j = 0, \quad \sum_{j=1}^7 a_j b_j = 0. \quad (8.5)$$

32 ADDONA Davide, CAPPÀ Gianluca, FERRARI Simone

We start by proving that Ef is well defined. Indeed for $j = 1, \dots, 7$ and $x \in X$ such that $G(x) > 0$ we have

$$\begin{aligned} G\left(x - (j+1)G(x)\frac{h_{x^*}}{|h_{x^*}|_H^2}\right) &= x^*\left(x - (j+1)(x^*(x) - r)\frac{h_{x^*}}{|h_{x^*}|_H^2}\right) - r = \\ &= x^*(x) - (j+1)(x^*(x) - r)\frac{x^*(h_{x^*})}{|h_{x^*}|_H^2} - r = -jG(x) \leq 0. \end{aligned}$$

We point out that (8.4) are the classical conditions to prove the continuity of Ef and its derivatives. (8.3) and (8.5) arise from the exponential term in (8.2), which is used to prove the continuity estimate for the extension operator.

The fact that Ef belongs to $\mathcal{F}\mathcal{C}_b^0(X)$ is obvious. Fix an orthonormal basis $\{h_i\}_{i \in \mathbb{N}}$ of H obtained by completing the set $\{h_{x^*}/|h_{x^*}|_H\}$, without loss of generality we let $h_1 = h_{x^*}/|h_{x^*}|_H$. Let $x_0 \in X$ such that $G(x_0) = 0$, then $G(x_0 + th_i) = t\delta_{1i}|h_{x^*}|_H$. We have for $i \neq 1$

$$\partial_i Ef(x_0) = \partial_i f(x_0),$$

while

$$\begin{aligned} \lim_{t \rightarrow 0^-} \frac{1}{t}(Ef(x_0 + th_1) - Ef(x_0)) &= \partial_1 f(x_0); \\ \lim_{t \rightarrow 0^+} \frac{1}{t}(Ef(x_0 + th_1) - Ef(x_0)) &= \sum_{j=1}^7 a_j \left(\partial_1 f(x_0)(1 - (j+1)) - f(x_0)\frac{c_j}{2} \right) = \partial_1 f(x_0). \end{aligned}$$

Thus, letting $T_j(x) := x - (j+1)G(x)\frac{h_{x^*}}{|h_{x^*}|_H^2}$ and $A_j(x) := \exp\left(-\frac{c_j G(x) + b_j G^2(x)}{2|h_{x^*}|_H}\right)$

$$\partial_i Ef(x) = \begin{cases} \partial_i f(x), & G(x) \leq 0; \\ \sum_{j=1}^7 a_j A_j(x) \left((1 - (j+1)\delta_{1i}) \partial_i f(T_j(x)) - f(T_j(x)) \frac{2b_j G(x) + c_j}{2} \delta_{1i} \right), & G(x) > 0. \end{cases}$$

In the same way it holds

$$\partial_j \partial_i Ef(x) = \begin{cases} \partial_j \partial_i f(x), & G(x) \leq 0; \\ B_{ij}(x), & G(x) > 0, \end{cases}$$

where

$$\begin{aligned}
 B_{ij}(x) &:= \sum_{l=1}^7 a_l \partial_{ij} f \left(x - (l+1)G(x) \frac{h_{x^*}}{|h_{x^*}|^2} \right) e^{\left(-\frac{c_l G(x) + b_l G^2(x)}{2|h_{x^*}|_H} \right)} (1 - (l+1)\delta_{1i})(1 - (l+1)\delta_{1j}) + \\
 &- \sum_{l=1}^7 a_l \partial_i f \left(x - (l+1)G(x) \frac{h_{x^*}}{|h_{x^*}|^2} \right) e^{\left(-\frac{c_l G(x) + b_l G^2(x)}{2|h_{x^*}|_H} \right)} (1 - (l+1)\delta_{1i}) \frac{2b_l G(x) + c_l}{2} \delta_{1j} + \\
 &- \sum_{l=1}^7 a_l \partial_j f \left(x - (l+1)G(x) \frac{h_{x^*}}{|h_{x^*}|^2} \right) e^{\left(-\frac{c_l G(x) + b_l G^2(x)}{2|h_{x^*}|_H} \right)} (1 - (l+1)\delta_{1j}) \frac{2b_l G(x) + c_l}{2} \delta_{1i} + \\
 &+ \sum_{l=1}^7 a_l f \left(x - (l+1)G(x) \frac{h_{x^*}}{|h_{x^*}|^2} \right) e^{\left(-\frac{c_l G(x) + b_l G^2(x)}{2|h_{x^*}|_H} \right)} \frac{(2b_l G(x) + c_l)^2}{4} \delta_{1i} \delta_{1j} + \\
 &+ |h_{x^*}|_H \sum_{l=1}^7 a_l b_l f \left(x - (l+1)G(x) \frac{h_{x^*}}{|h_{x^*}|^2} \right) e^{\left(-\frac{c_l G(x) + b_l G^2(x)}{2|h_{x^*}|_H} \right)} \delta_{1i} \delta_{1j}.
 \end{aligned}$$

So Ef belongs to $\mathcal{F}\mathcal{C}_b^2(X)$ and $Ef(x) = f(x)$, $\nabla_H Ef(x) = \nabla_H f(x)$, $\nabla_H^2 Ef(x) = \nabla_H^2 f(x)$ for every $x \in \Omega$. Without loss of generality we can assume that there exists $n \in \mathbb{N}$ and $\psi \in \mathcal{C}_b^2(\mathbb{R}^n)$ such that for every $x \in X$

$$f(x) = \psi(x^*(x), \widehat{h}_2(x), \dots, \widehat{h}_n(x)).$$

We remark that

$$Ef(x) = \begin{cases} \psi(x^*(x), \widehat{h}_2(x), \dots, \widehat{h}_n(x)), & x^*(x) \leq r, \\ \sum_{j=1}^7 a_j \psi(-jx^*(x) + (j+1)r, \widehat{h}_2(x), \dots, \widehat{h}_n(x)) \exp\left(-\frac{c_j(x^*(x)-r) + b_j(x^*(x)-r)^2}{2|h_{x^*}|_H}\right), & x^*(x) > r. \end{cases}$$

So we have

$$\begin{aligned}
 \int_X |Ef(x)|^2 d\mu(x) &\leq \int_{\xi_1 \leq r} |\psi(\xi_1, \xi_2, \dots, \xi_n)|^2 d\mu_n(\xi) + \\
 + 7 \sum_{j=1}^7 a_j^2 \int_{\xi_1 > r} &\left| \psi(-j\xi_1 + (j+1)r, \xi_2, \dots, \xi_n) \exp\left(-\frac{c_j(\xi_1 - r) + b_j(\xi_1 - r)^2}{2|h_{x^*}|_H}\right) \right|^2 d\mu_n(\xi).
 \end{aligned} \tag{8.6}$$

We remark that $d\mu_n(\xi) = \exp(-|\xi|^2/2|h_{x^*}|_H) dx$. For every $j = 1, \dots, 7$, consider the change of variable:

$$\begin{cases} \eta_1 = -j\xi_1 + (j+1)r; \\ \eta_i = \xi_i, & i = 2, \dots, 7. \end{cases} \tag{8.7}$$

34 ADDONA Davide, CAPPÀ Gianluca, FERRARI Simone

We use (8.7) in the second integral of (8.6), and we get

$$\begin{aligned}
 & \sum_{j=1}^7 a_j^2 \int_{\xi_1 > r} \left| \psi(-j\xi_1 + (j+1)r, \xi_2, \dots, \xi_n) \exp\left(-\frac{c_j(\xi_1 - r) + b_j(\xi_1 - r)^2}{2|h_{x^*}|_H}\right) \right|^2 d\mu_n(\xi) = \\
 & = \sum_{j=1}^7 a_j^2 \int_{\xi_1 > r} \left| \psi(-j\xi_1 + (j+1)r, \xi_2, \dots, \xi_n) \exp\left(-\frac{c_j(\xi_1 - r) + b_j(\xi_1 - r)^2}{2|h_{x^*}|_H}\right) \right|^2 e^{-\frac{|\xi|^2}{2|h_{x^*}|_H}} d\xi = \\
 & = \sum_{j=1}^7 a_j^2 \int_{\eta_1 \leq r} \left| \psi(\eta_1, \eta_2, \dots, \eta_n) \exp\left(-\frac{c_j\left(-\frac{\eta_1 - r}{j}\right) + b_j\left(-\frac{\eta_1 - r}{j}\right)^2}{2|h_{x^*}|_H}\right) \right|^2 e^{-\frac{\left(-\frac{\eta_1 - (j+1)r}{j}\right)^2 + \sum_{i=2}^n \eta_i^2}{2|h_{x^*}|_H}} d\xi.
 \end{aligned}$$

Using the definition of a_j, b_j and c_j we get

$$\begin{aligned}
 & \sum_{j=1}^7 a_j^2 \int_{\xi_1 > r} \left| \psi(-j\xi_1 + (j+1)r, \xi_2, \dots, \xi_n) \exp\left(-\frac{c_j(\xi_1 - r) + b_j(\xi_1 - r)^2}{2|h_{x^*}|_H}\right) \right|^2 d\mu_n(\xi) = \\
 & = C \int_{\xi_1 \leq r} |\psi(\xi_1, \xi_2, \dots, \xi_n)|^2 d\mu_n(\xi)
 \end{aligned}$$

for some constant $C > 0$. So

$$\int_X |Ef(x)|^2 d\mu(x) \leq K \int_{\xi_1 \leq r} |\psi(\xi_1, \xi_2, \dots, \xi_n)|^2 d\mu_n(\xi) = K \int_{\Omega} |f(x)|^2 d\mu(x),$$

where the constant $K > 0$ depend only on r and a_i for $i = 1, \dots, 7$. Using similar arguments on $\nabla_H Ef$ and $\nabla_H^2 Ef$ we get for every $f \in \mathcal{FC}_b^2(\Omega)$

$$\|Ef\|_{W^{2,2}(X, \mu)} \leq \tilde{K} \|f\|_{W^{2,2}(\Omega, \mu)},$$

where $\tilde{K} > 0$ is an adequate constant independent of f . A standard density argument gives the thesis of our lemma. \square

Using Lemma 8.1 and Theorem 1.6 we get a characterization of the domain of $L_{\mu, \Omega}$. In order to get Theorem 1.7 we need a further approximation argument.

Proposition 8.1. *Let $u \in W^{2,2}(\Omega, \mu)$ be such that $\langle \nabla_H u(x), h_{x^*} \rangle_H = 0$ for ρ -a.e. $x \in G^{-1}(0)$. There exists a sequence $(u_n)_{n \in \mathbb{N}}$ belonging to $\mathcal{FC}_b^2(\Omega)$ such that*

- (1) $\langle \nabla_H u_n(x), h_{x^*} \rangle_H = 0$ for every $n \in \mathbb{N}$ and ρ -a.e. $x \in G^{-1}(0)$;
- (2) $(u_n)_{n \in \mathbb{N}}$ converges to u in $W^{2,2}(\Omega, \mu)$.

Proof. Fix an orthonormal basis $\{h_i\}_{i \in \mathbb{N}}$ of H obtained by completing the set $\{h_{x^*}/|h_{x^*}|_H\}$, without loss of generality we let $h_1 = h_{x^*}/|h_{x^*}|_H$. Let $u \in W^{2,2}(\Omega, \mu)$ be such that

$$\partial_1 u(x) = \langle \nabla_H u(x), h_{x^*} \rangle_H = 0 \text{ for } \rho\text{-a.e. } x \in G^{-1}(0) \quad (8.8)$$

Let Eu be the extension defined in Lemma 8.1. We denote with $P_n, S_n : X \rightarrow H$ the functions defined as

$$P_n x = \sum_{i=1}^n \widehat{h}_i(x) h_i \quad \text{and} \quad S_n y = \sum_{i=n+1}^{+\infty} \widehat{h}_i(y) h_i;$$

for every $x, y \in X$. We recall that $P_n x$ converges pointwise μ -a.e. $x \in X$ to x (see⁸ Theorem 3.5.1). Let

$$v_n(x) = \int_X Eu(P_n x + S_n y) d\mu(y),$$

by⁸ Corollary 3.5.2 and Proposition 5.4.5, u_n converges to Eu in $W^{2,2}(X, \mu)$ as n goes to infinity and for every $i, n \in \mathbb{N}$

$$\partial_i v_n(x) = \begin{cases} \int_X \partial_i Eu(P_n x + S_n y) d\mu(y) & i \leq n \\ 0 & i > n \end{cases}$$

Observe that if $x \in G^{-1}(0)$, then for every $y \in X$ and $n \in \mathbb{N}$

$$\begin{aligned} G(P_n x + S_n y) &= x^*(P_n x + S_n y) - r = \sum_{i=1}^n \widehat{h}_i(x) \langle h_{x^*}, h_i \rangle_H + \sum_{i=n+1}^{+\infty} \widehat{h}_i(y) \langle h_{x^*}, h_i \rangle_H - r = \\ &= \widehat{h}_1(x) |h_{x^*}|_H - r = x^*(x) - r = 0. \end{aligned}$$

By (8.8) we get

$$\langle \nabla_H v_n(x), h_{x^*} \rangle_H = \int_X \partial_1 Eu(P_n x + S_n y) d\mu(y) = \int_X \partial_1 u(P_n x + S_n y) d\mu(y) = 0,$$

for ρ -a.e. $x \in G^{-1}(0)$.

We are almost done, but we need smoother function satisfying Proposition 8.1(1)-(2). Let $\psi_n(\xi) := v_n(\sum_{i=1}^n \xi_i h_i)$. We remind the reader that ψ_n belongs to $W^{2,2}(\mathbb{R}^n, \mu \circ P_n^{-1})$ and

$$\partial_1 \psi_n(\xi) = 0 \text{ for } \xi \in \mathbb{R}^n \text{ such that } \xi_1 = r.$$

Let \mathcal{L}_0^n be the generator of the m -dimensional Ornstein–Uhlenbeck operator with homogeneous Neumann condition in $L^2(\mathcal{O}_n, \mu \circ P_n^{-1})$, where $\mathcal{O}_n = \{\xi \in \mathbb{R}^n \mid \xi_1 \leq r\}$. By³¹ Theorem 12.4.9 we know that the domain of \mathcal{L}_0^n in $L^2(\mathcal{O}_n, \mu \circ P_n^{-1})$ is

$$D(\mathcal{L}_0^n) = \{\varphi \in W^{2,2}(\mathcal{O}_n, \mu \circ P_n^{-1}) \mid \langle \xi, \nabla \varphi \rangle \in L^2(\mathcal{O}_n, \mu \circ P_n^{-1}), \partial_1 \varphi(\xi) = 0 \text{ when } \xi_1 = r\}$$

and

$$\|D^k R(\lambda, \mathcal{L}_0^n)\|_{L(L^2(\mathcal{O}_n, \mu \circ P_n^{-1}))} \leq 2^k \lambda^{\frac{k}{2}-1},$$

where $R(\cdot, \mathcal{L}_0^n)$ is the resolvent operator associate to \mathcal{L}_0^n and $k = 0, 1, 2$. Let $f_n := \psi_n - \mathcal{L}_0^n \psi_n$, where the equality is meant in $L^2(\mathcal{O}_n, \mu \circ P_n^{-1})$. Let $(f_{n,k})_{k \in \mathbb{N}}$ be a sequence of bounded smooth function such that $f_{n,k}$ converges in $L^2(\mathcal{O}_n, \mu \circ P_n^{-1})$ to f_n as k goes to infinity. We let

$$\psi_{n,k} = R(1, \mathcal{L}_0^n) f_{n,k}.$$

36 ADDONA Davide, CAPPÀ Gianluca, FERRARI Simone

We recall that $\psi_{n,k}$ belongs to $D(\mathcal{L}_n^0)$ and to $\mathcal{C}_b^2(\mathbb{R}^n)$ (see³¹ Section 12). Let

$$F_{n,k}u(x) := \psi_{n,k}(\widehat{h}_1(x), \dots, \widehat{h}_n(x)).$$

We get that $F_{n,k}u$ belongs to $\mathcal{F}\mathcal{C}_b^2(\Omega)$ and satisfy the Neumann condition at the boundary. Let $\varepsilon > 0$ and consider $n_\varepsilon, k_\varepsilon \in \mathbb{N}$ such that

$$\|v_{n_\varepsilon} - Eu\|_{W^{2,2}(X,\mu)} \leq \frac{\varepsilon}{2}; \quad \|\psi_{n_\varepsilon, k_\varepsilon} - \psi_{n_\varepsilon}\|_{W^{2,2}(\mathcal{O}_{n_\varepsilon, \mu \circ P_{n_\varepsilon}^{-1}})} \leq \frac{\varepsilon}{2}$$

So

$$\begin{aligned} \|F_{n_\varepsilon, k_\varepsilon}u - u\|_{W^{2,2}(\Omega,\mu)} &\leq \|F_{n_\varepsilon, k_\varepsilon}u - Eu\|_{W^{2,2}(X,\mu)} \leq \\ &\|F_{n_\varepsilon, k_\varepsilon}u - v_{n_\varepsilon}\|_{W^{2,2}(X,\mu)} + \|v_{n_\varepsilon} - Eu\|_{W^{2,2}(X,\mu)} \leq \\ &\leq \|\psi_{n_\varepsilon, k_\varepsilon} - \psi_{n_\varepsilon}\|_{W^{2,2}(\mathcal{O}_{n_\varepsilon, \mu \circ P_{n_\varepsilon}^{-1}})} + \|v_{n_\varepsilon} - Eu\|_{W^{2,2}(X,\mu)} \leq \varepsilon. \end{aligned}$$

Thus the sequence $u_m := F_{n_{m-1}, k_{m-1}}u$ for $m \in \mathbb{N}$ is the sequence we were looking for. \square

As a consequence of Corollary 1.4 and Proposition 8.1, we get Theorem 1.7.

8.3. The case $\Omega = X$ when X is a Hilbert space

Let $X = L^2(0,1)$ endowed with the classical Wiener measure γ^W . It is well known that $H = H_0^1(0,1) := \{f \in W^{1,2}(0,1) : f(0) = 0\}$. An orthonormal basis of X is given by

$$e_n := \sqrt{2} \sin \frac{t}{\sqrt{\lambda_n}}, \quad \lambda_n := \frac{4}{\pi^2(2n-1)^2}, \quad n \in \mathbb{N}.$$

Let us choose

$$U(f) := \int_0^1 \Phi(f(x))dx,$$

where $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth convex function such that $|\Phi''(t)| \leq C(1+t^2)$, for some positive constant C . Then, arguing as in¹² Example 7.3, it follows that U satisfies Hypothesis 1.2 and

$$\partial_n U(f) = \sqrt{\lambda_n} \int_0^1 \Phi'(f(x))e_n(x)dx,$$

for any $n \in \mathbb{N}$. It remains to show that (7.1) holds true. To this aim we recall that, since Φ is convex, we have $(\Phi'(t) - \Phi'(0))t \geq 0$. Hence, it follows that

$$\langle \nabla_H U(f), f \rangle_{L^2(0,1)} = \sum_{n \in \mathbb{N}} \lambda_n \int_0^1 \Phi'(f(x))f(x)dx \geq \sum_{n \in \mathbb{N}} \lambda_n \Phi'(0) \int_0^1 f(x)dx.$$

This implies

$$-\langle \nabla_H U(f), f \rangle_{L^2(0,1)} \leq \sum_{n \in \mathbb{N}} \lambda_n |\Phi'(0)| \|f\|_{L^2(0,1)},$$

and therefore (7.1) holds with $C = \sum_{n \in \mathbb{N}} \lambda_n |\Phi'(0)|$.

Acknowledgment

The authors would like to thank Prof. Alessandra Lunardi, Prof. Diego Pallara and Prof. Leonardo Biliotti for many useful discussions and comments. The authors are members of GNAMPA of the Italian Istituto Nazionale di Alta Matematica (INdAM).

This research was partially supported by the PRIN 2015 grant: “Deterministic and stochastic evolution equations”, the GNAMPA 2017 project: “Equazioni e sistemi di equazioni di Kolmogorov in dimensione finita e non” and by the Italian “Istituto nazionale di alta matematica” (INdAM).

References

1. C. D. Aliprantis and K. C. Border, *Infinite dimensional analysis*, third edn. (Springer, Berlin, 2006). A hitchhiker’s guide.
2. V. Barbu, G. Da Prato and L. Tubaro, Kolmogorov equation associated to the stochastic reflection problem on a smooth convex set of a Hilbert space, *Ann. Probab.* **37**(4) (2009) 1427–1458.
3. V. Barbu, G. Da Prato and L. Tubaro, Kolmogorov equation associated to the stochastic reflection problem on a smooth convex set of a Hilbert space II, *Ann. Inst. Henri Poincaré Probab. Stat.* **47**(3) (2011) 699–724.
4. V. Barbu and T. Precupanu, *Convexity and optimization in Banach spaces* Springer Monographs in Mathematics, Springer Monographs in Mathematics, fourth edn. (Springer, Dordrecht, 2012).
5. H. H. Bauschke and P. L. Combettes, *Convex analysis and monotone operator theory in Hilbert spaces* CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC (Springer, New York, 2011).
6. M. Bertoldi and S. Fornaro, Gradient estimates in parabolic problems with unbounded coefficients, *Studia Math.* **165**(3) (2004) 221–254.
7. V. I. Bogachev, A. Y. Pilipenko and A. V. Shaposhnikov, Sobolev functions on infinite-dimensional domains, *J. Math. Anal. Appl.* **419** (2014) 1023–1044.
8. V. I. Bogachev, *Gaussian measures*, Mathematical Surveys and Monographs, Vol. 62 (American Mathematical Society, Providence, RI, 1998).
9. H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert* (North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973). North-Holland Mathematics Studies, No. 5. Notas de Matemática (50).
10. G. Cappa, Maximal L^2 regularity for Ornstein–Uhlenbeck equation in convex sets of Banach spaces, *J. Differential Equations* **260**(11) (2016) 8051–8071.
11. G. Cappa and S. Ferrari, Maximal Sobolev regularity for solutions of elliptic equations in infinite dimensional Banach spaces endowed with a weighted Gaussian measure., *J. Differential Equations* **261**(12) (2016) 7099–7131.
12. G. Cappa and S. Ferrari, Maximal Sobolev regularity for solutions of elliptic equations in Banach spaces endowed with a weighted Gaussian measure: the convex subset case, *J. Math. Anal. Appl.* **458**(1) (2018) 300–331.
13. P. Celada and A. Lunardi, Traces of Sobolev functions on regular surfaces in infinite dimensions, *J. Funct. Anal.* **266**(4) (2014) 1948–1987.
14. G. Da Prato, Regularity results for Kolmogorov equations in $L^2(H, \mu)$ spaces and applications, *Ukrain. Mat. Zh.* **49**(3) (1997) 448–457.

38 ADDONA Davide, CAPPÀ Gianluca, FERRARI Simone

15. G. Da Prato and B. Goldys, Elliptic operators on \mathbb{R}^d with unbounded coefficients, *J. Differential Equations* **172**(2) (2001) 333–358.
16. G. Da Prato and A. Lunardi, Elliptic operators with unbounded drift coefficients and Neumann boundary condition, *J. Differential Equations* **198**(1) (2004) 35–52.
17. G. Da Prato and A. Lunardi, On a class of self-adjoint elliptic operators in L^2 spaces with respect to invariant measures, *J. Differential Equations* **234**(1) (2007) 54–79.
18. G. Da Prato and A. Lunardi, On a class of elliptic and parabolic equations in convex domains without boundary conditions, *Discrete Contin. Dyn. Syst.* **22**(4) (2008) 933–953.
19. G. Da Prato and A. Lunardi, Sobolev regularity for a class of second order elliptic PDE's in infinite dimension, *Ann. Probab.* **42**(5) (2014) 2113–2160.
20. G. Da Prato and A. Lunardi, Maximal Sobolev regularity in Neumann problems for gradient systems in infinite dimensional domains, *Ann. Inst. Henri Poincaré Probab. Stat.* **51**(3) (2015) 1102–1123.
21. J. Diestel and J. J. Uhl, Jr., *Vector measures* (American Mathematical Society, Providence, R.I., 1977). With a foreword by B. J. Pettis, Mathematical Surveys, No. 15.
22. S. Ferrari, Sobolev spaces with respect to a weighted Gaussian measure in infinite dimensions, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **22**(4) (2019) 1950026, 32.
23. D. Feyel, Hausdorff-Gauss measures, *Stochastic analysis and related topics, VII (Kusadasi, 1998)*, *Progr. Probab.* **48** (Birkhäuser Boston, Boston, MA, 2001), pp. 59–76.
24. D. Feyel and A. de La Pradelle, Capacités gaussiennes, *Ann. Inst. Fourier (Grenoble)* **41**(1) (1991) 49–76.
25. D. Feyel and A. S. Üstünel, The notion of convexity and concavity on Wiener space, *J. Funct. Anal.* **176**(2) (2000) 400–428.
26. D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order* Classics in Mathematics, Classics in Mathematics (Springer-Verlag, Berlin, 2001). Reprint of the 1998 edition.
27. M. Hino, Dirichlet spaces on H -convex sets in Wiener space, *Bull. Sci. Math.* **135**(6-7) (2011) 667–683.
28. S. Kusuoka, The nonlinear transformation of Gaussian measure on Banach space and absolute continuity. I, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **29**(3) (1982) 567–597.
29. O. A. Ladyzhenskaya and N. N. Ural'tseva, *Linear and quasilinear elliptic equations* Translated from the Russian by Scripta Technica, Inc. Translation editor: Leon Ehrenpreis, Translated from the Russian by Scripta Technica, Inc. Translation editor: Leon Ehrenpreis (Academic Press, New York-London, 1968).
30. S. Lang, *Fundamentals of differential geometry*, Graduate Texts in Mathematics,, Vol. 191 (Springer-Verlag, New York, 1999).
31. L. Lorenzi and M. Bertoldi, *Analytical methods for Markov semigroups*, Pure and Applied Mathematics (Boca Raton),, Vol. 283 (Chapman & Hall/CRC, Boca Raton, FL, 2007).
32. L. Lorenzi and A. Lunardi, Elliptic operators with unbounded diffusion coefficients in L^2 spaces with respect to invariant measures, *J. Evol. Equ.* **6**(4) (2006) 691–709.
33. A. Lunardi, G. Metafune and D. Pallara, Dirichlet boundary conditions for elliptic operators with unbounded drift, *Proc. Amer. Math. Soc.* **133**(9) (2005) 2625–2635.
34. J. Maas and J. van Neerven, On the domain of nonsymmetric Ornstein-Uhlenbeck operators in Banach spaces, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **11**(4) (2008) 603–626.
35. G. Metafune, J. Prüss, A. Rhandi and R. Schnaubelt, The domain of the Ornstein-Uhlenbeck operator on an L^p -space with invariant measure, *Ann. Sc. Norm. Super.*

- Pisa Cl. Sci. (5)* **1**(2) (2002) 471–485.
36. N. Ovcharova, Second-order analysis of the Moreau-Yosida and the Lasry-Lions regularizations, *Optim. Methods Softw.* **25**(1) (2010) 109–116.
 37. N. Ovcharova and J. Gwinner, On Moreau-Yosida approximation and on stability of second-order subdifferentials of Clarke's type, *J. Convex Anal.* **14**(2) (2007) 413–432.
 38. R. R. Phelps, *Convex functions, monotone operators and differentiability*, Lecture Notes in Mathematics, Vol. 1364, second edn. (Springer-Verlag, Berlin, 1993).
 39. L. Qi, Second-order analysis of the Moreau-Yosida regularization, *Nonlinear analysis and convex analysis (Niigata, 1998)*, (World Sci. Publ., River Edge, NJ, 1999), pp. 16–25.
 40. A. S. Üstünel and M. Zakai, The Sard inequality on Wiener space, *J. Funct. Anal.* **149**(1) (1997) 226–244.