Stability analysis and Hopf bifurcation at high Lewis number in a combustion model with free interface

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INSTABILITIES IN A COMBUSTION MODEL
WITH TWO FREE INTERFACES

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ABSTRACT. We study in a strip of $\mathbb{R}^2$ a combustion model of flame propagation with stepwise temperature kinetics and zero-order reaction, characterized by two free interfaces, respectively the ignition and the trailing fronts. The latter interface presents an additional difficulty because the non-degeneracy condition is not met. We turn the system to a fully nonlinear problem which is thoroughly investigated. When the width ℓ of the strip is sufficiently large, we prove the existence of a critical value Le_c of the Lewis number Le, such that the one-dimensional, planar, solution is unstable for $0 < \mathrm{Le} < \mathrm{Le}_c$. Some numerical simulations confirm the analysis.

1. INTRODUCTION

This paper is devoted to the analysis of cellular instabilities of planar traveling fronts for a thermo-diffusive model of flame propagation with stepwise temperature kinetics and zero-order reaction. In non-dimensional form, the model reads:

$$\begin{cases} \Theta_t = \Delta\Theta + W(\Phi, \Theta), \\ \Phi_t = \frac{1}{\mathrm{Le}}\Delta\Phi - W(\Phi, \Theta), \end{cases} \tag{1.1}$$

where Θ and Φ are appropriately normalized temperature and concentration of deficient reactant, Le is the Lewis number and $W(\Phi, \Theta)$ is a reaction rate given by

$$W(\Theta, \Phi) = \begin{cases} A, & \text{if} \quad \Theta \geq \Theta_i \quad \text{and} \quad \Phi > 0, \\ 0, & \text{if} \quad \Theta < \Theta_i \quad \text{and/or} \quad \Phi = 0. \end{cases} \tag{1.2}$$

Here, $0 < \Theta_i < 1$ is the ignition temperature and $A > 0$ is a normalizing factor.

Combustion models involving discontinuous reaction terms, including the system (1.1)-(1.2), have been used by physicists and engineers since the very early stage of the development of the combustion science (see Mallard and Le Châtelier [34]), primarily due to their relative simplicity and mathematical tractability (see, e.g., [5, 18, 22], and more recently [2, 4]). These models have drawn several mathematical studies on systems with discontinuous nonlinearities and related Free Boundary Problems which include, besides the pioneering work of K.-C. Chang [16, 17], the references [1, 14, 24, 25, 26, 27, 36, 37], to mention a few of them. In particular, models with ignition temperature were introduced in the mathematical description of the propagation of premixed flames to solve the so-called "cold-boundary difficulty" (see, e.g., [15, Section 2.2], [3]).

More specifically, in this paper we consider the free interface problem associated with the model (1.1)-(1.2). The domain is the strip $\mathbb{R} \times (-\ell/2, \ell/2)$, the spatial coordinates are denoted by (x, y), $t > 0$ is the time. The free interfaces are respectively the *ignition interface* $x = F(t, y)$ and the *trailing interface* $x = G(t, y)$, $G(t, y) < F(t, y)$, defined by $\Theta(t, F(t, y), y) = \theta_i$, $\Phi(t, G(t, y), y) = 0$.

The system reads as follows, for $t > 0$ and $y \in (-\ell/2, \ell/2)$:

$$
\begin{cases}
\dfrac{\partial \Theta}{\partial t}(t, x, y) = \Delta \Theta(t, x, y), & x < G(t, y), \\[2mm]
\Phi(t, x, y) = 0, & x < G(t, y), \\[2mm]
\dfrac{\partial \Theta}{\partial t}(t, x, y) = \Delta \Theta(t, x, y) + A, & G(t, y) < x < F(t, y), \\[2mm]
\dfrac{\partial \Phi}{\partial t}(t, x, y) = (\mathrm{Le})^{-1} \Delta \Phi(t, x, y) - A, & G(t, y) < x < F(t, y), \\[2mm]
\dfrac{\partial \Theta}{\partial t} = \Delta \Theta(t, x, y), & x > F(t, y), \\[2mm]
\dfrac{\partial \Phi}{\partial t} = (\mathrm{Le})^{-1} \Delta \Phi(t, x, y), & x > F(t, y),
\end{cases}
\tag{1.3}
$$

where the normalizing factor A will be fixed below. The functions Θ and Φ are continuous across the interfaces for $t > 0$, as well as their normal derivatives. As $x \to \pm\infty$, it holds

$$
\Theta(t, -\infty, y) = \Phi(t, +\infty, y) = 1, \qquad \Theta(t, +\infty, y) = 0. \tag{1.4}
$$

Finally, periodic boundary conditions are assumed at $y = \pm\ell/2$.

As was noted in earlier studies (see [4, 6]), this system is very different from models arising in conventional thermo-diffusive combustion. Two are the principal differences. (i) The first one is that in the model considered here, the reaction zone is of order unity, whereas in the case of Arrhenius kinetics the reaction zone is infinitely thin. This fact suggests to refer to flame fronts for stepwise temperature kinetics as thick flames, in contrast to thin flames arising in Arrhenius kinetics. (ii) The second, even more important difference, is that, in the case of Arrhenius kinetics, there is a single interface separating burned and unburned gases. In contrast to that, in case of the stepwise temperature kinetics given by (1.2), there are two interfaces, namely the *ignition interface* where $\Theta = \Theta_i$ located at $x = F(t, y)$, and *trailing interface* at $x = G(t, y)$ being defined as a largest value of x where the concentration is equal to zero. As a consequence of (i), the normal derivatives are continuous across both interfaces, contrary to classical models with Arrhenius kinetics where jumps occur at the flame front (see e.g., [15, Section 11.8] and [35, 39, 40] for the related Kuramoto-Sivashinsky equation). There have been a number of mathematical works in the latter case based on the method of [9] that we are going to extend below, see in particular [7, 8, 12, 13, 30, 31, 32] for the flame front, and the references therein. Finally, note that Free Boundary Problems with two interfaces have already been considered in the literature, especially in Stefan problems, see e.g., [20, 21, 42] (one-dimensional problem) and [19] (radial solutions).

The above system admits a one-dimensional traveling wave (planar) solution $(\Theta^{(0)}, \Phi^{(0)})$ which propagates with constant positive velocity V (see [4, Section 4]). It is convenient to choose the normalizing factor $A = 1/R$ in such a way that $V = 1$, where the positive number $R = R(\theta_i)$ is given by:

$$
\theta_i R = 1 - e^{-R}, \quad 0 < \theta_i < 1. \tag{1.5}
$$

Thus, in the moving frame coordinate $x' = x - t$, the system for the traveling wave solution reads as follows:

$$
\begin{cases}
D_{x'} \Theta^{(0)} + D_{x'x'} \Theta^{(0)} = 0, & \text{in } (-\infty, 0], \\[1mm]
D_{x'} \Theta^{(0)} + D_{x'x'} \Theta^{(0)} = -R^{-1}, & \text{in } (0, R), \\[1mm]
\mathrm{Le}\, D_{x'} \Phi^{(0)} + D_{x'x'} \Phi^{(0)} = \mathrm{Le}\, R^{-1}, & \text{in } (0, R), \\[1mm]
D_{x'} \Theta^{(0)} + D_{x'x'} \Theta^{(0)} = 0, & \text{in } [R, +\infty), \\[1mm]
\mathrm{Le}\, D_{x'} \Phi^{(0)}_{x'} + D_{x'x'} \Phi^{(0)} = 0, & \text{in } [R, +\infty),
\end{cases}
$$

whose solution is

$$\Theta^{(0)}(x') = \begin{cases} 1, & x' \leq 0, \\ 1 + \dfrac{1 - x' - e^{-x'}}{R}, & x' \in (0, R), \\ \theta_i e^{R-x'}, & x' \geq R, \end{cases} \qquad \Phi^{(0)}(x') = \begin{cases} 0, & x' \leq 0, \\ \dfrac{e^{-\mathrm{Le}x'} - 1}{\mathrm{Le}\,R} + \dfrac{x'}{R}, & x' \in (0, R), \\ 1 + \dfrac{1 - e^{\mathrm{Le}R}}{\mathrm{Le}\,R\,e^{\mathrm{Le}x'}}, & x' \geq R. \end{cases}$$
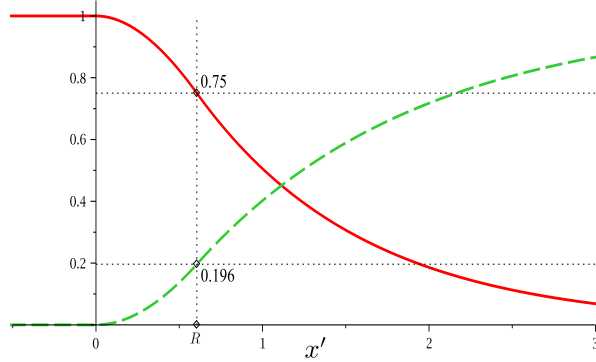
$$(1.6)$$

FIGURE 1. $\Theta^{(0)}$ (solid curve) and $\Phi^{(0)}$ (dashed curve) with $\theta_i = 0.75$, $\mathrm{Le} = 0.75$ ($R = 0.60586$).

The existence of traveling fronts poses a natural question of one and multidimensional stability, or especially instabilities of such fronts. It is known (see [35, 39]) that diffusional-thermal instabilities of planar flame fronts, when the Lewis number is less than unity, generates cellular flames and pattern formation. In this paper, we are not interested in high Lewis numbers, namely $\mathrm{Le} > 1$: in that case, it has been observed that large enough Lewis numbers give rise to pulsating instabilities, i.e., oscillatory behavior of the flame (see [4, 11]). We focus our attention on instabilities of the traveling wave $(\Theta^{(0)}, \Phi^{(0)})$, and thus for the ignition and the trailing interfaces. Earlier studies have shown (see [6]) that instabilities depend on the Lewis number and occur only when the width of the strip ℓ is large enough (in [4], ℓ is taken to infinity), which motivates the present study.

The main result of the paper is the following:

Main Theorem. *Let $0 < \theta_i < 1$ be fixed. There exist $\ell_0(\theta_i)$ sufficiently large, such that, whenever $\ell > \ell_0(\theta_i)$, there exists a critical value of the Lewis number, say $\mathrm{Le}_c \in (0, 1)$ (see (6.2)). If $\mathrm{Le} \in (0, \mathrm{Le}_c)$, then the traveling wave solution to problem (1.3)-(1.4) is unstable with respect to smooth and sufficiently small two dimensional perturbation. Further, also the ignition and the trailing interfaces are pointwise unstable.*

The paper is organized as follows. In Section 2, we introduce the main notation and the functional spaces. Section 3 is devoted to transforming problem (1.3)-(1.4) in the *fully nonlinear* problem for the perturbation of the traveling wave solution $(\Theta^{(0)}, \Phi^{(0)})$ in (1.6), set in a fixed domain, which reads

$$\begin{cases} D_t \mathbf{u}(t, \cdot, \cdot) = \mathscr{L}\mathbf{u}(t, \cdot, \cdot) + \mathscr{F}(\mathbf{u}(t, \cdot, \cdot)), & t \geq 0, \\ \mathscr{B}(\mathbf{u}(t, \cdot)) = \mathscr{H}(\mathbf{u}(t, \cdot)), & t \geq 0, \end{cases} \qquad (1.7)$$

where $\mathscr{L}$ is a second-order linear elliptic operator, $\mathscr{B}$ is a first-order linear boundary differential operator, $\mathscr{F}$, $\mathscr{H}$ are suitable nonlinear terms and $\mathscr{F}$ also depends on traces of second-order derivatives of the unknown $\mathbf{u}$. Here, $\mathbf{u}(t, \cdot) = (u_1(t, \cdot), u_2(t, \cdot))$ where $u_1(t, \cdot)$ is defined in the strip $\mathbb{R} \times [-\ell/2, \ell/2]$ and u_2 is defined in the half-strip $[0, +\infty) \times [-\ell/2, \ell/2]$. Finally, periodic boundary conditions are prescribed at $\pm\ell/2$. We determine that the ignition interface meets the transversality (or non-degeneracy) condition of [9]. Unfortunately, this is not the case of the trailing interface which is of

different nature. In short, the idea is to differentiate the mass fraction equation, taking advantage of the structure of the problems.

The theory of analytic semigroups plays a crucial role in all our analysis and is the core of Section 4. We prove (Theorem 4.4) that the realization L of the operator $\mathscr{L}$, with the homogeneous boundary conditions $\mathscr{B}\mathbf{u} = \mathbf{0}$ and periodic conditions on $y = \pm\ell/2$, generates an analytic semigroup e^{tL} in a suitable space of continuous functions. To prove such a result, we take advantage of the Fourier transform. Expanding $\mathbf{u}$ into a Fourier series, we are led to solve infinitely many ODE for its modes and, coming back, we can formally write $\mathbf{u}$ as a sum of finite number of terms (see (4.17)-(4.21)). The analysis of such terms is the content of Lemmata 4.1-4.3. As it has been already stressed, the nonlinear term $\mathscr{F}$ in (1.7) depends on traces of second-order derivatives of $\mathbf{u}$, which are not defined for functions in $D(L)$, since the second-order derivatives of functions belonging to this space are meant in the sense of distributions. To overcome this difficulty, we need to characterize the interpolation spaces $D_L(\alpha/2, \infty)$ and $D_L(1 + \alpha/2, \infty)$ ($\alpha \in (0, 1)$) and deal with the restriction of the semigroup e^{tL} to the space $D_L(\alpha/2, \infty)$. It is well known that such a restriction is still an analytic semigroup and it is associated with the part of L in $D_L(\alpha/2, \infty)$, which has $D_L(1 + \alpha/2, \infty)$ as domain. Finally, since the boundary conditions in (1.7) are nonhomogeneous, we also need to introduce two lifting operators (say $\mathscr{M}$ and $\mathscr{N}$). This is the content of the last part of Section 4. Theorem 4.4 contains also important results about the structure of the spectrum of operator L, in particular the definition of the dispersion relations.

The first lifting operator is used in Section 5 to show that the linearized version of problem (1.7), i.e., the problem

$$
\begin{cases}
D_t\mathbf{u}(t, \cdot, \cdot) = \mathscr{L}\mathbf{u}(t, \cdot, \cdot) + \mathbf{f}, & t \geq 0, \\
\mathscr{B}(\mathbf{u}(t, \cdot)) = \boldsymbol{\psi}, & t \geq 0, \\
\mathbf{u}(0, \cdot) = \mathbf{u}_0,
\end{cases}
\tag{1.8}
$$

is uniquely solvable in a suitable space of Hölder continuous functions and its solution $\mathbf{u}$ is given by the following formula:

$$
\mathbf{u}(t, \cdot) = e^{tL}(\mathbf{u}_0 - \mathscr{M}\boldsymbol{\psi}(0, \cdot)) + \int_0^t e^{(t-s)L}(\mathbf{f}(s, \cdot) + \mathscr{L}\mathscr{M}\boldsymbol{\psi}(s, \cdot))ds
$$
$$
- L\int_0^t e^{(t-s)L}(\mathscr{M}\boldsymbol{\psi}(s, \cdot) - \mathscr{M}\boldsymbol{\psi}(0, \cdot))ds + \mathscr{M}\boldsymbol{\psi}(0, \cdot),
$$

for $t \geq 0$. Based on this formula and via a fixed-point argument, we then solve problem (1.7) proving an optimal regularity result in Hölder spaces.

Section 6 contains the proof of Theorem 1. To prove instability, we need to determine a range of Lewis numbers Le which lead to eigenvalues of operator L with positive real part. In view of Theorem 4.4, the characterization of the spectrum of operator L plays a very crucial role. In Section 6.1, we analyze the reduced dispersion relation

$$
\mathcal{D}_{0,k}(\lambda, \text{Le}) = \exp\left(\frac{R}{2}(\text{Le} - 1 - X_k(\lambda) - Y_k(\lambda, \text{Le}))\right) - 1 + \theta_i R X_k(\lambda).
$$

where $X_k(\lambda) = \sqrt{1 + 4\lambda + 4\lambda_k}$, $Y_k(\lambda, \text{Le}) = Y_k(\lambda) = \sqrt{\text{Le}^2 + 4\lambda\text{Le} + 4\lambda_k}$ and $\lambda_k = 4\pi^2k^2\ell^{-2}$ for $k \in \mathbb{N} \cup \{0\}$. In Subsection 6.2, the lifting operator $\mathscr{N}$ is used to apply a variant of [28, Theorem 5.1.5] (see Lemma 6.5) in a suitable Banach space to prove instability of the free interfaces.

Finally, Section 7 is devoted to a brief presentation of a numerical method and computational results, which show two-cell patterns (see [6] for further results).

2. NOTATION, FUNCTIONAL SPACES AND PRELIMINARIES

In this section, we collect all the notation, the functional spaces and the preliminary results that we use throughout the paper

2.1. **Notation.** We find it convenient to set, for each $\tau > 0$,

$$S = \mathbb{R} \times (-\ell/2, \ell/2), \qquad S_\tau^+ = (\tau, +\infty) \times (-\ell/2, \ell/2), \qquad S_\tau^- = (-\infty, \tau) \times (-\ell/2, \ell/2),$$
$$H_\tau^- = (-\infty, \tau) \times \mathbb{R}, \qquad H_\tau^+ = (\tau, +\infty) \times \mathbb{R}, \qquad R_T = [0, T] \times [-\ell/2, \ell/2].$$

Functions. Given a function $f : (a, b) \to \mathbb{R}$ and a point $x_0 \in (a, b)$, we denote by $[f]_{x_0}$ the jump of f at x_0, i.e., the difference $f(x_0^+) - f(x_0^-)$ whenever defined. For each function $f : [-\ell/2, \ell/2) \to \mathbb{C}$ we denote by $f^\sharp$ its ℓ-periodic extension to $\mathbb{R}$. If f depends also on x running in some interval I, we still denote by $f^\sharp$ its periodic (with respect to y) extension to $I \times \mathbb{R}$. For every $f \in L^2((-\ell/2, \ell/2))$ and $k \in \mathbb{Z}$, we denote by $\hat{f}_k$ the k-th Fourier coefficient of f, i.e.,

$$\hat{f}_k = \frac{1}{\ell} \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} f \overline{e_k} dy,$$

where $e_h(y) = e^{\frac{2h\pi i}{\ell} y}$ for each $h \in \mathbb{Z}$ and $y \in \mathbb{R}$. When f depends also on the variable x running in some interval I, $\hat{f}_k(x)$ stands for the k-th Fourier coefficient of the function $f(x, \cdot)$.

The time and the spatial derivatives of a given function f are denoted by $D_t f$ ($= f_t$), $D_x f$ ($= f_x$), $D_y f$ ($= f_y$) $D_{xx} f$ ($= f_{xx}$) $D_{xy} f$ ($= f_{xy}$) and $D_{yy} f$ ($= f_{yy}$), respectively. If $\beta = (\beta_1, \beta_2)$ with $\beta_1, \beta_2 \in \mathbb{N} \cup \{0\}$, then we set $D^\gamma = D_x^{\gamma_1} D_y^{\gamma_2}$.

Finally, we denote by χ_A the characteristic function of the set $A \subset \mathbb{R}^d$ ($d \geq 1$).

Miscellanea. Throughout the paper, we denote by c_λ a positive constant, possibly depending on λ but being independent of k, n, x and the functions that we will consider, which may vary from line to line. We simply write c when the constant is independent also of λ.

The subscript "b" stands for bounded. For instance $C_b(\Omega; \mathbb{C})$ denotes the set of bounded and continuous function from Ω to $\mathbb{C}$. When we deal with spaces of real-valued functions we omit to write "$\mathbb{C}$".

Vector-valued functions are displayed in bold.

2.2. **Main function spaces.** Here, we collect the main function spaces used in the paper pointing out the (sub)section where they are used for the first time.

The spaces $\mathcal{X}$ and $\mathcal{X}_{k+\alpha}$ (Section 4). By $\mathcal{X}$ we denote the set of all pairs $\mathbf{f} = (f_1, f_2)$, where $f_1 : \overline{S} \to \mathbb{C}$ and $f_2 : \overline{S_0^+} \to \mathbb{C}$ are bounded functions, $f_1 \in C(\overline{S_0^-}; \mathbb{C}) \cap C([0, R] \times [-\ell/2, \ell/2]; \mathbb{C}) \cap C(\overline{S_R^+}; \mathbb{C})$, $f_2 \in C([0, R] \times [-\ell/2, \ell/2]; \mathbb{C}) \cap C(\overline{S_R^+}; \mathbb{C})$ and $\lim_{x \to \pm\infty} f_1(x, y) = \lim_{x \to +\infty} f_2(x, y) = 0$ for each $y \in [-\ell/2, \ell/2]$. It is endowed with the sup-norm, i.e., $\|\mathbf{f}\|_\infty = \|f_1\|_{L^\infty(S; \mathbb{C})} + \|f_2\|_{L^\infty(S_0^+; \mathbb{C})}$.

For each $\alpha \in (0, 1]$, $\mathcal{X}_\alpha$ denotes the subset of $\mathcal{X}$ of all $\mathbf{f}$ such that (i) $f_1 \in C_b^\alpha(\overline{S_0^-}; \mathbb{C}) \cap C_b^\alpha([0, R] \times [-\ell/2, \ell/2]; \mathbb{C}) \cap C_b^\alpha(\overline{S_R^+}; \mathbb{C})$, (ii) $f_2 \in C_b^\alpha([0, R] \times [-\ell/2, \ell/2]; \mathbb{C}) \cap C_b^\alpha(\overline{S_R^+}; \mathbb{C})$, (iii) $f_j(\cdot, -\ell/2) = f_j(\cdot, \ell/2)$ (and $\nabla f_j(\cdot, -\ell/2) = \nabla f_j(\cdot, \ell/2)$ if $\alpha = 1$) for $j = 1, 2$. It is endowed with the norm $\|\mathbf{f}\|_\alpha = \|f_1\|_{C_b^\alpha(\overline{S_0^-}; \mathbb{C})} + \sum_{j=1}^2 (\|f_j\|_{C_b^\alpha([0, R] \times [-\ell/2, \ell/2]; \mathbb{C})} + \|f_j\|_{C_b^\alpha(\overline{S_R^+}; \mathbb{C})})$.

For $k \in \mathbb{N}$ and $\alpha \in (0, 1)$, $\mathcal{X}_{k+\alpha}$ ($k \in \mathbb{N}$, $\alpha \in (0, 1)$) denotes the set of all $\mathbf{f} \in \mathcal{X}$ such that $D^\beta \mathbf{f} = (D^\gamma f_1, D^\gamma f_2) \in \mathcal{X}$, $D^\gamma f_j(\cdot, -\ell/2) = D^\gamma f_j(\cdot, \ell/2)$ for each $|\gamma| \leq k$, $j = 1, 2$, and $D^\gamma \mathbf{f} \in \mathcal{X}_\alpha$ for $|\gamma| = k$. It is endowed with the norm $\|\mathbf{f}\|_{k+\alpha} := \sum_{|\gamma| < k} \|D^\gamma \mathbf{f}\|_\infty + \sum_{|\gamma| = k} \|D^\gamma \mathbf{f}\|_\alpha$.

The spaces $\mathcal{Y}_\alpha(a, b)$ and $\mathcal{Y}_{2+\alpha}(a, b)$ (Section 5). For $\alpha \in (0, 1)$ and $0 \leq a < b$, we define by $\mathcal{Y}_\alpha(a, b)$ the space of all pairs $\mathbf{f} = (f_1, f_2)$ such that $f_1 : [a, b] \times \overline{S} \to \mathbb{R}$, $f_2 : [a, b] \times \overline{S_0^+} \to \mathbb{R}$ and

$$\|\mathbf{f}\|_{\mathcal{Y}_\alpha(a, b)} = \sup_{a < t < b} \|\mathbf{f}(t, \cdot, \cdot)\|_\alpha + \sup_{(x, y) \in S} \|f_1(\cdot, x, y)\|_{C^{\alpha/2}((a, b))} + \sup_{(x, y) \in S_0^+} \|f_2(\cdot, x, y)\|_{C^{\alpha/2}((a, b))} < +\infty.$$

Similarly, $\mathcal{Y}_{2+\alpha}(a, b)$ denotes the space of all the pairs $\mathbf{u}$ such that $D_t^{\gamma_1} D_x^{\gamma_2} D_y^{\gamma_3} \mathbf{u}$ belongs to $\mathcal{Y}_\alpha(a, b)$ for every $\gamma_1, \gamma_2, \gamma_3 \geq 0$ such that $2\gamma_1 + \gamma_2 + \gamma_3 \leq 2$. These are Banach spaces with the norms $\| \cdot \|_{\mathcal{Y}_\alpha(a, b)}$ and $\|\mathbf{u}\|_{\mathcal{Y}_{2+\alpha}(a, b)} = \sum_{2\gamma_1 + \gamma_2 + \gamma_3 \leq 2} \|D_t^{\gamma_1} D_x^{\gamma_2} D_y^{\gamma_3} \mathbf{u}\|_{\mathcal{Y}_\alpha(a, b)}$.

If $a = 0$ and $b = T$ then we simply write $\mathcal{Y}_\alpha$ and $\mathcal{Y}_{2+\alpha}$ instead of $\mathcal{Y}_\alpha(0, T)$ and $\mathcal{Y}_{2+\alpha}(0, T)$, respectively.

3. Derivation of the fully nonlinear problem

3.1. **The system on a fixed domain.** To begin with, we rewrite system (1.3) in the coordinates $t' = t$, $x' = x - t$, $y' = y$, $D_t = D_{t'} - D_{x'}$. Next, we look for the free interfaces respectively as:

$$G(t', y') = g(t', y'), \qquad F(t', y') = R + f(t', y'),$$

where f and g are small perturbations. In other words, the space variable x' varies from $-\infty$ to $g(t', y')$, from $g(t', y')$ to $R + f(t', y')$, and eventually from $R + f(t', y')$ to $+\infty$. As usual, it is convenient to transform the problem on a variable domain to a problem on a fixed domain. To this end, we define a coordinate transformation in the spirit of [9, Section 2.1], see also, in dimension one, [20, Section 5],

$$t' = \tau, \qquad x' = \xi + \beta(\xi)g(\tau, \eta) + \beta(\xi - R)f(\tau, \eta), \qquad y' = \eta,$$

where β is a smooth mollifier, equal to unity in a small neighborhood of $\xi = 0$, say $[-\delta, \delta]$, and has compact support contained in $(-2\delta, 2\delta) \subset (-R, R)$. When $x' = g$, $\xi = 0$, and $\xi = R$ when $x' = R + f$. Then, the trailing front and the ignition front are fixed at $\xi = 0$ and $\xi = R$, respectively. Thanks to the translation invariance, (1.6) holds with the variable ξ. For convenience, we introduce the notation

$$\varrho(\tau, \xi, \eta) = \beta(\xi)g(\tau, \eta) + \beta(\xi - R)f(\tau, \eta) \tag{3.1}$$

and we expand $(1 + \varrho_\xi)^{-1} = 1 - \varrho_\xi + (\varrho_\xi)^2(1 + \varrho_\xi)^{-1}$. It turns out that

$$D_{x'} = D_\xi - \varrho_\xi D_\xi + (\varrho_\xi)^2(1 + \varrho_\xi)^{-1}D_\xi,$$

$$D_{t'} = D_\tau - \varrho_\tau D_\xi + \varrho_\tau \varrho_\xi D_\xi - \varrho_\tau(\varrho_\xi)^2(1 + \varrho_\xi)^{-1}D_\xi,$$

$$D_{y'} = D_\eta - \varrho_\eta D_\xi + \varrho_\eta \varrho_\xi D_\xi - \varrho_\eta(\varrho_\xi)^2(1 + \varrho_\xi)^{-1}D_\xi$$

and system (1.3) reads:

$$\begin{aligned}
\Theta_\tau =&\, \Theta_\xi + \Delta\Theta + (\varrho_\eta^2 - \varrho_\xi^2 - 2\varrho_\xi)(1 + \varrho_\xi)^{-2}\Theta_{\xi\xi} - 2\varrho_\eta(1 + \varrho_\xi)^{-1}\Theta_{\xi\eta} \\
&+ \big[(\varrho_\tau - \varrho_\xi - \varrho_{\eta\eta})(1 + \varrho_\xi)^{-1} + 2\varrho_\eta\varrho_{\xi\eta}(1 + \varrho_\xi)^{-2} - \varrho_{\xi\xi}(1 + \varrho_\eta^2)(1 + \varrho_\xi)^{-3}\big]\Theta_\xi, \\
\Phi =&\, 0
\end{aligned} \tag{3.2}$$

in $(0, +\infty) \times (-\infty, 0) \times (-\ell/2, \ell/2)$,

$$\begin{aligned}
\Theta_\tau =&\, \Theta_\xi + \Delta\Theta + (\varrho_\eta^2 - \varrho_\xi^2 - 2\varrho_\xi)(1 + \varrho_\xi)^{-2}\Theta_{\xi\xi} - 2\varrho_\eta(1 + \varrho_\xi)^{-1}\Theta_{\xi\eta} + R^{-1} \\
&+ \big[(\varrho_\tau - \varrho_\xi - \varrho_{\eta\eta})(1 + \varrho_\xi)^{-1} + 2\varrho_\eta\varrho_{\xi\eta}(1 + \varrho_\xi)^{-2} - \varrho_{\xi\xi}(1 + \varrho_\eta^2)(1 + \varrho_\xi)^{-3}\big]\Theta_\xi, \\
\Phi_\tau =&\, \Phi_\xi + \mathrm{Le}^{-1}\Delta\Phi + \mathrm{Le}^{-1}(\varrho_\eta^2 - \varrho_\xi^2 - 2\varrho_\xi)(1 + \varrho_\xi)^{-2}\Phi_{\xi\xi} - 2\mathrm{Le}^{-1}\varrho_\eta(1 + \varrho_\xi)^{-1}\Phi_{\xi\eta} - R^{-1} \\
&+ \big[(\varrho_\tau - \varrho_\xi - \mathrm{Le}^{-1}\varrho_{\eta\eta})(1 + \varrho_\xi)^{-1} + 2\mathrm{Le}^{-1}\varrho_\eta\varrho_{\xi\eta}(1 + \varrho_\xi)^{-2} - \mathrm{Le}^{-1}\varrho_{\xi\xi}(1 + \varrho_\eta^2)(1 + \varrho_\xi)^{-3}\big]\Phi_\xi
\end{aligned} \tag{3.3}$$

in $(0, +\infty) \times (0, R) \times (-\ell/2, \ell/2)$ and

$$\begin{aligned}
\Theta_\tau =&\, \Theta_\xi + \Delta\Theta + (\varrho_\eta^2 - \varrho_\xi^2 - 2\varrho_\xi)(1 + \varrho_\xi)^{-2}\Theta_{\xi\xi} - 2\varrho_\eta(1 + \varrho_\xi)^{-1}\Theta_{\xi\eta} \\
&+ \big[(\varrho_\tau - \varrho_\xi - \varrho_{\eta\eta})(1 + \varrho_\xi)^{-1} + 2\varrho_\eta\varrho_{\xi\eta}(1 + \varrho_\xi)^{-2} - \varrho_{\xi\xi}(1 + \varrho_\eta^2)(1 + \varrho_\xi)^{-3}\big]\Theta_\xi, \\
\Phi_\tau =&\, \Phi_\xi + \mathrm{Le}^{-1}\Delta\Phi + \mathrm{Le}^{-1}(\varrho_\eta^2 - \varrho_\xi^2 - 2\varrho_\xi)(1 + \varrho_\xi)^{-2}\Phi_{\xi\xi} - 2\mathrm{Le}^{-1}\varrho_\eta(1 + \varrho_\xi)^{-1}\Phi_{\xi\eta} \\
&+ \big[(\varrho_\tau - \varrho_\xi - \mathrm{Le}^{-1}\varrho_{\eta\eta})(1 + \varrho_\xi)^{-1} + 2\mathrm{Le}^{-1}\varrho_\eta\varrho_{\xi\eta}(1 + \varrho_\xi)^{-2} - \mathrm{Le}^{-1}\varrho_{\xi\xi}(1 + \varrho_\eta^2)(1 + \varrho_\xi)^{-3}\big]\Phi_\xi
\end{aligned} \tag{3.4}$$

in $(0, +\infty) \times (R, +\infty) \times (-\ell/2, \ell/2)$. Moreover, Θ and Φ are continuous at the (fixed) interfaces $\xi = 0$ and $\xi = R$, and so are their first-order derivatives. Thus,

$$[\Theta(\tau, \cdot, \eta)]_0 = [\Theta_\xi(\tau, \cdot, \eta)]_0 = \Phi(\tau, 0, \eta) = \Phi_\xi(\tau, 0, \eta) = 0$$

and

$$\Theta(\tau, R, \eta) = \theta_i, \quad [\Theta(\tau, \cdot, \eta)]_R = [\Theta_\xi(\tau, \cdot, \eta)]_R = [\Phi(\tau, \cdot, \eta)] = [\Phi_\xi(\tau, \cdot, \eta)] = 0.$$

Conditions (1.4) hold at $\xi = \pm\infty$ and periodic boundary conditions are assumed at $\eta = \pm\ell/2$.

3.2. **Elimination of the interfaces.** From now on, with a slight abuse of notation, which does not cause confusion, we write t instead of τ and x, y instead of ξ and η. So, we warn the reader that, in the sections which follow, x and y are not the same as in the introduction.

In the spirit of [9, 31], we introduce the splitting:

$$\Theta(t, x, y) = \Theta^{(0)}(x) + \varrho(t, x, y)\Theta_x^{(0)}(x) + u(t, x, y), \tag{3.5}$$

$$\Phi(t, x, y) = \Phi^{(0)}(x) + \varrho(t, x, y)\Phi_x^{(0)}(x) + v(t, x, y), \tag{3.6}$$

which is a sort of Taylor expansion of (Θ, Φ) around the traveling wave solution $(\Theta^{(0)}, \Phi^{(0)})$. Thus, the pair (u, v) plays the role of a remainder and, since we are interested in stability issues, we can assume that u and v are "sufficiently small" in a sense which will be made precise later on.

A long but straightforward computation reveals that the pair (u, v) satisfies the differential equations

$$\begin{aligned}
u_t =& u_x + \Delta u + \varrho_t(1 + \varrho_x)^{-1}(\varrho\Theta_{xx}^{(0)} + u_x) - (1 + \varrho_x)^{-3}\varrho_{xx}(1 + \varrho_y^2)(\varrho\Theta_{xx}^{(0)} + u_x) \\
&- (1 + \varrho_x)^{-1}\big[(\varrho_x + \varrho_{yy})(\varrho\Theta_{xx}^{(0)} + u_x) + 2\varrho_y(\varrho_y\Theta_{xx}^{(0)} + u_{xy})\big] \\
&+ (1 + \varrho_x)^{-2}\big[2\varrho_y\varrho_{xy}(\varrho\Theta_{xx}^{(0)} + u_x) + (\varrho_y^2 - \varrho_x^2)(\varrho\Theta_{xxx}^{(0)} + \Theta_{xx}^{(0)} + u_{xx}) \\
&\qquad\qquad - 2\varrho_x(\varrho\Theta_{xxx}^{(0)} - \varrho_y^2\Theta_{xx}^{(0)} + u_{xx})\big],
\end{aligned} \tag{3.7}$$

in $(0, +\infty) \times (\mathbb{R} \setminus \{0, R\}) \times (-\ell/2, \ell/2)$ and

$$\begin{aligned}
v_t =& v_x + \mathrm{Le}^{-1}\Delta v + \varrho_t(1 + \varrho_x)^{-1}(\varrho\Phi_{xx}^{(0)} + v_x) - \mathrm{Le}^{-1}(1 + \varrho_x)^{-3}\varrho_{xx}(1 + \varrho_y^2)(\varrho\Phi_{xx}^{(0)} + v_x) \\
&- \mathrm{Le}^{-1}(1 + \varrho_x)^{-1}\big[(\mathrm{Le}\,\varrho_x + \varrho_{yy})(\varrho\Phi_{xx}^{(0)} + v_x) + 2\varrho_y(\varrho_y\Phi_{xx}^{(0)} + v_{xy})\big] \\
&+ \mathrm{Le}^{-1}(1 + \varrho_x)^{-2}\big[2\varrho_y\varrho_{xy}(\varrho\Phi_{xx}^{(0)} + v_x) + (\varrho_y^2 - \varrho_x^2)(\varrho\Phi_{xxx}^{(0)} + \Phi_{xx}^{(0)} + v_{xx}) \\
&\qquad\qquad - 2\varrho_x(\varrho\Phi_{xxx}^{(0)} - \varrho_y^2\Phi_{xx}^{(0)} + v_{xx})\big]
\end{aligned} \tag{3.8}$$

in $(0, +\infty) \times [(0, R) \cup (R, +\infty)] \times (-\ell/2, \ell/2)$ and $v = 0$ in $(0, +\infty) \times (-\infty, 0) \times (-\ell/2, \ell/2)$.

Two steps are still needed:
(a) we have to determine the jump conditions satisfied by u and v;
(b) again in the spirit of [9, 31], we have to get rid of the function ϱ from the right-hand sides of (3.7) and (3.8). As we will see, some difficulties appear and, to overcome them, we will differentiate the differential equation (3.8).

3.2.1. **The ignition interface $x = R$.** Note that $\Theta^{(0)}$, $\Phi^{(0)}$ belong to $C^1(\mathbb{R})$. Thus, $[u]_R = [v]_R = 0$. Moreover, $\Theta_x^{(0)}(R) = -\theta_i$ and $\Phi_x^{(0)}(R) = (1 - \exp(-\mathrm{Le}R))/R$, so that they do not vanish at the interface $x = R$. The latter is a kind of transversality or non-degeneracy condition (see [9]). Evaluating (3.5) at $x = R$, we get $u(R) = \theta_i f$.

Next, differentiating (3.5) and (3.6) for $x \neq R$, and taking the jumps across $x = R$, it is not difficult to show that $[u_x]_R = -R^{-1}f$ and $[v_x]_R = R^{-1}\mathrm{Le}\,f$. This is a key point since we are able to express f in terms of u and write

$$f(t, y) = \theta_i^{-1}u(t, R, y). \tag{3.9}$$

Summing up, the interface conditions at $x = R$ are the following:

$$[u]_R = [v]_R = 0, \quad u(R) + \theta_i R[u_x]_R = 0, \qquad \mathrm{Le}[u_x]_R + [v_x]_R = 0. \tag{3.10}$$

3.2.2. **The trailing interface $x = 0$.** Taking the jump at $x = 0$ of both sides of (3.5) and (3.6), we get the conditions $[u]_0 = v(0) = 0$ for u and v. The trailing interface has a different nature with respect to the ignition interface. Indeed, since $\Theta_x^{(0)}(0) = \Phi_x^{(0)}(0) = 0$, the non-degeneracy condition of [9] is not verified and we are able to express g in terms neither of u or v. On the other hand, $\Theta_{xx}^{(0)}(0^+) = -R^{-1}$ and $\Phi_{xx}^{(0)}(0^+) = R^{-1}\mathrm{Le}$, so that they do not vanish. Differentiating (3.5) and (3.6)

for $x \neq 0$, and taking the jumps yields: $[u_x]_0 = R^{-1}g$, $v_x(\cdot, 0^+, \cdot) = -R^{-1}g\mathrm{Le}$. Hence, we get the additional interface condition $\mathrm{Le}\,[u_x]_0 + v_x(\cdot, 0^+, \cdot) = 0$, so that the interface conditions at $x = 0$ are

$$[u]_0 = v(0) = 0, \quad \mathrm{Le}\,[u_x]_0 + v_x(\cdot, 0^+, \cdot) = 0. \tag{3.11}$$

We can also write

$$g(t, y) = -R\,\mathrm{Le}^{-1}v_x(t, 0^+, y). \tag{3.12}$$

Although the front g could be eliminated, the method of [9] is not applicable since, in contrast to (3.9), g is related to the derivative of v in the equation (3.12).

3.3. **Differentiation and new interface conditions.** To overcome the difficulty pointed out above, the trick is to differentiate (3.8) with respect to x, taking advantage of the structure of the system and consider the problem satisfied by the pair (u, v_x). From (3.10) and (3.11) we get the following interface conditions for u and $w = v_x$:

$$\begin{aligned}
&[u]_0 = 0, &&\mathrm{Le}\,[u_x]_0 + w(0^+) = 0, \\
&[u]_R = 0, &&u(R) + \theta_i R[u_x]_R = 0, &&\mathrm{Le}[u_x]_R + [w]_R = 0.
\end{aligned} \tag{3.13}$$

We missed two jump conditions: one at the trailing interface and the other one at the ignition interface. To obtain the additional condition at the trailing interface $x = 0$, we differentiate (3.6) twice in a neighborhood of $x = 0$ and take the trace at $x = 0^+$. Using (3.12), we get

$$\Phi_{xx}(\cdot, 0^+, \cdot) = \mathrm{Le}\,R^{-1} + \mathrm{Le}\,w(\cdot, 0^+, \cdot) + w_x(\cdot, 0^+, \cdot). \tag{3.14}$$

To get rid of $\Phi_{xx}(\cdot, 0^+, \cdot)$ from the left-hand side of (3.14), we observe that, for $x > 0$ sufficiently small, the second equation in (3.3) reduces to

$$\Phi_t = \Phi_x + \mathrm{Le}^{-1}\Delta\Phi + \mathrm{Le}^{-1}g_y^2\Phi_{xx} - 2\mathrm{Le}^{-1}g_y\Phi_{xy} - R^{-1} + (g_t - \mathrm{Le}^{-1}g_{yy})\Phi_x. \tag{3.15}$$

Taking the trace of (3.15) at $x = 0^+$ it is easy to check that $\Phi_{xx}(\cdot, 0^+, \cdot)(1 + g_y^2) = \mathrm{Le}\,R^{-1}$. Hence, using (3.12) and (3.14) we get the additional interface condition

$$\mathrm{Le}\,w(\cdot, 0^+, \cdot) + w_x(\cdot, 0^+, \cdot) = \mathrm{Le}\,R^{-1}\{[1 + R^2\mathrm{Le}^{-2}(w_y(\cdot, 0^+, \cdot))^2]^{-1} - 1\}. \tag{3.16}$$

We likewise identify the additional interface condition at the ignition interface $x = R$. Differentiating twice (3.6) in a neighborhood of $x = R$, taking the jump at $x = R$ and using (3.9), (3.10) gives

$$[\Phi_{xx}]_R = -R^{-1}\mathrm{Le} + \mathrm{Le}\,[w]_R + [w_x]_R. \tag{3.17}$$

We need to compute $[\Phi_{xx}]_R$: in a neighborhood of R^-, the second equation in (3.3) yields

$$\Phi_t = \Phi_x + \mathrm{Le}^{-1}\Delta\Phi + \mathrm{Le}^{-1}(f_y)^2\Phi_{xx} - 2\mathrm{Le}^{-1}f_y\Phi_{xy} - R^{-1} + (f_t - \mathrm{Le}^{-1}f_{yy})\Phi_x,$$

while in a neighborhood of R^+ from the second equation in (3.4) we get

$$\Phi_t = \Phi_x + \mathrm{Le}^{-1}\Delta\Phi + \mathrm{Le}^{-1}(f_y)^2\Phi_{xx} - 2\mathrm{Le}^{-1}f_y\Phi_{xy} + (f_t - \mathrm{Le}^{-1}f_{yy})\Phi_x.$$

Using the previous two equations it can be easily shown that $[\Phi_{xx}]_R(1 + (f_y)^2) = -R^{-1}\mathrm{Le}$, which, together with (3.9) and (3.17), gives

$$\mathrm{Le}[w]_R + [w_x]_R = -\mathrm{Le}\,R^{-1}\{[1 + \theta_i^{-2}(u_{\widetilde{y}}(\cdot, R, \cdot))^2]^{-1} - 1\}. \tag{3.18}$$

This is the additional condition we were looking for.

3.4. **Elimination of ϱ and its time and spatial derivatives.** Formulae (3.9) (3.12) enable the elimination of the fronts f and g from the differential equations satisfied by u and w. First, they allow to write the following formula for ϱ (see (3.1)):

$$\varrho(t,x,y) = \theta_i^{-1}\beta(x-R)u(t,R,y) - R\,\mathrm{Le}^{-1}\beta(x)w(t,0^+,y). \tag{3.19}$$

Differentiation of (3.19) with respect to x and y is benign. The right-hand sides of (3.7) and (3.8) depend also on ϱ_t. Hence, we need to compute such a derivative and express it in terms of (traces of) spatial derivatives of u and w. Since

$$\varrho_t(t,x,y) = \theta_i^{-1}\beta(x-R)u_t(t,R,y) - R\,\mathrm{Le}^{-1}\beta(x)w_t(t,0^+,y), \tag{3.20}$$

we need to get rid of $u_t(t,R,y)$ and $w_t(t,0^+,y)$. For simplicity, we forget the arguments t and y. We evaluate (3.7) at $x = R^+$ (it would be equivalent at $x = R^-$). Recalling that all the derivatives of ϱ with respect of x vanish and taking (3.19) into account, we get

$$u_t(R) = u_x(R^+) + \Delta u(R^+) + \theta_i^{-1}u_t(R)(u(R) + u_x(R^+)) - \theta_i^{-1}u_{yy}(R)u_x(R^+)$$
$$- 2\theta_i^{-1}u_{\widetilde{y}}(R)u_{xy}(R^+) + \theta_i^{-2}(u_y(R))^2(u_{xx}(R^+) - u(R) - \theta_i) - \theta_i^{-1}u(R)u_{yy}(R).$$

Since θ_i is fixed, assuming that the perturbations are small we may invert and write

$$u_t(R) = [1-\theta_i^{-1}(u(R)+u_x(R^+))]^{-1}[u_x(R^+) + \Delta u(R^+) - \theta_i^{-1}u_{yy}(R)u_x(R^+) - \theta_i^{-1}u(R)u_{yy}(R)$$
$$- 2\theta_i^{-1}u_{\widetilde{y}}(R)u_{xy}(R^+) + \theta_i^{-2}(u_y(R))^2(u_{xx}(R^+) - u(R) - \theta_i)]. \tag{3.21}$$

Similarly, differentiating and evaluating (3.8) at $x = 0^+$ we get

$$w_t(0^+) = w_x(0^+) + \mathrm{Le}^{-1}\Delta w(0^+) - R\mathrm{Le}^{-1}w_t(0^+)(\mathrm{Le}\,w(0^+) + w_x(0^+))$$
$$+ R\mathrm{Le}^{-2}\big[w_{yy}(0^+)(\mathrm{Le}\,w(0^+) + w_x(0^+)) + 2w_y(0^+)w_{xy}(0^+)\big]$$
$$+ R^2\mathrm{Le}^{-3}(w_y(0^+))^2(-\mathrm{Le}^2 w(0^+) + R^{-1}\mathrm{Le}^2 + w_{xx}(0^+)),$$

so that

$$w_t(0^+) = \{\mathrm{Le}\,w_x(0^+) + \Delta w(0^+) + R\mathrm{Le}^{-1}\big[w_{yy}(0^+)(\mathrm{Le}\,w(0^+) + w_x(0^+)) + 2w_y(0^+)w_{xy}(0^+)\big]$$
$$+ R^2\mathrm{Le}^{-2}(w_y(0^+))^2(-\mathrm{Le}^2 w(0^+) + R^{-1}\mathrm{Le}^2 + w_{xx}(0^+))\}$$
$$\times [\mathrm{Le} + R(\mathrm{Le}\,w(0^+) + w_x(0^+))]^{-1}. \tag{3.22}$$

A related remark is that equation (3.20) for ϱ_t, together with formulae (3.21)-(3.22), may be viewed as a *second-order Stefan condition*, see [10].

3.5. **The final system.** Using (3.7), (3.8), (3.13), (3.16), (3.18)-(3.22), we can write the final problem for $\mathbf{u} = (u,w)$, which is *fully nonlinear* since the nonlinear part of the differential equations contains traces at $x = 0^+$ and R of (first- and) second-order derivatives of the unknown $\mathbf{u}$ itself.

Summing up, the pair $\mathbf{u} = (u,w)$ solves the nonlinear system

$$\begin{cases} D_t\mathbf{u}(t,\cdot,\cdot) = \mathscr{L}\mathbf{u}(t,\cdot,\cdot) + \mathscr{F}(\mathbf{u}(t,\cdot,\cdot)), & t \geq 0, \\ \mathscr{B}(\mathbf{u}(t,\cdot)) = \mathscr{H}(\mathbf{u}(t,\cdot)), & t \geq 0, \end{cases} \tag{3.23}$$

and satisfies periodic boundary conditions at $y = \pm\ell/2$, where

$$\mathscr{L}\mathbf{v} = (\Delta\zeta + \zeta_x, \mathrm{Le}^{-1}\Delta\upsilon + \upsilon_x), \tag{3.24}$$

$$\mathscr{B}\mathbf{v} = \begin{pmatrix} \zeta(0^+,\cdot) - \zeta(0^-,\cdot) \\ \zeta(R^+,\cdot) - \zeta(R^-,\cdot) \\ \mathrm{Le}[\zeta_x(0^+,\cdot) - \zeta_x(0^-,\cdot)] + \upsilon(0^+,\cdot) \\ \mathrm{Le}\,\upsilon(0^+,\cdot) + \upsilon_x(0^+,\cdot) \\ \zeta(R^+,\cdot) + \theta_i R[\zeta_x(R^+,\cdot) - \zeta_x(R^-,\cdot)] \\ \mathrm{Le}[\zeta_x(R^+,\cdot) - \zeta_x(R^-,\cdot)] + \upsilon(R^+,\cdot) - \upsilon(R^-,\cdot) \\ \mathrm{Le}[\upsilon(R^+,\cdot) - \upsilon(R^-,\cdot)] + \upsilon_x(R^+,\cdot) - \upsilon_x(R^-,\cdot) \end{pmatrix}, \tag{3.25}$$

$$\mathscr{F}(\mathbf{v}) = (\Lambda(\mathbf{v})\mathscr{F}_1(\mathbf{v}) - \mathscr{F}_2(\mathbf{v}), \Lambda(\mathbf{v})D_x\mathscr{G}_1(\mathbf{v}) + D_x\Lambda(\mathbf{v})\mathscr{G}_1(\mathbf{v}) - \mathrm{Le}^{-1}D_x\mathscr{G}_2(\mathbf{v})),$$

$$\mathscr{F}_1(\mathbf{v}) = \frac{\theta_i^{-1}\beta_R\Theta_{xx}^{(0)}\zeta(R^+, \cdot) - R\mathrm{Le}^{-1}\beta\Theta_{xx}^{(0)}\upsilon(0, \cdot) + \zeta_x}{1 + \theta_i^{-1}\beta'(\cdot - R)\zeta(R^+, \cdot) - R\mathrm{Le}^{-1}\beta'\upsilon(0, \cdot)},$$

$$
\begin{aligned}
\mathscr{F}_2(\mathbf{v}) = &\{\left(\theta_i^{-1}\beta'(\cdot - R)\zeta(R^+, \cdot) - R\mathrm{Le}^{-1}\beta'\upsilon(0, \cdot) + \theta_i^{-1}\beta_R\zeta_{yy}(R^+, \cdot) - R\mathrm{Le}^{-1}\beta\upsilon_{yy}(0, \cdot)\right) \\
&\qquad \times (\theta_i^{-1}\beta_R\Theta_{xx}^{(0)}\zeta(R^+, \cdot) - R\mathrm{Le}^{-1}\beta\Theta_{xx}^{(0)}\upsilon(0, \cdot) + \zeta_x) \\
&\quad + 2(\theta_i^{-1}\beta_R\zeta_y(R^+, \cdot) - R\mathrm{Le}^{-1}\beta\upsilon_y(0, \cdot))^2\Theta_{xx}^{(0)} \\
&\quad + 2(\theta_i^{-1}\beta_R\zeta_y(R^+, \cdot) - R\mathrm{Le}^{-1}\beta\upsilon_y(0, \cdot))\zeta_{xy}\}(1 + \theta_i^{-1}\beta'(\cdot - R)\zeta(R^+, \cdot) - R\mathrm{Le}^{-1}\beta'\upsilon(0, \cdot))^{-1} \\
&- \big\{2\left(\theta_i^{-1}\beta_R\zeta_y(R^+, \cdot) - R\mathrm{Le}^{-1}\beta\upsilon_y(0, \cdot)\right)\left(\theta_i^{-1}\beta'(\cdot - R)\zeta_y(R^+, \cdot) - R\mathrm{Le}^{-1}\beta'\upsilon_y(0, \cdot)\right) \\
&\qquad \times \left(\theta_i^{-1}\beta_R\Theta_{xx}^{(0)}\zeta(R^+, \cdot) - R\mathrm{Le}^{-1}\beta\Theta_{xx}^{(0)}\upsilon(0, \cdot) + \zeta_x\right) \\
&\quad + \big[\left(\theta_i^{-1}\beta_R\zeta_y(R^+, \cdot) - R\mathrm{Le}^{-1}\beta\upsilon_y(0, \cdot)\right)^2 - \left(\theta_i^{-1}\beta'(\cdot - R)\zeta(R^+, \cdot) - R\mathrm{Le}^{-1}\beta'\upsilon(0, \cdot)\right)^2\big] \\
&\qquad \times \left(\theta_i^{-1}\beta_R\Theta_{xxx}^{(0)}\zeta(R^+, \cdot) - R\mathrm{Le}^{-1}\beta\Theta_{xxx}^{(0)}\upsilon(0, \cdot) + \Theta_{xx}^{(0)} + \zeta_{xx}\right) \\
&\quad - 2\left(\theta_i^{-1}\beta'(\cdot - R)\zeta(R^+, \cdot) - R\mathrm{Le}^{-1}\beta'\upsilon(0, \cdot)\right) \\
&\qquad \times \big[\theta_i^{-1}\beta_R\Theta_{xxx}^{(0)}\zeta(R^+, \cdot) - R\mathrm{Le}^{-1}\beta\Theta_{xxx}^{(0)}\upsilon(0, \cdot) \\
&\qquad\quad - \left(\theta_i^{-1}\beta_R\zeta_y(R^+, \cdot) - R\mathrm{Le}^{-1}\beta\upsilon_y(0, \cdot)\right)^2\Theta_{xx}^{(0)} + \zeta_{xx}\big]\big\} \\
&\qquad \times (1 + \theta_i^{-1}\beta'(\cdot - R)\zeta(R^+, \cdot) - R\mathrm{Le}^{-1}\beta'\upsilon(0, \cdot))^{-2} \\
&+ \big\{\left(\theta_i^{-1}\beta''(\cdot - R)\zeta(R^+, \cdot) - R\mathrm{Le}^{-1}\beta''\upsilon(0, \cdot)\right)\big[1 + \left(\theta_i^{-1}\beta_R\zeta_y(R^+, \cdot) - R\mathrm{Le}^{-1}\beta\upsilon_y(0, \cdot)\right)^2\big] \\
&\qquad \times (\theta_i^{-1}\beta_R\Theta_{xx}^{(0)}\zeta(R^+, \cdot) - R\mathrm{Le}^{-1}\beta\Theta_{xx}^{(0)}\upsilon(0, \cdot) + \zeta_x)\big\} \\
&\qquad \times (1 + \theta_i^{-1}\beta'(\cdot - R)\zeta(R^+, \cdot) - R\mathrm{Le}^{-1}\beta'\upsilon(0, \cdot))^{-3},
\end{aligned}
$$

$$\mathscr{G}_1(\mathbf{v}) = \frac{\theta_i^{-1}\beta_R\Phi_{xx}^{(0)}\zeta(R^+, \cdot) - R\mathrm{Le}^{-1}\beta\Phi_{xx}^{(0)}\upsilon(0, \cdot) + \upsilon}{1 + \theta_i^{-1}\beta'(\cdot - R)\zeta(R^+, \cdot) - R\mathrm{Le}^{-1}\beta'\upsilon(0, \cdot)},$$

$$
\begin{aligned}
\mathscr{G}_2(\mathbf{v}) = &\{[\mathrm{Le}\,\theta_i^{-1}\beta'(\cdot - R)\zeta(R^+, \cdot) - R\beta'\upsilon(0, \cdot) + \theta_i^{-1}\beta_R\zeta_{yy}(R^+, \cdot) - R\mathrm{Le}^{-1}\beta\upsilon_{yy}(0, \cdot)] \\
&\qquad \times [\theta_i^{-1}\beta_R\Phi_{xx}^{(0)}\zeta(R^+, \cdot) - R\mathrm{Le}^{-1}\beta\Phi_{xx}^{(0)}\upsilon(0, \cdot) + \upsilon] \\
&\quad + 2(\theta_i^{-1}\beta_R\zeta_y(R^+, \cdot) - R\mathrm{Le}^{-1}\beta\upsilon_y(0, \cdot))^2\Phi_{xx}^{(0)} \\
&\quad + 2(\theta_i^{-1}\beta_R\zeta_y(R^+, \cdot) - R\mathrm{Le}^{-1}\beta\upsilon_y(0, \cdot))\upsilon_y\}(1 + \theta_i^{-1}\beta'(\cdot - R)\zeta(R^+, \cdot) - R\mathrm{Le}^{-1}\beta'\upsilon(0, \cdot))^{-1} \\
&- \big\{2\left(\theta_i^{-1}\beta_R\zeta_y(R^+, \cdot) - R\mathrm{Le}^{-1}\beta\upsilon_y(0, \cdot)\right)\left(\theta_i^{-1}\beta'(\cdot - R)\zeta_y(R^+, \cdot) - R\mathrm{Le}^{-1}\beta'\upsilon_y(0, \cdot)\right) \\
&\qquad \times \left(\theta_i^{-1}\beta_R\Phi_{xx}^{(0)}\zeta(R^+, \cdot) - R\mathrm{Le}^{-1}\beta\Phi_{xx}^{(0)}\upsilon(0, \cdot) + \upsilon\right) \\
&\quad + \big[\left(\theta_i^{-1}\beta_R\zeta_y(R^+, \cdot) - R\mathrm{Le}^{-1}\beta\upsilon_y(0, \cdot)\right)^2 - \left(\theta_i^{-1}\beta'(\cdot - R)\zeta(R^+, \cdot) - R\mathrm{Le}^{-1}\beta'\upsilon(0, \cdot)\right)^2\big] \\
&\qquad \times \left(\theta_i^{-1}\beta_R\Phi_{xxx}^{(0)}\zeta(R^+, \cdot) - R\mathrm{Le}^{-1}\beta\Phi_{xxx}^{(0)}\upsilon(0, \cdot) + \Phi_{xx}^{(0)} + \upsilon_x\right) \\
&\quad - 2\left(\theta_i^{-1}\beta'(\cdot - R)\zeta(R^+, \cdot) - R\mathrm{Le}^{-1}\beta'\upsilon(0, \cdot)\right) \\
&\qquad \times \big[\theta_i^{-1}\beta_R\Phi_{xxx}^{(0)}\zeta(R^+, \cdot) - R\mathrm{Le}^{-1}\beta\Phi_{xxx}^{(0)}\upsilon(0, \cdot) \\
&\qquad\quad - \left(\theta_i^{-1}\beta_R\zeta_y(R^+, \cdot) - R\mathrm{Le}^{-1}\beta\upsilon_y(0, \cdot)\right)^2\Phi_{xx}^{(0)} + \upsilon_x\big]\big\} \\
&\qquad \times (1 + \theta_i^{-1}\beta'(\cdot - R)\zeta(R^+, \cdot) - R\mathrm{Le}^{-1}\beta'\upsilon(0, \cdot))^{-2} \\
&+ \big\{\left(\theta_i^{-1}\beta''(\cdot - R)\zeta(R^+, \cdot) - R\mathrm{Le}^{-1}\beta''\upsilon(0, \cdot)\right)\big[1 + \left(\theta_i^{-1}\beta_R\zeta_y(R^+, \cdot) - R\mathrm{Le}^{-1}\beta\upsilon_y(0, \cdot)\right)^2\big] \\
&\qquad \times (\theta_i^{-1}\beta_R\Phi_{xx}^{(0)}\zeta(R^+, \cdot) - R\mathrm{Le}^{-1}\beta\Phi_{xx}^{(0)}\upsilon(0, \cdot) + \upsilon)\big\} \\
&\qquad \times (1 + \theta_i^{-1}\beta'(\cdot - R)\zeta(R^+, \cdot) - R\mathrm{Le}^{-1}\beta'\upsilon(0, \cdot))^{-3},
\end{aligned}
$$

$$\Lambda(\mathbf{v}) = (\theta_i - \zeta(R^+, \cdot) - \zeta_x(R^+, \cdot))^{-1}$$
$$\times \left[\zeta_x(R^+, \cdot) + \Delta\zeta(R^+, \cdot) - \theta_i^{-1}\zeta_{yy}(R, \cdot)\zeta_x(R^+, \cdot) - 2\theta_i^{-1}\zeta_y(R, \cdot)\zeta_{xy}(R^+, \cdot) \right.$$
$$\left. + \theta_i^{-2}(\zeta_y(R, \cdot))^2(\zeta_{xx}(R^+, \cdot) - \zeta(R, \cdot) - \theta_i) - \theta_i^{-1}\zeta(R, \cdot)\zeta_{yy}(R, \cdot) \right]$$
$$- [\mathrm{Le}^2 + R\mathrm{Le}(\mathrm{Le}\,v(0^+, \cdot) + v_x(0^+, \cdot))]^{-1}$$
$$\times \left\{ \mathrm{Le}\,v_x(0^+, \cdot) + \Delta v(0^+, \cdot) \right.$$
$$+ R\mathrm{Le}^{-1}\left[v_{yy}(0^+, \cdot)(\mathrm{Le}\,v(0^+, \cdot) + v_x(0^+, \cdot)) + 2v_y(0^+, \cdot)v_{xy}(0^+, \cdot) \right]$$
$$\left. + R^2\mathrm{Le}^{-2}(v_y(0^+, \cdot))^2(-\mathrm{Le}^2 v(0^+, \cdot) + R^{-1}\mathrm{Le}^2 + v_{xx}(0^+, \cdot)) \right\},$$

$$\mathcal{H}_j(\mathbf{v}) = 0 \text{ if } j \neq 4, 7, \qquad \mathcal{H}_4(\mathbf{v}) = -\frac{R\mathrm{Le}(v_y(0, \cdot))^2}{\mathrm{Le}^2 + R^2(v_y(0, \cdot))^2}, \qquad \mathcal{H}_7(\mathbf{v}) = \frac{\mathrm{Le}(\zeta_y(R^+, \cdot))^2}{R(\theta_i^2 + (\zeta_y(R^+, \cdot))^2)},$$

on smooth enough functions $\mathbf{v} = (\zeta, v)$, where $\beta_R = \beta(\cdot - R)$.

Remark 3.1. Note that each smooth enough function $\mathbf{u}$, which solves problem (3.23), has its first component u_1 which is continuous on $\{R\} \times [-\ell/2, \ell/2]$. Therefore, the operator $\mathcal{B}$ can be replaced with the operator $\widetilde{\mathcal{B}}$ which is defined as $\mathcal{B}$ with the fifth equation being replaced by the condition $\frac{1}{2}(v_1(R^+, \cdot) + v_1(R^-, \cdot)) + \theta_i R[D_x v_1(R^+, \cdot) - D_x v_1(R^+, \cdot)] = 0$.

We will use the above remark in Subsection 4.3.

4. Tools

In this section we collect some technical results which are used in the next (sub)sections.

4.1. **Preliminary results needed to prove Theorems 4.4 and Proposition 4.5.** We find it convenient to set

$$(\mathcal{F}_{k,\rho}f)(x) := \frac{1}{Z_k} \int_{\mathbb{R}} e^{-\frac{\rho}{2}s} e^{-\frac{1}{2}Z_k|s|} \hat{f}_k(x - s)\,ds, \qquad x \in \mathbb{R}, \ f \in C_b(\overline{S}; \mathbb{C}), \ k \in \mathbb{Z},$$

where $Z_k = Z_k(\lambda, \rho) = \sqrt{\rho^2 + 4\lambda\rho + 4\lambda_k}$ for every $k \in \mathbb{Z}$ and $\lambda_k = (4\ell^2)^{-1}k^2\pi^2$. Moreover, we denote by Σ_0 the set of $\lambda \in \mathbb{C}$ with positive real part.

Lemma 4.1. *For every $\rho \in (0, +\infty)$, $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > -\rho^{-1}(\mathrm{Im}\lambda)^2$ and $f \in C_b(\overline{S}; \mathbb{C})$, the series $\ell^{-1}\sum_{k\in\mathbb{Z}}(\mathcal{F}_{k,\rho}f)e_k$ defines a bounded and continuous function $\mathcal{R}_{\lambda,\rho}f$ in $\mathbb{R}^2$ which, clearly, is periodic with respect to y. Moreover,*

(i) *$\mathcal{R}_{\lambda,\rho}f \in \bigcap_{p<+\infty} W^{2,p}_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{C})$ and $\lambda\mathcal{R}_{\lambda,\rho}f - \rho^{-1}\Delta\mathcal{R}_{\lambda,\rho}f - D_x\mathcal{R}_{\lambda,\rho}f = f$ in S;*

(ii) *$\nabla\mathcal{R}_{\lambda,\rho}f \in C_b(\mathbb{R}^2; \mathbb{C}) \times C_b(\mathbb{R}^2; \mathbb{C})$ and*

$$|\lambda|\|\mathcal{R}_{\lambda,\rho}f\|_\infty + \sqrt{|\lambda|}\|\nabla\mathcal{R}_{\lambda,\rho}f\|_\infty \leq c\|f\|_\infty, \qquad \lambda \in \Sigma_0; \tag{4.1}$$

(iii) *if further $\lim_{x\to-\infty} f(x, y) = 0$ (resp. $\lim_{x\to+\infty} f(x, y) = 0$) for every $y \in [-\ell/2, \ell/2]$, then $(\mathcal{R}_{\lambda,\rho}f)(\cdot, y)$ vanishes as $x \to -\infty$ (resp. $x \to +\infty$) for every $y \in \mathbb{R}$;*

(iv) *for every $f \in C_b^\alpha(S; \mathbb{C})$, such that $f(\cdot, -\ell/2) = f(\cdot, \ell/2)$, and $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > -\rho^{-1}(\mathrm{Im}\lambda)^2$, the function $\mathcal{R}_{\lambda,\rho}f$ admits classical derivatives up to the second-order which belong to $C_b^\alpha(\mathbb{R}^2; \mathbb{C})$. Moreover,*

$$\|\mathcal{R}_{\lambda,\rho}f\|_{C_b^{2+\alpha}(\mathbb{R}^2; \mathbb{C})} \leq c_\lambda\|f\|_{C_b^\alpha(S; \mathbb{C})}. \tag{4.2}$$

Proof. To begin with, we claim that, for each $f \in C_b(\overline{S}; \mathbb{C})$ and $\lambda \in \mathbb{C}$, such that $\rho\mathrm{Re}\lambda + (\mathrm{Im}\lambda)^2 > 0$, the series in the statement converges uniformly in $\mathbb{R}^2$. To prove the claim, we observe that $\mathrm{Re}(Z_k) > \rho$ and $|Z_k| \geq \mathrm{Re}(Z_k) \geq c_\lambda(k + 1)$ for every $k \in \mathbb{Z}$. Thus, we can estimate

$$|(\mathcal{F}_{k,\rho}f)(x)| \leq \sup_{x\in\mathbb{R}} |\hat{f}_k(x)|\frac{1}{|Z_k|} \int_{\mathbb{R}} e^{\frac{\rho-\mathrm{Re}(Z_k)}{2}|s|}\,ds \leq \frac{c_\lambda}{k^2 + 1}\|f\|_{C_b(\overline{S}; \mathbb{C})}, \qquad x \in \mathbb{R}, \ k \in \mathbb{Z}, \tag{4.3}$$

and this is enough to infer that the series converges locally uniformly on $\mathbb{R}^2$ and, as a byproduct, that the operator $\mathcal{R}_{\lambda,\rho}$ is bounded from $C_b(\overline{S}; \mathbb{C})$ into $C_b(\mathbb{R}^2; \mathbb{C})$. Moreover, if $f(\cdot, y)$ vanishes at

$-\infty$ (resp. $+\infty$) for each $y \in [-\ell/2, \ell/2]$, then by dominated convergence the function $\mathcal{F}_k f$ vanishes at $-\infty$ (resp. $+\infty$) and, in view of the uniform convergence of the series which defines the function $\mathscr{R}_{\lambda,\rho}$, this is enough to conclude that this latter function tends to 0 as $x \to -\infty$ (resp. $x \to +\infty$) for each $y \in \mathbb{R}$.

Now, we prove properties (i), (ii) and (iv).

(i) Let us prove that the function $\mathscr{R}_{\lambda,\rho} f$ is the unique solution to the equation $\lambda u - \rho^{-1}\Delta u - D_x u = f^\sharp$ in $\mathcal{D} = \{u \in C_b^1(\mathbb{R}^2; \mathbb{C}) \cap \bigcap_{p<+\infty} W_{loc}^{2,p}(\mathbb{R}^2; \mathbb{C}) : \Delta u \in L^\infty(\mathbb{R}^2; \mathbb{C}) \cap C_b(\overline{S}; \mathbb{C}), u(\cdot, \cdot + \ell) = u\}$. For this purpose, for every $n \in \mathbb{N}$ we introduce the functions $u_n = \ell^{-1}\sum_{k=-n}^{n}(\mathcal{F}_{k,\rho}f)e_k$ and $f_n = \ell^{-1}\sum_{k=-n}^{n}\hat{f}(\cdot, k)e_k$. Note that $\lambda u_n - \rho^{-1}\Delta u_n - D_x u_n = f_n$ in $\mathbb{R}^2$, for every $n \in \mathbb{N}$, since the function $\mathcal{F}_{k,\rho}f$ $(k \in \mathbb{Z})$ solves the differential equation $(\lambda + \rho^{-1}\lambda_k)w - \rho^{-1}w'' - w' = f_k$ in $\mathbb{R}$. Thus,

$$\langle u_n, \rho^{-1}\Delta\varphi - \varphi_x \rangle = \int_{\mathbb{R}^2} u_n(\rho^{-1}\Delta\varphi - \varphi_x)dxdy = \int_{\mathbb{R}^2}(\lambda u_n - f_n)\varphi dxdy =: \langle \lambda u_n - f_n, \varphi \rangle$$

for every $\varphi \in C_c^\infty(\mathbb{R}^2; \mathbb{C})$. Letting n tend to $+\infty$ and applying the dominated convergence theorem, it follows that $\mathscr{R}_{\lambda,\rho}f$ is a distributional solution to the equation $\lambda\mathscr{R}_{\lambda,\rho}f - \rho^{-1}\Delta\mathscr{R}_{\lambda,\rho}f - D_x\mathscr{R}_{\lambda,\rho}f = f^\sharp$. By elliptic regularity (see e.g., [23]), we can infer that $\mathscr{R}_{\lambda,\rho}f \in \bigcap_{p<+\infty} W_{loc}^{2,p}(\mathbb{R}^2; \mathbb{C})$. Since $\mathscr{R}_{\lambda,\rho}f$ is bounded and continuous in $\mathbb{R}^2$ and $\Delta\mathscr{R}_{\lambda,\rho}f + D_x\mathscr{R}_{\lambda,\rho}f = \lambda\mathscr{R}_{\lambda,\rho}f - f^\sharp$ belongs to $L^\infty(\mathbb{R}^2; \mathbb{C})$, again by classical results we can infer that $\mathscr{R}_{\lambda,\rho}f \in C_b^{1+\gamma}(\mathbb{R}^2; \mathbb{C})$ for each $\gamma \in (0,1)$ and, as a byproduct, that $\Delta\mathscr{R}_{\lambda,\rho}f \in L^\infty(\mathbb{R}^2; \mathbb{C}) \cap C_b(\overline{S}; \mathbb{C})$. We can thus conclude that $\mathscr{R}_{\lambda,\rho}f$ belongs to $\mathcal{D}$.

To prove uniqueness, we assume that v is another solution in $\mathcal{D}$ of the equation $\lambda u - \rho^{-1}\Delta u - u_x = f^\sharp$. The smoothness of v implies that, for each $k \in \mathbb{Z}$, the function $\hat{v}_k$ belongs to $C_b^1(\mathbb{R}; \mathbb{C})$. Moreover, integrating by parts we obtain that

$$\int_{\mathbb{R}}\hat{v}_k\varphi''dx = \frac{1}{\ell}\int_S \varphi''v\overline{e_k}dxdy = \frac{1}{\ell}\int_S \varphi v_{xx}\overline{e_k}dxdy$$

for each $\varphi \in C_c^\infty(\mathbb{R})$. By Fubini theorem, $\hat{v}_k$ belongs to $W_{loc}^{2,p}(\mathbb{R}; \mathbb{C})$. Since $\lambda v - \rho^{-1}\Delta v - v_x = f$ in S, we can write

$$\int_S \varphi(x)v_{xx}\overline{e_k}dxdy = \lim_{n\to+\infty}\int_S \varphi(x)v_{xx}(x,y)\psi_n(y)\overline{e_k(y)}dxdy$$

$$= \lambda\rho\int_{\mathbb{R}}\varphi v_k dx - \lim_{n\to+\infty}\int_S \varphi v_{yy}\psi_n\overline{e_k}dxdy - \rho\int_{\mathbb{R}}\varphi\hat{v}_k'dx - \rho\int_{\mathbb{R}}\varphi f_k dx, \quad (4.4)$$

where $\psi_n(y) = \psi(n|y|/\ell + 1 - n/2)$ for each $y \in [-\ell/2, \ell/2)$, $n \in \mathbb{N}$, and ψ is a smooth function such that $\psi = 1$ in $(-\infty, 1/2]$ and $\psi = 0$ outside $(-\infty, 3/4)$. Clearly, ψ_n converges to 1 in $L^1((-\ell/2, \ell/2))$ as n tends to $+\infty$. An integration by parts shows that

$$\lim_{n\to+\infty}\int_S \varphi(x)v_{yy}(x,y)\psi_n(y)\overline{e_k(y)}dxdy$$

$$= -\lim_{n\to+\infty}\frac{n}{\ell}\int_{\mathbb{R}^2}\varphi(x)\chi_A(x)\text{signum}(y)v_y(x,y)\psi'\left(\frac{n}{\ell}|y| + 1 - \frac{n}{2}\right)\chi_{B_n}(y)\overline{e_k(y)}dxdy$$

$$- \lambda_k\int_{\mathbb{R}}\varphi(x)v_k(x)dx, \quad (4.5)$$

where $A = \text{supp}(\varphi)$ and $B_n = \left\{y \in \mathbb{R} : \frac{\ell}{2} - \frac{\ell}{2n} \leq |y| \leq \frac{\ell}{2} - \frac{\ell}{4n}\right\}$ for every $n \in \mathbb{N}$. We claim that the first term in the last side of (4.5) is zero. For this purpose, we split

$$\frac{n}{\ell}\int_{\mathbb{R}^2}\varphi(x)\chi_A(x)\text{signum}(y)v_y(x,y)\psi'\left(\frac{n}{\ell}|y| + 1 - \frac{n}{2}\right)\chi_{B_n}(y)\overline{e_k(y)}dxdy$$

$$= \frac{n}{\ell}\int_{\mathbb{R}^2}\varphi(x)\chi_A(x)\text{signum}(y)(v_y(x,y) - v_y(x,\ell/2))\psi'\left(\frac{n}{\ell}|y| + 1 - \frac{n}{2}\right)\chi_{B_n}(y)\overline{e_k(y)}dxdy$$

$$+ \frac{n}{\ell}\int_A \varphi(x)v_y(x,\ell/2)dx\int_{\mathbb{R}}\text{signum}(y)\psi'\left(\frac{n}{\ell}|y| + 1 - \frac{n}{2}\right)\chi_{B_n}(y)\overline{e_k(y)}dy$$

$$=: \mathcal{K}_{1,n}(x,y) + \mathcal{K}_{2,n}(x,y)$$

for every $(x, y) \in \mathbb{R}^2$. Since v is ℓ-periodic with respect to y, $D_y v$ is ℓ-periodic with respect to y as well. Moreover, this latter function is $1/2$-Hölder continuous in $\mathbb{R}^2$ since it belongs to $\mathcal{D}$. Hence, we can estimate

$$|v_y(x, y) - v_y(x, \ell/2)| = |v_y(x, y) - v_y(x, -\ell/2)| \leq [v_y]_{C^{1/2}(A \times [-\ell/2, \ell/2]; \mathbb{C})} \min\{|y - \ell/2|, |y + \ell/2|\}^{\frac{1}{2}}$$

for every $x \in A$ and $y \in [-\ell/2, \ell/2]$. Thus,

$$|\mathcal{K}_{1,n}| \leq c\|\varphi\|_\infty \|\psi'\|_\infty n^{\frac{1}{2}} m(A \times B_n) \leq c\|\varphi\|_\infty \|\psi'\|_\infty n^{-\frac{1}{2}}, \qquad n \in \mathbb{N},$$

where $m(A \times B_n)$ denotes the Lebesgue measure of the set $A \times B_n$, so that $\mathcal{K}_{1,n}$ vanishes as n tends to $+\infty$. As far as $\mathcal{K}_{2,n}$ is concerned, we observe that

$$\mathcal{K}_{2,n}(x, y) = -\frac{2in}{\ell} \int_A D_y v(x, \ell/2) \varphi(x) dx \int_{\frac{\ell}{2} - \frac{\ell}{2n}}^{\frac{\ell}{2} - \frac{\ell}{4n}} \psi'\left(\frac{n}{\ell} y + 1 - \frac{n}{2}\right) \sin\left(\frac{2k\pi}{\ell} y\right) dy$$

$$= c\left[\frac{4k\pi i}{\ell} \int_{\frac{\ell}{2} - \frac{\ell}{2n}}^{\frac{\ell}{2} - \frac{\ell}{4n}} \psi\left(\frac{n}{\ell} y + 1 - \frac{n}{2}\right) \cos\left(\frac{2k\pi}{\ell} y\right) dy - 2(-1)^k \sin\left(\frac{k\pi}{n}\right)\right].$$

Hence, $\mathcal{K}_{2,n}$ vanishes as n tends to $+\infty$. The claim is so proved.

From (4.4) and (4.5), we can now infer that $\rho^{-1} \hat{v}_k'' = (\lambda + \rho^{-1}\lambda_k)\hat{v}_k - \hat{v}_k' - \hat{f}_k$. Thus, $v_k = \mathcal{F}_k f$ for every $k \in \mathbb{Z}$ and, as a byproduct, we deduce that $v = \mathcal{R}_{\lambda, \rho} f$.

(ii) By classical results, the realization A_ρ in $L^\infty(\mathbb{R}^2; \mathbb{C})$ of the operator $\rho^{-1}\Delta + D_x$, with domain $D(A_\rho) = \{u \in C_b^1(\mathbb{R}^2; \mathbb{C}) \cap \bigcap_{p < +\infty} W_{\text{loc}}^{2,p}(\mathbb{R}^2; \mathbb{C}) : \Delta u \in L^\infty(\mathbb{R}^2; \mathbb{C})\} \supset \mathcal{D}$, generates an analytic semigroup. Moreover, $\|R(\lambda, A_\rho)\|_{L(L^\infty(\mathbb{R}^2))} \leq c|\lambda|^{-1}$ for every $\lambda \in \Sigma_0$ and $\|\nabla v\|_\infty \leq c\|v\|_\infty^{1/2} \|A_\rho v\|_\infty^{1/2}$ for every $v \in D(A_\rho)$. Note that the function $R(\lambda, A_\rho)f^\sharp$ is ℓ-periodic with respect to y. Indeed, $R(\lambda, A_\rho)f^\sharp$ and $(R(\lambda, A_\rho)f^\sharp)(\cdot, \cdot + \ell)$ both solve (in $D(A_\rho)$) the equation $\lambda u - \rho^{-1}\Delta u - u_x = f^\sharp$ and, by uniqueness, they coincide. Finally, since $f^\sharp$ is continuous in $\overline{S}$, $\Delta R(\lambda, A_\rho)f^\sharp$ belongs to $C_b(\overline{S}; \mathbb{C})$. Hence, $R(\lambda, A_\rho)f^\sharp \in \mathcal{D}$ and, by (i), it coincides with $\mathcal{R}_{\lambda, \rho} f^\sharp$. Using the above estimates, inequality (4.1) follows immediately.

(iv) Since $f \in C_b^\alpha(S; \mathbb{C})$ and $f(\cdot, -\ell/2) = f(\cdot, \ell/2)$, the function $f^\sharp$ belongs to $C_b^\alpha(\mathbb{R}^2; \mathbb{C})$. Hence, $\Delta \mathcal{R}_{\lambda, \rho} f = \rho(\lambda \mathcal{R}_{\lambda, \rho} f - f^\sharp - D_x \mathcal{R}_{\lambda, \rho} f)$ is an element of $C_b^\alpha(\mathbb{R}^2; \mathbb{C})$. Classical results (see e.g., [29]) yield (4.2). □

Lemma 4.2. *For $g \in C_b(\overline{S_0^+}; \mathbb{C})$ and $\lambda \in \mathbb{C}$, such that $\text{Re}\lambda > -\text{Le}^{-1}(\text{Im}\lambda)^2$, the function $\mathscr{S}_\lambda g = \ell^{-1} \sum_{k \in \mathbb{Z}} (\mathcal{G}_k g) e_k$ is bounded and continuous in $\mathbb{R}^2$. Here, $\mathcal{G}_k g := \mathcal{F}_{k, \text{Le}}(\overline{g})$ for each $k \in \mathbb{Z}$, and $\overline{g}$ is the trivial extension of g to $\mathbb{R} \times [-\ell/2, \ell/2]$. Moreover,*

(i) *$\mathscr{S}_\lambda g$ belongs to $\bigcap_{p < +\infty} W_{\text{loc}}^{2,p}(\mathbb{R}^2; \mathbb{C})$ and $\lambda \mathscr{S}_\lambda g - \text{Le}^{-1}\Delta \mathscr{S}_\lambda g - D_x \mathscr{S}_\lambda g = g$ in S_0^+;*

(ii) *there exists a positive constant c_1, independent of λ, such that*

$$|\lambda| \|\mathscr{S}_\lambda g\|_\infty + \sqrt{|\lambda|} \|\nabla \mathscr{S}_\lambda g\|_\infty \leq c_1 \|g\|_\infty, \qquad \lambda \in \Sigma_0; \tag{4.6}$$

(iii) *if further $\lim_{x \to +\infty} g(x, y) = 0$ for each $y \in [-\ell/2, \ell/2]$, then $(\mathscr{S}_\lambda g)(\cdot, y)$ vanishes as $x \to +\infty$ for each $y \in \mathbb{R}$;*

(iv) *if $g \in C_b^\alpha(S_0^+; \mathbb{C})$ and $g(\cdot, -\ell/2) = g(\cdot, \ell/2)$, then $\mathscr{S}_\lambda g$ belongs to $C_b^{2+\alpha}(\mathbb{R}^2; \mathbb{C})$ for every $\lambda \in \mathbb{C}$ such that $\text{Re}\lambda > -\text{Le}^{-1}(\text{Im}\lambda)^2$. Moreover,*

$$\|\mathscr{S}_\lambda g\|_{C_b^{2+\alpha}(\mathbb{R}^2; \mathbb{C})} \leq c_{2, \lambda} \|g\|_{C_b^\alpha(S_0^+; \mathbb{C})}, \tag{4.7}$$

with the constant $c_{2, \lambda}$ being independent of g.

Proof. We split the proof into two steps: in the first one we prove properties (i), (ii) and (iii) and in the second step we prove property (iv).

Step 1. Let us fix g and λ as in the statement. Arguing as in the first part of the proof of Lemma 4.1, taking the continuity of the functions $\mathcal{G}_k g$ ($k \in \mathbb{Z}$) into account, it can be checked that the series in the statement converges uniformly in $\mathbb{R}^2$, so that the function $\mathscr{S}_\lambda g$ is well defined and it vanishes as $x \to +\infty$ for every $y \in \mathbb{R}$, if $\lim_{x \to +\infty} g(x, y) = 0$ for every $y \in [-\ell/2, \ell/2]$.

To check properties (i) and (ii), for each $n \in \mathbb{N}$ we set $g_n := \overline{g} \star_x \psi_n$, where $\star_x$ stands for convolution with respect to the variable x and (ψ_n) is a standard sequence of mollifiers. Clearly, $\mathscr{S}_\lambda g_n = \mathscr{R}_{\lambda,\mathrm{Le}} g_n$. The sequence (g_n) converges to $\overline{g}$ pointwise in $\mathbb{R}^2$ as $n \to +\infty$ and $\|g_n\|_\infty \leq \|g\|_\infty$ for each $n \in \mathbb{N}$. Thus, we can infer that $\mathcal{F}_{k,\rho} g_n$ converges to $\mathcal{G}_k g$ pointwise in $\mathbb{R}$ as n tends to $+\infty$, for every $k \in \mathbb{N}$ and, by dominated convergence, $\mathscr{R}_{\lambda,\mathrm{Le}} g_n$ tends to $\mathscr{S}_\lambda g$ pointwise in $\mathbb{R}^2$.

Applying the classical interior L^p-estimates for the operator $\mathrm{Le}^{-1}\Delta + D_x$ and using (4.1), which allows us to write

$$|\lambda| \|\nabla \mathscr{R}_{\lambda,\mathrm{Le}} g_n\|_\infty + |\lambda|^{\frac{1}{2}} \|\nabla \mathscr{R}_{\lambda,\mathrm{Le}} g_n\|_\infty \leq c \|g_n\|_\infty \leq c \|g\|_\infty, \qquad n \in \mathbb{N}, \tag{4.8}$$

we can estimate

$$\begin{aligned}
\|\mathscr{R}_{\lambda,\mathrm{Le}} g_n\|_{W^{2,p}(B(0,r);\mathbb{C})} &\leq c_{p,r}(\|\mathscr{R}_{\lambda,\mathrm{Le}} g_n\|_{L^p(B(0,2r);\mathbb{C})} + \|\mathrm{Le}^{-1}\Delta \mathscr{R}_{\lambda,\mathrm{Le}} g_n + D_x \mathscr{R}_{\lambda,\mathrm{Le}} g_n\|_{L^p(B(0,2r);\mathbb{C})}) \\
&\leq c_{p,r}[(1+|\lambda|)\|\mathscr{R}_{\lambda,\mathrm{Le}} g_n\|_{C(\overline{B(0,2r)};\mathbb{C})} + \|g_n\|_{C(\overline{B(0,2r)};\mathbb{C})}] \\
&\leq c_{p,r,\lambda}\|g_n\|_\infty \leq c_{p,r,\lambda}\|g\|_\infty
\end{aligned}$$

for every $p \in [1,+\infty)$ and $r > 0$. Hence, by compactness, we conclude that $\mathscr{R}_{\lambda,\mathrm{Le}} g_n$ converges to $\mathscr{S}_\lambda g$ in $C^1(\overline{B(0,r)};\mathbb{C})$ for each $r > 0$, $\mathscr{S}_\lambda g \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{C})$ for every $p \in [1,+\infty)$ and $\lambda \mathscr{S}_\lambda g - \mathrm{Le}^{-1}\Delta \mathscr{S}_\lambda g - D_x \mathscr{S}_\lambda g = g$ in S_0^+. Finally, estimate (4.6) follows at once from (4.8).

Step 2. To complete the proof, here we check property (iv), which demands some additional effort. We begin by checking that the function $\zeta = \mathscr{S}_\lambda g(0,\cdot)$ belongs to $C_b^{2+\alpha}(\mathbb{R};\mathbb{C})$. For this purpose, we set $\zeta_n = \ell^{-1} \sum_{k=-n}^{n} (\mathcal{G}_k g)(0) e_k$ for $n \in \mathbb{N}$. Clearly, each function ζ_n is smooth and

$$\begin{aligned}
\zeta_n' =& \frac{2\pi i}{\ell^2} \sum_{|k|<k_0} \frac{k}{Z_k} e_k \int_0^{+\infty} e^{\frac{\mathrm{Le}}{2}s} e^{-\frac{1}{2}Z_k s} \hat{g}_k(s) ds + \frac{2\pi i}{\ell^2} \sum_{k_0 \leq |k| \leq n} \left(\frac{k}{Z_k} - \frac{\ell}{\pi} \right) e_k \int_0^{+\infty} e^{\frac{\mathrm{Le}}{2}s} e^{-\frac{1}{2}Z_k s} \hat{g}_k(s) ds \\
&+ \frac{2i}{\ell} \sum_{|k|=k_0}^{n} e_k \int_0^{+\infty} e^{\frac{\mathrm{Le}}{2}s} \left(e^{-\frac{1}{2}Z_k s} - e^{-\frac{\pi|k|}{2\ell}s} \right) \hat{g}_k(s) ds + \frac{2i}{\ell} \sum_{|k|=k_0} e_k \int_0^{+\infty} e^{\frac{\mathrm{Le}}{2}s - \frac{\pi|k|}{2\ell}s} \hat{g}_k(s) ds \\
=&: \mathcal{I}_0 + \sum_{h=1}^{3} \mathcal{I}_{h,n},
\end{aligned}$$

where $k_0 \in \mathbb{N}$ is chosen so that $\pi k_0 > \ell \mathrm{Le}$. As it is easily seen,

$$\left| \frac{k}{Z_k} - \frac{\ell}{\pi} \right| \leq \frac{c_\lambda}{k^2+1}, \qquad k \in \mathbb{Z}, \tag{4.9}$$

so that $\mathcal{I}_{1,n}$ converges uniformly in $\mathbb{R}^2$ as $n \to +\infty$. On the other hand,

$$\left| e^{-\frac{1}{2}Z_k s} - e^{-\frac{\pi|k|}{2\ell}s} \right| = \left| \int_0^1 \frac{d}{dr} e^{-\frac{1}{2}Z_k(r)s} dr \right| \leq s \int_0^1 \left| \frac{\mathrm{Le}^2 + 4\lambda\mathrm{Le}}{4Z_k(r)} \right| e^{-\frac{1}{2}\mathrm{Re}(Z_k(r))s} dr,$$

where we have set $Z_k(r) = \left((\mathrm{Le}^2 + 4\lambda\mathrm{Le})r + \frac{k^2\pi^2}{\ell^2} \right)^{\frac{1}{2}}$. Note that $\mathrm{Re}(Z_k(r)) \geq c_\lambda |k|$ for $|k| \geq k_0$. For such values of k and for $\mathrm{Re}\lambda > 0$ (which implies that $\mathrm{Re}(Z_k(r)) > \mathrm{Le}$ for every $r \in [0,1]$ and $|k| \geq k_0$) we can estimate

$$\begin{aligned}
\left| \int_0^{+\infty} e^{\frac{\mathrm{Le}}{2}s} \left(e^{-\frac{1}{2}Z_k s} - e^{-\frac{\pi|k|}{2\ell}s} \right) \hat{g}_k(s) ds \right| &\leq \frac{|\mathrm{Le}^2 + 4\lambda\mathrm{Le}|}{4|k|} \|g\|_\infty \int_0^1 r^{-\frac{1}{2}} dr \int_0^{+\infty} s e^{\frac{\mathrm{Le}}{2}s - \frac{1}{2}(\mathrm{Re}(Z_k(r))s} ds \\
&\leq \frac{c_\lambda}{k^3} \|g\|_\infty, \tag{4.10}
\end{aligned}$$

so that the sequence $(\mathcal{I}_{2,n})$ converges uniformly in $\mathbb{R}^2$. Next, we observe that

$$\mathcal{I}_{3,n}(y) = \frac{2i}{\ell} \int_0^{+\infty} ds \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} K_n(s,\eta) g_n(s, y-\eta) d\eta, \qquad y \in \mathbb{R}, \ n \in \mathbb{N},$$

where $K_n(x,y) = H_n(x,y) + H_n(x,-y)$ and

$$H_n(x,y) = e^{\frac{\text{Le}}{2}x}\frac{e^{-\frac{\pi}{2\ell}(x-4iy)(k_0-1)} - e^{-\frac{\pi}{2\ell}(x-4iy)n}}{e^{\frac{\pi}{2\ell}(x-4iy)} - 1}, \qquad g_n(x,y) = \ell^{-1}\sum_{k=-n}^{n}\hat{g}(x,k)e^{\frac{2k\pi i}{\ell}y}$$

for $x \geq 0$, $y \in \mathbb{R}^2$ and $n \in \mathbb{N}$. We set

$$K(x,y) = e^{\left(\frac{\text{Le}}{2} - \frac{\pi(k_0-1)}{2\ell}\right)x}\left(\frac{e^{-\frac{2\pi(k_0-1)i}{\ell}y}}{e^{\frac{\pi}{2\ell}(x-4iy)} - 1} + \frac{e^{\frac{2\pi(k_0-1)i}{\ell}y}}{e^{\frac{\pi}{2\ell}(x+4iy)} - 1}\right), \qquad x \geq 0, \ \ y \in \mathbb{R},$$

and prove that $\mathcal{I}_{3,n}$ converges pointwise in $\mathbb{R}$ to the function $\mathcal{I}_3$, defined by[1]

$$\mathcal{I}_3(y) = \frac{i\ell}{\pi}\int_0^{+\infty}ds\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}}K(s,\eta)g^\sharp(s,y-\eta)d\eta, \qquad y \in \mathbb{R}.$$

For this purpose, we split

$$\mathcal{I}_{3,n} - \mathcal{I}_3 = \int_0^{+\infty}ds\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}}(g_n^\sharp(s,\cdot-\eta) - g^\sharp(s,\cdot-\eta))K_n(s,\eta)d\eta$$

$$+ \int_0^{+\infty}ds\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}}g^\sharp(s,\cdot-\eta)(K_n(s,\eta) - K(s,\eta))d\eta =: \mathcal{A}_{1,n} + \mathcal{A}_{2,n}$$

for every $n \in \mathbb{N}$ and observe that

$$\|\mathcal{A}_{1,n}\|_\infty \leq \int_0^{+\infty}\|g_n(s,\cdot) - g(s,\cdot)\|_{L^2((-\ell/2,\ell/2))}\|K_n(s,\cdot)\|_{L^2((-\ell/2,\ell/2))}ds, \qquad n \in \mathbb{N}.$$

Since (i) $\|g_n(0,\cdot)-g(0,\cdot)\|_{L^2((-\ell/2,\ell/2);\mathbb{C})}$ vanishes as $n \to +\infty$, (ii) $\|g_n(x,\cdot)-g(x,\cdot)\|_{L^2((-\ell/2,\ell/2);\mathbb{C})} \leq 2\|g(x,\cdot)\|_{L^2((-\ell/2,\ell/2);\mathbb{C})} \leq 2\ell\|g\|_\infty$, for $x \geq 0$ and $n \in \mathbb{N}$, and (iii) $|K_n| \leq 2|K|$ in $\mathbb{R}_+ \times \mathbb{R}$ for every $n \in \mathbb{N}$, the dominated convergence theorem shows that $\mathcal{A}_{1,n}$ converges to zero pointwise in $\mathbb{R}$ as n tends to $+\infty$. Moreover, $\|\mathcal{A}_{1,n}\|_\infty \leq c\|g\|_\infty$. That theorem also shows that $\mathcal{A}_{2,n}$ converges to zero pointwise in $\mathbb{R}$ as n tends to $+\infty$; moreover, $\|\mathcal{A}_{2,n}\|_\infty \leq c\|g\|_\infty$ for each $n \in \mathbb{N}$. Now, writing

$$\zeta_n(y) = \zeta_n(0) + \int_0^y\left(\mathcal{I}_0(r) + \sum_{h=1}^{3}\mathcal{I}_{h,n}(r)\right)dr, \qquad y \in \mathbb{R}, \ n \in \mathbb{N},$$

and letting n tend to $+\infty$, again by dominated convergence we conclude that

$$\zeta' = \frac{2\pi i}{\ell^2}\sum_{|k|<k_0}\frac{k}{Z_k}e_k\int_0^{+\infty}e^{\frac{\text{Le}}{2}s}e^{-\frac{1}{2}Z_ks}\hat{g}_k(s)ds + \frac{2\pi i}{\ell^2}\sum_{|k|\geq k_0}\left(\frac{k}{Z_k} - \frac{\ell}{\pi}\right)e_k\int_0^{+\infty}e^{\frac{\text{Le}}{2}s}e^{-\frac{1}{2}Z_ks}\hat{g}_k(s)ds$$

$$+ \frac{2i}{\ell}\sum_{|k|\geq k_0}e_k\int_0^{+\infty}e^{\frac{\text{Le}}{2}s}\left(e^{-\frac{1}{2}Z_ks} - e^{-\frac{\pi k}{2\ell}s}\right)\hat{g}_k(s)ds + \frac{i\ell}{\pi}\int_0^{+\infty}ds\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}}K(s,\eta)g^\sharp(s,\cdot-\eta)d\eta.$$

Denote by $\phi_1,\ldots,\phi_4$ the four terms in the right-hand side of the previous formula. Clearly, ϕ_1 belongs to $C_b^\infty(\mathbb{R};\mathbb{C})$. In particular, $\|\phi_1\|_{C_b^{1+\alpha}(\mathbb{R};\mathbb{C})} \leq c\|g\|_\infty$. As far as ϕ_2 and ϕ_3 are concerned, using (4.9), (4.10), the same arguments here above and in the first part of the proof of Lemma 4.1, it can be easily shown that such functions belong to $C_b^{1+\alpha}(\mathbb{R};\mathbb{C})$ and $\|\phi_2\|_{C_b^{1+\alpha}(\mathbb{R};\mathbb{C})} + \|\phi_3\|_{C_b^{1+\alpha}(\mathbb{R};\mathbb{C})} \leq c\|g\|_\infty$.

The function ϕ_4 is the limit in $C_b(\mathbb{R};\mathbb{C})$ of the sequence $(\phi_{4,n})$ defined by

$$\phi_{4,n} = \frac{i\ell}{\pi}\int_{\frac{1}{n}}^{+\infty}ds\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}}K(s,\eta)g^\sharp(s,\cdot-\eta)d\eta, \qquad n \in \mathbb{N}.$$

---

[1]$\mathcal{I}_3$ belongs to $C_b(\mathbb{R};\mathbb{C})$. Indeed, $H \in L^1(S_0^+;\mathbb{C})$ as it follows observing that $|e^{\frac{\pi}{2\ell}(x\pm4iy)} - 1| \geq e^{\frac{\pi}{2\ell}x} - 1$, which implies the inequality $|K(x,y)| \leq e^{\left(\frac{\text{Le}}{2} - \frac{\pi(k_0-1)}{2\ell}\right)x}\min\{2(e^{\frac{\pi}{2\ell}x} - 1)^{-1}, c(x^2+y^2)^{-1/2}\}$ for every $(x,y) \in S_0^+$. This shows that $\mathcal{I}_3$ is bounded in $\mathbb{R}^2$. Moreover, $\mathcal{I}_3$ is the uniform limit as $\varepsilon \to 0^+$ of the function $\int_\varepsilon^{+\infty}ds\int_{-\ell/2}^{\ell/2}K(s,\cdot-\eta)g(s,\eta)d\eta$, which is clearly continuous in $\mathbb{R}$ thanks to the above estimate for K. Hence, $\mathcal{I}_3$ is itself continuous in $\mathbb{R}$.

Clearly, each function $\phi_{4,n}$ is continuously differentiable in $\mathbb{R}$ and

$$\phi'_{4,n} = \frac{i\ell}{\pi} \int_{\frac{1}{n}}^{+\infty} ds \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} K_y(s,\eta) g^\sharp(s, \cdot - \eta) d\eta,$$

where

$$K_y(x,y) = \frac{2\pi i}{\ell}(k_0-1)e^{\left(\frac{\mathrm{Le}}{2} - \frac{\pi(k_0-1)}{2\ell}\right)x}\left(\frac{e^{\frac{2\pi(k_0-1)i}{\ell}y}}{e^{\frac{\pi}{2\ell}(x+4iy)}-1} - \frac{e^{-\frac{2\pi(k_0-1)i}{\ell}y}}{e^{\frac{\pi}{2\ell}(x-4iy)}-1}\right)$$
$$+ \frac{2\pi i}{\ell}e^{\left(\frac{\mathrm{Le}}{2} - \frac{\pi(k_0-1)}{2\ell}\right)x}\left(\frac{e^{\frac{\pi}{2\ell}(x-4ik_0y)}}{(e^{\frac{\pi}{2\ell}(x-4iy)}-1)^2} - \frac{e^{\frac{\pi}{2\ell}(x+4ik_0y)}}{(e^{\frac{\pi}{2\ell}(x+4iy)}-1)^2}\right) =: L_1(x,y) + L_2(x,y)$$

$$(4.11)$$

for $(x,y) \in \overline{H_0^+}$. Since K is ℓ-periodic with respect to y, it follows that $\int_{-\ell/2}^{\ell/2} K_y(s,\eta)d\eta = 0$. Hence, we can write

$$\phi'_{4,n}(y) = \frac{i\ell}{\pi} \int_{\frac{1}{n}}^{+\infty} ds \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} K_y(s,\eta)(g^\sharp(s,y-\eta) - g(s,y))d\eta, \qquad y \in \mathbb{R}, \ n \in \mathbb{N}.$$

By assumptions, $g \in C_b^\alpha(S_0^+;\mathbb{C})$ and this allows us to estimate

$$|K_y(s,\eta)(g^\sharp(s,y-\eta) - g^\sharp(s,y))| \leq c \min\{(s^2+\eta^2)^{\frac{\alpha}{2}-1}, (e^{\frac{\pi}{2\ell}|s|}-1)^{-1}\}\|g\|_{C_b^\alpha([0,+\infty)\times\mathbb{R};\mathbb{C})}$$

for every $(s,\eta) \in \mathbb{R} \times \mathbb{R}_+$ and $y \geq 0$. Thus, we can let n tend to $+\infty$ in (4.11) and conclude that $\phi'_{4,n}$ converges uniformly in $\mathbb{R}^2$. As a byproduct, ϕ_4 is continuously differentiable in $\mathbb{R}$,

$$\phi'_4 = \int_0^{+\infty} ds \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} K_y(s,\eta)(g^\sharp(s, \cdot - \eta) - g(s,\cdot))d\eta$$

and $\|\phi'_4\|_\infty \leq c\|g\|_{C_b^\alpha(S_0^+;\mathbb{C})}$.

To prove that ϕ'_4 belongs to $C_b^\alpha(\mathbb{R};\mathbb{C})$ we split

$$\phi'_4 = \int_0^{+\infty} ds \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} (L_1(s,\eta) + L_2(s,\eta)\chi_{(1,+\infty)}(s))(g^\sharp(s,\cdot - \eta) - g(s,\cdot))d\eta$$
$$+ \int_0^1 ds \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} L_2(s,\eta)(g^\sharp(s,\cdot - \eta) - g(s,\cdot))d\eta. \qquad (4.12)$$

Since the function $L_1 + L_2\chi_{(1,+\infty)} \in L^1(S_0^+;\mathbb{C})$, the first term in the right-hand side of (4.12) belongs to $C_b^\alpha(\mathbb{R};\mathbb{C})$ and its $C_b^\alpha(\mathbb{R};\mathbb{C})$-norm can be estimated from above by $c\|g\|_{C_b^\alpha(S_0^+;\mathbb{C})}$. To estimate the other term, which we denote by Ψ, we observe that

$$\frac{e^{\frac{\pi}{2\ell}(x-4ik_0y)}}{(e^{\frac{\pi}{2\ell}(x-4iy)}-1)^2} = \frac{\ell^2}{\pi^2}\frac{4x^2 + 32ixy - 64y^2}{(x^2+16y^2)^2} + \psi(x,y), \qquad (x,y) \in (0,1) \times (-\ell/2,\ell/2),$$

for some function $\psi \in L^1((0,1) \times (-\ell/2,\ell/2);\mathbb{C})$. Thus,

$$\Psi(y) = -\frac{128i\ell^2}{\pi^2} \int_0^1 ds \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} \frac{s(y-\eta)}{(s^2+16(y-\eta)^2)^2}(g^\sharp(s,\eta) - g(s,y))d\eta$$
$$- 2\int_0^1 ds \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} (\psi(s,\eta) - \psi(s,-\eta))(g^\sharp(s,y-\eta) - g(s,y))d\eta =: \Psi_1(y) + \Psi_2(y)$$

for every $y \in \mathbb{R}$. The function Ψ_2 is clearly α-Hölder continuous in $[-\ell/2,\ell/2]$ since $\psi \in L^1((0,1) \times (-\ell/2,\ell/2);\mathbb{C})$. Moreover, $\|\Psi_2\|_{C_b^\alpha(\mathbb{R};\mathbb{C})} \leq c\|g\|_{C_b^\alpha(\mathbb{R};\mathbb{C})}$. As far as the function Ψ_1 is concerned, we approximate it with the family of functions $\Psi_{1,h}$ defined by

$$\Psi_{1,h}(y) = -\frac{128i\ell^2}{\pi^2} \int_0^1 ds \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} \frac{s(y-\eta)}{(s^2+16(y-\eta)^2+h^2)^2}(g(s,\eta) - g(s,\cdot))d\eta, \qquad h > 0.$$

Each of these functions is continuously differentiable in $\mathbb{R}$ with bounded derivative, so that, we can estimate

$$|\Psi_1(y_2) - \Psi_1(y_1)| \leq |\Psi_1(y_2) - \Psi_{1,h}(y_2)| + |\Psi_{1,h}(y_2) - \Psi_{1,h}(y_1)| + |\Psi_{1,h}(y_1) - \Psi_1(y_1)|$$
$$\leq 2\|\Psi_1 - \Psi_{1,h}\|_\infty + \|\Psi'_{1,h}\|_\infty |y_2 - y_1| \tag{4.13}$$

for $y_1, y_2 \in [-\ell/2, \ell/2]$. Note that

$$\|\Psi_1 - \Psi_{1,h}\|_\infty \leq \frac{256\ell^2}{\pi^2}\|g\|_{C_b^\alpha(S_0^+;\mathbb{C})} \int_{\mathbb{R}^2} \frac{|s\eta|h^2}{(s^2 + 16\eta^2 + h^2)^2(s^2 + 16\eta^2)} ds d\eta$$

$$\leq ch^2\|g\|_{C_b^\alpha(S_0^+;\mathbb{C})} \int_0^{+\infty} \frac{\rho^{\alpha+1}}{(\rho^2 + h^2)^2} d\rho = ch^\alpha\|g\|_{C_b^\alpha(S_0^+;\mathbb{C})}$$

and

$$\|\Psi'_{1,h}\|_\infty \leq \frac{128\ell^2}{\pi^2}\|g\|_{C_b^\alpha(S_0^+;\mathbb{C})} \int_{\mathbb{R}^2} \frac{(s^2 + 48\eta^2 + h^2)|\eta|^\alpha}{(s^2 + 16\eta^2 + h^2)^3} ds d\eta$$

$$\leq c\|g\|_{C_b^\alpha(S_0^+;\mathbb{C})} \int_0^{+\infty} \frac{\rho^{2+\alpha}}{(\rho^2 + h^2)^2} d\rho \leq c\|g\|_{C_b^\alpha(S_0^+;\mathbb{C})} h^{\alpha-1}.$$

Replacing these inequalities into (4.13) and taking $h = |y_2 - y_1|$, we conclude that

$$|\Psi_1(y_2) - \Psi_1(y_1)| \leq c\|g\|_{C_b^\alpha(S_0^+;\mathbb{C})}|y_2 - y_1|^\alpha.$$

Therefore, $\phi'_4 \in C_b^\alpha(\mathbb{R};\mathbb{C})$ and $\|\phi'_4\|_{C_b^\alpha(\mathbb{R};\mathbb{C})} \leq c\|g\|_{C_b^\alpha(S_0^+;\mathbb{C})}$. Putting everything together it follows that $\zeta \in C_b^{2+\alpha}(\mathbb{R};\mathbb{C})$ and $\|\zeta\|_{C_b^{2+\alpha}(\mathbb{R};\mathbb{C})} \leq c\|g\|_{C_b^\alpha(S_0^+;\mathbb{C})}$.

Finally, we consider the function $\mathscr{S}_\lambda g - \zeta =: v \in C_b([0, +\infty) \times \mathbb{R};\mathbb{C}) \cap \bigcap_{p < +\infty} W_{\text{loc}}^{2,p}((0, +\infty) \times \mathbb{R};\mathbb{C})$. Since $\text{Le}^{-1}\Delta v + v_x \in C_b^\alpha((0, +\infty) \times \mathbb{R};\mathbb{C})$ and by construction $v(0, \cdot) = 0$, by classical results (see e.g., [29]) $v \in C_b^{2+\alpha}((0, +\infty) \times \mathbb{R})$ and

$$\|v\|_{C_b^{2+\alpha}(H_0^+;\mathbb{C})} \leq c(\|v\|_\infty + \|\text{Le}^{-1}\Delta v + v_x\|_{C_b^\alpha(H_0^+)} + \|\zeta\|_{C_b^{2+\alpha}(\mathbb{R};\mathbb{C})})$$

$$\leq c(\|v\|_\infty + \|\lambda\mathscr{S}_\lambda g - g\|_{C_b^\alpha(H_0^+;\mathbb{C})} + \|\zeta\|_{C_b^{2+\alpha}(\mathbb{R};\mathbb{C})}) \leq c\|g\|_{C_b^\alpha(H_0^+;\mathbb{C})}.$$

Formula (4.7) follows as once. $\qquad\square$

Lemma 4.3. *For each $\lambda \in \mathbb{C}$ such that $\text{Re}\lambda > -(\text{Im}\lambda)^2$, $f \in C_b(\overline{S};\mathbb{C})$ and $g \in C_b(\overline{S_0^+};\mathbb{C})$, we denote by $\mathscr{T}_\lambda^\pm f : \overline{H_R^\mp} \to \mathbb{C}$ and $\mathscr{U}_\lambda g : \overline{H_0^+} \to \mathbb{C}$, respectively, the functions defined by*

$$(\mathscr{T}_\lambda^\pm f)(x,y) = e^{-\frac{\text{Le}}{2}(x-R)} \sum_{k \in \mathbb{Z}} e^{\pm\frac{1}{2}Y_k(x-R)} \mathscr{F}_{k,1}f(R)e_k(y), \qquad (x,y) \in \overline{H_R^\mp},$$

$$(\mathscr{U}_\lambda g)(x,y) = e^{-\frac{\text{Le}}{2}x} \sum_{k \in \mathbb{Z}} e^{-\frac{Y_k}{2}x}(\mathcal{G}_k g)(0)e_k(y), \qquad (x,y) \in \overline{H_0^+}.$$

Then, the following properties are satisfied.

(i) *$\mathscr{T}_\lambda^\pm f$ belongs to $C_b^1(\overline{H_R^\mp};\mathbb{C}) \cap W_{\text{loc}}^{2,p}(H_R^\mp;\mathbb{C})$, for each $p < +\infty$, solves the equation $\lambda u - \text{Le}^{-1}\Delta u - u_x = 0$ in $H_R^\mp$ and, for each $M > 0$, there exists a constant $c_M > 0$ such that*

$$|\lambda|\|\mathscr{T}_\lambda^\pm f\|_\infty + \sqrt{|\lambda|}\|\nabla\mathscr{T}_\lambda^\pm f\|_\infty \leq c_M\|f\|_\infty, \qquad |\lambda| \geq M; \tag{4.14}$$

(ii) *$\lim_{x \to +\infty}(\mathscr{T}_\lambda^- f)(x,y) = \lim_{x \to -\infty}(\mathscr{T}_\lambda^+ f)(x,y) = 0$ for every $y \in \mathbb{R}$ and $f \in C_b(\overline{S};\mathbb{C})$;*

(iii) *if $f \in C_b^\alpha(S;\mathbb{C})$ and $f(\cdot, -\ell/2) = f(\cdot, -\ell/2)$, then the function $\mathscr{T}_\lambda^\pm f$ belongs to $C_b^{2+\alpha}(\overline{H_R^\mp};\mathbb{C})$ and $\|\mathscr{T}_\lambda^\pm f\|_{C_b^{2+\alpha}(H_R^\mp)} \leq c\|f\|_{C_b^\alpha(S;\mathbb{C})}$ for each $\text{Re}\lambda > -(\text{Im}\lambda)^2$;*

(iv) *$\mathscr{U}_\lambda g$ belongs to $C_b^1(\overline{H_0^+};\mathbb{C}) \cap W_{\text{loc}}^{2,p}(H_0^+;\mathbb{C})$, for each $p < +\infty$, solves the equation $\lambda u - \text{Le}^{-1}\Delta u - D_x u = 0$ and, for every $M > 0$ there exists a positive constant c'_M such that*

$$|\lambda|\|\mathscr{U}_\lambda g\|_\infty + \sqrt{|\lambda|}\|\nabla\mathscr{U}_\lambda g\|_\infty \leq c'_M\|g\|_\infty, \qquad |\lambda| \geq M;$$

(v) $\lim_{x\to+\infty}(\mathscr{U}_\lambda^- g)(x,y) = 0$ *for each* $y \in \mathbb{R}$ *and* $g \in C_b(\overline{S_0^+};\mathbb{C})$;

(vi) *if* $g \in C_b^\alpha(S_0^+;\mathbb{C})$ *is such that* $g(\cdot,-\ell/2) = g(\cdot,-\ell/2)$, *then the function* $\mathscr{U}_\lambda g$ *belongs to* $C_b^{2+\alpha}(H_0^+;\mathbb{C})$ *and* $\|\mathscr{U}_\lambda g\|_{C_b^{2+\alpha}(H_0^+;\mathbb{C})} \le c\|g\|_{C_b^\alpha(S_0^+;\mathbb{C})}$ *for each* $\lambda \in \mathbb{C}$ *such that* $\operatorname{Re}\lambda > -(\operatorname{Im}\lambda)^2$.

Proof. (i) The arguments as in the proof of Lemma 4.1 show that the function $\mathscr{T}_\lambda^\pm f$ is continuous in $\overline{H_R^\mp}$ and smooth in its interior, where it solves the equation $\lambda u - \mathrm{Le}^{-1}\Delta u - u_x = 0$. Further, the function $v^\pm = \mathscr{T}_\lambda^\pm f - \mathscr{R}_{\lambda,1}f$ is bounded, vanishes on $\{R\}\times\mathbb{R}$ and $\lambda v^\pm - \mathrm{Le}^{-1}\Delta v^\pm - v_x^\pm = h$ in $H_R^\mp$, where $L^\infty(H_R^\mp;\mathbb{C}) \ni h = -\mathrm{Le}^{-1}f^\sharp + (\mathrm{Le}^{-1}-1)(\lambda\mathscr{R}_{\lambda,1}f - D_x\mathscr{R}_{\lambda,1}f)$.

By classical results, the realization of the operator $\mathrm{Le}^{-1}\Delta + D_x$ in $L^\infty(H_R^\mp;\mathbb{C})$ with homogeneous Dirichlet boundary conditions generates an analytic semigroup with domain $\{u \in C_b(\overline{H_R^\mp};\mathbb{C}) \cap \bigcap_{p<+\infty} W^{2,p}_{\mathrm{loc}}(H_R^\mp;\mathbb{C}) : \mathrm{Le}^{-1}\Delta u + D_x u \in L^\infty(H_R^\mp;\mathbb{C})\}$. In particular, for every $\lambda \in \Sigma_0$ it holds that $|\lambda|\|v^\pm\|_\infty + \sqrt{|\lambda|}\|\nabla v^\pm\|_\infty \le c\|h\|_\infty$. From the definition of $v^\pm$ and taking (4.1) into account, estimate (4.14) follows immediately.

(ii) The proof of this property is immediate since the series defining $\mathscr{T}_\lambda^+ f$ (resp. $\mathscr{T}_\lambda^- f$) converges uniformly in $\overline{H_R^-}$ (resp. in $\overline{H_R^+}$) and each of its terms vanishes as $x \to -\infty$ (resp. $x \to +\infty$), uniformly with respect to $y \in \mathbb{R}$.

(iii) Fix $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > -(\operatorname{Im}\lambda)^2$. Since $f \in C_b^\alpha(S)$ and $f(\cdot,-\ell/2) = f(\cdot,\ell/2)$, thanks to Lemma 4.1(*iii*) we can infer that the function $h \in C_b^\alpha(\mathbb{R}^2;\mathbb{C})$. Hence, $\mathrm{Le}^{-1}\Delta v^\pm + v_x^\pm \in C_b^\alpha(H_R^\mp;\mathbb{C})$ and by classical results it follows that $v \in C_b^{2+\alpha}(H_R^\mp;\mathbb{C})$ and

$$\|v^\pm\|_{C_b^{2+\alpha}(H_R^\mp;\mathbb{C})} \le c(\|v^\pm\|_\infty + \|\mathrm{Le}^{-1}\Delta v^\pm + v_x^\pm\|_{C_b^\alpha(H_R^\mp;\mathbb{C})}).$$

From the definition of $v^\pm$ and the above estimate, the assertion follows at once.

(iv)-(vi) The proof of these three properties follows applying the procedure of the first part of the proof, with $\mathscr{R}_{\lambda,1}f$ being replaced by the function $\mathscr{S}_\lambda g$. The details are left to the reader. $\qquad\square$

4.2. **Analytic semigroups and interpolation spaces.**

To state the main result of this subsection, for each $k \in \mathbb{N} \cup \{0\}$ we introduce the functions (the so-called dispersion relations)

$$\mathcal{D}_k(\lambda) = (\mathrm{Le} - Y_k)\left[\exp\left(\frac{R}{2}(\mathrm{Le} - 1 - X_k - Y_k)\right) - 1 + \theta_i R X_k\right],$$

$$\widetilde{\mathcal{D}}_k(\lambda) = (\mathrm{Le} - Y_k)\left[\exp\left(\frac{R}{2}(\mathrm{Le} - 1 - X_k - Y_k)\right) - \exp\left(\frac{R}{2}(\mathrm{Le} - 1 + X_k - Y_k)\right) + \theta_i R X_k\right],$$

where $X_k = X_k(\lambda) = \sqrt{1 + 4\lambda + 4\lambda_k}$, $Y_k = Y_k(\lambda) = \sqrt{\mathrm{Le}^2 + 4\lambda\mathrm{Le} + 4\lambda_k}$, $\lambda_k = 4\pi^2 k^2 \ell^{-2}$, and the sets

$$\Omega_k = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \ge -(\operatorname{Im}\lambda)^2 - \lambda_k \text{ and } \mathcal{D}_k(\lambda) = 0\}, \tag{4.15}$$

$$\Omega_k' = \{\lambda \in \mathbb{C} : -\mathrm{Le}^{-1}((\operatorname{Im}\lambda)^2 + \lambda_k) < \operatorname{Re}\lambda < -(\operatorname{Im}\lambda)^2 - \lambda_k, \text{ and } \widetilde{\mathcal{D}}_k(\lambda) = 0\}.$$

Theorem 4.4. *The realization L of the operator $\mathscr{L}$ in $\mathfrak{X}$, with domain*

$$D(L) = \left\{\mathbf{u} \in \mathfrak{X} : u_j(\cdot,-\ell/2) = u_j(\cdot,\ell/2), j=1,2, u_1^\sharp \in C_b^1(\overline{H_0^-};\mathbb{C}) \cap \bigcap_{p<+\infty} W^{2,p}_{\mathrm{loc}}(H_0^-;\mathbb{C})\right.$$

$$\left. u_1^\sharp, u_2^\sharp \in C^1([0,R]\times\mathbb{R};\mathbb{C}) \cap C_b^1(\overline{H_R^+};\mathbb{C}) \cap \bigcap_{p<+\infty} W^{2,p}_{\mathrm{loc}}((\mathbb{R}_+\setminus\{R\})\times\mathbb{R};\mathbb{C}), \mathscr{L}\mathbf{u}\in\mathfrak{X}, \mathscr{B}\mathbf{u}=\mathbf{0}\right\},$$

(see (3.24) and (3.25)) generates an analytic semigroup in $\mathfrak{X}$. Moreover,

(i) *the spectrum $\sigma(L)$ of L is the set* $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \le -\mathrm{Le}^{-1}(\operatorname{Im}\lambda)^2\} \cup \bigcup_{k\in\mathbb{N}\cup\{0\}}(\Omega_k \cup \Omega_k')$;

(ii) *there exist two positive constants M and c such that* $\sqrt{|\lambda|}\|\nabla R(\lambda,L)\mathbf{f}\|_\infty \le c\|\mathbf{f}\|_\infty$ *for* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda \ge M$ *and* $\mathbf{f} \in \mathfrak{X}$;

(iii) *if $\mathbf{f} \in \mathfrak{X}_\alpha$ for some $\alpha \in (0,1)$, then for each $\lambda \in \rho(L)$ the function $R(\lambda,L)\mathbf{f}$ belongs to $\mathfrak{X}_{2+\alpha}$ and $\|R(\lambda,L)\mathbf{f}\|_{2+\alpha} \le c_\lambda\|\mathbf{f}\|_\alpha$.*

Proof. Since it is rather long, we split the proof into four steps. In Steps 1 and 2, we characterize $\sigma(L)$, whereas in Step 3 we prove that L generates an analytic semigroup in $\mathfrak{X}$ as well as the estimate for the spatial gradient for the resolvent operator. Finally, in Step 4, we prove property (iii).

Step 1. Fix $\mathbf{f} \in \mathfrak{X}$ and $\lambda \in \mathbb{C}$ such that $\mathrm{Re}\,\lambda > -(\mathrm{Im}\,\lambda)^2$ and $\lambda \notin \Omega_k$ for each $k \in \mathbb{N} \cup \{0\}$, and assume that the equation $\lambda \mathbf{u} - \mathscr{L}\mathbf{u} = \mathbf{f}$ admits a solution $\mathbf{u} = (u_1, u_2)$ in $D(L)$. The arguments in the proof of Lemma 4.1(i) show that for every $k \in \mathbb{Z}$ the functions $\hat{u}_{1,k}$ and $\hat{u}_{2,k}$ (see Subsection 2.1), solve, respectively, the differential equations $\lambda \hat{u}_k - \hat{u}'_k - \hat{u}''_k + \lambda_k \hat{u}_k = \hat{f}_{1,k}$ in $\mathbb{R} \setminus \{0, R\}$ and $\lambda \hat{v}_k - \hat{v}'_k - \mathrm{Le}^{-1}\hat{v}''_k + \mathrm{Le}^{-1}\lambda_k \hat{v}_k = \hat{f}_{2,k}$ in $\mathbb{R}_+ \setminus \{R\}$. Moreover, they belong to $C_b^1(\mathbb{R}; \mathbb{C})$ and $C_b^1([0, +\infty); \mathbb{C})$, respectively. Thus,

$$\hat{u}_{1,k}(x) = (\mathcal{F}_{k,1}f_1)(x) + c_{1,k}e^{\nu_k^+ x}\chi_{(-\infty,0]}(x) + (c_{2,k}e^{\nu_k^- x} + c_{3,k}e^{\nu_k^+ x})\chi_{(0,R)}(x) + c_{4,k}e^{\nu_k^- x}\chi_{[R,+\infty)}(x)$$

for every $x \in \mathbb{R}$, $k \in \mathbb{Z}$ and

$$\hat{u}_{2,k}(x) = (\mathcal{G}_k f_2)(x) + (d_{1,k}e^{\mu_k^- x} + d_{2,k}e^{\mu_k^+ x})\chi_{[0,R)}(x) + d_{3,k}e^{\mu_k^- x}\chi_{[R,+\infty)}(x) \qquad (4.16)$$

for every $x \geq 0$ and $k \in \mathbb{Z}$, where $\mu_k^{\pm} = -\frac{\mathrm{Le}}{2} \pm \frac{1}{2}Y_k$, $\nu_k^{\pm} = -\frac{1}{2} \pm \frac{1}{2}X_k$ and $c_{1,k}, c_{2,k}, c_{3,k}, c_{4,k}$, $d_{1,k}, d_{2,k}, d_{3,k}$ ($k \in \mathbb{Z}$) are complex constants determined imposing the conditions $\mathscr{B}(\hat{u}_{1,k}, \hat{u}_{2,k}) = \mathbf{0}$ that the infinitely many functions $\hat{u}_{1,k}$ and $\hat{u}_{2,k}$ have to satisfy. It turns out that the above constants are uniquely determined if and only if $\mathcal{D}_k(\lambda) \neq 0$, as we are assuming, and as a byproduct,

$$u_1 = \mathscr{R}_{\lambda,1}f_1 + \frac{1}{\ell}\sum_{k \in \mathbb{Z}} p_{1,k}(\mathcal{F}_{k,1}f_1)(R)e_k + \frac{2}{\mathrm{Le}\,\ell}\sum_{k \in \mathbb{Z}} p_{2,k}(\mathcal{G}_k f_2)(0)e_k, \qquad \text{in } \overline{H_0^-}, \qquad (4.17)$$

$$u_1 = \mathscr{R}_{\lambda,1}f_1 + \frac{1}{\ell}\sum_{k \in \mathbb{Z}} p_{3,k}(\mathcal{F}_{k,1}f_1)(R)e_k + \frac{2}{\mathrm{Le}\,\ell}\sum_{k \in \mathbb{Z}} p_{4,k}(\mathcal{G}_k f_2)(0)e_k, \qquad (4.18)$$

$$u_2 = \mathscr{S}_\lambda f_2 - \frac{\mathrm{Le}}{2\theta_i R\ell}\mathscr{T}_\lambda^+ f_1 + \frac{1}{\ell}\mathscr{U}_\lambda^- f_2 + \frac{\mathrm{Le}}{2\ell}\sum_{k \in \mathbb{Z}} q_{1,k}(\mathcal{F}_{k,1}f_1)(R)e_k + \frac{1}{\ell}\sum_{k \in \mathbb{Z}} q_{2,k}(\mathcal{G}_k f_2)(0)e_k, \quad (4.19)$$

in $(0, R) \times \mathbb{R}$, and

$$u_1 = \mathscr{R}_{\lambda,1}f_1 + \frac{1}{\ell}\sum_{k \in \mathbb{Z}} p_{5,k}(\mathcal{F}_{k,1}f_1)(R)e_k + \frac{2\theta_i R}{\mathrm{Le}\,\ell}\sum_{k \in \mathbb{Z}} p_{6,k}(\mathcal{G}_k f_2)(0)e_k, \qquad (4.20)$$

$$u_2 = \mathscr{S}_\lambda f_2 + \frac{\mathrm{Le}}{2\theta_i R\ell}\mathscr{T}_\lambda^- f_1 + \frac{\mathrm{Le}}{2\ell}\sum_{k \in \mathbb{Z}} q_{3,k}(\mathcal{F}_{k,1}f_1)(R)e_k + \frac{1}{\ell}\sum_{k \in \mathbb{Z}} q_{4,k}(\mathcal{G}_k f_2)(0)e_k, \qquad (4.21)$$

in $\overline{H_R^+}$, where $\mathscr{R}_{\lambda,1}f$ and $\mathscr{S}_\lambda g$ have been introduced in Lemmata 4.1 and 4.2,

$$p_{1,k}(x) = \frac{e^{-\nu_k^+ R} - e^{-\mu_k^+ R}}{W_k}e^{\nu_k^+ x}, \qquad p_{2,k}(x) = \frac{(\theta_i RX_k - 1 + e^{-X_k R})Y_k}{X_k W_k(Y_k - \mathrm{Le})}e^{\nu_k^+ x}$$

$$p_{3,k}(x) = \frac{e^{\nu_k^+ (x-R)} - e^{\nu_k^- x - \mu_k^+ R}}{W_k}, \qquad p_{4,k}(x) = \frac{Y_k[(\theta_i RX_k - 1)e^{\nu_k^- x} + e^{-X_k R}e^{\nu_k^+ x}]}{X_k W_k(Y_k - \mathrm{Le})},$$

$$p_{5,k}(x) = \frac{e^{-\nu_k^- R} - e^{-\mu_k^+ R}}{W_k}e^{\nu_k^- x}, \qquad p_{6,k}(x) = \frac{Y_k}{(Y_k - \mathrm{Le})W_k}e^{\nu_k^- x},$$

$$q_{1,k}(x) = \frac{e^{(\nu_k^- - \mu_k^+)R} - 1}{\theta_i RW_k}e^{\mu_k^+ (x-R)} - \frac{X_k e^{-\mu_k^+ R}}{Y_k W_k}[(\mathrm{Le} + Y_k)e^{\mu_k^- x} - \mathrm{Le}\,e^{\mu_k^+ x}],$$

$$q_{2,k}(x) = \frac{2\mathrm{Le}(\theta_i RX_k - 1)}{W_k(Y_k - \mathrm{Le})}e^{\mu_k^- x} - \frac{e^{(\nu_k^- - \mu_k^+)R}}{W_k}(e^{\mu_k^- x} + e^{\mu_k^+ x}),$$

$$q_{3,k}(x) = \frac{1 - e^{(\nu_k^- - \mu_k^+)R}}{\theta_i RW_k}e^{\mu_k^- (x-R)} - \frac{\mathrm{Le}(e^{\mu_k^- x - \mu_k^+ R} - e^{\mu_k^- (x-R)})X_k}{Y_k W_k} - \frac{X_k e^{\mu_k^- x - \mu_k^+ R}}{W_k},$$

$$q_{4,k}(x) = \frac{Y_k(\theta_i RX_k - 1)e^{\mu_k^- x} - \mathrm{Le}\,e^{\mu_k^- x + (\nu_k^- - \mu_k^+)R} + (Y_k + \mathrm{Le})e^{\nu_k^- R + \mu_k^+ (x-R)}}{W_k(Y_k - \mathrm{Le})}$$

and $W_k = W_k(\lambda) = \theta_i R X_k - 1 + e^{(\nu_k^- - \mu_k^+)R}$.

In view of Lemmata 4.1 to 4.3, to prove that the pair $\mathbf{u}$ defined by (4.17)-(4.21) belongs to $D(L)$ and $\lambda\mathbf{u} - \mathscr{L}\mathbf{u} = \mathbf{f}$ we just need to consider the series in the above formulae, which we denote, respectively by $\mathscr{P}_{\lambda,2k+j}f_j$ ($j = 1, 2$, $k = 0, 1, 2$), $\mathscr{Q}_{\lambda,2h+j}f_j$ ($j = 1, 2$, $h = 0, 1$). To begin with, we observe that, by (i) in the proof of Lemma 4.1, we already know that $\mathrm{Re}(X_k) + \mathrm{Re}(Y_k) \geq c_\lambda|k|$ and $|X_k| + |Y_k| \leq c_\lambda(|k| + 1)$ for each $k \in \mathbb{Z}$. As a byproduct, taking also (4.3) into account, we can infer that $|(\mathcal{F}_{k,1}f_1)(R)| \leq c_\lambda(1 + k^2)^{-1}\|f_1\|_\infty$ and $|(\mathcal{G}_k f_2)(0)| \leq c_\lambda(1 + k^2)^{-1}\|f_2\|_\infty$ for each $k \in \mathbb{Z}$. Moreover, we can also estimate

$$|W_k| \geq \theta_i R|X_k| - 1 - e^{\mathrm{Re}(\nu_k^- - \mu_k^+)R} \geq \theta_i R|X_k| - 1 - e^{\frac{\mathrm{Le}-1}{2}R} \geq c_\lambda|k|, \qquad k \in \mathbb{Z} \setminus \{0\}. \qquad (4.22)$$

Putting everything together, we conclude that $\|p_{1,k}\|_{C_b^h((-\infty,0];\mathbb{C})} \leq c_1 e^{-c_2 k}$ for each $h \in \mathbb{N}$ and

$$\|p_{2,k}\|_{C_b^j((-\infty,0];\mathbb{C})} + \sum_{i=1}^2 [\|(p_{i+2,k}, q_{i,k})\|_{C^j([0,R];\mathbb{C}^2)} + \|(p_{4+i,k}, q_{i+2,k})\|_{C_b^j([R,+\infty);\mathbb{C}^2)}] \leq c_\lambda|k|^{j-1}$$

for each $k \in \mathbb{Z}\setminus\{0\}$. Using these estimates, it is easy to check that $\mathscr{P}_{\lambda,1}f_1 \in C_b^\beta(\overline{H_0^-};\mathbb{C})$ for $\beta > 0$ and $\mathscr{P}_{\lambda,2}f_2 \in C^\infty(H_0^-;\mathbb{C}) \cap C_b^1(\overline{H_0^-};\mathbb{C})$. Moreover, they solve the equation $\lambda w - \Delta_x w - D_x w = 0$ in H_0^-. Since the series which define $\mathscr{P}_{\lambda,1}f_1$ and $\mathscr{P}_{\lambda,2}f_2$ converge uniformly in H_0^- and each term vanishes as $x \to -\infty$, uniformly with respect to $y \in \mathbb{R}$, we immediately infer that $\lim_{x\to-\infty}(\mathscr{P}_{\lambda,1}f_1)(x,y) = \lim_{x\to-\infty}(\mathscr{P}_{\lambda,1}f_2)(x,y) = 0$ for each $y \in \mathbb{R}$. On the other hand, the functions $\mathscr{P}_{\lambda,3}f_1$, $\mathscr{P}_{\lambda,4}f_2$ and $\mathscr{Q}_{\lambda,1}f_1$, $\mathscr{Q}_{\lambda,2}f_2$ belong to $C^\infty((0,R) \times \mathbb{R};\mathbb{C}) \cap C_b^1([0,R] \times \mathbb{R};\mathbb{C})$ and solve, in $(0,R) \times \mathbb{R}$, the equations $\lambda w_1 - \Delta_x w_1 - D_x w_1 = 0$ and $\lambda w_2 - \mathrm{Le}^{-1}\Delta_x w_2 - D_x w_2 = 0$, respectively. Finally, the functions $\mathscr{P}_{\lambda,5}f_1$, $\mathscr{P}_{\lambda,6}f_2$ and $\mathscr{Q}_{\lambda,3}f_1$, $\mathscr{Q}_{\lambda,4}f_2$ belong to $C^\infty(H_R^+;\mathbb{C}) \cap C_b^1(\overline{H_R^+};\mathbb{C})$ solves, in H_R^+, the equations $\lambda w_1 - \Delta_x w_1 - D_x w_1 = 0$ and $\lambda w_2 - \mathrm{Le}^{-1}\Delta_x w_2 - D_x w_2 = 0$, respectively, and vanish as x tends to $+\infty$ for each $y \in \mathbb{R}$. Therefore, the function $\mathbf{u}$ defined by (4.17)-(4.21) belongs to $\bigcap_{p<+\infty} W_{\mathrm{loc}}^{2,p}((\mathbb{R} \setminus \{0, R\}) \times \mathbb{R}) \times \bigcap_{p<+\infty} W_{\mathrm{loc}}^{2,p}((\mathbb{R}_+ \setminus \{R\}) \times \mathbb{R})$, solve the equation $\lambda\mathbf{u} - \mathscr{L}\mathbf{u} = \mathbf{f}$ and $\lim_{x\to\pm\infty} u_1(x,y) = \lim_{x\to+\infty} u_2(x,y) = 0$ for each $y \in \mathbb{R}$. Moreover, $\|\nabla\mathbf{u}\|_\infty \leq c\|\mathbf{f}\|_\infty$. To conclude that $\mathbf{u} \in D(L)$, we have to check that $\mathscr{B}\mathbf{u} = \mathbf{0}$, but this is an easy task taking into account that all the series appearing in the definition of $\mathbf{u}$ may be differentiated term by term and $\mathscr{B}(\hat{u}_{1,k}, \hat{u}_{2,k}) = \mathbf{0}$ for every $k \in \mathbb{Z}$. We have so proved that $\mathbf{u} \in D(L)$ and that

$$\bigcup_{k\in\mathbb{N}\cup\{0\}} \Omega_k \subset \sigma(L) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \leq -(\mathrm{Im}\lambda)^2\} \cup \bigcup_{k\in\mathbb{N}\cup\{0\}} \Omega_k.$$

Step 2. To complete the characterization of $\sigma(L)$, let us check that $\sigma(L) \supset \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \leq -\mathrm{Le}^{-1}(\mathrm{Im}\lambda)^2\} \cup \bigcup_{k\in\mathbb{N}\cup\{0\}} \Omega_k'$. Clearly, each $\lambda \in \mathbb{C}$ such that $\mathrm{Re}\lambda \leq -\mathrm{Le}^{-1}(\mathrm{Im}\lambda)^2$ belongs to $\sigma(L)$, since in this case $\nu_0^\pm$ and $\mu_0^\pm$ have nonpositive real parts so that the more general solution to the equation $\lambda\mathbf{u} - \mathcal{L}\mathbf{u} = \mathbf{0}$, which belongs to $\mathfrak{X}$ and is independent of y, is determined up to 8 arbitrary complex constants and we have just 7 boundary condition. Thus, the previous equation admits infinitely many solutions in $\mathfrak{X}$. Similarly, if $\lambda \in \Omega_k'$ for some $k \in \mathbb{N} \cup \{0\}$, then the pair $\mathbf{u} = (\hat{u}_{1,k}e_k, \hat{u}_{2,k}e_k)$, where

$$\hat{u}_{1,k}(x) = (\mathcal{F}_{k,1}f_1)(x) + (c_{1,k}e^{\nu_k^- x} + c_{2,k}e^{\nu_k^+ x})\chi_{(0,R)}(x) + (c_{3,k}e^{\nu_k^- x} + c_{4,k}e^{\nu_k^+ x})\chi_{[R,+\infty)}(x), \qquad x \in \mathbb{R},$$

and $\hat{u}_{2,k}$ is still given by (4.16), is smooth, belongs to $\mathfrak{X}$ and solves the differential equation $\lambda\mathbf{u} - \mathscr{L}\mathbf{u} = \mathbf{f}$ for every choice of $c_{i,k}$, $d_{j,k}$ ($i = 1, \ldots, 4$, $j = 1, 2, 3$). Imposing the condition $\mathscr{B}\mathbf{u} = \mathbf{0}$, we get to a linear system of 7 equations in 7 unknowns whose determinant is $\widetilde{\mathcal{D}}_k(\lambda)$. Since $\lambda \in \Omega_k'$, the above equation admits infinitely many solutions in $D(L)$.

Step 3. Since the roots of the dispersion relation have bounded from above real part (see also the forthcoming computations), Step 1 shows that the resolvent set $\rho(L)$ contains a right-halfplane. Hence, to prove that L generates an analytic semigroup it remains to prove that $|\lambda|\|R(\lambda, L)\mathbf{f}\|_\infty \leq c\|\mathbf{f}\|_\infty$ for each λ in a suitable right-halfplane. Again, in view of Lemmata 4.1, 4.2 and 4.3, we can limit ourselves to dealing with the functions $\mathscr{P}_{\lambda,2k+j}f_j$ and $\mathscr{Q}_{\lambda,2h+j}f_j$.

For each $\lambda \in \mathbb{C}$ with positive real part, we can refine the estimate for $\mathrm{Re}(X_k)$ and $\mathrm{Re}(Y_k)$; it turns out that

$$|X_k| \geq \mathrm{Re}(X_k) = \sqrt{\frac{|1+4\lambda+4\lambda_k|+\mathrm{Re}(1+4\lambda+4\lambda_k)}{2}} \geq \sqrt{2|\lambda|} \vee 1 \vee 2\sqrt{\lambda_k} \geq \frac{\sqrt{3}}{3}\sqrt{|\lambda|+1+\lambda_k},$$

$$|X_k| \leq 2\sqrt{1+|\lambda|+\lambda_k}$$

and, similarly, $c_1\sqrt{1+|\lambda|+\lambda_k} \leq \mathrm{Re}(Y_k) \leq |Y_k| \leq c_2\sqrt{1+|\lambda|+\lambda_k}$ for each $k \in \mathbb{Z}$ and $\lambda \in \mathbb{C}$ with positive real part. As a byproduct, we get $|(\mathcal{F}_{k,1}f_1)(R)| + |(\mathcal{G}_k f_2)(0)| \leq c_R(|\lambda|+1+k^2)^{-1}\|\mathbf{f}\|_{\mathcal{X}}$ for each k and λ as above. Moreover, using (4.22) we can also estimate $|W_k| \geq c_R\sqrt{1+|\lambda|+\lambda_k}$ for each $k \in \mathbb{Z}$ and $\lambda \in \Sigma_M := \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \geq M\}$ with M large enough. Finally,

$$|Y_k - \mathrm{Le}| = \left|\frac{4\mathrm{Le}\lambda + 4\lambda_k}{\sqrt{\mathrm{Le}^2 + 4\mathrm{Le}\lambda + \lambda_k} + \mathrm{Le}}\right| \geq \frac{\mathrm{Le}|\lambda| + \lambda_k}{\sqrt{\mathrm{Le}^2 + \mathrm{Le}|\lambda| + \lambda_k}} \geq \frac{\mathrm{Le}}{2}\sqrt{1+|\lambda|+\lambda_k}$$

for each $\lambda \in \Sigma_1$ and $k \in \mathbb{Z}$, since $\mathrm{Le} \in (0,1)$. Hence, up to replacing M with $M \vee 1$, if necessary, we can estimate

$$\sum_{j=1}^{2}(|p_{j,k}(x)| + |q_{j,k}(x')|) + \sum_{j=3}^{4}(|p_{j,k}(x')| + |q_{j,k}(x'')|) + \sum_{j=5}^{6}|p_{j,k}(x'')| \leq c_R(1+|\lambda|+\lambda_k)^{-\frac{1}{2}}$$

for each $k \in \mathbb{Z}$, $\lambda \in \Sigma_M$, $x < 0$, $x' \in (0,R)$ and $x'' > R$. We are almost done. Indeed, taking the above estimates and the fact that

$$\sum_{k \in \mathbb{Z}} \frac{\|\mathbf{f}\|_\infty}{(|\lambda|+1+k^2)^{3/2}} \leq c\|\mathbf{f}\|_\infty \int_0^{+\infty}(|\lambda|+1+r^2)^{-\frac{3}{2}}dr \leq \frac{c}{1+|\lambda|}\|\mathbf{f}\|_\infty$$

into account, we easily conclude that

$$\sum_{k=1}^{3}(\|\mathscr{P}_{\lambda,2k-1}f_1\|_\infty + \|\mathscr{P}_{\lambda,2k}f_2\|_\infty) + \sum_{k=1}^{2}(\|\mathscr{Q}_{\lambda,2k-1}f_1\|_\infty + \|\mathscr{Q}_{\lambda,2k}f_2\|_\infty) \leq c_M|\lambda|^{-1}\|\mathbf{f}\|_{\mathcal{X}}$$

for every $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda \geq M$ and some positive constant c_M independent of λ. Similarly,

$$k^{1-j}\{|D_x^{(j)}p_{i,k}(x)| + |D_{x'}^{(j)}p_{i+2,k}(x')| + |D_{x'}^{(j)}q_{i,k}(x')| + |D_{x''}^{(j)}q_{i,k}(x'')| + |D_{x''}^{(j)}p_{i+4,k}(x'')|\} \leq c_M,$$

for each $x \leq 0$, $x' \in [0,R]$, $x'' \geq 0$, $i = 1,2$, $j = 0,1$ and

$$\sum_{k \in \mathbb{Z}} \frac{\|\mathbf{f}\|_\infty}{|\lambda|+1+k^2} \leq c\|\mathbf{f}\|_\infty \int_0^{+\infty}(|\lambda|+1+r^2)^{-1}dr \leq \frac{c}{\sqrt{1+|\lambda|}}\|\mathbf{f}\|_\infty.$$

Thus, we deduce that

$$\sum_{k=1}^{3}(\|\nabla\mathscr{P}_{\lambda,2k-1}f_1\|_\infty + \|\nabla\mathscr{P}_{\lambda,2k}f_2\|_\infty) + \sum_{k=1}^{2}(\|\nabla\mathscr{Q}_{\lambda,2k-1}f_1\|_\infty + \|\nabla\mathscr{Q}_{\lambda,2k}f_2\|_\infty) \leq c_M|\lambda|^{-\frac{1}{2}}\|\mathbf{f}\|_\infty.$$

Step 4. Finally, we show that if $\mathbf{f} \in \mathcal{X}_\alpha$ then $\mathbf{u} \in \mathcal{X}_{2+\alpha}$. Again, in view of Lemmata 4.1-4.3 and the estimate $\|p_{1,k}\|_{C_b^h((-\infty,0];\mathbb{C})} \leq c_1 e^{-c_2 k}$ (for every $h \in \mathbb{N}$) in Step 3, which shows that the function $\mathscr{P}_{\lambda,1}f_1$ belongs to $C_b^\beta(H_0^-)$ for every $\beta > 0$ and $\|\mathscr{P}_{\lambda,1}f_1\|_{C_b^\beta(H_0^-)} \leq c_\lambda\|f_1\|_\infty$, it suffices to deal with the other functions $\mathscr{P}_{\lambda,2k+j}f_j$ and $\mathscr{Q}_{\lambda,2h+j}f_j$. We adapt the arguments in Step 2 of the proof of Lemma 4.2. To begin with, we consider the function $\mathscr{P}_{\lambda,2}f_2 \in C_b^2(\overline{H_0^-})$ which solves the equation $\lambda\mathscr{P}_{\lambda,2}f_2 - \Delta\mathscr{P}_{\lambda,2}f_2 - D_x\mathscr{P}_{\lambda,2}f_2 = 0$ in H_0^-. To prove that it belongs to $C_b^{2+\alpha}(H_0^-)$, we check that $(\mathscr{P}_{\lambda,2}f_2)(0,\cdot) \in C_b^{2+\alpha}(\mathbb{R})$. Note that $p_{2,k}(0) = \ell(\pi k)^{-1} + \widetilde{p}_{2,k}$ for every $k \in \mathbb{Z} \setminus \{0\}$, where $\widetilde{p}_{2,k} = O(k^{-2})$. Therefore, we can split

$$(\mathscr{P}_{\lambda,2}f_2)(0,\cdot) = \frac{2}{\mathrm{Le}\,\pi}\sum_{k \in \mathbb{Z}}k^{-1}(\mathcal{G}_k f_2)(0)e_k + \frac{2}{\mathrm{Le}\,\ell}\sum_{k \in \mathbb{Z}}\widetilde{p}_{2,k}(\mathcal{G}_k f_2)(0)e_k = \psi_1 + \psi_2.$$

Since $|\widetilde{p}_{2,k}(\mathcal{G}_k f_2)(0)| \leq c|k|^{-4}\|f_2\|_\infty$ for every $k \in \mathbb{Z} \setminus \{0\}$, it follows immediately that $\psi_2 \in C_b^{2+\alpha}(\mathbb{R})$ and $\|\psi_2\|_{C_b^{2+\alpha}(\mathbb{R})} \leq c\|f_2\|_\infty$. As far as ψ_1 is concerned, a straightforward computation reveals that $\psi_1' = c(\mathscr{S}_\lambda f_2)(0,\cdot)$ so that, by Lemma 4.2, $\psi_1' \in C_b^{1+\alpha}(\mathbb{R})$ and $\|\psi_1'\|_{C_b^{1+\alpha}(\mathbb{R})} \leq c\|\mathbf{f}\|_\alpha$. Thus, $(\mathscr{P}_{\lambda,2}f_2)(0,\cdot)$ belongs to $C_b^{2+\alpha}(\mathbb{R})$ and $\|(\mathscr{P}_{\lambda,2}f_2)(0,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R})} \leq c_\lambda\|\mathbf{f}\|_\alpha$. By classical results for elliptic problems (see [29]), $\mathscr{P}_{\lambda,2}f_2$ belongs to $C_b^{2+\alpha}(H_0^-)$ and $\|\mathscr{P}_{\lambda,2}f_2\|_{C_b^{2+\alpha}(H_0^-)} \leq c_\lambda\|\mathbf{f}\|_\alpha$.

Next, we split $\mathscr{P}_{\lambda,3}f_1$ into the sum of the functions

$$\mathscr{P}_{3,\lambda,1}f_1 = \frac{1}{\ell}\sum_{k\in\mathbb{Z}} \frac{e^{\nu_k^+(\cdot-R)}}{W_k}(\mathcal{F}_{k,1}f_1)(R)e_k, \qquad \mathscr{P}_{3,\lambda,2}f_1 = \frac{1}{\ell}\sum_{k\in\mathbb{Z}} \frac{e^{-\mu_k^+ R}}{W_k}(\mathcal{F}_{k,1}f_1)(R)e^{\nu_k^-\cdot}e_k.$$

The first function belongs to $C_b^2(\overline{H_R^-};\mathbb{C})$ and $\lambda\mathscr{P}_{3,\lambda,1}f_1 - \Delta\mathscr{P}_{3,\lambda,1}f_1 - D_x\mathscr{P}_{3,\lambda,1}f_1 = 0$ in H_R^-. Since

$$((\mathscr{P}_{3,\lambda,1}f_1)(R,\cdot))' = c(\mathscr{R}_{\lambda,1}f_1)(R,\cdot) + \frac{1}{\ell}\sum_{k\in\mathbb{Z}}\widetilde{p}_{3,k}(\mathcal{F}_{k,1}f_1)(R)e_k$$

and $|\widetilde{p}_{3,k}| \leq ck^{-1}$, the same arguments as above and Lemma 4.1 allow to show first that the function $((\mathscr{P}_{3,\lambda,1}f_1)(R,\cdot))'$ belongs to $C_b^{1+\alpha}(\mathbb{R})$ and then to conclude that $\mathscr{P}_{3,\lambda,1}f_1 \in C_b^{2+\alpha}(H_R^-)$ and $\|\mathscr{P}_{3,\lambda,1}f_1\|_{C_b^{2+\alpha}(H_R^-)} \leq c_\lambda\|\mathbf{f}\|_\alpha$. The smoothness of the function $\mathscr{P}_{3,\lambda,2}f_1$ is easier to prove, due to the uniform (in $[0,R]$) exponential decay to zero of the terms of the series. It turns out that $\|\mathscr{P}_{3,\lambda,2}f_1\|_{C_b^{2+\alpha}(H_R^-)} \leq c_\lambda\|f_1\|_\infty$.

Let us consider the function $\mathscr{P}_{4,\lambda}f_2$, which we split it into the sum of the functions $\mathscr{P}_{4,\lambda,1}f_2$ and $\mathscr{P}_{4,\lambda,2}f_2$ defined, respectively, by

$$\mathscr{P}_{4,\lambda,j}f_2 = \frac{2}{\mathrm{Le}\ell}\sum_{k\in\mathbb{N}} p_{4,j,k}(\mathcal{G}_k f_2)(0)e_k, \qquad j = 1,2,$$

where

$$p_{4,1,k}(x) = \frac{\theta_i R}{W_k}e^{\nu_k^- x}, \qquad p_{4,2,k}(x) = \frac{\theta_i R\mathrm{Le}\,e^{\nu_k^- x}}{W_k(Y_k-\mathrm{Le})} - \frac{Y_k e^{\nu_k^- x}}{X_k W_k(Y_k-\mathrm{Le})} + \frac{Y_k e^{-X_k R+\nu_k^+ x}}{X_k W_k(Y_k-\mathrm{Le})}.$$

Function $\mathscr{P}_{4,\lambda,1}f_2$ belongs to $C_b^2(\overline{H_0^+})$ and $\lambda\mathscr{P}_{4,\lambda,1}f_2 - \Delta\mathscr{P}_{4,\lambda,1}f_2 - D_x\mathscr{P}_{4,\lambda,1}f_2 = 0$ in H_0^+. Moreover, $(\mathscr{P}_{4,\lambda,1}f_2)(0,\cdot)$ is an element of $C_b^{2+\alpha}(\mathbb{R})$ and $\|(\mathscr{P}_{4,\lambda,1}f_2)(0,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R})} \leq c_\lambda\|\mathbf{f}\|_\alpha$, so that $\mathscr{P}_{4,\lambda,1}f_2 \in C_b^{2+\alpha}(H_0^+)$ and $\|\mathscr{P}_{4,\lambda,1}f_2\|_{C_b^{2+\alpha}(H_0^+)} \leq c_\lambda\|\mathbf{f}\|_\alpha$. On the other hand, the series, which defines $\mathscr{P}_{4,\lambda,2}f_2$ is easier to analyze since it converges in $C_b^{2+\alpha}(H_R^-)$ and $\|\mathscr{P}_{4,\lambda,2}f_2\|_{C_b^{2+\alpha}(H_R^-)} \leq c_\lambda\|\mathbf{f}\|_\alpha$.

All the remaining functions $\mathscr{P}_{\lambda,2k+j}f_j$ and $\mathscr{Q}_{\lambda,2h+j}f_j$ can be analyzed in the same way. The details are left to the reader. $\qquad\square$

Now, we characterize the interpolation spaces $D_L(\alpha/2,\infty)$ and $D_L(1+\alpha/2,\infty)$. To simplify the notation, we introduce the operator $\mathscr{B}_0$, defined by

$$\mathscr{B}_0\mathbf{u} = (u_1(0^+,\cdot) - u_1(0^-,\cdot), u_1(R^+,\cdot) - u_1(R^-,\cdot)), \qquad \mathbf{u} \in \mathfrak{X}, \tag{4.23}$$

and the sets $\mathfrak{X}_{\alpha,\mathscr{B}_0} = \{\mathbf{u} \in \mathfrak{X}_\alpha : \mathscr{B}_0\mathbf{u} = \mathbf{0}\}$ $(\alpha \in (0,1])$ and $\mathfrak{X}_{2+\alpha,\mathscr{B}} = \{\mathbf{u} \in \mathfrak{X}_{2+\alpha} : \mathscr{B}\mathbf{u} = \mathbf{0}, \mathscr{B}_0 L\mathbf{u} = \mathbf{0}, \lim_{x\to\pm\infty}(L\mathbf{u})_1(x,y) = \lim_{x\to+\infty}(L\mathbf{u})_2(x,y) = 0\}$ $(\alpha \in (0,1))$, equipped with the norm of $\mathfrak{X}_\alpha$ and $\mathfrak{X}_{2+\alpha}$, respectively.

Proposition 4.5. *For each $\alpha \in (0,1)$ the following characterizations hold:*

$$(i)\ D_L(\alpha/2,\infty) = \mathfrak{X}_{\alpha,\mathscr{B}_0}, \qquad (ii)\ D_L(1+\alpha/2,\infty) = \mathfrak{X}_{2+\alpha,\mathscr{B}}, \tag{4.24}$$

with equivalence of respective norms. Moreover,

$$\mathfrak{X}_{1,\mathscr{B}_0} \hookrightarrow D_L(1/2,\infty). \tag{4.25}$$

Proof. Throughout the proof, we assume that α is arbitrarily fixed in $(0,1)$.

Step 1: proof of (4.24)(i) and (4.25). Given $\mathbf{f} = (f_1, f_2) \in \mathfrak{X}_{\alpha, \mathscr{B}_0}$ and $t > 0$, we introduce the functions $f_{t,1}$ and $f_{t,2}$ defined by

$$f_{t,1}(x,y) = \int_{H_0^+} f_1^\sharp(x + tx', y + ty')\varphi(x', y')dx'dy', \qquad (x,y) \in \mathbb{R}^2,$$

$$f_{t,2}(x,y) = \int_{H_0^+} \widetilde{f}_2(x - tx', y + ty')\varphi(x', y')dx'dy', \qquad (x,y) \in [0, R] \times \mathbb{R},$$

$$f_{t,2}(x,y) = \int_{H_0^+} f_2^\sharp(x + tx', y + ty')\varphi(x', y')dx'dy', \qquad (x,y) \in H_R^+,$$

where φ is a positive smooth function with compact support in $(0, R) \times \mathbb{R}$, $\|\varphi\|_{L^1(\mathbb{R}^2)} = 1$ and $\widetilde{f}_2 : \overline{H_R^-} \to \mathbb{C}$ equals the function $f_2^\sharp \vartheta$ in $[0, R] \times \mathbb{R}$, whereas $\widetilde{f}_2(x, \cdot) = f_2^\sharp(0, \cdot)\vartheta(x)$ if $x < 0$ and ϑ is a smooth function compactly supported in $(-1, +\infty)$ and equal to 1 in $(-1/2, +\infty)$. Since $\mathscr{B}_0\mathbf{f} = \mathbf{0}$ and $f_j(\cdot, -\ell/2) = f_j(\cdot, \ell/2)$ for $j = 1, 2$, the function $f_1^\sharp$ belongs to $C_b^\alpha(\mathbb{R}^2; \mathbb{C})$. On the other hand, the functions $\widetilde{f}_2$ and $f_2^\sharp$ belong to $C^\alpha(H_R^-; \mathbb{C})$ and to $C_b^\alpha(H_R^+; \mathbb{C})$, respectively. Moreover, $f_1^\sharp(\cdot, y)$ and $f_2^\sharp(\cdot, y)$ vanish at $\pm\infty$ and at $+\infty$, respectively, for every $y \in \mathbb{R}$. Hence, if we set $\mathbf{f}_t = (f_{t,1}, f_{t,2})$, then $\mathbf{f}_t$ belongs to $\mathfrak{X}$, $\|\mathbf{f} - \mathbf{f}_t\|_\infty \leq ct^\alpha\|\mathbf{f}\|_\alpha$ and $\|\mathbf{f}_t\|_\infty \leq c\|\mathbf{f}\|_\alpha$ for every $t > 0$. Similarly, since $\int_{H_0^+} D^\gamma\varphi(x', y')dx'dy' = 0$ for every multi-index γ, we can write

$$D^\gamma f_{t,1}(x,y) = t^{-|\gamma|} \int_{H_0^+} (f_1^\sharp(x + tx', y + ty') - f_1^\sharp(x, y))D^\gamma\varphi(x', y')dx'dy', \qquad (x,y) \in \mathbb{R}^2,$$

so that $|D^\gamma f_{t,1}(x,y)| \leq ct^{\alpha - |\gamma|}\|\mathbf{f}\|_\alpha$ for every $(x, y) \in \mathbb{R}^2$. In the same way we can estimate the derivatives of the function $f_{t,2}$, and conclude that $t^{|\gamma|-\alpha}\|D^\gamma\mathbf{f}_t\|_\infty \leq c_T\|\mathbf{f}\|_\alpha$ for every $t \in (0, T]$ and $T > 0$.

Now, we split $\omega^{\alpha/2}LR(\omega, L)\mathbf{f} = \omega^{\alpha/2}LR(\omega, L)(\mathbf{f} - \mathbf{f}_{\omega^{-1/2}}) + \omega^{\alpha/2}LR(\omega, L)\mathbf{f}_{\omega^{-1/2}}$ for each $\omega \in \rho(L) \cap \mathbb{R}$. By Theorem 4.4, for $\omega \in \mathbb{R}$ sufficiently large (let us say $\omega \geq \omega_0 > 0$), it holds that $\|LR(\omega, L)\|_{L(\mathfrak{X})} \leq M$ for some positive constant M, independent of ω. Hence, using the above estimates, we can infer that $\|\omega^{\alpha/2}LR(\omega, L)(\mathbf{f} - \mathbf{f}_{\omega^{-1/2}})\|_\infty \leq c\|\mathbf{f}\|_\alpha$. Next, we consider the term $\mathbf{v} = LR(\omega, L)\mathbf{f}_{\omega^{-1/2}}$, which belongs to $D(L)$ and solves the equation $\omega\mathbf{v} - L\mathbf{v} = L\mathbf{f}_{\omega^{-1/2}}$. If $\omega \geq \omega_0$, then we can estimate $\omega\|\mathbf{v}\|_\infty \leq c\|L\mathbf{f}_{\omega^{-1/2}}\|_\infty \leq c\omega^{1-\alpha/2}\|\mathbf{f}\|_\alpha$. Putting everything together, we conclude that $\omega^{\alpha/2}\|LR(\omega, L)\mathbf{f}\|_\infty \leq c\|\mathbf{f}\|_\alpha$ for each $\omega \geq \omega_0$ and this shows that $\mathbf{f} \in D_L(\alpha/2, \infty)$ and $\|\mathbf{f}\|_{D_L(\alpha/2, \infty)} \leq c\|\mathbf{f}\|_\alpha$.

Formula (4.25) can be proved just in the same way observing that, if $\mathbf{f}$ belongs to $\mathfrak{X}_{1, \mathscr{B}_0}$, then the functions $f_1^\sharp$, $\widetilde{f}_2$ and $f_2^\sharp$ are bounded and Lipschitz continuous in $\mathbb{R}^2$ in $[0, R] \times \mathbb{R}$ and in H_R^+, respectively.

The embedding "$\hookrightarrow$" in (4.24)(i) is a straightforward consequence of two properties:

$$(a)\ \|\nabla\mathbf{u}\|_\infty \leq c\|\mathbf{u}\|_\infty^{\frac{1}{2}}\|\mathbf{u}\|_{D(L)}^{\frac{1}{2}}, \quad \mathbf{u} \in D(L), \qquad (b)\ (\mathfrak{X}, \mathfrak{X}_{1, \mathscr{B}_0})_{\alpha, \infty} \hookrightarrow \mathfrak{X}_{\alpha, \mathscr{B}_0}.$$

Indeed, property (a) shows that $\mathfrak{X}_{1, \mathscr{B}_0}$ belongs to the class $J_{1/2}$ between $\mathfrak{X}$ and $D(L)$, so that applying the reiteration theorem, we get $D_L(\alpha/2, \infty) = (\mathfrak{X}, D(L))_{\alpha/2, \infty} \subset (\mathfrak{X}, \mathfrak{X}_{1, \mathscr{B}_0})_{\alpha, \infty}$ and conclude using (b).

Proof of (a). It is an almost straightforward consequence of the estimate $\|\nabla R(\lambda, L)\mathbf{f}\|_\infty \leq c|\lambda|^{-1/2}\|\mathbf{f}\|_\infty$ for each $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq M$, contained in Theorem 4.4. Indeed, let $L_M = L - MI$. As it is easily seen, $\rho(L_M) \supset (0, +\infty)$ and $R(\lambda, L_M) = R(\lambda + M, L)$ for each $\lambda > 0$. It thus follows that $\sqrt{\lambda}\|\nabla R(\lambda, L_M)\|_{L(\mathfrak{X}; \mathfrak{X} \times \mathfrak{X})} \leq c$ for each $\lambda > 0$, so that we can estimate

$$\|\nabla\mathbf{u}\|_\infty = \|\nabla R(\lambda, L_M)(\lambda\mathbf{u} - L_M\mathbf{u})\|_\infty \leq c\lambda^{-\frac{1}{2}}\|\lambda\mathbf{u} - L_M\mathbf{u}\|_\infty \leq c(\lambda^{\frac{1}{2}}\|\mathbf{u}\|_\infty + \lambda^{-\frac{1}{2}}\|L_M\mathbf{u}\|_\infty)$$

for each $\mathbf{u} \in D(L) = D(L_M)$ and $\lambda > 0$. Minimizing with respect to $\lambda > 0$, we conclude the proof.

Proof of (b). Let us fix a nontrivial $\mathbf{f} \in (\mathfrak{X}, \mathfrak{X}_{1, \mathscr{B}_0})_{\alpha, \infty}$. Since $(\mathfrak{X}, \mathfrak{X}_{1, \mathscr{B}_0})_{\alpha, \infty} \hookrightarrow (\mathfrak{X}, \mathfrak{X}_{1, \mathscr{B}_0})_{\alpha/2, 1}$ and $\mathfrak{X}_{1, \mathscr{B}_0}$ is dense in $(\mathfrak{X}, \mathfrak{X}_{1, \mathscr{B}_0})_{\alpha/2, 1}$, we immediately deduce that $f_j(\cdot, -\ell/2) = f_j(\cdot, \ell/2)$ for $j = 1, 2$

and $\mathscr{B}_0 \mathbf{f} = \mathbf{0}$. Next, we recall that

$$\inf\left\{\|\mathbf{g}\|_\infty + t\|\mathbf{h}\|_1 : \mathbf{f} = \mathbf{g} + \mathbf{h},\, \mathbf{g} \in \mathfrak{X}, \mathbf{h} \in \mathfrak{X}_{1,\mathscr{B}_0}\right\} \leq t^\alpha \|\mathbf{f}\|_{(\mathfrak{X},\mathfrak{X}_{1,\mathscr{B}_0})_{\alpha,\infty}}, \qquad t > 0.$$

Fix $(x_j, y_j) \in S_0^-$, $j = 1, 2$, and take $t = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}$. Then, we can determine $\mathbf{g} \in \mathfrak{X}$ and $\mathbf{h} \in \mathfrak{X}_{1,\mathscr{B}_0}$ such that $\mathbf{f} = \mathbf{g} + \mathbf{h}$ and

$$\|\mathbf{g}\|_\infty + \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}\|\mathbf{h}\|_1 \leq 2\|\mathbf{f}\|_{(\mathfrak{X},\mathfrak{X}_{1,\mathscr{B}_0})_{\alpha,\infty}}(|x_2 - x_1|^2 + |y_2 - y_1|^2)^{\frac{\alpha}{2}}.$$

From this estimate we can infer that

$$|f_1(x_2, y_2) - f_1(x_1, y_1)|$$
$$\leq |g_1(x_2, y_2) - g_1(x_1, y_1)| + |h_1(x_2, y_2) - h_1(x_1, y_1)| \leq 2\|\mathbf{g}\|_\infty + \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}\|\mathbf{h}\|_1$$
$$\leq 4\|\mathbf{f}\|_{(\mathfrak{X},\mathfrak{X}_{1,\mathscr{B}_0})_{\alpha,\infty}}(|x_2 - x_1|^2 + |y_2 - y_1|^2)^{\frac{\alpha}{2}}.$$

Hence, $f_1 \in C_b^\alpha(S_0^-)$ and $\|f_1\|_{C_b^\alpha(S_0^-)} \leq 5\|\mathbf{f}\|_{(\mathfrak{X},\mathfrak{X}_{1,\mathscr{B}_0})_{\alpha,\infty}}$.

The same argument can be used to prove that $f_1, f_2 \in C^\alpha((0,R) \times (-\ell/2, \ell/2); \mathbb{C}) \cap C_b^\alpha(S_R^+; \mathbb{C})$ and

$$\|f_1\|_{C_b^\alpha(S_R^+; \mathbb{C})} + \sum_{j=1}^2 \|f_j\|_{C^\alpha((0,R) \times (-\ell/2, \ell/2); \mathbb{C})} + \|f_2\|_{C_b^\alpha(S_R^+; \mathbb{C})} \leq c\|\mathbf{f}\|_{(\mathfrak{X},\mathfrak{X}_{1,\mathscr{B}_0})_{\alpha,\infty}}.$$

The proof of (b) is now complete.

Step 2: proof of (4.24)(ii). The embedding "$\hookleftarrow$" easily follows from the first part of the proof. The other embedding follows from Theorem 4.4. Indeed, fix $\mathbf{u} \in D_L(1 + \alpha/2, \infty)$ and $\lambda \in \rho(L)$. Then, the function $\mathbf{f} := \lambda\mathbf{u} - L\mathbf{u}$ belongs to $D_L(\alpha/2, \infty) = \mathfrak{X}_{\alpha,\mathscr{B}_0}$ and $\|\mathbf{f}\|_\alpha \leq c\|\mathbf{u}\|_{D_L(1+\alpha/2,\infty)}$ for some positive constant c, independent of $\mathbf{u}$. Since $\mathbf{u} = R(\lambda, L)\mathbf{f}$, Theorem 4.4 implies that $\mathbf{u} \in \mathfrak{X}_{2+\alpha}$ and $\|\mathbf{u}\|_{2+\alpha} \leq c\|\mathbf{f}\|_\alpha \leq c\|\mathbf{u}\|_{D_L(1+\alpha/2,\infty)}$. Clearly, $\mathscr{B}\mathbf{u} = \mathbf{0}$, $\mathscr{B}_0 L\mathbf{u} = \mathbf{0}$ and $\lim_{x \to \pm\infty}(L\mathbf{u})_1(x, y) = \lim_{x \to +\infty}(L\mathbf{u})_2(x, y) = 0$, and this completes the proof. $\qquad\square$

Remark 4.6. From the classical theory of analytic semigroups (see e.g., [33]) and Proposition 4.5 it follows that the part L_α of L in $\mathfrak{X}_{\alpha,\mathscr{B}_0}$, i.e., the restriction of L to $\mathfrak{X}_{2+\alpha,\mathscr{B}_0}$, generates an analytic semigroup for each $\alpha \in (0, 1)$.

4.3. **The lifting operators.** In this subsection we introduce some lifting operators which are used in the proof of the Main Theorem and Theorem 5.1.

To begin with, we consider the operator $\mathscr{M}$ defined by $\mathscr{M}\boldsymbol{\psi} = (0, M\psi_1 + \frac{1}{2}(M\psi_2)(\cdot - R, \cdot))$ on functions $\boldsymbol{\psi} \in C([-\ell/2, \ell/2]; \mathbb{R}^2)$ such that $\boldsymbol{\psi}(-\ell/2) = \boldsymbol{\psi}(\ell/2)$, where

$$(M\psi_1)(x, y) := |x|\eta(x) \int_\mathbb{R} \varphi(\xi)\psi_1^\sharp(y + \xi x)d\xi, \qquad (x, y) \in \mathbb{R}^2.$$

Here, η and φ are smooth functions such that $\chi_{(-R/4, R/4)} \leq \eta \leq \chi_{(-R/2, R/2)}$, φ is an even nonnegative function compactly supported in $(-1, 1)$ with $\|\varphi\|_{L^1(\mathbb{R})} = 1$. As it is easily seen, $\mathscr{M}\boldsymbol{\psi} \in \mathfrak{X}_{2+\alpha}$, $\mathscr{B}\mathscr{M}\boldsymbol{\psi} = (0, 0, 0, \psi_1, 0, 0, \psi_2)$ for each $\boldsymbol{\psi}$ as above.

Next, we introduce the operator $\mathscr{N}$ defined by

$$(\mathscr{N}\boldsymbol{h})_1 = N_1 h_1 + (N_1 h_2) \circ \tau_R + \frac{N_2 h_3}{2\mathrm{Le}} + \frac{(N_2 h_5) \circ \tau_R}{2\theta_i R} + N_3\left(h_8 - \frac{1}{\mathrm{Le}}h_3 - h_1''\right)$$
$$+ \left[N_3\left(h_9 - \frac{1}{\theta_i R}h_5 - h_2''\right)\right] \circ \tau_R,$$
$$(\mathscr{N}\boldsymbol{h})_2 = \left[N_1\left(h_6 - \frac{\mathrm{Le}}{\theta_i R}h_5\right)\right] \circ \tau_R + N_2 h_4 + \frac{1}{2}\left[N_3\left(h_7 - \mathrm{Le}h_6 + \frac{\mathrm{Le}^2}{\theta_i R}h_5\right)\right] \circ \tau_R$$

on smooth enough functions $\boldsymbol{h} : [-\ell/2, \ell/2] \to \mathbb{R}^9$, where $\tau_R(x, y) = (x - R, y)$,

$$(N_1\zeta)(x, y) = \frac{1}{2}\eta(x)[\chi_{[0,+\infty)}(x) - \chi_{(-\infty,0]}(x)] \int_\mathbb{R} \varphi(\sigma)\zeta^\sharp(y + \sigma x^3)d\sigma,$$

$$(N_2\zeta)(x,y) = |x|\eta(x) \int_{\mathbb{R}} \varphi(\sigma)\zeta^\sharp(y+\sigma|x|)d\sigma, \qquad (N_3\zeta)(x,y) = \frac{x}{4}(N_2\zeta)(x,y),$$

for each $(x,y) \in \mathbb{R}^2$. Moreover, we set $\mathscr{B}_*\mathbf{v} = (\widetilde{\mathscr{B}}\mathbf{v}, \mathscr{B}_0\mathscr{L}\mathbf{v})$ for each $\mathbf{v} \in \mathbf{X}_{2+\alpha}$, where $\widetilde{\mathscr{B}}$ is the operator in Remark 3.1 and the operator $\mathscr{B}_0$ is defined in (4.23).

In the next lemma, we deal with real valued spaces. In particular, by $\widetilde{D}(L_\alpha)$ we denote the subset of $D(L_\alpha)$ of real valued functions.

Lemma 4.7. *The following properties are satisfied.*

(i) *The operator $\mathscr{N}$ is bounded from the set $(\mathbf{X}_{2+\alpha})^2 \times (\mathbf{X}_{1+\alpha})^5 \times (\mathbf{X}_\alpha)^2$ into $\mathbf{X}_{2+\alpha}$. Moreover, the operator $P = I - \mathscr{N}\mathscr{B}_* : \mathbf{X}_{2+\alpha} \to \mathbf{X}_{2+\alpha}$ is a projection onto the kernel of $\mathscr{B}_*$ which coincides with $\widetilde{D}(L_\alpha)$ [2].*

(ii) *Let $\mathbf{J}$ denote the set of all functions $\mathbf{u} \in \mathbf{X}_{2+\alpha}$ such that $\mathscr{B}\mathbf{u} = \mathscr{H}(\mathbf{u})$, $\mathscr{B}_0(\mathscr{L}\mathbf{u} + \mathscr{F}(\mathbf{u})) = \mathbf{0}$. Then, there exist $r_0, r_1 > 0$ such that $\mathbf{J} \cap B(0, r_0)$ is the graph of a smooth function $\Upsilon : B(0, r_1) \subset \widetilde{D}(L_\alpha) \to (I-P)(\mathbf{X}_{2+\alpha})$ such that $\Upsilon(\mathbf{0}) = \mathbf{0}$.*

Proof. (i) Showing that $\mathscr{N}$ is a bounded operator is an easy task. Some long but straightforward computations reveal that $\mathscr{B}_*\mathscr{N} = I$ on $(\mathbf{X}_{2+\alpha})^2 \times (\mathbf{X}_{1+\alpha})^5 \times (\mathbf{X}_\alpha)^2$ and allow to prove that P is a projection onto $\mathrm{Ker}(\mathscr{B}_*) = \widetilde{D}(L_\alpha)$. In particular, we can split $\mathbf{X}_{2+\alpha} = \widetilde{D}(L_\alpha) \oplus (I-P)(\mathbf{X}_{2+\alpha})$. The details are left to the reader.

(ii) Let $\mathscr{K} : B(0, r_0) \subset \widetilde{D}(L_\alpha) \oplus (I-P)(\mathbf{X}_{2+\alpha}) \to (\mathbf{X}_{2+\alpha})^2 \times (\mathbf{X}_{1+\alpha})^5 \times (\mathbf{X}_\alpha)^2$ be the operator defined by $\mathscr{K}(\mathbf{u}, \mathbf{v}) = (\mathscr{B}(\mathbf{u}+\mathbf{v}) - \mathscr{H}(\mathbf{u}+\mathbf{v}), \mathscr{B}_0(\mathscr{L}\mathbf{u}+\mathscr{L}\mathbf{v}+\mathscr{F}(\mathbf{u}+\mathbf{v})))$ for each $(\mathbf{u}, \mathbf{v}) \in B(0, r_0)$, with r_0 small enough to guarantee that $\mathscr{K}$ is well defined. Since the functions $\mathscr{F}$ and $\mathscr{H}$ are quadratic at $\mathbf{0}$, it follows that $\mathscr{K}(\mathbf{0}, \mathbf{0}) = \mathbf{0}$ and $\mathscr{K}$ is Fréchet differentiable at $(\mathbf{0}, \mathbf{0})$, with $\mathscr{K}_\mathbf{u}(\mathbf{0}, \mathbf{0}) = \mathscr{B}_*$. In view of (i), $\mathscr{B}_*$ is an isomorphism from $(I-P)(\mathbf{X}_{2+\alpha})$ to $(\mathbf{X}_{2+\alpha})^2 \times (\mathbf{X}_{1+\alpha})^5 \times (\mathbf{X}_\alpha)^2$. Thus, we can invoke the implicit function theorem to complete the proof. $\square$

5. Solving the nonlinear problem (3.23)

Now we are able to solve the nonlinear Cauchy problem

$$\begin{cases} D_t\mathbf{u} = \mathscr{L}\mathbf{u} + \mathscr{F}(\mathbf{u}), \\ \mathscr{B}\mathbf{u} = \mathscr{H}(\mathbf{u}), \\ D_x^{\gamma_1} D_y^{\gamma_2}\mathbf{u}(\cdot, \cdot, -\ell/2) = D_x^{\gamma_1} D_y^{\gamma_2}\mathbf{u}(\cdot, \cdot, \ell/2), \qquad \gamma_1 + \gamma_2 \leq 2, \end{cases} \tag{5.1}$$

for the unknown $\mathbf{u} = (u, w)$. Also in this section we assume that the function spaces that we deal with are real valued ones.

Theorem 5.1. *Fix $\alpha \in (0, 1)$ and $T > 0$. Then, there exists $r_0 = r_0(T) > 0$ such that, for each $\mathbf{u}_0 \in B(0, r_0) \subset \mathbf{X}_{2+\alpha}$ satisfying the compatibility conditions $\mathscr{B}\mathbf{u}_0 = \mathscr{H}(\mathbf{u}_0)$, $\mathscr{B}_0(\mathscr{L}\mathbf{u}_0 + \mathscr{F}(\mathbf{u}_0)) = \mathbf{0}$ and $D^\gamma\mathbf{u_0}(\cdot, -\ell/2) = D^\gamma\mathbf{u_0}(\cdot, \ell/2)$ for each multi-index γ with length at most two, problem (5.1) admits a unique solution $\mathbf{u} \in \mathbf{Y}_{2+\alpha}$ with $\mathbf{u}(0, \cdot) = \mathbf{u}_0$. Moreover, $\|\mathbf{u}\|_{\mathbf{Y}_{2+\alpha}} \leq c\|\mathbf{u}_0\|_{2+\alpha}$.*

Proof. The proof can be obtained arguing as in the proof of [30, Theorem 4.1]. For this purpose, we just sketch the main points.

We first need to prove optimal regularity results for the linear version of problem (5.1), i.e., with the problem

$$\begin{cases} D_t\mathbf{u}(t, \cdot, \cdot) = \mathscr{L}\mathbf{u}(t, \cdot, \cdot) + \mathbf{f}(t, \cdot, \cdot), & t \in [0, T], \\ \mathscr{B}(\mathbf{u}(t, \cdot, \cdot)) = \mathbf{h}(t, \cdot), & t \in [0, T], \\ D_x^{\gamma_1} D_y^{\gamma_2}\mathbf{u}(t, \cdot, -\ell/2) = D_x^{\gamma_1} D_y^{\gamma_2}\mathbf{u}(t, \cdot, \ell/2), & t \in [0, T], \quad \gamma_1 + \gamma_2 \leq 2, \\ \mathbf{u}(0, \cdot) = \mathbf{u}_0, \end{cases} \tag{5.2}$$

when $\mathbf{f} \in \mathbf{Y}_\alpha$, $\mathbf{u}_0 \in \mathbf{X}_{2+\alpha}$, when $h_j \equiv 0$ if $j \neq 4, 7$, $h_4 = \psi_1$, $h_7 = \psi_2$, $\boldsymbol{\psi} = (\psi_1, \psi_2) \in C^{(1+\alpha)/2, 1+\alpha}((0, T) \times (-\ell/2, \ell/2); \mathbb{R}^2)$ satisfy the compatibility conditions

- $\mathscr{B}\mathbf{u_0} = \mathbf{h}(0, \cdot)$, $\mathscr{B}_0(\mathscr{L}\mathbf{u_0}(0, \cdot) + \mathbf{f}(0, \cdot)) = \mathbf{0}$;

---

[2]see Remark 4.6

- $\mathbf{f}(0,\cdot,-\ell/2) = \mathbf{f}(0,\cdot,\ell/2)$, $D^\gamma \mathbf{u_0}(\cdot,-\ell/2) = D^\gamma \mathbf{u_0}(\cdot,\ell/2)$ and $D_y^{(j)}\boldsymbol{\psi}(\cdot,-\ell/2) = D_y^{(j)}\boldsymbol{\psi}(\cdot,\ell/2)$ for every multi-index γ with length at most two and $j = 0,1$.

We also need to show the estimate

$$\|\mathbf{u}\|_{\mathcal{Y}_{2+\alpha}} \le c_0(\|\mathbf{f}\|_{\mathcal{Y}_\alpha} + \|\mathbf{u_0}\|_{2+\alpha} + \|\boldsymbol{\psi}\|_{C^{(1+\alpha)/2,1+\alpha}((0,T)\times(-\ell/2,\ell/2);\mathbb{R}^2)}). \qquad (5.3)$$

for its unique solution $\mathbf{u} \in \mathcal{Y}_{2+\alpha}$. This is the content of Steps 1 to 3.

Step 1. To begin with, we note that $Mh \in C_b^{(1+\alpha)/2,2+\alpha}((0,T)\times H_0^-) \cap C_b^{(1+\alpha)/2,2+\alpha}((0,T)\times H_0^+)$ for all $h \in C^{(1+\alpha)/2,1+\alpha}((0,T)\times(-\ell/2,\ell/2))$ such that $D_y^{(j)}h(\cdot,-\ell/2) = D_y^{(j)}h(\cdot,\ell/2)$ ($j = 0,1$), and

$$\|Mh\|_{C_b^{(1+\alpha)/2,2+\alpha}((0,T)\times H_0^-)} + \|Mh\|_{C_b^{(1+\alpha)/2,2+\alpha}((0,T)\times H_0^+)} \le c\|h\|_{C^{(1+\alpha)/2,1+\alpha}((0,T)\times(-\ell/2,\ell/2))}.$$

Thus, the function $\mathbf{f} + \mathscr{L}\mathscr{M}\boldsymbol{\psi}$ belongs to $C^{\alpha/2}([0,T];\mathcal{X})$. Moreover, by Proposition 4.5 and the compatibility conditions on $\mathbf{u_0}$ it follows that $\mathbf{u_0} - \mathscr{M}(\boldsymbol{\psi}(0,\cdot)) \in D(L)$, $\mathscr{L}\mathbf{u_0} + \mathbf{f}(0,\cdot) \in D_L(\alpha/2,\infty)$. The theory of analytic semigroups (see e.g., [33, Chapter 4]), Theorem 4.4 and Proposition 4.5 show that there exists a unique function $\mathbf{v} \in C^{1+\alpha/2}([0,T];\mathcal{X}) \cap C([0,T];D(L))$ which solves the equation $D_t\mathbf{v} = \mathscr{L}\mathbf{v} + \mathbf{f} + \mathscr{L}\mathscr{M}\boldsymbol{\psi}$ and satisfies the condition $\mathbf{v}(0,\cdot) = \mathbf{u_0}$. In addition $D_t\mathbf{v}$ is bounded with values in $\mathcal{X}_{\alpha,\mathscr{B}_0}$ (which implies that $D_t\mathbf{v} \in \mathcal{Y}_\alpha$) and $\mathscr{L}\mathbf{v} \in C^\alpha([0,T];\mathcal{X})$. By difference, $\mathscr{L}\mathbf{v} = L\mathbf{v}$ is bounded in $[0,T]$ with values in $\mathcal{X}_\alpha$ and, in view of Theorem 4.4, $\mathbf{v}$ is bounded in $[0,T]$ with values in $\mathcal{X}_{2+\alpha}$. In particular, $\mathscr{B}\mathbf{v} = \mathbf{0}$ and $D^\gamma\mathbf{v}(\cdot,\cdot,-\ell/2) = D^\gamma\mathbf{v}(\cdot,\cdot,\ell/2)$ for every $|\gamma| \le 2$. Further, $\|\mathbf{v}(t,\cdot)\|_{2+\alpha} + \|D_t\mathbf{v}\|_{\mathcal{Y}_\alpha} \le c(\|\mathbf{f}\|_{\mathcal{Y}_\alpha} + \|\mathbf{u_0}\|_{2+\alpha} + \|\boldsymbol{\psi}\|_{C^{(1+\alpha)/2,1+\alpha}((0,T)\times(-\ell/2,\ell/2);\mathbb{R}^2)})$ for every $t \in [0,T]$.

Step 2. Let $\widetilde{\mathbf{w}}$ be the function defined by $\widetilde{\mathbf{w}}(t,\cdot,\cdot) = \int_0^t e^{(t-s)L}(\mathscr{M}\boldsymbol{\psi}(s,\cdot) - \mathscr{M}\boldsymbol{\psi}(0,\cdot))ds$ for every $t \in [0,T]$, where $\{e^{tL}\}$ is the analytic semigroup generated by L in $\mathcal{X}$. Taking (4.25) into account, it follows that $\|\mathscr{M}\boldsymbol{\psi}\|_{C^{(1+\alpha)/2}([0,T],D_L(1/2,\infty))} \le c\|\boldsymbol{\psi}\|_{C^{(1+\alpha)/2,1+\alpha}((0,T)\times(-\ell/2,\ell/2);\mathbb{R}^2)}$. Again by the theory of analytic semigroups we infer that $\widetilde{\mathbf{w}} \in C([0,T];D(L))$, $D_t\widetilde{\mathbf{w}}$ is bounded in $[0,T]$ with values in $\mathcal{X}_{2+\alpha,\mathscr{B}}$, $L\widetilde{\mathbf{w}} \in C^{1+\alpha/2}([0,T];\mathcal{X})$, $D_t\widetilde{\mathbf{w}} = L\widetilde{\mathbf{w}} + \mathscr{M}\boldsymbol{\psi} - \mathscr{M}\boldsymbol{\psi}(0,\cdot)$ and $\|D_t\widetilde{\mathbf{w}}\|_{C^{1+\alpha/2}([0,T];D(L))} + \sup_{t\in[0,T]}\|LD_t\widetilde{\mathbf{w}}(t,\cdot)\|_{\mathcal{X}_{\alpha,\mathscr{B}_0}} \le c\|\boldsymbol{\psi}\|_{C^{(1+\alpha)/2,1+\alpha}((0,T)\times(-\ell/2,\ell/2);\mathbb{R}^2)}$. From these properties and using the same arguments as above, it can be easily checked that the function $\mathbf{w} = -L\widetilde{\mathbf{w}} + \mathscr{M}(\boldsymbol{\psi}(0,\cdot))$ is as smooth as $\mathbf{v}$ is. Moreover, $D_t\mathbf{w} = \mathscr{L}\mathbf{w} - \mathscr{L}\mathscr{M}\boldsymbol{\psi}$, $\mathscr{B}\mathbf{w} = (0,0,0,\psi_1,0,0,\psi_2)$, $\mathbf{w}(0,\cdot,\cdot) = \mathscr{M}(\boldsymbol{\psi}(0,\cdot))$ and $D^\gamma\mathbf{w}(\cdot,-\ell/2) = D^\gamma\mathbf{w}(\cdot,\ell/2)$ for every $|\gamma| \le 2$.

Step 3. Clearly, the function $\mathbf{u} = \mathbf{v} + \mathbf{w} \in \mathcal{Y}_{2+\alpha}$ solves the Cauchy problem (5.2), satisfies (5.3) and it is the unique solution to the above problem in $C^1([0,T];\mathcal{X}) \cap C([0,T];D(L))$. Moreover,

$$\sup_{t\in[0,T]} \|\mathbf{u}(t,\cdot)\|_{2+\alpha} + \|D_t\mathbf{u}\|_{\mathcal{Y}_\alpha} \le c(\|\mathbf{f}\|_{\mathcal{Y}_\alpha} + \|\mathbf{u_0}\|_{2+\alpha} + \|\boldsymbol{\psi}\|_{C^{(1+\alpha)/2,1+\alpha}((0,T)\times(-\ell/2,\ell/2);\mathbb{R}^2)}). \quad (5.4)$$

To conclude that $\mathbf{u} \in \mathcal{Y}_{2+\alpha}$ and it satisfies estimate (5.3), we use an interpolation argument. It is well known that $\|\cdot\|_{C_b^2(H_R^+;\mathbb{R}^2)} \le c\|\cdot\|_{C_b^\alpha(H_R^+;\mathbb{R}^2)}^{\alpha/2}\|\cdot\|_{C_b^{2+\alpha}(H_R^+;\mathbb{R}^2)}^{1-\alpha/2}$. Using this estimate and the formula

$$\mathbf{u}(t,x,y) - \mathbf{u}(s,x,y) = \int_s^t D_t\mathbf{u}(r,x,y)dr, \qquad s,t \in [0,T], \ (x,y) \in H_R^+,$$

to show that $\|\mathbf{u}(t,\cdot) - \mathbf{u}(s,\cdot)\|_{C_b^\alpha(H_R^+;\mathbb{R}^2)} \le \|D_t\mathbf{u}\|_{\mathcal{Y}_\alpha}|t-s|$, we get $\|\mathbf{u}(t,\cdot) - \mathbf{u}(s,\cdot)\|_{C_b^2(H_R^+;\mathbb{R}^2)} \le c\|D_t\mathbf{u}\|_{\mathcal{Y}_\alpha}^{\alpha/2}\sup_{r\in[0,T]}\|\mathbf{u}(r,\cdot)\|_{2+\alpha}^{1-\frac{\alpha}{2}}|t-s|^{\alpha/2}$.

In the same way, we can show that $u_1 \in C^{\alpha/2}([0,T];C_b^2(\overline{H_R^-};\mathbb{R}^2))$, $\mathbf{u} \in C^{\alpha/2}([0,T];C_b^2([0,R] \times [-\ell/2,\ell/2];\mathbb{R}^2))$ and

$$\|\mathbf{u}\|_{C^{\alpha/2}([0,T];C_b^2([0,R]\times[-\ell/2,\ell/2];\mathbb{R}^2))} + \|u_1\|_{C^{\alpha/2}([0,T];C_b^2(\overline{H_R^-};\mathbb{R}^2))} \le c\left(\|D_t\mathbf{u}\|_{\mathcal{Y}_\alpha} + \sup_{t\in[0,T]}\|\mathbf{u}(t,\cdot)\|_{2+\alpha}\right).$$

Taking (5.4) into account we complete this step of the proof. In particular, from all the above results it follows that

$$\mathbf{u}(t,\cdot,\cdot) = e^{tL}\mathbf{u_0} + \int_0^t e^{(t-s)L}[\mathbf{f}(s,\cdot,\cdot) + \mathscr{L}\mathscr{M}(\boldsymbol{\psi}(s,\cdot))]ds - L\int_0^t e^{(t-s)L}\mathscr{M}(\boldsymbol{\psi}(s,\cdot))ds, \qquad t \in [0,T].$$
$$(5.5)$$

Step 4. Let $r > 0$ and $\mathfrak{C}_r$ be the space of all $\mathbf{u} \in \mathcal{Y}_{2+\alpha}$ such that $D_x^{\gamma_1} D_y^{\gamma_2} \mathbf{u}(\cdot, \cdot, -\ell/2) = D_x^{\gamma_1} D_y^{\gamma_2} \mathbf{u}(\cdot, \cdot, \ell/2)$, for every $0 \le \gamma_1, \gamma_2$ such that $\gamma_1 + \gamma_2 \le 2$, $\|\mathbf{u}\|_{\mathcal{Y}_{2+\alpha}} \le r$ and $\mathbf{u}(0, \cdot, \cdot) = \mathbf{u}_0$.

In view of Steps 1-3, for every $\mathbf{u}_0 \in B(0, r_0) \subset \mathcal{Y}_{2+\alpha}$ satisfying the compatibility conditions in the statement of the theorem, we can define the operator Γ, which to every $\mathbf{u} \in \mathfrak{C}_r$ (with r sufficiently small norm to guarantee that the nonlinear terms $\mathscr{F}(\mathbf{u}(t, \cdot, \cdot))$ and $\mathscr{H}(\mathbf{u}(t, \cdot, \cdot))$ are well defined for every $t \in [0, T]$) associates the unique solution $\mathbf{v}$ of the Cauchy problem (5.2) with $\mathbf{f} = \mathscr{F}(\mathbf{u})$ and $\boldsymbol{\psi} = \mathscr{H}(\mathbf{u})$. Since the maps $\mathbf{u} \mapsto \mathscr{F}(\mathbf{u})$, $\mathbf{u} \mapsto \mathscr{H}(\mathbf{u})$ are smooth in $\mathfrak{C}_r$ and quadratic at $\mathbf{u} = \mathbf{0}$, we can estimate

$$
\begin{aligned}
&\|\mathscr{F}(\mathbf{u})\|_{\mathcal{Y}_\alpha} \le c\|\mathbf{u}\|_{\mathcal{Y}_{2+\alpha}}^2, \qquad \|\mathscr{F}(\mathbf{u}) - \mathscr{F}(\mathbf{v})\|_{\mathcal{Y}_\alpha} \le cr\|\mathbf{u} - \mathbf{v}\|_{\mathcal{Y}_\alpha}, \\
&\|\mathscr{H}(\mathbf{u})\|_{C^{(1+\alpha)/2, 1+\alpha}((0,T) \times (-\ell/2, \ell/2); \mathbb{R}^2)} \le c\|\mathbf{u}\|_{\mathcal{Y}_{2+\alpha}}^2, \\
&\|\mathscr{H}(\mathbf{u}) - \mathscr{H}(\mathbf{v})\|_{C^{(1+\alpha)/2, 1+\alpha}((0,T) \times (-\ell/2, \ell/2); \mathbb{R}^2)} \le cr\|\mathbf{u} - \mathbf{v}\|_{\mathcal{Y}_\alpha}.
\end{aligned}
\tag{5.6}
$$

These estimates combined with (5.3) show that r and r_0 can be determined small enough such that Γ is a contraction in $\mathfrak{C}_\rho$.

Uniqueness of the solution $\mathbf{u}$ to (5.1) follows from standard arguments, which we briefly sketch here. At first, for every $t_0 \in [0, T]$, $R, \delta > 0$ and $\mathbf{u}_1 \in \mathcal{X}_{2+\alpha}$, which satisfies the compatibility conditions $\mathscr{B}(\mathbf{u}_1) = \mathscr{H}(\mathbf{u}_1)$, $\mathscr{B}_0(\mathscr{L}\mathbf{u}_1 + \mathscr{F}(\mathbf{u}_1)) = 0$ and $D^\gamma \mathbf{u}_1(\cdot, -\ell/2) = D^\gamma \mathbf{u}_1(\cdot, \ell/2)$ for each multi-index γ with length at most two, we set

$$
\mathcal{Z}_{\delta, R}^{t_0}(\mathbf{u}_1) := \{\mathbf{u} \in \mathcal{Y}_{2+\alpha}(t_0, t_0 + \delta) : \mathbf{u}(t_0, \cdot) = \mathbf{u}_1, \ \|\mathbf{u} - \mathbf{u}_1\|_{\mathcal{Y}_{2+\alpha}(t_0, t_0 + \delta)} \le R\}.
$$

Given $R > 0$ we can determine $r_1 > 0$ and $\delta > 0$ (independent of t_0) with $\delta^{\alpha/2} R$ sufficiently small such that, if $\mathbf{u}_1$ belongs to $B(0, r_1) \subset \mathcal{X}_{2+\alpha}$, then the Cauchy problem

$$
\begin{cases}
D_t \mathbf{w} = \mathscr{L}\mathbf{w} + \mathscr{F}(\mathbf{w}), \\
\mathscr{B}\mathbf{w} = \mathscr{H}(\mathbf{w}), \\
D_x^{\gamma_1} D_y^{\gamma_2} \mathbf{w}(\cdot, \cdot, -\ell/2) = D_x^{\gamma_1} D_y^{\gamma_2} \mathbf{w}(\cdot, \cdot, \ell/2), \qquad \gamma_1 + \gamma_2 \le 2 \\
\mathbf{w}(t_0, \cdot) = \mathbf{u}_1,
\end{cases}
\tag{5.7}
$$

admits a unique solution $\mathbf{w} \in \mathcal{X}_{\delta, R}^{t_0}(\mathbf{u}_1)$. We are almost done. Indeed, let $\mathbf{u} \in \mathfrak{C}_r$ be the unique fixed point of Γ, and take r_0 small enough such that $\mathbf{u} \in B(\mathbf{0}, \rho_1) \subset \mathcal{Y}_{2+\alpha}$. Assume that $\mathbf{v} \in \mathcal{Y}_{2+\alpha}$ is another solution to (5.1), and let $t_0 > 0$ denote the supremum of the set $\{\tau \in [0, T] : \mathbf{u}(t, \cdot) = \mathbf{v}(t, \cdot), t \in [0, \tau]\}$. Suppose by contradiction that $t_0 < T$. Then, both $\mathbf{u}$ and $\mathbf{v}$ are solutions in $\mathcal{Y}_{2+\alpha}(t_0, t_0 + \delta)$ to the Cauchy problem (5.7), with $\mathbf{u_1} := \mathbf{u}(t_0, \cdot) = \mathbf{v}(t_0, \cdot)$. Taking $R \ge 2\max\{\|\mathbf{u}\|_{\mathcal{Y}_{2+\alpha}}, \|\mathbf{v}\|_{\mathcal{Y}_{2+\alpha}}\}$ large enough and $\delta > 0$ small enough, it follows that $\mathbf{u}$ and $\mathbf{v}$ both belong to $\mathcal{Z}_{\delta, R}^{t_0}(\mathbf{u}_1)$, so that they do coincide, leading us to a contradiction. $\qquad\square$

Remark 5.2. Since problem (5.1) is autonomous, under the same assumptions as in Theorem 5.1, for each $a > 0$ and $T > 0$ there exists a unique solution $\mathbf{u} \in \mathcal{Y}_{2+\alpha}(a, a + T)$ such that $\mathbf{u}(a, \cdot) = \mathbf{u}_0$.

6. PROOF OF THE MAIN RESULT

6.1. **Study of the dispersion relation and the point spectrum.** Since we are interested in the instability of the traveling wave solution $(\Theta^{(0)}, \Phi^{(0)})$ to problem (1.3)-(1.4), we need to determine a range of Lewis numbers Le which lead to eigenvalues of L_α (see Remark 4.6) with positive real part. In view of Theorem 4.4, such eigenvalues will lie in Ω_k (see (4.15)). For simplicity, we will look for positive real elements of Ω_k. Note that $\mathrm{Le} - Y_k$ vanishes for no λ's with positive real part. To determine elements of $\sigma(L_\alpha)$ with positive real part we need to analyze the reduced dispersion relation

$$
\mathcal{D}_{0,k}(\lambda, \mathrm{Le}) = \exp\left(\frac{R}{2}(\mathrm{Le} - 1 - X_k(\lambda) - Y_k(\lambda, \mathrm{Le}))\right) - 1 + \theta_i R X_k(\lambda).
$$

We recall that $X_k(\lambda) = \sqrt{1 + 4\lambda + 4\lambda_k}$, $Y_k(\lambda, \mathrm{Le}) = Y_k(\lambda) = \sqrt{\mathrm{Le}^2 + 4\lambda \mathrm{Le} + 4\lambda_k}$ and $\lambda_k = 4\pi^2 k^2 \ell^{-2}$ for each $k \in \mathbb{N} \cup \{0\}$. Throughout this subsection we assume θ_i is fixed in $(0, 1)$; so is $R > 0$ via (1.5).

Lemma 6.1. *There exists $\ell_0(\theta_i)$ such that, for all $\ell > \ell_0(\theta_i)$, $\mathcal{D}_{0,1}(0,\mathrm{Le}) = 0$ has a unique root $\mathrm{Le}_c = \mathrm{Le}_c(1) \in (0,1)$. Moreover, for each ℓ fixed as above, there exists a maximal integer $K \geq 1$ such that, for $k \in \{1,\ldots,K\}$, $\mathcal{D}_{0,k}(0,\mathrm{Le}) = 0$ has a unique root $\mathrm{Le}_c(k) \in (0,1)$. Finally, it holds:*

$$0 < \mathrm{Le}_c(K) \leq \ldots \leq \mathrm{Le}_c(2) \leq \mathrm{Le}_c(1). \tag{6.1}$$

Proof. An easy but formal computation shows that, if $\mathrm{Le}_c(k)$ is a root of $\mathcal{D}_{0,k}(0,\cdot)$, then

$$\mathrm{Le}_c(k) = 1 + X_k(0) + Y_k(0,\mathrm{Le}_c(k)) + 2R^{-1}\ln(1 - \theta_i R X_k(0)),$$

or equivalently:

$$\mathrm{Le}_c(k) = \frac{(1 + X_k(0))[R^2 + 2R\log(1 - \theta_i R X_k(0))] + 2|\log(1 - \theta_i R X_k(0))|^2}{R^2(1 + X_k(0)) + 2R\log(1 - \theta_i R X_k(0))}. \tag{6.2}$$

However, Formula (6.2) makes sense only if $1 - \theta_i R X_k(0) > 0$. Hence, for each fixed $\ell > 0$ there exists $K \in \mathbb{N}$ such that $1 - \theta_i R X_k(0) > 0$ if and only if $k \leq K$.

Further, $\mathrm{Le}_c(k)$ is required to meet the physical requirement that $0 < \mathrm{Le}_c(k) < 1$. In this respect, ℓ should be large enough:

$$\mathrm{Le}_c(1) = \mathrm{Le}_0 + \frac{16\pi^2\theta_i e^R(1 - \theta_i e^R)}{(2\theta_i e^R - 1)\ell^2} + o(\ell^{-2})$$

as $\ell \to +\infty$, where $\mathrm{Le}_0 = R(2e^R - R - 2)^{-1}$ belongs to $(0,1)$ see [4, Formula (43), p. 2083]. Thus, there exists $\ell_0(\theta_i) > 0$ such that, if $\ell > \ell_0(\theta_i)$, then $\mathrm{Le}_c(1) \in (0,1)$.

Finally, it remains to prove property (6.1), for a fixed $\ell > \ell_0(\theta_i)$ which in turn defines the integer $K \geq 1$. The latter property follows from the following estimate (see [6, Proposition 3.1]):

$$\left(\sqrt{(\mathrm{Le}_c(k))^2 + 4\lambda_k} - \mathrm{Le}_c(k)\right)\frac{d\mathrm{Le}_c(k)}{d\lambda_k} < 4\left(1 - \frac{\theta_i}{1 - \theta_i R}\right).$$

Obviously, $\dfrac{\theta_i}{1 - \theta_i R} = \dfrac{1 - e^{-R}}{Re^{-R}} = \dfrac{e^R - 1}{R} > 1$, which implies that $\dfrac{d\mathrm{Le}_c(k)}{d\lambda_k} < 0$. $\square$

In view of Lemma 6.1 our focus will be on the case when $\mathrm{Le} \in (0,\mathrm{Le}_c(1))$. Hereafter we will simply denote the critical value $\mathrm{Le}_c(1)$ by Le_c, keeping in mind that Le_c at fixed $0 < \theta_i < 1$ depends on $\ell > \ell_0(\theta_i)$.

Lemma 6.2. *The function $\mathcal{D}_{0,1}$ is smooth in $[0,\sqrt{\lambda_1}] \times [0,\mathrm{Le}_c]$. Moreover, $\dfrac{\partial\mathcal{D}_{0,1}}{\partial\lambda}$ is positive in $[0,\sqrt{\lambda_1}] \times [0,\mathrm{Le}_c]$, $\dfrac{\partial\mathcal{D}_{0,1}}{\partial\mathrm{Le}}$ is positive in $[0,\sqrt{\lambda_1}) \times [0,\mathrm{Le}_c]$ and vanishes on $\sqrt{\lambda_1} \times [0,\mathrm{Le}_c]$.*

Proof. The proof of the positivity of $\dfrac{\partial\mathcal{D}_{0,1}}{\partial\mathrm{Le}}$ is straightforward and based on the observation that $Y_1(\lambda,\mathrm{Le}) - \mathrm{Le} - 2\lambda > 0$ for $\lambda > \sqrt{\lambda_1}$ and $Y_1(\sqrt{\lambda_1}),\mathrm{Le}) = \mathrm{Le} + 2\sqrt{\lambda_1}$. On the other hand, we observe that

$$\frac{\partial\mathcal{D}_{0,1}}{\partial\lambda}(\lambda,\mathrm{Le}) = -R\exp\left(\frac{R}{2}(\mathrm{Le} - 1 - X_1(\lambda) - Y_1(\lambda,\mathrm{Le}))\right)(X_1^{-1} + \mathrm{Le}\,Y_1^{-1}) + 2(1 - e^{-R})X_1^{-1}.$$

Since $\mathrm{Le}\,Y_1^{-1} < (X_1)^{-1}$ and $\mathrm{Le} - 1 - X_1(\lambda) - Y_1(\lambda,\mathrm{Le}) < -2$, we can estimate

$$\frac{\partial\mathcal{D}_{0,1}}{\partial\lambda}(\lambda,\mathrm{Le}) > 2(1 - (R+1)e^{-R})X_1^{-1}, \qquad (\lambda,\mathrm{Le}_c) \in [0,\sqrt{\lambda_1}] \times [0,\mathrm{Le}_c],$$

and the positivity of $\dfrac{\partial\mathcal{D}_{0,1}}{\partial\lambda}$ in $[0,\sqrt{\lambda_1}] \times [0,\mathrm{Le}_c]$ follows immediately. $\square$

We can now prove the following result.

Proposition 6.3. *Under the assumptions of Lemma 6.1, there exist $\lambda_* \in (0,\sqrt{\lambda_1})$ and a decreasing, continuously differentiable function $\widetilde{\varphi} : (0,\mathrm{Le}_c) \to (0,\lambda_*)$ such that $\mathcal{D}_{0,1}(\widetilde{\varphi}(\mathrm{Le}),\mathrm{Le}) = 0$ for all $\mathrm{Le} \in (0,\mathrm{Le}_c)$.*

Proof. From Lemma 6.2 it follows that $\frac{\partial \mathcal{D}_{0,1}}{\partial \text{Le}}(0, \text{Le}_c) > 0$. We can thus apply the implicit function theorem, which shows that there exist $\delta_1, \delta_2 > 0$ and a unique function $\varphi \in C^1([\text{Le}_c - \delta_1, \text{Le}_c + \delta_1])$ such that, if $(\lambda, \text{Le}) \in [\text{Le}_c - \delta_1, \text{Le}_c + \delta_1] \times [-\delta_2, \delta_2]$ is a root of $\mathcal{D}_{0,1}$, then $\lambda = \varphi(\text{Le})$. In view of the previous lemma, φ is a decreasing function. As a byproduct, taking the restriction of φ to $[\text{Le}_c - \delta_1, \text{Le}_c]$, we have constructed a (small) branch $\lambda = \varphi(\text{Le})$ of positive roots of $\mathcal{D}_{0,1}(\lambda, \text{Le}) = 0$. We may reiterate the implicit function theorem and continue this branch up to a left endpoint $(\lambda_*, \text{Le}_*) \in [0, \sqrt{\lambda_1}] \times [0, \text{Le}_c]$. By continuity, $\mathcal{D}_{0,1}(\lambda_*, \text{Le}_*) = 0$. This maximal extension of φ is a non-increasing, C^1-function $\widetilde{\varphi} : (\text{Le}_*, \text{Le}_c] \to [0, \lambda_*)$.

To complete the proof, we need to show that $\text{Le}_* = 0$. If $\text{Le}_* > 0$ then $\lambda_* = \sqrt{\lambda_1}$ otherwise, applying the implicit function theorem again, we could extend $\widetilde{\varphi}$ in a left-neighborhood of Le_*, contradicting the maximality of $\widetilde{\varphi}$. On the other hand, it is not difficult to check that $\mathcal{D}_{0,1}(\sqrt{\lambda_1}, \text{Le}_*) \neq 0$ whenever $R > 0$. Indeed, using condition (1.5) we can easily show that

$$\mathcal{D}_{0,1}(\sqrt{\lambda_1}, \text{Le}_*) = e^{-R(1+2\sqrt{\lambda_1})} - 1 + \theta_i R(1 + 2\sqrt{\lambda_1}) = e^{-R(1+2\sqrt{\lambda_1})} + 2\sqrt{\lambda_1} - e^{-R}(1 + 2\sqrt{\lambda_1})$$

and the function $x \mapsto f(x) = e^{-x(1+2\sqrt{\lambda_1})} + 2\sqrt{\lambda_1} - e^{-x}(1 + 2\sqrt{\lambda_1})$ vanishes at $x = 0$ and its derivative is positive in $(0, +\infty)$. $\qquad \square$
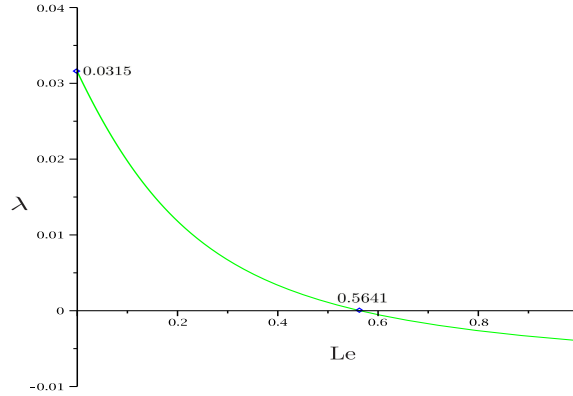

FIGURE 2. Numerical computation of the implicit curve $\lambda = \widetilde{\varphi}(\text{Le})$ for $\text{Le} \in (0, \text{Le}_c)$, extended beyond Le_c. Here $\theta_i = 0.75$, $\ell = 100$, $\text{Le}_c \simeq 0.5641$, $\lambda_* \simeq 0.0315$. Note that $\sqrt{\lambda_1} = \pi/50 \simeq 0.0628$.

Corollary 6.4. *The spectrum of the operator L contains elements with positive real parts. Moreover, the part of $\sigma(L)$ in the right halfplane $\{\lambda \in \mathbb{C} : \text{Re}\,\lambda \geq 0\}$ consist of 0 and a finite number of eigenvalues.*

Proof. By Theorem 4.4 we know that if $\lambda \neq 0$ is an element in the spectrum of L with nonnegative real part, then it is an eigenvalue and it belongs to Ω_k for some $k \in \mathbb{N} \cup \{0\}$. Hence, $D_k(\lambda) = 0$ or, equivalently, $\mathcal{D}_{0,k}(\lambda) = 0$ for some $k \in \mathbb{N} \cup \{0\}$. As it is immediately seen, each function $\lambda \mapsto \mathcal{D}_{0,k}(\lambda)$ is holomorphic in the halfplane $\Pi = \{\lambda \in \mathbb{C} : \text{Re}\,\lambda \geq 0\}$ and it does not identically vanish in it. Therefore, its zeroes in Π are at most finitely many. Moreover, for each $\lambda \in \Pi$ and $k \in \mathbb{N} \cup \{0\}$ we can estimate

$$\text{Re}\,X_k(\lambda) \geq \sqrt{\frac{1}{2} + 2\lambda_k}, \qquad \text{Re}\,Y_k(\lambda, \text{Le}) \geq \sqrt{\frac{\text{Le}^2}{2} + 2\lambda_k},$$

so that the real part of $\mathcal{D}_{0,k}(\cdot, \text{Le})$ diverges to $+\infty$, as $k \to +\infty$, uniformly with respect to $\lambda \in \Pi$. As a byproduct, we deduce that there exists $k_0 \in \mathbb{N}$ such that the nontrivial eigenvalues $\lambda \in \Pi$ lie in $\bigcup_{k=0}^{k_0} \Omega_k$ and this completes the proof. $\qquad \square$

6.2. **Proof of Main Theorem.** To prove the main result of this section, we also need the following result which is a variant of [28, Theorem 5.1.5] and [13, Theorem 4.3].

Lemma 6.5. *Let X be a complex Banach space, $r > 0$ and $T_n : B(0, r) \subset X \to X$ ($n \in \mathbb{N}$) be a bounded operator such that $T_n(x) = Mx + O(\|x\|^p)$ as $\|x\| \to 0$, for some $p > 1$ and some bounded linear operator M on X with spectral radius $\rho > 1$. Further, assume that there exists an eigenvector u of M with eigenvalue $\lambda \in \mathbb{C}$ such that $|\lambda|^p > \rho$ and that there exists $x' \in X'$ such that $x'(u) \neq 0$. Then, there exist $c > 0$ and, for any $\delta > 0$, $x_0 \in B(0, \delta)$ and $n_0 \in \mathbb{N}$ (depending on δ) such that the sequence $x_0, \ldots, x_{n_0}$, where $x_n = T_n(x_{n-1})$ for any $n = 1, \ldots, n_0$, is well defined and $|x'(x_{n_0})| \geq c|x'(u)|$.*

Proof. Without loss of generality, we assume that $\|u\| = 1$ and $\|x'\| \leq 1$. Moreover, we choose $a, b > 0$ such that $\|T_n(x) - Mx\| \leq b\|x\|^p$ for each $x \in B(0, a) \subset X$ and $n \in \mathbb{N}$. Since $|\lambda|^p > \rho$, we can fix $\eta > 0$ such that $|\lambda|^p > \rho + \eta$ and, from the definition of the spectral radius of a bounded operator, we can also determine a positive constant K such that $\|M^n\|_{L(X)} \leq K(\rho + \eta)^n$ for any $n \in \mathbb{N}$. Finally, we fix $\delta > 0$, choose $n_0 \in \mathbb{N}$ be such that $|\lambda|^{-n_0} < \delta$, and set $x_0 := \sigma u|\lambda|^{-n_0}$, where $\sigma \in (0, 1)$ is chosen so as to satisfy the conditions

$$\sigma \leq \frac{a}{2}, \qquad \frac{2^p bK}{|\lambda|^p - \rho - \eta}\sigma^{p-1} \leq \frac{1}{2}|x'(u)|. \tag{6.3}$$

To begin with, we prove that the sequence $x_0, \ldots, x_{n_0}$ is well defined. For this purpose, in view of the condition in (6.3) it suffices to check that, if x_k is well defined, then $\|x_k\| \leq 2\sigma|\lambda|^{k-n_0}$. We prove by recurrence. Clearly, x_0 satisfies this property. Suppose that the claim is true for $k = 0, \ldots, n-1$. Then, x_n is well defined and it easy to check that

$$x_n = M^n x_0 + \sum_{k=0}^{n-1} M^{n-1-k}(x_{k+1} - Mx_k) = M^n x_0 + \sum_{k=0}^{n-1} M^{n-1-k}(T_{k+1}(x_k) - Mx_k). \tag{6.4}$$

Thus, we can estimate

$$\|x_n\| \leq |\lambda|^n \|x_0\| + Kb \sum_{k=0}^{n-1} (\rho + \eta)^{n-1-k}\|x_k\|^p. \tag{6.5}$$

Let us consider the second term in the right-hand side of (6.5), which we denote by S_n. Since we are assuming that $\|x_k\| \leq 2\sigma|\lambda|^{k-n_0}$ for each $k = 0, \ldots, n-1$, we can write

$$S_n \leq 2^p Kb\sigma^p|\lambda|^{p(n-n_0-1)} \sum_{k=0}^{n-1} \left(\frac{\rho + \eta}{|\lambda|^p}\right)^{n-1-k} \leq \sigma|\lambda|^{n-n_0}\frac{2^p Kb}{|\lambda|^p - \rho - \eta}\sigma^{p-1}$$

and, using the second condition in (6.3) and the fact that x' has norm which does not exceed 1, we conclude that

$$S_n \leq \frac{1}{2}\sigma|\lambda|^{n-n_0}|x'(u)| \leq \frac{1}{2}\sigma|\lambda|^{n-n_0}. \tag{6.6}$$

Since $|\lambda|^n\|x_0\| \leq \sigma|\lambda|^{n-n_0}$, from (6.5) and (6.6) the claim follows at once.

To conclude the proof, it suffices to use (6.4) with $n = n_0$, as well as (6.5) and (6.6) again, to estimate

$$|x'(x_{n_0})| \geq |x'(M^{n_0}x_0)| - |x'(S_{n_0})| \geq \sigma|x'(u)| - \frac{1}{2}\sigma|x'(u)| = \frac{1}{2}\sigma|x'(u)|.$$

The assertion follows with $c = \sigma/2$. $\qquad\square$

Now, we can state and prove the following theorem.

Theorem 6.6. *Let $0 < \theta_i < 1$ be fixed, $\ell > \ell_0(\theta_i)$ as in Lemma 6.1, $\mathrm{Le}_c = \mathrm{Le}_c(1)$ defined by (6.2). Then, for each $\mathrm{Le} \in (0, \mathrm{Le}_c)$, the null solution $\mathbf{u}$ of problem (5.1) is pointwise unstable with respect to small perturbations in $\mathfrak{X}_{2+\alpha}$. More precisely, there exists a positive constant C such that for each $y_0 \in \mathbb{R}$ and $\delta > 0$ there exist $\widetilde{\mathbf{u}}_0, \mathbf{u}_0^* \in B(0, \delta) \subset \mathfrak{X}_{2+\alpha}$ and $\widetilde{n}, n_* \in \mathbb{N}$ depending on δ such that*

$\min\{|\widetilde{u}_2(\widetilde{n}, 0, y_0)|, |u_1^*(n_*, R, y_0)|\} \geq C$, *where* $\widetilde{\mathbf{u}} = (\widetilde{u}_1, \widetilde{u}_2)$ *and* $\mathbf{u}_* = (u_1^*, u_2^*)$ *denote the solution to the Cauchy problem with initial datum* $\widetilde{\mathbf{u}}_0$ *and* $\mathbf{u}_0^*$, *respectively.*

Proof. We split the proof into two steps. The first one is devoted to prove an estimate which will allow us to apply Lemma 6.5. Then, in Step 2, we prove the pointwise instability.

Step 1. The smoothness of Υ implies that there exists $c > 0$ such that $\|\Upsilon(\mathbf{v}_0)\|_{2+\alpha} \leq c\|\mathbf{v}_0\|_{2+\alpha}$, for each $\mathbf{v}_0 \in D(L_\alpha)$ with sufficiently small norm. Here, Υ is defined in Lemma 4.7(*ii*). Fix r so small such that $\|\mathbf{v}_0 + \Upsilon(\mathbf{v}_0)\|_{2+\alpha} \leq r_0$ for each $\mathbf{v}_0 \in \overline{B(0, r)} \subset D(L_\alpha)$, where $r_0 = r_0(1)$ is defined in the statement of Theorem 5.1.

For $n \in \mathbb{N}$, let $R_n : B(0, \rho) \subset \widetilde{D}(L_\alpha) \to \widetilde{D}(L_\alpha)$ be the map defined by $R_n(\mathbf{v}_0) = P(\mathbf{u}_n(n, \cdot, \cdot))$ for each $\mathbf{v_0} \in B(0, \rho)$ (see Lemma 4.7), where $\mathbf{u}_n$ is the solution to problem (5.1) with initial condition $\mathbf{u}_0 = \mathbf{v}_0 + \Upsilon(\mathbf{v}_0)$ at $t = n-1$. (Note that, by Lemma 4.7(*ii*), $\mathbf{u}_0$ satisfies the compatibility conditions in Theorem 5.1. Further, by Remark 5.2, $\mathbf{u}_n$ is well defined in the time domain $[n-1, n]$.) We claim that

$$\|R_n(\mathbf{v_0}) - e^{L_\alpha}\mathbf{v}_0\|_{2+\alpha} \leq c\|\mathbf{v}_0\|_{2+\alpha}^2, \qquad \mathbf{v}_0 \in B(0, r). \tag{6.7}$$

Estimate (6.7) follows from the integral representation of the solution of problem (5.1) and estimates (5.6). Indeed, again by Remark 5.2, $\mathbf{u}_n(n, \cdot, \cdot)$ is the value at $t = 1$ of the solution $\mathbf{u}$ to problem (5.1), with $\mathbf{u}(0, \cdot) = \mathbf{u}_0$ and, by the proof of Theorem 5.1 (see, in particular, formula (5.5)),

$$\mathbf{u}(1, \cdot, \cdot) - e^L \mathbf{u}_0 = \int_0^1 e^{(1-s)L}[\mathscr{F}(\mathbf{u}(s, \cdot, \cdot)) + \mathscr{L}\mathscr{M}(\mathscr{H}(\mathbf{u}(s, \cdot, \cdot)))]ds - L\int_0^1 e^{(1-s)L}\mathscr{M}(\mathscr{H}(\mathbf{u}(s, \cdot, \cdot)))ds.$$

Since $\mathscr{F}$ and $\mathscr{H}$ are quadratic at 0, it follows immediately that $\|\mathbf{u}(1, \cdot, \cdot) - e^L\mathbf{u}_0\|_{2+\alpha} \leq c\|\mathbf{v}_0\|_{2+\alpha}^2$. Noting that $P(\mathbf{u}(1, \cdot, \cdot) - e^{L_\alpha}\mathbf{u}_0) = R_n(\mathbf{v}_0) - e^{L_\alpha}\mathbf{v}_0$, formula (6.7) follows at once.

Step 2. Let us begin by proving that there exists $C > 0$ such that for any $y_0 \in \mathbb{R}$ and $\delta > 0$ there exists $\mathbf{u}_0 \in B(0, \delta) \subset \mathfrak{X}_{2+\alpha}$ and $n_0 \in \mathbb{N}$ depending on δ such that $|u_2(n_0, 0^+, y_0)| \geq C$, where $\mathbf{u} = (u_1, u_2)$ is the solution to (5.1) with initial datum $\mathbf{u}_0$ at time $t = 0$. For this purpose, we want to apply Lemma 6.5 with $X = D(L_\alpha)$ endowed with the norm of $\mathfrak{X}_{2+\alpha}$. To begin with, we observe that, by Corollary 6.4, there exists only a finite number of eigenvalues of L (and hence of L_α) with positive real part. From the spectral mapping theorem for analytic semigroups it thus follows that the spectral radius ρ of the operator $M = e^{L_\alpha}$ is larger than one and there exists an eigenvalue λ such that $|\lambda| = \rho$. Let us fix $y_0 \in \mathbb{R}$ and $\delta > 0$. It is not difficult to show that a corresponding eigenfunction is the function $\mathbf{w} = (w_1 e_1(\cdot - 2\pi\ell^{-1}y_0)), w_2 e_1(\cdot - 2\pi\ell^{-1}y_0)))$, where

$$w_1(x) = e^{\nu_1^+ x}\chi_{(-\infty, 0]}(x) + (c_1 e^{\nu_1^- x} + c_2 e^{\nu_1^+ x})\chi_{(0, R)}(x) + c_3 e^{\nu_1^- x}\chi_{[R, +\infty)}(x),$$

$$w_2(x) = (d_1 e^{\mu_1^- x} + d_2 e^{\mu_1^+ x})\chi_{[0, R)}(x) + d_3 e^{\mu_1^- x}\chi_{[R, +\infty)}(x),$$

for every $x \in \mathbb{R}$ and

$$c_1 = \frac{e^{(X_1 + \mu_1^+)R}(\theta_i RX_1 - 1)}{e^{\mu_1^+ R} - e^{\nu_1^+ R}}, \qquad c_2 = \frac{e^{\mu_1^+ R}}{e^{\mu_1^+ R} - e^{\nu_1^+ R}}, \qquad c_3 = \frac{\theta_i Re^{(X_1 + \mu_1^+)R}X_1}{e^{\mu_1^+ R} - e^{\nu_1^+ R}}$$

$$d_1 = -\frac{\text{Le}(\text{Le} + \mu_1^+)e^{\nu_1^+ R}X_1}{(e^{\mu_1^+ R} - e^{\nu_1^+ R})Y_1}, \qquad d_2 = -(\text{Le} + \mu_1^-)d_1, \qquad d_3 = (1 - e^{Y_1 R})d_1,$$

$$\nu_1^\pm = -\frac{1}{2} \pm \sqrt{1 + 4\widetilde{\lambda} + 4\lambda_1}, \qquad \mu_1^\pm = -\frac{\text{le}}{2} \pm \sqrt{\text{Le}^2 + 4\text{Le}\widetilde{\lambda} + 4\lambda_1}.$$

Note that

$$w_2(0^+, y_0) = (d_1 + d_2) = \frac{\text{Le}X_1 e^{\nu_1^+ R}X_1}{(e^{\mu_1^+ R} - e^{\nu_1^+ R})Y_1} \neq 0.$$

Hence, if we set $x'(\mathbf{f}) = f_2(0^+, y_0)$ for any $\mathbf{f} \in D(L_\alpha)$, then $|x'(\mathbf{w})| \neq 0$. As in Step 1, we fix $r > 0$ such that $\|\mathbf{v}_{0,j} + \Upsilon(\mathbf{v}_{0,j})\|_{2+\alpha} \leq r_0$ for $j = 1, 2$ for each $\mathbf{v}_0 = \mathbf{v}_{0,1} + i\mathbf{v}_{0,2} \in B(0, r) \subset D(L_\alpha)$. By Theorem 5.1 both $\mathbf{u}(n, \mathbf{v}_{0,1} + \Upsilon(\mathbf{v}_{0,1}), n - 1)$ and $\mathbf{u}(n, \mathbf{v}_{0,2} + \Upsilon(\mathbf{v}_{0,2}), n - 1)$ are well defined for any $n \in \mathbb{N}$. We can thus introduce the operator $T_n : B(0, r) \subset D(L_\alpha) \to D(L_\alpha)$ $(n \in \mathbb{N})$ by

setting $T_n(\mathbf{v}_0) = P\mathbf{u}(n, \mathbf{v}_{0,1} + \Upsilon(\mathbf{v}_{0,1}), n-1) + iP\mathbf{u}(n, \mathbf{v}_{0,2} + \Upsilon(\mathbf{v}_{0,2}), n-1)$ for each $\mathbf{v}_0 \in B(0, r)$, where P is the projection in Lemma 4.7(i). By the arguments in Step 1 we deduce that $\|T_n(\mathbf{v}_0) - e^{L_\alpha}\mathbf{v}_0\|_X \leq C\|\mathbf{v}_0\|_X^2$ for some positive constant C and each $\mathbf{v}_0 \in B(0, r)$. We can thus apply Lemma 6.5 with $M = e^{L_\alpha}$, $p = 2$ and conclude that there exist $c > 0$ and, for each $\delta > 0$, a function $\mathbf{v}_0 = \mathbf{v}_{0,1} + i\mathbf{v}_{0,2} \in B(0, \delta) \subset D(L_\alpha)$ and $n_0 \in \mathbb{N}$ such that $\mathbf{v}_n = T_n(\mathbf{v}_{n-1})$ is well defined for each $n \in \{1, \ldots, n_0\}$ and $|x'(\mathbf{v}_{n_0})| \geq c$. Since $\mathbf{v}_{n_0} = P\mathbf{u}(n_0, \mathbf{v}_{0,1} + \Upsilon(\mathbf{v}_{0,1}), 0) + iP\mathbf{u}(n_0, \mathbf{v}_{0,2} + \Upsilon(\mathbf{v}_{0,2}), 0)$, where $\mathbf{u}(n_0, \mathbf{u}_0, 0)$ denotes the value at n_0 of the unique solution to problem (3.23) with initial datum $\mathbf{u}_0$ at time $t = 0$, we have so proved that

$$|(P\mathbf{u}(n_0, \mathbf{v}_{0,1} + \Upsilon(\mathbf{v}_{0,1}), 0))_2(0^+, y_0)|^2 + |(P\mathbf{u}(n_0, \mathbf{v}_{0,2} + \Upsilon(\mathbf{v}_{0,2}), 0))_2(0^+, y_0)|^2 \geq c^2. \qquad (6.8)$$

By definition, $P = I - \mathcal{N}\mathcal{B}_*$ (see Lemma 4.7(i)) and $(\mathcal{N}\mathbf{u})_2(0^+, \cdot) = 0$ for any function $\mathbf{u}$. Hence, $(P\mathbf{u}(n_0, \mathbf{v}_{0,j} + \Upsilon(\mathbf{v}_{0,j}), 0))_2(0^+, y_0) = (\mathbf{u}(n_0, \mathbf{v}_{0,j} + \Upsilon(\mathbf{v}_{0,j}), 0))_2(0^+, y_0)$ for $j = 1, 2$. From (6.8) it thus follows that there exists $\bar{j}$ such that $|(\mathbf{u}(n_0, \mathbf{v}_{0,\bar{j}} + \Upsilon(\mathbf{v}_{0,\bar{j}}), 0))_2(0^+, y_0)| \geq c/2$ and the thesis follows with $C = c/2$, $\tilde{n} = n_0$ and $\tilde{\mathbf{u}} = \mathbf{u}(\tilde{n}, \mathbf{v}_{0,\bar{j}} + \Upsilon(\mathbf{v}_{0,\bar{j}}), 0)$.

To prove the existence of $\mathbf{u}_*$ as in the statement of the theorem, it suffices to take as x' the functional defined by $x'(\mathbf{f}) = f_1(R, y_0)$ for each $\mathbf{f} \in D(L_\alpha)$. The missing easy details are left to the reader. $\qquad \square$

Remark 6.7. Clearly, Theorem 6.6 implies the $C^{2+\alpha}$-instability of the null solution $\mathbf{u}$ to (5.1).

Remark 6.8. The $C^{2+\alpha}$-instability of the null solution $\mathbf{u}$ to (5.1) can be directly obtained by applying [28, Theorem 5.1.5], taking advantage of Step 1 of Theorem 6.6 and arguing as in [13, Corollary 4.5]. Finally, it can also be proved in a slightly different way adapting the arguments in [32, Theorem 3.4].

From Theorem 6.6 we can now easily derive the proof of the main result of this paper.

Proof of Main Theorem. Taking the changes of variables and unknown in Subsections 3.1 and 3.2 into account, the result in Theorem 6.6 allows us to conclude easily that the normalized temperature Θ and the normalized concentration of deficient reactant in problem (1.3)-(1.4) are unstable with respect to two dimensional $C^{2+\alpha}$ perturbations. Similarly, using formulae (3.9) and (3.12) and again Theorem 6.6, we can infer that there exist initial data $(\widetilde{\Theta}, \widetilde{\Phi})$ and (Θ_*, Φ_*) with $C^{2+\alpha}$-norm, arbitrarily close to the traveling wave solution (1.6) such that the trailing front G (resp. the ignition front F) to problem (1.3)-(1.4) with initial datum $(\Theta(0, \cdot), \Phi(0, \cdot)) = (\widetilde{\Theta}, \widetilde{\Phi})$ (resp. $(\Theta(0, \cdot), \Phi(0, \cdot)) = (\Theta_*, \Phi_*)$) is not arbitrarily close to 0 (resp. R). $\qquad \square$

7. Numerical simulation

In this section, we are going to use some high resolution numerical methods, including Chebyshev collocation and Fourier spectral method (see, e.g., [38, 41, 43]). We consider the problem (3.23) in the finite domain $\Omega = \Omega_- \cup \Omega_0 \cup \Omega_+ = ([-A, 0] \cup [0, R] \cup [R, B]) \times [-\ell/2, \ell/2]$, where $A > 0$ and $B > 0$ are large enough, see Figure 3. The independent variables are $-A \leq x \leq B$, $-\ell/2 \leq y \leq \ell/2$.
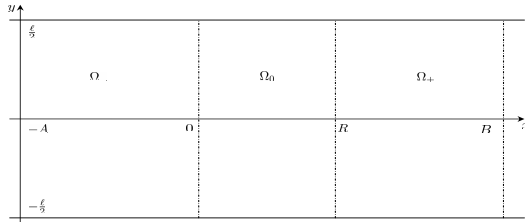

FIGURE 3. Computational domain.

7.1. **The linear system.** Here, we consider the linearized system around the null solution (5.2) with $\mathbf{f} = \mathbf{0}$ and $\mathbf{h} = \mathbf{0}$ and we map Ω_-, Ω_0 and Ω_+ to $\mathbb{D} = [-1,1] \times [0, 2\pi]$. Then, we consider in $\mathbb{D}$ the system for the three pairs of unknowns (u_1, w_1), (u_2, w_2) and (u_3, w_3), corresponding respectively to (u, w) in Ω_-, Ω_0 and Ω_+. The new independent variables are denoted by $\widetilde{x} \in [-1, 1]$ and $\widetilde{y} \in [0, 2\pi]$. Therefore, the system is equivalent to:

$$\begin{cases} D_t u_1 = \frac{2}{A} D_{\widetilde{x}} u_1 + \frac{4}{A^2} D_{\widetilde{x}\widetilde{x}} u_1 + \frac{4\pi^2}{\ell^2} D_{\widetilde{y}\widetilde{y}} u_1, \ w_1 \equiv 0, \\ D_t u_2 = \frac{2}{R} D_{\widetilde{x}} u_2 + \frac{4}{R^2} D_{\widetilde{x}\widetilde{x}} u_2 + 4\frac{\pi^2}{\ell^2} D_{\widetilde{y}\widetilde{y}} u_2, \\ D_t w_2 = \frac{2}{R} D_{\widetilde{x}} w_2 + \frac{4}{\mathrm{Le}\,R^2} D_{\widetilde{x}\widetilde{x}} w_2 + \frac{4\pi^2}{\mathrm{Le}\,\ell^2} D_{\widetilde{y}\widetilde{y}} w_2, \\ D_t u_3 = \frac{2}{B-R} D_{\widetilde{x}} u_3 + \frac{4}{(B-R)^2} D_{\widetilde{x}\widetilde{x}} u_3 + \frac{4\pi^2}{\ell^2} D_{\widetilde{y}\widetilde{y}} u_3, \\ D_t w_3 = \frac{2}{B-R} D_{\widetilde{x}} w_3 + \frac{4}{\mathrm{Le}\,(B-R)^2} D_{\widetilde{x}\widetilde{x}} w_3 + \frac{4\pi^2}{\mathrm{Le}\,\ell^2} D_{\widetilde{y}\widetilde{y}} w_3, \end{cases} \qquad (7.1)$$

together with the boundary conditions:

$$\begin{cases} u_1(\cdot, -1, \cdot) = u_3(\cdot, 1, \cdot) = w_3(\cdot, 1, \cdot) = 0, \quad u_1(\cdot, 1, \cdot) = u_2(\cdot, -1, \cdot), \\ w_2(\cdot, -1, \cdot) = \frac{2\mathrm{Le}}{A} D_{\widetilde{x}} u_1(\cdot, 1, \cdot) - \frac{2\mathrm{Le}}{R} D_{\widetilde{x}} u_2(\cdot, -1, \cdot), \quad D_{\widetilde{x}} w_2(\cdot, -1, \cdot) = -\frac{\mathrm{Le}R}{2} w_2(\cdot, -1, \cdot), \\ D_{\widetilde{x}} w_2(\cdot, 1, \cdot) = \frac{R}{B-R} D_{\widetilde{x}} w_3(\cdot, -1, \cdot) + \frac{\mathrm{Le}R}{2} (w_3(\cdot, -1, \cdot) - w_2(\cdot, 1, \cdot)), \\ D_{\widetilde{x}} u_3(\cdot, -1, \cdot) = \frac{B-R}{R} D_{\widetilde{x}} u_2(\cdot, 1, \cdot) - \frac{B-R}{2\mathrm{Le}} (w_3(\cdot, -1, \cdot) - w_2(\cdot, 1, \cdot)), \\ u_2(\cdot, 1, \cdot) = -\frac{2\theta_i R}{B-R} D_{\widetilde{x}} u_3(\cdot, -1, \cdot) + 2\theta_i D_{\widetilde{x}} u_2(\cdot, 1, \cdot), \quad u_3(\cdot, -1, \cdot) = u_2(\cdot, 1, \cdot). \end{cases} \qquad (7.2)$$

Let us give a brief overview of the numerical method. Hereafter, we denote by (u, w) any pair of unknowns (u_i, w_i), $1 \le i \le 3$. We discretize system (7.1)-(7.2) using a forward-Euler explicit scheme in time. Then, we use a discrete Fourier transform in the direction $\widetilde{y} \in (0, 2\pi)$, namely:

$$u(t, \widetilde{x}, \widetilde{y}) = \sum_{k=-N_{\widetilde{y}}/2}^{N_{\widetilde{y}}/2} \hat{u}_k(t, \widetilde{x}) e^{ik\widetilde{y}}, \qquad w(t, \widetilde{x}, \widetilde{y}) = \sum_{k=-N_{\widetilde{y}}/2}^{N_{\widetilde{y}}/2} \hat{w}_k(t, \widetilde{x}) e^{ik\widetilde{y}},$$

and

$$D_{\widetilde{y}\widetilde{y}} u(t, \widetilde{x}, \widetilde{y}) = -\sum_{k=-N_{\widetilde{y}}/2}^{N_{\widetilde{y}}/2} k^2 \hat{u}_k(t, \widetilde{x}) e^{ik\widetilde{y}}, \qquad D_{\widetilde{y}\widetilde{y}} w(t, \widetilde{x}, \widetilde{y}) = -\sum_{k=-N_{\widetilde{y}}/2}^{N_{\widetilde{y}}/2} k^2 \hat{w}_k(t, \widetilde{x}) e^{ik\widetilde{y}}.$$

Finally, we use a Chebyshev collocation method in $\widetilde{x} \in (-1, 1)$. Let $\{l_j(\widetilde{x})\}_{j=0}^{N_{\widetilde{x}}}$ be the Lagrange polynomials based on the Chebyshev-Gauss-Lobatto points $\{\widetilde{x}_j\}_{j=0}^{N_{\widetilde{x}}} = \{\cos(j\pi/N_{\widetilde{x}})\}_{j=0}^{N_{\widetilde{x}}}$. We set:

$$\hat{u}_k(t, \widetilde{x}) = \sum_{j=0}^{N_{\widetilde{x}}} \hat{u}_{kj} l_j(t, \widetilde{x}), \qquad \hat{w}_k(t, \widetilde{x}) = \sum_{j=0}^{N_{\widetilde{x}}} \hat{w}_{kj} l_j(t, \widetilde{x}).$$

Denoting the differential matrix of order m associated with $\{\widetilde{x}_j\}_{j=0}^{N_{\widetilde{x}}}$ by $D^m = (d_{ij}^{(m)})_{i,j=0,\cdots,N_{\widetilde{x}}}$, where $d_{ij}^{(m)} = l_j^{(m)}(\cdot, \widetilde{x}_i)$ and $l_j(\cdot, \widetilde{x}_i) = \delta_{ij}$, we eventually obtain

$$\hat{u}_k(\cdot, \widetilde{x}_i) = \sum_{j=0}^{N_{\widetilde{x}}} \hat{u}_{kj} \delta_{ij}, \qquad D_{\widetilde{x}} \hat{u}_k(\cdot, \widetilde{x}_i) = \sum_{j=0}^{N_{\widetilde{x}}} \hat{u}_{kj} d_{ij}^{(1)}, \qquad D_{\widetilde{x}\widetilde{x}} \hat{u}_k(\cdot, \widetilde{x}_i) = \sum_{j=0}^{N_{\widetilde{x}}} \hat{u}_{kj} d_{ij}^{(2)},$$

$$\hat{w}_k(\cdot, \widetilde{x}_i) = \sum_{j=0}^{N_{\widetilde{x}}} \hat{w}_{kj} \delta_{ij}, \qquad D_{\widetilde{x}} \hat{w}_k(\cdot, \widetilde{x}_i) = \sum_{j=0}^{N_{\widetilde{x}}} \hat{w}_{kj} d_{ij}^{(1)}, \qquad D_{\widetilde{x}\widetilde{x}} \hat{w}_k(\cdot, \widetilde{x}_i) = \sum_{j=0}^{N_{\widetilde{x}}} \hat{w}_{kj} d_{ij}^{(2)}.$$

As initial data, we take $w_2(0, \widetilde{y}) = \varepsilon(1 + \sin^2(\widetilde{y}))$, $\widetilde{y} \in [0, 2\pi]$, which corresponds to $x = R/2$; the other unknowns are taken as 0. The following pictures are for $\varepsilon = 10^{-2}$, $A = B = 10$, $\ell = 100$, $\Delta t = 10^{-3}$. As expected, the two profiles blow up for Lewis number below critical.
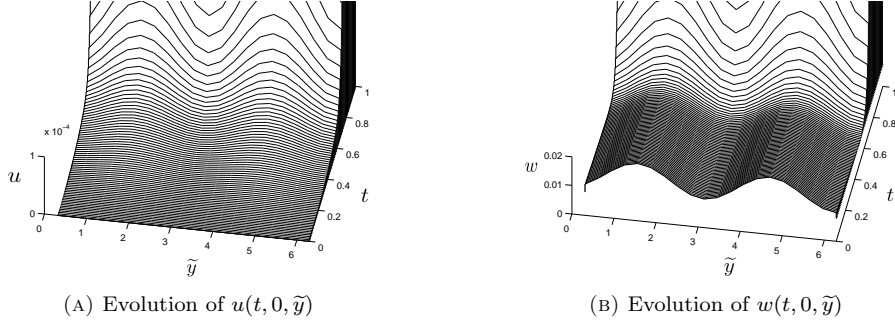
(A) Evolution of $u(t,0,\widetilde{y})$

(B) Evolution of $w(t,0,\widetilde{y})$

FIGURE 4. Evolution of (u,w) solution of the linear problem (3.23) (with $\mathbf{f} = \mathbf{0}$ and $\mathbf{h} = \mathbf{0}$) for $\theta_i = 0.75, \text{Le} = 0.3 < \text{Le}_c \simeq 0.56$. (A) $u(t,0,\widetilde{y})$ varies from 0 to 10^{-4}, $0 < t < 1, 0 < \widetilde{y} < 2\pi$, (B) $w(t,0,\widetilde{y})$ varies from 0 to 2.10^{-2}, $0 < t < 1, 0 < \widetilde{y} < 2\pi$.

7.2. **The fully nonlinear system.** By treating the nonlinearities explicitly, we can use the same algorithm as in the linear case. We approximate the mollifier β by the following trapezoid, see Figure 5:

$$\beta(x) = \begin{cases} 2 + x/\delta, & -2\delta < x < -\delta, \\ 1, & -\delta \leq x \leq \delta, \\ 2 - x/\delta, & \delta < x < 2\delta, \\ 0, & \text{elsewhere,} \end{cases}$$

Then, the fully nonlinear terms in system (3.23), namely $\mathscr{F}_1, \mathscr{F}_2, \mathscr{G}_1, \mathscr{G}_2, \mathscr{G}_3$, as well as $\varrho_\tau, \varrho_{\tau\xi}$, have
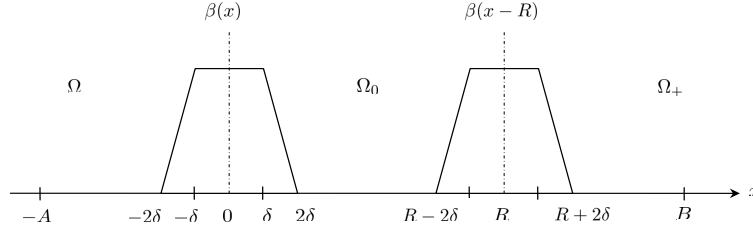

FIGURE 5. Approximation of the mollifier β.

to be computed separately in eight intervals, as they are zero elsewhere: $[-2\delta, -\delta] \cup \cdots \cup [R+\delta, R+2\delta]$, see Figure 5. We refer to the Appendix for the formulae.

Hereafter, we present some typical numerical results for the fully nonlinear problem. Simulations were performed using a standard pseudo-spectral method with small time step $\Delta t = 10^{-5}$ and small amplitude of initial perturbations (of order 10^{-4} to 10^{-3}), to ensure sufficient accuracy.

We consider the situation when ignition temperature is fixed at $\theta_i = 0.75$ and $\ell = 100$, in such a case $\text{Le}_c \simeq 0.5641$. Three significant values of the Lewis number have been chosen in the interval $(0, \text{Le}_c)$, namely $\text{Le} = 0.10$, $\text{Le} = 0.20$ and $\text{Le} = 0.50$. Figures 6 and 7 represent the interface patterns and temperature levels. Numerically, we observe that, after a rapid transition period, a steady configuration consisting of "two-cell" patterns for the ignition and trailing interfaces is established. These simulations confirm the theoretical analysis, that is instability of the planar fronts for $\text{Le} \in (0, \text{Le}_c)$.

ACKNOWLEGMENTS

(A) Le = 0.10   (B) Le = 0.20   (C) Le = 0.50
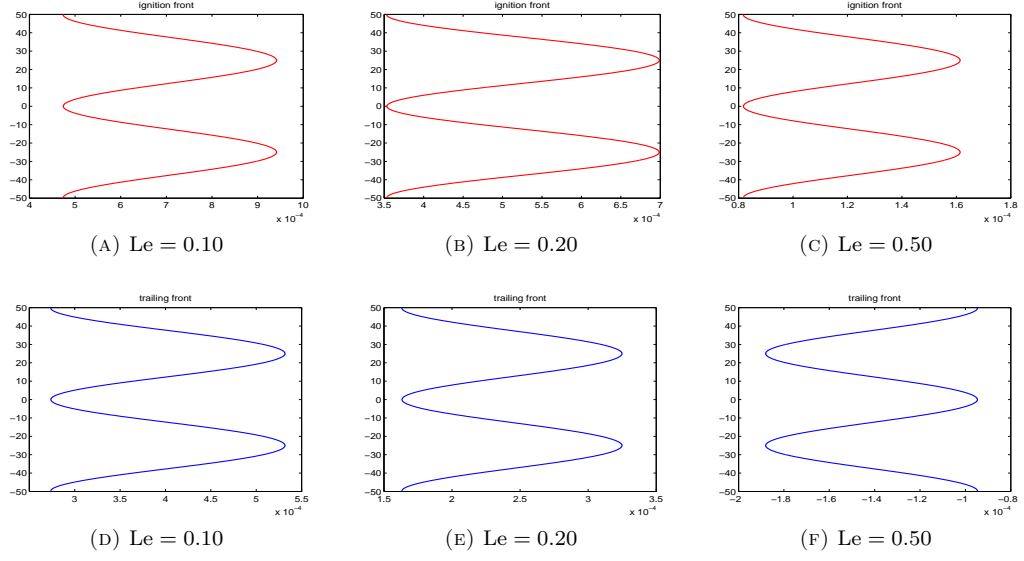
(D) Le = 0.10   (E) Le = 0.20   (F) Le = 0.50

FIGURE 6. Patterns of ignition (above) and trailing (below) interfaces. Here $\theta_i = 0.75$, $\ell = 100$, $\mathrm{Le}_c \simeq 0.5641$.
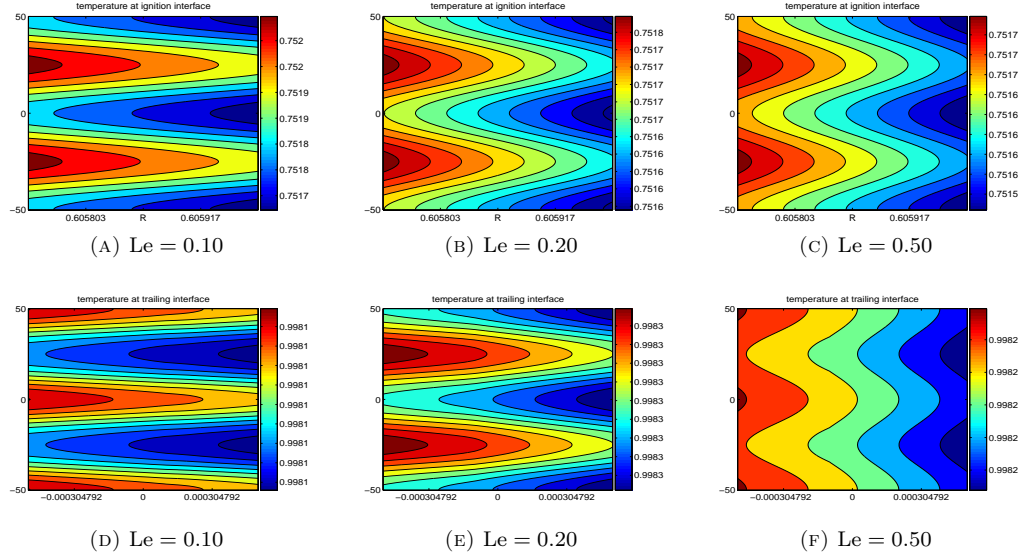

(A) Le = 0.10   (B) Le = 0.20   (C) Le = 0.50

(D) Le = 0.10   (E) Le = 0.20   (F) Le = 0.50

FIGURE 7. Temperature levels around the ignition (above) and trailing (below) interfaces. Parameters are as in Figure 6.

REFERENCES

[1] R.K. Alexander, B. Fleishman, *Perturbation and bifurcation in a free boundary problem*, J. Differential Equations **45** (1982), 34-52.

[2] A. Bayliss, E. M. Lennon, M. C. Tanzy, V. A. Volpert, *Solution of adiabatic and nonadiabatic combustion problems using step-function reaction models*, J. Eng. Math. **79** (2013), 101-124.

[3] H. Berestycki, B. Nicolaenko, B. Scheurer, *Traveling wave solutions to combustion models and their singular limits*, SIAM J. Math. Anal. **16** (1985), 1207-1242.

[4] I. Brailovsky, P. V. Gordon, L. Kagan, G.I. Sivashinsky, *Diffusive-thermal instabilities in premixed flames: stepwise ignition-temperature kinetics*, Combustion and Flame **162** (2015), 2077-2086.

[5] I. Brailovsky, G.I. Sivashinsky, *Momentum loss as a mechanism for deflagration to detonation transition*, Combust. Theory Model. **2** (1998), 429-447.

[6] C.-M Brauner, P.V. Gordon, W. Zhang, *An ignition-temperature model with two free interfaces in premixed flames*, Combustion Theory Model. **20** (2016), 976-994.

[7] C.-M. Brauner, L. Hu, L. Lorenzi, *Asymptotic analysis in a gas-solid combustion model with pattern formation*, Chin. Ann. Math. Ser. B **34** (2013), 65-88. See also: Partial differential equations: theory, control and approximation, 139-169, Springer, Dordrecht, 2014

[8] C.-M. Brauner, J. Hulshof, L. Lorenzi, G.I. Sivashinsky, *A fully nonlinear equation for the flame front in a quasi-steady combustion model*, Discrete Contin. Dyn. Syst. Ser. A **27** (2010), 1415-1446.

[9] C.-M. Brauner, J. Hulshof, A. Lunardi, *A general approach to stability in free boundary problems*, J. Differential Equations **164** (2000), 16-48.

[10] C.-M. Brauner, L. Lorenzi, *Local existence in free interface problems with underlyning second-order Stefan condition*, Rev. Roumaine Math. Pures Appl. **23** (2018), 339-359.

[11] C.-M. Brauner, L. Lorenzi, M. Zhang, *Stability analysis and Hopf bifurcation for large Lewis number in a combustion model with free interface* (submitted). Available on ArXiv: arXiv:1901.01123.

[12] C.-M. Brauner, L. Lorenzi, G.I. Sivashinsky, C.-J. Xu, *On a strongly damped wave equation for the flame front*, Chin. Ann. Math. Ser. B **31** (2010), 819-840.

[13] C.-M. Brauner, A. Lunardi, *Instabilities in a two-dimensional combustion model with free boundary*, Arch. Ration. Mech. Anal. **154** (2000), 157-182.

[14] C.-M. Brauner, J.-M. Roquejoffre, Cl. Schmidt-Lainé, *Stability of travelling waves in a parabolic equation with discontinuous source term*, Comm. Appl. Nonlinear Anal. **2** (1995), 83-100.

[15] J.D. Buckmaster, G.S.S. Ludford, Theory of Laminar Flames, Cambridge University Press, 1982.

[16] K.-C. Chang, *On the multiple solutions of the elliptic differential equations with discontinuous nonlinear terms*, Scientia Sinica **21** (1978), 139-158.

[17] K.-C. Chang, *The obstacle problem and partial differential equations with discontinuous nonlinearities*, Comm. Pure Appl. Math. **33** (1980), 117-146.

[18] P. Colella, A. Majda, V. Roytburd, *Theoretical and structure for reacting shock waves*, SIAM J. Sci. Statist. Comput. **7** (1986), 1059-1080.

[19] Y. Du, Z. Guo, *Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary, II*, J. Differential Equations **250** (2011), 4336-4366.

[20] Y. Du, Z. Lin, *Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary*, SIAM J. Math Anal. **42** (2010), 377-405.

[21] Y. Du, Z. Lin, *Erratum: Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary*. SIAM J. Math. Anal. **45** (2013), 1995-1996.

[22] J. H. Ferziger, T. Echekki, *A Simplified Reaction Rate Model and its Application to the Analysis of Premixed Flames*, Combust. Sci. Technol. **89** (1993), 293-315.

[23] D. Gilbarg, N. Trudinger, Elliptic partial differential equations of second order. Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001.

[24] R. Gianni, *Existence of the free boundary in a multi-dimensional combustion problem*, Proc. Roy. Soc. Edinburgh Sect. A **125** (1995), 525-544.

[25] R. Gianni, J. Hulshof, *The semilinear heat equation with a Heaviside source term*, EJAM **3**(1992), 367-379.

[26] R. Gianni, P. Mannucci, *Existence theorems for a free boundary problem in combustion theory*, Quart. Appl. Math. **51**(1993), 43-53.

[27] R. Gianni, P. Mannucci, *Some existence theorems for an N-dimensional parabolic equation with a discontinuous source term*, SIAM J. Math. Anal. **24**(1993), 618-633.

[28] D. Henry, Geometric theory of parabolic equations, Lect. Notes in Math. **840**, Springer, Berlin 1980.

[29] N.V. Krylov, Lectures on elliptic and parabolic equations in Sobolev spaces, Graduate Studies in Mathematics **96**, American Mathematical Society, Providence, RI, 2008

[30] L. Lorenzi, *Regularity and analyticity in a two-dimensional combustion model*, Adv. Differential Equations **7** (2002), 1343-1376.

[31] L. Lorenzi, *A free boundary problem stemmed from combustion theory. I. Existence, uniqueness and regularity results*, J. Math. Anal. Appl. **274** (2002), 505-535.

[32] L. Lorenzi, *A free boundary problem stemmed from combustion theory. II. Stability, instability and bifurcation results*, J. Math. Anal. Appl. **275** (2002), 131-160.

[33] A. Lunardi, Analytic Semigroups and Optimal Regularity in Parabolic Problems, Birkhäuser, Basel, 1996.

[34] E. Mallard, H.L. Le Châtelier, *Recherches expérimentales et théoriques sur la combustion des mélanges gazeux explosifs. Premier mémoire: Température d'inflammation*, Ann. Mines **4** (1883), pp. 274-295.

[35] B.J. Matkowski, G.I. Sivashinsky, *An asymptotic derivation of two models in flame theory associated with the constant density approximation*, SIAM J. Appl. Math. **37** (1979), 686-699.

[36] J. Norbury, A. M. Stuart, *Parabolic free boundary problems arising in porous medium combustion,* IMA J. Appl. Math. **39** (1987), 241-257.

[37] J. Norbury, A. M. Stuart, *A model for porous medium combustion*, Quart. J. Mech. Appl. Math. **42** (1989), 159-178.

[38] J. Shen, T. Tang, L.L. Wang, Spectral Methods. Algorithms, Analysis and Applications, Springer Series in Computational Mathematics **41**. Springer, Heidelberg, 2011.

[39] G.I. Sivashinsky, *On flame propagation under condition of stoichiometry*, SIAM J. Appl. Math. **39** (1980), 67-82.

[40] R. Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physics, Applied Mathematical Sciences **68**, 2nd ed., Springer-Verlag, New York, 1997.

[41] Z. Wang, H. Wang, S. Qiu, *A new method for numerical differentiation based on direct and inverse problems of partial differential equations*, Applied Mathematics Letters **43** (2015), 61-67.

[42] M. Wang, J. Zhao, *A Free Boundary Problem for a predator-prey model with double free boundaries*, J. Dynam. Differential Equations **29** (2017), 957-979.

[43] Z. Wang, W. Zhang, B. Wu, *Regularized optimization method for determining the space-dependent source in a parabolic equation without iteration*, Journal of Computational Analysis & Applications **20** (2016), 1107-1126.

8. Appendix

In this Appendix, we write explicitly the formulae for the fully nonlinear terms in system (3.23), namely $\mathscr{F}_1$, $\mathscr{F}_2$, $\mathscr{G}_1$, $\mathscr{G}_2$, $\mathscr{G}_3$. They have to be computed separately in 8 intervals, as they are zero elsewhere: $[-2\delta, -\delta] \cup [-\delta, 0] \cup [0, \delta] \cup [\delta, 2\delta] \cup [R - 2\delta, R - \delta] \cup [R - \delta, R] \cup [R, R + \delta] \cup [R + \delta, R + 2\delta]$, see Figure 5. The formulae for ϱ_t and ϱ_{tx} can be easily derived. We refer to Subsection 7.1 for the notation $\widetilde{x}$, $\widetilde{y}$. Moreover, we recall that $\beta_R = \beta(\cdot - R)$ where β is as in Subsection 7.2.

(1) For $x \in [-2\delta, -\delta]$ and $x = \frac{A}{2}(\widetilde{x} - 1)$,

$$\mathscr{F}_1(u, w) = \frac{2\text{Le}}{A\text{Le} - aRw(0^+)} u_{\widetilde{x}},$$

$$\mathscr{F}_2(u, w) = \frac{1}{\text{Le} - Rw(0^+)} \left\{ \frac{2Rw(0^+)}{A} u_{\widetilde{x}} + \frac{8\pi^2 R\beta}{A\ell^2} [w_{\widetilde{y}\widetilde{y}}(0^+)u_{\widetilde{x}} + 2w_{\widetilde{y}}(0^+)u_{\widetilde{x}\widetilde{y}}] \right\}$$
$$+ \frac{\text{Le}}{(\text{Le} - Rw(0^+))^2} \left\{ \frac{16\pi^2 R^2 \beta w_{\widetilde{y}}^2(0^+)}{\text{Le} A\ell^2} \left(u_{\widetilde{x}} + \frac{\beta}{A} u_{\widetilde{x}\widetilde{x}} \right) + \frac{4Rw(0^+)}{A^2} \left(2 - \frac{Rw(0^+)}{\text{Le}} u_{\widetilde{x}\widetilde{x}} \right) \right\}.$$

(2) For $x \in [-\delta, 0]$ and $x = \frac{A}{2}(\widetilde{x} - 1)$,

$$\mathscr{F}_1(u) = \frac{2}{A} u_{\widetilde{x}}, \qquad \mathscr{F}_2(u, w) = \frac{8R\pi^2}{A\ell^2\text{Le}} \left(w_{\widetilde{y}\widetilde{y}}(0^+)u_{\widetilde{x}\widetilde{x}} + w_{\widetilde{y}}(0^+)u_{\widetilde{x}\widetilde{y}} + \frac{2Rw_{\widetilde{y}}^2(0^+)}{A\text{Le}} u_{\widetilde{x}\widetilde{x}} \right).$$

(3) For $x \in [0, \delta]$ and $x = \frac{R}{2}(\widetilde{x} + 1)$,

$$\mathscr{F}_1(u) = \frac{1}{\text{Le}} e^{-\frac{R}{2}(\widetilde{x}+1)} + \frac{2}{R} u_{\widetilde{x}}, \qquad \mathscr{G}_1(w) = \text{Le } e^{-\frac{\text{Le}R}{2}(\widetilde{x}+1)} w(0^+) + \frac{2}{R} w_{\widetilde{x}},$$

$$\mathscr{F}_2(u, w) = \frac{4\pi^2 R}{\ell^2 \text{Le}^2} e^{-\frac{R}{2}(\widetilde{x}+1)} \left[-w_{\widetilde{y}\widetilde{y}}(0^+)w(0^+) + w_{\widetilde{y}}^2(0^+) \left(1 - \frac{Rw(0^+)}{\text{Le}} \right) \right]$$
$$+ \frac{8\pi^2}{\ell^2 \text{Le}} \left[w_{\widetilde{y}\widetilde{y}}(0^+)u_{\widetilde{x}} + 2w_{\widetilde{y}}(0^+) \left(u_{\widetilde{x}\widetilde{y}} + \frac{w_{\widetilde{y}}(0^+)}{\text{Le}} u_{\widetilde{x}\widetilde{x}} \right) \right],$$

$$\mathscr{G}_3(w) = \frac{4\pi^2 R}{\text{Le } \ell^2} e^{-\frac{\text{Le}R}{2}(\widetilde{x}+1)} \left[w_{\widetilde{y}\widetilde{y}}(0^+)w(0^+) + w_{\widetilde{y}}^2(0^+) \left(\frac{2\pi}{\ell} - w(0^+) - 1 \right) \right]$$
$$+ \frac{8\pi^2}{\text{Le}^2 \ell^2} \left(w_{\widetilde{y}\widetilde{y}}(0^+)w_{\widetilde{x}} + 2\pi^2 w_{\widetilde{y}}(0^+)w_{\widetilde{x}\widetilde{y}} + \frac{2w_{\widetilde{y}}^2(0^+)}{\text{Le}} w_{\widetilde{x}\widetilde{x}} \right).$$

(4) For $x \in [\delta, 2\delta]$ and $x = \frac{R}{2}(\widetilde{x} + 1)$,

$$\mathscr{F}_1(u, w) = \frac{\beta w(0^+)}{\text{Le} + Rw(0^+)} e^{\frac{-R(\widetilde{x}+1)}{2}} + \frac{2\text{Le}}{R(\text{Le} + Rw(0^+))} u_{\widetilde{x}},$$

$$\mathscr{F}_2(u, w) = \frac{1}{\text{Le} + Rw(0^+)} \left\{ \frac{R\beta}{\text{Le}} e^{-\frac{R}{2}(\widetilde{x}+1)} \left[\frac{4\pi^2\beta}{\ell^2} \left(2w_{\widetilde{y}}^2(0^+) - w(0^+)w_{\widetilde{y}\widetilde{y}}(0^+) \right) - w^2(0^+) \right] \right.$$
$$\left. + 8\pi^2\beta \left(2w_{\widetilde{y}}(0^+)u_{\widetilde{x}\widetilde{y}} - w_{\widetilde{y}\widetilde{y}}(0^+)u_{\widetilde{x}} \right) - 2w(0^+)u_{\widetilde{x}} \right\}$$

$$+ \frac{\text{Le}}{(\text{Le} + Rw(0^+))^2} \left\{ e^{-\frac{R}{2}(\widetilde{x}+1)} \left[-\frac{4\pi^2 R\beta}{\ell^2 \text{Le}} \left(\frac{4\beta w(0^+) w_{\widetilde{x}\widetilde{y}}(0^+) w_{\widetilde{y}}(0^+)}{\text{Le}} + R w_{\widetilde{y}}^2(0^+) \left(\frac{\beta w(0^+)}{\text{Le}} + \frac{1}{R} \right) \right. \right. \right.$$

$$\left. \left. + 2Rw(0^+) w_{\widetilde{y}}^2(0^+) \right) + \frac{R^2 w^2(0^+)}{\text{Le}} \left(\frac{\beta w(0^+)}{\text{Le}} + \frac{1}{R} \right) + 2Rw^2(0^+) \right]$$

$$\left. + \frac{16\pi^2 \beta w_{\widetilde{y}}(0^+)}{\text{Le}\,\ell^2} [-2w_{\widetilde{x}\widetilde{y}}(0^+) u_{\widetilde{x}} + w_{\widetilde{y}}(0^+) u_{\widetilde{x}\widetilde{x}}] - 4u_{\widetilde{x}\widetilde{x}} w(0^+) \left(\frac{w(0^+)}{\text{Le}} + \frac{2}{R} \right) \right\},$$

$$\mathscr{G}_1(w) = \frac{\text{Le}}{\text{Le} + Rw(0^+)} \left[e^{-\frac{\text{Le}\,R}{2}(\widetilde{x}+1)} \beta w(0^+) \left(\text{Le} - \frac{Rw(0^+)}{\text{Le}} \right) + \frac{2}{R} w_{\widetilde{x}} \right],$$

$$\mathscr{G}_2(w) = \frac{\text{Le}}{\text{Le} + Rw(0^+)} w - \frac{\beta w(0^+) \text{Le}}{\text{Le} + Rw(0^+)} e^{-\frac{\text{Le}\,R}{2}(\widetilde{x}+1)},$$

$$\mathscr{G}_3(w) = \frac{1}{\text{Le} + Rw(0^+)} \left\{ e^{-\frac{\text{Le}\,R}{2}(\widetilde{x}+1)} \left[\frac{4\pi^2 R\beta(2 + \beta\text{Le})}{\text{Le}\,\ell^2} [w(0^+) w_{\widetilde{y}\widetilde{y}}(0^+) + 2w_{\widetilde{y}}^2(0^+)] - Rw(0^+)(1 + \beta\text{Le}) \right] \right.$$

$$\left. + \frac{4\pi^2}{\ell^2 \text{Le}} \left[w_{\widetilde{y}\widetilde{y}}(0^+)(2\beta w_{\widetilde{x}} - Rw) + 2w_{\widetilde{y}}(0^+)(2\beta w_{\widetilde{x}\widetilde{y}} - Rw_{\widetilde{y}}) \right] - 2w(0^+) w_{\widetilde{x}} \right\}$$

$$+ \frac{\text{Le}}{(\text{Le} + Rw(0^+))^2} \left\{ e^{-\frac{\text{Le}\,R}{2}(x+1)} \left[\frac{4\pi^2 R\beta w_{\widetilde{y}}^2(0^+)}{\text{Le}^2 \ell^2} \left(Rw(0^+) \big(2\beta - 2 - 4\text{Le}\,\beta - \beta\text{Le}(1 + \beta\text{Le}) \big) + \text{Le}(\beta\text{Le} - 2) \right) \right. \right.$$

$$\left. + Rw^2(0^+) \left(1 + (1 + \beta\text{Le}) \left(2 + \frac{Rw(0^+)}{\text{Le}} \right) \right) \right]$$

$$\left. + \frac{4\pi^2 R w_{\widetilde{y}}^2(0^+)}{\text{Le}\,\ell^2} \left(\frac{2Rw}{\text{Le}} + \frac{4\beta^2}{\text{Le}\,R} w_{\widetilde{x}\widetilde{x}} - 8\beta w_{\widetilde{x}} \right) - \frac{4w(0^+)}{\text{Le}\,R} \left(2 + \frac{Rw(0^+)}{\text{Le}} \right) w_{\widetilde{x}\widetilde{x}} \right\},$$

(5) For $x \in [R - 2\delta, R - \delta]$ and $x = \frac{R}{2}(\widetilde{x}+1)$,

$$\mathscr{F}_1(u) = \frac{1}{R(\theta_i + u(R^+))} \left(2\theta_i u_{\widetilde{x}} - \beta_R u(R^+) e^{-\frac{R}{2}(\widetilde{x}+1)} \right),$$

$$\mathscr{F}_2(u) = \frac{1}{\theta_i + u(R^+)} \left\{ e^{-\frac{R}{2}(\widetilde{x}+1)} \left[\frac{4\pi^2 \beta_R^2}{\theta_i \ell^2 R} \big(u(R^+) u_{\widetilde{y}\widetilde{y}}(R^+) + 2u_{\widetilde{y}}^2(R^+) \big) + \frac{\beta_R u(R^+)}{\theta_i R} \right] \right.$$

$$\left. - \frac{8\pi^2 \beta_R}{\ell^2 R} \big(u_{\widetilde{y}\widetilde{y}}(R^+) u_{\widetilde{x}} + 2u_{\widetilde{y}}(R^+) u_{\widetilde{x}\widetilde{y}} \big) \right\}$$

$$+ \frac{\theta_i}{(\theta_i + u(R^+))^2} \left\{ e^{-\frac{R}{2}(\widetilde{x}+1)} \left[\frac{4\pi^2 \beta_R^2 u_{\widetilde{y}}^2(R^+)}{\theta_i \ell^2 R} \left(\frac{\beta_R - 4}{\theta_i} u(R^+) - 1 \right) \right. \right.$$

$$\left. \left. - \frac{u^2(R^+)}{\theta_i R} \left(\frac{\beta_R u(R^+)}{\theta_i} - 1 + 2\beta_R \right) \right] \right.$$

$$\left. + \frac{16\pi^2 \beta_R u_{\widetilde{y}}^2(R^+)}{\theta_i \ell^2 R} \left(u_{\widetilde{x}} + \frac{\beta_R}{R} u_{\widetilde{x}\widetilde{x}} \right) - \frac{4u(R^+)}{R^2} u_{\widetilde{x}\widetilde{x}} \left(\frac{u(R^+)}{\theta_i} + 2 \right) \right\},$$

$$\mathscr{G}_1(u, w) = \frac{\theta_i}{\theta_i + u(R^+)} \left[e^{-\frac{\text{Le}\,R}{2}(\widetilde{x}+1)} \frac{\text{Le}\,\beta_R u(R^+)}{\theta_i R} \left(\frac{u(R^+)}{\theta_i} - \text{Le} \right) + \frac{2}{R} w_{\widetilde{x}} \right],$$

$$\mathscr{G}_2(u, w) = \frac{\theta_i}{\theta_i + u(R^+)} \left(e^{-\frac{\text{Le}\,R}{2}(\widetilde{x}+1)} \frac{\text{Le}\,\beta_R u(R^+)}{\theta_i R} + w \right),$$

$$\mathscr{G}_3(u, w) = \frac{\theta_i}{\text{Le}(\theta_i + u(R^+))} \left\{ e^{-\frac{\text{Le}\,R}{2}(\widetilde{x}+1)} \left[\frac{4\pi^2 \beta_R \text{Le}}{\theta_i \ell^2 R} (\text{Le}\,\beta_R - 2) \big(u_{\widetilde{y}\widetilde{y}}(R^+) u(R^+) + 2u_{\widetilde{y}}^2(R^+) \big) \right. \right.$$

$$\left. + \frac{\text{Le}^2 u^2(R^+)}{\theta_i R} (\text{Le}\,\beta_R - 1) \right]$$

$$\left. - \frac{4\pi^2 u_{\widetilde{y}\widetilde{y}}(R^+)}{\ell} \left(\frac{w}{\ell} + \frac{2\beta_R}{R} w_{\widetilde{x}} \right) - \frac{2\text{Le}}{R} w_{\widetilde{x}} - \frac{8\pi^2 u_{\widetilde{y}}(R^+)}{\ell^2} \left(w_{\widetilde{y}} + \frac{2\beta_R}{R} w_{\widetilde{x}\widetilde{y}} \right) \right\}$$

$$+ \frac{\theta_i}{\text{Le}(\theta_i + u(R^+))^2} \left\{ e^{-\frac{\text{Le}\,R}{2}(\widetilde{x}+1)} \left[\frac{8\pi^2 \beta_R \text{Le}\,u_{\widetilde{y}}^2(R^+)}{\theta_i \ell^2 R} \frac{u(R^+)}{\theta_i} \left(1 + \frac{\text{Le}\,\beta_R}{2} (\text{Le}\,\beta_R - 5 + 2\text{Le}) \right) \right. \right.$$

$$\left. + \frac{8\pi^2 \beta_R \text{Le}\,u_{\widetilde{y}}^2(R^+)}{\theta_i \ell^2 R} \left(1 + \frac{\text{Le}^2 \beta_R}{2} \right) \right]$$

$$\left. - \frac{\text{Le}^2 u^2(R^+)}{\theta_i R} \left[\left(2 + \frac{u(R^+)}{\theta_i} \right) (\text{Le}\,\beta_R - 1) + \text{Le} \right] \right]$$

$$+\frac{8\pi^2 u_{\widetilde{y}}^2(R^+)}{\theta_i\ell^2}\left(w+\frac{4\beta_R}{R}w_{\widetilde{x}}+\frac{2\beta_R^2}{R^2}w_{\widetilde{x}\widetilde{x}}\right)-\frac{4u(R^+)}{R^2}\left(2+\frac{u(R^+)}{\theta_i}\right)w_{\widetilde{x}\widetilde{x}}\bigg\},$$

(6) For $x\in[R-\delta,R]$ and $x=\frac{R}{2}(\widetilde{x}+1)$,

$$\mathscr{F}_1(u)=-\frac{u(R^+)}{\theta_i R}e^{-\frac{R}{2}(\widetilde{x}+1)}+\frac{2}{R}u_{\widetilde{x}},\quad \mathscr{G}_1(u,w)=-\frac{\mathrm{Le}^2u(R^+)}{\theta_i R}e^{\frac{-\mathrm{Le}\,R(\widetilde{x}+1)}{2}}+\frac{2}{R}w_{\widetilde{x}},$$

$$\mathscr{F}_2(u)=e^{-\frac{R}{2}(\widetilde{x}+1)}\frac{4\pi^2}{\theta_i^2\ell^2 R}\left[u(R^+)u_{\widetilde{y}\widetilde{y}}(R^+)+u_{\widetilde{y}}^2(R^+)\left(1+\frac{u(R^+)}{\theta_i}\right)\right]$$

$$+\frac{8\pi^2}{\theta_i\ell^2 R}\left(\frac{2u_{\widetilde{y}}^2(R^+)}{\theta_i R}u_{\widetilde{x}\widetilde{x}}-u_{\widetilde{x}}u_{\widetilde{y}\widetilde{y}}(R^+)-2u_{\widetilde{y}}(R^+)u_{\widetilde{x}\widetilde{y}}\right),$$

$$\mathscr{G}_3(u,w)=e^{-\frac{\mathrm{Le}\,R}{2}(\widetilde{x}+1)}\frac{4\pi^2}{\theta_i^2\ell^2 R}\left[u(R^+)u_{\widetilde{y}\widetilde{y}}(R^+)+\mathrm{Le}\,u_{\widetilde{y}}^2(R^+)(1+\mathrm{Le}\,u(R^+))\right]$$

$$+\frac{8\pi^2}{\mathrm{Le}\,\theta_i\ell^2 R}\left(\frac{2u_{\widetilde{y}}^2(R^+)}{\theta_i R}w_{\widetilde{x}\widetilde{x}}-w_{\widetilde{x}}u_{\widetilde{y}\widetilde{y}}(R^+)-2u_{\widetilde{y}}(R^+)w_{\widetilde{x}\widetilde{y}}\right),$$

(7) For $x\in[R,R+\delta]$ and $x=\frac{B-R}{2}(\widetilde{x}+1)+R$,

$$\mathscr{F}_1(u)=u(R^+)e^{-\frac{B-R}{2}(\widetilde{x}+1)}+\frac{2}{B-R}u_{\widetilde{x}},$$

$$\mathscr{F}_2(u)=-e^{-\frac{B-R}{2}(\widetilde{x}+1)}\frac{4\pi^2}{\theta_i\ell^2}\left[u(R^+)u_{\widetilde{y}\widetilde{y}}(R^+)+\left(1+\frac{u(R^+)}{\theta_i}\right)u_{\widetilde{y}}^2(R^+)\right]$$

$$-\frac{8\pi^2}{\theta_i\ell^2(B-R)}\left[u_{\widetilde{y}\widetilde{y}}(R^+)u_{\widetilde{x}}+2u_{\widetilde{y}}(R^+)u_{\widetilde{x}\widetilde{y}}-\frac{2u_{\widetilde{y}}^2(R^+)}{\theta_i(B-R)}u_{\widetilde{x}\widetilde{x}}\right],$$

$$\mathscr{G}_1(u,w)=-\frac{\mathrm{Le}^2(e^{-\mathrm{Le}\,R}-1)u(R^+)}{\theta_i R}e^{-\frac{\mathrm{Le}(B-R)}{2}(\widetilde{x}+1)}+\frac{2}{B-R}w_{\widetilde{x}},$$

$$\mathscr{G}_3(u,w)=e^{-\frac{\mathrm{Le}(B-R)}{2}(\widetilde{x}+1)}\frac{4\pi^2\mathrm{Le}(e^{-\mathrm{Le}\,R}-1)}{\theta_i^2\ell^2 R}\left[u(R^+)u_{\widetilde{y}\widetilde{y}}(R^+)+\left(\frac{\mathrm{Le}\,u(R^+)}{\theta_i}+1\right)u_{\widetilde{y}}^2(R^+)\right]$$

$$+\frac{8\pi^2}{\mathrm{Le}\,\theta_i\ell^2(B-R)}\left[-u_{\widetilde{y}\widetilde{y}}(R^+)w_{\widetilde{x}}-2u_{\widetilde{y}}(R^+)w_{\widetilde{x}\widetilde{y}}+\frac{2u_{\widetilde{y}}^2(R^+)}{\theta_i(B-R)}w_{\widetilde{x}\widetilde{x}}\right]$$

(8) For $x\in[R+\delta,R+2\delta]$ and $x=\frac{B-R}{2}(\widetilde{x}+1)+R$,

$$\mathscr{F}_1(u)=\frac{\theta_i}{\theta_i-u(R^+)}\left(\beta_R u(R^+)e^{-\frac{B-R}{2}(\widetilde{x}+1)}+\frac{2}{B-R}u_{\widetilde{x}}\right),$$

$$\mathscr{F}_2(u)=\frac{1}{\theta_i-u(R^+)}\left\{e^{-\frac{B-R}{2}(\widetilde{x}+1)}\left[-\frac{4\pi^2\beta_R^2}{\ell^2}\left(u(R^+)u_{\widetilde{y}\widetilde{y}}(R^+)+2u_{\widetilde{y}}^2(R^+)\right)+\beta_R u^2(R^+)\right]\right.$$

$$-\frac{8\pi^2\beta_R}{\ell^2(B-R)}\left(u_{\widetilde{y}\widetilde{y}}(R^+)u_{\widetilde{x}}+2u_{\widetilde{y}}(R^+)u_{\widetilde{x}\widetilde{y}}\right)+\frac{2u(R^+)}{B-R}u_{\widetilde{x}}\bigg\}$$

$$+\frac{\theta_i^2}{(\theta_i-u(R^+))^2}\left\{e^{-\frac{B-R}{2}(\widetilde{x}+1)}\left[\frac{u^2(R^+)}{\theta_i}\left(\frac{\beta_R u(R^+)}{\theta_i}-1-2\beta_R\right)\right.\right.$$

$$\left.-\frac{4\pi^2\beta_R^2 u_{\widetilde{y}}^2(R^+)}{\theta_i^2\ell^2}\left(3\beta_R u(R^+)-\theta_i+2u(R^+)\right)\right]$$

$$\left.-\frac{16\pi^2\beta_R^2 u_{\widetilde{y}}^2(R^+)}{\theta_i^2\ell^2(B-R)}\left(u_{\widetilde{x}}-\frac{1}{B-R}u_{\widetilde{x}\widetilde{x}}\right)-\frac{4}{(B-R)^2}\left(\frac{u^2(R^+)}{\theta_i}-1\right)u_{\widetilde{x}\widetilde{x}}\right\},$$

$$\mathscr{G}_1(u,w)=\frac{\theta_i}{\theta_i-u(R^+)}\left[-e^{-\frac{\mathrm{Le}(B-R)}{2}(\widetilde{x}+1)}\frac{\beta_R\mathrm{Le}(e^{-\mathrm{Le}\,R}-1)u(R^+)}{\theta_i R}\left(\frac{u(R^+)}{\theta_i}+\mathrm{Le}\right)+\frac{2}{B-R}w_{\widetilde{x}}\right],$$

$$\mathscr{G}_2(u,w)=\frac{\theta_i}{\theta_i-u(R^+)}\left(e^{-\frac{\mathrm{Le}(B-R)}{2}(\widetilde{x}+1)}\frac{\beta_R\mathrm{Le}(e^{-\mathrm{Le}\,R}-1)u(R^+)}{\theta_i R}+w\right),$$

$$\mathscr{G}_3(u,w)=\frac{1}{\theta_i-u(R^+)}\left\{e^{-\frac{\mathrm{Le}(B-R)}{2}(\widetilde{x}+1)}\frac{e^{-\mathrm{Le}\,R}-1}{\theta_i R}\left[\frac{4\pi^2\beta_R(2+\beta_R\mathrm{Le})}{\ell^2}\left(u_{\widetilde{y}\widetilde{y}}(R^+)u(R^+)+2u_{\widetilde{y}}^2(R^+)\right)\right.\right.$$

$$\left.\left.-\mathrm{Le}\,(1+\beta_R\mathrm{Le})u^2(R^+)\right]\right.$$

$$+ \frac{4\pi^2}{\mathrm{Le}\,\ell^2}\left[u_{\widetilde{y}\widetilde{y}}(R^+)\left(w - \frac{2\beta_R}{B-R}w_{\widetilde{x}}\right) + 2u_{\widetilde{y}}(R^+)\left(w_{\widetilde{y}} - \frac{2\beta_R}{B-R}w_{\widetilde{x}\widetilde{y}}\right)\right] + \frac{2u(R^+)}{B-R}w_{\widetilde{x}}\bigg\}$$

$$+ \frac{\theta_i}{(\theta_i - u(R^+))^2}\bigg\{e^{-\frac{\mathrm{Le}(B-R)}{2}(\widetilde{x}+1)}\frac{e^{-\mathrm{Le}\,R} - 1}{\theta_i R}$$

$$\times\left[\frac{4\pi^2 u_{\widetilde{y}}^2(R^+)}{\ell^2}\left(\frac{\beta_R^2\mathrm{Le}(7 + \beta_R\mathrm{Le}) + \beta_R(2 - \mathrm{Le}^2) - \mathrm{Le}}{\theta_i}u(R^+) - 2\beta_R - \mathrm{Le}(\beta_R^2 - 1)\right)\right.$$

$$\left. + 2(1 + \beta_R\mathrm{Le})\mathrm{Le}\,u^2(R^+)\right]$$

$$+ \frac{8\pi^2 u_{\widetilde{y}}^2(R^+)}{\mathrm{Le}\,\theta_i\ell^2}\left(w - \frac{4\beta_R}{B-R}w_{\widetilde{x}} + \frac{2(\beta_R^2 - 1)}{(B-R)^2}w_{\widetilde{x}\widetilde{x}}\right) + \frac{8u(R^+)}{\mathrm{Le}(B-R)^2}w_{\widetilde{x}\widetilde{x}}\bigg\}.$$

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