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SPECIAL SECTION: AN HOMAGE TO MANFREDO P. DO CARMO

The Kempf–Ness Theorem and invariant theory for real reductive representations

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Abstract
This paper does not contain any new result. We give new proofs of the Kempf–Ness Theorem and Hilbert–Mumford criterion for real reductive representations avoiding any algebraic results.

Keywords Gradient map · Geometric invariant theory · Stability · Polystability · Semi-stability

Mathematics Subject Classification 53D20 · 14L24

1 Introduction

Let U be a compact connected Lie group acting on a finite dimensional complex vector space V. Write $U^{\mathbb{C}}$ the complexification of U. Luna [31] defined a surjective quotient morphism $\pi : V \longrightarrow V//U^{\mathbb{C}}$ where each fiber contains exactly one closed orbit. Kempf and Ness showed there exists a closed subset $\mathcal{M} \subset V$ such that the inclusion $\mathcal{M} \hookrightarrow V$ induces an homeomorphism from $\mathcal{M}/K$ to $V//U^{\mathbb{C}}$ [25]. This shows, in particular, that $V//U^{\mathbb{C}}$ is homeomorphic to a real semialgebraic set. In 1990 Richardson and Slodowoy [39] proved that the Kempf–Ness Theorem extends to the case of real reductive representations. Therefore the Kempf–Ness Theorem provides geometric criterion for the closedness of orbits of a real reductive representation and the existence of quotient [16,18,19,25,26,35,39,40]. The Kempf–Ness Theorem have also allowed to prove many results exploiting tools from geometric invariant theory. This is the perspective taken, amongst many others, in the papers [16,28,29]. Recently Böhm and Lafuente [9] proved the Kempf–Ness Theorem for linear actions of real

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reductive Lie groups avoiding any deep algebraic result. This motivated us to give a new proof of the Kempf–Ness Theorem and the Hilbert Mumford criterion for real reductive representations using original ideas from [3–6] and the Slice Theorem proved by Heinzner, Schwarz and Stötzel (see [20, Theorem 3.3, p. 8] and [19, Theorem 14.10, p. 233 and Theorem 14.21, p. 226]).

In the literature there exist several non equivalent definition of real reductive Lie group [1–4,14,17,23,27,32,33,43]. Since our interest is linear representation we restricted ourselves to linear groups, i.e., subgroup of $\mathrm{GL}(V)$ for some finite dimensional real vector space.

Let $\rho : G \longrightarrow \mathrm{GL}(V)$ be a faithful representation on a finite dimensional real vector space. We identify G with $\rho(G) \subset \mathrm{GL}(V)$ and we assume that G is closed and it is closed under transpose. This means there exists a scalar product $\langle \cdot, \cdot \rangle$ on V such that $G = K \exp(\mathfrak{p})$, where $K = G \cap \mathrm{O}(V)$ and $\mathfrak{p} = \mathfrak{g} \cap \mathrm{Sym}(V)$. Here we denote by $\mathrm{O}(V)$ the orthogonal group with respect to $\langle \cdot, \cdot \rangle$, by $\mathrm{Sym}(V)$ the set of symmetric endomorphisms of V and finally with $\mathfrak{g}$ the Lie algebra of G. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition, that is $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, K is a maximal compact subgroup of G and the map $K \times \mathfrak{p} \mapsto G$, $(k, \xi) \mapsto k \exp(\xi)$, is a diffeomorphism. Moreover, any two maximal Abelian subalgebras of $\mathfrak{p}$ are conjugate by an element of K and the decomposition $G = KTK$ holds, where $T = \exp(\mathfrak{t})$ is the connected Abelian subgroup corresponding to a maximal Abelian subalgebra $\mathfrak{t}$ contained in $\mathfrak{p}$ [23,27]. In this setting, the function

$$\Psi : G \times V \longrightarrow \mathbb{R}, \quad (g, x) \mapsto \frac{1}{2}(\langle gx, gx \rangle - \langle x, x \rangle).$$

is a Kempf–Ness function (see Sect. 3) and the corresponding gradient map is given by

$$\mathfrak{F}_{\mathfrak{p}} : V \longrightarrow \mathfrak{p}^*, \quad \mathfrak{F}_{\mathfrak{p}}(x)(\xi) = \langle \xi x, x \rangle.$$

If $\mathfrak{a} \subset \mathfrak{p}$ is an Abelian subalgebra, then $\Psi_{|_{A\times V}}$ is a Kempf–Ness function with respect to the $A = \exp(\mathfrak{a})$ action on V and the corresponding gradient map is given by $\mathfrak{F}_{\mathfrak{a}}(x) = \mathfrak{F}_{\mathfrak{p}}(x)_{|_{\mathfrak{a}}}$. Write $\mathcal{M} = \mathfrak{F}_{\mathfrak{p}}^{-1}(0)$. The aim of this paper is to give a new proof of the following well-known result [1,5,25,31,34,39].

Theorem 1 *Let $G \subset \mathrm{GL}(V)$ be as above and let $x \in V$. The following holds:*

(a) *$G \cdot x$ is closed if and only if $G \cdot x \cap \mathcal{M} \neq \emptyset$;*
(b) *$\overline{G \cdot x}$ contains exactly one closed orbit;*
(c) *$\overline{G \cdot x} \cap \mathcal{M} = K \cdot v$, v minimizes the distance to $0 \in V$ within $\overline{G \cdot x}$ and $G \cdot v$ is the unique closed orbit contained in $\overline{G \cdot x}$;*
(d) *if $G \cdot v$ is the unique closed orbit contained in $\overline{G \cdot x}$, then there exists $\xi \in \mathfrak{p}$ such that $\lim_{t \mapsto +\infty} \exp(t\xi)x$ exists and lies in $G \cdot v$.*

Moreover, the null cone $\mathcal{N} = \{x \in V : 0 \in \overline{G \cdot x}\}$ is closed.

We point out that in [22] the authors prove that $\mathcal{N}$ is real algebraic. If G is Abelian a proof avoiding any algebraic result is given in [7]. However, if G is real reductive, then we are not able to give a new proof of this result.

This paper is organized as follows.

In Sect. 2 we recall basic results on the Tits boundary of an Hadamard manifold.

In Sect. 3 we recall the abstract setting on which we are able to develop a geometrical invariant theory for actions of real reductive Lie groups. We also recall the definition of stable, semistable and polystable points and we prove some basic results.

In Sect. 4 we define the maximal weight on the Tits boundary of G/K and we discuss some properties of the maximal weight.

In Sect. 5 we characterize stable points (Theorem 16) for real reductive representations. We also prove that any point is semistable (Theorem 19) and $G \cdot x$ is closed if and only if $G \cdot x \cap \mathfrak{F}_{\mathfrak{p}}^{-1}(0) \neq \emptyset$ (Theorem 20). Using the Slice Theorem, we prove that for any $x \in V$, $\overline{G \cdot x} \cap \mathfrak{F}_{\mathfrak{p}}^{-1}(0) = K \cdot v$ and $G \cdot v$ is the unique closed orbit contained in $\overline{G \cdot x}$ (Theorem 22). We also prove that v minimize the distance to $0 \in V$ within the orbit $G \cdot x$ (Corollary 23). Using results proved in [7] and the Slice Theorem we prove the Hilbert Mumford criterion for real reductive representations (Theorems 27 and 28). Finally we prove that the null cone is closed (Theorem 29). Summing up, in Sect. 5 we proof our main result.

In Sect. 6 we briefly discuss the G action on $\mathbb{P}(V)$ giving a sketch of the proof of some well-known results.

2 Real reductive groups

Let G be a non-compact real reductive Lie group and denote by $\mathfrak{g}$ its Lie algebra. Recall that such G has a finite number of connected components and its algebra splits as $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{z}(\mathfrak{g})$, where $[\mathfrak{g}, \mathfrak{g}]$ is semisimple and $\mathfrak{z}(\mathfrak{g})$ is the center of $\mathfrak{g}$. Further, maximal compact subgroups of G always exist and meet every connected components, and any two of them are conjugate under an element of the identity component G^o of G. Assume that there exists a Cartan involution $\theta : G \longrightarrow G$ with fixed points set K and let us denote also by $\theta : \mathfrak{g} \longrightarrow \mathfrak{g}$ its differential. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and the map $f : K \times \mathfrak{p} \to G$, $f(g, v) = g \exp v$ is a diffeomorphism. This means that $G = K \exp(\mathfrak{p})$ and G/K is simply connected. Since $\theta_{|\mathfrak{k}} = \mathrm{Id}$ and $\theta_{|\mathfrak{p}} = -\mathrm{Id}$, we have $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. Therefore if $\mathfrak{a} \subset \mathfrak{p}$ is a Lie subalgebra, then it must be Abelian. Moreover, two maximal Abelian subalgebras contained in $\mathfrak{p}$ are conjugate with respect to the identity component K^o. We refer the reader to [13,14,17,23,27,43] for more details on real reductive Lie groups. Set

$$X := G/K.$$

Observe that G acts on X from the left by:

$$L_g : X \to X, \quad L_g(hK) := ghK, \quad g \in G.$$

To simplify the notation, we will often write gx instead of $L_g(x)$. The choice of an $\mathrm{Ad}(K)$-invariant scalar product on $\mathfrak{p}$ induces a G-invariant Riemannian metric on X. It is well known that X endowed with this metric is a symmetric space of non compact type and thus an Hadamard manifold, i.e. a simply connected complete Riemannian manifold of non positive sectional curvature [15,23]. The Riemannian exponential map arises by the exponential map of Lie groups. Hence a geodesic on X is given by $g\exp(tv)K$, where $g \in G$ and $v \in \mathfrak{p}$. In the sequel we denote by γ^v the geodesic $\gamma^v(t) = \exp(tv)K$.

Since X is an Hadamard manifold there is a natural notion of boundary at infinity $\partial_\infty X$ which can be described using geodesics.

Two unit speed geodesic rays $\gamma, \gamma' : (0, +\infty) \to X$ are equivalent, denoted by $\gamma \sim \gamma'$, if $\sup_{t\in(0,+\infty)} d(\gamma(t), \gamma'(t)) < +\infty$. The *Tits boundary* of X, denoted by $\partial_\infty X$, is the set of equivalence classes of unit speed geodesic ray in X.

Set $o := K \in X$. Mapping v to the tangent vector $\dot{\gamma}^v(0)$ yields an isomorphism $\mathfrak{p} \cong T_o X$. Any geodesic ray in X is equivalent to a unique ray starting from o, so the map:

$$\mathrm{e} : S(\mathfrak{p}) \to \partial_\infty X, \ \mathrm{e}(v) := [\gamma^v], \tag{1}$$

where $S(\mathfrak{p})$ is the unit sphere in $\mathfrak{p}$, is a bijection. The *sphere topology* is the topology on $\partial_\infty X$ such that e is a homeomorphism. (For more details on the Tits boundary see for example [14, §I.2] and [15].)

Since G acts by isometries on X, if γ is a unit speed geodesic in X, then for each $g \in G$ also $g\gamma$ is. Further, since $\gamma \sim \gamma'$ implies $g\gamma \sim g\gamma'$, we get a G-action on the Tits boundary $\partial_\infty X$ by:

$$g \cdot [\gamma] = [g\gamma],$$

which also induces by (1) a G-action on $S(\mathfrak{p})$ given by:

$$g \cdot v = \mathrm{e}^{-1}(g \cdot \mathrm{e}(v)).$$

This action is continuous with respect to the sphere topology on $\partial_\infty X$ (see [14], p. 41), but it is not smooth.

Definition 2 Let $H \subset G$ be a closed subgroup. Set $L := H \cap K$ and $\tilde{\mathfrak{p}} := \mathfrak{h} \cap \mathfrak{p}$. Following [20,21], we say that H is *compatible* if $H = L\exp(\tilde{\mathfrak{p}})$.

If H is a compatible subgroup of G, then it follows that it is a real reductive subgroup of G, the Cartan involution of G induces a Cartan involution of H, L is a maximal compact subgroup of H and finally $\mathfrak{h} = \mathfrak{l} \oplus \tilde{\mathfrak{p}}$. Note that H has finitely many connected components. Moreover, there are totally geodesic inclusions $X' := H/L \hookrightarrow X$ and $\partial_\infty X' \subset \partial_\infty X$.

3 Kempf–Ness functions

In this section we briefly recall the abstract setting introduced in [4] (see also [5,6]).

Let $\mathscr{M}$ be a Hausdorff topological space and let G be a connected non-compact real reductive group which acts continuously on $\mathscr{M}$. Observe that with these assumptions we can write $G = K\exp(\mathfrak{p})$, where K is a maximal compact subgroup of G. Starting with these data we consider a function $\Psi : \mathscr{M} \times G \to \mathbb{R}$, subject to four conditions.

($P1$) For any $x \in \mathscr{M}$ the function $\Psi(x, \cdot)$ is smooth on G.
($P2$) The function $\Psi(x, \cdot)$ is left–invariant with respect to K, i.e.: $\Psi(x, kg) = \Psi(x, g)$.
($P3$) For any $x \in \mathscr{M}$, and any $v \in \mathfrak{p}$ and $t \in \mathbb{R}$:

$$\frac{\mathrm{d}^2}{\mathrm{dt}^2}\Psi(x, \exp(tv)) \geq 0.$$

Moreover:

$$\left.\frac{\mathrm{d}^2}{\mathrm{dt}^2}\right|_{t=0}\Psi(x, \exp(tv)) = 0$$

if and only if $\exp(\mathbb{R}v) \subset G_x$.
($P4$) For any $x \in \mathscr{M}$, and any $g, h \in G$:

$$\Psi(x, g) + \Psi(gx, h) = \Psi(x, hg).$$

This equation is called the *cocycle condition*.

Remark 3 Taking $g = h = e$ in the cocycle condition ($P4$) we have $\Psi(x, e) = 0$. Hence $\Psi(x, k) = 0$ for every $k \in K$, since $\Psi(x, \cdot)$ is K-invariant on the second factor. Moreover, for any $x \in \mathscr{M}$ and for any $g, h \in G_x$ we have:

$$\Psi(x, hg) = \Psi(x, g) + \Psi(x, h), \tag{2}$$

which implies that $\Psi(x, \cdot) : G_x \longrightarrow \mathbb{R}$ is a homomorphism.

For $x \in \mathscr{M}$ define $\mathfrak{F}_{\mathfrak{p}}(x) \in \mathfrak{p}^*$ by requiring that:

$$\mathfrak{F}_{\mathfrak{p}}^v(x) = \mathfrak{F}_{\mathfrak{p}}(x)(v) = \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0}\Psi(x, \exp(tv)),$$

for any $v \in \mathfrak{p}$. We call $\mathfrak{F}_{\mathfrak{p}}$ the *gradient map* of $(\mathscr{M}, G, K, \Psi)$. As immediate consequence of the definition of $\mathfrak{F}_{\mathfrak{p}}$, we have the following result.

Proposition 4 *The map $\mathfrak{F}_{\mathfrak{p}} : \mathscr{M} \to \mathfrak{p}^*$ is K-equivariant.*

Proof It is an easy application of the cocycle condition and the left-invariance with respect to K of $\Psi(x, \cdot)$. Indeed,

$$\begin{aligned}\mathfrak{F}_{\mathfrak{p}}^v(kx) &= \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0}\Psi(x, \exp(tv)k) = \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0}\Psi(x, k^{-1}\exp(tv)k)\\ &= \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0}\Psi\left(x, \exp(t\mathrm{Ad}(k^{-1})(v))\right) = \mathrm{Ad}^*(k)(\mathfrak{F}_{\mathfrak{p}}(x))(v).\end{aligned}$$

□

The following definition summarizes the above discussion.

Definition 5 Let G be a non-compact real reductive Lie group, K a maximal compact subgroup of G and $\mathscr{M}$ a topological space with a continuous G–action. A *Kempf–Ness function* for $(\mathscr{M}, G, K)$ is a function

$$\Psi : \mathscr{M} \times G \to \mathbb{R},$$

that satisfies conditions $(P1)$–$(P4)$.

The Kempf–Ness function induces, in a natural way, a function $\psi_x : G/K \longrightarrow \mathbb{R}$ defined as follows

$$\psi_x(gK) = \Psi(x, g^{-1}).$$

Using ψ_x instead of Ψ the cocycle condition reads

$$\psi_x(ghK) = \psi_x(gK) + \psi_{g^{-1}x}(hK). \tag{3}$$

Remark 6 If $\mathscr{M}'$ is a G-invariant subspace of $\mathscr{M}$, the restriction of Ψ to $G \times \mathscr{M}'$ is a Kempf–Ness function for $(\mathscr{M}', G, K)$. The function $\mathfrak{F}_{\mathfrak{p}}$ for $(\mathscr{M}', G, K)$ is simply the restrictions of those for $\mathscr{M}$. If $G' \subset G$ is a compatible subgroup of G, i.e., $G' = K' \exp(\mathfrak{p}')$, then $K' \subset K$, $\mathfrak{p}' \subset \mathfrak{p}$ and $X' := G'/K' \hookrightarrow X$ is a totally geodesic inclusion. If Ψ is a Kempf–Ness function for $(G, K, \mathscr{M})$, then $\Psi^{G'} := \Psi|_{\mathscr{M} \times G'}$ is a Kempf–Ness function for $(G', K', \mathscr{M})$. The related functions are

$$\mathfrak{F}_{\mathfrak{p}'} : \mathscr{M} \to \mathfrak{p}'^{*}, \quad \mathfrak{F}_{\mathfrak{p}'}(x) := \mathfrak{F}_{\mathfrak{p}}(x)|_{\mathfrak{p}'}, \tag{4}$$

$$\psi_x^{G'} := \psi_x|_{X'}. \tag{5}$$

Let $(\mathscr{M}, G, K)$ be as above and let Ψ be a Kempf–Ness function.

Definition 7 Let $x \in \mathscr{M}$. Then:

(a) x is *polystable* if $G \cdot x \cap \mathfrak{F}_{\mathfrak{p}}^{-1}(0) \neq \emptyset$.
(b) x is *stable* if it is polystable and $\mathfrak{g}_x$ is conjugate to a subalgebra of $\mathfrak{k}$.
(c) x is *semistable* if $\overline{G \cdot x} \cap \mathfrak{F}_{\mathfrak{p}}^{-1}(0) \neq \emptyset$.
(d) x is *unstable* if it is not semi-stable.

Remark 8 The four conditions above are G-invariant in the sense that if a point x satisfies one of them, then every point in the orbit of x satisfy the same condition. This follows directly from the definition for polystability, semistability and unstability, while for stability it is enough to recall that $\mathfrak{g}_{gx} = \mathrm{Ad}(g)(\mathfrak{g}_x)$.

The following result establishes a relation between the Kempf–Ness function and polystable points.

Proposition 9 *Let $x \in \mathscr{M}$. The following conditions are equivalent:*

(a) $g \in G$ *is a critical point of* $\Psi(x, \cdot)$*;*
(b) $\mathfrak{F}_{\mathfrak{p}}(gx) = 0$*;*
(c) $g^{-1}K$ *is a critical point of* ψ_x.

Proof Let $v \in \mathfrak{p}$. Using the cocycle condition ($P4$), one gets:

$$\Psi(x, \exp(tv)g) = \Psi(x, g) + \Psi(gx, \exp(tv)).$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} \Psi(x, \exp(tv)g) = \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} \Psi(gx, \exp(tv)) = \mathfrak{F}_{\mathfrak{p}}^{v}(gx). \tag{6}$$

Since for any $k \in K$, $\Psi(x, kg) = \Psi(x, g)$, then $\mathfrak{F}_{\mathfrak{p}}(gx) = 0$ if and only if g is a critical point of $\Psi(x, \cdot)$ if and only if $g^{-1}K$ is a critical point of ψ_x. □

Proposition 10 *If* $\mathfrak{F}_{\mathfrak{p}}(x) = 0$*, then* G_x *is compatible and* $G \cdot x \cap \mathfrak{F}_{\mathfrak{p}}^{-1}(0) = K \cdot x$*. Moreover, if* $G = \exp(\mathfrak{p})$ *with* $\mathfrak{p}$ *Abelian, then* G_x *is compatible for any* $x \in \mathscr{M}$*.*

Proof Let $g \in G_x$. Then $g = k\exp(\xi)$ for some $k \in K$ and $\xi \in \mathfrak{p}$. Since $\mathfrak{F}_{\mathfrak{p}}$ is K-equivariant, it follows $\mathfrak{F}_{\mathfrak{p}}(\exp(\xi)x) = 0$. Let $f(t) := \mathfrak{F}_{\mathfrak{p}}^{v}(\exp(tv)x)$. Then $f(0) = f(1) = 0$ and

$$\frac{\mathrm{d}}{\mathrm{dt}} f(t) = \frac{\mathrm{d}}{\mathrm{dt}} \mathfrak{F}_{\mathfrak{p}}^{v}(\exp(tv)x) = \frac{\mathrm{d}^2}{\mathrm{dt}^2} \Psi(x, \exp(tv)) \geq 0.$$

Therefore $\frac{\mathrm{d}^2}{\mathrm{dt}^2}\Psi(x, \exp(tv)) = 0$ for $0 \leq t \leq 1$. It follows from ($P3$) that $\exp(tv)x = x$ for any $t \in \mathbb{R}$ and thus $\exp(t\xi) \in G_x$. This proves that G_x is compatible. The same computation proves $G \cdot x \cap \mathfrak{F}_{\mathfrak{p}}^{-1}(0) = K \cdot x$. Indeed, assume $gx \in \mathfrak{F}_{\mathfrak{p}}^{-1}(0)$ for some $g \in G$. Write $g = k\exp(\xi)$. Therefore $\exp(\xi)x \in \mathfrak{F}_{\mathfrak{p}}^{-1}(0)$ and so, as before, $\exp(\mathbb{R}\xi) \subset G_x$ which implies $gx = kx$. Finally, assume that $G = \exp(\mathfrak{p})$ is Abelian. By the above discussion it is easy to check that $g = \exp(\xi) \in G_x$ if and only if $\exp(\mathbb{R}\xi) \subset G_x$. Therefore G_x is compatible. □

4 Maximal weights

Let $\mathscr{M}$ be a Hausdorff topological space on which a connected reductive Lie group G acts continuously. Assume that the G action on $\mathscr{M}$ admits a Kempf–Ness function

$$\Psi : \mathscr{M} \times G \to \mathbb{R}.$$

Write $\mathfrak{F}_{\mathfrak{p}} : \mathscr{M} \longrightarrow \mathfrak{p}^*$ the associated gradient map. Given $\xi \in \mathfrak{p}$ for any $t \in \mathbb{R}$ we define $\lambda(x, \xi, t) = \mathfrak{F}_{\mathfrak{p}}^{\xi}(\exp(t\xi)x)$. Applying the cocycle condition we get

$$\mathfrak{F}_{\mathfrak{p}}^{\xi}(\exp(tv)x) = \frac{\mathrm{d}}{\mathrm{dt}} \Psi(x, \exp(t\xi))$$

and so

$$\frac{\mathrm{d}}{\mathrm{dt}} \mathfrak{F}_{\mathfrak{p}}^{\xi}(\exp(tv)x) = \frac{\mathrm{d}^2}{\mathrm{dt}^2} \Psi(x, \exp(tv)) \geq 0.$$

This means that

$$\lambda(x, \xi, t) = \mathfrak{F}_{\mathfrak{p}}^{\xi}(x) + \int_0^t \frac{\mathrm{d}^2}{\mathrm{ds}^2} \Psi(x, \exp(sv)) \mathrm{d}s$$

is a non decreasing function as a function of t. Moreover,

$$\Psi(x, \exp(t\xi)) = \int_0^t \lambda(x, \xi, \tau) \mathrm{d}\tau.$$

and so

$$\Lambda(x, \xi) := \lim_{t \mapsto +\infty} \frac{\mathrm{d}}{\mathrm{dt}} \Psi(x, \exp(t\xi)) \in \mathbb{R} \cup \{\infty\},$$

The function $\Lambda(x, \cdot)$ is called maximal weight of x in the direction ξ. For a reference see, amongst many others, [3,4,34,36,42]. The following useful Lemma is proved in [42, Lemma 2.10].

Lemma 11 *Let V be a subspace of $\mathfrak{p}$. For a point $x \in \mathscr{M}$ the following conditions are equivalent:*

(a) *The map $\Psi(x, \exp(\xi))$ is linearly proper on V, i.e. there exist positive constants C_1 and C_2 such that:*

$$||\xi||^2 \leq C_1 \Psi(x, \exp(\xi)) + C_2, \quad \forall \xi \in V.$$

(b) $\Lambda(x, \xi) > 0$ *for any* $\xi \in V - \{0\}$.

Let $x \in \mathscr{M}$ and let

$$\psi_x : G/K \longrightarrow \mathbb{R}, \quad \psi_x(gK) = \Psi(x, g^{-1})$$

In the sequel we denote by $X = G/K$.

Lemma 12 *The function ψ_x is geodesically convex on X. More precisely, if $v \in \mathfrak{p}$ and $\alpha(t) = g\exp(tv)K$ is a geodesic in X, then $\psi_x \circ \alpha$ is either strictly convex or affine. The latter case occurs if and only if $g\exp(\mathbb{R}v)g^{-1} \subset G_x$. In the case $g = e$, the function $\psi_x \circ \alpha$ is linear if $\exp(\mathbb{R}v) \subset G_x$ and strictly convex otherwise.*

Proof Fix $t_0 \in \mathbb{R}$. Set $h := g\exp(t_0 v)$. By (3)

$$\psi_x(\alpha(t_0 + s)) = \psi_x(h\exp(sv)K) = \psi_x(hK) + \psi_{h^{-1}x}(\exp(sv)K).$$

Hence

$$\left.\frac{d^2}{dt^2}\right|_{t=t_0}\psi_x(\alpha(t)) = \left.\frac{d^2}{ds^2}\right|_{s=0}\psi_{h^{-1}x}(\exp(sv)K)$$
$$= \left.\frac{d^2}{ds^2}\right|_{s=0}\Psi(h^{-1}x, \exp(-sv)).$$

From $(P3)$ yields convexity of $\psi_x \circ \alpha$. If $\psi_x \circ \alpha$ is not strictly convex at t_0, then again by $(P3)$ we conclude that $\exp(\mathbb{R}v) \subset G_{h^{-1}x}$. By Remark 3, the function $\psi_x(\exp(tv)K) = \Psi(h^{-1}x, \exp(-tv))$ is a linear function of t. By (3) we have $\psi_x(\alpha(t)) = \psi_x(hK) + \psi_x(\exp((t-t_0)v)K)$. This proves $\psi_x \circ \alpha$ is affine. Moreover from $\exp(\mathbb{R}v) \subset G_{h^{-1}x}$ it follows that $\exp(\mathbb{R}v) \subset G_{g^{-1}x}$ and so $g\exp(\mathbb{R}v)g^{-1} \subset G_x$. The same computation shows that conversely if $g\exp(\mathbb{R}v)g^{-1} \subset G_x$ then $\psi_x \circ \alpha$ is affine. In case $g = e$ we know that $\psi_x(K) = 0$, so if the function is affine, then it is in fact linear. $\square$

Now, assume that $\mathscr{M}$ is a connected Riemannian manifold and there exists $x_o \in \mathscr{M}$ and a constant $C > 0$ such that for any $x \in \mathscr{M}$ and any $\xi \in \mathfrak{p}$, we have

$$\parallel \xi^{\#}(x) \parallel \leq C \parallel \xi \parallel (1 + d_{\mathscr{M}}(x_o, x)), \tag{7}$$

where $\xi^{\#}(x) = \left.\frac{d}{dt}\right|_{t=0}\exp(t\xi)x$, i.e., the fundamental vector induced by the G action on $\mathscr{M}$ and $\parallel \xi \parallel$ is the norm of ξ with respect to any $\mathrm{Ad}(K)$-invariant scalar product on $\mathfrak{p}$. This condition is satisfied if $\mathscr{M}$ is compact or if $\mathscr{M}$ is a vector space and G acts linearly on $\mathscr{M}$. Under this assumption Mundet, see [36], proves that the function

$$\lambda_x : \partial_\infty X \longrightarrow \mathbb{R}, \quad \lambda_x([\gamma]) = \lim_{t \mapsto +\infty}\frac{\psi_x(\gamma(t))}{t} \in \mathbb{R} \cup \{\infty\}$$

is well-defined. Moreover, if $[\gamma] \in \partial_\infty X$ and $g \in G$, then

$$\lambda_{g^{-1}x}([\gamma]) = \lambda_x([g\gamma]). \tag{8}$$

In the sequel we denote by

$$\lambda_x(\xi) = \lambda_x([\exp(-t\xi)]).$$

$\lambda_x(\xi)$ is also called maximal weight in the direction ξ [36–38]. We point out that if ψ_x is globally Lipschitz continuous, then

$$\lambda_x([\gamma]) := \lim_{t \to +\infty}(u \circ \gamma)'(t), \tag{9}$$

see, amongst many others, [24, §3.1] [3,4,34,36,42]. Therefore if ψ_x is Lipschitz continuous then $\lambda_x(\xi) = \Lambda(x, \xi)$.

5 Real reductive representations

Let G be a connected real reductive Lie group and let $\rho : G \longrightarrow \mathrm{GL}(V)$ be a faithful representation on a finite dimensional real vector space V. We identify G with $\rho(G) \subset \mathrm{GL}(V)$ and we assume that G is closed and it is closed under transpose. This means that there exists a scalar product $\langle \cdot, \cdot \rangle$ on V such that $G = K \exp(\mathfrak{p})$, where $K \subset \mathrm{O}(V)$ and $\mathfrak{p} \subset \mathfrak{g} \cap \mathrm{Sym}(V)$. In the sequel, we denote by $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$. We define

$$\Psi : V \times G \longrightarrow \mathbb{R} \quad \Psi(x, g) = \frac{1}{2}\left(\| gx \|^2 - \| x \|^2\right).$$

Lemma 13 $\Psi : V \times G \longrightarrow \mathbb{R}$ *is a Kempf–Ness function and the corresponding gradient map* $\mathfrak{F}_{\mathfrak{p}} : V \longrightarrow \mathfrak{p}^*$ *is given by* $\mathfrak{F}_{\mathfrak{p}}^{\xi}(x) = \langle \xi x, x \rangle$.

Proof $(P1)$ and $(P2)$ are easy to check. Let $\xi \in \mathfrak{p}$ and let $f(t) = \Psi(x, \exp(t\xi))$. Then

$$f'(t) = \langle \exp(t\xi)\xi x, \exp(t\xi)x \rangle, \quad f''(t) = \langle \exp(t\xi)\xi x, \exp(t\xi)\xi x \rangle.$$

Hence $f''(t) \geq 0$ and $f''(0) = 0$ if and only if $\xi x = 0$ and so if and only if $\exp(\mathbb{R}\xi) \subset G_x$. Now,

$$\begin{aligned} \Psi(x, hg) &= \frac{1}{2}\left(\| hgx \|^2 - \| x \|^2\right) \\ &= \frac{1}{2}\left(\| hgx \|^2 - \| gx \|^2\right) + \frac{1}{2}\left(\| gx \|^2 - \| x \|^2\right) \\ &= \Psi(gx, h) + \Psi(x, g), \end{aligned}$$

proving the cocycle condition. Finally

$$\mathfrak{F}_{\mathfrak{p}}^{\xi}(x) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} \langle \exp(t\xi)x, \exp(t\xi)x \rangle = \langle \xi x, x \rangle,$$

concluding the proof. □

We may restrict on $\mathfrak{g}$ the scalar product

$$\langle X, Y \rangle = \mathrm{Tr}(XY^T), \tag{10}$$

which is $\mathrm{Ad}(K)$-invariant. Moreover $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is an orthogonal splitting. In particular, identifying $\mathfrak{p}$ with $\mathfrak{p}^*$ by means of $\langle \cdot, \cdot \rangle$, we may think the gradient map as a $\mathfrak{p}$-valued map,

$$\mathfrak{F}_{\mathfrak{p}} : V \longrightarrow \mathfrak{p}.$$

Let $\xi \in \mathfrak{p}$ and let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of eigenvectors of ξ with respect to the eigenvalues $\alpha_1, \ldots, \alpha_n$. If $x \in V$ then $x = x_1 e_1 + \cdots + x_n e_n$. In the sequel we denote by $\mathrm{supp}_x = \{i \in \{1, \ldots, n\} : x_i \neq 0\}$.

Lemma 14 (a) $\lambda_x(\xi) = 0$ *or* $\lambda_x(\xi) = \infty$ *for any* $\xi \in \mathfrak{p}$*;*
(b) $\lambda_x(\xi) = 0$ *if and only if* $\alpha_i \leq 0$ *for any* $i \in I$*, where* $I = \mathrm{supp}_x$*;*
(c) $\lambda_x(\xi) = \infty$ *if and only if there exists* $i \in \mathrm{supp}_x$ *such that* $\alpha_i > 0$.

Proof Write $x = \sum_{i=1}^n x_i e_i$. It is easy to check that

$$\Psi(x, \exp(t\xi)) = \frac{1}{2}\left(\sum_{i \in I} e^{2t\alpha_i} \parallel x_i \parallel^2 - \parallel x \parallel^2\right).$$

Therefore

$$\lambda_x(\xi) = \lim_{t \mapsto +\infty} \frac{1}{2} \sum_{i \in I} \frac{e^{2t\alpha_i}}{t} \parallel x_i \parallel^2 = \begin{cases} 0 & \text{if } \alpha_i \leq 0 \text{ for any } i \in I \\ +\infty & \text{if } \alpha_i > 0 \text{ for some } i \in I \end{cases}$$

□

Since $\Psi(x, \cdot)$ is not globally Lipschitz, $\lambda_x(\xi)$ does not coincide in general with $\Lambda(x, \xi)$.

Lemma 15 *Let* $x \in V$ *and let* $\xi \in \mathfrak{p}$*. Then*

(a) $\lambda_x(\xi) = +\infty$ *if and only if* $\Lambda(x, \xi) = +\infty$*;*
(b) $\lambda_x(\xi) = 0$ *if and only if* $\exp(t\xi) \in G_x$ *or* $\Lambda(x, \xi) < 0$.

Proof It is easy to check that $\frac{\mathrm{d}}{\mathrm{dt}}\Psi(x, \exp(t\xi)) = 2\sum_{i=1}^n \alpha_i e^{2t\alpha_i} \parallel x_i \parallel^2$. Therefore $\Lambda(x, \xi) > 0$ if and only if there exists $i \in \mathrm{supp}_x$ such that $\alpha_i > 0$ and so, by the above Lemma, if and only if $\lambda(x, \xi) > 0$.

Assume $\lambda_x(\xi) = 0$. By the above Lemma it follows $\alpha_i \leq 0$ for any $i \in \mathrm{supp}_x$. If $\alpha_i = 0$ for any $i \in \mathrm{supp}_x$, then $\exp(\mathbb{R}\xi) \subset G_x$. Otherwise $\alpha_i \leq 0$ and $\alpha_{i_o} < 0$ for some $i_o \in \mathrm{supp}_x$ and so $\Lambda(x, \xi) < 0$. The vice-versa might be proved similarly. □

Theorem 16 *The element* $x \in V$ *is stable if and only if* $\lambda_x > 0$ *for any* $\xi \in \mathfrak{p} - \{0\}$.

Proof Assume that x is stable. Since λ_x is G-equivariant, we may assume that $\mathfrak{F}_{\mathfrak{p}}(x) = 0$. By Proposition 10 the Lie algebra of $\mathfrak{g}_x$ is compatible and so it is contained in $\mathfrak{k}$. Let $\xi \in \mathfrak{p} - \{0\}$ and let $f(t) = \Psi(x, \exp(t\xi))$. Since $\mathfrak{F}_{\mathfrak{p}}(x) = 0$, it follows $f'(0) = \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} \Psi(x, \exp(t\xi)) = 0$ and so, keeping in mind that $\frac{\mathrm{d}}{\mathrm{dt}}\Psi(x, \exp(t\xi))$ increases, $f(t) = 0$ for any $t \geq 0$ or $\Lambda(x, \xi) > 0$. If $f(t) = 0$, then $\exp(\mathbb{R}\xi) \subset G_x$ which is a contradiction. Therefore $\Lambda(x, \xi) > 0$ and so by Lemma 15 $\lambda_x(\xi) > 0$. Vice-versa, assume $\lambda_x(\xi) > 0$ for any $\xi \in \mathfrak{p} - \{0\}$. By Lemma 15 $\Lambda(x, \xi) > 0$ for any $\xi \in \mathfrak{p} - \{0\}$. Hence, keeping in mind that $\frac{\mathrm{d}^2}{\mathrm{dt}^2}\Psi(x, \exp(t\xi)) \geq 0$, it follows there exist $t(\xi) > 0$ and $C_o > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{dt}}\Psi(x, \exp(t(\xi)) \geq C_o > 0,$$

for any $t \geq t(\xi)$. Hence there exists a neighborhood U_ξ of ξ in $\mathrm{S}(\mathfrak{p})$ such that $\frac{\mathrm{d}}{\mathrm{dt}}\Psi(x, \exp(t\nu)) > \frac{C_o}{2}$ for any $t \geq t(\xi)$ and for any $\nu \in U_\xi$. By usual compactness argument, there exist two constants $C > 0$ and $t_o > 0$ such that $\frac{\mathrm{d}}{\mathrm{dt}}\Psi(x, \exp(t\xi)) \geq C$, for any $\xi \in \mathrm{S}(\mathfrak{p})$ and for any $t \geq t_o$. Therefore, for any $v \in \mathfrak{p}$ such that $\| v \| \geq t_o$, we get

$$\Psi(x, \exp(t(v))) = \Psi\left(x, \exp\left(t_o \frac{v}{\| v \|}\right)\right) + \int_{t_o}^{\|v\|} \frac{\mathrm{d}}{\mathrm{dt}}\Psi\left(x, \exp\left(t\frac{v}{\| v \|}\right)\right)\mathrm{d}t.$$

Hence $\Psi(x, \exp(v)) \geq \min_{\|w\|=t_o} \Psi(x, \exp(w))$ for any $v \in \mathfrak{p}$ such that $\| v \| \geq t_o$. This means that Ψ_x has a critical point which is a global minimum. By Proposition 9, x is polystable and so there exits $g \in G$ such that $\mathfrak{F}_\mathfrak{p}(gx) = 0$. Write $y = gx$. By the G-equivariance of the maximal weight, it follows $\lambda_y > 0$. We claim $\mathfrak{g}_y \subset \mathfrak{k}$. By Proposition 10 $\mathfrak{g}_y$ is compatible. Let $\xi \in \mathfrak{g}_y \cap \mathfrak{p}$. By Lemma 15 it follows $\lambda_x(\xi) = 0$. Therefore $\mathfrak{g}_y \cap \mathfrak{p} = \{0\}$ and so $\mathfrak{g}_y \subset \mathfrak{k}$ concluding the proof. □

Corollary 17 *If $x \in V$ is stable, then G_x is compact.*

Proof Let $g \in G$ be such that $\mathfrak{F}_\mathfrak{p}(gx) = 0$ and set $y = gx$. By Proposition 10 G_y is compatible and so it has only finitely many connected components. Since G_y^0 is compact and compatible, it follows that $\mathfrak{g}_y \subset \mathfrak{k}$ and so $G_y^0 \subset K$. Therefore G_y and $G_x = g^{-1}G_y g$ are both compact. □

Corollary 18 *A point $x \in V$ is G-stable if and only if it is A-stable for any Abelian group $A = \exp(\mathfrak{a})$, where $\mathfrak{a}$ is a subalgebra of $\mathfrak{g}$ contained in $\mathfrak{p}$.*

Proof By Remark 6, if $\xi \in \mathfrak{a}$, then $\lambda_x^A(\xi) = \lambda_x(\xi)$ and so the maximal weight of x in the direction $\xi \in \mathfrak{a}$ with respect to $A = \exp(\mathfrak{a})$ coincides with the maximal weight of x in the direction ξ with respect to G. Therefore, by Theorem 16, x is G stable if and only if x is $A = \exp(\mathfrak{a})$ stable for any Abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$. □

Theorem 19 *Any $x \in V$ is semistable.*

Proof The proof of this result is not new but for the sake of completeness we write it down for the reader.

Let $\| \cdot \|^2 \colon \overline{G \cdot x} \longrightarrow \mathbb{R}$ the norm square restricted to the closure of the G orbit throughout x. Since this function is proper it has a minimum and so it has a critical at some point $y \in \overline{G \cdot x}$. Since $G \cdot y \subset \overline{G \cdot x}$, the function

$$G \longrightarrow \mathbb{R} \quad g \mapsto \| gy \|^2$$

has a minimum at $e \in G$. Therefore the function

$$\Psi(y, \cdot) : G \longrightarrow \mathbb{R} \quad g \mapsto \frac{1}{2}\left(\| gy \|^2 - \| y \|^2\right)$$

has a critical point at e. By Proposition 9 we get $\mathfrak{F}_\mathfrak{p}(y) = 0$ concluding the proof. □

Theorem 20 *A point $x \in V$ is polystable if and only if $G \cdot x$ is closed.*

Proof Assume that $x \in V$ is polystable. We may assume that $x \in \mathfrak{F}_{\mathfrak{p}}^{-1}(0)$. By Proposition 10 the stabilizer G_x is compatible. Hence $\mathfrak{g}_x = \mathfrak{k}_x \oplus \mathfrak{q}$ with $\mathfrak{k}_x \subset \mathfrak{k}$ and $\mathfrak{q} \subset \mathfrak{p}$. We shall prove that $v \in S(\mathfrak{q})$ if and only if $\lambda_x(v) = 0$, where $S(\mathfrak{q})$ denotes the sphere of $\mathfrak{q}$. Let first $v \in S(\mathfrak{q})$. By Remark 3 the function:

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \quad t \mapsto \Psi(x, \exp(tv)),$$

is linear. Hence $\lambda_x(v) = \Lambda(x, v) \geq 0$. The same holds for $\lambda_x(-v) = \Lambda(x, -v) \geq 0$. Therefore $\lim_{t\to+\infty} f'(t) = a \geq 0$ and $\lim_{t\to+\infty} f'(-t) = -a \geq 0$. Thus, $f(t) = \Psi(x, \exp(tv)) = 0$ and condition $(P3)$ implies $\lambda_x(v) = 0$.

Vice-versa, assume $\lambda_x(v) = 0$. by Lemma 15 $\exp(\mathbb{R}v) \in G_x$ or $\Lambda(v, \xi) < 0$. Since

$$\Lambda(x, v) \geq \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} \psi_x(\exp(-tv)) = \mathfrak{F}_{\mathfrak{p}}^{v}(x) = 0,$$

it follows $\exp(\mathbb{R}v) \in G_x$. Summing up, we have proved $\lambda_x(v) \geq 0$ and $\lambda_x(v) = 0$ if and only if $v \in \mathfrak{q}$. Write $\mathfrak{p} = \mathfrak{q} \oplus \mathfrak{q}^{\perp}$. By a Mostow decomposition, see [19, Th. 9.3, p. 211], any $g \in G$ can be written as $g = k\exp(\theta)h$, where $k \in K$, $h \in G_x$ and $\theta \in \mathfrak{q}^{\perp}$.

Let g_n be a sequence in G such that $g_n x \mapsto y$. Write $g_n = k_n \exp(\theta_n)h_n$, where $k_n \in K$, $\theta_n \in \mathfrak{q}^{\perp}$ and $h_n \in G_x$. Then $g_n x = k_n \exp(\theta_n)x$. Since $\Psi(x, \exp(\theta)) = \frac{1}{2}\left(\| \exp(\theta)x \|^2 - \| x \|^2\right)$ for any $\theta \in \mathfrak{q}$ and $\lambda_x(v) > 0$ for any $v \in \mathfrak{q} - \{0\}$, applying Lemmata 11 and 15, there exist C_1, C_2 such that

$$\| \theta_n \|^2 \leq C_1 \| \exp(\theta_n)x \|^2 + C_2.$$

Since $\| g_n x \|^2 = \| \exp(\theta_n)x \|^2$, it follows $\| \exp(\theta_n) \|^2$ is bounded and so $\| \theta_n \|^2$ is. Therefore, up to subsequence, $\theta_n \mapsto \theta$, $k_n \mapsto k$ and so $y = k\exp(\theta)x \in G \cdot x$.

Viceversa, assume that $G \cdot x$ is closed. By Theorem 19, the vector x is semistable. Hence $G \cdot x \cap \mathfrak{F}_{\mathfrak{p}}^{-1}(0) \neq \emptyset$ and so x is polystable. □

Remark 21 If $G \cdot x$ is closed orbit then x is polystable and so, by Proposition 10 the isotropy is reductive. The vice-versa does not hold. Indeed, let $\mathbb{R}$ acting on $\mathbb{R}^2$ as follows

$$\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \quad (t, (x, y)) \mapsto (e^t x, e^{-t} y).$$

Then the orbit throughout $(1, 0)$ is $\{(x, 0) : x > 0\}$ and so it is not closed but $\mathbb{R}_{(1,0)} = \{e\}$.

Applying Theorem 19 and the Slice Theorem [20], we are able to prove that $\overline{G \cdot x}$ contains exactly one closed orbit.

Theorem 22 *For any $x \in V$, $\overline{G \cdot x} \cap \mathfrak{F}_{\mathfrak{p}}^{-1}(0) = K \cdot v$ and $G \cdot v$ is the unique closed orbit contained in $\overline{G \cdot x}$.*

Proof Let $x \in V$. By Theorem 19 it follows $\overline{G \cdot x} \cap \mathfrak{F}_{\mathfrak{p}}^{-1}(0) \neq \emptyset$ and so, by Theorem 20, $\overline{G \cdot x}$ contains a closed orbit. Assume by contradiction $\overline{G \cdot x}$ contains at least two closed orbits that we denote by $G \cdot v_1$ and $G \cdot v_2$ respectively. We may assume that $v_1, v_2 \in \mathfrak{F}_{\mathfrak{p}}^{-1}(0)$. Since $v_1 \in \mathfrak{F}_{\mathfrak{p}}^{-1}(0)$, respectively $v_2 \in \mathfrak{F}_{\mathfrak{p}}^{-1}(0)$, the Slice Theorem holds ([19,20]). Hence there exists a G_{v_1}-invariant decomposition $V = \mathfrak{g} \cdot v_1 \oplus W_1$ where $\mathfrak{g} \cdot v_1 = \{\xi v_1 : \xi \in \mathfrak{g}\}$, respectively a G_{v_2}-invariant decomposition $V = \mathfrak{g} \cdot v_2 \oplus W_2$ where $\mathfrak{g} \cdot v_2 = \{\xi v_2 : \xi \in \mathfrak{g}\}$, an open G_{v_1}-invariant subsets $S_1 \subset W_1$, respectively an open G_{v_2}-invariant subset $S_2 \subset W_2$, a G-invariant open subset Ω_1 containing v_1 and a G-equivariant diffeomorphism $F_1 : G \times_{G_{v_1}} S_1 \longrightarrow \Omega_1$ such that $0 \in S_1$, $v_1 \in \Omega_1$ and $F_1([e, 0]) = v_1$, respectively a G-invariant open subset Ω_2 containing v_2 and a G-equivariant diffeomorphism $F_2 : G \times_{G_{v_2}} S_2 \longrightarrow \Omega_2$ such that $0 \in S_2$, $v_2 \in \Omega_2$ and $F_2([e, 0]) = v_2$. Since both $G \cdot z_1$ and $G \cdot z_2$ are closed, we may choose S_1 and S_2 such that $\Omega_1 \cap \Omega_2 = \emptyset$. Otherwise there exist $x_n \in S_1$, $y_n \in S_2$ and $g_n \in G$ such that $x_n \mapsto 0$, $y_n \mapsto 0$ and $F_1([g_n, x_n]) = [e, y_n] \mapsto v_2$. Keeping in mind that $G \cdot v_1$ is closed, it follows $v_2 \in G \cdot v_1$. A contradiction. On the other hand $G \cdot x \subset \Omega_1 \cap \Omega_2$ providing a contradiction on the existence of two closed orbits. Therefore $\overline{G \cdot x}$ contains exactly one closed orbit. $\square$

Corollary 23 *Let $x \in V$ and let $v \in V$ such that $\overline{G \cdot x} \cap \mathfrak{F}_{\mathfrak{p}}^{-1}(0) = K \cdot v$. Then*

$$\parallel v \parallel^2 = \inf_{z \in \overline{G \cdot x}} \parallel z \parallel^2 .$$

Proof Since $\overline{G \cdot x}$ is closed there exists $z \in \overline{G \cdot x}$ such that

$$\parallel z \parallel^2 = \inf_{z \in \overline{G \cdot x}} \parallel z \parallel^2 .$$

This implies that the restricted Kempf–Ness function $\Psi(z, \cdot) : G \longrightarrow \mathbb{R}$ has a critical point at e. By Proposition 9 $z \in \mathfrak{F}_{\mathfrak{p}}^{-1}(0)$. By Theorems 20 and 22 the orbit $G \cdot z$ is closed and $z \in K \cdot v$ concluding the proof. $\square$

Example 24 Let $G = \mathrm{SL}(n, \mathbb{R}) = \mathrm{SO}(n, \mathbb{R}) \exp(\mathrm{Sym}_o(n))$ where $\mathrm{Sym}_o(n)$ is the set of symmetric matrices of trace 0. $\mathrm{SL}(n, \mathbb{R})$ acts on its lie algebra $\mathfrak{sl}(n\, R)$ by means of the adjoint action:

$$\mathrm{SL}(n, \mathbb{R}) \times \mathfrak{sl}(n\, R) \longrightarrow \mathfrak{sl}(n\, R), \quad (A, X) \mapsto AXA^{-1}.$$

An $\mathrm{SO}(n, \mathbb{R})$ invariant scalar product on $\mathfrak{sl}(n, \mathbb{R})$ is given by $\langle X, Y \rangle = \mathrm{Tr}(XY^T)$.

Let $X = A + B \in \mathfrak{sl}(n, \mathbb{R})$, where $A \in \mathfrak{so}(n)$ and $B \in \mathrm{Sym}_o(n)$. It is easy to check that

$$\mathfrak{F}(A + B)(\xi) = \langle [\xi, A] + [\xi, B], A + B \rangle.$$

Hence, Keeping in mind that $\langle \mathfrak{so}(n), \mathrm{Sym}_o(n) \rangle = 0$, we get

$$\mathfrak{F}(A + B)(\xi) = \langle [\xi, A], B \rangle + \langle [\xi, B], A \rangle = 2\langle [A, B], \xi \rangle.$$

If $X = A + B \in \mathfrak{F}^{-1}(0)$ then $[A, B] = 0$ and so $X = A + B$ is diagonalizable over $\mathbb{C}$. By Theorem 20 it follows that $\mathrm{SL}(n, \mathbb{R}) \cdot X$ is closed. Vice-versa, assume that $X \in \mathfrak{sl}(n, \mathbb{R})$ is diagonalize over $\mathbb{C}$. By Theorem 22 it follows $\mathfrak{F}^{-1}(0) \cap \overline{\mathrm{SL}(n, \mathbb{R}) \cdot X} = \mathrm{SO}(n) \cdot A$. This implies that A and X have the same characteristic polynomial and they are both diagonalizable over $\mathbb{C}$. Hence, there exists $H \in \mathrm{SL}(n, \mathbb{C})$ such that $HXH^{-1} = A$. Write $H = C + iD$. Therefore $(C + iD)X = A(C + iD)$, and so $CX = AC$ and $DX = AD$. This implies

$$(C + tD)X = A(C + tD).$$

for every $t \in \mathbb{R}$. Since $\det(C + tD)$ is a polynomial of degree n, it follows $A \in \mathrm{GL}(n, \mathbb{R}) \cdot X$ and so $A \in \mathrm{SL}(n, \mathbb{R}) \cdot X$. Summing up we have proved that $\mathrm{SL}(n, \mathbb{R}) \cdot X$ is closed if and only if X is diagonalizable over $\mathbb{C}$.

Now we shall prove the Hilbert–Mumford criterion for real reductive Lie groups. Given $\beta \in \mathfrak{p}$, we define the following subgroups:

$$\begin{aligned} G^{\beta} &= \{g \in G : \mathrm{Ad}(g)(\beta) = \beta\} \\ G^{\beta-} &:= \left\{g \in G : \lim_{t \mapsto +\infty} \exp(t\beta) g \exp(-t\beta) \text{exists}\right\} \\ R^{\beta-} &:= \left\{g \in G : \lim_{t \mapsto +\infty} \exp(t\beta) g \exp(-t\beta) = e\right\} \end{aligned}$$

The next lemma is well-known. A proof is given in [1] (see also [2,21]).

Lemma 25 *If $g \in G^{\beta-}$ then $\lim_{t \mapsto +\infty} \exp(t\beta) g \exp(-t\beta) \in G^{\beta}$. Moreover, $G^{\beta-}$ is a parabolic subgroup of G with Lie algebra $\mathfrak{g}^{\beta-} = \mathfrak{g}^{\beta} \oplus \mathfrak{r}^{\beta-}$ and $G = G^{\beta-}K$. Every parabolic subgroup of G equals $G^{\beta-}$ for some $\beta \in \mathfrak{p}$. $R^{\beta-}$ is the unipotent radical of $G^{\beta-}$ and G^{β} is a Levi factor. Finally, $G = KG^{\beta-}$.*

In a recent paper [7], the author gives a proof of the following result avoiding any algebraic result.

Theorem 26 *Let $x \in V$. If $0 \in \overline{G \cdot x}$, then there exists $\xi \in \mathfrak{p}$ such that $\lim_{t \mapsto +\infty} \exp(t\xi)x = 0$.*

Applying Theorem 26 and the Slice Theorem given in [21], see also [30,41], we are able to proof the Hilbert–Mumford criterion for reductive groups.

Theorem 27 (Hilbert–Mumford for reductive groups) *Let $x \in V$ and let $u \in \overline{G \cdot x}$ be such that $G \cdot u$ is closed. Then there exists $\xi \in \mathfrak{p}$ such that $\lim_{t \mapsto +\infty} \exp(t\xi)x$ lies in $G \cdot u$.*

Proof Let $u \in \overline{G \cdot x}$ be such that $G \cdot u$ is closed. We may assume $u \in \mathfrak{F}_{\mathfrak{p}}^{-1}(0)$. By Proposition 10 $G_u = K_u \exp(\mathfrak{p}_u)$ is compatible and reductive and so the G_u action V is completely reducible [19, 14.9]. Therefore, there exists G_u stable decomposition $V = \mathfrak{g} \cdot u \oplus W$. By the Slice Theorem [20, Theorem 3.1], there exists a G-stable neighborhood Ω of u, a G_u-invariant open neighborhood S of $0 \in W$ and G-equivariant diffeomorphism $F_u : G \times_{G_u} S \longrightarrow \Omega$, where $F_u([e,0]) = u$. In particular $G \cdot x \subset \Omega$ and $G \cdot x \cap S = G_u \cdot s$ for some $s \in G \cdot x \cap S$. This means that the condition $u \in \overline{G \cdot x}$ is equivalent to $0 \in \overline{G_u \cdot s}$. Moreover, since $K_u \subset \mathrm{O}(V)$, it follows that $0 \in \overline{G_u^o \cdot s}$. By Theorem 15 there exists $\nu \in \mathfrak{p}_u$ such that

$$\lim_{t \mapsto +\infty} \exp(t\nu)s = u.$$

Write, $s = gkx$, where $g \in G^{\nu-}$ and $k \in K$. Since

$$\exp(t\nu)gkx = (\exp(t\nu)g\exp(-t\nu))\, k^{-1}\exp(t\mathrm{Ad}(k^{-1})(\xi))x,$$

keeping in mind that $\lim_{t \mapsto +\infty} \exp(t\nu)g\exp(-t\nu)$ exists, it follows

$$\lim_{t \mapsto +\infty} \exp(t\mathrm{Ad}(k^{-1})(\xi))x \in G \cdot u.$$

□

Now assume that $G = \exp(\mathfrak{p})$ where $\mathfrak{p}$ is an Abelian subalgebra.

Theorem 28 (Hilbert–Munford for Abelian groups) *Let $x \in V$ and let $u \in \overline{G \cdot x}$. Then there exists $\xi \in \mathfrak{p}$ and $a \in G$ such that* $\lim_{t \mapsto +\infty} \exp(t\xi)ax = u$.

Proof Let $u \in \overline{G \cdot x}$. Since G is Abelian the Slice Theorem works for every G orbit [21]. Hence there exists a G-stable neighborhood Ω of u, a G_u-invariant open neighborhood S of $0 \in W$ and G-equivariant diffeomorphism $F_u : G \times_{G_u} S \longrightarrow \Omega$, where $F_u([e,0]) = u$. In particular $G \cdot x \subset \Omega$ and $G \cdot x \cap S = G_u \cdot s$, where $s \in G \cdot x \cap S$. Therefore $u \in \overline{G \cdot x}$ is equivalent to $0 \in \overline{G_u \cdot s}$. By Theorem 15 there exists $\nu \in \mathfrak{p}_u$ such that

$$\lim_{t \mapsto +\infty} \exp(t\nu)s = u.$$

Now, $s = ax$ and so

$$\lim_{t \mapsto +\infty} \exp(t\nu)ax = u$$

concluding the proof. □

Let $A = \exp(\mathfrak{a})$, where $\mathfrak{a} \subset \mathfrak{p}$ is Abelian. In [7] the author proves that

$$\mathcal{N}_A = \{x \in V : 0 \in \overline{A \cdot x}\}$$

is real algebraic, and so it is closed, avoiding any algebraic result. As a consequence, we proof that the null cone with respect to G is closed.

Theorem 29 *The set $\mathcal{N} = \{x \in V : 0 \in \overline{G \cdot x}\}$ is closed.*

Proof Let $x_n \in \mathcal{N}$ such that $x_n \mapsto x$. By Theorem 26 there exists $\xi_n \in \mathfrak{p}$ such that

$$\lim_{t \mapsto +\infty} \exp(t\xi_n)x_n = 0.$$

Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal Abelian subalgebra. Since $K\mathfrak{a} = \mathfrak{p}$, i.e., the $\mathrm{Ad}(K)$ action on $\mathfrak{p}$ is polar and $\mathfrak{a}$ is a section [27], there exists $k_n \in K$ such that $\mathrm{Ad}(k_n)(\xi_n) \in \mathfrak{a}$. Therefore

$$0 \in \overline{A \cdot k_n x_n},$$

for every $n \in \mathbb{N}$. Up to subsequence, we may assume $k_n \mapsto k$. Since $\mathcal{N}_A$ is closed, it follows $k_n x_n \mapsto kx \in \mathcal{N}_A$ and so $x \in \mathcal{N}$ concluding the proof. □

Corollary 30 *In the above notation we have*

$$\mathcal{N} = K\mathcal{N}_A.$$

Proof Let $x \in \mathcal{N}$. By Theorem 26, there exists $\xi \in \mathfrak{p}$ such that $\lim_{t \mapsto +\infty} \exp(t\xi)x = 0$. Let $k \in K$ such that $\mathrm{Ad}(k)(\xi) \in \mathfrak{a}$. Then $0 \in \overline{A \cdot kx}$ and so $x \in K\mathcal{N}_A$. Vice-versa, let $x \in \mathcal{N}_A$ and let $k \in K$. Then there exists $\xi \in \mathfrak{a}$ such that $\exp(t\xi)x = 0$. Therefore

$$\lim_{t \mapsto +\infty} \exp(t\mathrm{Ad}(k^{-1})(\xi))kx = 0,$$

and so $kx \in \mathcal{N}$. □

Example 31 Let $\mathrm{SL}(n, \mathbb{R}) = \mathrm{SO}(n, \mathbb{R}) \exp(\mathrm{Sym}_o(n))$ where $\mathrm{Sym}_o(n)$ is the set of symmetric matrices of trace 0. $\mathrm{SL}(n, \mathbb{R})$ acts on its lie algebra $\mathfrak{sl}(n\ R)$ by means of the adjoint action:

$$\mathrm{SL}(n, \mathbb{R}) \times \mathfrak{sl}(n\ R) \longrightarrow \mathfrak{sl}(n\ R), \quad (A, X) \mapsto AXA^{-1}.$$

An $\mathrm{SO}(n, \mathbb{R})$ invariant scalar product on $\mathfrak{sl}(n, \mathbb{R})$ is given by $\langle X, Y \rangle = \mathrm{Tr}(XY^T)$. Let $X \in \mathfrak{sl}(n, \mathbb{R})$ be a nilpotent matrix. By Theorem 19 and Example 19 we get that $\overline{G \cdot X}$ contains a matrices A which is diagonalizable over $\mathbb{C}$. Since the characteristic polynomial of X coincides with the polynomial characteristic of A, it follows that $A = 0$ and so $X \in \mathcal{N}$, i.e., X belongs to the Null cone. Vice-versa, $X \in \mathcal{N}$ if and only if there exists $\xi \in \mathrm{Sym}_o(n)$ such that $\exp(t\xi)X\exp(-t\xi) \mapsto 0$. Then the characteristic polynomial of X is given by $P_X(t) = t^n$ and so X in Nilpotent. Summing up we have proved that $\mathcal{N} = \{X \in \mathfrak{sl}(n, \mathbb{R}) : X \text{ is nilpotent}\}$.

6 Real reductive representations on projective spaces

The G action on V induces, in a natural way, an action on the projective space $\mathbb{P}(V)$ as follows

$$G \times \mathbb{P}(V) \longrightarrow \mathbb{P}(V) \quad (g, [v]) \mapsto [gv].$$

Lemma 32 *The function*

$$\tilde{\Psi} : G \times \mathbb{P}(V) \longrightarrow \mathbb{R} \quad (g, [x]) \mapsto \log\left(\frac{\parallel gx \parallel}{\parallel x \parallel}\right),$$

is a Kempf–Ness function and the corresponding gradient map $\tilde{\mathfrak{F}}_{\mathfrak{p}} : \mathbb{P}(V) \longrightarrow \mathfrak{p}^*$ *is given by*

$$\tilde{\mathfrak{F}}_{\mathfrak{p}}([x])(\xi) = \frac{\langle \xi x, x\rangle}{\parallel x \parallel^2}.$$

Proof $P(1)$ and $P(2)$ are easy to check. Let $\xi \in \mathfrak{p}$ and let $f(t) = \tilde{\Psi}(x, \exp(t\xi))$. Then

$$f'(t) = \frac{\langle \exp(t\xi)\xi x, \exp(t\xi)x\rangle}{\langle \exp(t\xi)x, \exp(t\xi)x\rangle}$$
$$f''(t) = 2\frac{\langle \exp(t\xi)\xi x, \exp(t\xi)\xi x\rangle^2 \langle \exp(t\xi)x, \exp(t\xi)x\rangle^2 - \langle \exp(t\xi)\xi x, \exp(t\xi)x\rangle^2}{\langle \exp(t\xi)x, \exp(t\xi)x\rangle^2}.$$

Hence, by the Cauchy-Schwartz's inequality, $f''(t) \geq 0$ and $f''(0) = 0$ if and only if ξx and x are linearly dependent vector and so if and only if $\exp(\mathbb{R}\xi) \subset G_{[x]}$. Finally

$$\begin{aligned}\Psi(x, hg) &= \frac{1}{2}\log\left(\frac{\parallel hgx \parallel}{\parallel x \parallel}\right)\\ &= \frac{1}{2}\log\left(\frac{\parallel hgx \parallel}{\parallel gx \parallel}\right) + \frac{1}{2}\log\left(\frac{\parallel gx \parallel}{\parallel x \parallel}\right)\\ &= \tilde{\Psi}(gx, h) + \tilde{\Psi}(x, g).\end{aligned}$$

□

In the sequel we denote by

$$\pi : V - \{0\} \longrightarrow \mathbb{P}(V),$$

the projection which is an open map. Now we recall the following results. A proof can be found [4]

Theorem 33 *Let* $x \in \mathbb{P}(V)$. *Then*

(a) x *is semistable if and only if for any* $\xi \in \mathfrak{p}$, $\lambda_x(\xi) \geq 0$;
(b) x *is stable if and only if for any* $\xi \in \mathfrak{p} - \{0\}$, $\lambda_x(\xi) > 0$ *for a*

Since $\mathbb{P}(V)$ is compact, by (9) we have $\lambda_x(\xi) = \Lambda(x, \xi)$, for any $x \in \mathbb{P}(V)$ and any $\xi \in \mathfrak{p}$. Applying the above theorem we get the following well-known result.

Proposition 34 *Let* $x \in \mathbb{P}(V)$ *and let* $\tilde{x} \in V - \{0\}$ *such that* $\pi(\tilde{x}) = x$. *Then*

(a) x *is semistable if and only if* $0 \notin \overline{G \cdot \tilde{x}}$;

(b) *x is stable if and only if $\tilde{x}$ is stable.*
(c) *x is polystable if and only if $\tilde{x}$ is polystable;*

In particular the set of the semistable points is empty or open and dense.

Proof By Theorem 33, x is semistable if and only if $\Lambda(x,\xi) \geq 0$ for any $\xi \in \mathfrak{p}$. Let $e_1,\dots,e_n$ be an orthonormal basis of eigenvectors of ξ and let $\alpha_1,\dots,\alpha_n$ being the corresponding eigenvalues. We may assume $\alpha_1 \geq \cdots \geq \alpha_n$ and there exists $s \in \{1,\dots n\}$ such that either $\alpha_s \geq 0$ and $s=n$ or $\alpha_{s+1}<0$. It is easy to check that

$$\begin{aligned}\Lambda(x,\xi) &= \lim_{t\mapsto+\infty} \frac{\mathrm{d}}{\mathrm{dt}}\tilde{\Psi}(x,\exp(t\xi)) \\ &= \lim_{t\mapsto+\infty} \frac{\alpha_1 e^{2\alpha_1 t} \parallel x_1 \parallel^2 + \cdots + \alpha_n e^{2\alpha_n t} \parallel x_n \parallel^2}{e^{2\alpha_1 t} \parallel x_1 \parallel^2 + \cdots + e^{2\alpha_n t} \parallel x_n \parallel^2}.\end{aligned}$$

Therefore, $\Lambda(x,\xi) \geq 0$ if and only if $\exists i \in \{1,\dots,s\}$ such that $x_i \neq 0$. It is not difficult to proof that x is semistable if and only if $\lim_{t\mapsto+\infty}\exp(t\xi)\tilde{x}$ does not converge to 0. Since it holds for any ξ, by the Hilbert–Mumford criterion for reductive Lie groups, it follows that $x \in \mathbb{P}(V)$ is semistable if and only if $0 \notin \overline{G\cdot\tilde{x}}$. Denoting by $\mathbb{P}(V)^{ss}$ the set of semistable points we have proved a well-known fact that $\mathbb{P}(V)^{ss} = \pi^{-1}(V-\mathcal{N})$, Since $\mathcal{N}$ is closed and real algebraic, see [19] for a proof, it follows, keeping in mind that π is open, $\mathbb{P}(V)^{ss}$ is empty or open and dense.

The same argument proves $\Lambda(x,\xi) > 0$ if and only if there exists $i \in \{1,\dots,n\}$ such that $\alpha_i > 0$ and so, by Lemma 15, if and only if $\lambda_{\tilde{x}}(\xi) > 0$. By Theorem 16 and 33, we get x is stable if and only if $\tilde{x}$ is stable. Since $x \in \tilde{\mathfrak{F}}_{\mathfrak{p}}^{-1}(0)$ if and only if $\tilde{x} \in \mathfrak{F}_{\mathfrak{p}}^{-1}(0)$, the last item follows easily. □

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