Proceedings of the 8th GACM Colloquium on Computational Mechanics
for Young Scientists from Academia and Industry
August 28–30, 2019 in Kassel, Germany

A Brief Survey on Non-standard Constraints: Simulation and Optimal Control

Dominik Kern * [1] and Ulrich J. Römer[2]

[1]TU Chemnitz, Germany
[2]Karlsruhe Institute of Technology (KIT), Germany

Abstract. In terms of simulation and control holonomic constraints are well documented and thus termed standard. As non-standard constraints, we understand non-holonomic and unilateral constraints. We limit this survey to mechanical systems with a finite number of degrees of freedom. The long-term behavior of non-holonomic integrators as compared to structure-preserving integrators for holonomically constrained systems is briefly discussed. Some recent research regarding the treatment of unilaterally constrained systems by event-driven or time-stepping schemes for time integration and in the context of optimal control problems is outlined.

1 Introduction

Many technical systems are modeled as multibody systems which are comprised of rigid or elastic bodies, joints, and force elements or forcing. We restrict the discussion to discrete systems, meaning elastic bodies are presumed to be discretized by shape functions or finite elements and the system is represented by a finite number of degrees of freedom. Joints are the typical pictorialization of constraints; for example they constrain a body to move only along or around a certain axis. Generally, constraints may either account for kinematic restrictions of a body, such as with redundant coordinates, or for interactions between bodies and the environment. Such interactions may be described by an equation, such as a pivot, or by an inequation, such as a non-penetration condition.

Without any claim for completeness, we report about some active research related to simulation and control. We adopt the common classification of constraints into: holonomic or non-holonomic (e. g. massless rigid connectors or rolling without sliding), scleronomic or rheonomic (e. g. static or moving environment), and unilateral or bilateral (e. g. limiters or bearings). Bilateral holonomic, particularly holonomic-scleronomic, constraints are the simplest type of constraints and their treatment has been well studied and is covered in many textbooks [12, 32]. Therefore, we refer to them as *standard*. There are some ongoing discussions regarding the remaining constraint types, non-holonomic and unilateral, which is why we refer to them as *non-standard*.

2 Non-holonomic constraints

We need to recall the definition of a holonomic constraint to give meaning to the term *non-holonomic*. Having a multibody system with generalized coordinates $\mathbf{q}$ in mind, a holonomic-scleronomic constraint is given by an algebraic equation $\Psi(\mathbf{q}) = 0$ on position level, whereas a holonomic-rheonomic constraint would explicitly depend on time t. For further details on non-holonomic systems we refer to the classical textbook of Neimark & Fufaev [25] and more recently Rabier & Rheinboldt and Bloch [5, 26]. A popular formulation for the numerical solution of holonomically constrained systems was proposed by Gear, Gupta and Leimkuhler [11]

***Corresponding author:** dominik.kern@mb.tu-chemnitz.de **ORCID iD:** 0000-0002-1958-2982

$$\dot{\mathbf{q}} = \mathbf{v} - \mathbf{G}^{\mathsf{T}}(\mathbf{q})\boldsymbol{\eta} \ , \tag{1a}$$

$$\mathbf{M}(\mathbf{q})\dot{\mathbf{v}} = \mathbf{f}(\mathbf{q}, \mathbf{v}) - \mathbf{G}^{\mathsf{T}}(\mathbf{q})\boldsymbol{\lambda} \ , \tag{1b}$$

$$\boldsymbol{\Psi}(\mathbf{q}) = \mathbf{0} \ , \tag{1c}$$

$$\mathbf{G}(\mathbf{q})\mathbf{v} = \mathbf{0} \ , \tag{1d}$$

with mass matrix $\mathbf{M}(\mathbf{q})$, generalized forces $\mathbf{f}$, Lagrange multipliers $\boldsymbol{\lambda}$ and $\boldsymbol{\eta}$, and $\mathbf{G} = \partial\boldsymbol{\Psi}/\partial\mathbf{q}$. It enforces the constraints on both, position and velocity level, and is also referred to as stabilized index-2 formulation. A combination of differential and algebraic equations like in the example given by eq. (1) is called *differential algebraic equation (DAE)* or *descriptor system*.

Non-holonomic constraints are usually understood as equations between differentials of $d\mathbf{q}$ that cannot be integrated to an equation on position level. A general criteria for non-integrability gives the Frobenius-Theorem [14]. In mechanical systems, where most constraints appear as linear forms $\mathbf{A}(\mathbf{q})\dot{\mathbf{q}} = \mathbf{0}$, there is a simple necessary condition. For a non-holonomic constraint the derivative of the corresponding row in $\mathbf{A}$ must be a non-symmetric matrix.

From the point of view of numerics, the inclusion of non-holonomic constraints into eq. (1) is straightforward: they are added to the holonomic constraints on velocity level. This formulation works well for moderate time spans. The robustness and the excellent long-time behavior of structure-preserving integrators for unconstrained and holonomically constrained systems [13] suggest taking advantage of the geometrical properties of non-holonomic systems. However, non-holonomic systems do not, in general, preserve a symplectic structure. There is the open question whether energy conservation is the only geometric property of non-holonomic systems. Consequently, an energy-consistent scheme has already been proposed [4] to include both, holonomic and non-holonomic constraints. Recent research indicates that the good long-term behavior of current non-holonomic integrators is due to the special properties of popular benchmark systems and does not hold in general [22].

Lie-group methods [8, 24] and their successful application for the description of large rotations are not considered further, since there is no hope to identify a Lie-group for any possible constraint.

The consideration of non-holonomic constraints in optimal control problems is also more challenging than for holonomic ones. We restrict the discussion to feed-forward control, meaning we calculate a nominal trajectory that is optimal in the sense of a given cost functional. These are constrained optimizations, because the equations of motion, or more generally the system dynamics, are imposed as constraints on the optimal control problem. For a general discussion, we refer to the comprehensive textbook [12] and take a glance at concepts motivated by structure preservation. Both, the discretization-based approach *Discrete Mechanics and Optimal Control of Constrained systems (DMOCC)* [19] and the functional-based approach [15], are designed for holonomic constraints. Nevertheless, both allow different time discretizations than variational integrators. Consequently, it will be interesting to find out how an energy-consistent time discretization like the one proposed by Betsch [4] performs in this context.

3 Unilateral constraints

While bilateral constraints describe relationships between some or all state variables via equations, *unilateral constraints* introduce one-sided limits on the admissible domain which result in inequations $\phi(\mathbf{q}) \geq \mathbf{0}$ or inclusions. Consequently, solutions may be non-smooth or even discontinuous. In mechanics, typical origins of unilateral constraints are collisions and friction. Several approaches exist to describe such systems, we refer to [6, 17] for further details. Numerical methods for approximate solutions have to deal with the (piecewise) non-smooth dynamics and follow one of two basic philosophies: event-driven or time-stepping, cf. figure 1. *Event-driven schemes* rely on algorithms for ODEs or DAEs to solve the smooth

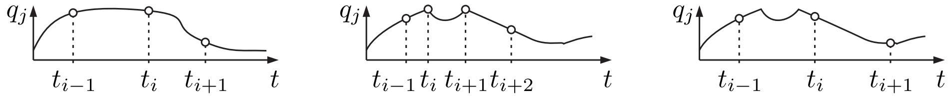

Figure 1. Numerical approximation of time behavior of state q_j: **left** smooth solution (variable time step); **middle** non-smooth solution with event-driven approach (variable time step); **right** non-smooth solution with time-stepping scheme (fixed time step).

parts of the solution trajectory [3], discrete events where inequalities become active are determined iteratively [29], a discrete state update is evaluated and the smooth solver is restarted. *Time-stepping schemes* approximate the solution trajectory at fixed time steps where constraints are satisfied exactly (subject only to numerical precision), but might be (slightly) violated in between [1, 6]. Event-driven schemes are standard in most commercial software tools, since high-order methods can be used for the smooth parts of the solution and good performance is achieved if there is a moderate number of events. They perform badly if there is a high number of events or even accumulation points (Zeno behavior), since this leads to very small time steps and the algorithm gets stuck. Furthermore, existence and uniqueness of a solution of the ODE or DAE after an event is not guaranteed (cf. the Painlevé paradox) [31]. Time-stepping schemes overcome these problems by "averaging" over the influence of all events during fixed time steps [23]. Velocity jumps due to collisions are treated by a coefficient of restitution which acts as though all collisions within a time step occur simultaneously [30]. They can be formulated as implicit time integration methods which require the solution of a linear or nonlinear complementary problem at every time step [2].

In terms of structure-preservation, event-driven collision integrators follow naturally from Hamilton's principle, when the collision time is an additional variable [9]. Note, that this use of Hamilton's principle in equality form [9] does not allow trajectories to approach or to exit motions along the admissible set boundary, i.e. contacts being opened or closed. In addition, the adjusted time steps may slightly violate the conservation properties of variational integrators and thus accumulate errors. The extension of classical analytical mechanics to perfect unilateral constraints, with possible persistent contact, requires a reformulation of the variational principles of mechanics in terms of variational inequalities [18]. It is thus natural to consider a nonsmooth, discrete Variational Principle for inequality constrained systems [16]. Since variational integrators are formulated in terms of momenta and avoid direct computation of forces they are a convenient choice for time-stepping schemes. There is ongoing research in the development of higher order methods to increase time step size and performance [28]. In the context of transient simulations, the competition between event-driven and time-stepping schemes is not decided yet and maybe never will.

For the optimal control problem (feed-forward), we distinguish two cases: firstly one only requires compliance with the unilateral constraints, secondly there are collisions with the admissible boundary set. The first case, pure compliance with unilateral constraints such as actuator saturation, is comparatively simple and commonly accounted for by the Karush-Kuhn-Tucker conditions [7]. The second case, when the events of interaction with the boundary of the admissible domain become subject of the optimization, requires more sophistication since then the related dynamics are non-smooth and the events, their number and their states, are additional unknown variables. As an example, consider a pendulum-on-cart-system where the cart moves between two limiters and the collisions with these boundaries are to be utilized for a swing-up of the pendulum. Since classical control theory presumes smooth dynamics, the direct approach, not to say brute-force, splits this problem into two layers. The inner layer finds the optimal solution between two collisions. The earlier collision is given by a fixed post-collision state and the later one by a fixed pre-collision state, defining already the subsequent post-collision state. The outer layer is used to find the optimal collision times and states [10]. A further outer layer may optimize the contact mode sequence, which is an integer programming problem growing exponentially with the number of contact constraints. Alternatively, time transformations [27] or contact-implicit trajectory optimization [21] can be used to avoid this mixed integer optimal control problem. We further mention that there is a close relation to the theory of switched systems [20].

References

[1] V. Acary and B. Brogliato. *Numerical methods for nonsmooth dynamical systems: applications in mechanics and electronics*. Berlin: Springer, 2008.

[2] M. Anitescu. Optimization-based simulation of nonsmooth rigid multibody dynamics. *Math Program* 105.1 (2006), pp. 113–143.

[3] M. Arnold and K. Frischmuth. Solving problems with unilateral constraints by DAE methods. *Math Comput Simulat* 47.1 (1998), pp. 47–67.

[4] P. Betsch. A Unified Approach to the Energy-Consistent Numerical Integration of Nonholonomic Mechanical Systems and Flexible Multibody Dynamics. *GAMM-Mitteilungen* 27.1 (2004), pp. 66–87.

[5] A. M. Bloch. Nonholonomic mechanics. In: *Nonholonomic mechanics and control*. Vol. 24. New York: Springer, 2003, pp. 207–276.

[6] B. Brogliato. *Nonsmooth Mechanics*. 2nd ed. London: Springer, 1999.

[7] R. Cottle and C. E. Lemke. *Nonlinear programming*. Vol. 9. SIAM-AMS Proceedings. American Mathematical Soc., 1976.

[8] P. E. Crouch and R. Grossman. Numerical integration of ordinary differential equations on manifolds. *J Nonlinear Sci* 3.1 (1993), pp. 1–33.

[9] R. C. Fetecau, J. E. Marsden, M. Ortiz, and M. West. Nonsmooth Lagrangian mechanics and variational collision integrators. *SIAM J Appl Dyn Syst* 2.3 (2003), pp. 381–416.

[10] K. Flaßkamp and S. Ober-Blöbaum. Variational formulation and optimal control of hybrid Lagrangian systems. In: *P Int Conf HSCC*. ACM. 2011, pp. 241–250.

[11] C. W. Gear, B. Leimkuhler, and G. K. Gupta. Automatic integration of Euler-Lagrange equations with constraints. *J Comput Appl Math* 12 (1985), pp. 77–90.

[12] M. Gerdts. *Optimal control of ODEs and DAEs*. Berlin: De Gruyter, 2011.

[13] E. Hairer, C. Lubich, and G. Wanner. *Geometric numerical integration: structure-preserving algorithms for ordinary differential equations*. Berlin: Springer, 2006.

[14] T. Hawkins. Frobenius, Cartan, and the problem of Pfaff. *Arch Hist Exact Sci* 59.4 (2005), pp. 381–436.

[15] E. R. Johnson and T. D. Murphey. Scalable variational integrators for constrained mechanical systems in generalized coordinates. *IEEE T Robot* 25.6 (2009), pp. 1249–1261.

[16] D. M. Kaufman and D. K. Pai. Geometric numerical integration of inequality constrained, nonsmooth Hamiltonian systems. *SIAM J Sci Comput* 34.5 (2012), A2670–A2703.

[17] J. Kleinert and B. Simeon. Differential-algebraic equations and beyond: from smooth to nonsmooth constrained dynamical systems. *arXiv:1811.07658 (preprint)* (2018).

[18] R. I. Leine, U. Aeberhard, and C. Glocker. Hamilton's principle as variational inequality for mechanical systems with impact. *J Nonlin Sci* 19.6 (2009), p. 633.

[19] S. Leyendecker, S. Ober-Blöbaum, J. E. Marsden, and M. Ortiz. Discrete mechanics and optimal control for constrained systems. *Optim Contr Appl Met* 31.6 (2010), pp. 505–528.

[20] D. Liberzon. *Switching in systems and control*. Boston: Birkhäuser, 2003.

[21] Z. Manchester, N. Doshi, R. J Wood, and S. Kuindersma. Contact-implicit trajectory optimization using variational integrators. *Int J Rob Res* (2019).

[22] K. Modin and O. Verdier. What makes nonholonomic integrators work? *arXiv:1707.04514 (preprint)* (2019).

[23] J. J. Moreau. Unilateral contact and dry friction in finite freedom dynamics. In: *Nonsmooth Mechanics and Applications*. Vienna: Springer, 1988, pp. 1–82.

[24] H. Munthe-Kaas. Runge-Kutta methods on Lie groups. *BIT* 38.1 (1998), pp. 92–111.

[25] J. I. Neimark and N. A. Fufaev. *Dynamics of nonholonomic systems*. American Mathematical Society, 2004.

[26] P. J. Rabier and W. C. Rheinboldt. *Nonholonomic Motion of Rigid Mechanical Systems from a DAE Viewpoint*. Philadelphia: SIAM, 2000.

[27] M. Ringkamp, S. Ober-Blöbaum, and S. Leyendecker. On the time transformation of mixed integer optimal control problems using a consistent fixed integer control function. *Math Program* 161.1-2 (2017), pp. 551–581.

[28] T. Schindler and V. Acary. Timestepping schemes for nonsmooth dynamics based on discontinuous Galerkin methods: definition and outlook. *Math Comput Simulat* 95 (2014), pp. 180–199.

[29] L. F. Shampine and S. Thompson. Event location for ordinary differential equations. *Comput Math Appl* 39.5-6 (2000), pp. 43–54.

[30] B. Smith, D. M. Kaufman, E. Vouga, R. Tamstorf, and E. Grinspun. Reflections on simultaneous impact. *ACM T Graphic* 31.4 (2012), pp. 1–12.

[31] D. E. Stewart. Rigid-body dynamics with friction and impact. *SIAM Rev* 42.1 (2000), pp. 3–39.

[32] G. Wanner and E. Hairer. *Solving ordinary differential equations II*. Berlin: Springer, 1996.