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CRC Preprint 2022/6, January 2022

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# CONSTRUCTION AND ANALYSIS OF AN ADI SPLITTING SCHEME FOR MAXWELL EQUATIONS WITH LOW REGULARITY IN HETEROGENEOUS MEDIA

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**ABSTRACT.** We construct a dimension splitting scheme for the time integration of linear Maxwell equations in a heterogeneous cuboid. The domain contains several homogeneous subcuboids, and serves as a model for a rectangular embedded waveguide. Due to discontinuities of the material parameters and irregular initial data, the solution of the Maxwell system has regularity below  $H^1$ . The splitting scheme is adapted to the arising singularities, and is shown to converge with order one in  $L^2$ . The error result only imposes assumptions on the model parameters and the initial data, but not on the unknown solution. To achieve this result, the regularity of the Maxwell system is analyzed in detail, giving rise to sharp explicit regularity statements. In particular, the regularity parameters are given in explicit terms of the largest jump of the material parameters.

## 1. INTRODUCTION

Maxwell equations belong to the fundamental equations in physics, and are in particular used to describe a large number of phenomena in optics, see [32, 25, 9, 17]. Their solutions are hence of great interest in many applications, like the design of waveguides, see Section 9.3 in [44]. To model waveguides, heterogeneous media are often studied that consist of several homogeneous submedia. This approach leads to material parameters that are discontinuous at the interfaces between different submedia. Maxwell equations with discontinuous material parameters, however, usually have irregular solutions, see [15, 8, 7, 11, 12, 13] for instance. This poses severe difficulties for the analysis of numerical schemes for the considered Maxwell equations.

On domains with tensor-structure, alternating direction implicit (ADI) schemes are very attractive methods for the time integration of linear isotropic Maxwell equations. In the ADI splitting from [56, 42], the Maxwell operator is split according to the spatial dimensions in which derivatives arise. The split system is then integrated in time by means of the Peaceman-Rachford scheme, see [43]. The splitting from [56, 42] can also be integrated in an energy conserving way, see [10]. These schemes are implicit and can be shown to be unconditionally stable, see [56, 42, 10, 29, 31, 38] for instance. Despite being implicit, the mentioned ADI schemes are also computationally cheap. In particular, the implicit steps can be shown to decouple into essentially one-dimensional problems amounting to linear complexity, see [56, 42, 10, 29, 30, 38]. In [46, 47], the Peaceman-Rachford ADI

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2020 *Mathematics Subject Classification.* 35Q61, 47D06, 65M15, 35J05, 65J08.

*Key words and phrases.* Maxwell equations, Heterogeneous media, Splitting method, Error bound, Regularity analysis, Elliptic transmission problem.

Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 258734477 – SFB 1173.

scheme is transformed into an even more efficient formulation, being called fundamental ADI-FDTD scheme. There is also a modified ADI scheme that uniformly preserves the exponential decay behavior of the Maxwell equations with interior damping, see [53].

Despite their practical relevance, it seems to the best of our knowledge that only few rigorous error results are known about ADI schemes. In [29, 20, 21, 19, 18], the material parameters are required to be  $W^{1,\infty}$  respectively  $W^{1,\infty} \cap W^{2,3}$  regular on the entire cuboidal domain. In presence of appropriate initial data, the ADI schemes from [56, 42, 10] are then shown to be of order two in  $H^{-1}$  and  $L^2$ , respectively. While the mentioned error statements focus only on the time discretization, a fully discrete error analysis is performed in [38, 31] for the Peaceman-Rachford ADI scheme in combination with a discontinuous Galerkin discretization in space. [31] moreover provides estimates on time- and space-derivative errors. In [54], the Maxwell equations are considered with positive material parameters being piecewise constant on two adjacent cuboids. Assuming appropriate initial data, time discrete approximations of the Maxwell system provided by the Peaceman-Rachford ADI scheme from [56, 42] are here shown to be of order 3/2 in  $L^2$ . The error analysis of ADI schemes on the heterogeneous medium from the current paper is not covered by the existing literature, to the best of our knowledge. The major reason is that the solution of the considered Maxwell system has lower regularity than required in the above mentioned literature.

We study the time dependent linear isotropic Maxwell equations

$$\begin{aligned} \partial_t \mathbf{E} &= \frac{1}{\varepsilon} \operatorname{curl} \mathbf{H} - \frac{1}{\varepsilon} \mathbf{J}, & \partial_t \mathbf{H} &= -\frac{1}{\mu} \operatorname{curl} \mathbf{E}, \\ \mathbf{E}(0) &= \mathbf{E}_0, & \mathbf{H}(0) &= \mathbf{H}_0, \end{aligned} \tag{1}$$

for  $t \geq 0$  on the cuboid

$$Q = (a_1^-, a_1^+) \times (a_2^-, a_2^+) \times (a_3^-, a_3^+)$$

with the boundary conditions of a perfect conductor

$$\mathbf{E} \times \nu = 0, \quad \mu \mathbf{H} \cdot \nu = 0$$

on the boundary  $\partial Q$ . Conditions on the divergence of  $\mathbf{E}$  and  $\mathbf{H}$  are incorporated in an appropriate state space for (1), see (13) and Remark 1. The vector  $\nu$  denotes the unit exterior normal vector at  $\partial Q$ ,  $\mathbf{E} = \mathbf{E}(x, t) \in \mathbb{R}^3$  stands for the electric field,  $\mathbf{H} = \mathbf{H}(x, t) \in \mathbb{R}^3$  for the magnetic field, and  $\mathbf{J} = \mathbf{J}(x, t) \in \mathbb{R}^3$  is a given external electric current. The functions  $\varepsilon = \varepsilon(x) > 0$  and  $\mu = \mu(x) > 0$  are the electric permittivity and magnetic permeability, respectively, and describe the properties of the material  $Q$  consists of.

The following assumptions on the parameters  $\varepsilon$  and  $\mu$  are essential throughout the paper. The conditions are inspired by a model of a rectangular embedded waveguide, see Section 9.3 in [44] for instance. To formulate the preconditions, we make the following geometric constructions. The cuboid  $Q$  is divided into a chain of smaller cuboids  $\tilde{Q}_1, \dots, \tilde{Q}_L$ , where the interfaces between adjacent cuboids should be parallel to the  $x_2$ - $x_3$ -plane. We collect these interfaces in a set  $\tilde{\mathcal{F}}_{\text{int}}$ . Each cuboid  $\tilde{Q}_i$  further contains smaller subcuboids  $\tilde{Q}_{i,1}, \dots, \tilde{Q}_{i,K}$ , that are separated from each other, and touch the planes  $\{x_3 = a_3^-\}$  and  $\{x_3 = a_3^+\}$ . The smaller subcuboids  $\tilde{Q}_{i,1}, \dots, \tilde{Q}_{i,K}$  are, however, not allowed to touch an interface in  $\tilde{\mathcal{F}}_{\text{int}}$ . The remainder of  $\tilde{Q}_i$  is then denoted by  $\tilde{Q}_{i,0}$ . The resulting partition of  $Q$  corresponds to a specific composition of materials. The subcuboids  $\tilde{Q}_{i,1}, \dots, \tilde{Q}_{i,K}$  play the role of embedded waveguide structures, while  $\tilde{Q}_{i,0}$  serves as the surrounding medium. Note that our analysis can in a straightforward way also be transferred to the case that each cuboid  $\tilde{Q}_{i,j}$ ,  $j \in \{1, \dots, K\}$ , contains further embedded subcuboids that

again touch the planes  $\{x_3 = a_3^-\}$  and  $\{x_3 = a_3^+\}$ , but no other face of  $\tilde{Q}_{i,j}$ . For the sake of a clear presentation, we however omit this extension.

For the material parameters  $\varepsilon$  and  $\mu$ , we throughout impose the assumptions

$$\begin{aligned} \varepsilon|_{\tilde{Q}_{i,j}}, \mu|_{\tilde{Q}_{i,j}} &\in \mathbb{R}_{>0}, \\ \varepsilon|_{\tilde{Q}_{i,0}} &\leq \varepsilon|_{\tilde{Q}_{i,l}} < \frac{1}{1 + 2\sqrt{2} - 2\sqrt{2 + \sqrt{2}}} \varepsilon|_{\tilde{Q}_{i,0}}, \\ \frac{\varepsilon|_{\tilde{Q}_{i,l}}}{\varepsilon|_{\tilde{Q}_{i,0}}} &\neq 1 - \frac{2 \cos(\frac{15}{7}\pi) \sin(\frac{6}{7}\pi)}{\cos(\frac{12}{7}\pi) \sin(\frac{9}{7}\pi)}, \\ \mu|_{\tilde{Q}_{i,0}} &= \mu|_{\tilde{Q}_{i,l}}, \end{aligned} \tag{2}$$

for  $i \in \{1, \dots, L\}$ ,  $j \in \{0, \dots, K\}$ , and  $l \in \{1, \dots, K\}$ . These assumptions mean that each subdomain  $\tilde{Q}_{i,j}$  should consist of a homogeneous medium. Additionally, the relative jumps of the parameter  $\varepsilon$  inside the cuboids  $\tilde{Q}_i$  must not be too large, while  $\mu$  is assumed to be constant in each  $\tilde{Q}_i$ . Note, however, that the difference between the material parameters  $\varepsilon$  and  $\mu$  on  $\tilde{Q}_{i,0}$  and  $\tilde{Q}_{k,0}$  is allowed to be arbitrarily large for distinct  $i, k \in \{1, \dots, L\}$ . The condition in the second line of (2) will ensure that the solutions of the Maxwell system (1) do not become too singular at the interior edges. In fact, the regularity of the solutions can be expressed in terms of the largest relative jump of  $\varepsilon$  in a cuboid  $\tilde{Q}_i$ , see Corollary 2 and Remark 2. The third line of (2) is used to avoid technical difficulties, see the proof of Lemma 3.5.

Due to low regularity in the  $x_1$ - $x_2$ -plane of the solutions of (1), see Remark 2, we use a different directional splitting of the Maxwell operator than the standard one from [56, 42]. (In fact, the solution of (1) is not contained in the domains of the standard splitting operators. Hence the standard Peaceman-Rachford ADI scheme is not applicable to the original solution, see [29].) The idea behind the directional splitting, we consider, is to treat the  $x_3$ -direction independently, and to leave the  $x_1$ - $x_2$ -directions coupled, see Section 6.1. The split system is then integrated in time by means of the Peaceman-Rachford scheme [43], see (101). The resulting scheme is shown to be unconditionally stable, see Lemma 6.3. For the implicit steps in the scheme (101), decoupled two-dimensional elliptic problems have to be solved for the third components of the approximations to the electric and magnetic fields, see Remark 3. All other components of the electromagnetic field approximations are obtained by solving only one-dimensional elliptic problems.

Our main result is given in Theorem 6.4, stating that the directional splitting scheme (101) converges with order one in  $L^2$  to the solution of (1). The error result is rigorous in the sense that we impose assumptions only on the material parameters and the initial data. Furthermore, we can deal with less regular initial data than comparative literature [29, 21, 20, 19]. For these irregular data, we can, however, only show convergence of order one instead of order two. Indeed, the local error can only be expanded to terms of second order in the time step size, since higher order error terms cannot be controlled properly in our regularity setting. We are also going to provide a rigorous convergence result of (expected) order two for scheme (101) in a subsequent work in preparation. There we, however, have to impose stronger assumptions on the initial data.

To establish Theorem 6.4, we study the regularity of (1) in detail. The regularity of the time-harmonic counterpart of (1) on more general heterogeneous polyhedral domains has been analyzed in several papers, see [8, 15, 13, 7, 11, 12] for instance. We provide a regularity analysis here to have sharp regularity statements for our model problem that explicitly link the size of the jumps of the material parameters to the regularity of the problem, see Corollary 2 and Remark 2. Moreover, we obtain that the components of the electric and magnetic field have differing regularity. This

turns out to be crucial for the numerical approximation scheme. The regularity statement and the associated reasoning will additionally be employed in the above mentioned follow-up work to derive higher regularity statements. For the sake of a clear presentation, we hence give a detailed account of the arguments. In particular, we localize at the interior edges in our medium, and study elliptic transmission problems in a neighborhood of these edges, see Section 3 and [14, 15, 12]. To obtain the desired sharp and explicit statement of Corollary 2 and Remark 2, we determine the first nonzero eigenvalue of a one-dimensional transmission problem. This turns out to be quite involved, see Lemma 3.5. The actual regularity and wellposedness statement in Corollary 2 is then deduced by constructing a regular state space  $X_1$  in (13) and by applying semigroup theory on the latter space in Proposition 5.

**Structure of the paper.** In Section 2 we recall useful function spaces, and introduce an analytical framework for the Maxwell system (1). In particular, we construct a space  $X_1$  in (13) that turns out to be a regular state space for (1). In the spirit of [15], we then study the regularity of a transmission problem for the Laplacian in Section 3. Using these findings, the space  $X_1$  is shown to embed into a certain space of piecewise fractional Sobolev regularity, see Section 4. In Section 5 we then prove the wellposedness of (1) in  $X_1$ , and in this way the desired regularity statement. A directional splitting scheme is constructed in Section 6. It is shown to be unconditionally stable, and a rigorous error estimate is established there, see Theorem 6.4.

**Notation.** For convenience, we use a partition of  $Q$  that is different from the above  $\bar{Q} = \bigcup_{i=1}^L \bigcup_{j=0}^K \bar{Q}_{i,j}$ . The new one is subordinate to the above, and obtained by appropriate refinement. In particular, the material parameters  $\varepsilon$  and  $\mu$  are assumed to be constant on each element of the new partition. We arrive at  $N$  smaller open cuboids  $Q_1, \dots, Q_N$  with  $\bar{Q} = \bigcup_{i=1}^N \bar{Q}_i$ . These cuboids should not overlap and again touch both planes  $\{x_3 = a_3^-\}$  and  $\{x_3 = a_3^+\}$ . It is furthermore assumed that if two subcuboids share an interface, that the edges of the corresponding faces then coincide.

We denote the open faces of  $Q$  by

$$\Gamma_j^\pm := \{x \in \partial Q \mid x_j = a_j^\pm, x_l \in (a_l^-, a_l^+) \text{ for } l \neq j\}, \quad \Gamma_j := \Gamma_j^+ \cup \Gamma_j^- \quad (3)$$

for  $j \in \{1, 2, 3\}$ . The set of interfaces of the fine partition  $Q_1, \dots, Q_N$  is called  $\mathcal{F}_{\text{int}}$ , and the set of exterior faces is  $\mathcal{F}_{\text{ext}}$ . We also assign a unit normal vector  $\nu_F \in \mathbb{R}^3$  to every face  $F \in \mathcal{F}_{\text{int}} \cup \mathcal{F}_{\text{ext}}$  in the following way. In case  $F$  is an interface being parallel to the  $x_j$ - $x_3$ -plane, we choose  $\nu_F$  as the canonical unit vector  $e_l$ ,  $l \neq j \in \{1, 2\}$ . Otherwise,  $F$  is an exterior face, and  $\nu_F$  coincides with the outer unit normal vector  $\nu$  of  $\partial Q$ . We also employ a set of effective interfaces  $\mathcal{F}_{\text{int}}^{\text{eff}}$  that contains all physical interfaces. It is defined via

$$\mathcal{F}_{\text{int}}^{\text{eff}} := \{\mathcal{F} \subseteq Q \text{ is a face of } \bar{Q}_{i,j}, i \in \{1, \dots, L\}, j \in \{1, \dots, K\}\} \cup \tilde{\mathcal{F}}_{\text{int}}. \quad (4)$$

Unit normal vectors for interfaces in  $\mathcal{F}_{\text{int}}^{\text{eff}}$  are defined in the same way as for interfaces in  $\mathcal{F}_{\text{int}}$ .

The restriction of a function  $f \in L^2(Q)$  to a subcuboid  $Q_i$  is denoted by  $f^{(i)}$  for  $i \in \{1, \dots, N\}$ . We also need a notation for jumps of functions at interfaces in  $Q$ . To that end, let  $F$  be an interface between two cuboids  $Q_{i1}$  and  $Q_{i2}$  with face vector  $\nu_F$  pointing from  $Q_{i1}$  to  $Q_{i2}$ . Assume additionally that the restrictions  $f^{(i1)}$  and  $f^{(i2)}$  have well defined traces  $\text{tr}_F f^{(i1)}$  and  $\text{tr}_F f^{(i2)}$  at  $F$ . The jump  $[[f]]_F$  of  $f$  at  $F$  is then defined as

$$[[f]]_F := \text{tr}_F f^{(i2)} - \text{tr}_F f^{(i1)}.$$

For a linear operator  $A$  on a normed vector space  $(X, \|\cdot\|)$ , we denote its domain by  $\mathcal{D}(A)$ , and its graph norm by  $\|x\|_{\mathcal{D}(A)}^2 := \|x\|^2 + \|Ax\|^2$ ,  $x \in \mathcal{D}(A)$ .

## 2. ANALYTICAL PRELIMINARIES

This section is structured into two parts. The first one collects useful analytical concepts and results about several function spaces that will throughout be employed without further notice. We then proceed in the second part by interpreting the Maxwell system (1) as an evolution equation on an appropriate state space.

**2.1. Important function spaces.** For our reasoning, the divergence operator  $\operatorname{div}$ , and the two- and three-dimensional  $\operatorname{curl}_2$  and  $\operatorname{curl}$  are essential. Formally, they are defined by

$$\begin{aligned} \operatorname{div} \phi &= \sum_{i=1}^3 \partial_i \phi_i, & \operatorname{curl}_2 v &= \partial_1 v_2 - \partial_2 v_1, \\ \operatorname{curl} \phi &= (\partial_2 \phi_3 - \partial_3 \phi_2, \partial_3 \phi_1 - \partial_1 \phi_3, \partial_1 \phi_2 - \partial_2 \phi_1), \end{aligned}$$

for distributions  $\phi = (\phi_1, \phi_2, \phi_3)$  on a Lipschitz domain  $\Omega \subseteq \mathbb{R}^3$  and  $v = (v_1, v_2)$  on a Lipschitz domain  $S \subseteq \mathbb{R}^2$ .

For the sake of a clear presentation, we subsequently introduce only spaces and trace operators related to the  $\operatorname{curl}_2$ ,  $\operatorname{curl}$ , and  $\operatorname{div}$  operators on the cuboid  $Q$  and a rectangle  $S$ . The definitions and results, however, can be transferred to the subdomains  $Q_1, \dots, Q_N$  by appropriate adaptations. We first recall the Banach spaces

$$\begin{aligned} H(\operatorname{curl}_2, S) &:= \{v \in L^2(S)^2 \mid \operatorname{curl}_2 v \in L^2(S)\}, & \|v\|_{\operatorname{curl}_2}^2 &:= \|v\|_{L^2}^2 + \|\operatorname{curl}_2 v\|_{L^2}^2, \\ H(\operatorname{curl}, Q) &:= \{\phi \in L^2(Q)^3 \mid \operatorname{curl} \phi \in L^2(Q)^3\}, & \|\phi\|_{\operatorname{curl}}^2 &:= \|\phi\|_{L^2}^2 + \|\operatorname{curl} \phi\|_{L^2}^2, \\ H(\operatorname{div}, Q) &:= \{\phi \in L^2(Q)^3 \mid \operatorname{div} \phi \in L^2(Q)\}, & \|\phi\|_{\operatorname{div}}^2 &:= \|\phi\|_{L^2}^2 + \|\operatorname{div} \phi\|_{L^2}^2. \end{aligned}$$

We further use the subspaces  $H_0(\operatorname{curl}_2, S)$ ,  $H_0(\operatorname{curl}, Q)$  and  $H_0(\operatorname{div}, Q)$ , being the completion of the space of test functions on  $S$  and  $Q$  with respect to the norms  $\|\cdot\|_{\operatorname{curl}_2}$ ,  $\|\cdot\|_{\operatorname{curl}}$  and  $\|\cdot\|_{\operatorname{div}}$ , respectively. For these spaces, Theorems I.2.4–I.2.6 in [24] state the following. The space  $C^\infty(\overline{Q})^3$  is dense in  $H(\operatorname{div}, Q)$ , and the normal trace operator  $\gamma_n : v \mapsto v \cdot \nu|_{\partial Q}$  extends from  $C^\infty(\overline{Q})^3$  in a linear and continuous way to the space  $H(\operatorname{div}, Q)$ , now mapping into  $H^{-1/2}(\partial Q)$ . In the following, we will simply write  $v \cdot \nu$  instead of  $\gamma_n(v)$  for  $v \in H(\operatorname{div}, Q)$ . As a consequence of the density and extension result, Green's formula can be extended to  $H(\operatorname{div}, Q)$ , stating

$$\int_Q v \cdot \nabla \varphi \, dx + \int_Q (\operatorname{div} v) \varphi \, dx = \langle v \cdot \nu, \varphi \rangle_{H^{-1/2}(\partial Q) \times H^{1/2}(\partial Q)}$$

for functions  $v \in H(\operatorname{div}, Q)$  and  $\varphi \in H^1(Q)$ . Moreover, the subspace  $H_0(\operatorname{div}, Q)$  coincides with the kernel of  $\gamma_n$  on  $H(\operatorname{div}, Q)$ .

Concerning the curl operator, Theorems I.2.10–I.2.12 in [24] establish similar results. The space  $C^\infty(\overline{Q})^3$  is also dense in  $H(\operatorname{curl}, Q)$ , and the tangential trace operator  $\gamma_t : v \mapsto v \times \nu|_{\partial Q}$  has a unique linear and continuous extension to the space  $H(\operatorname{curl}, Q)$  with kernel  $H_0(\operatorname{curl}, Q)$  and range  $H^{-1/2}(\partial Q)^3$ . Again, we write only  $v \times \nu$  instead of  $\gamma_t(v)$  for  $v \in H(\operatorname{curl}, Q)$ . Here, Green's formula has the representation

$$\int_Q (\operatorname{curl} v) \cdot \varphi \, dx - \int_Q v \cdot \operatorname{curl} \varphi \, dx = \langle v \times \nu, \varphi \rangle_{H^{-1/2}(\partial Q) \times H^{1/2}(\partial Q)}$$

for vectors  $v \in H(\operatorname{curl}, Q)$  and  $\varphi \in H^1(Q)^3$ .

To cover the two-dimensional case, we additionally introduce the unit tangent  $\nu_t$  on  $\partial S$ . Denoting by  $\nu_S = (\nu_1, \nu_2)$  the unit exterior normal vector of  $\partial S$ , it is

defined by  $\nu_t = (-\nu_2, \nu_1)$ . For the two-dimensional case, Theorems I.2.10–I.2.12 in [24] then yield that  $C^\infty(\overline{S})^2$  is dense in  $H(\text{curl}_2, S)$ , and the tangential trace  $\gamma_t : v \mapsto v \cdot \nu_t|_{\partial S}$  extends continuously to  $H(\text{curl}_2, S)$  with kernel  $H_0(\text{curl}_2, S)$  and range  $H^{-1/2}(\partial S)$ . In this setting, the Green's formula is given by

$$\int_S (\text{curl}_2 v) \phi \, dx - \int_S v \cdot (\partial_2 \phi, -\partial_1 \phi) \, dx = \langle v \cdot \nu_t, \phi \rangle_{H^{-1/2}(\partial S) \times H^{1/2}(\partial S)}$$

for  $v \in H(\text{curl}_2, S)$  and  $\phi \in H^1(S)$ . We simply call the application of all three Green's formulas integration by parts.

Closely related are intersections of the above spaces, that are useful to derive regularity statements. We define the spaces

$$\begin{aligned} H_T(\text{curl}, \text{div}, Q) &:= H(\text{curl}, Q) \cap H_0(\text{div}, Q), \\ H_N(\text{curl}, \text{div}, Q) &:= H_0(\text{curl}, Q) \cap H(\text{div}, Q). \end{aligned}$$

We equip both with the complete norm

$$\|\cdot\|_{H_T}^2 := \|\text{curl} \cdot\|_{L^2(Q)}^2 + \|\text{div} \cdot\|_{L^2(Q)}^2.$$

The spaces  $H_T(\text{curl}, \text{div}, Q)$  and  $H_N(\text{curl}, \text{div}, Q)$  then continuously embed into  $H^1(Q)^3$ , meaning that there is a constant  $C_T > 0$  with

$$\|\mathbf{H}\|_{H^1(Q)}^2 \leq C_T \|\mathbf{H}\|_{H_T}^2 = C_T (\|\text{curl} \mathbf{H}\|_{L^2(Q)}^2 + \|\text{div} \mathbf{H}\|_{L^2(Q)}^2) \quad (5)$$

for all  $\mathbf{H} \in H_T(\text{curl}, \text{div}, Q) \cup H_N(\text{curl}, \text{div}, Q)$ , see for example Lemmas I.3.4, I.3.6 and Theorems I.3.7, I.3.9 in [24].

During the proof of the global error bound in Theorem 6.4, we also use extrapolation theory, see Section V.1.3 in [1] and Section 2.10 in [50]. Let  $A$  be a closed and densely defined operator on a Banach space  $(X, \|\cdot\|_X)$  with nonempty resolvent set. Let additionally  $\lambda$  be an element of the resolvent set of  $A$ . Then the extrapolation space  $X_{-1}^A$  with respect to  $A$  is defined as the completion of  $X$  in the norm  $\|\cdot\|_{X_{-1}^A} = \|(\lambda I - A)^{-1} \cdot\|_X$ . Note that this definition is independent of the choice of the resolvent value  $\lambda$ . The operator  $A$  then has a unique and bounded extension  $A_{-1}$  from  $X$  to  $X_{-1}^A$ . It is called the extrapolation operator of  $A$  to  $X$ . The resolvent operator  $(\lambda I - A)^{-1}$  moreover extends to the bounded operator  $(\lambda I - A_{-1})^{-1}$  from  $X_{-1}^A$  to  $X$ .

Interpolation theory is another important tool for our analysis. Throughout, we only employ real interpolation on Hilbert spaces, which can be defined via the K-method, see Section 1.1 in [41] for instance. By means of interpolation spaces, we in particular define fractional order Sobolev spaces, see [40, 48]. These spaces throughout serve as a measure for regularity statements. Let  $s \in [0, 2]$ ,  $k \in \{1, 2\}$ ,  $\theta \in (0, 1) \setminus \{1/2\}$ ,  $d \in \mathbb{N}$ ,  $O \subseteq \mathbb{R}^d$  open with a Lipschitz boundary, and define

$$H^s(O) := (L^2(O), H^2(O))_{s/2, 2}, \quad H_0^\theta(O) := (L^2(O), H_0^1(O))_{\theta, 2}. \quad (6)$$

We additionally note that the spaces  $H^\theta(O)$  and  $H_0^\theta(O)$  coincide for  $\theta \in (0, 1/2)$  (this can be verified by means of Corollary 1.4.4.5 in [27] for instance).

The spaces of functions with piecewise Sobolev regularity are also important. Let  $\Gamma^*$  be a union of some faces of  $Q$ . Define the spaces

$$\begin{aligned} PH^q(Q) &:= \{f \in L^2(Q) \mid f^{(i)} \in H^q(Q_i), i \in \{1, \dots, N\}\}, & q \in [0, 2], \\ PH_{\Gamma^*}^s(Q) &:= \{f \in PH^s(Q) \mid f^{(i)} = 0 \text{ on } \partial Q_i \cap \Gamma^*, i \in \{1, \dots, N\}\}, & s \in (1/2, 2], \end{aligned}$$

equipped with the norms

$$\|f\|_{PH^q}^2 := \sum_{i=1}^N \left\| f^{(i)} \right\|_{H^q(Q_i)}^2, \quad \|g\|_{PH_{\Gamma^*}^s} := \|g\|_{PH^s},$$



for  $f \in PH^q(Q)$  and  $g \in PH_{\Gamma^*}^s(Q)$ .

The next lemma serves as a technical tool, establishing a useful density result for function spaces related to the electric and the magnetic field. It uses to approximate with piecewise regular functions, that satisfy prescribed transmission conditions, and that vanish in a neighborhood of all exterior and interior edges of  $Q$ . The result is applied in the proof for Lemma 3.1, and it will play a crucial role in a subsequent work that is in preparation. For the statement, let  $\Gamma^*$  be a (possibly empty) union of opposite faces of the cuboid  $Q$ , and let  $\mathcal{F}_{\text{int},j}$  denote the set of all interfaces whose normal vector is parallel to the  $j$ -th canonical unit vector  $e_j$ ,  $j \in \{1, 2\}$ .

**Lemma 2.1.** *Let  $\varepsilon$  satisfy (2). Define the spaces*

$$\begin{aligned} V &:= \{\varphi \in PH_{\Gamma^*}^1(Q) \mid \llbracket \varepsilon \varphi \rrbracket_{\mathcal{F}} = 0, \llbracket \varphi \rrbracket_{\mathcal{F}'} = 0 \text{ for all } \mathcal{F} \in \mathcal{F}_{\text{int},j}, \\ &\quad \mathcal{F}' \in \mathcal{F}_{\text{int}} \setminus \mathcal{F}_{\text{int},j}\}, \\ W &:= \{\varphi \in PH^2(Q) \cap V \mid \varphi^{(i)} \text{ is smooth, } \text{supp}(\varphi) \cap \overline{\Gamma^*} = \emptyset, \\ &\quad \varphi \text{ vanishes in a neighborhood of all edges of } Q_1, \dots, Q_N, \\ &\quad \partial_{\nu_{\mathcal{F}}} \varphi^{(i)} = 0 \text{ for faces } \mathcal{F} \subseteq \partial Q_i, i \in \{1, \dots, N\}\}. \end{aligned}$$

*The space  $W$  is dense in  $V$  with respect to the norm in  $PH^1(Q)$ .*

*Proof.* We show only the density of  $W$  in  $V$  in the case  $\Gamma^* = \Gamma_2 \cup \Gamma_3$ , and assume  $j = 2$ . All remaining settings can be established with the same techniques, up to appropriate modifications.

1) Let  $\varphi \in V$  and  $\delta > 0$ . Applying Lemma 2.5 in [15] to every interior and exterior edge of  $Q$ , there is a function  $\hat{\varphi} \in V$ , that vanishes in an open neighborhood of all edges of  $Q_1, \dots, Q_N$  and satisfies

$$\|\hat{\varphi} - \varphi\|_{PH^1(Q)} \leq \delta. \quad (7)$$

Hence, there is a union  $\mathcal{T}$  of tubes of inner radius  $\zeta > 0$  around all edges with  $\hat{\varphi}$  vanishing on  $Q \cap \mathcal{T}$ .

We next construct a piecewise smooth function fulfilling the required transmission, support, and normal derivative conditions. We only deal with the cuboid

$$Q_1 = (a_1^{-,1}, a_1^{+,1}) \times (a_2^{-,1}, a_2^{+,1}) \times (a_3^{-,1}, a_3^{+,1}),$$

(setting  $a_3^{\pm,1} := a_3^{\pm}$ ) and we assume that  $Q_1$  touches the faces  $\Gamma_1^+$  and  $\Gamma_2^+$  of  $Q$ , see (3). All other cuboids can be treated in the same way with slight modifications. Let  $l \in \{1, 2, 3\}$ , and  $\chi_{m,l} : \mathbb{R} \rightarrow [0, 1]$  be a smooth cut-off function with  $\text{supp } \chi_{m,l} \subseteq [a_l^{-,1}, a_l^{-,1} + \frac{1}{m}] \cup [a_l^{+,1} - \frac{1}{m}, a_l^{+,1}]$ ,  $\chi_{m,l} = 1$  on  $[a_l^{-,1}, a_l^{-,1} + \frac{1}{2m}] \cup [a_l^{+,1} - \frac{1}{2m}, a_l^{+,1}]$ , and  $\|\chi'_{m,l}\|_{\infty} \leq Cm$  for a uniform constant  $C > 0$  for all  $m \geq m_l \in \mathbb{N}$ . Let

$$\Gamma_l^{\pm,1} = \{x \in \partial Q_1 \mid x_l \in \{a_l^{\pm,1}\}, x_j \in (a_j^{-,1}, a_j^{+,1}) \text{ for } j \neq l\},$$

and denote the pyramid with basis  $\Gamma_l^{\pm,1}$  and peak  $(\frac{a_1^{-,1} + a_1^{+,1}}{2}, \frac{a_2^{-,1} + a_2^{+,1}}{2}, \frac{a_3^- + a_3^+}{2})$  by  $P_l^{\pm,1}$ . Its reflection at the face  $\Gamma_l^{\pm,1}$  is called  $\check{P}_l^{\pm,1}$ . Let further  $Q_{i_k}$  be the adjacent cuboid of  $Q_1$  in coordinate direction  $k \in \{1, 2\}$ .

We then define the larger set  $\check{Q}_1 := \bar{Q}_1 \cup \bigcup_{l=1}^3 (\check{P}_l^{+,1} \cup \check{P}_l^{-,1})$ , and put

$$g_{m,(1)}(x) := \begin{cases} \check{\varphi}^{(1)}(x) & \text{for } x \in P_1^{\pm,1} \cup P_2^{-,1}, \\ (1 - \chi_{m,3}(x_3))\check{\varphi}^{(1)}(x) & \text{for } x = (x_1, x_2, x_3) \in P_3^{\pm,1}, \\ (1 - \chi_{m,2}(x_2))\check{\varphi}^{(1)}(x) & \text{for } x = (x_1, x_2, x_3) \in P_2^{+,1}, \\ \check{\varphi}^{(i_1)}(x) & \text{for } x \in \check{P}_1^{-,1}, \\ \check{\varphi}^{(1)}(-x_1 + 2a_1^{+,1}, x_2, x_3) & \text{for } x = (x_1, x_2, x_3) \in \check{P}_1^{+,1}, \\ \frac{\varepsilon^{(i_2)}}{\varepsilon^{(1)}}\check{\varphi}^{(i_2)}(x) & \text{for } x \in \check{P}_2^{-,1}, \\ 0 & \text{for } x \in \check{P}_3^{\pm,1} \cup \check{P}_2^{+,1}, \\ 0 & \text{for } x \in \mathbb{R}^3 \setminus \check{Q}_1. \end{cases}$$

Since  $\check{\varphi}$  is an element of  $V$  and vanishes on  $\bar{Q} \cap \mathcal{T}$ , there is a number  $m_4 \in \mathbb{N}$  and an open superset  $\check{Q}_1$  of  $\bar{Q}_1$  with  $g_{m,(1)}|_{\check{Q}_1} \in H^1(\check{Q}_1)$  for  $m \geq m_4$ . We then repeat the same reasoning for all other subcuboids, by appropriately changing the definition of the function  $g_{m,(i)}$  for each subcuboid  $Q_i$ . Define then a function  $g_m$  on  $Q$  by  $g_m|_{Q_i} := g_{m,(i)}|_{Q_i}$  for  $i \in \{1, \dots, N\}$ .

Taking the exterior face conditions for  $\check{\varphi}$  into account, the arguments from the proof of Lemma 2.1 in [21] show that  $g_m^{(i)}$  converges to  $\check{\varphi}^{(i)}$  in  $H^1(P)$  for  $P \in \{P_l^{\pm,i} \mid l \in \{1, 2, 3\}\}$  as  $m \rightarrow \infty$ . There consequently is a number  $\check{m} \geq m_4$  with

$$\|g_{\check{m}} - \check{\varphi}\|_{PH^1(Q)} \leq \delta. \quad (8)$$

We next employ the standard mollifier  $\rho_{n,l}$  that acts on the  $l$ -th coordinate, and that is supported within  $[-\frac{1}{n}, \frac{1}{n}]$ . Let

$$\tilde{\psi}_{n,i} := \rho_{n,3} * \rho_{n,2} * \rho_{n,1} * g_{\check{m},(i)}, \quad n \in \mathbb{N}, i \in \{1, \dots, N\}.$$

By construction, the function  $\tilde{\psi}_{n,i}$  is smooth, and it vanishes in a union  $\tilde{\mathcal{T}}$  of tubes with radius  $\frac{3}{4}\zeta$  around all edges as well as in a neighborhood of all exterior faces in  $\Gamma_2 \cup \Gamma_3$ , provided that  $n \geq n_0 \in \mathbb{N}$ . We also remark that the function  $\tilde{\psi}_n$ , being defined by  $\tilde{\psi}_n|_{Q_i} := \tilde{\psi}_{n,i}$ , satisfies all required transmission conditions in  $Q$  for sufficiently large  $n$ . As a consequence of standard mollifier theory, the sequence  $(\tilde{\psi}_{n,i})_n$  furthermore converges in  $H^1(\check{Q}_i)$  to  $g_{\check{m},(i)}$ . There consequently is a number  $\check{n} \geq n_0$  with

$$\|\tilde{\psi}_{\check{n},i} - g_{\check{m},(i)}\|_{H^1(\check{Q}_i)} \leq \delta, \quad i \in \{1, \dots, N\}. \quad (9)$$

2) It remains to incorporate also the Neumann boundary conditions at the faces of  $Q_1$ . This is done by transferring a technique from the proof of Lemma 3.3 in [21] to our setting. Let  $\kappa \in (0, \frac{a_1^{+,1} - a_1^{-,1}}{2})$  be a fixed number. Let  $\tilde{\alpha} : [a_1^{-,1}, a_1^{+,1}] \rightarrow [0, 1]$  be a smooth function with  $\text{supp } \tilde{\alpha} \subseteq [a_1^{-,1}, a_1^{-,1} + \frac{\kappa}{2}]$ , and  $\tilde{\alpha} = 1$  on  $[a_1^{-,1}, a_1^{-,1} + \frac{\kappa}{4}]$ . Define then the function

$$\begin{aligned} h_{k,1}^-(x_1, x_2, x_3) &:= \tilde{\psi}_{\check{n},1}(x_1, x_2, x_3) - \tilde{\alpha}(x_1) \int_{a_1^{-,1}}^{x_1} \chi_{k,1}(s) \partial_1 \tilde{\psi}_{\check{n},1}(s, x_2, x_3) ds \\ &=: \tilde{\psi}_{\check{n},1}(x) - r_k(x) \end{aligned}$$

for  $x = (x_1, x_2, x_3) \in P_1^{-,1}$  and  $k \in \mathbb{N}$ . By construction of  $\tilde{\psi}_{\check{n},1}$ , the functions  $h_{k,1}^-$  and  $r_k$  are smooth. We next deduce that  $r_k$  tends to zero in  $H^1(P_1^{-,1})$  as  $k \rightarrow \infty$ . The integrand of  $r_k$  is uniformly bounded in  $k$ , and converges pointwise to zero. Thus,  $(r_k)_k$  is uniformly bounded. Applying now Lebesgue's theorem of dominated convergence twice, we infer that  $r_k$  converges pointwise and in  $L^2(P_1^{-,1})$  to zero as

$k \rightarrow \infty$ . A simple computation further gives rise to the formulas

$$\begin{aligned}\partial_1 r_k &= (\partial_1 \tilde{\alpha}) \int_{a_1^-}^{x_1} \chi_{k,1}(s) \partial_1 \tilde{\psi}_{\tilde{n},1}(s, \cdot) ds + \tilde{\alpha} \chi_{k,1} \partial_1 \tilde{\psi}_{\tilde{n},1}, \\ \partial_l r_k &= \tilde{\alpha} \int_{a_1^-}^{x_1} \chi_{k,1}(s) \partial_l \tilde{\psi}_{\tilde{n},1}(s, \cdot) ds, \quad l \in \{2, 3\}.\end{aligned}$$

Similar arguments to the ones above now imply that  $(\partial_1 r_k)_k$  and  $(\partial_l r_k)_k$  are null sequences in  $L^2(P_1^{-,1})$ . As a result,  $(h_{k,1}^-)_k$  converges to  $\tilde{\psi}_{\tilde{n},1}$  in  $H^1(P_1^{-,1})$ , and  $\partial_1 h_{k,1}^- = 0$  on  $\Gamma_1^{-,1}$ . By analogous constructions on all other pyramids  $P_1^{+,1}$ ,  $P_2^{\pm,1}$ , and  $P_3^{\pm,1}$ , we further obtain similar functions  $h_{k,1}^+$ ,  $h_{k,2}^{\pm}$  and  $h_{k,3}^{\pm}$  for  $k \in \mathbb{N}$ . They are in particular smooth and coincide with  $\tilde{\psi}_{\tilde{n},1}$ , provided that the distance to the associated face is larger than  $\frac{\kappa}{2}$ . Define now a new mapping  $\psi_{k,1}$  on  $Q_1$  via its restrictions  $\psi_{k,1}|_{P_j^{\pm,1}} := h_{k,j}^{\pm}$ . As the function  $\tilde{\psi}_{\tilde{n},1}$  vanishes in  $\tilde{\mathcal{T}}$  (union of tubes around all edges with radius  $3/4\zeta$ ), we can choose  $\kappa > 0$  so small that  $\psi_{k,1}$  is smooth on  $Q_1$ . We then repeat the analogous construction for all remaining cuboids  $Q_2, \dots, Q_N$ , obtaining functions  $\psi_{k,2}, \dots, \psi_{k,N}$  for  $k \in \mathbb{N}$ . Finally, we define the mapping  $\psi_k$  elementwise by  $\psi_k^{(i)} := \psi_{k,i}$ , for  $i \in \{1, \dots, N\}$ .

By construction,  $\psi_k$  is smooth on every cuboid, and it vanishes in an open neighborhood of  $\Gamma_2 \cup \Gamma_3$  and of all edges of the subcuboids. It further satisfies the required normal derivative condition at all faces for sufficiently large  $k$ . Using finally that the function  $\tilde{\psi}_{\tilde{n}}$  satisfies the required transmission conditions, we conclude that  $\psi_k$  also fulfills by definition the transmission conditions  $\llbracket \varepsilon \psi_k \rrbracket_{\mathcal{F}} = 0$ ,  $\llbracket \psi_k \rrbracket_{\mathcal{F}'} = 0$  for all  $\mathcal{F} \in \mathcal{F}_{\text{int},2}$  and  $\mathcal{F}' \in \mathcal{F}_{\text{int},1}$ . Taking also (7)–(9) into account,  $\psi_k$  is contained in  $W$ , and the estimate  $\|\psi_k - \varphi\|_{PH^1(Q)} \leq 4\delta$  is valid for sufficiently large  $k$ .  $\square$

**2.2. Analytical framework for the Maxwell system.** Throughout, we consider the Maxwell equations (1) as an evolution equation on the space  $X := L^2(Q)^6$ . The space is equipped with the weighted inner product

$$\left( \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}, \begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{H}} \end{pmatrix} \right) := \int_Q \varepsilon \mathbf{E} \cdot \tilde{\mathbf{E}} + \mu \mathbf{H} \cdot \tilde{\mathbf{H}} dx, \quad \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}, \begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{H}} \end{pmatrix} \in X,$$

inducing the norm  $\|\cdot\|$  on  $X$ . The positivity and boundedness assumption on  $\varepsilon$  and  $\mu$  implies that  $\|\cdot\|$  is equivalent to the standard  $L^2$ -norm.

On  $X$  we consider the Maxwell operator

$$M := \begin{pmatrix} 0 & \frac{1}{\varepsilon} \text{curl} \\ -\frac{1}{\mu} \text{curl} & 0 \end{pmatrix}$$

with domain

$$\begin{aligned}\mathcal{D}(M) &:= H_0(\text{curl}, Q) \times H(\text{curl}, Q) \\ &= \{(\mathbf{E}, \mathbf{H}) \in L^2(Q)^6 \mid \text{curl } \mathbf{E}^{(i)}, \text{curl } \mathbf{H}^{(i)} \in L^2(Q_i)^3, \llbracket \mathbf{E} \times \nu_F \rrbracket_F = 0, \\ &\quad \llbracket \mathbf{H} \times \nu_F \rrbracket_F = 0, \mathbf{E} \times \nu = 0 \text{ on } \partial Q, \\ &\quad i \in \{1, \dots, N\}, F \in \mathcal{F}_{\text{int}}\},\end{aligned}\tag{10}$$

involving transmission conditions in tangential direction.

We next incorporate the boundary conditions for the magnetic field, as well as divergence and normal transmission conditions. Recall to that end the set of effective interfaces  $\mathcal{F}_{\text{int}}^{\text{eff}}$ . The latter contains all interfaces between the submedia  $\tilde{Q}_{i,l}$ ,  $i \in \{1, \dots, L\}$ ,  $l \in \{0, \dots, K\}$ . For each effective interface  $\mathcal{F} \in \mathcal{F}_{\text{int}}^{\text{eff}}$ , we put

$$V(\mathcal{F}) := (L^2(\mathcal{F}), H_{\mathcal{F}}^1)_{\frac{1}{2}, 2}, \quad H_{\mathcal{F}}^1 := \{u \in H^1(\mathcal{F}) \mid u = 0 \text{ on } \mathcal{F} \cap \partial Q\}.\tag{11}$$

We then define the subspace

$$X_0 := \{(\mathbf{E}, \mathbf{H}) \in L^2(Q)^6 \mid \operatorname{div}(\varepsilon \mathbf{E}|_{\tilde{Q}_{i,l}}) \in L^2(\tilde{Q}_{i,l}), \llbracket \varepsilon \mathbf{E} \cdot \nu_{\mathcal{F}} \rrbracket_{\mathcal{F}} \in V(\mathcal{F}), \quad (12)$$

$$\operatorname{div}(\mu \mathbf{H}) = 0, \quad \mu \mathbf{H} \cdot \nu = 0 \text{ on } \partial Q, \quad \mathcal{F} \in \mathcal{F}_{\text{int}}^{\text{eff}}, \quad i \in \{1, \dots, L\}, \quad l \in \{0, \dots, K\}\},$$

of  $X$ , which is inspired by the spaces  $X_{\text{div}}$  and  $X_0$  in [29, 21, 20]. The space  $X_0$  is complete with respect to the norm

$$\|(\mathbf{E}, \mathbf{H})\|_{X_0}^2 := \|(\mathbf{E}, \mathbf{H})\|^2 + \sum_{i=1}^N \left\| \operatorname{div}(\varepsilon^{(i)} \mathbf{E}^{(i)}) \right\|_{L^2(Q_i)}^2 + \sum_{\mathcal{F} \in \mathcal{F}_{\text{int}}^{\text{eff}}} \|\llbracket \varepsilon \mathbf{E} \cdot \nu_{\mathcal{F}} \rrbracket_{\mathcal{F}}\|_{V(\mathcal{F})}^2.$$

Indeed, let  $(\mathbf{E}^n, \mathbf{H}^n)_n$  be a Cauchy-sequence in  $X_0$ , and fix numbers  $i \in \{1, \dots, L\}$ ,  $l \in \{0, \dots, K\}$ , as well as an interface  $\mathcal{F} \in \mathcal{F}_{\text{int}}^{\text{eff}}$  that is a face of  $\tilde{Q}_{i,l}$ . The given sequence converges to a limit  $(\mathbf{E}, \mathbf{H})$  in  $L^2(Q)^6$  with respect to the norm  $\|\cdot\|$ . Since also the sequence  $(\operatorname{div} \mu \mathbf{H}^n)_n$  converges in  $L^2(Q)$ , we infer from the closedness of the divergence operator and the continuity of the normal trace operator that  $\operatorname{div}(\mu \mathbf{H}) = 0$  and  $\mu \mathbf{H} \cdot \nu = 0$  on  $\partial Q$ . For the vector  $\mathbf{E}$ , we observe that also the sequence  $(\varepsilon \mathbf{E}^n|_{\tilde{Q}_{i,l}})_n$  converges with respect to the graph norm of the divergence operator on  $\tilde{Q}_{i,l}$ , whence  $\operatorname{div}(\varepsilon \mathbf{E}|_{\tilde{Q}_{i,l}})$  is an element of  $L^2(\tilde{Q}_{i,l})$ . Employing now the continuity of the normal trace operator for  $\mathcal{F}$ , we conclude that  $(\operatorname{tr}_{\mathcal{F}}(\varepsilon \mathbf{E}^n)|_{\tilde{Q}_{i,l}})_n$  converges in  $H^{-1/2}(\mathcal{F})$  to  $\operatorname{tr}_{\mathcal{F}}(\varepsilon \mathbf{E})|_{\tilde{Q}_{i,l}}$ . We repeat this reasoning on every other submedium. By definition of the norm in  $X_0$  and the uniqueness of limits, the jump  $\llbracket \varepsilon \mathbf{E} \cdot \nu_{\mathcal{F}} \rrbracket_{\mathcal{F}}$  is consequently contained in  $V(\mathcal{F})$ . Altogether,  $(\mathbf{E}, \mathbf{H})$  is an element of  $X_0$ , and the limit of  $(\mathbf{E}^n, \mathbf{H}^n)$  in  $X_0$ .

To equip the Maxwell operator with the magnetic boundary as well as the electric and magnetic divergence conditions, we introduce the restriction  $M_0$  of the Maxwell operator to the space  $X_0$ , and consider it on the space

$$X_1 := \mathcal{D}(M_0) := \mathcal{D}(M) \cap X_0, \quad (13)$$

which is equipped with the norm

$$\left\| \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \right\|_{X_1}^2 := \left\| \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \right\|_{X_0}^2 + \left\| M \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \right\|^2, \quad \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \in X_1.$$

**Remark 1.** By interpreting the Maxwell equations (1) on  $X_1$ , we only assume that the divergence of the electric field is an  $L^2$ -function on every submedium  $\tilde{Q}_{i,l}$ ,  $i \in \{1, \dots, L\}$ ,  $l \in \{0, \dots, K\}$ . In particular, we allow for nonzero jumps of the normal component of the field  $\varepsilon \mathbf{E}$  across effective interfaces in  $\mathcal{F}_{\text{int}}^{\text{eff}}$ . These discontinuities represent surface charges on the interfaces, see Section 3.5 in [25].  $\diamond$

Although the space  $X_1$  is mainly defined by means of the domains of the divergence and curl operators, which themselves allow for irregular functions, the space  $X_1$  indeed embeds into a space of functions with piecewise fractional Sobolev regularity above  $1/2$ , see Proposition 4.

The next lemma deals with  $M_0$ , and it shows that  $M_0$  is not only the restriction of  $M$  to  $X_0$ , but also its part in this space. The statement corresponds to relation (2.5) in [21].

**Lemma 2.2.** *The identity  $\mathcal{D}(M_0^k) = \mathcal{D}(M^k) \cap X_0$  is valid for all  $k \in \mathbb{N}$ , and  $M(\mathcal{D}(M))$  is a subset of  $X_0$ . In particular,  $M_0$  is the part of  $M$  in  $X_0$ , and the space  $X_1$  is complete.*

*Proof.* We show only that the space  $X_1$  is complete. The remaining statements can be established in the same way as identity (2.5) in [21].

To deduce the completeness of  $X_1$ , we first note that  $M_0$  is closed in  $X_0$  as the part of a closed operator, and thus its domain  $X_1$  is complete with respect to the

graph norm of  $M_0$ . It hence suffices to show that the graph norm of  $M_0$  coincides with the standard norm on  $X_1$ .

Let  $(\mathbf{E}, \mathbf{H}) \in X_1 = \mathcal{D}(M) \cap X_0$ . Combining the relation  $\operatorname{div}(\varepsilon(M(\mathbf{E}, \mathbf{H}))_1) = \operatorname{div}(\operatorname{curl} \mathbf{H}) = 0$  with the transmission conditions in  $H(\operatorname{curl}, Q)$  and  $H(\operatorname{div}, Q)$  and the definition of the norms on  $X_0$  and  $X_1$ , the identities

$$\left\| \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \right\|_{X_1}^2 = \left\| \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \right\|_{X_0}^2 + \left\| M \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \right\|_{X_0}^2 = \left\| \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \right\|_{\mathcal{D}(M_0)}^2$$

immediately follow.  $\square$

The part of  $M$  in  $X_1$  is denoted by  $M_1$ , and it is shown to generate a strongly continuous semigroup on  $X_1$ . Thus, the space  $X_1$  serves as a state space for the Maxwell equations (1), see Proposition 5. Using a regularity statement for the space  $X_1$ , we can then conclude that the system (1) possesses solutions of piecewise  $H^{1-\theta}$ -regularity,  $\theta \in (0, 1)$  appropriate, see Corollary 2 and Remark 2. As a starting point, the following result states the generator property of the Maxwell operator on  $X$ . The statement is part of Proposition 3.5 in [29].

**Proposition 1.** *Let  $\varepsilon$  and  $\mu$  satisfy (2). The Maxwell operator  $M$  generates a unitary  $C_0$ -group  $(e^{tM})_{t \in \mathbb{R}}$  on  $X$ .*

### 3. ANALYSIS OF AN ELLIPTIC TRANSMISSION PROBLEM

This section is concerned with investigations of transmission problems for the Laplacian on the cuboid  $Q$  with homogeneous transmission conditions, see (14). By *homogeneous* we mean that the solution and its normal derivative are required to be continuous at all interfaces up to multiplication with the discontinuous parameters  $\varepsilon$  and  $\mu$ . The considered elliptic transmission problem arises several times in literature, see [15, 36, 39, 34, 11, 12, 13] for instance. Note, however, that there are no explicit regularity statements for our particular application of the embedded waveguide at hand, to the best of our knowledge. In other words, we are interested in precise results in terms of the size of jumps of the parameters  $\varepsilon$  and  $\mu$ . This is because the below system (14) arises naturally when analyzing the regularity of the electric and magnetic field, see the proof of Lemma 4.2 and [15, 11, 12]. Because we are also going to transfer some arguments from the analysis of (14) to a different elliptic transmission problem in a subsequent work, we analyze the problem here in detail to have a self-contained presentation.

Let  $\eta \in \{\varepsilon, \mu\}$  satisfy the assumptions (2). The function  $\eta$  will throughout serve as a placeholder for the material parameters  $\varepsilon$  and  $\mu$ . Let further  $\Gamma^*$  be a nonempty union of some of the sets  $\Gamma_1, \Gamma_2, \Gamma_3$ , consisting of opposite boundary faces of  $Q$ , see (3). Consider the elliptic transmission problem

$$\begin{aligned} -\Delta \psi^{(i)} &= f^{(i)} && \text{on } Q_i \text{ for } i \in \{1, \dots, N\}, \\ \psi &= 0 && \text{on } \Gamma^*, \\ \nabla \psi \cdot \nu &= 0 && \text{on } \partial Q \setminus \Gamma^*, \\ \llbracket \psi \rrbracket_{\mathcal{F}} &= 0 = \llbracket \eta \nabla \psi \cdot \nu_{\mathcal{F}} \rrbracket_{\mathcal{F}} && \text{on } \mathcal{F} \in \mathcal{F}_{\text{int}}, \end{aligned} \tag{14}$$

involving a given function  $f \in L^2(Q)$ .

We next recall the decomposition  $\overline{Q} = \bigcup_{i=1}^L \bigcup_{j=0}^K \overline{Q_{ij}}$  from Section 1. To measure the regularity of the solution of (14) in the case  $\eta = \varepsilon$ , we introduce the number  $\overline{\kappa} \in (0, 1]$  with

$$\max_{\substack{i \in \{1, \dots, L\}, \\ l \in \{1, \dots, K\}}} \frac{(\varepsilon|_{\overline{Q_{i,l}}} - \varepsilon|_{\overline{Q_{i,0}}})^2}{\varepsilon|_{\overline{Q_{i,l}}} \varepsilon|_{\overline{Q_{i,0}}}} = -\frac{4 \sin^2(\overline{\kappa}\pi)}{\sin(\frac{\overline{\kappa}}{2}\pi) \sin(\frac{3\overline{\kappa}}{2}\pi)}. \tag{15}$$

As the function on the right is monotonically decreasing in  $\bar{\kappa}$  on  $(\frac{2}{3}, 1]$ , we infer that  $\bar{\kappa}$  decreases if the relative discontinuities of the material parameter  $\varepsilon$  in the subcuboids  $\tilde{Q}_1, \dots, \tilde{Q}_L$  become stronger, meaning the material becomes more heterogeneous. In the limit case of homogeneous subcuboids  $\tilde{Q}_1, \dots, \tilde{Q}_L$ , on the other hand, the number  $\bar{\kappa}$  is one. We further note that the assumptions (2) imply the crucial lower estimate

$$\bar{\kappa} > 3/4. \quad (16)$$

Indeed, using (2) twice, we can estimate the left hand side of (15) via

$$\begin{aligned} \frac{(\varepsilon|\tilde{Q}_{i,l} - \varepsilon|\tilde{Q}_{i,0})^2}{\varepsilon|\tilde{Q}_{i,l}\varepsilon|\tilde{Q}_{i,0}} &< \frac{\varepsilon|\tilde{Q}_{i,l}|^2}{\varepsilon|\tilde{Q}_{i,l}\varepsilon|\tilde{Q}_{i,0}} \left(2\sqrt{2} - 2\sqrt{2 + \sqrt{2}}\right)^2 < \frac{(2\sqrt{2} - 2\sqrt{2 + \sqrt{2}})^2}{1 + 2\sqrt{2} - 2\sqrt{2 + \sqrt{2}}} \\ &= 4\sqrt{2}. \end{aligned}$$

The central result of this section is the following regularity statement for (14). To state it, we employ the following notation. We define  $\tilde{Q} := Q \cap \{x_3 = 1/2\}$ ,  $\tilde{Q}_i := Q_i \cap \{x_3 = 1/2\}$ ,  $i \in \{1, \dots, N\}$ , and interpret them as rectangles in  $\mathbb{R}^2$ . Piecewise Sobolev regularity on  $\tilde{Q}$  is then defined with respect to the partition  $\tilde{Q}_1, \dots, \tilde{Q}_N$ . We put

$$\mathcal{V}_{2-\kappa} := H_{x_3}^2((0, 1), L^2(\tilde{Q})) \cap H_{x_3}^1((0, 1), H^1(\tilde{Q})) \cap L_{x_3}^2((0, 1), PH^{2-\kappa}(\tilde{Q})) \quad (17)$$

for  $\kappa \in [0, 1)$ . This space is canonically equipped with the sum of the norms.

**Proposition 2.** *Let  $\eta \in \{\varepsilon, \mu\}$ , and let  $\varepsilon$  and  $\mu$  satisfy (2). Let further  $\kappa > 1 - \bar{\kappa}$  if  $\eta = \varepsilon$ , and  $\kappa = 0$  if  $\eta = \mu$ . Assume also that  $f \in L^2(Q)$ , and let  $\Gamma^*$  be nonempty. There is a unique solution  $\psi \in \mathcal{V}_{2-\kappa}$  of (14) with  $\|\psi\|_{\mathcal{V}_{2-\kappa}} \leq C \|f\|_{L^2(Q)}$  for a constant  $C = C(Q, \eta, \kappa) > 0$ .*

The remainder of this section is concerned with the proof of Proposition 2. The structure of the argument is oriented towards the papers [36, 39, 15].

We express system (14) equivalently by the formula

$$\Delta_{0, \Gamma^*} u = f, \quad (18)$$

involving the Laplacian

$$\begin{aligned} (\Delta_{0, \Gamma^*} u)^{(i)} &:= \Delta u^{(i)}, \quad \text{on } Q_i, \quad i \in \{1, \dots, N\}, \\ u \in \mathcal{D}(\Delta_{0, \Gamma^*}) &:= \{v \in H_{\Gamma^*}^1(Q) \mid \Delta v^{(i)} \in L^2(Q_i), \nabla v \cdot \nu = 0 \text{ on } \partial Q \setminus \Gamma^*, \\ &\quad \llbracket \eta \nabla v \cdot \nu_{\mathcal{F}} \rrbracket_{\mathcal{F}} = 0 \text{ for } \mathcal{F} \in \mathcal{F}_{\text{int}}, \quad i \in \{1, \dots, N\}\}. \end{aligned} \quad (19)$$

**3.1. Energy estimates for the Laplacian with transmission conditions.** In the next two lemmas, we provide a useful energy identity and an a priori estimate for the Laplace operator on  $\mathcal{D}(\Delta_{0, \Gamma^*}) \cap PH^2(Q)$ . This is done in the spirit of Grisvard, see [26].

**Lemma 3.1.** *Let  $\eta \in \{\varepsilon, \mu\}$  satisfy (2). The identity*

$$\begin{aligned} \sum_{i=1}^N \eta^{(i)} \left( \left\| \partial_1^2 u^{(i)} \right\|_{L^2(Q_i)}^2 + \left\| \partial_2^2 u^{(i)} \right\|_{L^2(Q_i)}^2 + \left\| \partial_3^2 u^{(i)} \right\|_{L^2(Q_i)}^2 + 2 \left\| \partial_1 \partial_2 u^{(i)} \right\|_{L^2(Q_i)}^2 \right. \\ \left. + 2 \left\| \partial_1 \partial_3 u^{(i)} \right\|_{L^2(Q_i)}^2 + 2 \left\| \partial_2 \partial_3 u^{(i)} \right\|_{L^2(Q_i)}^2 \right) = \sum_{i=1}^N \eta^{(i)} \left\| \Delta u^{(i)} \right\|_{L^2(Q_i)}^2 \end{aligned}$$

is valid for  $u \in \mathcal{D}(\Delta_{0, \Gamma^*}) \cap PH^2(Q)$ .

*Proof.* 1) We only treat the case  $\Gamma^* = \Gamma_1$ . The remaining cases are proved in a similar way. A simple calculation first leads to the equation

$$\begin{aligned} \left\| \Delta u^{(i)} \right\|_{L^2(Q_i)}^2 &= \left\| \partial_1^2 u^{(i)} \right\|_{L^2(Q_i)}^2 + \left\| \partial_2^2 u^{(i)} \right\|_{L^2(Q_i)}^2 + \left\| \partial_3^2 u^{(i)} \right\|_{L^2(Q_i)}^2 \\ &+ 2 \int_{Q_i} (\partial_1^2 u^{(i)})(\partial_2^2 u^{(i)}) dx + 2 \int_{Q_i} (\partial_1^2 u^{(i)})(\partial_3^2 u^{(i)}) dx + 2 \int_{Q_i} (\partial_2^2 u^{(i)})(\partial_3^2 u^{(i)}) dx \end{aligned} \quad (20)$$

for  $i \in \{1, \dots, N\}$ . It now remains to consider the last three terms on the right hand side.

2) By Lemma 2.1, there are two sequences  $(\varphi_n)_n$  and  $(\psi_n)_n$  in  $PH^2(Q)$  satisfying  $\varphi_n^{(i)} \rightarrow \partial_3 u^{(i)}$ ,  $\psi_n^{(i)} \rightarrow \partial_2 u^{(i)}$  in  $H^1(Q_i)$  as  $n \rightarrow \infty$ , and fulfilling the boundary and transmission conditions  $\varphi_n^{(i)} = 0$  on  $\Gamma_3^{(i)}$ ,  $\psi_n^{(i)} = 0$  on  $\Gamma_2^{(i)} \cap \partial Q$ , and  $[\![\varphi_n]\!]_{\mathcal{F}} = 0 = [\![\eta\psi_n]\!]_{\mathcal{F}}$  for  $\mathcal{F} \in \mathcal{F}_{\text{int},2}$  for all  $i \in \{1, \dots, N\}$  and  $n \in \mathbb{N}$ . Employing Lemma 2.1 of [21] and Lemma 7.1 of [54], the relations

$$\begin{aligned} \partial_2 \varphi_n^{(i)} &= 0 \text{ on } \Gamma_3^{(i)}, & \partial_3 \psi_n^{(i)} &= 0 \text{ on } \Gamma_2^{(i)} \cap \partial Q, \\ [\![\partial_3 \varphi_n^{(i)}]\!]_{\mathcal{F}} &= 0 = [\![\eta \partial_3 \psi_n^{(i)}]\!]_{\mathcal{F}} \text{ for } \mathcal{F} \in \mathcal{F}_{\text{int},2}, \end{aligned}$$

are furthermore valid. An integration by parts then leads to the formula

$$\sum_{i=1}^N \int_{Q_i} \eta^{(i)} (\partial_3 \varphi_n^{(i)}) (\partial_2 \psi_n^{(i)}) dx = \sum_{i=1}^N \int_{Q_i} \eta^{(i)} (\partial_2 \varphi_n^{(i)}) (\partial_3 \psi_n^{(i)}) dx.$$

Taking limits, we infer the formula

$$\sum_{i=1}^N \int_{Q_i} \eta^{(i)} (\partial_3^2 u^{(i)}) (\partial_2^2 u^{(i)}) dx = \sum_{i=1}^N \int_{Q_i} \eta^{(i)} (\partial_2 \partial_3 u^{(i)})^2 dx \quad (21)$$

for the last term on the right hand side of (20). Similar reasoning also gives rise to the equations

$$\sum_{i=1}^N \eta^{(i)} \int_{Q_i} (\partial_1^2 u^{(i)}) (\partial_3^2 u^{(i)}) dx = \sum_{i=1}^N \eta^{(i)} \int_{Q_i} (\partial_1 \partial_3 u^{(i)})^2 dx, \quad (22)$$

$$\sum_{i=1}^N \eta^{(i)} \int_{Q_i} (\partial_1^2 u^{(i)}) (\partial_2^2 u^{(i)}) dx = \sum_{i=1}^N \eta^{(i)} \int_{Q_i} (\partial_1 \partial_2 u^{(i)})^2 dx. \quad (23)$$

Inserting (21)–(23) into (20), we finally arrive at the desired statement.  $\square$

**Lemma 3.2.** *Let  $u \in \mathcal{D}(\Delta_{0,\Gamma^*}) \cap PH^2(Q)$ , and  $\eta \in \{\varepsilon, \mu\}$  satisfy (2). The estimate*

$$\sum_{i=1}^N \eta^{(i)} \left\| u^{(i)} \right\|_{H^2(Q_i)} \leq C \sum_{i=1}^N \eta^{(i)} \left\| \Delta u^{(i)} \right\|_{L^2(Q_i)}$$

*is valid with a uniform constant  $C = C(\eta, Q) > 0$ .*

*Proof.* The interface and boundary conditions for  $u$  lead in an integration by parts to the relation

$$\sum_{i=1}^N \int_{Q_i} \eta^{(i)} |\nabla u^{(i)}|^2 dx \leq \sum_{i=1}^N \left\| u^{(i)} \right\|_{L^2(Q_i)} \eta^{(i)} \left\| \Delta u^{(i)} \right\|_{L^2(Q_i)}.$$

Since  $u$  is an element of  $H_{\Gamma^*}^1(Q)$ , the Poincaré inequality, see Theorem 13.6.9 in [50], leads to the inequalities

$$\sum_{i=1}^N \int_{Q_i} \eta^{(i)} |\nabla u^{(i)}|^2 dx \leq \left( \sum_{i=1}^N \eta^{(i)} \left\| \Delta u^{(i)} \right\|_{L^2(Q_i)}^2 \right)^{1/2} \left( \sum_{i=1}^N \eta^{(i)} \left\| u^{(i)} \right\|_{L^2(Q_i)}^2 \right)^{1/2}$$

$$\leq C_P \left( \sum_{i=1}^N \eta^{(i)} \left\| \Delta u^{(i)} \right\|_{L^2(Q_i)}^2 \right)^{1/2} \left( \sum_{i=1}^N \eta^{(i)} \left\| \nabla u^{(i)} \right\|_{L^2(Q_i)}^2 \right)^{1/2},$$

with a uniform constant  $C_P > 0$ . Similar reasoning also shows that the  $L^2$ -norm of  $u$  can be estimated by means of the  $L^2$ -norm of the Laplacian of  $u$ . In view of Lemma 3.1, we have derived the asserted statement.  $\square$

**3.2. Geometric constructions for the elliptic transmission problem.** Having classical elliptic regularity theory in mind, it is natural that the behavior of functions in  $\mathcal{D}(\Delta_{0,\Gamma^*})$  near interior edges is most important. This subsection introduces appropriate geometric objects for the analysis close to an interior edge. We denote the union of all edges of the interfaces by  $\mathcal{S}$ . The latter is also called skeleton.

**Definition 3.3.** Let  $e \subseteq \mathcal{S} \cap Q$  be an interior edge, and let  $Q_{\text{in},1}, \dots, Q_{\text{in},4}$  be the four adjacent cuboids to  $e$ . The material parameter  $\varepsilon$  has a **strong discontinuity** at  $e$  if  $\varepsilon|_{Q_{\text{in},1} \cup \dots \cup Q_{\text{in},4}}$  has a strictly larger value on one cuboid than on the remaining three.

In the following, we fix an interior edge  $e_{\text{in}} \subseteq \mathcal{S} \cap Q$ . After translation and scaling we assume the identity

$$e_{\text{in}} = \{(0, 0)\} \times [0, 1]. \quad (24)$$

We moreover assume that  $\varepsilon$  has a strong discontinuity at  $e_{\text{in}}$ , and fix four cuboids  $Q_{\text{in},1}, \dots, Q_{\text{in},4}$  having  $e_{\text{in}}$  as a common edge. We denote by  $\varepsilon_{\text{in}}$  the restriction of  $\varepsilon$  to the latter cuboids. The notation  $\varepsilon_{\text{in}}^{(i)}$  then refers to  $\varepsilon_{\text{in}}|_{Q_{\text{in},i}}$ . As  $\varepsilon$  satisfies (2), it then suffices to treat the configuration

$$\begin{aligned} \varepsilon_{\text{in}}^{(1)} = \varepsilon_{\text{in}}^{(2)} = \varepsilon_{\text{in}}^{(3)} < \varepsilon_{\text{in}}^{(4)} < \frac{1}{1 + 2\sqrt{2} - 2\sqrt{2} + \sqrt{2}} \varepsilon_{\text{in}}^{(1)}, \\ \frac{\varepsilon_{\text{in}}^{(4)}}{\varepsilon_{\text{in}}^{(1)}} \neq 1 - 2 \frac{\cos(\frac{15}{7}\pi) \sin(\frac{6}{7}\pi)}{\cos(\frac{12}{7}\pi) \sin(\frac{9}{7}\pi)}. \end{aligned} \quad (25)$$

All other cases (such as  $\varepsilon_{\text{in}}^{(1)}$  having the largest value) are then covered by symmetry.

As in [14, 15], we use a cylindrical coordinate system to deal with the behavior of functions in  $\mathcal{D}(\Delta_{0,\Gamma^*})$  near the interior edge  $e_{\text{in}}$ . To that end, we employ a cylinder  $\mathcal{Z}_{\text{in}}$  around  $e_{\text{in}}$  with radius 1, that touches the faces  $\Gamma_3^+$  and  $\Gamma_3^-$  of  $Q$ . After scaling, we can assume that  $\mathcal{Z}$  touches no interior edge (except  $e_{\text{in}}$ , of course). We set

$$\mathcal{Z}_{\text{in},i} := \mathcal{Z}_{\text{in}} \cap Q_{\text{in},i}, \quad i \in \{1, \dots, 4\},$$

and transfer the notion of restrictions of functions and piecewise regularity to this partition of  $\mathcal{Z}_{\text{in}}$ . Also  $\varepsilon_{\text{in}}$  is defined accordingly on  $\mathcal{Z}_{\text{in}}$ . The interfaces

$$\mathcal{F}_k^{\mathcal{Z}} := \overline{\mathcal{Z}_{\text{in},k}} \cap \overline{\mathcal{Z}_{\text{in},k+1}}, \quad \mathcal{F}_4^{\mathcal{Z}} := \overline{\mathcal{Z}_{\text{in},1}} \cap \overline{\mathcal{Z}_{\text{in},4}}, \quad k \in \{1, 2, 3\},$$

are furthermore employed. After rotating, the representation

$$\begin{aligned} \mathcal{Z}_{\text{in},i} &= \{(x, y, z) \mid (x, y) \in D_{\text{in},i}, z \in [0, 1]\}, \\ D_{\text{in},i} &= \{(r \cos \varphi, r \sin \varphi) \mid r \in (0, 1), \varphi \in I_{\text{in},i}\} \end{aligned} \quad (26)$$

is valid for  $i \in \{1, \dots, 4\}$ , using the intervals

$$I_{\text{in},1} := (0, \frac{\pi}{2}), \quad I_{\text{in},2} := (\frac{\pi}{2}, \pi), \quad I_{\text{in},3} := (\pi, \frac{3}{2}\pi), \quad I_{\text{in},4} := (\frac{3}{2}\pi, 2\pi).$$

By  $(r, \varphi, z)$  we throughout denote cylindrical coordinates. Note that  $D_{\text{in},1}, \dots, D_{\text{in},4}$  give rise to a partition of the unit disc  $D$ . The partition represents the regions, where  $\varepsilon_{\text{in}}$  is constant.



The interfaces near  $e_{\text{in}}$  are then represented by the interfaces between the subdomains  $D_{\text{in},i}$ . To address them, we set

$$\check{A}_{\text{in},k} := \partial D_{\text{in},k} \cap \partial D_{\text{in},k+1}, \quad \check{A}_{\text{in},4} := \partial D_{\text{in},1} \cap \partial D_{\text{in},4},$$

for  $i \in \{1, 2, 3\}$ . Also the definition of the jump  $\llbracket \cdot \rrbracket$  is transferred to the interfaces  $\mathcal{F}_i^Z$  and  $\check{A}_{\text{in},i}$ .

**3.3. Analysis of a Laplacian on the disc with transmission conditions.** In this Subsection, we use the notation from Subsection 3.2 without further notice. Our goal is a precise spectral knowledge of the two-dimensional Laplacian

$$\begin{aligned} (\check{L}_{\text{in}}\psi)^{(i)} &:= \Delta\psi^{(i)} \quad \text{on } D_{\text{in},i}, \\ \psi &\in \mathcal{D}(\check{L}_{\text{in}}) := \{\psi \in H_0^1(D) \mid \Delta\psi^{(i)} \in L^2(D_{\text{in},i}), \\ &\quad \llbracket \varepsilon_{\text{in}} \partial_\nu \psi \rrbracket_{\check{A}_{\text{in},i}} = 0 \text{ for } i \in \{1, \dots, 4\}\} \end{aligned} \quad (27)$$

with transmission conditions on the unit disc  $D$ . Note that the transmission conditions fit to the ones of the Laplacian  $\Delta_{0,\Gamma^*}$ , see (19).

The next statement provides the selfadjointness and invertibility of  $\check{L}_{\text{in}}$  on  $L^2(D)$ . Although the statement is well known, see [36] for instance, we provide the proof for the sake of a self-contained presentation.

**Lemma 3.4.** *Let  $\varepsilon_{\text{in}}$  satisfy (25). The operator  $\check{L}_{\text{in}}$  is invertible, has a compact resolvent, and is selfadjoint on  $L^2(D)$  with respect to the inner product*

$$(f, g)_{\varepsilon_{\text{in}}, D} := \int_D \varepsilon_{\text{in}} f g \, dx, \quad f, g \in L^2(D).$$

*Proof.* We first show that  $\check{L}_{\text{in}}$  is surjective. Let  $f \in L^2(D)$ . We consider the associated variational formulation

$$\sum_{i=1}^4 \int_{D_{\text{in},i}} \varepsilon_{\text{in}}^{(i)} \nabla u^{(i)} \nabla \varphi^{(i)} \, dx = \int_D f \varphi \, dx, \quad \varphi \in H_0^1(D). \quad (28)$$

The Lax-Milgram Lemma then yields a unique function  $u \in H_0^1(D)$  satisfying (28) for all  $\varphi \in H_0^1(D)$ . Inserting smooth test functions  $\varphi$  with compact support in  $D$  into (28), we further infer that  $\text{div}(\varepsilon_{\text{in}} \nabla u)$  is an element of  $L^2(D)$ . Altogether,  $u$  is an element of  $\mathcal{D}(\check{L}_{\text{in}})$ , and  $\check{L}_{\text{in}}$  is surjective. The injectivity of  $\check{L}_{\text{in}}$  can be established with similar reasoning.

Let now  $v, w \in \mathcal{D}(\check{L}_{\text{in}})$ . Employing the boundary and interface conditions from (27) in an integration by parts, we infer the relations

$$\begin{aligned} - \sum_{i=1}^4 \int_{D_{\text{in},i}} \varepsilon_{\text{in}}^{(i)} v^{(i)} \Delta w^{(i)} \, dx &= \sum_{i=1}^4 \int_{D_{\text{in},i}} \varepsilon_{\text{in}}^{(i)} \nabla v^{(i)} \nabla w^{(i)} \, dx \\ &= - \sum_{i=1}^4 \int_{D_{\text{in},i}} \varepsilon_{\text{in}}^{(i)} (\Delta v^{(i)}) w^{(i)} \, dx. \end{aligned}$$

This shows the symmetry of the operator  $\check{L}_{\text{in}}$ . The remaining asserted statements follow now from the closedness of  $\check{L}_{\text{in}}$ .  $\square$

The eigenvalue problem for  $\check{L}_{\text{in}}$  can be handled by transferring the reasoning in [49]. This means that we switch into polar coordinates  $(r, \varphi)$  on the disc  $D$ , and proceed with a separation of variables. Recall the intervals  $I_{\text{in},1}, \dots, I_{\text{in},4}$  from (26). As the coefficient  $\varepsilon_{\text{in}}$  depends only on the angle  $\varphi$ , it can be interpreted as a piecewise constant function on the union  $I_{\text{in},1} \cup \dots \cup I_{\text{in},4}$ .

We study the eigenvalue problem

$$\begin{aligned}
(\psi_{\text{in}}^{(i)})''(\varphi) &= -\kappa_{\text{in}}^2 \psi_{\text{in}}^{(i)}(\varphi) \quad \text{for } \varphi \in I_{\text{in},i}, \quad i \in \{1, \dots, 4\}, \\
\psi_{\text{in}}^{(1)}(0) &= \psi_{\text{in}}^{(4)}(2\pi), \quad \varepsilon_{\text{in}}^{(1)}(\psi_{\text{in}}^{(1)})'(0) = \varepsilon_{\text{in}}^{(4)}(\psi_{\text{in}}^{(4)})'(2\pi), \\
\psi_{\text{in}}^{(1)}\left(\frac{\pi}{2}\right) &= \psi_{\text{in}}^{(2)}\left(\frac{\pi}{2}\right), \quad (\psi_{\text{in}}^{(1)})'\left(\frac{\pi}{2}\right) = (\psi_{\text{in}}^{(2)})'\left(\frac{\pi}{2}\right), \\
\psi_{\text{in}}^{(2)}(\pi) &= \psi_{\text{in}}^{(3)}(\pi), \quad (\psi_{\text{in}}^{(2)})'(\pi) = (\psi_{\text{in}}^{(3)})'(\pi), \\
\psi_{\text{in}}^{(3)}\left(\frac{3}{2}\pi\right) &= \psi_{\text{in}}^{(4)}\left(\frac{3}{2}\pi\right), \quad \varepsilon_{\text{in}}^{(1)}(\psi_{\text{in}}^{(3)})'\left(\frac{3}{2}\pi\right) = \varepsilon_{\text{in}}^{(4)}(\psi_{\text{in}}^{(4)})'\left(\frac{3}{2}\pi\right).
\end{aligned} \tag{29}$$

For the formulation of the derivative transmission condition in the third and fourth line of (29), we employ the condition  $\varepsilon_{\text{in}}^{(1)} = \varepsilon_{\text{in}}^{(2)} = \varepsilon_{\text{in}}^{(3)}$ . By Lemma 4.2 in [37], system (29) has countably many eigenvalues  $0 = \kappa_{\text{in},0}^2 < \kappa_{\text{in},1}^2 \leq \dots \rightarrow \infty$ , and associated piecewise smooth eigenfunctions  $\psi_{\text{in},0}, \psi_{\text{in},1}, \dots$ . The latter form an orthonormal basis of  $L^2(0, 2\pi)$  with respect to the inner product

$$(f, g)_{\varepsilon_{\text{in}}} = \int_0^{2\pi} \varepsilon_{\text{in}} f g \, d\varphi, \quad f, g \in L^2(0, 2\pi).$$

We next derive a sharp lower bound for the square root of the first nonzero eigenvalue of (29). To that end, we employ the number  $\bar{\kappa}$  from (15).

**Lemma 3.5.** *Let  $\varepsilon_{\text{in}}$  satisfy (25). Then  $\bar{\kappa} \leq \kappa_{\text{in},1} < 1 \leq \kappa_{\text{in},2}$ .*

*Proof.* 1) We first assume that  $\lambda \in (0, 1)$  is an eigenvalue of (29) with an associated eigenfunction  $\psi \neq 0$ . The first line of (29) then implies the representation

$$\psi^{(i)}(\varphi) = a^{(i)} \cos(\sqrt{\lambda}\varphi) + b^{(i)} \sin(\sqrt{\lambda}\varphi), \quad a^{(i)}, b^{(i)} \in \mathbb{R}, \quad \varphi \in I_{\text{in},i} \tag{30}$$

for  $i \in \{1, \dots, 4\}$ . The third and fourth line of (29) lead to the relations  $a^{(1)} = a^{(2)} = a^{(3)}$  and  $b^{(1)} = b^{(2)} = b^{(3)}$ . The second and fifth lines of (29) further result in the formulas

$$a^{(1)} = a^{(4)} \cos(\sqrt{\lambda}2\pi) + b^{(4)} \sin(\sqrt{\lambda}2\pi), \tag{31}$$

$$b^{(1)} = \frac{\varepsilon_{\text{in}}^{(4)}}{\varepsilon_{\text{in}}^{(1)}} (-a^{(4)} \sin(\sqrt{\lambda}2\pi) + b^{(4)} \cos(\sqrt{\lambda}2\pi)),$$

$$\begin{aligned}
a^{(4)} \cos(\sqrt{\lambda}\frac{3}{2}\pi) + b^{(4)} \sin(\sqrt{\lambda}\frac{3}{2}\pi) &= a^{(1)} \cos(\sqrt{\lambda}\frac{3}{2}\pi) + b^{(1)} \sin(\sqrt{\lambda}\frac{3}{2}\pi) \\
&= a^{(4)} \cos(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\frac{3}{2}\pi) + b^{(4)} \sin(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\frac{3}{2}\pi) \\
&\quad - \frac{\varepsilon_{\text{in}}^{(4)}}{\varepsilon_{\text{in}}^{(1)}} a^{(4)} \sin(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\frac{3}{2}\pi) \\
&\quad + \frac{\varepsilon_{\text{in}}^{(4)}}{\varepsilon_{\text{in}}^{(1)}} b^{(4)} \cos(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\frac{3}{2}\pi).
\end{aligned} \tag{32}$$

Reformulating the last identity, the equation

$$\begin{aligned}
a^{(4)} & \left( \cos(\sqrt{\lambda}\frac{3}{2}\pi) - \cos(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\frac{3}{2}\pi) + \frac{\varepsilon_{\text{in}}^{(4)}}{\varepsilon_{\text{in}}^{(1)}} \sin(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\frac{3}{2}\pi) \right) \\
&= b^{(4)} \left( \sin(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\frac{3}{2}\pi) - \sin(\sqrt{\lambda}\frac{3}{2}\pi) + \frac{\varepsilon_{\text{in}}^{(4)}}{\varepsilon_{\text{in}}^{(1)}} \cos(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\frac{3}{2}\pi) \right) \\
&=: b^{(4)} A_1(\lambda)
\end{aligned} \tag{33}$$

is derived. Relating the derivative condition in the fifth line of (29) to (31), we conclude the formulas

$$\begin{aligned}
a^{(4)} & \left( -\varepsilon_{\text{in}}^{(4)} \sin(\sqrt{\lambda}\frac{3}{2}\pi) + \varepsilon_{\text{in}}^{(1)} \cos(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\frac{3}{2}\pi) + \varepsilon_{\text{in}}^{(4)} \sin(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\frac{3}{2}\pi) \right) \\
&= b^{(4)} \left( -\varepsilon_{\text{in}}^{(4)} \cos(\sqrt{\lambda}\frac{3}{2}\pi) - \varepsilon_{\text{in}}^{(1)} \sin(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\frac{3}{2}\pi) + \varepsilon_{\text{in}}^{(4)} \cos(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\frac{3}{2}\pi) \right) \\
&=: b^{(4)} A_2(\lambda).
\end{aligned} \tag{34}$$

2) We next show by contradiction that  $a^{(4)}$  is nonzero. Assume hence that  $a^{(4)} = 0$ . Equations (31) and (33) then imply that  $b^{(4)}$  is nonzero and that  $A_1(\lambda)$  vanishes. In the following, the numbers

$$\omega := \varepsilon_{\text{in}}^{(4)} / \varepsilon_{\text{in}}^{(1)} - 1, \quad \xi := \varepsilon_{\text{in}}^{(4)} - \varepsilon_{\text{in}}^{(1)}$$

are employed. Manipulating (33) by means of trigonometric identities, we infer the relations

$$\begin{aligned} 0 = A_1(\lambda) &= \sin(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\frac{3}{2}\pi) - \sin(\sqrt{\lambda}\frac{3}{2}\pi) + \left(\frac{\varepsilon_{\text{in}}^{(4)}}{\varepsilon_{\text{in}}^{(1)}} - 1\right) \cos(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\frac{3}{2}\pi) \\ &\quad + \cos(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\frac{3}{2}\pi) \\ &= \omega \cos(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\frac{3}{2}\pi) + 2 \cos(\sqrt{\lambda}\frac{5}{2}\pi) \sin(\sqrt{\lambda}\pi). \end{aligned}$$

Since the two summands in the last line have no common zeros on  $(0, 1)$ , the last line gives rise to the formula

$$\omega = \omega(\lambda) = -\frac{2 \cos(\sqrt{\lambda}\frac{5}{2}\pi) \sin(\sqrt{\lambda}\pi)}{\cos(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\frac{3}{2}\pi)}. \quad (35)$$

From (34) we further deduce that the expression  $A_2(\lambda)$  vanishes. Manipulating the defining relation for  $A_2(\lambda)$  by means of trigonometric identities, the equations

$$\begin{aligned} 0 &= -\frac{\varepsilon_{\text{in}}^{(4)}}{\varepsilon_{\text{in}}^{(1)}} \cos(\sqrt{\lambda}\frac{3}{2}\pi) - \sin(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\frac{3}{2}\pi) + \frac{\varepsilon_{\text{in}}^{(4)}}{\varepsilon_{\text{in}}^{(1)}} \cos(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\frac{3}{2}\pi) \\ &= -\frac{\varepsilon_{\text{in}}^{(4)}}{\varepsilon_{\text{in}}^{(1)}} \cos(\sqrt{\lambda}\frac{3}{2}\pi) - \sin(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\frac{3}{2}\pi) + \omega \cos(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\frac{3}{2}\pi) \\ &\quad + \cos(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\frac{3}{2}\pi) \\ &= -\frac{\varepsilon_{\text{in}}^{(4)}}{\varepsilon_{\text{in}}^{(1)}} \cos(\sqrt{\lambda}\frac{3}{2}\pi) + \cos(\sqrt{\lambda}\frac{7}{2}\pi) + \omega \cos(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\frac{3}{2}\pi) \\ &= -\omega \cos(\sqrt{\lambda}\frac{3}{2}\pi) - 2 \sin(\sqrt{\lambda}\frac{5}{2}\pi) \sin(\sqrt{\lambda}\pi) + \omega \cos(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\frac{3}{2}\pi) \end{aligned}$$

follow. The right hand side is next multiplied with the factor  $\cos(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\frac{3}{2}\pi)$ . Using also (35), we then infer the formulas

$$\begin{aligned} 0 &= -\omega \cos(\sqrt{\lambda}\frac{3}{2}\pi) \cos(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\frac{3}{2}\pi) \\ &\quad - 2 \sin(\sqrt{\lambda}\frac{5}{2}\pi) \sin(\sqrt{\lambda}\pi) \cos(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\frac{3}{2}\pi) \\ &\quad + \omega \cos^2(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\frac{3}{2}\pi) \sin(\sqrt{\lambda}\frac{3}{2}\pi) \\ &= 2 \cos(\sqrt{\lambda}\frac{5}{2}\pi) \sin(\sqrt{\lambda}\pi) \cos(\sqrt{\lambda}\frac{3}{2}\pi) \\ &\quad - 2 \sin(\sqrt{\lambda}\frac{5}{2}\pi) \sin(\sqrt{\lambda}\pi) \cos(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\frac{3}{2}\pi) \\ &\quad - 2 \cos(\sqrt{\lambda}\frac{5}{2}\pi) \sin(\sqrt{\lambda}\pi) \cos(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\frac{3}{2}\pi). \end{aligned} \quad (36)$$

We next divide (36) by  $\sin(\sqrt{\lambda}\pi) \neq 0$ , and we use besides the angle sum formula for cosine the trigonometric relations

$$\begin{aligned} \cos(\sqrt{\lambda}\frac{5}{2}\pi) \cos(\sqrt{\lambda}\frac{3}{2}\pi) &= \frac{1}{2}(\cos(\sqrt{\lambda}4\pi) + \cos(\sqrt{\lambda}\pi)), \\ \cos(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\pi) &= \frac{1}{2}(\cos(\sqrt{\lambda}3\pi) + \cos(\sqrt{\lambda}\pi)). \end{aligned}$$

In this way, we arrive at the equations

$$\begin{aligned} 0 &= \cos(\sqrt{\lambda}\frac{5}{2}\pi) \cos(\sqrt{\lambda}\frac{3}{2}\pi) - \sin(\sqrt{\lambda}\frac{5}{2}\pi) \cos(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\frac{3}{2}\pi) \\ &\quad - \cos(\sqrt{\lambda}\frac{5}{2}\pi) \cos(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\frac{3}{2}\pi) \\ &= \cos(\sqrt{\lambda}\frac{5}{2}\pi) \cos(\sqrt{\lambda}\frac{3}{2}\pi) - \cos(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\pi) \\ &= \frac{1}{2}(\cos(\sqrt{\lambda}4\pi) - \cos(\sqrt{\lambda}3\pi)) = -\sin(\sqrt{\lambda}\frac{7}{2}\pi) \sin(\sqrt{\lambda}\frac{1}{2}\pi). \end{aligned}$$

As  $\sin(\sqrt{\lambda}\frac{1}{2}\pi) \neq 0$ , we conclude that  $\lambda$  is an element of the set  $\{\frac{4}{49}, \frac{16}{49}, \frac{36}{49}\}$ . Plugging these values for  $\lambda$  into the formula (35) for  $\omega(\lambda)$ , we obtain, however, that  $\omega(4/49)$  and  $\omega(16/49)$  are negative (thus contradicting (25)), while  $\omega(36/49)$  is excluded in (25). We conclude that  $a^{(4)}$  is different from zero.

3) Taking the results of part 2) into account, we can assume that  $a^{(4)} = 1$ . In the following, we distinguish the cases of  $A_2(\lambda)$  being zero and nonzero, see (34).

3.i) Suppose  $A_2(\lambda) = 0$ , and proceed similar to case 2). The formula for  $A_2(\lambda)$  in (34) is divided by  $\varepsilon_{\text{in}}^{(4)}$ , and the number

$$\omega_0 := 1 + \varepsilon_{\text{in}}^{(1)}/\varepsilon_{\text{in}}^{(4)}$$

is introduced. By means of trigonometric identities, the equations

$$\begin{aligned} 0 &= -\cos(\sqrt{\lambda}\frac{3}{2}\pi) - \frac{\varepsilon_{\text{in}}^{(1)}}{\varepsilon_{\text{in}}^{(4)}} \sin(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\frac{3}{2}\pi) + \cos(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\frac{3}{2}\pi) \\ &= -\cos(\sqrt{\lambda}\frac{3}{2}\pi) - \omega_0 \sin(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\frac{3}{2}\pi) + \cos(\sqrt{\lambda}\frac{1}{2}\pi) \\ &= 2 \sin(\sqrt{\lambda}\pi) \sin(\sqrt{\lambda}\frac{1}{2}\pi) - \omega_0 \sin(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\frac{3}{2}\pi) \end{aligned} \quad (37)$$

are then obtained. Since the summands on the right hand side of (37) have no common zero on  $(0, 1)$ , the formula

$$\omega_0 = \frac{2 \sin(\sqrt{\lambda}\pi) \sin(\sqrt{\lambda}\frac{1}{2}\pi)}{\sin(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\frac{3}{2}\pi)} \quad (38)$$

follows. Treating the left hand side of (34) in the same way, we further arrive at

$$0 = -2 \cos(\sqrt{\lambda}\pi) \sin(\sqrt{\lambda}\frac{1}{2}\pi) + \omega_0 \cos(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\frac{3}{2}\pi).$$

Multiplying the right hand side by  $\sin(\sqrt{\lambda}2\pi)$  and inserting (38), we deduce

$$0 = -2 \sin(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\pi) \sin(\sqrt{\lambda}\frac{1}{2}\pi) + 2 \cos(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\pi) \sin(\sqrt{\lambda}\frac{1}{2}\pi).$$

Dividing by  $\sin(\sqrt{\lambda}\frac{1}{2}\pi)$  and using the sum formula for sine, we arrive at the identity  $0 = \sin(\sqrt{\lambda}\pi)$ . Since  $\lambda$  is assumed to belong to  $(0, 1)$ , this is a contradiction.

3.ii) In consideration of the results in 3.i), we infer that  $A_2(\lambda)$  has to be nonzero. Dividing in (34) by  $A_2(\lambda)$ , and using trigonometric identities as well as the number  $\xi = \varepsilon_{\text{in}}^{(4)} - \varepsilon_{\text{in}}^{(1)}$ , we then obtain the equations

$$\begin{aligned} b^{(4)} &= \frac{-\varepsilon_{\text{in}}^{(4)} \sin(\sqrt{\lambda}\frac{3}{2}\pi) + \varepsilon_{\text{in}}^{(1)} \cos(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\frac{3}{2}\pi) + \varepsilon_{\text{in}}^{(4)} \sin(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\frac{3}{2}\pi)}{-\varepsilon_{\text{in}}^{(4)} \cos(\sqrt{\lambda}\frac{3}{2}\pi) - \varepsilon_{\text{in}}^{(1)} \sin(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\frac{3}{2}\pi) + \varepsilon_{\text{in}}^{(4)} \cos(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\frac{3}{2}\pi)} \\ &= \frac{-\varepsilon_{\text{in}}^{(4)} \sin(\sqrt{\lambda}\frac{3}{2}\pi) + \varepsilon_{\text{in}}^{(4)} \sin(\sqrt{\lambda}\frac{7}{2}\pi) - \xi \cos(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\frac{3}{2}\pi)}{-\varepsilon_{\text{in}}^{(4)} \cos(\sqrt{\lambda}\frac{3}{2}\pi) + \varepsilon_{\text{in}}^{(4)} \cos(\sqrt{\lambda}\frac{7}{2}\pi) + \xi \sin(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\frac{3}{2}\pi)} \\ &= -\frac{2\varepsilon_{\text{in}}^{(4)} \cos(\sqrt{\lambda}\frac{5}{2}\pi) \sin(\sqrt{\lambda}\pi) - \xi \cos(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\frac{3}{2}\pi)}{2\varepsilon_{\text{in}}^{(4)} \sin(\sqrt{\lambda}\frac{5}{2}\pi) \sin(\sqrt{\lambda}\pi) - \xi \sin(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\frac{3}{2}\pi)}. \end{aligned} \quad (39)$$

We next reformulate (33) algebraically with the number  $\omega = \varepsilon_{\text{in}}^{(4)}/\varepsilon_{\text{in}}^{(1)} - 1$  and the relation  $a^{(4)} = 1$ . We derive the identities

$$\begin{aligned} 0 &= \cos(\sqrt{\lambda}\frac{3}{2}\pi) - \cos(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\frac{3}{2}\pi) + \frac{\varepsilon_{\text{in}}^{(4)}}{\varepsilon_{\text{in}}^{(1)}} \sin(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\frac{3}{2}\pi) \\ &\quad - b^{(4)} \left( \sin(\sqrt{\lambda}2\pi) \cos(\sqrt{\lambda}\frac{3}{2}\pi) - \sin(\sqrt{\lambda}\frac{3}{2}\pi) + \frac{\varepsilon_{\text{in}}^{(4)}}{\varepsilon_{\text{in}}^{(1)}} \cos(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\frac{3}{2}\pi) \right) \\ &= \cos(\sqrt{\lambda}\frac{3}{2}\pi) - \cos(\sqrt{\lambda}\frac{7}{2}\pi) + \omega \sin(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\frac{3}{2}\pi) \\ &\quad - b^{(4)} \left( -\sin(\sqrt{\lambda}\frac{3}{2}\pi) + \sin(\sqrt{\lambda}\frac{7}{2}\pi) + \omega \cos(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\frac{3}{2}\pi) \right) \\ &= 2 \sin(\sqrt{\lambda}\frac{5}{2}\pi) \sin(\sqrt{\lambda}\pi) + \omega \sin(\sqrt{\lambda}2\pi) \sin(\sqrt{\lambda}\frac{3}{2}\pi) \end{aligned}$$

$$- b^{(4)} \left( 2 \cos(\sqrt{\lambda} \frac{5}{2} \pi) \sin(\sqrt{\lambda} \pi) + \omega \cos(\sqrt{\lambda} 2\pi) \sin(\sqrt{\lambda} \frac{3}{2} \pi) \right). \quad (40)$$

The representation (39) for  $b^{(4)}$  is next inserted into the right hand side of (40), and all arising expressions are multiplied with the denominator in (39). In this way, we deduce the equations

$$\begin{aligned} 0 &= 4\varepsilon_{\text{in}}^{(4)} \sin^2(\sqrt{\lambda} \frac{5}{2} \pi) \sin^2(\sqrt{\lambda} \pi) + 2\omega\varepsilon_{\text{in}}^{(4)} \sin(\sqrt{\lambda} 2\pi) \sin(\sqrt{\lambda} \frac{3}{2} \pi) \sin(\sqrt{\lambda} \frac{5}{2} \pi) \sin(\sqrt{\lambda} \pi) \\ &\quad - 2\xi \sin(\sqrt{\lambda} \frac{5}{2} \pi) \sin(\sqrt{\lambda} \pi) \sin(\sqrt{\lambda} 2\pi) \sin(\sqrt{\lambda} \frac{3}{2} \pi) - \omega\xi \sin^2(\sqrt{\lambda} 2\pi) \sin^2(\sqrt{\lambda} \frac{3}{2} \pi) \\ &\quad + 4\varepsilon_{\text{in}}^{(4)} \cos^2(\sqrt{\lambda} \frac{5}{2} \pi) \sin^2(\sqrt{\lambda} \pi) - 2\xi \cos(\sqrt{\lambda} \frac{5}{2} \pi) \sin(\sqrt{\lambda} \pi) \cos(\sqrt{\lambda} 2\pi) \sin(\sqrt{\lambda} \frac{3}{2} \pi) \\ &\quad + 2\omega\varepsilon_{\text{in}}^{(4)} \cos(\sqrt{\lambda} 2\pi) \sin(\sqrt{\lambda} \frac{3}{2} \pi) \cos(\sqrt{\lambda} \frac{5}{2} \pi) \sin(\sqrt{\lambda} \pi) \\ &\quad - \omega\xi \cos^2(\sqrt{\lambda} 2\pi) \sin^2(\sqrt{\lambda} \frac{3}{2} \pi) \end{aligned} \quad (41)$$

$$\begin{aligned} &= 4\varepsilon_{\text{in}}^{(4)} \sin^2(\sqrt{\lambda} \pi) + 2(\omega\varepsilon_{\text{in}}^{(4)} - \xi) \cos(\sqrt{\lambda} \frac{1}{2} \pi) \sin(\sqrt{\lambda} \pi) \sin(\sqrt{\lambda} \frac{3}{2} \pi) \\ &\quad - \omega\xi \sin^2(\sqrt{\lambda} \frac{3}{2} \pi). \end{aligned} \quad (42)$$

To further simplify the expressions on the right hand side, we use the formulas  $\cos(\sqrt{\lambda} \frac{1}{2} \pi) \sin(\sqrt{\lambda} \pi) = \frac{1}{2} (\sin(\sqrt{\lambda} \frac{1}{2} \pi) + \sin(\sqrt{\lambda} \frac{3}{2} \pi))$ ,

$$\begin{aligned} \omega\varepsilon_{\text{in}}^{(4)} - \xi &= \frac{(\varepsilon_{\text{in}}^{(4)})^2}{\varepsilon_{\text{in}}^{(1)}} - 2\varepsilon_{\text{in}}^{(4)} + \varepsilon_{\text{in}}^{(1)} = \frac{(\varepsilon_{\text{in}}^{(4)} - \varepsilon_{\text{in}}^{(1)})^2}{\varepsilon_{\text{in}}^{(1)}} = \left( \frac{\varepsilon_{\text{in}}^{(4)}}{\varepsilon_{\text{in}}^{(1)}} - 1 \right) (\varepsilon_{\text{in}}^{(4)} - \varepsilon_{\text{in}}^{(1)}) \\ &= \omega\xi. \end{aligned}$$

Inserting these relations in (42), we arrive at the identities

$$\begin{aligned} 0 &= 4\varepsilon_{\text{in}}^{(4)} \sin^2(\sqrt{\lambda} \pi) + 2\omega\xi \cos(\sqrt{\lambda} \frac{1}{2} \pi) \sin(\sqrt{\lambda} \pi) \sin(\sqrt{\lambda} \frac{3}{2} \pi) - \omega\xi \sin^2(\sqrt{\lambda} \frac{3}{2} \pi) \\ &= 4\varepsilon_{\text{in}}^{(4)} \sin^2(\sqrt{\lambda} \pi) + \omega\xi \sin(\sqrt{\lambda} \frac{1}{2} \pi) \sin(\sqrt{\lambda} \frac{3}{2} \pi). \end{aligned}$$

As the two summands on the right hand side have no common zeros on  $(0, 1)$ , we conclude the representation

$$\frac{(\varepsilon_{\text{in}}^{(4)} - \varepsilon_{\text{in}}^{(1)})^2}{\varepsilon_{\text{in}}^{(1)}} = \omega\xi = - \frac{4\varepsilon_{\text{in}}^{(4)} \sin^2(\sqrt{\lambda} \pi)}{\sin(\sqrt{\lambda} \frac{1}{2} \pi) \sin(\sqrt{\lambda} \frac{3}{2} \pi)}. \quad (43)$$

Note that  $\lambda$  is uniquely determined by (43), and that it is greater or equal than  $\bar{\kappa}^2$ , see (15). Altogether, there is at most one eigenvalue of (29) in  $(0, 1)$ , and if it exists, it is greater or equal than  $\bar{\kappa}^2$ . The associated eigenspace is furthermore one-dimensional.

4) Let  $\lambda \in (0, 1)$  satisfy (43). Then  $\sqrt{\lambda} \geq \bar{\kappa} > \frac{3}{4}$ , see (16). It remains to show that  $\lambda$  is indeed an eigenvalue of (29). We first prove that the denominator in (39), being  $A_2(\lambda)$ , is nonzero. So, assume  $A_2(\lambda)$  was zero. By definition of  $\lambda$ , (41) is still valid. Rewriting (41) in product formula, we then deduce the identities

$$\begin{aligned} 0 &= (2 \sin(\sqrt{\lambda} \frac{5}{2} \pi) \sin(\sqrt{\lambda} \pi) + \omega \sin(\sqrt{\lambda} 2\pi) \sin(\sqrt{\lambda} \frac{3}{2} \pi)) \\ &\quad \cdot (2\varepsilon_{\text{in}}^{(4)} \sin(\sqrt{\lambda} \frac{5}{2} \pi) \sin(\sqrt{\lambda} \pi) - \xi \sin(\sqrt{\lambda} 2\pi) \sin(\sqrt{\lambda} \frac{3}{2} \pi)) \\ &\quad + (2 \cos(\sqrt{\lambda} \frac{5}{2} \pi) \sin(\sqrt{\lambda} \pi) + \omega \cos(\sqrt{\lambda} 2\pi) \sin(\sqrt{\lambda} \frac{3}{2} \pi)) \\ &\quad \cdot (2\varepsilon_{\text{in}}^{(4)} \cos(\sqrt{\lambda} \frac{5}{2} \pi) \sin(\sqrt{\lambda} \pi) - \xi \cos(\sqrt{\lambda} 2\pi) \sin(\sqrt{\lambda} \frac{3}{2} \pi)) \\ &= (2 \cos(\sqrt{\lambda} \frac{5}{2} \pi) \sin(\sqrt{\lambda} \pi) + \omega \cos(\sqrt{\lambda} 2\pi) \sin(\sqrt{\lambda} \frac{3}{2} \pi)) \\ &\quad \cdot (2\varepsilon_{\text{in}}^{(4)} \cos(\sqrt{\lambda} \frac{5}{2} \pi) \sin(\sqrt{\lambda} \pi) - \xi \cos(\sqrt{\lambda} 2\pi) \sin(\sqrt{\lambda} \frac{3}{2} \pi)). \end{aligned}$$

Since the second factor on the right hand side is positive (as  $\sqrt{\lambda} > \frac{3}{4}$ ), we conclude that the first one has to be zero. This expression is, however, equal to  $A_1(\lambda)$ , see (33). (This can be seen by reversing the reasoning in (40)). Now the arguments in part 2) lead to a contradiction. This means that  $A_2(\lambda)$  is nonzero.

We then define  $b^{(4)}$  according to (39), set  $a^{(4)} = 1$ , define  $a^{(1)} = a^{(2)} = a^{(3)}$ ,  $b^{(1)} = b^{(2)} = b^{(3)}$  by (31), and choose  $\psi$  as in (30).

Altogether, only the required transmission conditions need to be validated for  $\psi$ . By definition of  $b^{(4)}$ , formula (34) is satisfied. Due to (43), identity (42) is also true. Dividing the right hand side of (42) by  $A_2(\lambda)$ , we then conclude that (40) holds. This finally means that also the first transmission condition (33) is fulfilled.  $\square$

In the following, we construct eigenfunctions for the Dirichlet Laplacian  $\check{L}_{\text{in}}$  from (27). This is done by means of Bessel functions and the eigenvalues of (29). The spectral analysis is concluded in Lemma 3.6. We recall for  $\nu \geq 0$  the Bessel function  $J_\nu$  of order  $\nu$  as

$$J_\nu(t) := \sum_{j=0}^{\infty} (-1)^j \frac{\left(\frac{t}{2}\right)^{\nu+2j}}{j! \Gamma(\nu+j+1)}, \quad t \geq 0, \quad (44)$$

involving the Gamma function  $\Gamma(\cdot)$ . The mapping  $J_\nu$  is smooth on  $(0, \infty)$ , see for example the Theorem in Section 5.5.1 of [49]. The positive zeros of  $J_\nu$  are denoted by  $0 < \mu_1^{(\nu)} < \mu_2^{(\nu)} < \dots \rightarrow \infty$ .

Recall that  $\kappa_{\text{in},l}^2$  is an eigenvalue of system (29) with associated eigenfunction  $\psi_{\text{in},l}$ . We define the numbers

$$\check{\lambda}_{k,l}^{\text{in}} := (\mu_k^{(\kappa_{\text{in},l})})^2, \quad k \in \mathbb{N}, l \in \mathbb{N}_0,$$

and the associated mappings

$$\Psi_{k,l}^{\text{in}}(r, \varphi) := J_{\kappa_{\text{in},l}}(\sqrt{\check{\lambda}_{k,l}^{\text{in}}} r) \psi_{\text{in},l}(\varphi), \quad r \in (0, 1), \varphi \in (0, 2\pi), \quad (45)$$

with  $k \in \mathbb{N}$  and  $l \in \mathbb{N}_0$ . Due to the choice of  $\check{\lambda}_{k,l}^{\text{in}}$ , the function  $\Psi_{k,l}^{\text{in}}$  vanishes on the boundary of  $D$ . We further note that the functions  $\Psi_{k,1}^{\text{in}}$  have second weak derivatives with singularities at  $r = 0$ . This eventually causes the weaker regularity statement than  $H^2$  in Proposition 2. These singular functions are hence incorporated separately by means of the spaces

$$\check{M}_{\text{in}} := \text{span}\{\Psi_{k,l}^{\text{in}} \mid k \in \mathbb{N}, l \in \mathbb{N}_0 \setminus \{1\}\}, \quad \check{N}_{\text{in}} := \text{span}\{\Psi_{k,1}^{\text{in}} \mid k \in \mathbb{N}\}. \quad (46)$$

In the next lemma, we derive useful spectral properties of the Laplace operator  $\check{L}_{\text{in}}$ . The proof employs ideas from Theorem 2 in Section 5.5.2, Lemma 1 in Section 6.4.2, and Theorem 1 in Section 6.4.2 of [49]. Recall for the statement definition (27).

**Lemma 3.6.** *Let  $\varepsilon_{\text{in}}$  satisfy (25).*

a) *The family  $\{\Psi_{k,l}^{\text{in}} \mid k \in \mathbb{N}, l \in \mathbb{N}_0\}$  is an orthonormal basis of  $L^2(D)$  with respect to the inner product  $(\cdot, \cdot)_{\varepsilon_{\text{in}}, D}$  from Lemma 3.4.*

b) *The sets  $\check{M}_{\text{in}}$  and  $\check{N}_{\text{in}}$  are contained in the domain  $\mathcal{D}(\check{L}_{\text{in}})$ . Furthermore,  $\check{M}_{\text{in}}$  is a subspace of  $PH^2(D)$ . The eigenvector relation*

$$\check{L}_{\text{in}} \Psi_{k,l}^{\text{in}} = -\check{\lambda}_{k,l}^{\text{in}} \Psi_{k,l}^{\text{in}}, \quad k \in \mathbb{N}, l \in \mathbb{N}_0,$$

*is satisfied.*

*Proof.* a) The asserted orthogonality follows by combining the choice of the functions  $\{\psi_{\text{in},l} \mid l \in \mathbb{N}_0\}$  with Theorem 2 in Section 5.5.2 of [49]. The completeness of the system  $\{\Psi_{k,l}^{\text{in}} \mid k \in \mathbb{N}, l \in \mathbb{N}_0\}$  can be concluded in the same manner as in the proof of Lemma 1 in Section 6.4.2 of [49], now employing the completeness of  $\{\psi_{\text{in},l} \mid l \in \mathbb{N}_0\}$  in  $L^2(-\frac{\pi}{2}, \frac{3}{2}\pi) \cong L^2(0, 2\pi)$ .

b.i) Let  $k \in \mathbb{N}$  and  $l \in \mathbb{N}_0$ . We first focus on the transmission and boundary conditions. The mapping  $\Psi_{k,l}^{\text{in}}$  satisfies the required transmission conditions as a consequence of the choice of  $\psi_{\text{in},l}$ , see (29). The function  $\Psi_{k,l}^{\text{in}}$  furthermore satisfies

homogeneous Dirichlet boundary conditions on  $\partial D$ , as  $\sqrt{\check{\lambda}_{k,l}^{\text{in}}} = \mu_k^{(\kappa_{\text{in},l})}$  is a zero of  $J_{\kappa_{\text{in},l}}$ .

We next show that every function  $\Psi_{k,l}^{\text{in}}$  is better than  $H^1$ -regular. Denote by  $(r, \varphi)$  polar coordinates, and let  $i \in \{1, \dots, 4\}$ . Since  $\psi_{\text{in},l}$  solves (29), it has the representation

$$\psi_{\text{in},l}^{(i)}(\varphi) = a_l^{(i)} \cos(\kappa_{\text{in},l}\varphi) + b_l^{(i)} \sin(\kappa_{\text{in},l}\varphi), \quad \varphi \in I_{\text{in},i}, \quad (47)$$

with real numbers  $a_l^{(i)}, b_l^{(i)}$ . We then infer the formula

$$\begin{aligned} (\Psi_{k,l}^{\text{in}})^{(i)}(r, \varphi) &= 2^{-\kappa_{\text{in},l}} (a_l^{(i)} \cos(\kappa_{\text{in},l}\varphi) + b_l^{(i)} \sin(\kappa_{\text{in},l}\varphi)) r^{\kappa_{\text{in},l}} \\ &\quad \cdot \sum_{j=0}^{\infty} (-1)^j (\mu_k^{(\kappa_{\text{in},l})})^{\kappa_{\text{in},l}+2j} \frac{\left(\frac{r^2}{4}\right)^j}{j! \Gamma(\nu + j + 1)}. \end{aligned}$$

As the series on the right hand side converges uniformly in  $r \in [0, 1]$ , the function  $\Psi_{k,l}^{\text{in}}$  is as regular as the function  $(a_l^{(i)} \cos(\kappa_{\text{in},l}\varphi) + b_l^{(i)} \sin(\kappa_{\text{in},l}\varphi)) r^{\kappa_{\text{in},l}}$ . If  $l = 0$ , this means that  $\Psi_{k,0}^{\text{in}}$  is piecewise smooth. In case  $l \in \mathbb{N}$ ,  $\Psi_{k,l}^{\text{in}}$  then belongs to the space  $H^{1+\kappa}(D_{\text{in},i})$  for every  $\kappa < \min\{1, \kappa_{\text{in},l}\}$ , see [4, 5, 3, 55] for instance.

It hence suffices to show the stated eigenvalue-eigenvector relations to conclude that  $\check{M}_{\text{in}}$  and  $\check{N}_{\text{in}}$  are contained in  $\mathcal{D}(\check{L}_{\text{in}})$ . Applying Theorem 2 in Section 5.5.2 of [49] together with the choice of  $\psi_{\text{in},l}$ , see (29), we arrive at the desired relations

$$\begin{aligned} (\check{L}_{\text{in}} \Psi_{k,l}^{\text{in}})^{(i)}(r, \varphi) &= \Delta(\Psi_{k,l}^{\text{in}})^{(i)}(r, \varphi) \\ &= \frac{1}{r} \partial_r (r \partial_r J_{\kappa_{\text{in},l}}(\sqrt{\check{\lambda}_{k,l}^{\text{in}}} r)) \psi_{\text{in},l}^{(i)}(\varphi) + \frac{1}{r^2} (\partial_\varphi^2 \psi_{\text{in},l}^{(i)}(\varphi)) J_{\kappa_{\text{in},l}}(\sqrt{\check{\lambda}_{k,l}^{\text{in}}} r) \\ &= \left( \frac{1}{r} \partial_r (r \partial_r J_{\kappa_{\text{in},l}}(\sqrt{\check{\lambda}_{k,l}^{\text{in}}} r)) - \frac{1}{r^2} \kappa_{\text{in},l}^2 J_{\kappa_{\text{in},l}}(\sqrt{\check{\lambda}_{k,l}^{\text{in}}} r) \right) \psi_{\text{in},l}^{(i)}(\varphi) \\ &= -\check{\lambda}_{k,l}^{\text{in}} \psi_{\text{in},l}^{(i)}(\varphi) J_{\kappa_{\text{in},l}}(\sqrt{\check{\lambda}_{k,l}^{\text{in}}} r) = -\check{\lambda}_{k,l}^{\text{in}} (\Psi_{k,l}^{\text{in}})^{(i)}(r, \varphi). \end{aligned}$$

b.ii) It remains to show that every function in  $\check{M}_{\text{in}}$  is at least piecewise  $H^2$ -regular. Let  $l \in \mathbb{N}$  with  $\kappa_{\text{in},l} = 1$ , where  $\kappa_{\text{in},l}^2$  is an eigenvalue of (29). Note that there might be no  $l \in \mathbb{N}$  with this property. With formula (47) and cartesian coordinates, we arrive at the formula

$$\begin{aligned} (\Psi_{k,l}^{\text{in}})^{(i)}(r, \varphi) &= \frac{1}{2} (a_l^{(i)} x_1 + b_l^{(i)} x_2) \sum_{j=0}^{\infty} (-1)^j (\mu_k^{(1)})^{2j+1} \frac{\left(\frac{x_1^2 + x_2^2}{4}\right)^j}{j! \Gamma(j+2)} \\ &=: (\Phi_{k,l}^{\text{in}})^{(i)}(x_1, x_2) \end{aligned}$$

for  $(x_1, x_2) \in D_{\text{in},i}$ . As a result of the uniform convergence of the series and its derivatives, we conclude that  $(\Phi_{k,l}^{\text{in}})^{(i)}$  is smooth on  $\overline{D_{\text{in},i}}$ . This means that  $\Phi_{k,l}^{\text{in}}$  and consequently also  $\Psi_{k,l}^{\text{in}}$  are elements of  $PH^2(D)$ . Similar reasoning shows that also the functions  $\Psi_{k,0}^{\text{in}}$  belong to  $PH^2(D)$ .

It remains to consider the case  $l \in \mathbb{N}$  with  $\kappa_{\text{in},l} > 1$  (the case  $\kappa_{\text{in},l} < 1$  is excluded by definition of the space  $\check{M}_{\text{in}}$ ). Then,  $\Psi_{k,l}^{\text{in}}$  has the representation

$$\Psi_{k,l}^{\text{in}}(r, \varphi) = \frac{1}{2^{\kappa_{\text{in},l}}} \psi_{\text{in},l}(\varphi) r^{\kappa_{\text{in},l}} \sum_{j=0}^{\infty} (-1)^j \frac{(\sqrt{\check{\lambda}_{k,l}^{\text{in}}})^{2j+\kappa_{\text{in},l}} \left(\frac{r}{2}\right)^{2j}}{j! \Gamma(\kappa_{\text{in},l} + j + 1)}.$$

Taking the uniform convergence of the series and its derivatives as well as the piecewise smoothness of  $\psi_{\text{in},l}$  into account, the function  $\Psi_{k,l}^{\text{in}}$  satisfies the estimate

$$\int_0^1 \int_{I_{\text{in},i}} \left( \frac{1}{r} |\partial_r (\Psi_{k,l}^{\text{in}})^{(i)}|^2 + r |\partial_r^2 (\Psi_{k,l}^{\text{in}})^{(i)}|^2 + \frac{1}{r} |\partial_r \partial_\varphi (\Psi_{k,l}^{\text{in}})^{(i)}|^2 \right)$$

$$+ \frac{1}{r^3} |\partial_\varphi(\Psi_{k,l}^{\text{in}})^{(i)}|^2 + \frac{1}{r^3} |\partial_\varphi^2(\Psi_{k,l}^{\text{in}})^{(i)}|^2) d\varphi dr < \infty,$$

proving that  $\Psi_{k,l}^{\text{in}}$  is an element of  $PH^2(D)$ . Altogether, the space  $\check{M}_{\text{in}}$  is contained in  $PH^2(D)$ .  $\square$

We next present a useful a-priori energy estimate for the Laplacian  $\check{L}_{\text{in}}$  for functions in  $\check{M}_{\text{in}}$ . The estimate is established in Lemma 2.2 and the subsequent Remark in [36]. The statement uses the space

$$\check{Y}_{\text{in}} := \left\{ \psi \in \bigcap_{i=1}^4 H^2(D_{\text{in},i}) \mid \psi = 0 \text{ on } \partial D, \right. \\ \left. \llbracket \psi \rrbracket_{\check{A}_{\text{in},k}} = 0 = \llbracket \varepsilon_{\text{in}} \partial_\nu \psi \rrbracket_{\check{A}_{\text{in},k}} \text{ for } k \in \{1, \dots, 4\} \right\}. \quad (48)$$

**Lemma 3.7.** *Let  $\varepsilon_{\text{in}}$  satisfy (25). There is a constant  $C = C(\varepsilon_{\text{in}}) > 0$  with*

$$\|\psi\|_{PH^2(D)} \leq C (\|\psi\|_{L^2(D)} + \left( \sum_{i=1}^4 \|\Delta \psi^{(i)}\|_{L^2(D_{\text{in},i})}^2 \right)^{1/2})$$

for  $\psi \in \check{Y}_{\text{in}}$ .

Note that Lemma 3.6 shows that  $\check{M}_{\text{in}}$  is a subspace of  $\check{Y}_{\text{in}}$ . Standard reasoning, see (67) for instance, then leads to the inequality

$$\|\psi\|_{PH^2(D)} \leq C \|\check{L}_{\text{in}} \psi\|_{L^2(D)}, \quad \psi \in \check{M}_{\text{in}}, \quad (49)$$

with a uniform constant  $C = C(\varepsilon_{\text{in}}) > 0$ .

To derive a counterpart of Lemma 3.7 for functions in the space  $\check{N}_{\text{in}}$  from (46), we transfer ideas by Kellogg in the next two lemmas to our setting, see Theorem 5.2 and Lemma 5.6 in [36].

Let  $\nu \in (1/2, 1)$  and  $f \in C([0, 1])$ . In a first step, a norm estimate is derived for the one-dimensional equation

$$r^{1/2} \psi''(r) + r^{-1/2} \psi'(r) - \nu^2 r^{-3/2} \psi(r) = f(r), \quad r \in (0, 1), \quad \psi \in L^2(0, 1). \quad (50)$$

We note that the expression on the left hand side corresponds to the radial part of the operator Laplacian  $\check{L}_{\text{in}}$ , acting on functions in  $\check{N}_{\text{in}}$ . The inequality provided by the next lemma will thus be crucial for an energy estimate for  $\check{L}_{\text{in}}$ , see the proof of Lemma 3.9.

**Lemma 3.8.** *Let  $\kappa > 2(1 - \nu)$  with parameter  $\nu \in (1/2, 1)$  from (50). The solution  $\psi$  of (50) with boundary conditions  $\psi(0) = \psi(1) = 0$  satisfies the inequality*

$$\int_0^1 r^\kappa (r^{-1} (\psi')^2 + r^{-3} \psi^2) dr \leq C \int_0^1 f^2 dr$$

with a uniform constant  $C = C(\kappa, \nu) > 0$ .

*Proof.* We first note that there is at most one solution  $\psi \in L^2(0, 1)$  to (50). (The homogeneous counterpart has the fundamental system  $\{r^\nu, r^{-\nu}\}$ . The integrability constraint rules out the latter basis function. As a result, the endpoint condition at  $r = 1$  suffices to ensure uniqueness.) Using the technique of variation of parameters, we obtain the solution formula

$$\psi(r) = \alpha r^\nu + \frac{1}{2\nu} r^\nu \int_0^r t^{1/2-\nu} f(t) dt - \frac{1}{2\nu} r^{-\nu} \int_0^r t^{1/2+\nu} f(t) dt =: \sum_{j=1}^3 \phi_j(r)$$

for  $r \in (0, 1)$ , involving the number

$$\alpha := -\frac{1}{2\nu} \int_0^1 (t^{1/2-\nu} - t^{1/2+\nu}) f(t) dt.$$



We establish the desired estimate separately for the three functions  $\phi_1, \phi_2, \phi_3$ . The choice of  $\kappa > 2(1 - \nu)$  first implies the identities

$$\int_0^1 \frac{1}{r^{3-\kappa}} \phi_1^2 dr = \int_0^1 \frac{\alpha^2}{r^{3-2\nu-\kappa}} dr = \frac{\alpha^2}{\kappa + 2\nu - 2}.$$

Applying the Cauchy-Schwarz inequality to the defining formula for  $\alpha$ , the relations

$$\alpha^2 \leq \frac{1}{4\nu^2} \int_0^1 (t^{1-2\nu} - 2t + t^{1+2\nu}) dt \int_0^1 f^2 dt =: \frac{C_1(\nu)}{\nu^2} \int_0^1 f^2 dt \quad (51)$$

are also obtained. This implies the first result

$$\int_0^1 \frac{1}{r^{3-\kappa}} \phi_1^2(r) dr \leq \frac{C_1(\nu)}{(\kappa + 2(\nu - 1))\nu^2} \int_0^1 f^2 dt. \quad (52)$$

For the functions  $\phi_2, \phi_3$ , the analogous estimates

$$\int_0^1 \frac{1}{r^{3-\kappa}} \phi_2^2 dr \leq \frac{1}{8\nu^2(1-\nu)(\kappa + 2(\nu - 1))} \int_0^1 f^2 dt, \quad (53)$$

$$\int_0^1 \frac{1}{r^{3-\kappa}} \phi_3^2 dr \leq \frac{1}{8\nu^2(\nu + 1)\kappa} \int_0^1 f^2 dr \quad (54)$$

are valid. Inequalities (52)–(54) then lead to the result

$$\int_0^1 \frac{1}{r^{3-\kappa}} \psi^2(r) dr \leq C_2(\nu, \kappa) \int_0^1 f^2 dr. \quad (55)$$

It remains to estimate the first derivative of  $\psi$ . Employing (51), the relations

$$\int_0^1 \frac{1}{r^{1-\kappa}} (\phi_1')^2 dr = \int_0^1 \frac{1}{r^{1-\kappa}} \alpha^2 \nu^2 r^{2\nu-2} dr \leq \frac{C_1(\nu)}{(\kappa + 2(\nu - 1))} \int_0^1 f^2 dr \quad (56)$$

are immediately obtained. For  $\phi_2$  and  $\phi_3$ , the analogous inequalities

$$\begin{aligned} \int_0^1 \frac{1}{r^{1-\kappa}} (\phi_2')^2 dr &= \frac{1}{4\nu^2} \int_0^1 \frac{1}{r^{1-\kappa}} (\nu r^{\nu-1} \int_0^r t^{1/2-\nu} f(t) dt + r^{1/2} f(r))^2 dr \\ &\leq \frac{1}{2} \left( \frac{1}{2(1-\nu)(2(\nu-1)+\kappa)} + \frac{1}{\nu^2} \right) \int_0^1 f^2 dr, \end{aligned} \quad (57)$$

$$\begin{aligned} \int_0^1 \frac{1}{r^{1-\kappa}} (\phi_3')^2 dr &= \frac{1}{4\nu^2} \int_0^1 \frac{1}{r^{1-\kappa}} (-\nu r^{-\nu-1} \int_0^r t^{1/2+\nu} f(t) dt + r^{1/2} f(r))^2 dr \\ &\leq \frac{1}{2} \left( \frac{1}{2(\nu+1)\kappa} + \frac{1}{\nu^2} \right) \int_0^1 f^2 dr \end{aligned} \quad (58)$$

are true. The asserted statement is now a consequence of (55)–(58).  $\square$

We are now in the position to establish the desired a-priori estimate in fractional order Sobolev spaces for the operator  $\check{L}_{\text{in}}$  on the space  $\check{N}_{\text{in}}$  from (46). Recall the number  $\bar{\kappa}$  from (15).

**Lemma 3.9.** *Let  $\varepsilon_{\text{in}}$  satisfy (25),  $\kappa_0 \in (2(1 - \bar{\kappa}), 1)$ , and  $\phi \in \overline{\check{N}_{\text{in}}}^{\|\cdot\|_{\mathcal{D}(\check{L}_{\text{in}})}}$ . The inequality*

$$\|\phi\|_{PH^{2-\kappa_0/2}(D)} \leq C \|\check{L}_{\text{in}} \phi\|_{L^2(D)}$$

is valid with a uniform constant  $C = C(\kappa_0) > 0$ .

*Proof.* 1) Let  $\phi \in \check{N}_{\text{in}}$ . We use the sets

$$D_{i,\xi} := \{(x, y) \in D_{\text{in},i} \mid |(x, y)| \geq \xi\}$$

for  $\xi > 0$ , and denote by  $(r, \varphi)$  polar coordinates. Recall the definition of  $D_{\text{in},i}$  in (26). Note moreover that  $\phi$  is smooth on each  $D_{i,\xi}$  by definition of  $\check{N}_{\text{in}}$  in (46).

Combining the relation  $|x_1|^{\kappa_0/2}|x_2|^{\kappa_0/2} \leq \sqrt{x_1^2 + x_2^2}^{\kappa_0}$ ,  $(x_1, x_2) \in \mathbb{R}^2$ , with Lemma 2.12 from [6], it suffices to prove the estimate

$$\|r^{\frac{\kappa_0}{2}} \phi\|_{L^2(D)} + \sum_{j=1}^2 \|r^{\frac{\kappa_0}{2}} \partial_j \phi\|_{L^2(D)}^2 + \sum_{j,k=1}^2 \|r^{\frac{\kappa_0}{2}} \partial_j \partial_k \phi\|_{L^2(D)}^2 \leq C \|\check{L}_{\text{in}} \phi\|_{L^2(D)}$$

with a uniform constant  $C$ .

Transforming to polar coordinates, we infer the formula

$$\begin{aligned} & \sum_{i=1}^4 \int_{D_{i,\xi}} r^{\kappa_0} \varepsilon_{\text{in}}^{(i)} \left[ (\partial_x^2 \phi^{(i)}) (\partial_y^2 \phi^{(i)}) - (\partial_x \partial_y \phi^{(i)})^2 \right] d(x, y) \\ &= \sum_{i=1}^4 \int_{\xi}^1 \int_{I_{\text{in},i}} \varepsilon_{\text{in}}^{(i)} \left[ -(\partial_r \partial_{\varphi} \phi^{(i)})^2 \frac{1}{r^{1-\kappa_0}} - (\partial_{\varphi} \phi^{(i)})^2 \frac{1}{r^{3-\kappa_0}} + (\partial_r^2 \phi^{(i)}) (\partial_r \phi^{(i)}) r^{\kappa_0} \right. \\ & \quad \left. + (\partial_r^2 \phi^{(i)}) (\partial_{\varphi}^2 \phi^{(i)}) \frac{1}{r^{1-\kappa_0}} + 2(\partial_r \partial_{\varphi} \phi^{(i)}) (\partial_{\varphi} \phi^{(i)}) \frac{1}{r^{2-\kappa_0}} \right] d\varphi dr, \end{aligned} \quad (59)$$

see Section 1.5.4 in [52] for instance. Integrating the fourth expression on the right hand side with respect to the  $r$ - and  $\varphi$ -variable by parts, the identity

$$\begin{aligned} & \sum_{i=1}^4 \int_{\xi}^1 \int_{I_{\text{in},i}} \varepsilon_{\text{in}}^{(i)} r^{\kappa_0-1} (\partial_r^2 \phi^{(i)}) (\partial_{\varphi}^2 \phi^{(i)}) d\varphi dr \\ &= \sum_{i=1}^4 \int_{\xi}^1 \int_{I_{\text{in},i}} \varepsilon_{\text{in}}^{(i)} [r^{\kappa_0-1} (\partial_{\varphi} \partial_r \phi^{(i)})^2 - (1-\kappa_0) r^{\kappa_0-2} (\partial_{\varphi} \partial_r \phi^{(i)}) (\partial_{\varphi} \phi^{(i)})] d\varphi dr \\ & \quad - \sum_{i=1}^4 \int_{I_{\text{in},i}} \varepsilon_{\text{in}}^{(i)} [r^{\kappa_0-1} (\partial_r \phi^{(i)}) (\partial_{\varphi}^2 \phi^{(i)})] |_{r=\xi} d\varphi \end{aligned} \quad (60)$$

is obtained. Inserting (60) into (59) and manipulating the arising expressions algebraically, the equations

$$\begin{aligned} & \sum_{i=1}^4 \int_{D_{i,\xi}} r^{\kappa_0} \varepsilon_{\text{in}}^{(i)} [(\partial_x^2 \phi^{(i)}) (\partial_y^2 \phi^{(i)}) - (\partial_x \partial_y \phi^{(i)})^2] d(x, y) \\ &= \sum_{i=1}^4 \int_{\xi}^1 \int_{I_{\text{in},i}} \varepsilon_{\text{in}}^{(i)} [-r^{\kappa_0-3} (\partial_{\varphi} \phi^{(i)})^2 + r^{\kappa_0} (\partial_r^2 \phi^{(i)}) (\partial_r \phi^{(i)}) \\ & \quad + (1+\kappa_0) r^{\kappa_0-2} (\partial_{\varphi} \partial_r \phi^{(i)}) (\partial_{\varphi} \phi^{(i)})] d\varphi dr \\ & \quad - \sum_{i=1}^4 \int_{I_{\text{in},i}} \varepsilon_{\text{in}}^{(i)} [r^{\kappa_0-1} (\partial_r \phi^{(i)}) (\partial_{\varphi}^2 \phi^{(i)})] |_{r=\xi} d\varphi \\ &= \sum_{i=1}^4 \int_{\xi}^1 \int_{I_{\text{in},i}} \varepsilon_{\text{in}}^{(i)} \left[ -\frac{(2-\kappa_0)(1+\kappa_0)}{2} r^{\kappa_0-3} (\partial_{\varphi} \phi^{(i)})^2 + (1+\kappa_0) r^{\kappa_0-2} (\partial_{\varphi} \partial_r \phi^{(i)}) (\partial_{\varphi} \phi^{(i)}) \right. \\ & \quad \left. + r^{\kappa_0} (\partial_r^2 \phi^{(i)}) (\partial_r \phi^{(i)}) + \frac{(2-\kappa_0)(1+\kappa_0)-2}{2} r^{\kappa_0-3} (\partial_{\varphi} \phi^{(i)})^2 \right] d\varphi dr \\ & \quad - \sum_{i=1}^4 \int_{I_{\text{in},i}} \varepsilon_{\text{in}}^{(i)} [r^{\kappa_0-1} (\partial_r \phi^{(i)}) (\partial_{\varphi}^2 \phi^{(i)})] |_{r=\xi} d\varphi \\ &= \sum_{i=1}^4 \int_{\xi}^1 \int_{I_{\text{in},i}} \varepsilon_{\text{in}}^{(i)} \left[ \frac{1}{2} \partial_r ((1+\kappa_0) r^{\kappa_0-2} (\partial_{\varphi} \phi^{(i)})^2) + r^{\kappa_0} (\partial_r^2 \phi^{(i)}) (\partial_r \phi^{(i)}) \right. \\ & \quad \left. + \frac{\kappa_0(1-\kappa_0)}{2} r^{\kappa_0-3} (\partial_{\varphi} \phi^{(i)})^2 \right] d\varphi dr \end{aligned}$$

$$-\sum_{i=1}^4 \int_{I_{in,i}} \varepsilon_{in}^{(i)} [r^{\kappa_0-1} (\partial_r \phi^{(i)}) (\partial_\varphi^2 \phi^{(i)})] |_{r=\xi} d\varphi$$

follow. An integration by parts with respect to the  $r$ -variable then leads to

$$\begin{aligned} & \sum_{i=1}^4 \int_{D_{i,\xi}} r^{\kappa_0} \varepsilon_{in}^{(i)} [(\partial_x^2 \phi^{(i)}) (\partial_y^2 \phi^{(i)}) - (\partial_x \partial_y \phi^{(i)})^2] d(x, y) \\ &= \sum_{i=1}^4 \int_{\xi}^1 \int_{I_{in,i}} \varepsilon_{in}^{(i)} [r^{\kappa_0} (\partial_r^2 \phi^{(i)}) (\partial_r \phi^{(i)}) + \frac{\kappa_0(1-\kappa_0)}{2} r^{\kappa_0-3} (\partial_\varphi \phi^{(i)})^2] d\varphi dr \\ &+ \sum_{i=1}^4 \int_{I_{in,i}} \varepsilon_{in}^{(i)} [(\frac{1+\kappa_0}{2} (\partial_\varphi \phi^{(i)})^2) |_{r=1} - (r^{\kappa_0-1} (\partial_r \phi^{(i)}) (\partial_\varphi^2 \phi^{(i)})) \\ &+ \frac{1+\kappa_0}{2} r^{\kappa_0-2} (\partial_\varphi \phi^{(i)})^2 |_{r=\xi}] d\varphi. \end{aligned} \quad (61)$$

2) The first term on the right hand side of (61) is next treated separately by means of Lemma 3.8. We first note that  $\phi$  has the representation

$$\phi(r \cos \varphi, r \sin \varphi) = \sum_{k=1}^Z \alpha_k J_{\kappa_{in,1}} (\sqrt{\check{\lambda}_{k,1}^{in}} r) \psi_{in,1}(\varphi), \quad (62)$$

for  $r \in (0, 1)$ ,  $\varphi \in (0, 2\pi)$ , with numbers  $Z \in \mathbb{N}$  and  $\alpha_1, \dots, \alpha_Z \in \mathbb{R}$ . (The function  $\phi$  is an element of  $\tilde{N}_{in}$ , see (46).) Since  $\psi_{in,1}$  is an eigenfunction of (29) to the eigenvalue  $\kappa_{in,1}^2$ , we deduce the identity

$$r^{1/2} \Delta \phi^{(i)} = r^{1/2} \partial_r^2 \phi^{(i)} + r^{-1/2} \partial_r \phi^{(i)} - \kappa_{in,1}^2 r^{-3/2} \phi^{(i)} =: f^{(i)}, \quad r \in (0, 1). \quad (63)$$

To apply Lemma 3.8, we note that the function  $f$  is continuous in  $r$  and that  $\phi$  belongs to  $L^2(D)$ . Indeed, using Lemma 3.6, the formula

$$\Delta \phi^{(i)}(r, \varphi) = - \sum_{k=1}^Z \check{\lambda}_{k,1}^{in} \alpha_k J_{\kappa_{in,1}} (\sqrt{\check{\lambda}_{k,1}^{in}} r) \psi_{in,1}(\varphi)$$

is valid. Combining (63) with Lemma 3.8, we then deduce the estimate

$$\sum_{i=1}^4 \int_0^1 \int_{I_{in,i}} r^{\kappa_0-1} (\partial_r \phi^{(i)})^2 d\varphi dr \leq \tilde{C} \sum_{i=1}^4 \int_0^1 \int_{I_{in,i}} r (\Delta \phi^{(i)})^2 d\varphi dr$$

with a constant  $\tilde{C} = \tilde{C}(\kappa_0) > 0$ . For the first term on the right hand side of (61), we consequently arrive at the inequalities

$$\begin{aligned} & \sum_{i=1}^4 \int_{\xi}^1 \int_{I_{in,i}} \varepsilon_{in}^{(i)} r^{\kappa_0} (\partial_r^2 \phi^{(i)}) (\partial_r \phi^{(i)}) d\varphi dr \\ & \geq - \sum_{i=1}^4 \left( \frac{1}{16} \int_{\xi}^1 \int_{I_{in,i}} \varepsilon_{in}^{(i)} r^{\kappa_0+1} (\partial_r^2 \phi^{(i)})^2 d\varphi dr + 4 \int_0^1 \int_{I_{in,i}} \varepsilon_{in}^{(i)} r^{\kappa_0-1} (\partial_r \phi^{(i)})^2 d\varphi dr \right) \\ & \geq - \left( \sum_{i=1}^4 \frac{1}{16} \int_{\xi}^1 \int_{I_{in,i}} \varepsilon_{in}^{(i)} r^{\kappa_0+1} (\partial_r^2 \phi^{(i)})^2 d\varphi dr + 4\tilde{C} \|\sqrt{\varepsilon_{in}} \check{L}_{in} \phi\|_{L^2(D)}^2 \right), \end{aligned} \quad (64)$$

additionally using the Young and Cauchy-Schwarz estimates.

3) We next focus on the face integrals on the right hand side of (61). To that end, we analyze the behavior of  $\phi$  near the center of  $D$ . Since  $\phi$  has the representation (62), it suffices to treat in the following only the function  $\tilde{\phi}(r, \varphi) := J_{\kappa_{in,1}}(r) \psi_{in,1}(\varphi)$ . As  $\psi_{in,1}$  is an eigenfunction of (29), definition (44) leads to the identity

$$(\partial_r \tilde{\phi}^{(i)}) (\partial_\varphi^2 \tilde{\phi}^{(i)})(r, \varphi)$$

$$= -\frac{(\kappa_{\text{in},1})^2}{2} \left( \sum_{j=0}^{\infty} (-1)^j (\kappa_{\text{in},1} + 2j) \frac{\left(\frac{r}{2}\right)^{\kappa_{\text{in},1}+2j-1}}{j! \Gamma(\kappa_{\text{in},1}+j+1)} \right) (\psi_{\text{in},1}^{(i)}(\varphi))^2 \sum_{j=0}^{\infty} (-1)^j \frac{\left(\frac{r}{2}\right)^{\kappa_{\text{in},1}+2j}}{j! \Gamma(\kappa_{\text{in},1}+j+1)}.$$

Since  $\kappa_0 > 2(1 - \kappa_{\text{in},1})$ , this implies that the function  $r^{\kappa_0-1}(\partial_r \tilde{\phi}^{(i)})(\partial_\varphi^2 \tilde{\phi}^{(i)})$  possesses a continuous extension to  $[0, 1] \times I_{\text{in},i}$ , and that it tends to zero as  $r \rightarrow 0$ . Similar reasoning further shows the same statements for the function  $r^{\kappa_0-2}(\partial_\varphi \tilde{\phi}^{(i)})^2$ . The Lebesgue theorem of dominated convergence hence yields the result

$$\lim_{\xi \rightarrow 0} \sum_{i=1}^4 \int_{I_{\text{in},i}} \varepsilon_{\text{in}}^{(i)} [r^{\kappa_0-1}(\partial_r \phi^{(i)})(\partial_\varphi^2 \phi^{(i)}) + \frac{1+\kappa_0}{2} r^{\kappa_0-2}(\partial_\varphi \phi^{(i)})^2]_{|r=\xi} d\varphi = 0. \quad (65)$$

4) For the next step, the formula

$$\partial_r^2 \phi^{(i)} = \frac{x^2}{r^2} (\partial_x^2 \phi^{(i)}) + 2 \frac{xy}{r^2} \partial_x \partial_y \phi^{(i)} + \frac{y^2}{r^2} \partial_y^2 \phi^{(i)}$$

is useful. Combining (61) and (64), we derive the estimates

$$\begin{aligned} & \sum_{i=1}^4 \int_{D_{i,\xi}} r^{\kappa_0} \varepsilon_{\text{in}}^{(i)} (\Delta \phi^{(i)})^2 d(x, y) + 4\tilde{C} \|\sqrt{\varepsilon_{\text{in}}} \check{L}_{\text{in}} \phi\|_{L^2(D)}^2 \\ & \geq \sum_{i=1}^4 \left( \int_{D_{i,\xi}} r^{\kappa_0} \varepsilon_{\text{in}}^{(i)} [(\partial_x^2 \phi^{(i)})^2 + (\partial_y^2 \phi^{(i)})^2 + 2(\partial_x \partial_y \phi^{(i)})^2] d(x, y) \right. \\ & \quad - \frac{1}{16} \int_{\xi}^1 \int_{I_{\text{in},i}} \varepsilon_{\text{in}}^{(i)} r^{\kappa_0+1} (\partial_r^2 \phi^{(i)})^2 d\varphi dr \\ & \quad \left. - \int_{I_{\text{in},i}} \varepsilon_{\text{in}}^{(i)} [r^{\kappa_0-1}(\partial_r \phi^{(i)})(\partial_\varphi^2 \phi^{(i)}) + \frac{1+\kappa_0}{2} r^{\kappa_0-2}(\partial_\varphi \phi^{(i)})^2]_{|r=\xi} d\varphi \right) \\ & \geq \frac{1}{2} \sum_{i=1}^4 \left( \int_{D_{i,\xi}} r^{\kappa_0} \varepsilon_{\text{in}}^{(i)} [(\partial_x^2 \phi^{(i)})^2 + (\partial_y^2 \phi^{(i)})^2 + 2(\partial_x \partial_y \phi^{(i)})^2] d(x, y) \right. \\ & \quad \left. - 2 \int_{I_{\text{in},i}} \varepsilon_{\text{in}}^{(i)} [r^{\kappa_0-1}(\partial_r \phi^{(i)})(\partial_\varphi^2 \phi^{(i)}) + \frac{1+\kappa_0}{2} r^{\kappa_0-2}(\partial_\varphi \phi^{(i)})^2]_{|r=\xi} d\varphi \right). \end{aligned}$$

In the limit  $\xi \rightarrow 0$ , the monotone convergence principle and (65) lead to the relation

$$\begin{aligned} & (1 + 4\tilde{C}) \|\sqrt{\varepsilon_{\text{in}}} \check{L}_{\text{in}} \phi\|_{L^2(D)}^2 \\ & \geq \frac{1}{2} \sum_{i=1}^4 \int_{D_{\text{in},i}} r^{\kappa_0} \varepsilon_{\text{in}}^{(i)} [(\partial_x^2 \phi^{(i)})^2 + (\partial_y^2 \phi^{(i)})^2 + 2(\partial_x \partial_y \phi^{(i)})^2] d(x, y). \quad (66) \end{aligned}$$

Combining the Cauchy-Schwarz inequality with an integration by parts, we furthermore infer the estimate

$$\begin{aligned} \|\sqrt{\varepsilon_{\text{in}}} \check{L}_{\text{in}} \phi\|_{L^2(D)} \cdot \|\sqrt{\varepsilon_{\text{in}}} \phi\|_{L^2(D)} & \geq - \sum_{i=1}^4 \int_{D_{\text{in},i}} \varepsilon_{\text{in}}^{(i)} (\check{L}_{\text{in}} \phi)^{(i)} \phi^{(i)} d(x, y) \\ & = \sum_{i=1}^4 \int_{D_{\text{in},i}} \varepsilon_{\text{in}}^{(i)} |\nabla \phi^{(i)}|^2 d(x, y). \end{aligned}$$

Taking additionally the Poincaré inequality into account, we conclude the remaining estimate

$$\|\sqrt{\varepsilon_{\text{in}}} \check{L}_{\text{in}} \phi\|_{L^2(D)} \geq C \|\sqrt{\varepsilon_{\text{in}}} \phi\|_{PH^1(D)}. \quad (67)$$

Altogether, (66) and (67) imply the desired energy inequality.  $\square$

The following corollary is an important consequence of Lemmas 3.6 and 3.9, as well as (49).

**Corollary 1.** *Let  $\varepsilon_{\text{in}}$  satisfy (25), and  $\kappa > 1 - \bar{\kappa}$ . The domain  $\mathcal{D}(\check{L}_{\text{in}})$  is a subspace of  $PH^{2-\kappa}(D)$  with  $\|u\|_{PH^{2-\kappa}(D)} \leq C\|\check{L}_{\text{in}}u\|_{L^2(D)}$ ,  $u \in \mathcal{D}(\check{L}_{\text{in}})$ , for a constant  $C = C(\kappa) > 0$ .*

*Proof.* Combining Lemmas 3.6 with (49), the estimate

$$\|u\|_{PH^{2-\kappa}(D)} \leq C\|\sqrt{\varepsilon_{\text{in}}}\check{L}_{\text{in}}u\|_{L^2(D)}$$

is valid for  $u \in \check{M}_{\text{in}}$  with a uniform constant  $C > 0$ . Lemma 3.9 implies the analogous inequality on  $\check{N}_{\text{in}}$  (with a constant  $C = C(\kappa) > 0$ ). Let  $v+w \in \check{M}_{\text{in}} \oplus \check{N}_{\text{in}}$ . The triangle inequality and Lemma 3.6 then imply the desired estimates

$$\begin{aligned} \|v+w\|_{PH^{2-\kappa}(D)}^2 &\leq 2(\|v\|_{PH^{2-\kappa}(D)}^2 + \|w\|_{PH^{2-\kappa}(D)}^2) \\ &\leq 2C^2(\|\sqrt{\varepsilon_{\text{in}}}\check{L}_{\text{in}}v\|_{L^2(D)}^2 + \|\sqrt{\varepsilon_{\text{in}}}\check{L}_{\text{in}}w\|_{L^2(D)}^2) \\ &= 2C^2\|\sqrt{\varepsilon_{\text{in}}}\check{L}_{\text{in}}(v+w)\|_{L^2(D)}^2. \end{aligned} \quad (68)$$

By Lemmas 3.4 and 3.6, the identity  $\overline{\check{M}_{\text{in}} \oplus \check{N}_{\text{in}}}^{\mathcal{D}(\check{L}_{\text{in}})} = \mathcal{D}(\check{L}_{\text{in}})$  is valid. The stated embedding hence is a consequence of (68).  $\square$

**3.4. Conclusion of the regularity statement.** We now establish the desired regularity statement for functions in the domain of the operator  $\Delta_{0,\Gamma^*}$  from (19), resulting in a regularity result for the solution to the interface problem (14). To that end, we first use a cut-off argument to focus on thin cylinders around interior edges. This principle is well known to experts in the field, see [16, 14, 15] for instance. To have a self-contained presentation, we however sketch the arguments.

Let us first fix some notation for the next statements. Recall that  $\mathcal{S}$  is the union of all edges of the interfaces. Let  $e_{\text{in}} \subseteq \mathcal{S}$  be an interior edge, and  $\delta > 0$  be so small, that all cylinders around the interior edges with radius  $\delta$  are disjoint from each other. We denote by  $\text{dist}(e_{\text{in}}, \cdot) : \bar{Q} \rightarrow [0, \infty]$  the distance function to  $e_{\text{in}}$ . Let additionally  $\chi_\delta : [0, \infty) \rightarrow [0, 1]$  be a smooth cut-off function with  $\chi_\delta = 1$  on  $[0, \delta^2/4]$  and  $\text{supp } \chi_\delta \subseteq [0, 9\delta^2/16]$ . Note that  $\chi_\delta(\text{dist}(e_{\text{in}}, \cdot)^2)$  is cylindrically symmetric on  $\bar{Q}$  with respect to  $e_{\text{in}}$  and smooth.

**Lemma 3.10.** *Let  $\varepsilon$  satisfy (2), and let  $e_{\text{in}} \subseteq \mathcal{S}$  be an interior edge. Let furthermore  $u \in \mathcal{D}(\Delta_{0,\Gamma^*})$ . The function  $\chi_\delta(\text{dist}(e_{\text{in}}, \cdot)^2)u$  belongs to  $\mathcal{D}(\Delta_{0,\Gamma^*})$ .*

*Proof.* After translating, scaling and rotating, we can assume that  $\delta = 1$ , and that  $e_{\text{in}}$  satisfies (24). We abbreviate  $v := \chi_1(\text{dist}(e_{\text{in}}, \cdot)^2)u$ .

Recall definition (19). As the mapping  $\chi_1(\text{dist}(e_{\text{in}}, \cdot)^2)$  is smooth and  $u \in H^1(Q)$ ,  $v$  is also contained in  $H^1(Q)$ . Furthermore,  $v$  satisfies the same boundary conditions as  $u$ , since the factor  $\chi_1(\text{dist}(e_{\text{in}}, \cdot)^2)$  does not depend on  $x_3$ . Using the product rule for the Laplacian, we furthermore infer  $\Delta v^{(i)} \in L^2(Q_i)$ .

It consequently remains to verify the first order transmission condition for  $v$ . By construction, it suffices to focus on the interfaces that touch  $e_{\text{in}}$ . Let  $\mathcal{F}$  be such an interface. We assume the representation

$$\mathcal{F} = \{0\} \times [0, 1] \times [0, 1],$$

as the other interfaces can be treated in the same way. Let  $x = (x_1, x_2, x_3) = (0, x_2, x_3) \in \mathcal{F}$ . We calculate

$$\nabla \chi_\delta(\text{dist}(e_{\text{in}}, x)^2) \cdot \nu_{\mathcal{F}} = \chi'_\delta(\text{dist}(e_{\text{in}}, x)^2) \partial_1(x_1^2 + x_2^2)|_{x_1=0} = 0.$$

This means that  $v$  fulfills the same interface conditions as  $u$ , whence  $v$  is an element of the domain  $\mathcal{D}(\Delta_{0,\Gamma^*})$ .  $\square$

Recall for the next statement Definition 3.3, (17), and (15).

**Lemma 3.11.** *Let  $\varepsilon$  satisfy (2), and let  $e_{\text{in}} \subseteq \mathcal{S}$  be an interior edge, where  $\varepsilon$  has a strong discontinuity. Let furthermore  $u \in \mathcal{D}(\Delta_{0,\Gamma^*})$  and  $\kappa > 1 - \bar{\kappa}$ . The function  $\chi_\delta(\text{dist}(e_{\text{in}}, \cdot)^2)u$  belongs to  $\mathcal{V}_{2-\kappa} \cap \mathcal{D}(\Delta_{0,\Gamma^*})$  and*

$$\|\chi_\delta(\text{dist}(e_{\text{in}}, \cdot)u)\|_{\mathcal{V}_{2-\kappa}} \leq C\|\Delta_{0,\Gamma^*}(\chi_\delta(\text{dist}(e_{\text{in}}, \cdot)u))\|_{L^2(Q)}$$

with a number  $C = C(\delta, \varepsilon, \kappa)$ .

*Proof.* 1) After translating, scaling and rotating, we can assume that  $\delta = 1$ , and  $e_{\text{in}}$  satisfies (24). We moreover adopt the constructions in Section 3.2, and assume that  $\Gamma_3 \subseteq \Gamma^*$ . (The case  $\Gamma_3 \not\subseteq \Gamma^*$  can be handled with the usual modifications for homogeneous Neumann boundary conditions.) Throughout,  $C = C(\delta, \varepsilon, \kappa)$  is a constant that changes from line to line. As in the proof of Lemma 3.10, we set

$$v := \chi_1(\text{dist}(e_{\text{in}}, \cdot)^2)u \in \mathcal{D}(\Delta_{0,\Gamma^*}).$$

By construction of  $v$ , it suffices to prove the inequality

$$\begin{aligned} \|v\|_{L^2((0,1),PH^{2-\kappa}(D))} + \|v\|_{H^1((0,1),H^1(D))} + \|v\|_{H^2((0,1),L^2(D))} \\ \leq C\|\Delta_{0,\Gamma^*}v\|_{L^2(\tilde{\mathcal{Z}}_{\text{in}})}. \end{aligned} \quad (69)$$

2) The function  $v$  is odd reflected at  $\Gamma_3$  to the large cylinder  $\tilde{\mathcal{Z}}_{\text{in}} := D \times (-1, 2)$ , and the resulting mapping is still denoted by  $v$ . The parameter  $\varepsilon_{\text{in}}$  is reflected in an even way. Note that  $v$  belongs to  $H^1(\tilde{\mathcal{Z}}_{\text{in}})$ , and that  $\varepsilon_{\text{in}}\nabla v|_{D_{\text{in},i} \times (-1,2)}$  is an element of  $H(\text{div}, \tilde{\mathcal{Z}}_{\text{in}})$ .

3) Let  $\check{\chi}_3 : \mathbb{R} \rightarrow [0, 1]$  be a smooth cut-off function with  $\check{\chi}_3 = 1$  on  $[0, 1]$  and  $\text{supp } \check{\chi}_3 \subseteq [-1/2, 3/2]$ . We analyze the product  $\check{\chi}_3(x_3)v$  in the following, and thereby use ideas and techniques from [16, 14]. To that end, we extend the function  $\check{\chi}_3(x_3)v$  trivially by zero in  $x_3$ -direction to the infinite cylinder  $D \times \mathbb{R}$ . The extended function is denoted by the same symbol. Note that this extension argument does not change the transmission behavior.

Put now

$$-\Delta(\check{\chi}_3 v^{(i)}) =: \tilde{f}^{(i)} \in L^2(D_{\text{in},i} \times \mathbb{R}), \quad (70)$$

for  $i \in \{1, 2, 3, 4\}$ . The above extension procedure then implies the fact

$$\|\tilde{f}\|_{L^2(D \times \mathbb{R})} \leq C\|f\|_{L^2(\tilde{\mathcal{Z}}_{\text{in}})}. \quad (71)$$

Next we apply a partial Fourier-Transform with respect to the  $x_3$ -variable, and we denote the resulting function by  $\hat{w}$  for  $w \in L^2(D \times \mathbb{R})$ . The inverse transform of a function  $v \in L^2(D \times \mathbb{R})$  is denoted by  $\check{v}$ . We moreover call the new variable in Fourier space  $\xi$ . Relation (70) then gives rise to the formula

$$(\xi^2 - \partial_{x_1}^2 - \partial_{x_2}^2)\hat{(\check{\chi}_3 v^{(i)})}(x_1, x_2, \xi) = \hat{(\tilde{f}^{(i)})}, \quad (x_1, x_2, \xi) \in D_{\text{in},i} \times \mathbb{R}. \quad (72)$$

The variable  $\xi$  is considered to be fixed in the next steps (the statements are then tacitly valid for almost all  $\xi$ ). Fubini's Theorem throughout provides  $L^2$ -integrability of the arising expressions. Equation (72) in this respect means that the mapping  $(\partial_{x_1}^2 + \partial_{x_2}^2)\hat{(\check{\chi}_3^{(i)} v^{(i)})}(\cdot, \xi)$  is an element of  $L^2(D_{\text{in},i})$ . Furthermore,  $\hat{(\check{\chi}_3 v)}(\cdot, \xi)$  belongs to  $H_0^1(D)$ . Combining the fact  $\text{div}(\varepsilon_{\text{in}}\nabla\check{\chi}_3 v) \in L^2(D \times \mathbb{R})$  with the reasoning for (72), we conclude that  $\hat{(\check{\chi}_3 v)}(\cdot, \xi)$  fulfills on  $D$  the boundary and transmission conditions that are required in  $\mathcal{D}(\check{L}_{\text{in}})$ , see (27). Altogether, we arrive at the fact  $\hat{(\check{\chi}_3 v)}(\cdot, \xi) \in \mathcal{D}(\check{L}_{\text{in}})$ .

4) Corollary 1, the triangle inequality and (72) provide the relations

$$\begin{aligned} \|\hat{(\check{\chi}_3 v)}(\cdot, \xi)\|_{PH^{2-\kappa}(D)}^2 &\leq C\|\check{L}_{\text{in}}\hat{(\check{\chi}_3 v)}(\cdot, \xi)\|_{L^2(D)}^2 \\ &\leq C\left(\|\xi^2\hat{(\check{\chi}_3 v)}(\cdot, \xi)\|_{L^2(D)}^2 + \|\hat{(\tilde{f})}(\cdot, \xi)\|_{L^2(D)}^2\right). \end{aligned} \quad (73)$$

We next take also the estimate

$$0 \leq -\Re(\varepsilon_{\text{in}} \xi^{2\wedge} (\dot{\chi}_3 v)(\cdot, \xi), \check{L}_{\text{in}} \hat{(\dot{\chi}_3 v)}(\cdot, \xi))_{L^2(D)}, \quad (74)$$

into account, that is a consequence of the positivity of  $-\check{L}_{\text{in}}$ . Combining (72)–(74), we arrive at the inequalities

$$\begin{aligned} \|\hat{(\dot{\chi}_3 v)}(\cdot, \xi)\|_{PH^{2-\kappa}(D)}^2 &\leq C \left( \|\sqrt{\varepsilon_{\text{in}}} \xi^{2\wedge} (\dot{\chi}_3 v)(\cdot, \xi)\|_{L^2(D)}^2 \right. \\ &\quad - 2\Re(\varepsilon_{\text{in}} \xi^{2\wedge} (\dot{\chi}_3 v)(\cdot, \xi), \check{L}_{\text{in}} \hat{(\dot{\chi}_3 v)}(\cdot, \xi))_{L^2(D)} \\ &\quad \left. + \|\sqrt{\varepsilon_{\text{in}}} \check{L}_{\text{in}} \hat{(\dot{\chi}_3 v)}(\cdot, \xi)\|_{L^2(D)}^2 + \|\sqrt{\varepsilon_{\text{in}}} \hat{(\tilde{f})}(\cdot, \xi)\|_{L^2(D)}^2 \right) \\ &\leq C \|\sqrt{\varepsilon_{\text{in}}} \hat{(\tilde{f})}(\cdot, \xi)\|_{L^2(D)}^2. \end{aligned} \quad (75)$$

Integrating now with respect to  $\xi$  and using Plancherel's Theorem, we conclude the estimate

$$\|\dot{\chi}_3 v\|_{L^2(\mathbb{R}, PH^{2-\kappa}(D))}^2 \leq C \|\tilde{f}\|_{L^2(D \times \mathbb{R})}^2$$

follows. In view of (71), we consequently derive the relation

$$\|\dot{\chi}_3 v\|_{L^2(\mathbb{R}, PH^{2-\kappa}(D))} \leq C \|f\|_{L^2(\tilde{\mathcal{Z}}_{\text{in}})}. \quad (76)$$

This means that the function  $v$  belongs to the space  $L^2((0, 1), PH^{2-\kappa}(D))$ .

5) The reasoning in (75) further yields the inequality

$$\|\xi^{2\wedge} (\dot{\chi}_3 v)(\cdot, \xi)\|_{L^2(D)}^2 \leq C \|\sqrt{\varepsilon_{\text{in}}} \hat{(\tilde{f})}(\cdot, \xi)\|_{L^2(D)}^2.$$

Integrating with respect to  $\xi$  and using (71), we infer the estimate

$$\|\dot{\chi}_3 v\|_{H^2(\mathbb{R}, L^2(D))} \leq C \|f\|_{L^2(\tilde{\mathcal{Z}}_{\text{in}})}. \quad (77)$$

In the following, we employ the equivalence of the piecewise  $H^1$ -norm and the graph norm in  $\mathcal{D}(-\check{L}_{\text{in}})^{1/2}$  several times. With the selfadjointness of the operator  $(-\check{L}_{\text{in}})^{1/2}$  and the Cauchy-Schwarz estimate, we next deduce the inequalities

$$\begin{aligned} &\|\sqrt{\varepsilon_{\text{in}}} |\xi| (-\check{L}_{\text{in}})^{1/2\wedge} (\dot{\chi}_3 v)(\cdot, \xi)\|_{L^2(D)} \\ &\leq \left( (\varepsilon_{\text{in}} |\xi| (-\check{L}_{\text{in}})^{1/2\wedge} (\dot{\chi}_3 v)(\cdot, \xi), |\xi| (-\check{L}_{\text{in}})^{1/2\wedge} (\dot{\chi}_3 v)(\cdot, \xi))_{L^2(D)} \right)^{1/2} \\ &= \left( (\varepsilon_{\text{in}} |\xi|^2 (\dot{\chi}_3 v)(\cdot, \xi), (-\check{L}_{\text{in}})^{\wedge} (\dot{\chi}_3 v)(\cdot, \xi))_{L^2(D)} \right)^{1/2} \\ &\leq \|\sqrt{\varepsilon_{\text{in}}} |\xi|^2 (\dot{\chi}_3 v)(\cdot, \xi)\|_{L^2(D)}^{1/2} \|\sqrt{\varepsilon_{\text{in}}} \check{L}_{\text{in}} \hat{(\dot{\chi}_3 v)}(\cdot, \xi)\|_{L^2(D)}^{1/2}. \end{aligned}$$

We now integrate with respect to  $\xi$ . The statements in (71) and (75), and the representation of the space  $H^1(\mathbb{R}, H^1(D))$  by means of the partial Fourier Transform in  $x_3$  now give rise to the relation

$$\|\sqrt{\varepsilon_{\text{in}}} \dot{\chi}_3 v\|_{H^1(\mathbb{R}, H^1(D))} \leq C \|f\|_{L^2(\tilde{\mathcal{Z}}_{\text{in}})}. \quad (78)$$

Combining (76)–(78), inequality (69) is valid.  $\square$

For the next statement, we collect all interior edges  $e_{\text{in}}$ , at which  $\varepsilon$  has a strong discontinuity, into a set  $\mathcal{E}(\varepsilon)$ , see Definition 3.3. We also set  $\mathcal{E}(\mu) := \emptyset$ , and recall that  $\delta$  is introduced at the beginning of this Subsection. The below lemma then states that functions in the domain  $\mathcal{D}(\Delta_{0, \Gamma^*})$  from (19) are  $H^2$ -regular near every edge of an interface at which  $\varepsilon$  has no strong discontinuity.

**Lemma 3.12.** *Let  $\eta \in \{\varepsilon, \mu\}$  satisfy (2), and let  $u \in \mathcal{D}(\Delta_{0, \Gamma^*})$ . The function  $w := (1 - \sum_{e \in \mathcal{E}(\eta)} \chi_\delta(\text{dist}(e, \cdot)^2))u$  is contained in  $PH^2(Q)$  with*

$$\|w\|_{PH^2(Q)} \leq \|\Delta_{0, \Gamma^*} w\|_{L^2(Q)}.$$

*Proof.* 1) We only treat the case  $\eta = \varepsilon$  and  $\Gamma_3 \subseteq \Gamma^*$ , as the remaining can be handled with similar arguments. By Lemma 3.10, the function  $w$  is an element of  $\mathcal{D}(\Delta_{0,\Gamma^*})$ . In view of Lemma 3.2, it suffices to show that  $w$  is piecewise  $H^2$ -regular. To reach this goal, we first analyze  $w$  on two adjacent cuboids  $Q_1$  and  $Q_2$  that share an interface  $\mathcal{F}$  with two interior edges. After appropriate coordinate transformations, we can assume the identities

$$Q_1 = (-1, 0) \times (-1, 1)^2, \quad Q_2 = (0, 1) \times (-1, 1)^2, \quad \mathcal{F} = \{0\} \times [-1, 1]^2.$$

A smooth cut-off function  $\chi : [-1, 1] \rightarrow [0, 1]$  is furthermore employed. It satisfies  $\text{supp } \chi \subseteq [-1 + \delta/8, 1 - \delta/8]$  and  $\chi = 1$  on  $[-1 + \delta/4, 1 - \delta/4]$  for the number  $\delta$  from the beginning of the current Subsection 3.4. Set also  $\tilde{Q} := (-1, 1)^3$ .

By construction, the function

$$f(x_1, x_2, x_3) := \eta \chi(x_1) \chi(x_2) w(x_1, x_2, x_3)$$

is then an element of the space

$$\begin{aligned} \{f \in PH^1(\tilde{Q}) \mid \Delta f|_{Q_i} \in L^2(Q_i), \ i \in \{1, 2\}, \llbracket \frac{1}{\eta} f \rrbracket_{\mathcal{F}} = \llbracket \partial_1 f \rrbracket_{\mathcal{F}} = 0, \\ f(\cdot, \pm 1, \cdot) = 0, \ f(\pm 1, \cdot, \cdot) = 0, \ \partial_3 f(\cdot, \cdot, \pm 1) = 0\}. \end{aligned}$$

By Proposition 8.1 in [54], the mapping  $f$  is then  $H^2$ -regular on  $Q_1$  and  $Q_2$ .

2) Note that  $w$  is also piecewise  $H^2$ -regular in a neighborhood of the boundary faces  $\Gamma_1$  and  $\Gamma_2$ . (The cuboids that touch the exterior faces  $\Gamma_1$  or  $\Gamma_2$  are handled as in part 1) but the cut-off procedure in the definition of  $f$  is not applied near the respective boundary face.) Taking the definition of  $w$  into account, the function  $w$  is altogether piecewise  $H^2$ -regular on  $Q$ .  $\square$

Combining Lemmas 3.11 and 3.12, we derive the desired regularity statement for functions in the domain  $\mathcal{D}(\Delta_{0,\Gamma^*})$ . Recall for the statement definitions (15) and (17).

**Lemma 3.13.** *Let  $u \in \mathcal{D}(\Delta_{0,\Gamma^*})$ , and let  $\eta \in \{\varepsilon, \mu\}$  satisfy (2). Choose further  $\kappa = 0$  if  $\eta = \mu$ , and  $\kappa > 1 - \bar{\kappa}$  if  $\eta = \varepsilon$ . The estimate*

$$\|u\|_{\mathcal{V}_{2-\kappa}} \leq C \|\Delta_{0,\Gamma^*} u\|_{L^2(Q)}$$

is valid with a uniform constant  $C = C(\kappa, \eta, Q)$ .

*Proof.* 1) In the following,  $C = C(\kappa, \varepsilon, Q) > 0$  is a constant that changes from line to line. Integration by parts and the Poincaré inequality imply the relations

$$\begin{aligned} \|\Delta_{0,\Gamma^*} u\|_{L^2(Q)} \|\nabla u\|_{L^2(Q)} &\geq C \|\Delta_{0,\Gamma^*} u\|_{L^2(Q)} \|u\|_{L^2(Q)} \geq -C \int_Q (\Delta_{0,\Gamma^*} u) u \, dx \\ &= C \|\nabla u\|_{L^2(Q)}^2 \geq C \|u\|_{L^2(Q)}^2. \end{aligned} \quad (79)$$

2) For  $e \in \mathcal{E}(\eta)$ , we set  $v_e := \chi_\delta(\text{dist}(e, \cdot)^2) u$ . Combining the triangle inequality with Lemmas 3.11 and 3.12, we infer the inequalities

$$\begin{aligned} \|u\|_{\mathcal{V}_{2-\kappa}} &\leq \sum_{e \in \mathcal{E}(\eta)} \|v_e\|_{\mathcal{V}_{2-\kappa}} + \|u - \sum_{e \in \mathcal{E}(\eta)} v_e\|_{\mathcal{V}_{2-\kappa}} \\ &\leq C \left( \sum_{e \in \mathcal{E}(\eta)} \|\Delta_{0,\Gamma^*} v_e\|_{L^2(Q)} + \|\Delta_{0,\Gamma^*} (u - \sum_{e \in \mathcal{E}(\eta)} v_e)\|_{L^2(Q)} \right). \end{aligned}$$

With Young's inequality, we then infer the estimate

$$\|u\|_{\mathcal{V}_{2-\kappa}}^2 \leq C \left( \sum_{e \in \mathcal{E}(\eta)} \|\Delta_{0,\Gamma^*} v_e\|_{L^2(Q)}^2 + \|\Delta_{0,\Gamma^*} (u - \sum_{e \in \mathcal{E}(\eta)} v_e)\|_{L^2(Q)}^2 \right). \quad (80)$$



3) Let  $e \in \mathcal{E}(\eta)$ , and abbreviate  $w_e := \chi_\delta(\text{dist}(e, \cdot)^2)$ . Employing the product rule for the Laplacian as well as Young's inequality, we deduce the relations

$$\begin{aligned} \|\Delta_{0,\Gamma^*} v_e\|_{L^2(Q)}^2 &= \sum_{i=1}^N \|(\Delta w_e)u^{(i)} + 2(\nabla w_e) \cdot (\nabla u^{(i)}) + w_e \Delta u^{(i)}\|_{L^2(Q_i)}^2 \\ &\leq 16 \sum_{i=1}^N (\|(\Delta w_e)u^{(i)}\|_{L^2(Q_i)}^2 + \|(\nabla w_e) \cdot (\nabla u^{(i)})\|_{L^2(Q_i)}^2 + \|w_e \Delta u^{(i)}\|_{L^2(Q_i)}^2). \end{aligned}$$

We next take into account that all functions  $w_e$  have disjoint support. In view of the regularity of  $w_e$  on  $\bar{Q}$  and inequality (79), we arrive at the result

$$\begin{aligned} \sum_{e \in \mathcal{E}(\eta)} \|\Delta_{0,\Gamma^*} v_e\|_{L^2(Q)}^2 &\leq C \left( \sum_{e \in \mathcal{E}(\eta)} \|(\Delta w_e)u\|_{L^2(Q)}^2 + \left\| \sum_{e \in \mathcal{E}(\eta)} (\nabla w_e) \cdot (\nabla u) \right\|_{L^2(Q)}^2 \right. \\ &\quad \left. + \left\| \sum_{e \in \mathcal{E}(\eta)} w_e \Delta_{0,\Gamma^*} u \right\|_{L^2(Q)}^2 \right) \\ &\leq C \|\Delta_{0,\Gamma^*} u\|_{L^2(Q)}^2. \end{aligned} \quad (81)$$

Analogous reasoning also establishes the statement

$$\|\Delta_{0,\Gamma^*} (u - \sum_{e \in \mathcal{E}(\eta)} v_e)\|_{L^2(Q)}^2 \leq C \|\Delta_{0,\Gamma^*} u\|_{L^2(Q)}^2. \quad (82)$$

The desired estimate is a consequence of (80)–(82).  $\square$

Proposition 2 is a direct consequence of Lemma 3.13.

*Proof of Proposition 2.* System (14) has the weak formulation

$$\int_Q \eta(\nabla \psi) \cdot (\nabla \varphi) \, dx = \int_Q \eta f \varphi \, dx, \quad \varphi \in H_{\Gamma^*}^1(Q). \quad (83)$$

In view of the Lax-Milgram Lemma (and the Poincaré estimate), (83) has a unique (weak) solution  $\psi \in H_{\Gamma^*}^1(Q)$ . Relation (83) being valid for all elements  $\varphi$  of  $H_0^1(Q)$ , we infer that  $\text{div}(\eta \nabla \psi) = \eta f$  is an element of  $L^2(Q)$ . This means that  $\eta \nabla \psi$  satisfies the derivative interface conditions. Standard arguments further show that  $\psi$  fulfills homogeneous Neumann boundary conditions on the boundary part  $\partial Q \setminus \Gamma^*$ , see part 2 of the proof for Proposition 8.12 in [54] for instance. This means that  $\psi$  belongs to the domain of the operator  $\Delta_{0,\Gamma^*}$ , see (19). Lemma 3.13 now implies the asserted regularity and energy statements.  $\square$

We can also treat the pure Neumann case  $\Gamma^* = \emptyset$ , as the difference only arises in the energy estimates. For the statement, recall the space  $\mathcal{V}_{2-\kappa}$  from (17) and the number  $\bar{\kappa}$  from (15).

**Proposition 3.** *Let  $\eta \in \{\varepsilon, \mu\}$  satisfy (2), and let  $f \in L^2(Q)$ . We set  $\kappa = 0$  if  $\eta = \mu$ , and  $\kappa > 1 - \bar{\kappa}$  if  $\eta = \varepsilon$ . There is a unique function  $\psi \in \mathcal{V}_{2-\kappa}$  solving*

$$\begin{aligned} (1 - \Delta)\psi^{(i)} &= f^{(i)} && \text{on } Q_i \text{ for } i \in \{1, \dots, N\}, \\ \nabla \psi \cdot \nu &= 0 && \text{on } \partial Q, \\ \llbracket \psi \rrbracket_{\mathcal{F}} &= 0 = \llbracket \eta \nabla \psi \cdot \nu_{\mathcal{F}} \rrbracket_{\mathcal{F}} && \text{for } \mathcal{F} \in \mathcal{F}_{\text{int}}. \end{aligned} \quad (84)$$

*It satisfies  $\|\psi\|_{\mathcal{V}_{2-\kappa}} \leq C \|f\|_{L^2(Q)}$  with a constant  $C = C(\kappa, \eta, Q) > 0$ .*

*Proof.* To unify the arguments, we introduce the appropriate Neumann-Laplacian

$$\begin{aligned} (\Delta_{0,\emptyset} v)^{(i)} &:= \Delta v^{(i)}, \\ v \in \mathcal{D}(\Delta_{0,\emptyset}) &:= \{v \in H^1(Q) \mid \Delta v^{(i)} \in L^2(Q_i), \nabla v \cdot \nu = 0 \text{ on } \partial Q, \\ &\quad \llbracket \psi \rrbracket_{\mathcal{F}} = 0 = \llbracket \eta \nabla \psi \rrbracket_{\mathcal{F}} \text{ for } \mathcal{F} \in \mathcal{F}_{\text{int}}, i \in \{1, \dots, N\}\}. \end{aligned}$$

As the reasoning in Lemmas 3.11–3.12 focuses only on the local behavior of functions in the domain of  $\Delta_{0,\Gamma^*}$  around the interior edges and also allows homogeneous Neumann boundary conditions, the mentioned statements are also valid for functions in the domain  $\mathcal{D}(\Delta_{0,\emptyset})$ . (In the proof of Lemma 3.12, one uses Proposition 8.2 from [54] instead of Proposition 8.1.)

Adapting the arguments in the proofs of Lemmas 3.2 and 3.13 to the current setting of Neumann boundary conditions, we furthermore derive the energy estimate  $\|u\|_{\mathcal{V}_{2-\kappa}} \leq C\|(I - \Delta_{0,\emptyset})u\|_{L^2(Q)}$  for  $u \in \mathcal{D}(\Delta_{0,\emptyset})$  with a uniform constant  $C = C(\kappa, \eta, Q) > 0$ .

We moreover note that system (84) has the weak formulation

$$\int_Q \eta \psi \varphi + \eta (\nabla \psi) \cdot (\nabla \varphi) \, dx = \int_Q \eta f \varphi \, dx, \quad \varphi \in H^1(Q).$$

Employing the Lax-Milgram Lemma, (84) has a unique weak solution  $\psi \in H^1(Q)$ . Combining the reasoning in the proof for Proposition 2 with the above regularity statement and energy estimate, we finally arrive at the asserted result.  $\square$

#### 4. REGULARITY RESULT FOR THE SPACE $X_1$

This section is devoted to an embedding result for the space  $X_1$  from (13). To this end, we extend the well known regularity results for the spaces  $H_N(\text{curl}, \text{div}, Q)$  and  $H_T(\text{curl}, \text{div}, Q)$ , see Sections I.3.4 and I.3.5 in [24] for instance. The corresponding spaces for our setting of discontinuous coefficients are

$$\begin{aligned} H_{N,00}(\text{curl}, \text{div}, \varepsilon, Q) &:= \{\mathbf{E} \in H_0(\text{curl}, Q) \mid \text{div}(\varepsilon \mathbf{E}) = 0\}, \\ H_{N,0}(\text{curl}, \text{div}, \varepsilon, Q) &:= \{\mathbf{E} \in H_0(\text{curl}, Q) \mid \text{div}(\varepsilon \mathbf{E}) \in L^2(Q)\}, \\ H_{T,00}(\text{curl}, \text{div}, \mu, Q) &:= \{\mathbf{H} \in H(\text{curl}, Q) \mid \text{div}(\mu \mathbf{H}) = 0, \mu \mathbf{H} \cdot \nu = 0 \text{ on } \partial Q\}. \end{aligned} \quad (85)$$

The first and last space are already complete with respect to the norm in  $H(\text{curl}, Q)$  (making use of the bounded normal trace operator from  $H(\text{div}, Q)$  into  $H^{-1/2}(\partial Q)$ ). The second space in (85) is complete with respect to the norm

$$\|\mathbf{E}\|_{H_{N,0}}^2 := \|\mathbf{E}\|_{L^2(Q)}^2 + \|\text{curl } \mathbf{E}\|_{L^2(Q)}^2 + \|\text{div}(\varepsilon \mathbf{E})\|_{L^2(Q)}^2.$$

Our first goal is to establish embeddings of the spaces from (85) into appropriate fractional Sobolev spaces. In a next step, we then derive the desired embedding of  $X_1$ , see Proposition 4. In literature, we could detect neither the precise dependence of  $\kappa$  on  $\varepsilon$  and  $\mu$ , nor the distinction between the regularity of the single components of the electric and magnetic field. These results, however, turn out to be essential for the error analysis in Section 6.2 and another paper that is in preparation. For a clear presentation, we hence deduce the desired embeddings in a sequence of lemmas. Note that [7, 11, 12, 8] contain regularity statements for the above or related spaces in a more general setting, allowing general polyhedral domains for instance. Our plan is to transfer parts of the reasoning in paragraphs I.3.3–I.3.5 in [24] to our setting of a transmission problem.

We start with the study of  $H_{N,00}(\text{curl}, \text{div}, \varepsilon, Q)$ .

**Lemma 4.1.** *Let  $\varepsilon$  satisfy (2). The curl-operator is injective on  $H_{N,00}(\text{curl}, \text{div}, \varepsilon, Q)$ .*

*Proof.* Let  $\mathbf{E} \in H_{N,00}(\text{curl}, \text{div}, \varepsilon, Q)$  with  $\text{curl } \mathbf{E} = 0$ . Theorem I.3.4 in [24] provides a potential  $\Phi \in H^1(Q)^3$  with  $\mathbf{E} = \frac{1}{\varepsilon} \text{curl } \Phi$  and  $\text{div } \Phi = 0$  on  $Q$ . Integrating by parts, we obtain the result

$$\int_Q \varepsilon |\mathbf{E}|^2 \, dx = \int_Q (\text{curl } \Phi) \cdot \mathbf{E} \, dx = \int_Q \Phi \cdot \text{curl } \mathbf{E} \, dx = 0. \quad \square$$

We next introduce the space

$$H_\varepsilon := \{\mathbf{E} \in L^2(Q)^3 \mid \operatorname{div}(\varepsilon \mathbf{E}) = 0, \varepsilon \mathbf{E} \cdot \nu = 0 \text{ on } \partial Q\}.$$

Note that  $H_\varepsilon$  is a closed subspace of  $L^2(Q)^3$ . (The divergence operator is closed in  $L^2(Q)^3$  on its maximal domain  $H(\operatorname{div}, Q)$ , and the normal trace operator is bounded on  $H(\operatorname{div}, Q)$ .)

The following statement characterizes the preimage of the curl-operator for the space  $H_\eta$ . The result corresponds to Theorem I.3.6 in [24], and extends Lemma 6.3 in [8] to our setting of multiple submedia in a cuboid. For the statement, we recall the number  $\bar{\kappa}$  from (15), and introduce the space

$$\begin{aligned} \mathcal{V}_{1-\kappa} &:= (PH^{1-\kappa}(Q)^2 \times H^1(Q)) \cap \{v \in L^2(Q)^3 \mid \partial_3 v \in L^2(Q)^3\}, \\ \|v\|_{\mathcal{V}_{1-\kappa}}^2 &:= \|v\|_{PH^{1-\kappa}(Q)^2 \times H^1(Q)}^2 + \|\partial_3 v\|_{L^2(Q)^3}^2, \quad v \in \mathcal{V}_{1-\kappa}. \end{aligned} \quad (86)$$

**Lemma 4.2.** *Let  $\varepsilon$  satisfy (2), and let  $\kappa > 1 - \bar{\kappa}$ . Each function  $\mathbf{E} \in H_\varepsilon$  has the representation*

$$\mathbf{E} = \frac{1}{\varepsilon} \operatorname{curl} \Phi$$

with a unique function  $\Phi \in H_{N,00}(\operatorname{curl}, \operatorname{div} \varepsilon, Q)$ . Moreover,  $\Phi$  belongs to the space  $\mathcal{V}_{1-\kappa}$ , and it satisfies the estimate  $\|\Phi\|_{\mathcal{V}_{1-\kappa}} \leq C \|\mathbf{E}\|_{L^2(Q)}$  with a uniform constant  $C > 0$  depending only on  $\varepsilon, \kappa, Q$ .

*Proof.* 1) Throughout the proof,  $C = C(\varepsilon, \kappa, Q) > 0$  is a constant that is allowed to change from line to line. Lemma 4.1 already implies that there is at most one function  $\Phi$  with the required properties. Consequently, it remains to show the existence of the desired vector  $\Phi$  as well as its regularity.

Using Theorem I.3.6 in [24], there is a vector  $\tilde{\Phi} \in H^1(Q)^3 \cap H_0(\operatorname{curl}, Q)$  with  $\frac{1}{\varepsilon} \operatorname{curl} \tilde{\Phi} = \mathbf{E}$  and  $\operatorname{div} \tilde{\Phi} = 0$  on  $Q$ . The parameter  $\varepsilon$  being piecewise constant on the subcuboids  $Q_1, \dots, Q_N$ , this implies the formula

$$\operatorname{div}(\varepsilon^{(i)} \tilde{\Phi}^{(i)}) = 0. \quad (87)$$

In general,  $\tilde{\Phi}$  does, however, not satisfy the additional transmission condition  $[[\varepsilon \tilde{\Phi} \cdot \nu_{\mathcal{F}}]]_{\mathcal{F}} = 0$  for all interfaces  $\mathcal{F}$ .

2) We next extend the traces  $[[\varepsilon \tilde{\Phi} \cdot \nu_{\mathcal{F}}]]_{\mathcal{F}}$  for the effective interfaces  $\mathcal{F} \in \mathcal{F}_{\text{int}}^{\text{eff}}$ , see the notation paragraph in Section 1. There is a function  $\hat{\psi} \in H^1(Q) \cap PH^2(Q)$  with  $\nabla \hat{\psi} \times \nu = 0$  on  $\partial Q$ ,  $[[\varepsilon \nabla \hat{\psi} \cdot \nu_{\mathcal{F}}]]_{\mathcal{F}} = [[\varepsilon \tilde{\Phi} \cdot \nu_{\mathcal{F}}]]_{\mathcal{F}}$  for  $\mathcal{F} \in \mathcal{F}_{\text{int}}^{\text{eff}}$ , and

$$\|\hat{\psi}\|_{PH^2(Q)} \leq C \sum_{\mathcal{F} \in \mathcal{F}_{\text{int}}^{\text{eff}}} \|[[\varepsilon \tilde{\Phi} \cdot \nu_{\mathcal{F}}]]_{\mathcal{F}}\|_{V(\mathcal{F})} \leq C \|\mathbf{E}\|_{L^2(Q)}. \quad (88)$$

Recall that  $V(\mathcal{F})$  is defined in (11). To show this claim, we consider the model case of four subcuboids

$$\begin{aligned} Q_1 &= (-1, 0)^2 \times (0, 1), & Q_2 &= (0, 1) \times (-1, 0) \times (0, 1), & Q_3 &= (0, 1)^3, \\ Q_4 &= (-1, 0) \times (0, 1)^2, & \mathcal{F}_j &= \overline{Q_j} \cap \overline{Q_{j+1}}, \quad j \in \{1, 2, 3\}, & \mathcal{F}_4 &= \overline{Q_1} \cap \overline{Q_4}, \\ \varepsilon^{(1)} &> \varepsilon^{(2)} = \varepsilon^{(3)} = \varepsilon^{(4)}, \end{aligned}$$

and construct a function  $\hat{\psi}$  on  $\tilde{Q} := (-1, 1)^2 \times (0, 1)$  that satisfies the extension property  $[[\varepsilon \nabla \hat{\psi} \cdot \nu_{\mathcal{F}_1}]]_{\mathcal{F}_1} = [[\varepsilon \tilde{\Phi} \cdot \nu_{\mathcal{F}_1}]]_{\mathcal{F}_1}$ , homogeneous Neumann boundary conditions on  $\partial \tilde{Q} \setminus \Gamma_3$ , homogeneous Dirichlet boundary conditions on  $\partial \tilde{Q} \cap \Gamma_3$ , and the required regularity and energy properties of  $\hat{\psi}$ . Due to symmetry, the trace  $[[\varepsilon \tilde{\Phi} \cdot \nu_{\mathcal{F}_4}]]_{\mathcal{F}_4}$  can be extended in a similar way. The desired function  $\hat{\psi}$  is then obtained by combining this reasoning with a cut-off argument around the edges in  $Q$  and the extension result from Propositions 2.2 and 2.3 in [2].

In the following, we use techniques from the proof of Lemma 3.1 in [20] and Lemma 8.13 in [54]. Set  $g := \tilde{\Phi}_1|_{\mathcal{F}_1}$ , with  $\tilde{\Phi}_1$  denoting the first component of  $\tilde{\Phi}$ . Identify  $\mathcal{F}_1$  with  $[-1, 0] \times [0, 1]$ , and consider the Laplacian  $\Delta_{\mathcal{F}_1}$  on  $\mathcal{F}_1$  with domain

$$\mathcal{D}(\Delta_{\mathcal{F}_1}) := \{u \in H^2(\mathcal{F}_1) \mid u(\cdot, 0) = u(\cdot, 1) = 0, \partial_2 u(0, \cdot) = \partial_2 u(1, \cdot) = 0\}.$$

The operator  $-\Delta_{\mathcal{F}_1}$  is then selfadjoint and positive definite on  $L^2(\mathcal{F}_1)$ . We can hence define positive definite and selfadjoint fractional powers  $(-\Delta_{\mathcal{F}_1})^\gamma$ ,  $\gamma > 0$ , of  $-\Delta_{\mathcal{F}_1}$ . Hence,  $-(-\Delta_{\mathcal{F}_1})^\gamma$  generates an analytic semigroup  $(e^{-t(-\Delta_{\mathcal{F}_1})^\gamma})_{t \geq 0}$ . Note further that the domain of  $(-\Delta_{\mathcal{F}_1})^{1/2}$  coincides with the domain of the bilinear form

$$a(\varphi, \tilde{\varphi}) = \int_{\mathcal{F}_1} \nabla \varphi \cdot \nabla \tilde{\varphi} dx, \quad \mathcal{D}(a) = \{\varphi \in H^1(\mathcal{F}_1) \mid \varphi(\cdot, 0) = \varphi(\cdot, 1) = 0\},$$

that is associated with  $-\Delta_{\mathcal{F}_1}$ , see Theorem VI.2.23 in [35] for instance. Combining furthermore the trace theorem with the boundary conditions for  $\tilde{\Phi}$ , we conclude that  $g$  is an element of the real interpolation space  $(L^2(\mathcal{F}_1), \mathcal{D}(-\Delta_{\mathcal{F}_1})^{1/2})_{1/2, 2}$  with

$$\|g\|_{(L^2(\mathcal{F}_1), \mathcal{D}(-\Delta_{\mathcal{F}_1})^{1/2})_{1/2, 2}} \leq C \|\tilde{\Phi}_1\|_{H^1(Q)}. \quad (89)$$

Let  $\chi : [-1, 1] \rightarrow [0, 1]$  be a smooth cut-off function with  $\chi = 1$  on  $[-1/2, 1/2]$  and support in  $[-3/4, 3/4]$ . We then set

$$\dot{\psi}^{(1)}(x_1, x_2, x_3) := \chi(x_1)x_1(e^{-x_1(-\Delta_{\mathcal{F}_1})^{1/2}}g)(x_2, x_3), \quad (x_1, x_2, x_3) \in Q_1.$$

In consideration of the analyticity of  $(e^{-t(-\Delta_{\mathcal{F}_1})^{1/2}})_{t \geq 0}$ , we conclude the identities

$$\dot{\psi}^{(1)}|_{\mathcal{F}_1} = 0, \quad \partial_1 \dot{\psi}^{(1)}|_{\mathcal{F}_1} = g, \quad \dot{\psi}^{(1)}|_{\Gamma_3} = 0,$$

as well as homogeneous Neumann boundary conditions on all other faces of  $Q_1$ . We further calculate

$$\begin{aligned} \partial_1 \dot{\chi}^{(1)} &= \left( \chi'(x_1)x_1 + \chi(x_1) - \chi(x_1)x_1(-\Delta_{\mathcal{F}_1})^{1/2} \right) e^{-x_1(-\Delta_{\mathcal{F}_1})^{1/2}}g, \\ \partial_1^2 \dot{\psi}^{(1)} &= \left( \chi''(x_1)x_1 + 2\chi'(x_1) - 2\chi'(x_1)x_1(-\Delta_{\mathcal{F}_1})^{1/2} - 2\chi(x_1)(-\Delta_{\mathcal{F}_1})^{1/2} \right. \\ &\quad \left. - \chi(x_1)x_1\Delta_{\mathcal{F}_1} \right) e^{-x_1(-\Delta_{\mathcal{F}_1})^{1/2}}g. \end{aligned}$$

We moreover note that the  $H^1$ - and  $H^2$ -norm on  $\mathcal{F}_1$  are equivalent to the norms  $\|(-\Delta_{\mathcal{F}_1})^{1/2} \cdot\|_{L^2(\mathcal{F}_1)}$  and  $\|\Delta_{\mathcal{F}_1} \cdot\|_{L^2(\mathcal{F}_1)}$  on  $\mathcal{D}(-\Delta_{\mathcal{F}_1})^{1/2}$  and  $\mathcal{D}(\Delta_{\mathcal{F}_1})$ , respectively. Using Remark 6.3 and Proposition 6.4 in [41], we hence conclude that  $\dot{\psi}^{(1)}$  belongs to  $H^2(Q_1)$  with

$$\|\dot{\psi}^{(1)}\|_{H^2(Q_1)} \leq C \|g\|_{(L^2(\mathcal{F}_1), \mathcal{D}(-\Delta_{\mathcal{F}_1})^{1/2})_{1/2, 2}} \leq C \|\tilde{\Phi}_1\|_{H^1(Q)},$$

see (89). Define now

$$\begin{aligned} \dot{\psi}^{(2)}(x_1, x_2, x_3) &:= -\dot{\psi}^{(1)}(-x_1, x_2, x_3), & (x_1, x_2, x_3) \in Q_2, \\ \dot{\psi}^{(3)}(x_1, x_2, x_3) &:= \dot{\psi}^{(2)}(x_1, -x_2, x_3), & (x_1, x_2, x_3) \in Q_3, \\ \dot{\psi}^{(4)}(x_1, x_2, x_3) &:= \dot{\psi}^{(1)}(x_1, -x_2, x_3), & (x_1, x_2, x_3) \in Q_4. \end{aligned}$$

By construction,  $\dot{\psi}$  belongs to  $PH^2(\tilde{Q}) \cap H^1(\tilde{Q})$ , and satisfies the extension property  $\llbracket \varepsilon \nabla \dot{\psi} \cdot \nu_{\mathcal{F}_1} \rrbracket_{\mathcal{F}_1} = \llbracket \varepsilon \tilde{\Phi} \cdot \nu_{\mathcal{F}_1} \rrbracket_{\mathcal{F}_1}$  as well as the continuity relation  $\llbracket \varepsilon \nabla \dot{\psi} \cdot \nu_{\mathcal{F}_j} \rrbracket_{\mathcal{F}_j} = 0$  for  $j \in \{2, 3, 4\}$ . Taking also (5) into account, we obtain the energy estimate

$$\|\dot{\psi}\|_{PH^2(\tilde{Q})} \leq C \|\tilde{\Phi}_1\|_{H^1(Q)} \leq C \|\mathbf{E}\|_{L^2(Q)}.$$

Altogether,  $\dot{\psi}$  is the desired extension on  $\tilde{Q}$ .

3) Proposition 2 provides a unique function  $\tilde{\psi} \in \mathcal{D}(\Delta_{0,\partial Q}) \hookrightarrow \mathcal{V}_{2-\kappa}$  with  $\Delta\tilde{\psi}^{(i)} = \Delta\hat{\psi}^{(i)}$  on  $Q_i$  and

$$\|\tilde{\psi}\|_{\mathcal{V}_{2-\kappa}} \leq C\|\hat{\psi}\|_{PH^2(Q)} \leq C\|\mathbf{E}\|_{L^2(Q)}. \quad (90)$$

Altogether,  $\Phi := \tilde{\Phi} - \nabla\hat{\psi} + \nabla\tilde{\psi}$  is the desired function. The asserted norm estimate is a consequence of (5), (87)–(90) and the definition of  $\mathcal{V}_{2-\kappa}$  in (17).  $\square$

The next proposition summarizes the results of the last two lemmas. The proof is a modification of the one for Theorem I.3.7 in [24]. As an intermediate result of the proof is crucial for the below reasoning, we elaborate the argument.

**Lemma 4.3.** *Let  $\varepsilon$  satisfy (2), and choose  $\kappa > 1 - \bar{\kappa}$ . Then  $H_{N,00}(\text{curl}, \text{div } \varepsilon, Q)$  embeds continuously into  $\mathcal{V}_{1-\kappa}$ .*

*Proof.* Let  $\mathbf{E} \in H_{N,00}(\text{curl}, \text{div } \varepsilon, Q)$ . Lemma 4.2 yields that the operator  $\frac{1}{\varepsilon} \text{curl}$  is bijective from  $H_{N,00}(\text{curl}, \text{div } \varepsilon, Q)$  into  $H_\varepsilon$ . Since it is also bounded and both mentioned spaces are complete, we infer by the open mapping principle that  $\frac{1}{\varepsilon} \text{curl}$  is an isomorphism between these spaces. Lemma 4.2 and Remark I.2.5 in [24] further lead to the identities

$$\begin{aligned} \frac{1}{\varepsilon} \text{curl} (H_{N,00}(\text{curl}, \text{div } \varepsilon, Q)) &= H_\varepsilon \\ &= \frac{1}{\varepsilon} \text{curl} (H_{N,00}(\text{curl}, \text{div } \varepsilon, Q) \cap \mathcal{V}_{1-\kappa}). \end{aligned}$$

As  $\frac{1}{\varepsilon} \text{curl}$  is an isomorphism, this implies that  $H_{N,00}(\text{curl}, \text{div } \varepsilon, Q)$  is a subspace of  $\mathcal{V}_{1-\kappa}$ . The estimate from Lemma 4.2 furthermore yields the relation

$$\|\mathbf{E}\|_{H(\text{curl}, Q)} + \|\mathbf{E}\|_{\mathcal{V}_{1-\kappa}} \leq C\|\frac{1}{\varepsilon} \text{curl} \mathbf{E}\|_{L^2(Q)}, \quad (91)$$

with a uniform constant  $C = C(\varepsilon, \kappa, Q) > 0$ . This means that the inverse  $(1/\varepsilon \text{curl})^{-1}$  is bounded from  $H_\varepsilon$  into  $H_{N,00}(\text{curl}, \text{div } \varepsilon, Q) \cap \mathcal{V}_{1-\kappa}$ . Altogether, the identity  $I = (\frac{1}{\varepsilon} \text{curl})^{-1} \circ \frac{1}{\varepsilon} \text{curl}$  is bounded from  $H_{N,00}(\text{curl}, \text{div } \varepsilon, Q)$  into  $\mathcal{V}_{1-\kappa}$ .  $\square$

In order to show the embedding property of the space  $X_1$  from (13) into  $\mathcal{V}_{1-\kappa} \times PH^1(Q)^3$ , we prove next that one can omit the  $L^2$ -norm in the definition of  $\|\cdot\|_{H_{N,0}}$ .

**Lemma 4.4.** *Let  $\varepsilon$  satisfy (2). The estimate*

$$\|\mathbf{E}\|_{L^2(Q)} \leq C_{N0}(\|\text{curl} \mathbf{E}\|_{L^2(Q)} + \|\text{div}(\varepsilon \mathbf{E})\|_{L^2(Q)})$$

*is valid for all functions  $\mathbf{E} \in H_{N,0}(\text{curl}, \text{div } \varepsilon, Q)$  with a uniform constant  $C_{N0} = C_{N0}(\varepsilon, Q) > 0$ .*

*Proof.* 1) Let  $\mathbf{E} \in H_{N,0}(\text{curl}, \text{div } \varepsilon, Q)$ . The main tool is an appropriate decomposition of  $\mathbf{E}$  into a vector we can apply (91) to, and a remainder. For that purpose, we consider the transmission problem

$$\begin{aligned} \Delta\phi^{(i)} &= \text{div}(\mathbf{E}^{(i)}) && \text{on } Q_i \text{ for } i \in \{1, \dots, N\}, \\ \phi &= 0 && \text{on } \partial Q, \\ \llbracket \phi \rrbracket_{\mathcal{F}} &= \llbracket \varepsilon \nabla \phi \cdot \nu_{\mathcal{F}} \rrbracket_{\mathcal{F}} = 0 && \text{on } \mathcal{F} \in \mathcal{F}_{\text{int}}. \end{aligned} \quad (92)$$

By Proposition 2, this system has a unique solution  $\phi \in \mathcal{V}_{2-\kappa}$  for  $\kappa > 1 - \bar{\kappa}$ . Employing the boundary conditions, we obtain the formula  $\nabla\phi \times \nu = 0$  on  $\partial Q$ , see Lemma 2.1 in [21] for instance. The transmission conditions further imply that the function  $\nabla\phi$  is an element of  $H_{N,0}(\text{curl}, \text{div } \varepsilon, Q)$ . Consequently, the mapping  $\psi := \nabla\phi - \mathbf{E}$  belongs to  $H_{N,00}(\text{curl}, \text{div } \varepsilon, Q)$ , and we can apply inequality (91) to it. In this way, we obtain the relations

$$\|\mathbf{E}\|_{L^2(Q)} \leq \|\psi\|_{L^2(Q)} + \|\nabla\phi\|_{L^2(Q)} \leq \frac{C}{\min \varepsilon} \|\text{curl} \mathbf{E}\|_{L^2(Q)} + \|\nabla\phi\|_{L^2(Q)},$$

where  $C$  is the uniform constant from (91). In view of the weak formulation of system (92) and the Poincaré inequality, we infer the estimates

$$\begin{aligned} \|\nabla\phi\|_{L^2(Q)}^2 &\leq -\frac{1}{\min \varepsilon} \int_Q \phi \operatorname{div}(\varepsilon \mathbf{E}) \, dx \leq \frac{1}{\min \varepsilon} \|\phi\|_{L^2(Q)} \|\operatorname{div}(\varepsilon \mathbf{E})\|_{L^2(Q)} \\ &\leq \frac{C_P}{\min \varepsilon} \|\nabla\phi\|_{L^2(Q)} \|\operatorname{div}(\varepsilon \mathbf{E})\|_{L^2(Q)}, \end{aligned} \quad (93)$$

employing the Poincaré constant  $C_P > 0$  for  $Q$ .  $\square$

In view of the assumptions (2), the parameter  $\mu$  is piecewise constant on the chain  $\tilde{Q}_1, \dots, \tilde{Q}_L$  of cuboids. As the setting of two cuboids from [54] transfers to the partition  $\bigcup_{l=1}^L \tilde{Q}_L$  in a straightforward way, the reasoning for Proposition 9.7 in [54] yields the following statement.

**Lemma 4.5.** *Let  $\mu$  satisfy (2). The space  $H_{T,00}(\operatorname{curl}, \operatorname{div} \mu, Q)$  embeds continuously into  $PH^1(Q)^3$ .*

We now deduce the desired regularity statement for functions in the space  $X_1$ . For the statement, recall the number  $\bar{\kappa}$  from (15) and the space  $\mathcal{V}_{1-\bar{\kappa}}$  from (86).

**Proposition 4.** *Let  $\varepsilon, \mu$  satisfy (2), and  $\kappa > 1 - \bar{\kappa}$ . The space  $X_1$  embeds continuously into  $\mathcal{V}_{1-\kappa} \times PH^1(Q)^3$ .*

*Proof.* 1) Let  $(\mathbf{E}, \mathbf{H}) \in X_1 = \mathcal{D}(M) \cap X_0$ . We first show the asserted regularity of  $(\mathbf{E}, \mathbf{H})$ . In view of Lemma 4.5, it remains to deal with the electric field  $\mathbf{E}$ .

Consider the elliptic transmission problem

$$\begin{aligned} \Delta\psi^{(i)} &= \operatorname{div} \mathbf{E}^{(i)} && \text{on } Q_i \text{ for } i \in \{1, \dots, N\}, \\ \psi &= 0 && \text{on } \partial Q, \\ \llbracket \psi \rrbracket_{\mathcal{F}} &= 0 && \text{for } \mathcal{F} \in \mathcal{F}_{\text{int}}, \\ \llbracket \varepsilon \nabla \psi \cdot \nu_{\mathcal{F}} \rrbracket_{\mathcal{F}} &= \llbracket \varepsilon \mathbf{E} \cdot \nu_{\mathcal{F}} \rrbracket_{\mathcal{F}} && \text{for } \mathcal{F} \in \mathcal{F}_{\text{int}}, \end{aligned} \quad (94)$$

which has a unique solution  $\psi \in \mathcal{V}_{2-\kappa} \cap H_0^1(Q)$ . (The space  $\mathcal{V}_{2-\kappa}$  is defined in (17).) Indeed, a modification of the reasoning in the proof for Lemma 4.2 and the precondition  $\llbracket \varepsilon \mathbf{E} \cdot \nu_{\mathcal{F}} \rrbracket_{\mathcal{F}} \in V(\mathcal{F})$ ,  $\mathcal{F} \in \mathcal{F}_{\text{int}}^{\text{eff}}$ , see (12), yield a unique mapping  $\tilde{\psi} \in \mathcal{V}_{2-\kappa} \cap H_0^1(Q)$  with  $\Delta\tilde{\psi}^{(i)} = 0$  on  $Q_i$ , satisfying the required boundary and transmission conditions in (94). We then arrive at the transmission problem (92), having a unique solution  $\check{\psi} \in \mathcal{V}_{2-\kappa} \cap H_0^1(Q)$ , see Proposition 2. Altogether,  $\psi := \tilde{\psi} + \check{\psi} \in \mathcal{V}_{2-\kappa} \cap H_0^1(Q)$  is the unique solution of (94).

Hence,  $\mathbf{E} - \nabla\psi$  is an element of  $H_{N,00}(\operatorname{curl}, \operatorname{div} \varepsilon, Q) \subseteq \mathcal{V}_{1-\kappa}$ , see Lemma 4.3. The vector  $\nabla\psi$  being an element of  $\mathcal{V}_{1-\kappa}$ , we infer the stated regularity result.

2) It remains to show the asserted embedding property. In the following,  $C = C(\varepsilon, \kappa, Q) > 0$  is a constant that changes from line to line. Lemma 4.5 yields the required estimate for  $\mathbf{H}$ , whence we again only treat the electric field component  $\mathbf{E}$ . Proposition 2 yields the estimate

$$\|\check{\psi}\|_{\mathcal{V}_{2-\kappa}} \leq C \sum_{i=1}^N \|\operatorname{div}(\varepsilon^{(i)} \mathbf{E}^{(i)})\|_{L^2(Q_i)}. \quad (95)$$

The reasoning for (88) and (90) furthermore leads to the bound

$$\|\nabla\check{\psi}\|_{\mathcal{V}_{1-\kappa}} \leq C_2 \sum_{\mathcal{F} \in \mathcal{F}_{\text{int}}^{\text{eff}}} \|\llbracket \varepsilon \mathbf{E} \cdot \nu_{\mathcal{F}} \rrbracket_{\mathcal{F}}\|_{V(\mathcal{F})}. \quad (96)$$

Applying Lemma 4.3 to  $\mathbf{E} - \nabla\psi$ , the relations

$$\|\mathbf{E}\|_{\mathcal{V}_{1-\kappa}} \leq \|\mathbf{E} - \nabla\psi\|_{\mathcal{V}_{1-\kappa}} + \|\nabla\psi\|_{\mathcal{V}_{1-\kappa}}$$

$$\begin{aligned}
 &\leq C(\|\mathbf{E}\|_{L^2(Q)} + \|\operatorname{curl} \mathbf{E}\|_{L^2(Q)} + \|\nabla \psi\|_{L^2(Q)}) + \|\nabla \psi\|_{V_{1-\kappa}} \\
 &\leq C(\|M(\mathbf{E}, \mathbf{H})\| + \|\mathbf{E}\|_{L^2(Q)} + \|\check{\psi}\|_{V_{2-\kappa}} + \|\nabla \check{\psi}\|_{V_{1-\kappa}})
 \end{aligned}$$

follow. The desired embedding is a consequence of (95) and (96).  $\square$

## 5. WELLPOSEDNESS OF THE MAXWELL SYSTEM IN $X_1$

The main result of Section 4 establishes a regularity statement for the space  $X_1$ , see Proposition 4. To conclude a corresponding regularity result for the solutions of the Maxwell system (1), we show in this Section that  $X_1$  is a state space of (1). This is done by means of semigroup theory.

The proof of the next proposition transfers techniques from the proof of Proposition 2.3 in [21] to the current setting. Recall for the statement that  $M_1$  is the part of  $M$  in  $X_1$ .

**Proposition 5.** *Let  $\varepsilon$  and  $\mu$  satisfy (2). The part  $M_1$  of  $M$  generates a contractive  $C_0$ -semigroup  $(e^{tM_1})_{t \geq 0}$  on  $X_1$ . The family  $(e^{tM_1})_{t \geq 0}$  is the restriction of  $(e^{tM})_{t \geq 0}$  to  $X_1$ .*

*Proof.* 1) Employing the theory of subspace semigroups, see for instance Paragraph II.2.3 in [22], it suffices to show that the family  $(e^{tM})_{t \geq 0}$  leaves the space  $X_1$  invariant, and that it is strongly continuous on it.

We first note that semigroup theory implies the inclusion  $e^{tM}(\mathcal{D}(M)) \subseteq \mathcal{D}(M)$  for  $t \geq 0$ . Regarding the magnetic conditions, the arguments in the proof of Proposition 2.3 in [21] apply also here. This reasoning results in the invariance of the space

$$X_{\text{mag}} := \{(u, v) \in X \mid \operatorname{div}(\mu v) = 0, (\mu v) \cdot \nu = 0 \text{ on } \partial Q\}$$

under the resolvent map  $R(\lambda, M)$  for  $\lambda > 0$ , and in the invariance of  $X_{\text{mag}}$  with respect to the family  $(e^{tM})_{t \geq 0}$ .

2) Let  $(\tilde{u}, \tilde{v}) \in X_1$ , and set  $(u(t), v(t)) := e^{tM}(\tilde{u}, \tilde{v})$  for  $t \geq 0$ . Semigroup theory then yields that the function  $(u, v)$  belongs to  $C([0, \infty), \mathcal{D}(M))$ . The Maxwell equations (1) with  $\mathbf{J} = 0$  lead to the formula

$$\partial_t u = \frac{1}{\varepsilon} \operatorname{curl} v, \quad t \geq 0.$$

Taking the divergence of this equation, the relation  $\partial_t \operatorname{div}(\varepsilon u(t)) = 0$  follows in  $L^2(\tilde{Q}_{i,l})$ ,  $i \in \{1, \dots, L\}$ ,  $l \in \{0, \dots, K\}$ , for the subdomains  $(\tilde{Q}_{i,l})$  from Section 1. This is equivalent to the identity

$$\operatorname{div}(\varepsilon u(t)) = \operatorname{div}(\varepsilon \tilde{u}) \tag{97}$$

on  $\tilde{Q}_{i,l}$ . As a result, the mapping  $[0, \infty) \rightarrow H(\operatorname{div}, \tilde{Q}_{i,k})$ ,  $t \mapsto \varepsilon u(t)$  is continuously differentiable (employing here the continuous differentiability in time of  $u$  on  $X$ ). Due to the continuity of the normal trace operator on  $H(\operatorname{div}, \tilde{Q}_{i,l})$ , the relations

$$\partial_t \llbracket \varepsilon u(t) \cdot \nu_{\mathcal{F}} \rrbracket_{\mathcal{F}} = \llbracket \operatorname{curl} \tilde{v} \cdot \nu_{\mathcal{F}} \rrbracket_{\mathcal{F}} = 0, \quad t \geq 0,$$

follow in  $H^{-1/2}(\mathcal{F})$  for every effective interface  $\mathcal{F} \in \mathcal{F}_{\text{int}}^{\text{eff}}$ , see the notation paragraph in Section 1. This shows that the function

$$\llbracket \varepsilon u(t) \cdot \nu_{\mathcal{F}} \rrbracket_{\mathcal{F}} = \llbracket \varepsilon \tilde{u} \cdot \nu_{\mathcal{F}} \rrbracket_{\mathcal{F}} \tag{98}$$

belongs to the space  $V(\mathcal{F})$  from (11) for  $\mathcal{F} \in \mathcal{F}_{\text{int}}^{\text{eff}}$ , and that the mapping  $[0, \infty) \rightarrow V(\mathcal{F})$ ,  $t \mapsto \llbracket \varepsilon u(t) \cdot \nu_{\mathcal{F}} \rrbracket_{\mathcal{F}}$  is continuously differentiable.

Altogether, we have derived that the vector  $(u(t), v(t))$  belongs to  $X_1$  for every  $t \geq 0$ , and that  $(u, v)$  is continuous on  $X_1$ . The contractivity of  $(e^{tM})_{t \geq 0}$  on  $X$ , as well as (97) and (98) imply the contractivity of  $(e^{tM}|_{X_1})_{t \geq 0}$  on  $X_1$ .  $\square$

The next statement is a conclusion of Proposition 5, and it transfers parts of Proposition 2.3 from [21] to our setting of discontinuous coefficients. Although the proof basically follows the lines of the one for Corollary 9.24 in [54], we present it here for the sake of a self-contained presentation. Note that the formula for  $\rho_{\mathcal{F}}$  is also deduced in Section 1 of [45]. For the external current density  $\mathbf{J}$ , the space

$$W := L^1([0, T], \mathcal{D}(M_1)) + W^{1,1}([0, T], X_1),$$

$$\|f\|_W := \inf_{\substack{f=f_1+f_2, \\ f_1 \in L^1([0, T], \mathcal{D}(M_1)), \\ f_2 \in W^{1,1}([0, T], X_1)}} (\|f_1\|_{L^1([0, T], \mathcal{D}(M_1))} + \|f_2\|_{W^{1,1}([0, T], X_1)}), \quad f \in W,$$

is employed for fixed  $T > 0$ .

**Corollary 2.** *Let  $\varepsilon$  and  $\mu$  satisfy (2). Let  $T > 0$ ,  $w_0 = (\mathbf{E}_0, \mathbf{H}_0)$  be initial data from  $\mathcal{D}(M_1) = \mathcal{D}(M^2) \cap X_0$ , and let  $g := (\frac{1}{\varepsilon}\mathbf{J}, 0) : [0, T] \rightarrow X_1$  be the weighted external current density that is continuous, and an element of  $W$ . The following items are valid.*

a) *The Maxwell system (1) possesses a unique classical solution  $w = (\mathbf{E}, \mathbf{H})$ , belonging to  $C([0, T], \mathcal{D}(M_1)) \cap C^1([0, T], X_1)$ . It satisfies the bounds*

$$\|w(t)\|_{X_1} \leq \|w_0\|_{X_1} + \|g\|_{L^1([0, t], X_1)},$$

$$\|Mw(t)\|_{X_1} \leq \|w_0\|_{\mathcal{D}(M_1)} + \left(\frac{2}{T} + 3\right) \|g\|_W,$$

for  $t \in [0, T]$ .

b) *The volume charge density  $\rho^{(i)}$  on  $Q_i$  and the surface charge  $\rho_{\mathcal{F}}$  are given via*

$$\rho^{(i)}(t) = \operatorname{div}(\varepsilon^{(i)} \mathbf{E}^{(i)}(t)) = \operatorname{div}(\varepsilon^{(i)} \mathbf{E}_0^{(i)}) - \int_0^t \operatorname{div}(\mathbf{J}^{(i)}(s)) \, ds,$$

$$\rho_{\mathcal{F}}(t) = \llbracket \varepsilon \mathbf{E}(t) \cdot \nu_{\mathcal{F}} \rrbracket_{\mathcal{F}} = \llbracket \varepsilon \mathbf{E}_0 \cdot \nu_{\mathcal{F}} \rrbracket_{\mathcal{F}} - \int_0^t \llbracket \mathbf{J}(s) \cdot \nu_{\mathcal{F}} \rrbracket_{\mathcal{F}} \, ds,$$

for  $t \in [0, T]$ ,  $i \in \{1, \dots, N\}$ , and  $\mathcal{F} \in \mathcal{F}_{\text{int}}^{\text{eff}}$ .

*Proof.* a) The stated classical wellposedness of (1) on  $X_1$  follows from Proposition 5 and semigroup theory, see Theorem 8.1.4 in [51] for instance. Duhamel's formula leads to the representation

$$w(t) = e^{tM_1} w_0 + \int_0^t e^{(t-s)M_1} g(s) \, ds = e^{tM_1} w_0 + \int_0^t e^{(t-s)M_1} \left(\frac{1}{\varepsilon}\mathbf{J}(s), 0\right) \, ds.$$

Taking the contractivity of  $(e^{tM_1})_{t \geq 0}$  into account, the relations

$$\|w(t)\|_{X_1} \leq \|w_0\|_{X_1} + \int_0^t \left\| \left(\frac{1}{\varepsilon}\mathbf{J}(s), 0\right) \right\|_{X_1} \, ds$$

$$= \|w_0\|_{X_1} + \left\| \left(\frac{1}{\varepsilon}\mathbf{J}, 0\right) \right\|_{L^1([0, T], X_1)}$$

follow.

Let  $(\frac{1}{\varepsilon}\mathbf{J}, 0) \in W$ ,  $\zeta > 0$ , and  $\mathbf{J}_1 \in L^1([0, T], \mathcal{D}(M_1))$ ,  $\mathbf{J}_2 \in W^{1,1}([0, T], X_1)$  with  $(\frac{1}{\varepsilon}\mathbf{J}, 0) = \mathbf{J}_1 + \mathbf{J}_2$  and

$$\left\| \left(\frac{1}{\varepsilon}\mathbf{J}, 0\right) \right\|_W \geq \|\mathbf{J}_1\|_{L^1([0, T], \mathcal{D}(M_1))} + \|\mathbf{J}_2\|_{W^{1,1}([0, T], X_1)} - \zeta.$$

An integration by parts in the above Duhamel formula leads to the identities

$$Mw(t) = e^{tM} Mw_0 + \int_0^t M e^{(t-s)M} (\mathbf{J}_1(s) + \mathbf{J}_2(s)) \, ds$$

$$= e^{tM} Mw_0 + \int_0^t e^{(t-s)M} M \mathbf{J}_1(s) \, ds - \int_0^t \left(\frac{d}{ds} e^{(t-s)M}\right) \mathbf{J}_2(s) \, ds$$



$$\begin{aligned}
 &= e^{tM} M w_0 + \int_0^t e^{(t-s)M} M \mathbf{J}_1(s) \, ds - \mathbf{J}_2(t) + e^{tM} \mathbf{J}_2(0) \\
 &\quad + \int_0^t e^{(t-s)M} \mathbf{J}'_2(s) \, ds.
 \end{aligned}$$

Combining Lemma 7.6 in [54] with Proposition 5, the relations

$$\begin{aligned}
 \|Mw(t)\|_{X_1} &\leq \|w_0\|_{\mathcal{D}(M_1)} + \|\mathbf{J}_1\|_{L^1([0,T],\mathcal{D}(M_1))} + \left(\frac{2}{T} + 3\right) \|\mathbf{J}_2\|_{W^{1,1}([0,T],X_1)} \\
 &\leq \|w_0\|_{\mathcal{D}(M_1)} + \left(\frac{2}{T} + 3\right) (\|(\frac{1}{\varepsilon}\mathbf{J}, 0)\|_{W_1} + \zeta)
 \end{aligned}$$

are derived. Letting  $\zeta$  tend to zero, we infer the second stated estimate.

b) The representation for the current density is obtained by modifying the arguments from Proposition 2.3 in [21] and part 2) from the proof of Proposition 5. The linear mapping  $X_1 \rightarrow L^2(Q_i)$ ,  $(u, v) \mapsto \operatorname{div}(\varepsilon^{(i)}u^{(i)})$  being continuous for  $i \in \{1, \dots, N\}$ , the regularity of  $w$  implies that  $\rho^{(i)} : [0, T] \rightarrow L^2(Q_i)$  is continuously differentiable. Similar reasoning further shows that  $[0, T] \rightarrow L^2(Q_i)$ ,  $s \mapsto \operatorname{div}(\mathbf{J}^{(i)}(s))$  is continuous. Taking the divergence in (1), leads to

$$\partial_t \operatorname{div}(\varepsilon^{(i)}\mathbf{E}^{(i)}(t)) = -\operatorname{div}(\mathbf{J}^{(i)}(t)), \quad t \in [0, T],$$

in  $L^2(Q_i)$ . The first asserted formula is obtained by integration with respect to  $t$ . Analogously, the arguments in part 2) of the proof for Proposition 5 result in the stated formula for the surface charge  $\rho_{\mathcal{F}}$  in  $V(\mathcal{F})$  for every effective interface  $\mathcal{F} \in \mathcal{F}_{\text{int}}^{\text{eff}}$ .  $\square$

**Remark 2.** In view of Proposition 4, Corollary 2 provides a classical solution of the Maxwell system (1) in the space  $C^1([0, T], \mathcal{V}_{1-\kappa} \times PH^1(Q)^3)$  for  $\kappa > 1 - \bar{\kappa}$  with the number  $\bar{\kappa}$  from (15) and the space  $\mathcal{V}_{1-\kappa}$  from (86).  $\diamond$

## 6. ANALYSIS OF A DIRECTIONAL SPLITTING SCHEME

This section is concerned with the construction and analysis of a directional splitting scheme for (1). The scheme can deal with the low regularity of the solution of the Maxwell system, see Remark 2. In particular, the regularity requirement for the initial data is weaker than for the ADI schemes from [56, 42, 10], see [29, 21, 19, 20, 23]. In Section 6.1, we introduce the splitting and analyze the splitting operators. We furthermore comment on the efficiency of the scheme. Subsequently, we bound the error of the scheme in Section 6.2. Here the regularity results from Section 5 are essential.

**6.1. Construction of a directional splitting scheme.** In view of the  $H^1$ -regularity in  $x_3$ -direction of the solution to (1), see Remark 2, we split the  $x_3$ -coordinate off and leave the  $x_1, x_2$  coordinates coupled. This strategy leads to the splitting

$$\begin{aligned}
 M \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} &= \begin{pmatrix} \frac{1}{\varepsilon} \operatorname{curl} \mathbf{H} \\ -\frac{1}{\mu} \operatorname{curl} \mathbf{E} \end{pmatrix} = \begin{pmatrix} \frac{1}{\varepsilon} \partial_2 \mathbf{H}_3 \\ -\frac{1}{\varepsilon} \partial_1 \mathbf{H}_3 \\ \frac{1}{\varepsilon} \operatorname{curl}_2(\mathbf{H}_1, \mathbf{H}_2) \\ -\frac{1}{\mu} \partial_2 \mathbf{E}_3 \\ \frac{1}{\mu} \partial_1 \mathbf{E}_3 \\ -\frac{1}{\mu} \operatorname{curl}_2(\mathbf{E}_1, \mathbf{E}_2) \end{pmatrix} + \begin{pmatrix} -\frac{1}{\varepsilon} \partial_3 \mathbf{H}_2 \\ \frac{1}{\varepsilon} \partial_3 \mathbf{H}_1 \\ 0 \\ \frac{1}{\mu} \partial_3 \mathbf{E}_2 \\ -\frac{1}{\mu} \partial_3 \mathbf{E}_1 \\ 0 \end{pmatrix} \\
 &=: A \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} + B \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \tag{99}
 \end{aligned}$$

involving the  $\operatorname{curl}_2$ -operator from Section 2.1. To define appropriate domains for the operators  $A$  and  $B$ , we denote  $S := (a_1^-, a_1^+) \times (a_2^-, a_2^+)$ , using the representation

$Q = (a_1^-, a_1^+) \times (a_2^-, a_2^+) \times (a_3^-, a_3^+)$ . Employing also the space  $H_0(\text{curl}_2, S)$  from Section 2.1, we consider the splitting operators  $A$  and  $B$  on the domains

$$\begin{aligned} \mathcal{D}(A) &:= \{(\mathbf{E}, \mathbf{H}) \in L^2(Q)^6 \mid (\mathbf{E}_1, \mathbf{E}_2) \in L^2((a_3^-, a_3^+), H_0(\text{curl}_2, S)), \\ &\quad (\mathbf{H}_1, \mathbf{H}_2) \in L^2((a_3^-, a_3^+), H(\text{curl}_2, S)), \\ &\quad \partial_1 \mathbf{E}_3, \partial_2 \mathbf{E}_3, \partial_1 \mathbf{H}_3, \partial_2 \mathbf{H}_3 \in L^2(Q), \\ &\quad \mathbf{E}_3 = 0 \text{ on } \Gamma_1 \cup \Gamma_2\}, \\ \mathcal{D}(B) &:= \{(\mathbf{E}, \mathbf{H}) \in L^2(Q)^6 \mid \partial_3 \mathbf{E}_1, \partial_3 \mathbf{E}_2, \partial_3 \mathbf{H}_1, \partial_3 \mathbf{H}_2 \in L^2(Q), \\ &\quad \mathbf{E}_1 = 0 = \mathbf{E}_2 \text{ on } \Gamma_3\}. \end{aligned} \quad (100)$$

With these domains, the operators  $A$  and  $B$  are closed and densely defined on  $X = L^2(Q)^6$ . Note additionally that Corollary 2 provides a classical solution of the Maxwell system (1) that is contained in  $\mathcal{D}(A) \cap \mathcal{D}(B)$ . (This follows from Remark 2 and the embedding of  $X_1$  into  $\mathcal{D}(M)$ .)

Let  $\tau \in (0, T)$  be a fixed time step size,  $n \in \mathbb{N}$  with  $n\tau \leq T$ , and  $(\frac{1}{\varepsilon} \mathbf{J}, 0) \in C([0, T], X_1)$ . We then approximate the solution  $(\mathbf{E}, \mathbf{H})$  of (1) with initial datum  $(\mathbf{E}_0, \mathbf{H}_0)$  at time  $t_n := \tau n \leq T$  by means of the Peaceman-Rachford directional splitting

$$\begin{aligned} \begin{pmatrix} \mathbf{E}^n \\ \mathbf{H}^n \end{pmatrix} &= \mathcal{T}_{\tau, n} \left[ \begin{pmatrix} \mathbf{E}^{n-1} \\ \mathbf{H}^{n-1} \end{pmatrix} \right] = (I - \frac{\tau}{2} B)^{-1} (I + \frac{\tau}{2} A) \left[ (I - \frac{\tau}{2} A)^{-1} (I + \frac{\tau}{2} B) \begin{pmatrix} \mathbf{E}^{n-1} \\ \mathbf{H}^{n-1} \end{pmatrix} \right. \\ &\quad \left. - \frac{\tau}{2\varepsilon} \begin{pmatrix} \mathbf{J}(t_{n-1}) + \mathbf{J}(t_n) \\ 0 \end{pmatrix} \right] \end{aligned} \quad (101)$$

with exact initial data  $(\mathbf{E}^0, \mathbf{H}^0) = (\mathbf{E}_0, \mathbf{H}_0) \in X_1$ . For a different operator splitting, this Peaceman-Rachford time integrator is employed in [43, 56, 42, 21, 29, 30, 46, 38, 18, 20, 31, 47, 54], for instance.

In the next two lemmas we derive that both splitting operators are skewadjoint. This implies that the scheme (101) is well defined and unconditionally stable, see Lemma 6.3. Recall that the inner product on  $X = L^2(Q)^6$  is defined in Section 2.2.

**Lemma 6.1.** *Let  $\varepsilon$  and  $\mu$  satisfy (2). The operators  $A$  and  $B$  are skewsymmetric on  $X$ .*

*Proof.* 1) Let  $(\mathbf{E}, \mathbf{H}), (\tilde{\mathbf{E}}, \tilde{\mathbf{H}}) \in \mathcal{D}(A)$ . We next employ Green's identities from Section 2.1. Taking the boundary conditions in  $\mathcal{D}(A)$  into account, we infer the equations

$$\begin{aligned} \left( A \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}, \begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{H}} \end{pmatrix} \right) &= \int_Q [(\partial_2 \mathbf{H}_3) \tilde{E}_1 - (\partial_1 \mathbf{H}_3) \tilde{E}_2 + (\text{curl}_2(\mathbf{H}_1, \mathbf{H}_2)) \tilde{E}_3 - (\partial_2 \mathbf{E}_3) \tilde{H}_1 \\ &\quad + (\partial_1 \mathbf{E}_3) \tilde{H}_2 - (\text{curl}_2(\mathbf{E}_1, \mathbf{E}_2)) \tilde{H}_3] dx \\ &= \int_Q [\mathbf{H}_3 \text{curl}_2(\tilde{\mathbf{E}}_1, \tilde{\mathbf{E}}_2) + \mathbf{H}_1 \partial_2 \tilde{E}_3 - \mathbf{H}_2 \partial_1 \tilde{E}_3 - \mathbf{E}_3 \text{curl}_2(\tilde{\mathbf{H}}_1, \tilde{\mathbf{H}}_2) \\ &\quad - \mathbf{E}_1 \partial_2 \tilde{H}_3 + \mathbf{E}_2 \partial_1 \tilde{H}_3] dx \\ &= - \left( \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}, A \begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{H}} \end{pmatrix} \right). \end{aligned}$$

This shows that  $A$  is skewsymmetric.

2) Let  $(\check{\mathbf{E}}, \check{\mathbf{H}}), (\hat{\mathbf{E}}, \hat{\mathbf{H}}) \in \mathcal{D}(B)$ . Using the boundary conditions in  $\mathcal{D}(B)$  for an integration by parts, we arrive at the identities

$$\left( B \begin{pmatrix} \check{\mathbf{E}} \\ \check{\mathbf{H}} \end{pmatrix}, \begin{pmatrix} \hat{\mathbf{E}} \\ \hat{\mathbf{H}} \end{pmatrix} \right) = \int_Q [ - (\partial_3 \check{\mathbf{H}}_2) \hat{E}_1 + (\partial_3 \check{\mathbf{H}}_1) \hat{E}_2 + (\partial_3 \check{\mathbf{E}}_2) \hat{H}_1 - (\partial_3 \check{\mathbf{E}}_1) \hat{H}_2 ] dx$$

$$\begin{aligned}
 &= \int_Q [\check{\mathbf{H}}_2 \partial_3 \hat{\mathbf{E}}_1 - \check{\mathbf{H}}_1 \partial_3 \hat{\mathbf{E}}_2 - \check{\mathbf{E}}_2 \partial_3 \hat{\mathbf{H}}_1 + \check{\mathbf{E}}_1 \partial_3 \hat{\mathbf{H}}_2] dx \\
 &= - \left( \begin{pmatrix} \check{\mathbf{E}} \\ \check{\mathbf{H}} \end{pmatrix}, B \begin{pmatrix} \hat{\mathbf{E}} \\ \hat{\mathbf{H}} \end{pmatrix} \right).
 \end{aligned}$$

Hence the operator  $B$  is also skewsymmetric.  $\square$

Using arguments from the proof of Lemma 4.1 in [29], we next conclude that both splitting operators are skewadjoint.

**Lemma 6.2.** *Let  $\varepsilon$  and  $\mu$  satisfy (2). The operators  $A$  and  $B$  are skewadjoint on  $X$ . In particular, the operators  $(I - \tau L)^{-1}$  and  $(I + \tau L)(I - \tau L)^{-1}$  are contractive on  $X$  for  $L \in \{A, B\}$  and  $\tau > 0$ .*

*Proof.* 1) As  $A$  and  $B$  are densely defined, closed, and skewsymmetric, see Lemma 6.1, it suffices to show that the operators  $I \pm A$  and  $I \pm B$  have dense range in  $X$ . We only consider the operators  $I - A$  and  $I - B$ , and show that the space of test functions  $C_c^\infty(Q)^6$  is contained in their range. (The operators  $I + A$  and  $I + B$  can be treated with the same arguments.)

2) Let  $(\tilde{\mathbf{E}}, \tilde{\mathbf{H}}) \in C_c^\infty(Q)^6$ . We want to show the existence of a vector  $(\mathbf{E}, \mathbf{H}) \in \mathcal{D}(A)$  with  $(I - A)(\mathbf{E}, \mathbf{H}) = (\tilde{\mathbf{E}}, \tilde{\mathbf{H}})$ . This is equivalent to the system

$$\begin{aligned}
 \mathbf{E}_1 - \frac{1}{\varepsilon} \partial_2 \mathbf{H}_3 &= \tilde{\mathbf{E}}_1, & \mathbf{H}_1 + \frac{1}{\mu} \partial_2 \mathbf{E}_3 &= \tilde{\mathbf{H}}_1, \\
 \mathbf{E}_2 + \frac{1}{\varepsilon} \partial_1 \mathbf{H}_3 &= \tilde{\mathbf{E}}_2, & \mathbf{H}_2 - \frac{1}{\mu} \partial_1 \mathbf{E}_3 &= \tilde{\mathbf{H}}_2, \\
 \mathbf{E}_3 - \frac{1}{\varepsilon} \partial_1 \mathbf{H}_2 + \frac{1}{\varepsilon} \partial_2 \mathbf{H}_1 &= \tilde{\mathbf{E}}_3, & \mathbf{H}_3 - \frac{1}{\mu} \partial_2 \mathbf{E}_1 + \frac{1}{\mu} \partial_1 \mathbf{E}_2 &= \tilde{\mathbf{H}}_3.
 \end{aligned} \tag{102}$$

By formally inserting the left equations of the first and second line into the right equation of the third line, we derive the formula

$$\mu \mathbf{H}_3 - \partial_1 \left( \frac{1}{\varepsilon} \partial_1 \mathbf{H}_3 \right) - \partial_2 \left( \frac{1}{\varepsilon} \partial_2 \mathbf{H}_3 \right) = \mu \tilde{\mathbf{H}}_3 + \partial_2 \tilde{\mathbf{E}}_1 - \partial_1 \tilde{\mathbf{E}}_2 =: f \in L^2(Q). \tag{103}$$

2) Recall the rectangle  $S := (a_1^-, a_1^+) \times (a_2^-, a_2^+)$ , being the projection of  $Q = (a_1^-, a_1^+) \times (a_2^-, a_2^+) \times (a_3^-, a_3^+)$  to the  $x_1 - x_2$  plane. We consider the equation

$$\begin{aligned}
 &\int_{a_3^-}^{a_3^+} \int_S [\mu w \varphi + \frac{1}{\varepsilon} (\nabla_{x_1, x_2} w) \cdot (\nabla_{x_1, x_2} \varphi)] d(x_1, x_2) dx_3 \\
 &= \int_{a_3^-}^{a_3^+} \int_S f \varphi d(x_1, x_2) dx_3, \quad \varphi \in L^2((a_3^-, a_3^+), H^1(S)),
 \end{aligned} \tag{104}$$

being the weak formulation of (103). The Lax-Milgram Lemma provides a unique solution  $w \in L^2((a_3^-, a_3^+), H^1(S))$  of (104). Taking  $\varphi \in L^2((a_3^-, a_3^+), C_c^\infty(S))$  in (104), we moreover obtain that  $H_3 := w$  satisfies (103) and that  $\frac{1}{\varepsilon} \nabla_{x_1, x_2} \mathbf{H}_3$  is an element of  $L^2((a_3^-, a_3^+), H_0(\text{div}, S))$ . Put

$$\mathbf{E}_1 := \tilde{\mathbf{E}}_1 + \frac{1}{\varepsilon} \partial_2 \mathbf{H}_3, \quad \mathbf{E}_2 := \tilde{\mathbf{E}}_2 - \frac{1}{\varepsilon} \partial_1 \mathbf{H}_3. \tag{105}$$

By construction, the left equations in the first and second line of (102) are then fulfilled. Using (103), we then derive the relation

$$\partial_2 \mathbf{E}_1 - \partial_1 \mathbf{E}_2 = \partial_2 \tilde{\mathbf{E}}_1 - \partial_1 \tilde{\mathbf{E}}_2 + \text{div}_{x_1, x_2} \left( \frac{1}{\varepsilon} \nabla_{x_1, x_2} \mathbf{H}_3 \right) = \mu (\mathbf{H}_3 - \tilde{\mathbf{H}}_3) \tag{106}$$

in  $H^{-1}(Q)$ . As the right hand side belongs to  $L^2(Q)$ , we infer that  $\text{curl}_2(\mathbf{E}_1, \mathbf{E}_2)$  is an element of  $L^2(Q)$ .

We next deal with the boundary conditions for  $(\mathbf{E}_1, \mathbf{E}_2)$ . Let  $\phi$  be an element of  $L^2((a_3^-, a_3^+), H^1(S))$ . By means of (105)–(106) and the fact  $\frac{1}{\varepsilon} \nabla_{x_1, x_2} \mathbf{H}_3 \in$

$L^2((a_3^-, a_3^+), H_0(\operatorname{div}, S))$ , we calculate

$$\begin{aligned}
& \int_{a_3^-}^{a_3^+} \int_S (\mathbf{E}_1, \mathbf{E}_2) \cdot (\partial_2 \phi, -\partial_1 \phi) \, d(x_1, x_2) \, dx_3 \\
&= \int_{a_3^-}^{a_3^+} \int_S \tilde{\mathbf{E}}_1 \partial_2 \phi + \frac{1}{\varepsilon} (\partial_2 \mathbf{H}_3) \partial_2 \phi - \tilde{\mathbf{E}}_2 \partial_1 \phi + \frac{1}{\varepsilon} (\partial_1 \mathbf{H}_3) \partial_1 \phi \, d(x_1, x_2) \, dx_3 \\
&= \int_{a_3^-}^{a_3^+} \int_S -(\partial_2 \tilde{\mathbf{E}}_1) \phi + (\partial_1 \tilde{\mathbf{E}}_2) \phi - \operatorname{div}(\frac{1}{\varepsilon} \nabla_{x_1, x_2} \mathbf{H}_3) \phi \, d(x_1, x_2) \, dx_3 \\
&= \int_{a_3^-}^{a_3^+} \int_S \operatorname{curl}_2(\mathbf{E}_1, \mathbf{E}_2) \phi \, d(x_1, x_2) \, dx_3.
\end{aligned}$$

With Lemma I.2.4 in [24] we conclude  $(\mathbf{E}_1, \mathbf{E}_2) \in L^2((a_3^-, a_3^+), H_0(\operatorname{curl}_2, S))$ .

3) Treating the remaining equations in (102) in a similar fashion, we arrive at a desired vector  $(\mathbf{E}, \mathbf{H}) \in \mathcal{D}(A)$  with  $(I - A)(\mathbf{E}, \mathbf{H}) = (\tilde{\mathbf{E}}, \tilde{\mathbf{H}})$ .

4) We next deal with the splitting operator  $B$ , and proceed similar to the above case for  $A$ . Solving the formula  $(I - B)(\check{\mathbf{E}}, \check{\mathbf{H}}) = (\tilde{\mathbf{E}}, \tilde{\mathbf{H}})$  for  $(\check{\mathbf{E}}, \check{\mathbf{H}}) \in \mathcal{D}(B)$  amounts to determining the solution of the system

$$\begin{aligned}
\check{\mathbf{E}}_1 + \frac{1}{\varepsilon} \partial_3 \check{\mathbf{H}}_2 &= \tilde{\mathbf{E}}_1, & \check{\mathbf{H}}_1 - \frac{1}{\mu} \partial_3 \check{\mathbf{E}}_2 &= \tilde{\mathbf{H}}_1, \\
\check{\mathbf{E}}_2 - \frac{1}{\varepsilon} \partial_3 \check{\mathbf{H}}_1 &= \tilde{\mathbf{E}}_2, & \check{\mathbf{H}}_2 + \frac{1}{\mu} \partial_3 \check{\mathbf{E}}_1 &= \tilde{\mathbf{H}}_2, \\
\check{\mathbf{E}}_3 &= \tilde{\mathbf{E}}_3, & \check{\mathbf{H}}_3 &= \tilde{\mathbf{H}}_3.
\end{aligned} \tag{107}$$

Formally inserting the equation on the right hand side of the second line of (107) into the one on the left hand side of the first line yields

$$\check{\mathbf{E}}_1 - \frac{1}{\varepsilon \mu} \partial_3^2 \check{\mathbf{E}}_1 = \tilde{\mathbf{E}}_1 - \frac{1}{\varepsilon} \partial_3 \tilde{\mathbf{H}}_2 \in L^2(Q). \tag{108}$$

As in the proof of Lemma 4.3 in [29], we obtain a unique  $\check{\mathbf{E}}_1 \in L^2(S, H^2(a_3^-, a_3^+))$  solving (108). (We use here the fact that  $\varepsilon$  and  $\mu$  are constant in  $x_3$ -direction.) It satisfies the boundary condition  $\check{\mathbf{E}}_1 = 0$  on  $\Gamma_3$ . We put

$$\check{\mathbf{H}}_2 := \tilde{\mathbf{H}}_2 - \frac{1}{\mu} \partial_3 \check{\mathbf{E}}_1.$$

Then,  $\partial_3 \check{\mathbf{H}}_2$  is an element of  $L^2(Q)$ , and (108) leads to the identity

$$\check{\mathbf{E}}_1 + \frac{1}{\varepsilon} \partial_3 \check{\mathbf{H}}_2 = \check{\mathbf{E}}_1 + \frac{1}{\varepsilon} \partial_3 \tilde{\mathbf{H}}_2 - \frac{1}{\varepsilon \mu} \partial_3^2 \check{\mathbf{E}}_1 = \tilde{\mathbf{E}}_1.$$

The remaining relations of (107) can be handled in the same way. Altogether, we obtain a vector  $(\check{\mathbf{E}}, \check{\mathbf{H}}) \in \mathcal{D}(B)$  with  $(I - B)(\check{\mathbf{E}}, \check{\mathbf{H}}) = (\tilde{\mathbf{E}}, \tilde{\mathbf{H}})$ .  $\square$

Combining formula (4.5) in [21] with Lemma 6.2, we can now conclude the unconditional stability of scheme (101).

**Lemma 6.3.** *Let  $\varepsilon$  and  $\mu$  satisfy (2),  $\tau > 0$ , and  $T > n\tau$ . Let also  $(\mathbf{E}^0, \mathbf{H}^0) \in \mathcal{D}(B)$ , and  $(\frac{1}{\varepsilon} \mathbf{J}, 0) \in C([0, T], \mathcal{D}(A))$ . Then the estimate*

$$\|(\mathbf{E}^n, \mathbf{H}^n)\| \leq \|(\mathbf{E}^0, \mathbf{H}^0)\|_{\mathcal{D}(B)} + T \max_{t \in [0, T]} \|(\frac{1}{\varepsilon} \mathbf{J}, 0)\|_{\mathcal{D}(A)}$$

*is valid.*

Using the reasoning in the proof of Lemma 6.2, we can also draw an important conclusion on the complexity of scheme (101).

**Remark 3.** Let  $\varepsilon$  and  $\mu$  satisfy (2), and let  $\tau > 0$ . Each application of scheme (101) essentially amounts to solving only two-dimensional decoupled elliptic transmission problems for  $\mathbf{E}_3$  and  $\mathbf{H}_3$ , and one-dimensional decoupled elliptic problems for  $\mathbf{E}_1$  and  $\mathbf{E}_2$ . To show this claim, we first note that the main effort for (101) consists in

evaluating the resolvents of  $A$  and  $B$ . In the following, we analyze both resolvent operators separately.

1) Let  $(\tilde{\mathbf{E}}, \tilde{\mathbf{H}}) \in X = L^2(Q)^6$ , and  $(\mathbf{E}, \mathbf{H}) = (I - \frac{\tau}{2}A)^{-1}(\tilde{\mathbf{E}}, \tilde{\mathbf{H}})$ . We then arrive at system (102) (with  $\frac{\tau}{2\varepsilon}$  instead of  $\frac{1}{\varepsilon}$  and  $\frac{\tau}{2\mu}$  instead of  $\frac{1}{\mu}$ ). From the identity on the right hand side of the third line in (102), we obtain the relations

$$\begin{aligned} \int_Q \mu \tilde{\mathbf{H}}_3 \varphi \, dx &= \int_{a_3^-}^{a_3^+} \int_S \mu \mathbf{H}_3 \varphi + \frac{\tau}{2} \operatorname{curl}_2(\mathbf{E}_1, \mathbf{E}_2) \varphi \, d(x_1, x_2) \\ &= \int_{a_3^-}^{a_3^+} \int_S [\mu \mathbf{H}_3 \varphi + \frac{\tau}{2} \mathbf{E}_1 \partial_2 \varphi - \frac{\tau}{2} \mathbf{E}_2 \partial_1 \varphi] \, d(x_1, x_2) \, dx_3 \end{aligned}$$

for all  $\varphi \in L^2((a_3^-, a_3^+), H^1(S))$ . Inserting the equations on the left hand side of the first and second line of (102), we arrive at the relation

$$\begin{aligned} \int_Q \mu \tilde{\mathbf{H}}_3 \varphi - \frac{\tau}{2} \tilde{\mathbf{E}}_1 \partial_2 \varphi + \frac{\tau}{2} \tilde{\mathbf{E}}_2 \partial_1 \varphi \, dx \\ = \int_{a_3^-}^{a_3^+} \int_S \mu \mathbf{H}_3 \varphi + \frac{\tau^2}{4\varepsilon} (\nabla_{x_1, x_2} \mathbf{H}_3) \cdot (\nabla_{x_1, x_2} \varphi) \, d(x_1, x_2) \, dx_3 \end{aligned} \quad (109)$$

for all  $\varphi \in L^2((a_3^-, a_3^+), H^1(S))$ . Having solved the essentially two-dimensional problem (109),  $\mathbf{E}_1$  and  $\mathbf{E}_2$  are directly obtained via the formulas on the left hand side of the first and second line of (102). A similar statement is true for  $\mathbf{E}_3$ .

2) Let  $(\check{\mathbf{E}}, \check{\mathbf{H}}) = (I - \frac{\tau}{2}B)^{-1}(\tilde{\mathbf{E}}, \tilde{\mathbf{H}})$ . Then system (107) is valid with  $\frac{\tau}{2\varepsilon}$  instead of  $\frac{1}{\varepsilon}$  and  $\frac{\tau}{2\mu}$  instead of  $\frac{1}{\mu}$ . The identity on the left hand side of the first line in (107) leads to the equation

$$\int_Q \varepsilon \check{\mathbf{E}}_1 \phi \, dx = \int_{a_3^-}^{a_3^+} \int_S \varepsilon \check{\mathbf{E}}_1 \phi - \frac{\tau}{2} \check{\mathbf{H}}_2 \partial_3 \phi \, d(x_1, x_2) \, dx_3, \quad \phi \in H_0^1((a_3^-, a_3^+), L^2(S)).$$

Plugging in the formula on the right hand side of the second line of (107), we conclude the weak formulation

$$\int_Q \varepsilon \check{\mathbf{E}}_1 \phi + \frac{\tau}{2} \check{\mathbf{H}}_2 \partial_3 \phi \, dx = \int_{a_3^-}^{a_3^+} \int_S \varepsilon \check{\mathbf{E}}_1 \phi + \frac{\tau^2}{4\mu} (\partial_3 \check{\mathbf{E}}_1) (\partial_3 \phi) \, d(x_1, x_2) \, dx_3,$$

$\phi \in H_0^1((a_3^-, a_3^+), L^2(S))$ , of (108). Having solved this one-dimensional elliptic problem,  $\check{\mathbf{H}}_2$  is directly obtained as  $\check{\mathbf{H}}_2 = \tilde{\mathbf{H}}_2 - \frac{\tau}{2\mu} \partial_3 \check{\mathbf{E}}_1$ . Similar statements are true for  $\check{\mathbf{E}}_2$  and  $\check{\mathbf{H}}_1$ .  $\diamond$

**6.2. Error bound for the directional splitting scheme.** This Section is devoted to a first order convergence result for scheme (101). The statement is proved by combining the regularity results from Section 5 with the statements about the splitting operators from Section 6.1.

In order to expand the semigroup  $(e^{tM})_{t \geq 0}$  for positive times, we additionally employ the functions

$$\Lambda_j(t)w := \frac{1}{t^j(j-1)!} \int_0^t (t-s)^{j-1} e^{sM} w \, ds, \quad \Lambda_0(t) := e^{tM}, \quad (110)$$

for  $w \in X, t \geq 0$  and  $j \in \mathbb{N}$ , see [28, 29] for instance. Note that Proposition 5 implies that  $\Lambda_j(t)$  leaves the space  $X_1$  invariant for  $j \in \mathbb{N}_0$ , and  $t \geq 0$ .

Standard semigroup theory and Proposition 5 moreover lead to the useful relations

$$\|\Lambda_j(t)\|_{\mathcal{L}(X_1)} \leq \frac{1}{j!}, \quad \|\Lambda_j(t)\|_{\mathcal{L}(\mathcal{D}(M_1))} \leq \frac{1}{j!}, \quad (111)$$

$$tM\Lambda_{j+1}(t) = \Lambda_j(t) - \frac{1}{j!}I \quad \text{on } \mathcal{D}(M), \quad j \in \mathbb{N}_0, \quad (112)$$

for  $t \geq 0$ , see Section 4 in [29]. The operator  $\Lambda_j(t)$  furthermore maps  $\mathcal{D}(M_1)$  into  $\mathcal{D}(M_1)$  for all  $t \geq 0$ .

We next demonstrate an error bound for the directional splitting scheme (101). Here arguments from the proofs of Theorem 4.2 in [29], Theorem 5.1 in [21], and Theorem 10.7 in [54] are employed. Throughout the statement and the associated proof, the solution of the Maxwell system (1) is denoted by  $w = (\mathbf{E}, \mathbf{H})$ , while the approximate solution at time  $t_n = n\tau$  is  $w_n$ . For the external current  $\mathbf{J}$  in (1), we also use the space

$$W_T := W^{1,1}([0, T], X_1) \cap C([0, T], \mathcal{D}(M_1))$$

with corresponding norm

$$\|\cdot\|_{W_T} := \|\cdot\|_{W^{1,1}([0, T], X_1)} + \|\cdot\|_{C([0, T], \mathcal{D}(M_1))},$$

for a fixed final time  $T > 0$ , see Section 2.2. Note the relation  $\mathcal{D}(M_1) = \mathcal{D}(M^2) \cap X_0$ .

**Theorem 6.4.** *Let  $\varepsilon$  and  $\mu$  satisfy (2),  $T \geq 1$ , and  $w_0 = w(0) \in \mathcal{D}(M^2) \cap X_0$ . Let also  $(\frac{1}{\varepsilon}\mathbf{J}, 0) \in W_T$ , and  $\tau \in (0, T)$ . There is a constant  $C > 0$  with*

$$\|w(t_n) - w_n\|_{L^2(Q)} \leq C\tau T (\|w_0\|_{\mathcal{D}(M_1)} + \|(\frac{1}{\varepsilon}\mathbf{J}, 0)\|_{W_T})$$

for all  $n \in \mathbb{N}_0$  with  $n\tau \leq T$ . The number  $C$  depends only on  $\varepsilon, \mu$ , and  $Q$ .

*Proof.* 1) We first estimate the local error. Throughout the proof,  $C > 0$  denotes a constant that depends only on  $\varepsilon, \mu$ , and  $Q$ . It is allowed to change from line to line. Let  $k \in \mathbb{N}_0$  with  $(k+1)\tau \leq T$ , and recall the notation  $t_k = k\tau$ . Inserting the identity

$$\frac{1}{\varepsilon}\mathbf{J}(t_k + s) = \frac{1}{\varepsilon}\mathbf{J}(t_k) + \int_0^s \frac{1}{\varepsilon}\mathbf{J}'(t_k + r) \, dr, \quad s \in [0, \tau], \quad (113)$$

into the Duhamel formula for  $w$ , we infer the representation

$$\begin{aligned} w(t_{k+1}) &= e^{\tau M} w(t_k) + \int_0^\tau e^{(\tau-s)M} (-\frac{1}{\varepsilon}\mathbf{J}(t_k + s), 0) \, ds \\ &= e^{\tau M} w(t_k) + \int_0^\tau e^{rM} (-\frac{1}{\varepsilon}\mathbf{J}(t_k), 0) \, dr \\ &\quad + \int_0^\tau e^{(\tau-s)M} \int_0^s (-\frac{1}{\varepsilon}\mathbf{J}'(t_k + r), 0) \, dr \, ds \\ &= e^{\tau M} w(t_k) + \tau\Lambda_1(\tau) (-\frac{1}{\varepsilon}\mathbf{J}(t_k), 0) + R_k(\tau), \end{aligned}$$

involving the remainder term

$$R_k(\tau) := \int_0^\tau e^{(\tau-s)M} \int_0^s (-\frac{1}{\varepsilon}\mathbf{J}'(t_k + r), 0) \, dr \, ds.$$

Using (113) in scheme (101), we on the other hand obtain the equations

$$\begin{aligned} \mathcal{T}_{\tau, k+1}(w(t_k)) &= (I - \frac{\tau}{2}B)^{-1} (I + \frac{\tau}{2}A) \left[ (I - \frac{\tau}{2}A)^{-1} (I + \frac{\tau}{2}B) w(t_k) + \tau (-\frac{1}{\varepsilon}\mathbf{J}(t_k), 0) \right. \\ &\quad \left. + \frac{\tau}{2} \int_0^\tau (-\frac{1}{\varepsilon}\mathbf{J}'(t_k + r), 0) \, dr \right] \\ &= (I - \frac{\tau}{2}B)^{-1} \left[ (I - \frac{\tau}{2}A_{-1})^{-1} (I + \frac{\tau}{2}A_{-1}) (I + \frac{\tau}{2}B) w(t_k) \right. \\ &\quad \left. + \tau (I + \frac{\tau}{2}A) (-\frac{1}{\varepsilon}\mathbf{J}(t_k), 0) + \frac{\tau}{2} (I + \frac{\tau}{2}A) \int_0^\tau (-\frac{1}{\varepsilon}\mathbf{J}'(t_k + r), 0) \, dr \right]. \end{aligned}$$

Note that  $A$  is extrapolated in the second identity, as  $Bw(t_k)$  is in general not contained in  $\mathcal{D}(A)$ , see Remark 2. (The original solution  $w(t)$  of (1) belongs, however,

to  $\mathcal{D}(B)$  for every  $t > 0$ , as  $X_1$  embeds into  $\mathcal{D}(B)$ , see Corollary 2 and Proposition 4.) Note furthermore that the functions  $(-\frac{1}{\varepsilon}\mathbf{J}(t), 0)$ ,  $\Lambda_1(\tau)(-\frac{1}{\varepsilon}\mathbf{J}(t), 0)$  and  $(-\frac{1}{\varepsilon}\mathbf{J}'(t), 0)$  belong to  $\mathcal{D}(A) \cap \mathcal{D}(B)$  for every  $t \in [0, T]$ , see Propositions 4 and 5, as well as (13).

Subtracting the representations for  $w(t_{k+1})$  and  $\mathcal{T}_{\tau, k+1}(w(t_k))$ , we conclude

$$\begin{aligned} & \mathcal{T}_{\tau, k+1}(w(t_k)) - w(t_{k+1}) \\ &= (I - \frac{\tau}{2}B)^{-1}(I - \frac{\tau}{2}A_{-1})^{-1} \left[ (I + \frac{\tau}{2}A_{-1})(I + \frac{\tau}{2}B) - (I - \frac{\tau}{2}A_{-1})(I - \frac{\tau}{2}B)e^{\tau M} \right] w(t_k) \\ & \quad + \tau(I - \frac{\tau}{2}B)^{-1} \left[ (I + \frac{\tau}{2}A) - (I - \frac{\tau}{2}B)\Lambda_1(\tau) \right] (-\frac{1}{\varepsilon}\mathbf{J}(t_k), 0) \\ & \quad + \frac{\tau}{2}(I - \frac{\tau}{2}B)^{-1}(I + \frac{\tau}{2}A) \int_0^\tau (-\frac{1}{\varepsilon}\mathbf{J}'(t_k + r), 0) dr - R_k(\tau) \\ &=: e_{1,k}(\tau) + e_{2,k}(\tau) + e_{3,k}(\tau) - R_k(\tau). \end{aligned} \tag{114}$$

In the next steps, we separately estimate the summands on the right hand side of (114).

2) We first deal with  $e_{1,k}(\tau)$ . Recall that the operators  $\Lambda_1(\tau)$  and  $\Lambda_2(\tau)$  leave the spaces  $\mathcal{D}(M_1)$  and  $X_1$  invariant. Thus  $M\Lambda_1(\tau)w(t_k)$  is an element of  $\mathcal{D}(B)$ , and  $\Lambda_j(\tau)w(t_k)$  belongs to  $\mathcal{D}(M^2)$  for  $j \in \{1, 2\}$ . Algebraic manipulations and (112) hence lead to the relations

$$\begin{aligned} & e_{1,k}(\tau) \\ &= (I - \frac{\tau}{2}B)^{-1}(I - \frac{\tau}{2}A_{-1})^{-1} \left[ (I + \frac{\tau}{2}A_{-1})(I + \frac{\tau}{2}B) - (I - \frac{\tau}{2}A_{-1})(I - \frac{\tau}{2}B)e^{\tau M} \right] w(t_k) \\ &= (I - \frac{\tau}{2}B)^{-1}(I - \frac{\tau}{2}A_{-1})^{-1} \left[ I + \frac{\tau}{2}M + \frac{\tau^2}{4}A_{-1}B - (I - \frac{\tau}{2}M + \frac{\tau^2}{4}A_{-1}B)e^{\tau M} \right] w(t_k) \\ &= (I - \frac{\tau}{2}B)^{-1}(I - \frac{\tau}{2}A_{-1})^{-1} \left[ I - e^{\tau M} + \frac{\tau}{2}M(I + e^{\tau M}) + \frac{\tau^2}{4}A_{-1}B(I - e^{\tau M}) \right] w(t_k) \\ &= (I - \frac{\tau}{2}B)^{-1}(I - \frac{\tau}{2}A_{-1})^{-1} \left[ -\tau M\Lambda_1(\tau) + \tau M + \frac{\tau^2}{2}M^2\Lambda_1(\tau) \right. \\ & \quad \left. - \frac{\tau^3}{4}A_{-1}BM\Lambda_1(\tau) \right] w(t_k) \\ &= (I - \frac{\tau}{2}B)^{-1}(I - \frac{\tau}{2}A_{-1})^{-1} \left[ -\tau^2M^2\Lambda_2(\tau) + \frac{\tau^2}{2}M^2\Lambda_1(\tau) - \frac{\tau^3}{4}A_{-1}BM\Lambda_1(\tau) \right] w(t_k). \end{aligned}$$

Combining Corollary 2, Remark 2 and (111), we conclude the estimates

$$\begin{aligned} & \|e_{1,k}(\tau)\| \leq C\tau^2 \|w(t_k)\|_{\mathcal{D}(M^2) \cap X_0}, \\ & \|(I + \frac{\tau}{2}B)e_{1,k}(\tau)\| \leq C\tau^2 \|w(t_k)\|_{\mathcal{D}(M^2) \cap X_0}. \end{aligned} \tag{115}$$

3) We next deal with the second term on the right hand side of (114). We note that the vector  $M\Lambda_2(\tau)(-\frac{1}{\varepsilon}\mathbf{J}(t), 0)$  is contained in  $\mathcal{D}(B)$  for every  $t \in [0, T]$ , as  $W_T \hookrightarrow C([0, T], \mathcal{D}(M_1))$ ,  $\Lambda_2(\tau)$  leaves  $\mathcal{D}(M_1)$  invariant, and  $X_1 \hookrightarrow \mathcal{D}(B)$ , see Proposition 4. With (112), algebraic manipulations then lead to the relations

$$\begin{aligned} e_{2,k}(\tau) &= \tau(I - \frac{\tau}{2}B)^{-1} \left[ (I + \frac{\tau}{2}A) - (I - \frac{\tau}{2}B)\Lambda_1(\tau) \right] (-\frac{1}{\varepsilon}\mathbf{J}(t_k), 0) \\ &= \tau^2(I - \frac{\tau}{2}B)^{-1} \left[ \frac{1}{2}M - M\Lambda_2(\tau) + \frac{\tau}{2}BM\Lambda_2(\tau) \right] (-\frac{1}{\varepsilon}\mathbf{J}(t_k), 0). \end{aligned}$$

Proposition 4 and (111) then lead to the estimate

$$\|e_{2,k}(\tau)\| + \|(I + \frac{\tau}{2}B)e_{2,k}(\tau)\| \leq C\tau^2 \|(-\frac{1}{\varepsilon}\mathbf{J}, 0)\|_{W_T}. \tag{116}$$

4) To bound  $e_{3,k}(\tau)$  and  $R_k(\tau)$ , we employ the embedding of  $X_1$  into  $\mathcal{D}(A) \cap \mathcal{D}(B)$ , see Proposition 4, as well as the contractivity of  $(e^{tM})_{t \geq 0}$  in  $X_1$ , see Proposition 5. We then infer the inequalities

$$\|e_{3,k}(\tau)\| + \|(I + \frac{\tau}{2}B)e_{3,k}(\tau)\| \leq C\tau \|(-\frac{1}{\varepsilon}\mathbf{J}, 0)\|_{W^{1,1}([t_k, t_{k+1}], X_1)}, \tag{117}$$

$$\|R_k(\tau)\| + \|(I + \frac{\tau}{2}B)R_k(\tau)\| \leq C\tau \|(-\frac{1}{\varepsilon}\mathbf{J}, 0)\|_{W^{1,1}([t_k, t_{k+1}], X_1)}. \tag{118}$$

5) The stated bound on the global error is now obtained in the standard way from the above results for the local error and the stability of scheme (101). Using the Lady Winderemere's fan argument, we first infer the global error formula

$$w_n - w(t_n) = \sum_{k=0}^{n-1} \left[ (I - \frac{\tau}{2}B)^{-1} (I + \frac{\tau}{2}A) (I - \frac{\tau}{2}A)^{-1} (I + \frac{\tau}{2}B) \right]^{n-1-k} \cdot \left( \mathcal{T}_{\tau, k+1} w(t_k) - w(t_{k+1}) \right).$$

We next combine (114)–(118) with the stability statement in Lemma 6.2. Abbreviating the Cayley transform  $(I + \frac{\tau}{2}L)(I - \frac{\tau}{2}L)^{-1}$  of  $L \in \{A, B\}$  by  $\gamma_\tau(L)$ , we then infer the stated bound

$$\begin{aligned} \|w_n - w(t_n)\| &\leq \sum_{k=0}^{n-2} \|(I - \frac{\tau}{2}B)^{-1}\| \|(\gamma_\tau(A)\gamma_\tau(B))^{n-2-k} \gamma_\tau(A)\| \\ &\quad \cdot \|(I + \frac{\tau}{2}B)(\mathcal{T}_{\tau, k+1} w(t_k) - w(t_{k+1}))\| + \|\mathcal{T}_{\tau, n} w(t_{n-1}) - w(t_n)\| \\ &\leq C \sum_{k=0}^{n-1} (\tau^2 \|w(t_k)\|_{\mathcal{D}(M^2) \cap X_0} + \tau^2 \|(-\frac{1}{\varepsilon} \mathbf{J}, 0)\|_{W_T} + \tau \|(-\frac{1}{\varepsilon} \mathbf{J}, 0)\|_{W^{1,1}([t_k, t_{k+1}], X_1)}) \\ &\leq C\tau T (\|w_0\|_{\mathcal{D}(M_1)} + \|(-\frac{1}{\varepsilon} \mathbf{J}, 0)\|_{W_T}). \end{aligned}$$

For the last estimate we employ Corollary 2 and the relation  $n\tau \leq T$ .  $\square$

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