A FORMULA FOR THE FIRST POSITIVE EIGENVALUE OF A ONE-DIMENSIONAL TRANSMISSION PROBLEM

Konstantin Zerulla

Department of Mathematics, Karlsruhe Institute of Technology
Englerstr. 2
76131 Karlsruhe, Germany

Abstract. A formula for the first positive eigenvalue of a one-dimensional elliptic transmission problem is derived. The eigenvalue problem arises during the spectral analysis of a Laplacian on the disc with angular transmission conditions. The formula for the first eigenvalue provides an explicit link to the transmission conditions in the problem.

In this note, we determine the first eigenvalue of the one-dimensional system

$$\begin{aligned} (\psi^{(i)})''(\varphi) &= -\kappa^2\psi^{(i)}(\varphi) \qquad \text{for } \varphi \in I_i,\ i \in \{1,\dots,4\},\\ \psi^{(1)}(0) &= \psi^{(4)}(2\pi), \quad \varepsilon^{(1)}(\psi^{(1)})'(0) = \varepsilon^{(4)}(\psi^{(4)})'(2\pi),\\ \psi^{(1)}(\tfrac{\pi}{2}) &= \psi^{(2)}(\tfrac{\pi}{2}), \quad (\psi^{(1)})'(\tfrac{\pi}{2}) = (\psi^{(2)})'(\tfrac{\pi}{2}),\\ \psi^{(2)}(\pi) &= \psi^{(3)}(\pi), \quad (\psi^{(2)})'(\pi) = (\psi^{(3)})'(\pi),\\ \psi^{(3)}(\tfrac{3}{2}\pi) &= \psi^{(4)}(\tfrac{3}{2}\pi), \quad \varepsilon^{(1)}(\psi^{(3)})'(\tfrac{3}{2}\pi) = \varepsilon^{(4)}(\psi^{(4)})'(\tfrac{3}{2}\pi), \end{aligned} \tag{1}$$

where

$$I_1 := (0, \tfrac{\pi}{2}), \quad I_2 := (\tfrac{\pi}{2}, \pi), \quad I_3 := (\pi, \tfrac{3}{2}\pi), \quad I_4 := (\tfrac{3}{2}\pi, 2\pi),$$

and ε is a piecewise constant function on the partition $I_1 \cup \cdots \cup I_4$. By $f^{(i)}$ we denote the restriction of a function $f \in L^2(0, 2\pi)$ to the interval I_i, $i \in \{1,\dots,4\}$. We furthermore assume the relation

$$\varepsilon^{(1)} = \varepsilon^{(2)} = \varepsilon^{(3)} < \varepsilon^{(4)}. \tag{2}$$

System (1) naturally arises during the analysis of the Laplacian

$$\begin{aligned} \Delta u &:= \tfrac{1}{\tilde{\varepsilon}} \operatorname{div}(\tilde{\varepsilon}\nabla u),\\ u \in \mathcal{D}(\Delta) &:= \{v \in H_0^1(D) \mid \operatorname{div}(\tilde{\varepsilon}\nabla u) \in L^2(D)\} \end{aligned}$$

on the unit disc D with transmission conditions, see [2, 3, 5, 1, 4] for instance. Here $\tilde{\varepsilon}(r, \varphi) := \varepsilon(\varphi)$ for $r \in [0, 1]$ and $\varphi \in [0, 2\pi)$. In fact, (1) corresponds to the angular part of the eigenvalue problem for the Laplacian on the disc. Note that the final result of this note in Lemma 2 is essential for the explicit nature of the regularity results in [5].

In the next two lemmas, we calculate the first nonzero eigenvalue of (1). Due to technical issues, we distinguish between two cases for the ratio $\varepsilon^{(4)}/\varepsilon^{(1)}$.

Lemma 1. *Let*

$$\frac{\varepsilon^{(4)}}{\varepsilon^{(1)}} \neq 1 - 2\frac{\cos(\frac{15}{7}\pi)\sin(\frac{6}{7}\pi)}{\cos(\frac{12}{7}\pi)\sin(\frac{9}{7}\pi)}.$$

Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 258734477 – SFB 1173.

Then the first nonzero eigenvalue κ_1^2 of (1) satisfies the identity

$$\frac{(\varepsilon^{(4)} - \varepsilon^{(1)})^2}{\varepsilon^{(4)}\varepsilon^{(1)}} = -\frac{4\sin^2(\kappa_1\pi)}{\sin(\frac{\kappa_1}{2}\pi)\sin(\frac{3}{2}\kappa_1\pi)}. \tag{3}$$

It has a one-dimensional eigenspace, and the next eigenvalue is greater than one.

Proof. 1) We first assume that $\lambda \in (0,1)$ is an eigenvalue of (1) with an associated eigenfunction $\psi \neq 0$. The first line of (1) then implies the representation

$$\psi^{(i)}(\varphi) = a^{(i)}\cos(\sqrt{\lambda}\varphi) + b^{(i)}\sin(\sqrt{\lambda}\varphi), \quad a^{(i)}, b^{(i)} \in \mathbb{R},\ \varphi \in I_i \tag{4}$$

for $i \in \{1, \dots, 4\}$. The third and fourth line of (1) lead to the relations $a^{(1)} = a^{(2)} = a^{(3)}$ and $b^{(1)} = b^{(2)} = b^{(3)}$. The second and fifth lines of (1) further result in the formulas

$$\begin{aligned} a^{(1)} &= a^{(4)}\cos(\sqrt{\lambda}2\pi) + b^{(4)}\sin(\sqrt{\lambda}2\pi), \\ b^{(1)} &= \tfrac{\varepsilon^{(4)}}{\varepsilon^{(1)}}\big(-a^{(4)}\sin(\sqrt{\lambda}2\pi) + b^{(4)}\cos(\sqrt{\lambda}2\pi)\big), \end{aligned} \tag{5}$$

$$\begin{aligned} a^{(4)}\cos(\sqrt{\lambda}\tfrac{3}{2}\pi) + b^{(4)}\sin(\sqrt{\lambda}\tfrac{3}{2}\pi) &= a^{(1)}\cos(\sqrt{\lambda}\tfrac{3}{2}\pi) + b^{(1)}\sin(\sqrt{\lambda}\tfrac{3}{2}\pi) \\ &= a^{(4)}\cos(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi) + b^{(4)}\sin(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi) \\ &\quad - \tfrac{\varepsilon^{(4)}}{\varepsilon^{(1)}}a^{(4)}\sin(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi) \\ &\quad + \tfrac{\varepsilon^{(4)}}{\varepsilon^{(1)}}b^{(4)}\cos(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi). \end{aligned} \tag{6}$$

Reformulating the last identity, the equation

$$\begin{aligned} &a^{(4)}\big(\cos(\sqrt{\lambda}\tfrac{3}{2}\pi) - \cos(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi) + \tfrac{\varepsilon^{(4)}}{\varepsilon^{(1)}}\sin(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi)\big) \\ &\quad = b^{(4)}\big(\sin(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi) - \sin(\sqrt{\lambda}\tfrac{3}{2}\pi) + \tfrac{\varepsilon^{(4)}}{\varepsilon^{(1)}}\cos(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi)\big) \\ &\quad =: b^{(4)}A_1(\lambda) \end{aligned} \tag{7}$$

is derived. Relating the derivative condition in the fifth line of (1) to (5), we conclude the formulas

$$\begin{aligned} &a^{(4)}\big(-\varepsilon^{(4)}\sin(\sqrt{\lambda}\tfrac{3}{2}\pi) + \varepsilon^{(1)}\cos(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi) + \varepsilon^{(4)}\sin(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi)\big) \\ &= b^{(4)}\big(-\varepsilon^{(4)}\cos(\sqrt{\lambda}\tfrac{3}{2}\pi) - \varepsilon^{(1)}\sin(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi) + \varepsilon^{(4)}\cos(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi)\big) \\ &=: b^{(4)}A_2(\lambda). \end{aligned} \tag{8}$$

2) We next show by contradiction that $a^{(4)}$ is nonzero. Assume hence that $a^{(4)} = 0$. Equations (5) and (7) then imply that $b^{(4)}$ is nonzero and that $A_1(\lambda)$ vanishes. In the following, the numbers

$$\omega := \varepsilon^{(4)}/\varepsilon^{(1)} - 1, \qquad \xi := \varepsilon^{(4)} - \varepsilon^{(1)}$$

are employed. Manipulating (7) by means of trigonometric identities, we infer the relations

$$\begin{aligned} 0 = A_1(\lambda) &= \sin(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi) - \sin(\sqrt{\lambda}\tfrac{3}{2}\pi) + (\tfrac{\varepsilon^{(4)}}{\varepsilon^{(1)}} - 1)\cos(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi) \\ &\quad + \cos(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi) \\ &= \omega\cos(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi) + 2\cos(\sqrt{\lambda}\tfrac{5}{2}\pi)\sin(\sqrt{\lambda}\pi). \end{aligned}$$

Since the two summands in the last line have no common zeros on $(0,1)$, the last line gives rise to the formula

$$\omega = \omega(\lambda) = -\frac{2\cos(\sqrt{\lambda}\frac{5}{2}\pi)\sin(\sqrt{\lambda}\pi)}{\cos(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\frac{3}{2}\pi)}. \tag{9}$$

From (8) we further deduce that the expression $A_2(\lambda)$ vanishes. Manipulating the defining relation for $A_2(\lambda)$ by means of trigonometric identities, the equations

$$\begin{aligned}
0 &= -\tfrac{\varepsilon^{(4)}}{\varepsilon^{(1)}}\cos(\sqrt{\lambda}\tfrac{3}{2}\pi) - \sin(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi) + \tfrac{\varepsilon^{(4)}}{\varepsilon^{(1)}}\cos(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi)\\
&= -\tfrac{\varepsilon^{(4)}}{\varepsilon^{(1)}}\cos(\sqrt{\lambda}\tfrac{3}{2}\pi) - \sin(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi) + \omega\cos(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi)\\
&\quad + \cos(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi)\\
&= -\tfrac{\varepsilon^{(4)}}{\varepsilon^{(1)}}\cos(\sqrt{\lambda}\tfrac{3}{2}\pi) + \cos(\sqrt{\lambda}\tfrac{7}{2}\pi) + \omega\cos(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi)\\
&= -\omega\cos(\sqrt{\lambda}\tfrac{3}{2}\pi) - 2\sin(\sqrt{\lambda}\tfrac{5}{2}\pi)\sin(\sqrt{\lambda}\pi) + \omega\cos(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi) \qquad (10)
\end{aligned}$$

follow. The right hand side is next multiplied with the factor $\cos(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\frac{3}{2}\pi)$. Using also (9), we then infer the formulas

$$\begin{aligned}
0 &= -\omega\cos(\sqrt{\lambda}\tfrac{3}{2}\pi)\cos(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi)\\
&\quad - 2\sin(\sqrt{\lambda}\tfrac{5}{2}\pi)\sin(\sqrt{\lambda}\pi)\cos(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi)\\
&\quad + \omega\cos^2(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi)\\
&= 2\cos(\sqrt{\lambda}\tfrac{5}{2}\pi)\sin(\sqrt{\lambda}\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi)\\
&\quad - 2\sin(\sqrt{\lambda}\tfrac{5}{2}\pi)\sin(\sqrt{\lambda}\pi)\cos(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi)\\
&\quad - 2\cos(\sqrt{\lambda}\tfrac{5}{2}\pi)\sin(\sqrt{\lambda}\pi)\cos(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi). \qquad (11)
\end{aligned}$$

We next divide (11) by $\sin(\sqrt{\lambda}\pi) \neq 0$, and we use besides the angle sum formula for cosine the trigonometric relations

$$\begin{aligned}
\cos(\sqrt{\lambda}\tfrac{5}{2}\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi) &= \tfrac{1}{2}(\cos(\sqrt{\lambda}4\pi) + \cos(\sqrt{\lambda}\pi)),\\
\cos(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\pi) &= \tfrac{1}{2}(\cos(\sqrt{\lambda}3\pi) + \cos(\sqrt{\lambda}\pi)).
\end{aligned}$$

In this way, we arrive at the equations

$$\begin{aligned}
0 &= \cos(\sqrt{\lambda}\tfrac{5}{2}\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi) - \sin(\sqrt{\lambda}\tfrac{5}{2}\pi)\cos(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi)\\
&\quad - \cos(\sqrt{\lambda}\tfrac{5}{2}\pi)\cos(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi)\\
&= \cos(\sqrt{\lambda}\tfrac{5}{2}\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi) - \cos(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\pi)\\
&= \tfrac{1}{2}(\cos(\sqrt{\lambda}4\pi) - \cos(\sqrt{\lambda}3\pi)) = -\sin(\sqrt{\lambda}\tfrac{7}{2}\pi)\sin(\sqrt{\lambda}\tfrac{1}{2}\pi).
\end{aligned}$$

As $\sin(\sqrt{\lambda}\frac{1}{2}\pi) \neq 0$, we conclude that λ is an element of the set $\{\frac{4}{49}, \frac{16}{49}, \frac{36}{49}\}$. Plugging these values for λ into the formula (9) for $\omega(\lambda)$, we obtain, however, that $\omega(4/49)$ and $\omega(16/49)$ are negative (thus contradicting (2)), while $\omega(36/49)$ is excluded in the assumption. We conclude that $a^{(4)}$ is different from zero.

3) Taking the results of part 2) into account, we can assume that $a^{(4)} = 1$. In the following, we distinguish the cases of $A_2(\lambda)$ being zero and nonzero, see (8).

3.i) Suppose $A_2(\lambda) = 0$, and proceed similar to part 2). The formula for $A_2(\lambda)$ in (8) is divided by $\varepsilon^{(4)}$, and the number

$$\omega_0 := 1 + \varepsilon^{(1)}/\varepsilon^{(4)}$$

is introduced. By means of trigonometric identities, the equations

$$\begin{aligned}
0 &= -\cos(\sqrt{\lambda}\tfrac{3}{2}\pi) - \tfrac{\varepsilon^{(1)}}{\varepsilon^{(4)}}\sin(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi) + \cos(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi)\\
&= -\cos(\sqrt{\lambda}\tfrac{3}{2}\pi) - \omega_0\sin(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi) + \cos(\sqrt{\lambda}\tfrac{1}{2}\pi)\\
&= 2\sin(\sqrt{\lambda}\pi)\sin(\sqrt{\lambda}\tfrac{1}{2}\pi) - \omega_0\sin(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi) \qquad (12)
\end{aligned}$$

are then obtained. Since the summands on the right hand side of (12) have no common zero on $(0,1)$, the formula

$$\omega_0 = \frac{2\sin(\sqrt{\lambda}\pi)\sin(\sqrt{\lambda}\frac{1}{2}\pi)}{\sin(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\frac{3}{2}\pi)} \tag{13}$$

follows. Treating the left hand side of (8) in the same way, we further arrive at

$$0 = -2\cos(\sqrt{\lambda}\pi)\sin(\sqrt{\lambda}\tfrac{1}{2}\pi) + \omega_0\cos(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi).$$

Multiplying by $\sin(\sqrt{\lambda}2\pi)$ and inserting (13), we deduce

$$0 = -2\sin(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\pi)\sin(\sqrt{\lambda}\tfrac{1}{2}\pi) + 2\cos(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\pi)\sin(\sqrt{\lambda}\tfrac{1}{2}\pi).$$

Dividing by $\sin(\sqrt{\lambda}\frac{1}{2}\pi)$ and using the sum formula for sine, we arrive at the identity $0 = \sin(\sqrt{\lambda}\pi)$. Since λ is assumed to belong to $(0,1)$, this is a contradiction.

3.ii) In consideration of the results in 3.i), we infer that $A_2(\lambda)$ has to be nonzero. Dividing in (8) by $A_2(\lambda)$, and using trigonometric identities as well as the number $\xi = \varepsilon^{(4)} - \varepsilon^{(1)}$, we then obtain the equations

$$\begin{aligned} b^{(4)} &= \frac{-\varepsilon^{(4)}\sin(\sqrt{\lambda}\frac{3}{2}\pi) + \varepsilon^{(1)}\cos(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\frac{3}{2}\pi) + \varepsilon^{(4)}\sin(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\frac{3}{2}\pi)}{-\varepsilon^{(4)}\cos(\sqrt{\lambda}\frac{3}{2}\pi) - \varepsilon^{(1)}\sin(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\frac{3}{2}\pi) + \varepsilon^{(4)}\cos(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\frac{3}{2}\pi)} \\ &= \frac{-\varepsilon^{(4)}\sin(\sqrt{\lambda}\frac{3}{2}\pi) + \varepsilon^{(4)}\sin(\sqrt{\lambda}\frac{7}{2}\pi) - \xi\cos(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\frac{3}{2}\pi)}{-\varepsilon^{(4)}\cos(\sqrt{\lambda}\frac{3}{2}\pi) + \varepsilon^{(4)}\cos(\sqrt{\lambda}\frac{7}{2}\pi) + \xi\sin(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\frac{3}{2}\pi)} \\ &= -\frac{2\varepsilon^{(4)}\cos(\sqrt{\lambda}\frac{5}{2}\pi)\sin(\sqrt{\lambda}\pi) - \xi\cos(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\frac{3}{2}\pi)}{2\varepsilon^{(4)}\sin(\sqrt{\lambda}\frac{5}{2}\pi)\sin(\sqrt{\lambda}\pi) - \xi\sin(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\frac{3}{2}\pi)}. \end{aligned} \tag{14}$$

We next reformulate (7) algebraically with the number $\omega = \varepsilon^{(4)}/\varepsilon^{(1)} - 1$ and the relation $a^{(4)} = 1$. We derive the identities

$$\begin{aligned} 0 &= \cos(\sqrt{\lambda}\tfrac{3}{2}\pi) - \cos(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi) + \tfrac{\varepsilon^{(4)}}{\varepsilon^{(1)}}\sin(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi) \\ &\quad - b^{(4)}\big(\sin(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\tfrac{3}{2}\pi) - \sin(\sqrt{\lambda}\tfrac{3}{2}\pi) + \tfrac{\varepsilon^{(4)}}{\varepsilon^{(1)}}\cos(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi)\big) \\ &= \cos(\sqrt{\lambda}\tfrac{3}{2}\pi) - \cos(\sqrt{\lambda}\tfrac{7}{2}\pi) + \omega\sin(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi) \\ &\quad - b^{(4)}\big(-\sin(\sqrt{\lambda}\tfrac{3}{2}\pi) + \sin(\sqrt{\lambda}\tfrac{7}{2}\pi) + \omega\cos(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi)\big) \\ &= 2\sin(\sqrt{\lambda}\tfrac{5}{2}\pi)\sin(\sqrt{\lambda}\pi) + \omega\sin(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi) \\ &\quad - b^{(4)}\big(2\cos(\sqrt{\lambda}\tfrac{5}{2}\pi)\sin(\sqrt{\lambda}\pi) + \omega\cos(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi)\big). \end{aligned} \tag{15}$$

The representation (14) for $b^{(4)}$ is next inserted into the right hand side of (15), and all arising expressions are multiplied with the denominator in (14). In this way, we deduce the equations

$$\begin{aligned} 0 &= 4\varepsilon^{(4)}\sin^2(\sqrt{\lambda}\tfrac{5}{2}\pi)\sin^2(\sqrt{\lambda}\pi) + 2\omega\varepsilon^{(4)}\sin(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi)\sin(\sqrt{\lambda}\tfrac{5}{2}\pi)\sin(\sqrt{\lambda}\pi) \\ &\quad - 2\xi\sin(\sqrt{\lambda}\tfrac{5}{2}\pi)\sin(\sqrt{\lambda}\pi)\sin(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi) - \omega\xi\sin^2(\sqrt{\lambda}2\pi)\sin^2(\sqrt{\lambda}\tfrac{3}{2}\pi) \\ &\quad + 4\varepsilon^{(4)}\cos^2(\sqrt{\lambda}\tfrac{5}{2}\pi)\sin^2(\sqrt{\lambda}\pi) - 2\xi\cos(\sqrt{\lambda}\tfrac{5}{2}\pi)\sin(\sqrt{\lambda}\pi)\cos(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi) \\ &\quad + 2\omega\varepsilon^{(4)}\cos(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi)\cos(\sqrt{\lambda}\tfrac{5}{2}\pi)\sin(\sqrt{\lambda}\pi) \\ &\quad - \omega\xi\cos^2(\sqrt{\lambda}2\pi)\sin^2(\sqrt{\lambda}\tfrac{3}{2}\pi) \end{aligned} \tag{16}$$

$$\begin{aligned} &= 4\varepsilon^{(4)}\sin^2(\sqrt{\lambda}\pi) + 2(\omega\varepsilon^{(4)} - \xi)\cos(\sqrt{\lambda}\tfrac{1}{2}\pi)\sin(\sqrt{\lambda}\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi) \\ &\quad - \omega\xi\sin^2(\sqrt{\lambda}\tfrac{3}{2}\pi). \end{aligned} \tag{17}$$

To further simplify the expressions on the right hand side, we use the formulas

$$\cos(\sqrt{\lambda}\tfrac{1}{2}\pi)\sin(\sqrt{\lambda}\pi) = \tfrac{1}{2}(\sin(\sqrt{\lambda}\tfrac{1}{2}\pi) + \sin(\sqrt{\lambda}\tfrac{3}{2}\pi)),$$

$$\begin{aligned}\omega\varepsilon^{(4)} - \xi &= \tfrac{(\varepsilon^{(4)})^2}{\varepsilon^{(1)}} - 2\varepsilon^{(4)} + \varepsilon^{(1)} = \tfrac{(\varepsilon^{(4)}-\varepsilon^{(1)})^2}{\varepsilon^{(1)}} = (\tfrac{\varepsilon^{(4)}}{\varepsilon^{(1)}} - 1)(\varepsilon^{(4)} - \varepsilon^{(1)}) \\ &= \omega\xi.\end{aligned}$$

Inserting these relations in (17), we arrive at the identities

$$\begin{aligned}0 &= 4\varepsilon^{(4)} \sin^2(\sqrt{\lambda}\pi) + 2\omega\xi \cos(\sqrt{\lambda}\tfrac{1}{2}\pi)\sin(\sqrt{\lambda}\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi) - \omega\xi \sin^2(\sqrt{\lambda}\tfrac{3}{2}\pi) \\ &= 4\varepsilon^{(4)} \sin^2(\sqrt{\lambda}\pi) + \omega\xi \sin(\sqrt{\lambda}\tfrac{1}{2}\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi).\end{aligned}$$

As the two summands on the right hand side have no common zeros on $(0,1)$, we conclude the representation

$$\frac{(\varepsilon^{(4)} - \varepsilon^{(1)})^2}{\varepsilon^{(1)}} = \omega\xi = -\frac{4\varepsilon^{(4)} \sin^2(\sqrt{\lambda}\pi)}{\sin(\sqrt{\lambda}\tfrac{1}{2}\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi)}. \tag{18}$$

Note that λ is uniquely determined by (18). Altogether, there is at most one eigenvalue of (1) in $(0,1)$. The associated eigenspace is furthermore one-dimensional.

4) Let $\lambda \in (4/9,1)$ satisfy (18). It remains to show that λ is indeed an eigenvalue of (1). We first prove by contradiction that the denominator in (14), being $A_2(\lambda)$, is nonzero. So, assume $A_2(\lambda)$ was zero. Due to the choice of λ, (16) is still valid. Rewriting (16) in product formula, we then deduce the identities

$$\begin{aligned}0 =& \big(2\sin(\sqrt{\lambda}\tfrac{5}{2}\pi)\sin(\sqrt{\lambda}\pi) + \omega\sin(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi)\big) \\ &\cdot \big(2\varepsilon^{(4)}\sin(\sqrt{\lambda}\tfrac{5}{2}\pi)\sin(\sqrt{\lambda}\pi) - \xi\sin(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi)\big) \\ &+ \big(2\cos(\sqrt{\lambda}\tfrac{5}{2}\pi)\sin(\sqrt{\lambda}\pi) + \omega\cos(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi)\big) \\ &\cdot \big(2\varepsilon^{(4)}\cos(\sqrt{\lambda}\tfrac{5}{2}\pi)\sin(\sqrt{\lambda}\pi) - \xi\cos(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi)\big) \\ &= \big(2\cos(\sqrt{\lambda}\tfrac{5}{2}\pi)\sin(\sqrt{\lambda}\pi) + \omega\cos(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi)\big) \\ &\cdot \big(2\varepsilon^{(4)}\cos(\sqrt{\lambda}\tfrac{5}{2}\pi)\sin(\sqrt{\lambda}\pi) - \xi\cos(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi)\big).\end{aligned} \tag{19}$$

Suppose the second factor on the right hand side of (19) is zero. Algebraic manipulation of the latter factor then gives rise to the formula

$$\frac{\varepsilon^{(4)}}{\varepsilon^{(1)}} = \frac{\cos(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi)}{\cos(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\tfrac{3}{2}\pi) - 2\cos(\sqrt{\lambda}\tfrac{5}{2}\pi)\sin(\sqrt{\lambda}\pi)}. \tag{20}$$

Since the right hand side of (20) is, however, less or equal than 1, we arrive at a contradiction to (2). As a result, the first factor on the right hand side of (19) has to be zero. Note that this expression coincides with $A_1(\lambda)$, see (7). (This can be seen by reversing the reasoning in (15)). Now the arguments in part 2) lead to a contradiction. This means that $A_2(\lambda)$ is nonzero.

We then define $b^{(4)}$ according to (14), set $a^{(4)} = 1$, and define $a^{(1)} = a^{(2)} = a^{(3)}$ as well as $b^{(1)} = b^{(2)} = b^{(3)}$ by (5). The function ψ is chosen as in (4).

Altogether, it only remains to validate the required transmission conditions for ψ. By definition of $b^{(4)}$, formula (8) is satisfied. Due to the choice of λ, identity (17) is also true. Dividing the right hand side of (17) by $A_2(\lambda)$, we then conclude that (15) holds. This finally means that also the first transmission condition (7) is fulfilled. □

Based on Lemma 1, we can now conclude formula (3) without an additional restriction on the ratio $\varepsilon^{(4)}/\varepsilon^{(1)}$.

Lemma 2. *Let ε satisfy (2). Then the statements of Lemma 1 on the first positive eigenvalue κ_1^2 of (1) are valid.*

Proof. 1) In view of Lemma 1, it remains to treat the case

$$\frac{\varepsilon^{(4)}}{\varepsilon^{(1)}} = 1 - 2\frac{\cos(\frac{15}{7}\pi)\sin(\frac{6}{7}\pi)}{\cos(\frac{12}{7}\pi)\sin(\frac{9}{7}\pi)}.$$

Assume first that $\lambda \in (0,1)$ is an eigenvalue of (1) with associated eigenfunction $\psi \neq 0$. The reasoning in part 1) of the proof for Lemma 1 applies also here, whence we employ the constructions and formulas from there.

2) We first show by contradiction that the second factor on the left hand side of (8) is not zero. So assume it is zero. We then arrive at the formula

$$\frac{\varepsilon^{(4)}}{\varepsilon^{(1)}} = -\frac{\cos(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\frac{3}{2}\pi)}{\sin(\sqrt{\lambda}2\pi)\cos(\sqrt{\lambda}\frac{3}{2}\pi) - \sin(\sqrt{\lambda}\frac{3}{2}\pi)}. \tag{21}$$

With (8) we further conclude that $b^{(4)} = 0$ or $A_2(\lambda) = 0$. We distinguish between these cases.

2.i) Let first $b^{(4)} = 0$. From (7) we then conclude the relation

$$\frac{\varepsilon^{(4)}}{\varepsilon^{(1)}} = \frac{\cos(\sqrt{\lambda}\frac{3}{2}\pi)(\cos(\sqrt{\lambda}2\pi) - 1)}{\sin(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\frac{3}{2}\pi)}. \tag{22}$$

Studying the signs of the right hand side expressions in (21) and (22), we infer that $\lambda \in (4/9, 1)$. Then the right hand side of (21) is, however, smaller than 1, and we obtain a contradiction to (2).

2.ii) Let $A_2(\lambda) = 0$. Then (10) is again valid and leads to the relation

$$\omega = \frac{2\sin(\sqrt{\lambda}\frac{5}{2}\pi)\sin(\sqrt{\lambda}\pi)}{(\cos(\sqrt{\lambda}2\pi) - 1)\cos(\sqrt{\lambda}\frac{3}{2}\pi)} \tag{23}$$

for the number $\omega = \varepsilon^{(4)}/\varepsilon^{(1)} - 1$. In consideration of the signs of the expressions on the right hand side of (21) and (23), we deduce $\lambda \in (16/25, 1)$. Again the right hand side of (21) is now smaller than 1, and we arrive at a contradiction. Altogether, the second factor on the left hand side of (8) is not zero.

3) We next show by contradiction that the expression $A_2(\lambda)$ from (8) is zero. So, assume $A_2(\lambda) \neq 0$. Relation (8) then shows that $a^{(4)}$ and $b^{(4)}$ are not zero, whence we can choose $a^{(4)} = 1$ without loss of generality. The arguments in part 3.ii) of the proof for Lemma 1 are then also valid in the current setting, and lead to relation (18). The assumption on $\varepsilon^{(4)}/\varepsilon^{(1)}$ then implies that $\lambda = \frac{36}{49}$, and that relation (9) is valid. Reversing the reasoning in part 2) of the proof for Lemma 1, we additionally infer that formula (11) holds true. Dividing (11) by $\cos(\sqrt{\lambda}2\pi)\sin(\sqrt{\lambda}\frac{3}{2}\pi)$, we then conclude that $A_2(\lambda) = 0$, see (10). This leads to a contradiction.

4) By part 3), $A_2(\lambda) = 0$ and $a^{(4)} = 0$. The reasoning in part 2) of the proof for Lemma 1 then implies that $\lambda = 36/49$. Taking the assumption on $\varepsilon^{(4)}/\varepsilon^{(1)}$ into account, formula (3) is valid for $\kappa_1 = \sqrt{\lambda}$.

5) It remains to show that $\lambda = 36/49$ is indeed an eigenvalue of (1). The above reasoning in this proof already implies that the associated eigenspace is then one-dimensional. We choose $a^{(4)} = 0$, $b^{(4)} = 1$, and define $a^{(1)} = a^{(2)} = a^{(3)}$ as well as $b^{(1)} = b^{(2)} = b^{(3)}$ by (5). A desired eigenfunction ψ of (1) is defined via (4).

It then remains to show that ψ satisfies the transmission condition in the fifth line of (1). This is equivalent to the validity of (7) and (8). Due to the assumption on $\varepsilon^{(4)}/\varepsilon^{(1)}$, relation (9) is valid, implying that $A_1(\lambda) = 0$. As $a^{(4)} = 0$, we infer that (7) is true. As $\lambda = 36/49$, we obtain that also $A_2(\lambda) = 0$ by repeating the arguments in part 2) of the proof for Lemma 1 in reverse order. Thus, also (8) is fulfilled. □

References

[1] M. Costabel and M. Dauge: Singularities of electromagnetic fields in polyhedral domains, Arch. Ration. Mech. Anal. 151 (3) (2000), 221–276.

[2] R.B. Kellogg: Singularities in interface problems, in: Numerical solution of partial differential equations-II. SYNSPADE 1970 (ed. B. Hubbard), Academic Press, New York (1971), 351–400.

[3] R.B. Kellogg: On the Poisson equation with intersecting interfaces, Appl. Anal. 4 (1974/75), 101–129.

[4] H. Triebel: Higher analysis, Johann Ambrosius Barth, Leipzig, 1992.

[5] K. Zerulla: Analysis of a dimension splitting scheme for Maxwell equations with low regularity in heterogeneous media, Preprint 2022/6 of CRC 1173. URL: `https://doi.org/10.5445/IR/1000142206`

Email address: `konstantin.zerulla@kit.edu`