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PAPER

Integer and fractionalized vortex lattices and off-diagonal long-range order

Michael A Rampp[1] and Jörg Schmalian[1,2]

1   Institut für Theorie der Kondensierten Materie, Karlsruher Institut für Technologie, 76131 Karlsruhe, Germany
2   Institut für QuantenMaterialien und Technologien, Karlsruher Institut für Technologie, 76344 Karlsruhe, Germany

E-mail: mrampp@pks.mpg.de

Abstract

We analyze the implication of off-diagonal long-range order (ODLRO) for inhomogeneous periodic field configurations and multi-component order parameters. For single component order parameters we show that the only static, periodic field configuration consistent with ODLRO is a vortex lattice with integer flux in units of the flux quantum in each unit cell. For a superconductor with g degenerate components, fractional vortices are allowed. Depending on the precise order-parameter manifold, they tend to occur in units of $1/g$ of the flux quantum. These results are well known to emerge from the Ginzburg-Landau or BCS theories of superconductivity. Our results imply that they are valid even if these theories no-longer apply. Integer and fractional vortex lattices are transparently seen to emerge as a consequence of the macroscopic coherence and single valuedness of the condensate.

1. Introduction

The Meissner effect [1] and the quantization of the magnetic flux in multiply connected samples [2, 3] belong to the most fundamental aspects of superconductivity. These phenomena follow from the phenomenological Ginzburg-Landau theory [4] and from the microscopy theory of superconductivity developed by Bardeen, Cooper, and Schrieffer (BCS) [5, 6]. They are, however, phenomena that occur beyond the regime of applicability of the BCS theory. This was anticipated by London, based on the concept of macroscopic coherence [7]. The formal framework to demonstrate that these phenomena are caused by a coherent condensate and are valid more generally was provided by C. N. Yang [8] who analyzed the two-particle density matrix

$$\rho^{(2)}_{\alpha\beta\gamma\delta}(\boldsymbol{r}_1, \boldsymbol{r}_2; \boldsymbol{r}_3, \boldsymbol{r}_4) = \langle \psi^{\dagger}_{\alpha}(\boldsymbol{r}_1)\psi^{\dagger}_{\beta}(\boldsymbol{r}_2)\psi_{\gamma}(\boldsymbol{r}_3)\psi_{\delta}(\boldsymbol{r}_4) \rangle \tag{1}$$

of a many-fermion system. Here $\psi^{\dagger}_{\alpha}(\boldsymbol{r})$ and $\psi_{\alpha}(\boldsymbol{r})$ are fermion creation and annihilation operators at position $\boldsymbol{r}$ and with spin α, respectively. Yang generalized the concept of off-diagonal long-range order (ODLRO), initially proposed for interacting bosons by Penrose and Onsager [9, 10], to fermionic systems and demonstrated that ODLRO implies flux quantization with elementary flux $\Phi_0 = \frac{hc}{2e}$. The beauty of the result is that it can be made without reference to the Hamiltonian and merely relies of the presence of a macroscopic pair condensate. More recently, it was shown in [11, 12] that a homogeneous magnetic field cannot exist in the bulk of a charged system, i.e. $\boldsymbol{B} = 0$ if it is spatially constant and if we ignore surface effects. This amounts to the Meissner effect as it occurs in type-I superconductors. In [13] the argumentation was then generalized to inhomogeneous fields with cylindrical symmetry and fields that are slowly varying in space.

In this paper we generalize previous conclusions that follow from ODLRO with regards to two aspects. On the one hand, we consider periodic magnetic fields without the restriction of slow variation in space. We show that the only static, periodic field configuration consistent with superconductivity is a vortex lattice with integer flux in each unit cell. On the other hand, we consider multi-component superconducting states and find that for a superconductor with g-component order parameter the elementary flux quantum changes to $\Phi_0 \rightarrow \frac{1}{g}\frac{hc}{2e}$. Hence, fractional vortices and fractional vortex lattices become possible. Both results are known within the regime of validity of the Ginzburg-Landau and BCS approaches. The former corresponds, of course, to

Abrikosov's vortex lattice of the mixed state [14–17], while fractional vortices were discussed in the context of superfluid ^{3}He [18], two-gap superconductors [19–21], $p_x \pm i p_y$ triplet superconductors [22–28], or spin-orbit-coupled Bose–Einstein condensates [28]. The composite of a half-flux vortex and the Majorana fermions bound at its core led to significant interest given the resulting non-Abelian fractional statistics [29–32]. Experimentally, Abrikosov vortex lattices were observed via small-angle neutron diffraction [33] and the Bitter decoration technique [34] in the 1960s. Evidence for fractional vortices is much sparser. In superfluid ^{3}He in a porous medium vortices with half the quantum unit of fluid flow have indeed been generated in the laboratory [35] and single fractional vortices have been observed in two-gap superconductors [36–38] in which a lattice might be stabilized by a periodic pinning array [39]. The extreme vortex pinning in the non-centrosymmetric superconductor CePt$_3$Si was also interpreted in terms of fractionalized vortices [40], but unambiguous evidence for a fractionalized vortex lattice does not exist thus far, even though there are strong arguments to expect such a state in triplet superconductors at high magnetic field [26]. Moreover, fractional vortices may also form a vortex lattice with a non-trivial unit cell consisting of multiple fractional defects that add up to an integer flux [41].

Our ODLRO analysis shows that these established results do not rely on the validity of the Ginzburg-Landau and BCS theories. They reveal, using rather straightforward reasoning, that integer and fractional vortex lattices are tied to macroscopic coherence and the single valuedness of the condensate. On the other hand, we can only make statements about what is quantum mechanically allowed, not what is energetically most stable.

2. Summary of off-diagonal long-range order

We first summarize some of the main aspects of ODLRO for fermionic systems. This brief summary follows closely [8, 11, 12]. We analyze the density matrix $\rho^{(2)}$ of equation (1) and consider the combined two-particle coordinates ($\mathbf{r}_1$, α, $\mathbf{r}_2$, β) and ($\mathbf{r}_3$, γ, $\mathbf{r}_4$, δ). The matrix structure of interest should be understood with respect to these combined indices. We then expand $\rho^{(2)}$ with respect to its eigenfunctions $\phi_{p,\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2)$:

$$\rho^{(2)}_{\alpha\beta\gamma\delta}(\mathbf{r}_1, \mathbf{r}_2; \mathbf{r}_3, \mathbf{r}_4) = \sum_p n_p \phi^*_{p,\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2) \phi_{p,\gamma\delta}(\mathbf{r}_3, \mathbf{r}'_4) \tag{2}$$

with eigenvalues n_p. ODLRO is a state where the largest eigenvalue n_0 is of the order of the particle number N. In this case holds

$$\rho^{(2)}_{\alpha\beta\gamma\delta}(\mathbf{r}_1, \mathbf{r}_2; \mathbf{r}_3, \mathbf{r}_4) \rightarrow n_0 \phi^*_{0,\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2) \phi_{0,\gamma\delta}(\mathbf{r}_3, \mathbf{r}_4) \tag{3}$$

in the limit where $|\mathbf{r}_{1,2} - \mathbf{r}_{3,4}| \rightarrow \infty$ while $|\mathbf{r}_1 - \mathbf{r}_2|$ and $|\mathbf{r}_3 - \mathbf{r}_4|$ remain finite. Hence, the long-distance physics of two-particle correlations are dominated by the condensate with condensate wave function $\phi_{0,\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2)$ (see figure 1). If one analyses the BCS ground-state wave function with gap Δ and density of states at the Fermi level ρ_F, it follows that $n_0 \sim \rho_F |\Delta| \, N$ [42], as expected. ODLRO was also shown rigorously to occur in the negative-U Hubbard model [43], including in its ground state [44], and in closely related models [45, 46].

From the antisymmetry under the exchange of the operators $\psi^\dagger_\alpha(\mathbf{r}_1) \leftrightarrow \psi^\dagger_\beta(\mathbf{r}_2)$ and $\psi_\gamma(\mathbf{r}_3) \leftrightarrow \psi_\delta(\mathbf{r}_4)$ in $\rho^{(2)}$ follows that $\phi_{0,\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2) = -\phi_{0,\beta\alpha}(\mathbf{r}_2, \mathbf{r}_1)$. It has the properties of a two-particle fermion wave function. In full analogy to the usual classification of anomalous expectation values, see e.g. [47], one can now expand

$$\phi_{0,\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2) = \varphi_{0,s}(\mathbf{r}_1, \mathbf{r}_2) i\sigma^y_{\alpha\beta} + \boldsymbol{\varphi}_{0,t}(\mathbf{r}_1, \mathbf{r}_2) \cdot i(\boldsymbol{\sigma}\sigma^y)_{\alpha\beta} \tag{4}$$

in terms of singlet and triplet contributions in spin space. Here σ^l stand for the Pauli matrices in spin space. Hence, all our conclusions apply equally to singlet or triplet superconductors or to combinations thereof as they occur in inversion symmetry breaking systems.

Important insights about the magnetic field behavior of superconductors, such as the Meissner effect and flux quantization follow from ODLRO because of a gauge argument. To see this we first consider a spatially homogeneous magnetic field $\mathbf{B} = \text{const.}$ [11, 12] and couple it to the charged fermions via minimal substitution. In particular this implies that the many-body wave function has to transform covariantly under local gauge transformations. The vector potential can be written as

$$\mathbf{A}(\mathbf{r}) = \mathbf{A}_0(\mathbf{r}) + \nabla\varphi(\mathbf{r}), \tag{5}$$

where $\mathbf{A}_0(\mathbf{r}) = \frac{1}{2}\mathbf{B} \times \mathbf{r}$ and $\varphi(\mathbf{r})$ is an arbitrary function. A spatial translation $\mathbf{r} \rightarrow \mathbf{r} - \mathbf{a}$ can be understood as a gauge transformation since the vector potential transforms as

$$\mathbf{A}(\mathbf{r}) \rightarrow \mathbf{A}(\mathbf{r} + \mathbf{a}) = \mathbf{A}(\mathbf{r}) + \nabla\chi_{\mathbf{a}}(\mathbf{r}), \tag{6}$$

with

$$\chi_{\mathbf{a}}(\mathbf{r}) = \mathbf{a} \cdot \mathbf{A}_0(\mathbf{r}) + \varphi(\mathbf{r} - \mathbf{a}) - \varphi(\mathbf{r}). \tag{7}$$

Since fermionic operators have to transform covariantly under local gauge transformations it follows $\psi_\alpha(\mathbf{r}) = e^{i\frac{e}{\hbar c}\chi_{\mathbf{a}}(\mathbf{r})}\psi_\alpha(\mathbf{r} - \mathbf{a})$. If the system is translation symmetric, it follows from expressing the two-particle

Figure 1. ODLRO in the two-particle density matrix is defined as the presence of one (or multiple) eigenvalues that scale linearly with system size. (a) As a result, the amplitude of creating and destroying two pairs of electrons is non-zero even in the limit of infinite separation of the pairs. (b) This can be understood as the presence of a macroscopic two-particle wave function describing the superconducting condensate.

density matrix as an expectation value of fermion operators (equation (1)) that

$$
\rho^{(2)}_{\alpha\beta\gamma\delta}(\boldsymbol{r}_1, \boldsymbol{r}_2; \boldsymbol{r}_3, \boldsymbol{r}_4) = e^{-i\frac{e}{\hbar c}(\chi_{\boldsymbol{a}}(\boldsymbol{r}_1) + \chi_{\boldsymbol{a}}(\boldsymbol{r}_2) - \chi_{\boldsymbol{a}}(\boldsymbol{r}_3) - \chi_{\boldsymbol{a}}(\boldsymbol{r}_4))}
$$
$$
\times \rho^{(2)}_{\alpha\beta\gamma\delta}(\boldsymbol{r}_1 - \boldsymbol{a}, \boldsymbol{r}_2 - \boldsymbol{a}; \boldsymbol{r}_3 - \boldsymbol{a}, \boldsymbol{r}_4 - \boldsymbol{a}). \tag{8}
$$

Without ODLRO this behavior of $\rho^{(2)}$ under gauge transformations or translations does not allow to make strong statements about the eigenfunctions $\phi_{p,\gamma\delta}(\boldsymbol{r}_3, \boldsymbol{r}_4)$. We perform now two consecutive displacements by two non-collinear vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ in alternate order. For a general two-particle density matrix this leads to the condition

$$
i\frac{e}{\hbar c}(\chi_{\boldsymbol{a}_1}(\boldsymbol{r}_1) - \chi_{\boldsymbol{a}_1}(\boldsymbol{r}_3) + \chi_{\boldsymbol{a}_2}(\boldsymbol{r}_1 - \boldsymbol{a}_1) - \chi_{\boldsymbol{a}_2}(\boldsymbol{r}_3 - \boldsymbol{a}_1) - \chi_{\boldsymbol{a}_2}(\boldsymbol{r}_1)
$$
$$
+ \chi_{\boldsymbol{a}_2}(\boldsymbol{r}_3) - \chi_{\boldsymbol{a}_1}(\boldsymbol{r}_1 - \boldsymbol{a}_2) + \chi_{\boldsymbol{a}_1}(\boldsymbol{r}_3 - \boldsymbol{a}_2)
$$
$$
\chi_{\boldsymbol{a}_1}(\boldsymbol{r}_2) - \chi_{\boldsymbol{a}_1}(\boldsymbol{r}_4) + \chi_{\boldsymbol{a}_2}(\boldsymbol{r}_2 - \boldsymbol{a}_1) - \chi_{\boldsymbol{a}_2}(\boldsymbol{r}_4 - \boldsymbol{a}_1) - \chi_{\boldsymbol{a}_2}(\boldsymbol{r}_2)
$$
$$
+ \chi_{\boldsymbol{a}_2}(\boldsymbol{r}_4) - \chi_{\boldsymbol{a}_1}(\boldsymbol{r}_2 - \boldsymbol{a}_2) + \chi_{\boldsymbol{a}_1}(\boldsymbol{r}_4 - \boldsymbol{a}_2)) \in \mathbb{Z}. \tag{9}
$$

This condition is automatically fulfilled, since the left-hand side is identically zero. However, once we have a macroscopic condensate and can use equation (3) it follows

$$
\phi_{0,\alpha\beta}(\boldsymbol{r}_1, \boldsymbol{r}_2) = f_{\boldsymbol{a}} e^{i\frac{e}{\hbar c}(\chi_{\boldsymbol{a}}(\boldsymbol{r}_1) + \chi_{\boldsymbol{a}}(\boldsymbol{r}_2))} \phi_{0,\alpha\beta}(\boldsymbol{r}_1 - \boldsymbol{a}, \boldsymbol{r}_2 - \boldsymbol{a}), \tag{10}
$$

where $f_{\boldsymbol{a}}$ is an $\boldsymbol{r}$-independent but displacement dependent phase factor $|f_{\boldsymbol{a}}| = 1$. And requiring successive displacements to commute yields a condition on the phases

$$
\chi_{\boldsymbol{a}_2}(\boldsymbol{r}) + \chi_{\boldsymbol{a}_1}(\boldsymbol{r} - \boldsymbol{a}_2) - \chi_{\boldsymbol{a}_1}(\boldsymbol{r}) - \chi_{\boldsymbol{a}_2}(\boldsymbol{r} - \boldsymbol{a}_1) = \frac{hc}{e}n \tag{11}
$$

with integer n. Abstractly speaking, this condition is equivalent to the requirement that the projective representation of the group of translations equation (10) preserves the commutativity of translations. The expression equation (7) allows to write condition equation (11) as:
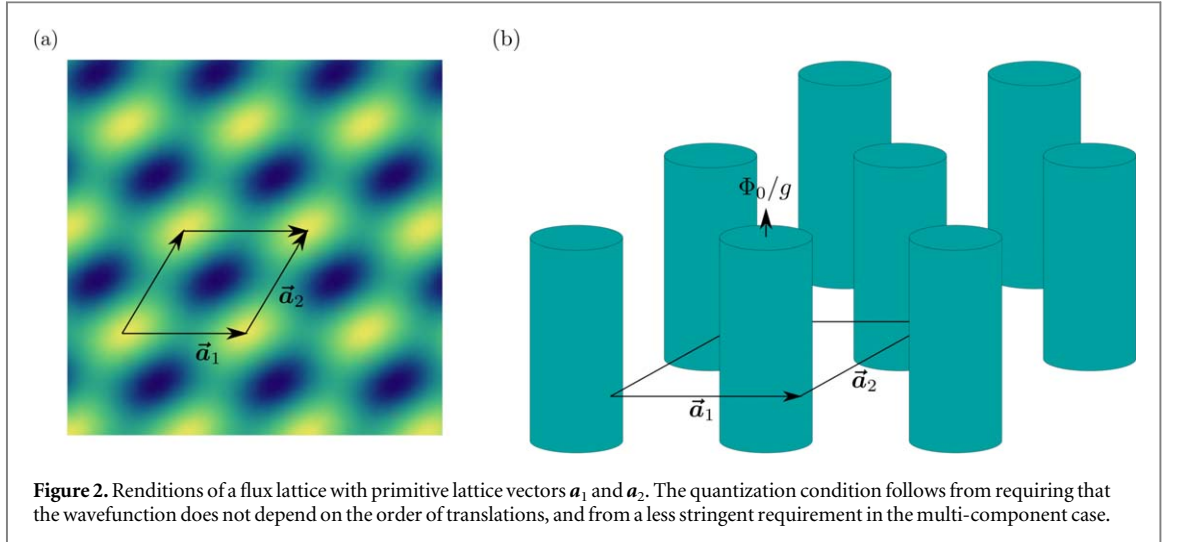
$$
\boldsymbol{B} \cdot (\boldsymbol{a}_1 \times \boldsymbol{a}_2) = n\Phi_0. \tag{12}
$$

In the continuum, where any displacement $\boldsymbol{a}_i$ is allowed, we can continuously change the left hand side of this equation. Since the right hand side cannot be changed continuously, the only solution is $\boldsymbol{B} = 0$, which yields the Meissner effect for homogeneous fields. In a periodic solid, the $\boldsymbol{a}_i$ must be integer multiples of the primitive lattice vectors. The smallest non-zero field allowed would then have to place a flux quantum in the unit cell of the system as discussed in [48, 49]. While this excludes currently achievable fields for ordinary solids, this regime becomes relevant for moiré materials, where the unit cells can be much larger. For a recent discussion of the related Hofstadter superconductors, see [50].

Using equation (10) and considering a continuum description, we can also perform an infinite sequence of infinitesimal displacements along a path

$$
\phi_{0,\alpha\beta}(\boldsymbol{r}'_1, \boldsymbol{r}'_2) = f_{\int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} d\boldsymbol{r}} e^{-i\frac{e}{\hbar c}\left(\int_{\boldsymbol{r}_1}^{\boldsymbol{r}'_1} \boldsymbol{A}(\boldsymbol{r}) \cdot d\boldsymbol{r} + \int_{\boldsymbol{r}_2}^{\boldsymbol{r}'_2} \boldsymbol{A}(\boldsymbol{r}) \cdot d\boldsymbol{r}\right)} \phi_{0,\alpha\beta}(\boldsymbol{r}_1, \boldsymbol{r}_2). \tag{13}
$$

Here, the path that connects $\boldsymbol{r}_1$ with $\boldsymbol{r}'_1$ must be the same as the one that connects $\boldsymbol{r}_2$ with $\boldsymbol{r}'_2$. In the case of a closed loop follows

Figure 2. Renditions of a flux lattice with primitive lattice vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$. The quantization condition follows from requiring that the wavefunction does not depend on the order of translations, and from a less stringent requirement in the multi-component case.

$$\phi_{0,\alpha\beta}(\boldsymbol{r}_1, \boldsymbol{r}_2) = e^{-i\frac{2e}{\hbar c}\oint \boldsymbol{A}(\boldsymbol{r})\cdot d\boldsymbol{r}}\phi_{0,\alpha\beta}(\boldsymbol{r}_1, \boldsymbol{r}_2). \tag{14}$$

The result for the quantization of the flux follows from the single-valuedness of the wave function and is

$$\Phi = \oint \boldsymbol{A}(\boldsymbol{r}) \cdot d\boldsymbol{r} = n\Phi_0. \tag{15}$$

This analysis of [8, 11, 12] reveals very transparently that macroscopic coherence in fermionic systems, reflected in a single large eigenvalue n_0 of $\rho^{(2)}$ of the order of the system size N, is the crucial ingredient that leads to the Meissner effect and to flux quantization.

3. ODLRO and integer and fractional vortex lattice states

3.1. Integer flux vortex lattices

In this section we allow for periodic magnetic fields subject to the following properties: $\boldsymbol{B}$ points in the $z-$ direction and is periodic in the xy-plane, i.e.

$$\boldsymbol{B}(\boldsymbol{r}) = \boldsymbol{B}(\boldsymbol{r} + \boldsymbol{a}_i), \tag{16}$$

with $i = 1, 2$, where $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ that are both orthogonal to $\boldsymbol{e}_z$, the unit vector along the z-direction; see figure 2. Moreover $\boldsymbol{B}(\boldsymbol{r})$ shall be independent of the z-coordinate. The question is now, what restriction does the presence of ODLRO pose on the magnetic field configuration? We have already seen that for a homogeneous field that is not too large, the only possible choice is a vanishing field and are now seeking to find the corresponding restriction for a periodic field.

In the homogeneous case, a spatial translation of the system was recognized with equation (6) as a gauge transformation. This is physically transparent, since the magnetic field configuration viewed from the displaced position is identical and therefore the vector potential $\boldsymbol{A}$ can differ at most by a gauge transformation. For a periodic field, this is only the case for a subset of translations, namely the discrete lattice translations. To show this, let us calculate explicitly the gauge transformation associated with such a displacement. As the magnetic field is periodic it can be expanded in a Fourier series using the reciprocal lattice vectors $\boldsymbol{K}$ of the periodic field configuration. Then we can perform the Fourier expansion

$$\boldsymbol{B}(\boldsymbol{r}) = \sum_{\boldsymbol{K}} e^{i\boldsymbol{K}\cdot\boldsymbol{r}}B_{\boldsymbol{K}}\boldsymbol{e}_z. \tag{17}$$

We assume that $\boldsymbol{a}_{1,2}$ are multiples of the underlying crystalline lattice. One can now explicitly generate a general expression for the vector potential

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{1}{2}B_0\boldsymbol{e}_z \times \boldsymbol{r} + \sum_{\boldsymbol{K}\neq 0} e^{i\boldsymbol{K}\cdot\boldsymbol{r}}\frac{iB_{\boldsymbol{K}}}{|\boldsymbol{K}|}\boldsymbol{K}' + \nabla\varphi(\boldsymbol{r}), \tag{18}$$

where $\boldsymbol{K}'$ is defined for every reciprocal lattice vector $\boldsymbol{K}$ as the unique unit vector that satisfies $\frac{\boldsymbol{K}}{|\boldsymbol{K}|} \times \boldsymbol{K}' = -\boldsymbol{e}_z$. The function φ ensures that the gauge choice is still arbitrary. Let us now show that a translation by a lattice vector $\boldsymbol{a}_i$ can be represented by a gauge transformation:

$$A(\boldsymbol{r} - \boldsymbol{a}_i) = \frac{1}{2} B_0 \boldsymbol{e}_z \times (\boldsymbol{r} - \boldsymbol{a}_i) + \sum_{K \neq 0} e^{iK \cdot (\boldsymbol{r} - \boldsymbol{a}_i)} \frac{iB_K}{|K|} K' + \nabla \varphi (\boldsymbol{r} - \boldsymbol{a}_i)$$

$$= A(\boldsymbol{r}) - \frac{1}{2} B_0 \boldsymbol{e}_z \times \boldsymbol{a}_i + \nabla \varphi (\boldsymbol{r} - \boldsymbol{a}_i) - \nabla \varphi (\boldsymbol{r})$$

$$= A(\boldsymbol{r}) + \nabla \chi_{\boldsymbol{a}_i}(\boldsymbol{r}), \tag{19}$$

with

$$\chi_{\boldsymbol{a}_i}(\boldsymbol{r}) = \frac{1}{2} B_0 \boldsymbol{a}_i \cdot (\boldsymbol{e}_z \times \boldsymbol{r}) + \varphi (\boldsymbol{r} - \boldsymbol{a}_i) - \varphi (\boldsymbol{r}). \tag{20}$$

We have used that for a lattice vector $\boldsymbol{a}_i$ holds that $\boldsymbol{a}_i \cdot K = 2\pi k_i$ with $k_i \in \mathbb{Z}$. As was shown in the previous section, the presence of ODLRO gives rise to equation (10). Consider again subsequent lattice translations around the unit cell spanned by $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$. The condition that the wave function be single valued leads to

$$f_{\boldsymbol{a}_1} f_{\boldsymbol{a}_2} e^{i\frac{2e}{\hbar c}(\chi_{\boldsymbol{a}_2}(\boldsymbol{r} - \boldsymbol{a}_1) + \chi_{\boldsymbol{a}_1}(\boldsymbol{r}))} = f_{\boldsymbol{a}_1} f_{\boldsymbol{a}_2} e^{i\frac{2e}{\hbar c}(\chi_{\boldsymbol{a}_1}(\boldsymbol{r} - \boldsymbol{a}_2) + \chi_{\boldsymbol{a}_2}(\boldsymbol{r}))}. \tag{21}$$

Since the $f_{\boldsymbol{a}_1}$ and $f_{\boldsymbol{a}_2}$ are just complex numbers they can be cancelled and we obtain

$$e^{-i\frac{2e}{\hbar c}(\chi_{\boldsymbol{a}_1}(\boldsymbol{r} - \boldsymbol{a}_2) + \chi_{\boldsymbol{a}_2}(\boldsymbol{r}) - \chi_{\boldsymbol{a}_2}(\boldsymbol{r} - \boldsymbol{a}_1) - \chi_{\boldsymbol{a}_1}(\boldsymbol{r}))} = 1. \tag{22}$$

Using $\chi_{\boldsymbol{a}_i}(\boldsymbol{r})$ of equation (20) we obtain for the combined gauge functions in the exponent

$$\chi_{\boldsymbol{a}_1}(\boldsymbol{r} - \boldsymbol{a}_2) + \chi_{\boldsymbol{a}_2}(\boldsymbol{r}) - \chi_{\boldsymbol{a}_2}(\boldsymbol{r} - \boldsymbol{a}_1) - \chi_{\boldsymbol{a}_1}(\boldsymbol{r}) = -B_0 \boldsymbol{e}_z \cdot (\boldsymbol{a}_1 \times \boldsymbol{a}_2). \tag{23}$$

This last expression has a clear physical meaning as minus the magnetic flux that passes through the unit cell. To show this we use $\boldsymbol{B}(\boldsymbol{r})$ of equation (17) and determine

$$\Phi = \int \boldsymbol{B}(\boldsymbol{r}) \cdot d\boldsymbol{S}$$

$$= B_0 \boldsymbol{e}_z \cdot (\boldsymbol{a}_1 \times \boldsymbol{a}_2) + \sum_{K \neq 0} B_K \int e^{iK \cdot \boldsymbol{r}} \boldsymbol{e}_z \cdot d\boldsymbol{S}. \tag{24}$$

One easily sees that the second term vanishes since the integration is over the parallelogram spanned by lattice vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ with $d\boldsymbol{S} \propto \boldsymbol{a}_1 \times \boldsymbol{a}_2$. Therefore we obtain

$$e^{i\frac{2e}{\hbar c}\Phi} = 1, \tag{25}$$

which implies that the flux through a unit cell of the lattice is quantized in integer units of the flux quantum Φ_0. In other words, a vortex lattice with integer flux in each unit cell is the only static, periodic field configuration of the type discussed above that is consistent with off-diagonal long-range order.

3.2. Fractionalized vortex lattices

So far an implicit assumption for ODLRO has been that the largest eigenvalue n_0 of the two-particle density matrix in equation (3) is unique. Next we address what happens when there are g degenerate eigenstates $\phi_{0,\alpha\beta}^{(i)}(\boldsymbol{r}_1, \boldsymbol{r}_2)$ with $i = 1, \cdots, g$ of the two-particle density matrix $\rho^{(2)}$. For the long distance behavior $|\boldsymbol{r}_{1,2} - \boldsymbol{r}_{3,4}| \to \infty$ with $|\boldsymbol{r}_1 - \boldsymbol{r}_2|$ and $|\boldsymbol{r}_3 - \boldsymbol{r}_4|$ finite, it follows now

$$\rho_{\alpha\beta\gamma\delta}^{(2)}(\boldsymbol{r}_1, \boldsymbol{r}_2; \boldsymbol{r}_3, \boldsymbol{r}_4) \to n_0 \sum_{i=1}^{g} \phi_{0,\alpha\beta}^{(i)*}(\boldsymbol{r}_1, \boldsymbol{r}_2) \phi_{0,\gamma\delta}^{(i)}(\boldsymbol{r}_3, \boldsymbol{r}_4). \tag{26}$$

We arrange these eigenstates in the g-component vector $\boldsymbol{\phi}_{0,\alpha\beta}(\boldsymbol{r}_1, \boldsymbol{r}_2)$ such that

$$\rho_{\alpha\beta\gamma\delta}^{(2)}(\boldsymbol{r}_1, \boldsymbol{r}_2; \boldsymbol{r}_3, \boldsymbol{r}_4) \to n_0 \boldsymbol{\phi}_{0,\alpha\beta}^{*}(\boldsymbol{r}_1, \boldsymbol{r}_2) \cdot \boldsymbol{\phi}_{0,\gamma\delta}(\boldsymbol{r}_3, \boldsymbol{r}_4). \tag{27}$$

The generic behavior equation (8) of $\rho^{(2)}$ under gauge transformations is of course unchanged. With multi-component ODLRO we then obtain the following transformation behavior of the wave function under translations by a lattice vector $\boldsymbol{a}_i$

$$\boldsymbol{\phi}_{0,\alpha\beta}(\boldsymbol{r}_1, \boldsymbol{r}_2) = e^{i\frac{e}{\hbar c}(\chi_{\boldsymbol{a}_i}(\boldsymbol{r}_1) + \chi_{\boldsymbol{a}_i}(\boldsymbol{r}_2))} \hat{f}_{\boldsymbol{a}_i} \cdot \boldsymbol{\phi}_{0,\alpha\beta}(\boldsymbol{r}_1 - \boldsymbol{a}_i, \boldsymbol{r}_2 - \boldsymbol{a}_i), \tag{28}$$

where we consider again a magnetic field periodic in the xy-plane and independent on the z-coordinate. $\hat{f}_{\boldsymbol{a}}$, which was formerly a phase factor, is now a unitary $g \times g$ matrix. It expresses the fact that the choice of basis at each point in space is arbitrary. Different components of the order parameter mix under gauge transformations and translations. If we now use the gauge function of equation (20) for periodic field configurations, the single-valuedness of the eigenfunctions implies

$$\hat{f}_{\boldsymbol{a}_1} \cdot \hat{f}_{\boldsymbol{a}_2} = \hat{f}_{\boldsymbol{a}_2} \cdot \hat{f}_{\boldsymbol{a}_1} e^{i\frac{2e}{\hbar c}\Phi}. \tag{29}$$

where Φ is again the flux through the parallelogram spanned by $\boldsymbol{a}_{1,2}$. Taking the determinant of this expression on both sides and using $\det \hat{f}_{\boldsymbol{a}_i} \cdot \hat{f}_{\boldsymbol{a}_j} = \det \hat{f}_{\boldsymbol{a}_i} \det \hat{f}_{\boldsymbol{a}_j}$ and $\det (e^{i\alpha} \hat{f}_{\boldsymbol{a}_i}) = e^{ig\alpha} \det \hat{f}_{\boldsymbol{a}_i}$, finally yields $e^{ig\frac{2e}{\hbar c}\Phi} = 1$. This leads to the quantization condition

$$\Phi = \frac{n}{g}\Phi_0. \tag{30}$$

The magnetic flux through the unit cell of the vortex lattice is thus quantized in fractional values of the flux quantum, where the denominator is given by the degree of degeneracy g.

One has to be somewhat careful with this argument. Strictly speaking, we find that such fractionalized vortices cannot be excluded if one only considers the determinant of the above condition equation (30) and there could be other, more stringent conditions in the full equation. An example, where we can confirm equation (30) is a g-component order parameter manifold that transforms like $U(g) = U(1) \times SU(g)$. The g eigenfunctions of the two-particle density matrix introduced above in fact transform under an irreducible representation of this group. The presence of stable line defects with a quantized integer index is guaranteed by the fundamental group being isomorphic to the integers $\pi_1(U(g)) \simeq \mathbb{Z}$ [51]. In a 1-component condensate these are vortices carrying an integer multiple of the flux quantum. Generally, the fundamental group does not carry any information about the physical meaning of the quantized index. The connection of the index to observable quantities has to be provided by identifying the properties of the actual defects. In the case of flux quantization we know that only the global phase couples to the electromagnetic field and that the trapped flux is proportional to the winding number. Thus, by identifying the fundamental defect and computing the winding number of the global phase the unit of flux quantization can be found.

Take as an example a two-fold degenerate state, i.e. $\hat{f}_a \in U(2)$. A general element $\hat{f}$ of $U(2)$ can be written as

$$\hat{f} = e^{i\theta}(n_0 1 + \boldsymbol{n} \cdot \boldsymbol{\sigma}), \quad n_0^2 + \boldsymbol{n}^2 = 1. \tag{31}$$

Parametrizing the defect by $\phi \in [0, 2\pi)$ we can construct

$$\hat{f} = e^{i\phi/2}\left(\cos\frac{\phi}{2}1 + \sin\frac{\phi}{2}\sigma^z\right). \tag{32}$$

This corresponds to a defect that carries one half of a flux quantum, because the global phase winds by $1/2$. Essentially, this is possible because $(-1) \in SU(2)$. We can generalize this by noting that $e^{i\frac{2\pi}{g}(g-1)}1 \in SU(g)$. The fundamental defect can be constructed by connecting 1 and $e^{i\frac{2\pi}{g}(g-1)}1$ in $SU(g)$ and simultaneously connecting 1 and $e^{i\frac{2\pi}{g}}$ in $U(1)$. The former is always possible, because $SU(g)$ is simply connected. The resulting loop cannot be deformed to a point in $U(g)$ and it carries a flux of Φ_0/g. This is fully consistent with equation (30). Notice, this conclusion relies on our assumption that the order parameter of the problem transforms under $U(g)$. Other order-parameter manifolds require their own, but analogous analysis.

4. Summary

In summary, we generalized the implications of off-diagonal long-range order in superconductors to periodically inhomogeneous magnetic fields. For single component superconductors one finds that a condition for the existence of finite fields is that the flux per unit cell is a multiple of the elementary flux quantum. This is of course the established Abrikosov vortex lattice. Still, our derivation has the appeal that it is valid for situations where the BCS or Ginzburg-Landau theories of superconductivity may not apply. Moreover, the rather simple nature of the proof may be of some appeal on its own right. The generalization to multi-component superconductors is relevant whenever the order parameter transforms according to a higher-dimensional irreducible representation of the symmetry group. It shows that now fractionalized vortex lattices become generally allowed inhomogeneous magnetic field states.

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Data availability statement

No new data were created or analysed in this study.

ORCID iDs

Michael A Rampp ⓘ https://orcid.org/0000-0002-8455-8460

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