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# Corner the Knight Game

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**Abstract**

In this paper, we shall introduce a game called *Corner the Knight Game*, which is a game changed the chess queen of *Corner the Queen Game* to the chess knight. Although this game is very simple, we could not find any literature on this game. Hence it should be of some meaning to investigate this elementary game.

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## 1 Introduction

Wythoff's Nim game is a two player's game which was introduced by W. A. Wythoff in his paper [8](1907). Later in 1960's R. P. Isaacs gave an equivalent description of Wythoff's nim game, which is now called *Corner the Queen Game*. In corner the queen game, a single chess queen is placed somewhere on a large grid of squares(or an infinite chess board) and the queen is restricted to move horizontally left, vertically downward and diagonally lower left. Each player can move the queen toward the lower left(or south west) corner of the grid. Let $(m, n)$ be the position of the queen. Then the winner of this game is the player who moves the queen to the lower left corner $(0, 0)$.

In this short note, we shall change the queen to the knight and call this new game *Corner the Knight Game*. In the following , we shall study the elementary properties of the grundy numbers of this game. It is a natural problem to change the queen to other chess pieces and the properties are very simple. We wonder this new game *Corner the Knight Game* has been known to the specialists, but we could not find any literature referred to this game. Hence it should be of some interest to write down these elementary results explicitly. In corner the knight game, the movement of the knight is also restricted as the queen of the corner the queen game as follows.

The Rule of Corner the Knight Game:

1) It is a two player game and each player can move the knight on the (generalized) chess board as follows.
2) Movement.
The knight at the position $(m, n)$ is restricted to move to the places $(m-2, n-1)$, $(m-1, n-2)$, $(m-2, n+1)$ or (m+1, n-2), where $m-i, n-j$ must be $\geq 0$.
3) Winner
The winner of this game is the player who moves the queen to the lower left corners $(0, 0), (1, 0), (0, 1)$ or $(1, 1)$.

One can easily obtain the following table of grundy numbers for small $(m, n)$ by checking the *mex*, i.e., minimum excluded value inductively:

**Table 1** Table of the grundy numbers $g_S(m, n)$ of corner the knight game for small $m$ and $n$.

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | |
| 11 | 1 | 1 | 2 | 3 | 1 | 1 | 2 | 3 | 1 | 1 | 2 | 1 | 4 | 3 | 2 | 3 | ... |
| 10 | 1 | 1 | 2 | 2 | 1 | 1 | 2 | 2 | 1 | 2 | 2 | 2 | 3 | 2 | 2 | 2 | ... |
| 9 | 0 | 0 | 2 | 3 | 0 | 0 | 2 | 3 | 0 | 0 | 2 | 1 | 0 | 0 | 1 | 1 | ... |
| 8 | 0 | 0 | 3 | 3 | 0 | 0 | 3 | 4 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | ... |
| 7 | 1 | 1 | 2 | 3 | 1 | 1 | 2 | 1 | 4 | 3 | 2 | 3 | 3 | 3 | 2 | 3 | ... |
| 6 | 1 | 1 | 2 | 2 | 1 | 2 | 2 | 2 | 3 | 2 | 2 | 2 | 3 | 2 | 2 | 2 | ... |
| 5 | 0 | 0 | 2 | 3 | 0 | 0 | 2 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | ... |
| 4 | 0 | 0 | 3 | 4 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | ... |
| 3 | 1 | 1 | 2 | 1 | 4 | 3 | 2 | 3 | 3 | 3 | 2 | 3 | 3 | 3 | 2 | 3 | ... |
| 2 | 1 | 2 | 2 | 2 | 3 | 2 | 2 | 2 | 3 | 2 | 2 | 2 | 3 | 2 | 2 | 2 | ... |
| 1 | 0 | 0 | 2 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | ... |
| 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | ... |
| $n/m$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | ... |

## 2 Proof of the Main Theorem

In the first place, we shall verify the following preparatory proposition.

**Proposition 2.1** *With the above notation, we have*

$$g_S(m,0) = \left[\frac{m}{2}\right] \pmod 2,$$

$$g_S(m,1) = \begin{cases} 2 & \textit{for the case } m=2, \\ \left[\frac{m}{2}\right] \pmod 2 & \textit{otherwise}, \end{cases}$$

$$g_S(m,2) = \begin{cases} 1 & \textit{for the case } m=1, \\ 2 & \textit{for the case } m=2, \\ 3 & \textit{for the cases} \equiv 0 \pmod 4 \textit{ and } m \geq 4, \\ 2 & \textit{otherwise}, \end{cases}$$

$$g_S(m,3) = \begin{cases} 1 & \textit{for the case } m=0,1,3, \\ 4 & \textit{for the case } m=4, \\ 2 & \textit{for the cases } m \equiv 2 \pmod 4, \\ 3 & \textit{otherwise}. \end{cases}$$

*Moreover* $g_S(m,4) = g_S(m-4,0)$ *and* $g_S(m,5) = g_S(m-4,1)$ *for the cases* $m \geq 4$.

Proof. One can verify the prposition's claims are true for $0 \leq m \leq 15$ as in Table1. In the first place, we shall concentrate on proving the periodicity of $g_S(m,0), g_S(m,1)$ for $m \geq 3$. Assume $g_S(4k,0) = g_S(4k,1) = g_S(4k+1,0) = g_S(4k+1,1) = 0$

and $g_S(4k+2,0) = g_S(4k+2,1) = g_S(4k+3,0) = g_S(4k+3,1) = 1$. Then the set of grundy numbers of the next places of the each knight at $(4k+1,2),(4k+1,3),(4K+2,2),(4k+2,3)$ contains the set $\{0,1\}$. Hence each grundy number of $g_S(4k+1,2), g_S(4k+1,3), g_S(4K+2,2), g_S(4k+2,3)$ must be $\geq 2$. $g_S(4k+2,2) \geq 2$ and $g_S(4k+2,0) = 1$ imply $g_S(4k+4,1) = 0$ and $g_S(4k+2,1) = 1$ implies $g_S(4k+4,0) = 0$. $g_S(4k+2,0) = g_S(4k+2,1) = 1$ and $g_S(4k+4,0) = g_S(4k+4) = 0$ imply $g_S(4k+3,2), g_S(4k+3,3)$ are also $\geq 2$. Hence we know $g_S(m,0) = g_S(m,1) \in \{0,1\}$ have the period 4 for $m \geq 3$ inductively. At the same time we have also verified $g_S(m,2), g_S(m,3) \geq 2$ for $m \geq 4$.

In the next place, we shall prove the facts $g_S(m,2), g_S(m,3) \geq 2$ for $m \geq 4$ imply $g_S(m,4) = g_S(m-4,0)$ and $g_S(m,5) = g_S(m-4,1)$. and hence $g_S(m,4), g_S(m,5)$ are also periodic with the period 4. In the cases $m = 4,5,6,7,8,9$, $g_S(m,4) = g_S(m-4,0)$ and $g_S(m,5) = g_S(m-4,1), g_S(m,6) = g_S(m-4,2)$ can be verified case by case. In the case $m \geq 10$ and $n = 4,5,6$, we shall assume $g_S(m',n) = g_S(m'-4,n-4)$ for any $m'$ $(4 \geq m' < m)$. By definition, $g_S(m,4) = \text{mex}\ \{g_S(m-2,5), g_S(m-2,3), g_S(m-1,2), g_S(m+1,4)\}$. $g_S(m-2,3), g_S(m-1,2), g_S(m+1,4) \geq 2$ imply $g_S(m,4)(\neq g_S(m-2,5) \in \{0,1\}$. By assumption $g_S(m-2,5) = g_S(m-6,1)$ and $g_S(m-4,0)(\neq g_S(m-6,1) \in \{0,1\}$. Hence $g_S(m,4) = g_S(m-4,0)$. Similarly the facts $g_S(m,2), g_S(m,3) \geq 2$ for $m \geq 4$ and the assumptions on $g_S(m',4) = g_S(m'-4,0)$ and $g_S(m',6) = g_S(m'-4,2)$for any $m'$ $(4 \geq m' < m)$ imply $g_S(m,5) = g_S(m-4,1)$.

In the next place we shall show the periodicty of $g_S(m,2)$ and $g_S(m,3)$. One can verify the grundy numbers for $0 \leq m \leq 6$. Since $g_S(m',0), g_S(m',1), g_S(m',4) \in \{0,1\}$ for $4 \leq m'$, we see $g_S(m,2)+g_S(m+2,3) = g_S(m,3)+g_S(m+2,2) = 5$. Hence we have verified the properties of $g_S(m,2)$ and $g_S(m,3)$, which completes the proof of tis proposition.

Now we shall prove the following main theorem using the induction on the value $p = m+n$.

**Theorem 2.2** *With the above notation, we have* $g_S(i,j) = g_S(j,i)$ *and* $g_S(i+4, j+$

4) $= g_S(i,j)$ *for any* $i, j \geq 0$. *More precisely we have*

$$\begin{aligned}
g_S(m,4n) &= \left[\frac{m}{2}\right] \pmod 2,\\
g_S(m,4n+1) &= \begin{cases} 2 & \textit{for the case } m = 4n+2,\\ \left[\frac{m}{2}\right] & \pmod 2 \textit{ otherwise,} \end{cases}\\
g_S(m,4n+2) &= \begin{cases} 3 & \textit{for the case } m \equiv 0 \pmod 4,\\ 2 & \textit{otherwise,} \end{cases}\\
g_S(m,4n+3) &= \begin{cases} 1 & \textit{for the case } m = 4n+3,\\ 4 & \textit{for the case } m = 4n+4,\\ 2 & \textit{for the case } m \equiv 2 \pmod 4,\\ 3 & \textit{otherwise,} \end{cases}
\end{aligned}$$

*where each* $g_S(i,j)$ *is restricted for* $i \geq j$*, because of the symmetry of grundy numbers* $g_S(i,j) = g_S(j,i)$.

Proof. From the proposition, we have already shown the periodicity $g_S(i,+4,j+4) = g_S(i,j)$ is true for the cases $i = 0,1$ or $j = 0,1$. Hence we have to verify $i,j \geq 2$, i.e., $m,n \geq 6$. Put $p = m+n$. Then we can verify the periodicity $g_S(m',n') = g_S(m'-4,n'-4)$ for any $4 \leq m'.n'$ and $p \leq 12$ as in Table 1. Assume the periodicity $g_S(m',n') = g_S(m'-4,n'-4)$ for any $4 \leq m'.n' \leq 6$ and $p$. Then by definition of the grundy number

$$g_S(m,n) = \text{mex}\,\{g_S(m-2,n-1), g_S(m-2,n+1), g_S(m-1,n-2), g_S(m+1,n-2)\}$$

for any $m \geq 6, n \geq 6$ with $m+n = p+1$. Then $p' = m'+n' = p-3$ or $p-1$ for the cases $(m',n') = (m-2,n-1), (m-2,n+1), (m-1,n-2)$ and $(m+1,n-2)$. From the assumption on the periodicity of $g_S(m',n')$, we obtain $g_S(m',n') = g_S(m'-4,n'-4)$ for above $m',n'$. Thus
$g_S(m,n) = \text{mex}\,\{g_S(m-2,n-1), g_S(m-2,n+1), g_S(m-1,n-2), g_S(m+1,n-2)\}$
$= \text{mex}\,\{g_S(m-6,n-5), g_S(m-6,n-3), g_S(m-5,n-6), g_S(m-3,n-6)\} = g_S(m-4,n-4)$.
Hence the periodicity holds for $m+n = p+1$, which completes the induction. Finally we give the following modified table with shadows, the periodicity with the period 4 was visualized as the L-shaped hook structure of the grundy numbers.

**Table 4** Table of the grundy numbers $g_S(m,n)$ of Corner the Knight game for small $m$ and $n$.

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | |
| 11 | 1 | 1 | 2 | 3 | 1 | 1 | 2 | 3 | 1 | 1 | 2 | 1 | 4 | 3 | 2 | 3 | ... |
| 10 | 1 | 1 | 2 | 2 | 1 | 1 | 2 | 2 | 1 | 2 | 2 | 2 | 3 | 2 | 2 | 2 | ... |
| 9 | 0 | 0 | 2 | 3 | 0 | 0 | 2 | 3 | 0 | 0 | 2 | 1 | 0 | 0 | 1 | 1 | ... |
| 8 | 0 | 0 | 3 | 3 | 0 | 0 | 3 | 4 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | ... |
| 7 | 1 | 1 | 2 | 3 | 1 | 1 | 2 | 1 | 4 | 3 | 2 | 3 | 3 | 3 | 2 | 3 | ... |
| 6 | 1 | 1 | 2 | 2 | 1 | 2 | 2 | 2 | 3 | 2 | 2 | 2 | 3 | 2 | 2 | 2 | ... |
| 5 | 0 | 0 | 2 | 3 | 0 | 0 | 2 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | ... |
| 4 | 0 | 0 | 3 | 4 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | ... |
| 3 | 1 | 1 | 2 | 1 | 4 | 3 | 2 | 3 | 3 | 3 | 2 | 3 | 3 | 3 | 2 | 3 | ... |
| 2 | 1 | 2 | 2 | 2 | 3 | 2 | 2 | 2 | 3 | 2 | 2 | 2 | 3 | 2 | 2 | 2 | ... |
| 1 | 0 | 0 | 2 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | ... |
| 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | ... |
| $n/m$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | ... |

## References


[ 1 ] C. L. Bouton, Nim, A Game with a Complete Mathematical Theory, Ann. of Math., **3** (1901–1902), 35–39.

[ 2 ] S. Katayama and T. Kubo, On Restricted Wythoff's Nim, Journal of the Mathematics, Tokushima University **52**, (2018), 53–57.

[ 3 ] S. Katayama and Y. Koyama, On Square and Rectangular Nim Games, Journal of the Mathematics, Tokushima University **54**, (2020) 93–104.

[ 4 ] R. Miyadera, M. Fukui, T. Inoue, Y, Nakaya and Y. Tokuni, A Variant of Corner the Queen Game, IPSJ Technical Report. 2016-GI-3506.

[ 5 ] R. Miyadera, M. Fukui, Y, Nakaya and Y. Tokuni, A Generalized Ryuoh-Nim:A Variant of the Classical Game of Wythoff Nim, IPSJ Technical Report. 2016-GI-3604.

[ 6 ] F. Sato, On the Mathematics of Extracting Games of Stones-Wonderful Relations between Games and Algebra, Sugaku-Shobou, 2014 (in Japanese).

[ 7 ] K. Suetsugu and M. Fukui, Reseach on a Variant of Wythoff Nim Which Restricts the Moves of Tokens, IPSJ Tecnical Report 2017-GI-3803(in Japanese).

[ 8 ] W. A. Wythoff, A Modification of the Game of Nim, Nieuw Arch, Wisk. **7** (1907), 199–202.