

Global Solvability for Mildly Degenerate Kirchhoff Type Dissipative Wave Equations in Bounded Domains

By

Kosuke ONO

*Department of Mathematical Sciences,
Tokushima University, Tokushima 770-8502, JAPAN*

e-mail : k.ono@tokushima-u.ac.jp

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Abstract

Consider the initial boundary value problem for degenerate dissipative wave equations of Kirchhoff type. When the wave coefficient $\rho > 0$ or the initial energy $E(0)$ is small, we show the global existence theorem.

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1 Introduction

In this paper, we study on the existence of global solutions to the initial boundary value problem for the following degenerate dissipative wave equations of Kirchhoff type :

$$\left\{ \begin{array}{l} \rho u'' + \|A^{1/2}u(t)\|^{2\gamma} Au + u' = 0 \quad \text{in } \Omega \times [0, \infty), \\ u(x, 0) = u_0(x) \quad \text{and} \quad u'(x, 0) = u_1(x) \quad \text{in } \Omega, \\ u(x, t) = 0 \quad \text{on } \partial\Omega \times [0, \infty), \end{array} \right. \quad (1.1)$$

where $u = u(x, t)$ is an unknown real value function, Ω is a bounded domain in \mathbb{R}^N with smooth boundary $\partial\Omega$, $' = \partial/\partial t$, $A = -\Delta = -\sum_{j=1}^N \partial^2/\partial x_j^2$ is the Laplace operator with the domain $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$, $\|\cdot\|$ is the norm of $L^2(\Omega)$, and $\rho > 0$ and $\gamma > 0$ are positive constants.

It is well known that Equation (1.1) describes the damped small amplitude vibrations of an elastic, stretched string when the dimension N is one or membrane when the dimension N is two (see Kirchhoff [6] and Carrier [2]).

The unique global solvability has been considered for the initial data $[u_0, u_1]$ belonging to $\mathcal{D}(A) \times \mathcal{D}(A^{1/2})$ and $\|A^{1/2}u_0\| \neq 0$. When $\gamma \geq 1$, under the

assumption that the initial data $[u_0, u_1]$ are small Nishihara and Yamada [9] have shown global existence theorems.

Under the assumption that the coefficient $\rho > 0$ is small, Ghisi and Gobino [4] have derived some decay estimates such that

$$C'(1+t)^{-\frac{1}{\gamma}} \leq \|A^{m/2}u(t)\|^2 \leq C(1+t)^{-\frac{1}{\gamma}} \quad \text{for } m = 1, 2$$

(see Ghisi [3] for weak dissipative cases, and Nishihara [8], Ono [11] for lower decay estimates, also [5], [7], [12] for upper decay estimates).

In this paper, we discuss to another smallness condition on the coefficient $\rho > 0$ or the initial energy $E(0)$, related to the unique global existence theorem.

We introduce an energy $E(t)$ as

$$E(t) \equiv \rho \|u'(t)\|^2 + \frac{1}{\gamma+1} M(t)^{\gamma+1} \quad \text{with } M(t) \equiv \|A^{1/2}u(t)\|^2. \quad (1.2)$$

By simple calculation, we see that the energy $E(t)$ has the so-called energy identity such that

$$\frac{d}{dt}E(t) + 2\|u'(t)\|^2 = 0 \quad (1.3)$$

or

$$E(t) + 2 \int_0^t \|u'(s)\|^2 ds = E(0) \quad (1.4)$$

where

$$E(0) = \rho \|u_1\|^2 + \frac{1}{\gamma+1} \|A^{1/2}u_0\|^{2(\gamma+1)}. \quad (1.5)$$

Our main result is as follows.

Theorem 1.1 *Let the initial data $[u_0, u_1]$ belong to $\mathcal{D}(A) \times \mathcal{D}(A^{1/2})$ and $\|A^{1/2}u_0\| \neq 0$. Suppose that the coefficient $\rho > 0$ or the initial energy $E(0)$ is small in the following sense*

$$2(\gamma+1) \frac{2\gamma+1}{\gamma+1} G(0)^{\frac{1}{2}} B(0)^{\frac{1}{2}} \rho E(0)^{\frac{\gamma}{\gamma+1}} < 1 \quad (1.6)$$

(equivalent to (3.2)) with $G(0)$ and $B(0)$ given by (2.3) and (2.8), respectively. Then, the problem (1.1) admits a unique global solution $u(t)$ in the class $C^0([0, \infty); \mathcal{D}(A)) \cap C^1([0, \infty); \mathcal{D}(A^{1/2})) \cap C^2([0, \infty); L^2(\Omega))$.

Theorem 1.1 follows from Theorem 3.1 in the continuing sections.

The notations we use in this paper are standard. The symbol (\cdot, \cdot) means the inner product in $L^2(\Omega)$ or sometimes duality between the space X and its dual X' . Positive constants will be denoted by C and will change from line to line.

2 Preliminaries

We obtain the following local existence theorem by standard arguments and we omit the proof here (see [1], [10], [13], [14], and the references cited therein).

Proposition 2.1 *Suppose that the initial data $[u_0, u_1]$ belong to $\mathcal{D}(A) \times \mathcal{D}(A^{1/2})$ and $\|A^{1/2}u_0\| \neq 0$. Then the problem (1.1) admits a unique local solution $u(t)$ in the class $C^0([0, T]; \mathcal{D}(A)) \cap C^1([0, T]; \mathcal{D}(A^{1/2})) \cap C^2([0, T]; L^2(\Omega))$ for some $T = T(\|Au_0\|, \|A^{1/2}u_1\|) > 0$.*

Moreover, if $\|A^{1/2}u(t)\| \neq 0$ and $\|Au(t)\| + \|A^{1/2}u'(t)\| < \infty$ for $t \geq 0$, then we can take that $T = \infty$.

In what follows in this section, we assume that $M(0) > 0$ and the function $u = u(t)$ is a solution of (1.1) and satisfies

$$\rho \frac{|M'(t)|}{M(t)} \leq \frac{1}{\gamma + 1}. \quad (2.1)$$

Proposition 2.2 *Under the assumption (2.1), it holds that*

$$\frac{\|Au(t)\|^2}{M(t)} \leq G(t) \leq G(0) \quad (2.2)$$

where

$$G(t) \equiv \frac{\|Au(t)\|^2}{M(t)} + \rho Q(t), \quad (2.3)$$

$$Q(t) \equiv \frac{1}{M(t)^{\gamma+1}} \left(M(t) \|A^{1/2}u'(t)\|^2 - \frac{1}{4} |M'(t)|^2 \right). \quad (2.4)$$

Proof. From Equation (1.1), we observe

$$\begin{aligned} & \frac{d}{dt} \frac{\|Au(t)\|^2}{M(t)} \\ &= \frac{1}{M(t)^{\gamma+2}} \left(2(M(t)^\gamma Au(t), Au'(t))M(t) - (M(t)^\gamma Au(t), Au(t))M'(t) \right) \\ &= \frac{-1}{M(t)^{\gamma+1}} \left(2 \left(\|A^{1/2}u'(t)\|^2 + \rho(A^{1/2}u''(t), A^{1/2}u'(t)) \right) M(t) \right. \\ & \quad \left. - \left(\frac{1}{2}M'(t) + \rho \left(\frac{1}{2}M''(t) - \|A^{1/2}u'(t)\|^2 \right) \right) M'(t) \right) \\ &= -2Q(t) - \rho R(t) \end{aligned} \quad (2.5)$$

where

$$\begin{aligned} R(t) \equiv & \frac{1}{M(t)^{\gamma+2}} \left(2M(t)(A^{1/2}u''(t), A^{1/2}u'(t)) \right. \\ & \left. + M'(t) \left(\|A^{1/2}u'(t)\|^2 - \frac{1}{2}M''(t) \right) \right). \end{aligned}$$

Moreover, we observe

$$\frac{d}{dt}Q(t) = -(\gamma + 2)\frac{M'(t)}{M(t)}Q(t) + R(t) \quad \text{and} \quad Q(t) \geq 0. \quad (2.6)$$

From (2.1), (2.5), and (2.6), we have

$$\frac{d}{dt}G(t) + 2\left(1 + \frac{\gamma + 2}{2}\rho\frac{M'(t)}{M(t)}\right)Q(t) \leq 0,$$

and hence, we obtain the desired estimate (2.2). \square

Proposition 2.3 *Under the assumption (2.1), it holds that*

$$\frac{\|u'(t)\|^2}{M(t)^{2\gamma+1}} \leq B(0) \quad (2.7)$$

where

$$B(0) \equiv \max\left\{\frac{\|u_1\|^2}{M(0)^{2\gamma+1}}, (2(\gamma + 1))^2G(0)\right\}. \quad (2.8)$$

Proof. Multiplying (1.1) by $2M(t)^{-2\gamma-1}u'$ and integrating it over Ω , we have from the Young inequality that

$$\begin{aligned} \rho\frac{d}{dt}\frac{\|u'(t)\|^2}{M(t)^{2\gamma+1}} + 2\left(1 + \frac{2\gamma + 1}{2}\rho\frac{M'(t)}{M(t)}\right)\frac{\|u'(t)\|^2}{M(t)^{2\gamma+1}} &= -\frac{M'(t)}{M(t)^{\gamma+1}} \\ &\leq \frac{1}{2(\gamma + 1)}\frac{\|u'(t)\|^2}{M(t)^{2\gamma+1}} + 2(\gamma + 1)\frac{\|Au(t)\|^2}{M(t)}. \end{aligned} \quad (2.9)$$

Since it follows from (2.1) that

$$1 + \frac{2\gamma + 1}{2}\rho\frac{M'(t)}{M(t)} \geq \frac{1}{2(\gamma + 1)}, \quad (2.10)$$

we observe from (2.2) and (2.9) that

$$\begin{aligned} \rho\frac{d}{dt}\frac{\|u'(t)\|^2}{M(t)^{2\gamma+1}} + \frac{1}{2(\gamma + 1)}\frac{\|u'(t)\|^2}{M(t)^{2\gamma+1}} &\leq 2(\gamma + 1)\frac{\|Au(t)\|^2}{M(t)} \\ &\leq 2(\gamma + 1)G(0). \end{aligned}$$

Thus, by standard calculation for ODE, we obtain the desired estimate (2.7).

\square

Proposition 2.4 *Under the assumption (2.1), it holds that*

$$M(t) \geq C'(1 + t)^{-\frac{1}{\gamma}} \quad (2.11)$$

with some positive constant C' .

Proof. Multiplying (1.1) by $2M(t)^{-2\gamma-1}u'$ and integrating it over Ω , we have from the Young inequality that

$$\begin{aligned} & \frac{d}{dt} \left(\rho \frac{\|u'(t)\|^2}{M(t)^{2\gamma+1}} + \frac{1}{M(t)^\gamma} \right) + 2 \left(1 + \frac{2\gamma+1}{2} \rho \frac{M'(t)}{M(t)} \right) \frac{\|u'(t)\|^2}{M(t)^{2\gamma+1}} \\ &= -(\gamma+1) \frac{M'(t)}{M(t)^{\gamma+1}} \\ &\leq \frac{1}{\gamma+1} \frac{\|u'(t)\|^2}{M(t)^{2\gamma+1}} + (\gamma+1)^3 \frac{\|Au(t)\|^2}{M(t)}, \end{aligned}$$

and from (2.10) that

$$\frac{d}{dt} \left(\rho \frac{\|u'(t)\|^2}{M(t)^{2\gamma+1}} + \frac{1}{M(t)^\gamma} \right) \leq (\gamma+1)^3 \frac{\|Au(t)\|^2}{M(t)} \leq (\gamma+1)^3 G(0)$$

Thus, immediately we obtain

$$\rho \frac{\|u'(t)\|^2}{M(t)^{2\gamma+1}} + \frac{1}{M(t)^\gamma} \leq C(1+t) \quad \text{or} \quad M(t)^\gamma \geq C^{-1}(1+t)^{-1}$$

which implies the desired estimate (2.11). \square

3 Global Solvability

We introduce the function $H(t)$ (a second order energy) as

$$H(t) \equiv \rho \frac{\|A^{1/2}u'(t)\|^2}{M(t)^\gamma} + \|Au(t)\|^2. \quad (3.1)$$

Theorem 3.1 *Let the initial data $[u_0, u_1]$ belong to $\mathcal{D}(A) \times \mathcal{D}(A^{1/2})$ and $M(0) > 0$. Suppose that*

$$2\rho B(0)^{\frac{1}{2}} G(0)^{\frac{1}{2}} ((\gamma+1)E(0))^{\frac{\gamma}{\gamma+1}} < \frac{1}{\gamma+1}. \quad (3.2)$$

Then, the problem (1.1) admits a unique global solution $u(t)$ in the class

$$C^0([0, \infty); \mathcal{D}(A)) \cap C^1([0, \infty); \mathcal{D}(A^{1/2})) \cap C^2([0, \infty); L^2(\Omega))$$

and this solution $u(t)$ satisfies

$$\rho \frac{|M'(t)|}{M(t)} < \frac{1}{\gamma+1} \quad \text{and} \quad H(t) \leq H(0), \quad (3.3)$$

$$\frac{\|Au(t)\|^2}{M(t)} \leq G(0) \quad \text{and} \quad \frac{\|u'(t)\|^2}{M(t)^{2\gamma+1}} \leq B(0), \quad (3.4)$$

$$C'(1+t)^{-\frac{1}{\gamma}} \leq M(t) \leq ((\gamma+1)E(0))^{\frac{1}{\gamma+1}} \quad \text{for } t \geq 0 \quad (3.5)$$

with $E(0)$, $G(0)$, and $B(0)$ given by (1.5), (2.3), and (2.8), respectively, where C' is some positive constant.

Proof. Let $u(t)$ be a solution on $[0, T]$. Since $M(0) > 0$, putting

$$T_1 \equiv \sup \{t \in [0, \infty) \mid M(s) > 0 \text{ for } 0 \leq s < t\},$$

we see that $T_1 > 0$. If $T_1 < T$, then

$$M(t) > 0 \text{ for } 0 \leq t < T_1 \text{ and } M(T_1) = 0. \quad (3.6)$$

Since it follows from (1.4) and (3.2) that

$$\begin{aligned} \rho \frac{|M'(0)|}{M(0)} &\leq 2\rho \left(\frac{\|u_1\|^2}{M(0)^{2\gamma+1}} \right)^{\frac{1}{2}} \left(\frac{\|Au_0\|^2}{M(0)} \right)^{\frac{1}{2}} M(0)^\gamma \\ &\leq 2\rho B(0)^{\frac{1}{2}} G(0)^{\frac{1}{2}} ((\gamma+1)E(0))^{\frac{1}{\gamma+1}} < \frac{1}{\gamma+1}, \end{aligned}$$

putting

$$T_2 \equiv \sup \left\{ t \in [0, \infty) \mid \rho \frac{|M'(s)|}{M(s)} < \frac{1}{\gamma+1} \text{ for } 0 \leq s < t \right\},$$

we see that $T_2 > 0$. If $T_2 < T_1$, then

$$\rho \frac{|M'(t)|}{M(t)} < \frac{1}{\gamma+1} \text{ for } 0 \leq t < T_2 \text{ and } \rho \frac{|M'(T_2)|}{M(T_2)} = \frac{1}{\gamma+1}. \quad (3.7)$$

From Proposition 2.2 and Proposition 2.3 we observe

$$\begin{aligned} \rho \frac{|M'(t)|}{M(t)} &\leq 2\rho \left(\frac{\|u'(t)\|^2}{M(t)^{2\gamma+1}} \right)^{\frac{1}{2}} \left(\frac{\|Au(t)\|^2}{M(t)} \right)^{\frac{1}{2}} M(t)^\gamma \\ &\leq 2\rho B(0)^{\frac{1}{2}} G(0)^{\frac{1}{2}} ((\gamma+1)E(0))^{\frac{1}{\gamma+1}} < \frac{1}{\gamma+1} \end{aligned} \quad (3.8)$$

for $0 \leq t \leq T_2$, which is a contradiction to (3.7), and hence, we have that $T_2 \geq T_1$. Moreover, from Proposition 2.4 we observe

$$M(t) \geq C'(1+t)^{-\frac{1}{\gamma}} > 0 \text{ for } 0 \leq t \leq T_1,$$

which is a contradiction to (3.6), and hence, we have that $T_1 \geq T$.

Multiplying (1.1) by $2M(t)^{-\gamma}Au'$ and integrating it over Ω we have

$$\frac{d}{dt}H(t) + 2 \left(1 + \frac{\gamma}{2} \rho \frac{M'(t)}{M(t)} \right) \frac{\|A^{1/2}u'(t)\|^2}{M(t)^\gamma} = 0.$$

Since it follows from (3.8) that

$$1 + \frac{\gamma}{2} \rho \frac{M'(t)}{M(t)} \geq 0,$$

we observe

$$\frac{d}{dt}H(t) \leq 0 \quad \text{and} \quad H(t) \leq H(0) \quad (3.9)$$

for $0 \leq t \leq T$. Thus, from above argument we see that $M(t) > 0$ and $\|Au(t)\| + \|A^{1/2}u'(t)\| \leq C$ for $t \geq 0$. Therefore, by the second argument of Proposition 2.1, we conclude that the problem (1.1) admits a unique global solution. Moreover, from Propositions 2.2–2.4, we obtain the desired estimates (3.3)–(3.5). \square

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