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On Polygonal Square Triangular Numbers Ⅱ

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Abstract

A pentagonal square triangular number is a number which is a pentagonal number, a square and a triangular number at the same time. In our previous paper [10], we have shown the only pentagonal square triangular number is 1. In this note, we shall continue to investigate several related problems and give more detailed results on these subjects.

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1 Introduction

The following explanation has been written in the on-line article "Pentagonal Square Triangular Number" of Wolfram Mathworld [18].

"It is almost certain that no other solution exists except for 1, although no proof of this fact appears to have yet appeared in print."

In that article, it is noted that, in 2003 and 2006, J. Silcox has pointed out the determination of pentagonal square triangular number is equivalent to solve the following simultaneous Pell equations

$$x^2 - 2y^2 = 1, z^2 - 6y^2 = 1 \text{ with even } y.$$

Moreover he noted that this problem has been already solved as the special case $R = 2, S = 6$ of W. S. Anglin's more general results. Actually, in 1996, W. S. Anglin determined all the positive integer solutions of simultaneous Pell equations $x^2 - Ry^2 = 1, z^2 - Sy^2 = 1$ for all the cases $0 < R < S \leq 200$ in his paper [2].

Since there has been no simple reference for the determination of the pentagonal square triangular number as above, we have given two different detailed proofs of this fact and also studied several related problems in our previous paper [10]. In this paper, we shall continue to investigate several related problems in somewhat general setting, which include the problem of the determination of polygonal square triangular numbers.

Let $P_k(n)$ be the $n-$th $k-$gonal number, i.e., the number of dots arranged as a regular $k-$gon with n dots on each side. Then $P_k(n)$ is written in the form

$$P_k(n) = \frac{n((k-2)n-(k-4))}{2}.$$

In the following, we will simply denote the set of all the $k-$gonal numbers $\{P_k(n) \mid n \in \mathbb{N}\}$ by P_k.

2 Known results

In our previous paper [10], we have shown the following results:

Proposition 2.1 *$P_a \cap P_b \cap P_c = \{1\}$ for the case $(a,b,c) = (3,4,5)$. Moreover $\{P_a \cap P_b \cap P_c\}$ is also $\{1\}$ for the other cases $(a,b,c) = (3,4,7), (3,4,8), (3,4,9), (3,4,10), (3,4,11), (3,4,12), (3,5,8)$ and $(4,5,8)$.*

Remark 2.2 *The case $(a,b,c) = (3,4,5)$ mentioned above is nothing but the case of pentagonal square triangular number.*

Here we recall some historical facts related to polygonal numbers. Euler is the first mathematician who considered the problem of determining square triangular numbers. He has treated this problem in his text “Algebra” 1774 (for example, see the section 1 of Dickson’s book [8] “Polygonal, Pyramidal, and Figurate Numbers”). Euler also verified the infiniteness of square triangular numbers $P_3 \cap P_4 = \{1, 36, 1225, 41616, 1413721, \ldots\}$, which is labeled as A001110 in the On-line Encyclopedia of Integer Sequences [15]. From now on, we shall abbreviate the On-line Encyclopedia of Integer Sequences to OEIS. It is also known that both of the pentagonal square numbers $P_4 \cap P_5 = \{1, 960400, 94109401, 903638458801, 8676736387298001\ldots\}$ (OEIS A036353) and the pentagonal triangular numbers $P_3 \cap P_5 = \{1, 210, 40755, 7906276, 1533776805, \ldots\}$ (OEIS A014979) are infinite. On the contrary, one can show $P_4 \cap P_{10} = \{1\}$ and more generally $|P_4 \cap P_{2a^2+2}| < \infty$ for any positive integer a as in [10]. In this paper, we denote the number of elements contained in a set S by $|S|$ as usual. Now we shall generalize this fact as follows.

Theorem 2.3 *Assume $a \neq b$, then*

$$|P_a \cap P_b| < \infty \Longleftrightarrow (a-2)(b-2) = \square \text{ except for the case } (a,b) = (3,6)$$

Proof. Without loss of generality, we may assume $3 \leq a < b$. For the sake of simplicity, we shall put $A = a-2, B = b-2$ and $C = c-2$. Assume

$$P_a(m) = P_b(n) \Longleftrightarrow \frac{m(Am-(A-2))}{2} = \frac{n(Bn-(B-2))}{2}.$$

Multiplying the both sides by $8AB^2$, we have

$$8AB^2P_a(m) = (2ABm-(A-2)B)^2-(A-2)^2B^2$$

$$= 8AB^2P_b(n) = AB((2Bn-(B-2))^2-AB(B-2)^2.$$

Substitute X and Y for $2ABm-(A-2)B$ and $(2Bn-(B-2))$, respectively. Then we have the following Pell equation

$$X^2-ABY^2 = (b-2)(a^2(b-2)-b^2(a-2)).$$

From the decomposition $a^2(b-2)-b^2(a-2) = (a-b)((a-2)(b-2)-4)$, the right hand side of the above equation $B(A-B)(AB-4) = (b-2)(a-b)((a-2)(b-2)-4) \neq 0$ except for the case $(a,b) = (3,6)$. In the case $(a,b) = (3,6)$, one sees $P_6(n) = n(2n-1) = P_3(2n-1)$. Therefore $P_6 \subset P_3$ and $|P_3 \cap P_6| = \infty$. If $(a,b) \neq (3,6)$ and $AB = (a-2)(b-2) = \square = M^2$, the left hand side of the above equation decomposes $(X+MY)(X-MY)$. Thus $|P_a \cap P_b| < \infty$ for these cases. Now assume $AB = (a-2)(b-2) \neq \square$. Then the above equation is a norm equation from the real quadratic field $\mathbb{Q}(\sqrt{AB})$ to $\mathbb{Q}$ with a special solution $(X,Y) = (a(b-2),b)$. Let (t,u) be the smallest positive integer solution of the Pell equation $x^2-ABy^2 = 1$. Then all the positive integer solutions (x,y) of this Pell equation are given by $(x,y) = (t_k,u_k)$, where t_k, u_k satisfies the following binary recurrence sequences.

$$t_{k+1} = 2tt_k - t_{k-1}, u_{k+1} = 2tu_k - u_{k-1},$$

with $t_0 = 1, t_1 = t, u_0 = 0, u_1 = u$. From this recurrence relation, we see $t_k \equiv t^k \pmod{t^2-1}$. Combing the facts $AB|(t^2-1)$ and $(t,AB) = 1$, one sees $t_k \equiv t^k \equiv 1 \pmod{AB}$. Thus we have $t_{2k} = 2t_k^2-1 \equiv 1 \pmod{2AB}$ and $u_2 = 2t_ku_k \equiv 0 \pmod 2$. Calculating the following equation

$$X_k + Y_k\sqrt{AB} = ((b-2)a + b\sqrt{AB})(t_{2k} + u_{2k}\sqrt{AB}),$$

we have a solution of $X_k^2 - ABY_k^2 = (b-2)(a^2(b-2)-b^2(a-2))$, where

$$X_k = (b-2)(at_{2k} + bu_2(a-2)(b-2)), Y_k = bt_{2k} + a(b-2)u_{2k}.$$

Since $at_2 + bu_{2k}(a-2)(b-2) \equiv a \equiv -(a-4) \pmod{2(a-2)}$ and $bt_{2k} + a(b-2)u_{2k} \equiv b \equiv -(b-4) \pmod{2(b-2)}$, each solution (X_k, Y_k) corresponds to a different pair (m,n) with $P_a(m) = P_b(n)$. Thus the solutions (m,n) of $P_a(m) = P_b(n)$ are infinite for this case, which completes the proof.

2.1 Simultaneous Pell Equations

In the following, we shall restrict ourselves to the cases $6 \notin \{a, b, c\}$ from the above argument. Then one can show the following fact.

Theorem 2.4 *Assume* $(a-2)(b-2) \neq \square, (b-2)(c-2) \neq \square$ *and* $(c-2)(a-2) \neq \square$. *Then* $|P_a \cap P_b| = |P_b \cap P_c| = |P_c \cap P_a| = \infty$, *but* $|P_a \cap P_b \cap P_c| < \infty$.

Proof. Assume $P_a(m) = P_b(n) = P_c(\ell)$. $P_a(m) = P_b(n)$ implies $A((2Bn-(B-2))^2 - (B-2)^2) = B((2Am-(A-2))^2 - (A-2)^2)$ and hence

$$A(2Bn-(B-2))^2 = B(2Am-(A-2))^2 - (A-B)(AB-4).$$

Similarly, $P_a(m) = P_c(\ell)$ implies

$$A(2C\ell-(B-2))^2 = C(2Am-(A-2))^2 - (A-C)(AC-4),$$

where we put $A = a-2, B = b-2, C = c-2$ as above. Therefore we have $A^2(2Bn-(B-2))^2(2C\ell-(B-2))^2$
$= (B(2Am-(A-2))^2-(A-B)(AB-4))(C(2Am-(A-2))^2-(A-C)(AC-4))$.
Putting $X = BC(2Am-(A-2))^2, Y = ABC(2Am-(A-2))(2Bn-(B-2))(2C\ell-(C-2))$, we have

$$E_{(a,b,c)} : Y^2 = X(X - C(A-B)(AB-4))(X - B(A-C)(AC-4)).$$

Since we may assume $3 \leq a < b < c$ and $6 \notin \{a, b, c\}$, we know $1 \leq A < B < C$ and $AB, BC, CA \neq 4$. Then we obtain $C(A-B)(AB-4) \neq 0$ and $B(A-C)(AC-4) \neq 0$. Moreover $C(A-B)(AB-4) - B(A-C)(AC-4) = A(B-C)(BC-4) \neq 0$. Therefore $E_{(a,b,c)}$ is a modular elliptic curve with integer coefficients and the number of integral points on $E_{(a,b,c)}$ is finite from Siegel's theorem. Since each triple (m, n, ℓ) which satisfies the condition $P_a(m) = P_b(n) = P_c(\ell)$ corresponds to an integer points on $E_{(a,b,c)}$ as above, we have proved this theorem.

Moreover, in the special case $(a, b) = (3, 4)$, that is, the case of polygonal square triangular number, we have the following corollary.

Corollary 2.5 *Assume* $c-2 \neq \square$ *and* $c-2 \neq 2\square$. *Then* $|P_3 \cap P_c| = |P_4 \cap P_c| = \infty$, *and* $|P_3 \cap P_4 \cap P_c| < \infty$.

In general, it has been proved that if R and S are distinct positive integers then the simultaneous Pell equations

$$x^2 - Ry^2 = 1, z^2 - Sy^2 = 1$$

possess at most two solutions in positive integers (x, y, z) ([3] and [4]). Since it is known that there exist infinite families of pairs (R, S) for which the above equations have two solutions, this result is the best possible results on the number of solutions of the simultaneous Pell equations. Moreover, it has been proved that there exists an upper estimate for the number of positive integer solutions of the following more general equations as in [3].

Proposition 2.6 (Bennett) *Let R, S be distinct positive integers with U and V are nonzero integers with $RV - SU \neq 0$. Let $N(R, S, U, V)$ be the number of positive integer solutions of following simultaneous diophantine equations*

$$x^2 - Ry^2 = U, z^2 - Sy^2 = V.$$

Then $N(R, S, U, V) \ll 2\min\{\omega(U), \omega(V)\}\log(|U| + |V|)$, where $\omega(t)$ denotes the number of distinct prime factors of t.

These results follow from a combination of simultaneous Padé approximation to binomial functions, the theory of linear forms in two logarithms and some gap principles introduced by Bennett.
Now we shall apply this proposition to our cases. Assume

$$P_a(m) = P_b(n) = P_c(\ell).$$

Put $A = a - 2, B = b - 2$ and $C = c - 2$. Then we know

$$(2BAm - B(A - 2))^2 - BA(2Bn - (B - 2))^2 = B(B - A)(4 - BA).$$

Similarly we have

$$(2BC\ell - B(C - 2))^2 - BC(2Bn - (B - 2))^2 = B(B - C)(4 - BC).$$

Put $x = 2BAm - B(A - 2)$, $y = 2Bn - (B - 2)$, $z = 2BC\ell - B(C - 2)$, $U = B(B - A)(4 - BA)$ and $V = B(B - C)(4 - BC)$. Also put $R = BA$ and $S = BC$.

Then we have the simultaneous Pell equations

$$x^2 - Ry^2 = U, z^2 - Sy^2 = Y, \text{with } RV - SU = B^3(A - C)(4 - AC) \neq 0.$$

Hence we have the following upper estimate for the number of elements in $P_a \cap P_b \cap P_c$.

Theorem 2.7 *Let U be $(b - 2)(b - a)(4 - (b - 2)(a - 2))$ and V be $(b - 2)(b - c)(4 - (b - 2)(c - 2))$, then*

$$|P_a \cap P_b \cap P_c| \ll 2\min\{\omega(U), \omega(V)\}\log(|U| + |V|),$$

where $\omega(t)$ denotes the number of distinct prime factors of t.

In our previous paper [10], we could not find any example (a, b, c) with $|P_a \cap P_b \cap P_c| > 1$. Now we note there are infinitely many (a, b, c) which satisfy $|P_a \cap P_b \cap P_c| \geq 2$.

For the cases $(a - 2)(b - 2) \neq \square$, there are infinitely many numbers in $P_a \cap P_b$. Hence, we may put $P_a \cap P_b = \{c_1, c_2, \ldots, c_k, \ldots\}$. Then it is obvious that at least two elements $1, c_k \in P_a \cap P_b \cap P_{c_k}$ and hence we have shown the following theorem:

Theorem 2.8 *There are infinitely many (a, b, c) with $|P_a \cap P_b \cap P_c| \geq 2$.*

In the following section, we shall investigate the case $a = 3, b = 4$, i.e., the cases of polygonal square triangular numbers more precisely.

3 Polygonal square triangular numbers

Let us recall the definition of Pell number p_k and Pell-Lucas number q_k. Pell number p_k is defined by the following recurrence relations:

$$p_0 = 0, p_1 = 1, \quad \text{and} \quad p_{k+1} = 2p_k + p_{k-1}, \text{ where } k \geq 1.$$

Pell-Lucas number q_k is defined by the following recurrence relations:

$$q_0 = 2, p_1 = 2, \quad \text{and} \quad q_{k+1} = 2q_k + q_{k-1}, \text{ where } k \geq 1.$$

For the sake of convenience, we shall list a few examples of Pell numbers $\{p_k\}$ and Pell-Lucas numbers $\{q_k\}$ as follows.

$\{p_k\} = \{0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 541, 13860, \ldots, \}$ (OEIS A000129),

$\{q_k\} = \{2, 2, 6, 14, 34, 82, 198, 478, 1154, 2786, 6726, 16238, \ldots, \}$ (OEIS A002203).

It is well known that Pell numbers and Pell-Lucas numbers satisfy the following Pell equation

$$(q_k/2)^2 - 2p_k^2 = (-1)^k.$$

Now le us recall the relations

$$P_3(m) = P_4(n) \Longleftrightarrow \frac{m(m+1)}{2} = n^2 \Longleftrightarrow (2m+1)^2 - 2(2n)^2 = 1.$$

Therefore we see that square triangular number n^2 can be represented by $2k-$th Pell number p_{2k} as $n = p_{2k}/2$ for some k. Put $c = (p_{2k}/2)^2$. Then one can verify the following precise version of Theorem 2.7 and 2.8.

Theorem 3.1 *For the special case $c = p_{2k}^2/4$, we have*

$$2 \leq |P_3 \cap P_4 \cap P_c| \ll 2\omega(2(c-2)) \log(4 + (c-2)^2)$$

Proof. One sees that $A = 3 - 2 = 1$ and $B = 4 - 2 = 2$ and $C = c - 2$. Hence $AB = 2 \neq \square, AC = c-2 = (p_{2k}/2)^2 - 2 \neq \square$. Assume $BC = 2(c-2) = \square$. Then $c-2 = (p_{2k}/2)^2 - 2 = 2x^2$ for some integer x. Then $2|c$ and $x^2 - 2(p_{2k}/4)^2 = -1$. Therefore we know $p_{2k}/4 = p_{2h+1}$ for some non-negative integer h. Since $n = p_{2k}/2$, we have the simultaneous equations

$$(2m+1)^2 = 8n^2 + 1, x^2 = 2(n/2)^2 - 1.$$

Putting $Y = 2 \times 16 \times x \times (2m+1)$ and $X = 16n^2 - 10$, one has the equation

$$E : Y^2 = X^3 - 364X - 2640.$$

We note the solution of the simultaneous Pell equations corresponds to the integer point $(X, Y) = (16n^2 - 10, 32x(2m+1))$ on the elliptic curve E. Then the Cremona's label of this curve E is 1088e2 and this curve has only 3 integer points $(-10,0),(-12,0),(22,0)$, which imply $n=0,\sqrt{22}/4,\sqrt{2}\notin\mathbb{N}$. It contradicts to the definition $n\in\mathbb{N}$ and we know $BC=2(c-2)\neq\square$. Hence we have shown $|P_3\cap P_4\cap P_c|<\infty$ from Corollary 2.5. Finally one can easily calculate $U=2^2$ and $V=2^2(c-2)^2$ for this case and hence can get the upper estimate of $|P_3\cap P_4\cap P_c|$ from Theorem 2.7, which completes the proof of this theorem.

3.1 Tridecagonal Square Triangular Numbers

Here we shall show $|P_3\cap P_4\cap P_{13}|$ is exactly 2. Let us recall the fact

$$P_3(m)=P_4(n)=P_{13}(\ell)\Longleftrightarrow\frac{m(m+1)}{2}=n^2,\frac{\ell(11\ell-9)}{2}=n^2.$$

Thus we have

$$(2m+1)^2=8n^2+1,(22\ell-9)^2=88n^2+81.$$

It follows that

$$88^2n^2(22\ell-9)^2(2m+1)^2=176n^2(176n^2+22)(176n^2+162).$$

Putting $Y=88n(22\ell-9)(2m+1),X=176n^2+61$, we obtain the following modular elliptic curve

$$E:Y^2=X^3+X^2-7721X+24079.$$

Here $\dfrac{m(m+1)}{2}=n^2=\dfrac{\ell(11\ell-9)}{2}\in P_3\cap P_4\cap P_{13}$ corresponds to an integer points $(176n^2+61,88n(22\ell-9)(2m+1))$ on $E(\mathbb{Z})$. This elliptic curve E's Cremona label is 73920cc2 and the Mordell Weil group $E(\mathbb{Q})$ is isomorphic to $\mathbb{Z}^2\times(\mathbb{Z}/2\mathbb{Z})^2$. The generators of $E(\mathbb{Q})$ are given by $\mathbb{Z}=\langle P_1=(25,252)\rangle$, $\mathbb{Z}=\langle P_2=(34,135)\rangle$. The torsion groups are generated by $\mathbb{Z}/2\mathbb{Z}=\langle P_3=(39,0)\rangle$, $\mathbb{Z}/2\mathbb{Z}=\langle P_4=(61,0)\rangle$. Finally all the integral points on E are given by

$$\begin{aligned}E(\mathbb{Z})=\{&(-101,0),(-89,\pm480),(-86,\pm525),(-71,\pm660),(-38.\pm683),\\&(-5,\pm528),(7,\pm432),(25.\pm252),(34,\pm135),(39,0),(61,0),(67,\pm168),\\&(79,\pm360),(109,\pm840),(237.\pm3432),(259,\pm3960),(399,\pm7800),\\&(655.\pm16632),(1411,\pm52920),(1789.\pm75600),(6397,\pm511632),\\&(112399.\pm3768300)\}.\end{aligned}$$

$X=176n^2+61$ yields that $X=237=176\times1^2+61$ and $X=6397=176\times6^2+61$, i.e., $n=1$ or $n=6$. Hence we have shown the following theorem.

Theorem 3.2 *There exist only two tridecagonal square triangular number* 1 *and* 36.

3.2 Further problems

In this paper, we have given a rough estimate on the values of $|P_a \cap P_b \cap P_c|$ and shown that there exist (a, b, c) with $|P_a \cap P_b \cap P_c| = 1$ and 2. Here we shall propose several next problems concerning the values of $|P_a \cap P_b \cap P_c|$.

Problem 3.3 *Let (a, b, c) be the integers (≥ 3) which satisfy all of $(a-2)(b-2), (b-2)(c-2), (c-2)(a-2) \neq \square$. Are there any triple (a, b, c) with $|P_a \cap P_b \cap P_c| = k$ for some positive integer $k \geq 3$? More precisely, are there any triple (a, b, c) with $|P_a \cap P_b \cap P_c| = k$ for any positive integer $k \geq 3$?*

After the investigations of the above quantitative problems, one may expect more qualitative problem as follows.

Problem 3.4 *Let (a, b, c) be the integers (≥ 3) which satisfy all of $(a-2)(b-2), (b-2)(c-2), (c-2)(a-2) \neq \square$. Can we characterize triple (a, b, c) with $|P_a \cap P_b \cap P_c| = k$ for given positive integer k?*

On the other hand, we have calculated the structure of the elliptic curves $E_{(a,b,c)}$ for small a, b, c, as by product. Since the examples are very few, we could not find any characteristic property for these examples but it may be a natural new problem to investigate the set of these elliptic curves $E_{(a,b,c)}$.

Problem 3.5 *Let (a, b, c) be the integers (≥ 3) which satisfy all of $(a-2)(b-2), (b-2)(c-2), (c-2)(a-2) \neq \square$. Are there any characteristic properties on the structure of the corresponding elliptic curves $E_{(a,b,c)}$?*

3.3 Relations between polygonal numbers and generalized polygonal numbers

The number $P_a(m) = \dfrac{m((a-2)m-(a-4))}{2}$ with $m \in \mathbb{Z}$ is called the generalized $a-$gonal number. By abuse of notation, we shall denote the generalized $a-$gonal number by $GP_a(m)$ and the set of all the generalized $a-$gon numbers by GP_a. Then $P_a(m) \subset GP_a(m)$ for any m and $P_a \subset GP_a$ by definition. We note it is obvious that $0 \notin P_a \cap P_b \cap P_c$ and $0 \in GP_a \cap GP_b \cap GP_c$. Hence $|P_a \cap P_b \cap P_c| < |GP_a \cap GP_b \cap GP_c|$ in general. We have calculated $|P_a \cap P_b \cap P_c|$ and also $|GP_a \cap GP_b \cap GP_c|$ for small values a, b, c and verified $|GP_a \cap GP_b \cap GP_c| - |P_a \cap P_b \cap P_c| = 1$ for many cases and obtained the following exceptional example.

Example 3.6 *In case of $(a, b, c) = (4, 5, 9)$, we have*

$$GP_4(m) = GP_5(n) = GP_7(\ell) \Longleftrightarrow (m, n, \ell) = (10, -8, -5), (0, 0, 0), (1, 1, 1).$$

Problem 3.7 *Let (a, b, c) be the integers (≥ 3). Are there any triple (a, b, c) with $|GP_a \cap GP_b \cap GP_c| - |P_a \cap P_b \cap P_c| > 2$?*

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