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Fully Localised Three-Dimensional Gravity-Capillary Solitary Waves on Water of Infinite Depth

B. Buffoni, M. D. Groves and E. Wahlén

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Abstract. *Fully localised three-dimensional solitary waves* are steady water waves which are evanescent in every horizontal direction. Existence theories for fully localised three-dimensional solitary waves on water of finite depth have recently been published, and in this paper we establish their existence on deep water. The governing equations are reduced to a perturbation of the two-dimensional nonlinear Schrödinger equation, which admits a family of localised solutions. Two of these solutions are symmetric in both horizontal directions and an application of a suitable variant of the implicit-function theorem shows that they persist under perturbations.

1. Introduction

The classical hydrodynamic problem for waves on the surface $\{y = \eta(x,z,t)\}$ of a three-dimensional body of deep water subject to gravity and capillarity is usually expressed in terms of an Eulerian velocity potential φ satisfying Laplace's equation

$$\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0, \qquad -\infty < y < \eta, \tag{1.1}$$

the kinematic boundary conditions

$$\varphi_y \to 0, \qquad y \to -\infty, \tag{1.2}$$

$$\eta_t = \varphi_y - \eta_x\varphi_x - \eta_z\varphi_z, \qquad y = \eta, \tag{1.3}$$

and the dynamical boundary condition

$$\varphi_t + \frac{1}{2}(\varphi_x^2 + \varphi_y^2 + \varphi_z^2) + \eta - \left[\frac{\eta_x}{\sqrt{1+\eta_x^2+\eta_z^2}}\right]_x - \left[\frac{\eta_z}{\sqrt{1+\eta_x^2+\eta_z^2}}\right]_z = 0, \qquad y = \eta; \tag{1.4}$$

here (x, y, z) are dimensionless spatial coordinates and t is time. In this paper we discuss waves of the form $\eta(x,z,t) = \eta(x-ct,z)$, $\varphi(x,y,z,t) = \varphi(x-ct,y,z)$ (propagating with permanent shape and constant speed in the x-direction) with $\eta(x-ct,z) \to 0$ as $|(x-ct,z)| \to \infty$ (decaying in every horizontal direction) and refer to them as *fully localised solitary waves.*

Theorem 1.1. *Suppose that $c^2 = 2(1-\varepsilon^2)$. For each sufficiently small value of $\varepsilon > 0$ equations (1.1)–(1.4) possess two fully localised solitary-wave solutions for which $\eta \in H^3(\mathbb{R}^2)$ is symmetric in x and z, that is $\eta(-x,z) = \eta(x,z)$ and $\eta(x,-z) = \eta(x,z)$. Moreover*

$$\eta(x,z) = \pm\varepsilon\zeta_0(\varepsilon x, \varepsilon z)\cos x + o(\varepsilon)$$

uniformly over $(x,z) \in \mathbb{R}^2$, where ζ_0 is the unique symmetric, positive (real) solution of the two-dimensional nonlinear Schrödinger equation

$$-\tfrac{1}{2}\zeta_{xx} - \zeta_{zz} + \zeta - \tfrac{11}{16}|\zeta|^2\zeta = 0. \tag{1.5}$$

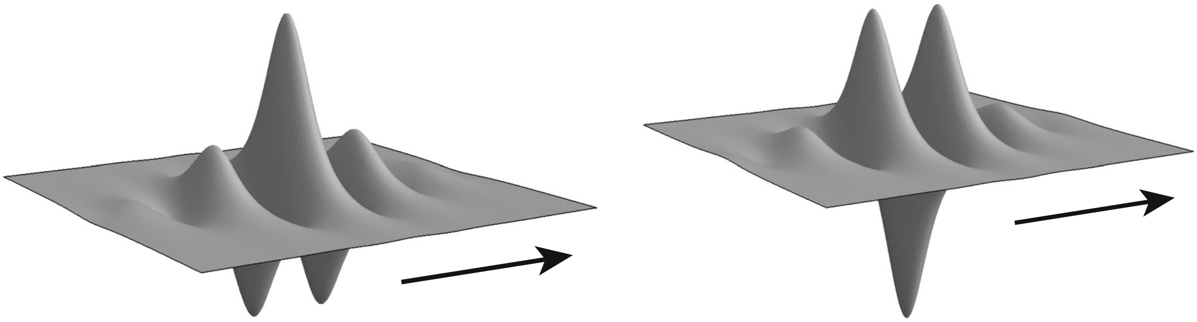

Fig. 1. Illustration of a symmetric fully localised solitary wave of elevation (left) and depression (right); the waves propagate in the direction shown by the arrow

Theorem 1.1 confirms the prediction made by formal weakly nonlinear analysis (see below) and numerical computations by Parau et al. [17] (see Fig. 1 for illustrations of typical free surfaces in their simulations). Qualitative properties of deep-water solitary waves (in two and three dimensions) have been discussed by Wheeler [23].

We begin by introducing the Zakharov–Craig–Sulem formulation of Eqs. (1.1)–(1.4) with independent variables η and $\Phi = \varphi|_{y=\eta}$, namely

$$\eta_t - G(\eta)\Phi = 0,$$

$$\Phi_t + \eta + \frac{1}{2}\Phi_x^2 + \frac{1}{2}\Phi_z^2 - \frac{(G(\eta)\Phi + \eta_x\Phi_x + \eta_z\Phi_z)^2}{2(1+\eta_x^2+\eta_z^2)} - \left[\frac{\eta_x}{\sqrt{1+\eta_x^2+\eta_z^2}}\right]_x - \left[\frac{\eta_z}{\sqrt{1+\eta_x^2+\eta_z^2}}\right]_z = 0,$$

where $G(\eta)\Phi = \varphi_y - \eta_x\varphi_x - \eta_z\varphi_z\big|_{y=\eta}$ and φ is the (unique) solution of the boundary-value problem

$$\begin{aligned} &\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0, && y < \eta, \\ &\varphi_y \to 0, && y \to -\infty, \\ &\varphi = \Phi, && y = \eta \end{aligned}$$

(Zakharov [24], Craig and Sulem [6]). Steady waves are nontrivial solutions of these equations of the form $\eta(x,z,t) = \eta(x-ct,z)$, $\Phi(x,z,t) = \Phi(x-ct,z)$; they satisfy

$$-c\eta_x - G(\eta)\Phi = 0, \tag{1.6}$$

$$-c\Phi_x + \eta + \frac{1}{2}\Phi_x^2 + \frac{1}{2}\Phi_z^2 - \frac{(G(\eta)\Phi + \eta_x\Phi_x + \eta_z\Phi_z)^2}{2(1+\eta_x^2+\eta_z^2)} - \left[\frac{\eta_x}{\sqrt{1+\eta_x^2+\eta_z^2}}\right]_x - \left[\frac{\eta_z}{\sqrt{1+\eta_x^2+\eta_z^2}}\right]_z = 0. \tag{1.7}$$

Equations (1.6), (1.7) can in fact be reduced to one equation for the variable η. Equation (1.6) implies that $\Phi = -cG(\eta)^{-1}\eta_x$, whereby (1.7) yields

$$\mathcal{K}'(\eta) - c^2\mathcal{L}'(\eta) = 0, \tag{1.8}$$

in which

$$\mathcal{K}'(\eta) = \eta - \left[\frac{\eta_x}{\sqrt{1+\eta_x^2+\eta_z^2}}\right]_x - \left[\frac{\eta_z}{\sqrt{1+\eta_x^2+\eta_z^2}}\right]_z, \tag{1.9}$$

$$\mathcal{L}'(\eta) = -\frac{1}{2}(K(\eta)\eta)^2 - \frac{1}{2}(L(\eta)\eta)^2 + \frac{(\eta_x - \eta_x K(\eta)\eta - \eta_z L(\eta)\eta)^2}{2(1+\eta_x^2+\eta_z^2)} + K(\eta)\eta \tag{1.10}$$

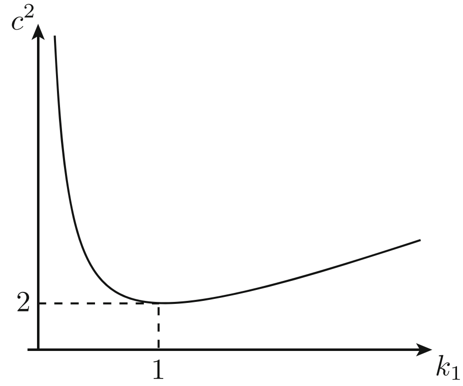

FIG. 2. Dispersion relation for a sinusoidal wave train with wave number $k_1 > 0$ and speed $c > 0$

and

$$K(\eta)\xi = -(G(\eta)^{-1}\xi_x)_x, \qquad L(\eta)\xi = -(G(\eta)^{-1}\xi_x)_z.$$

Notice that

$$K(\eta)\xi = -(\varphi|_{y=\eta})_x, \qquad L(\eta)\xi = -(\varphi|_{y=\eta})_z, \tag{1.11}$$

where φ solves the boundary-value problem

$$\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0, \qquad y < \eta, \tag{1.12}$$

$$\varphi_y \to 0, \qquad y \to -\infty, \tag{1.13}$$

$$\varphi_y - \eta_x\varphi_x - \eta_z\varphi_z = \xi_x, \qquad y = \eta \tag{1.14}$$

(and is therefore unique up to an additive constant); the operators K and L are studied in Sect. 2 below. Observe that (1.8) is invariant under the reflections $\eta(x,z) \mapsto \eta(-x,z)$ and $\eta(x,z) \mapsto \eta(x,-z)$; a solution which is invariant under these transformations is termed *symmetric*. Although this fact is not used in the present paper, let us also note that (1.8) is in fact the Euler–Lagrange equation for the functional

$$\mathcal{J}(\eta) := \mathcal{K}(\eta) - c^2\mathcal{L}(\eta),$$

in which

$$\mathcal{K}(\eta) = \int_{\mathbb{R}^2} \left(\frac{1}{2}\eta^2 + \sqrt{1+\eta_x^2+\eta_z^2} - 1\right) \mathrm{d}x\,\mathrm{d}z, \qquad \mathcal{L}(\eta) = \frac{1}{2}\int_{\mathbb{R}^2} \eta\, K(\eta)\eta \,\mathrm{d}x\,\mathrm{d}z;$$

the functions $\mathcal{K}'$ and $\mathcal{L}'$ are the $L^2(\mathbb{R}^2)$-gradients of respectively $\mathcal{K}$ and $\mathcal{L}$ (see Buffoni *et al.* [3,4]).

Let us now review the formal weakly nonlinear analysis used to derive the nonlinear Schrödinger equation for steady waves (see Ablowitz and Segur [1, §2.2]), beginning with sinusoidal wave trains. The linearised version of (1.8) admits a solution of the form

$$\eta(x,z) = A\cos k_1 x$$

whenever $c > 0$ and $k_1 > 0$ satisfy the linear dispersion relation

$$c^2 = k_1 + \frac{1}{k_1}$$

(see Fig. 2); note that the mapping $k_1 \mapsto c(k_1)$, $k_1 > 0$ has a unique global minimum $c_{\min} = \sqrt{2}$ at $k_1 = 1$. According to Dias and Kharif [7, §3] equality of the linear group and phase speeds (which occurs when $c'(k_1) = 0$) is associated with bifurcation of small-amplitude water waves. We therefore seek small-amplitude solitary waves with speed near $\sqrt{2}$ bifurcating from a linear sinusoidal wave train with unit wavenumber. Substituting $c^2 = 2(1-\varepsilon^2)$ and the formal expansion

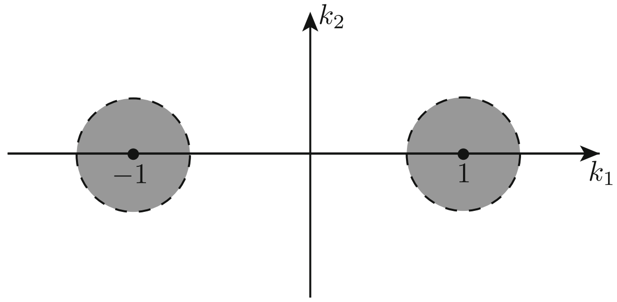

FIG. 3. The support of $\hat{\eta}_1$ is contained in the set $B = B_\delta(1,0) \cup B_\delta(-1,0)$

$$\eta(x,z) = \tfrac{1}{2}\varepsilon\big(A_1(X,Z)\mathrm{e}^{\mathrm{i}x} + \overline{A_1(X,Z)}\mathrm{e}^{-\mathrm{i}x}\big) \\ +\varepsilon^2 A_0(X,Z) + \tfrac{1}{2}\varepsilon^2\big(A_2(X,Z)\mathrm{e}^{2\mathrm{i}x} + \overline{A_2(X,Z)}\mathrm{e}^{-2\mathrm{i}x}\big) + \cdots,$$

where $X = \varepsilon x$, $Z = \varepsilon z$, into equation (1.8), one finds that A_1 satisfies the stationary nonlinear Schrödinger equation (1.5). This equation has a unique symmetric, positive (real) solution $\zeta_0 \in \mathcal{S}(\mathbb{R}^2)$ which is characterised as the ground state of the functional $\tilde{\mathcal{J}} : H^1(\mathbb{R}^2) \to \mathbb{R}$ with

$$\tilde{\mathcal{J}}(\zeta) = \int_{\mathbb{R}^2} \left(\frac{1}{4}|\zeta_x|^2 + \frac{1}{2}|\zeta_z|^2 + \frac{1}{2}|\zeta|^2 - \frac{11}{64}|\zeta|^4\right) \mathrm{d}x\,\mathrm{d}z$$

(see Sulem and Sulem [21, §4.2] and the references therein).

The above Ansatz indicates that the Fourier transform $\hat{\eta}$ of the surface-profile function η for a fully localised solitary wave is localised near the points $(\pm 1, 0)$. We therefore write $\eta = \eta_1 + \eta_2$, where $\mathrm{supp}(\hat{\eta}_1)$ and $\mathrm{supp}(\hat{\eta}_2)$ are contained in respectively $B = B_\delta(1,0) \cup B_\delta(-1,0)$ (with $\delta \in (0, \frac{1}{5})$) and its complement (see Fig. 3). Here the Fourier transform $\hat{\eta} = \mathcal{F}[\eta]$ of η is defined by

$$\hat{\eta}(k) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \eta(x,z)\mathrm{e}^{-\mathrm{i}(k_1 x + k_3 z)}\,\mathrm{d}x\,\mathrm{d}z, \qquad k = (k_1, k_3),$$

and we denote the Fourier multiplier with symbol m by $m(D)$ with $D = (-\mathrm{i}\partial_x, -\mathrm{i}\partial_z)$, so that $m(D)\eta = \mathcal{F}^{-1}[m\hat{\eta}]$; in particular $\eta_1 = \chi(D)\eta$, $\eta_2 = (1-\chi(D))\eta$, where χ is the indicator function of the set B. Writing $c^2 = 2(1-\varepsilon^2)$ and equation (1.8) as

$$\chi(D)\big(\mathcal{K}'(\eta_1+\eta_2) - 2(1-\varepsilon^2)\mathcal{L}'(\eta_1+\eta_2)\big) = 0, \\ (1-\chi(D))\big(\mathcal{K}'(\eta_1+\eta_2) - 2(1-\varepsilon^2)\mathcal{L}'(\eta_1+\eta_2)\big) = 0,$$

we find that the second equation is solvable for η_2 as a function of η_1 for sufficiently small values of ε; the first therefore reduces to

$$\chi(D)\big(\mathcal{K}'(\eta_1+\eta_2(\eta_1)) - 2(1-\varepsilon^2)\mathcal{L}'(\eta_1+\eta_2(\eta_1))\big) = 0$$

upon inserting $\eta_2 = \eta_2(\eta_1)$. Finally, the scaling

$$\eta_1(x,z) = \tfrac{1}{2}\varepsilon\zeta(X,Z)\mathrm{e}^{\mathrm{i}x} + \tfrac{1}{2}\varepsilon\overline{\zeta(X,Z)}\mathrm{e}^{-\mathrm{i}x} \tag{1.15}$$

transforms the reduced equation into a perturbation of the equation

$$\varepsilon^{-2}g(e+\varepsilon D)\zeta + 2f(e+\varepsilon D)\zeta - \tfrac{11}{8}|\zeta|^2\zeta = 0, \tag{1.16}$$

where $e = (1,0)$ and

$$g(s) = 1 + |s|^2 - 2f(s), \quad f(s) = \frac{s_1^2}{|s|}$$

(see Sects. 3 and 4; the reduced equation is stated precisely in equation (4.2)).

Equation (1.16) is termed a *full-dispersion* version of the stationary nonlinear Schrödinger equation (1.5) since it retains the linear part of the original equation (1.8); noting that

$$\varepsilon^{-2}g(e+\varepsilon k)+2f(e+\varepsilon k)=2+k_1^2+2k_3^2+O(\varepsilon),$$

we recover the fully reduced model equation in the formal limit $\varepsilon=0$ (see Obrecht and Saut [16] for a discussion of related full-dispersion model equations for three-dimensional water waves). In Sect. 5 we demonstrate that equation (1.16) for ζ has two symmetric solutions $\zeta_\varepsilon^\pm$ which satisfy $\zeta_\varepsilon^\pm \to \pm\zeta_0$ in $H^1(\mathbb{R}^2)$ as $\varepsilon\to 0$. The key step is a nondegeneracy result for the solution ζ_0 of (1.5) (see Weinstein [22], Kwong [13] and Chang et al. [5]) in a symmetric setting which allows one to apply a suitable variant of the implicit-function theorem. For this purpose we exploit the fact that the reduction procedure preserves the invariance of equation (1.8) under $\eta(x,z)\mapsto\eta(-x,z)$ and $\eta(x,z)\mapsto\eta(x,-z)$, so that equation (1.16) is invariant under the reflections $\zeta(x,z)\mapsto\overline{\zeta(-x,z)}$ and $\zeta(x,z)\mapsto\zeta(x,-z)$.

The scaling (1.15) implies that our waves have small amplitude but finite energy. When splitting our basic function space $\mathcal{X}=H^3(\mathbb{R}^2)$ into two parts $\mathcal{X}_1=\chi(D)\mathcal{X}$, $\mathcal{X}_2=(1-\chi(D))\mathcal{X}$ for η_1 and η_2, we respect this scaling by equipping $\mathcal{X}_1$ with the scaled norm $|\!|\!|\cdot|\!|\!|$ defined by

$$|\!|\!|\eta_1|\!|\!| := \left(\int_{\mathbb{R}^2}\left(1+\varepsilon^{-2}\left(\left(|k_1|-1\right)^2+k_3^2\right)\right)|\hat{\eta}_1(k)|^2\,\mathrm{d}k_1\,\mathrm{d}k_3\right)^{1/2}=\|\zeta\|_{H^1(\mathbb{R}^2)} \tag{1.17}$$

and taking ζ in a ball $B_R(0)\subseteq H^1(\mathbb{R}^2)$ which is large enough to contain ζ_0; solving the equation for η_2 yields the estimate

$$\|\eta_2(\eta_1)\|_{H^3(\mathbb{R}^2)}\lesssim\varepsilon^\theta|\!|\!|\eta_1|\!|\!|^2,$$

where θ is a fixed number in the interval $(0,1)$. Equation (1.17) shows that our waves have finite $H^3(\mathbb{R}^2)$-norm, while the estimates

$$\|\eta_1\|_\infty\lesssim\varepsilon^\theta|\!|\!|\eta_1|\!|\!|,\qquad \|\eta_2\|_\infty\lesssim\|\eta_2\|_{H^3(\mathbb{R}^2)},$$

show that they have small amplitude.

Comparison with previous results

The corresponding problem for water of finite depth is obtained from (1.1)–(1.4) by replacing the condition (1.2) by $\phi_y|=0$ at $y=-1$ and multiplying the terms in square brackets on the left-hand side of (1.4) by a dimensionless parameter β (called the *Bond number*) measuring the strength of surface-tension effects. That problem was examined for strong capillarity ($\beta>1/3$) by Groves and Sun [9] and Buffoni *et al.* [3] and, more relevantly for us, for weak capillarity ($\beta<1/3$) by Buffoni et al. [4]. In the latter case one obtains a dispersion relation with a global minimum $\sqrt{\Lambda}\in(0,1)$ obtained at a unique wave number $\omega\in(0,\infty)$ depending upon β. The above weakly nonlocal analysis leads to the Davey–Stewartson equation

$$-a_1\zeta_{xx}-a_2\zeta_{zz}+a_3\zeta-2C_1\mathcal{F}^{-1}\left[\frac{k_1^2}{(1-\Lambda)k_1^2+k_2^2}\mathcal{F}[|\zeta|^2]\right]\zeta-2C_2|\zeta|^2\zeta=0,$$

where $a_i>0$, $C_j>0$ are further constants depending on β. The Davey–Stewartson equation is the Euler–Lagrange equation for the functional

$$\tilde{\mathcal{J}}(\zeta)=\int_{\mathbb{R}^2}\left(a_1|\zeta_x|^2+a_2|\zeta_z|^2+a_3|\zeta|^2\right)\mathrm{d}x\,\mathrm{d}z$$
$$-C_1\int_{\mathbb{R}^2}\frac{k_1^2}{(1-\Lambda)k_1^2+k_2^2}|\mathcal{F}[|\zeta|^2]|^2\,\mathrm{d}k_1\,\mathrm{d}k_2-C_2\int_{\mathbb{R}^2}|\zeta|^4\,\mathrm{d}x\,\mathrm{d}z.$$

Buffoni et al. rigorously reduce the variational principle for fully localised three-dimensional solitary waves to a locally equivalent reduced variational principle whose variational functional is a perturbation of $\tilde{\mathcal{J}}$ and complete their existence theory by finding a nontrivial critical point of their reduced functional.

The variational method is also applicable to the infinite-depth case, but in the present paper we take a different approach, directly reducing the hydrodynamic equations themselves. The methods do however have common features, particularly in solving for $\eta_2=\eta_2(\eta_1)$ and approximating Fourier multipliers by differential operators, and for conciseness we simply state those results here and refer to reference [4] for their proofs. This direct technique was not available to Buffoni *et al.* since no corresponding nondegeneracy result for the Davey–Stewartson equation is currently available. However, by combining the method in the present paper with the computations in reference [4], one can obtain the existence of two families of fully localised solitary-wave solutions when $\beta > 0$ is sufficiently small (which corresponds to assuming that the depth is sufficiently large in physical variables). Indeed, noting that $\omega \to \infty$ as $\beta \to 0$ and setting $\zeta(x,z) = \omega^{-1}\tilde{\zeta}(\omega x, \omega z)$, one finds after multiplying by ω that the Davey–Stewartson equation limits to the nonlinear Schrödinger equation (1.5) for $\tilde{\zeta}$ as $\omega \to \infty$. The Davey–Stewartson equation therefore also has two families of nondegenerate fully localised solitary-wave solutions for large ω. Repeating the proof of Theorem 1.1, we thus arrive at the following result.

Theorem 1.2. *Fix $\beta > 0$ small enough. For each sufficiently small $\varepsilon > 0$ the gravity-capillary water wave problem with finite depth and $c^2 = (1-\varepsilon^2)\Lambda$ possesses two fully localised solitary-wave solutions which are symmetric in x and z.*

The approach in the present paper does not use any variational arguments, relying instead upon the nondegeneracy of two particular symmetric solutions to the limiting equation (the nonlinear Schrödinger equation) to enable an application of the implicit-function theorem. It is 'self-improving' in that it would apply equally well to any other symmetric solution once its nondegeneracy is established; see Strauss [20], Berestycki and Lions [2], Jones and Küpper [12] and McLeod, Troy and Weissler [15] for existence theories for radial solutions to equations of nonlinear Schrödinger type (and Iaia and Warchall [11] for a discussion of nonradial solutions). By contrast, variational approaches typically require more sophisticated tools to handle multiple solutions. Our method also applies to other problems which can be reduced to a limiting equation with a nondegenerate symmetric solitary-wave solution. The limiting equation for water with finite depth and strong surface tension is the KP-I equation, which has an (explicit) symmetric solitary-wave solution. This solution has recently been shown to be nondegenerate by Liu and Wei [14], and the present method would therefore presumably lead to a simpler existence proof than the variational theory in references [9] and [3]. A similar method has also been used by Stefanov andWright [19] to establish the existence of solitary-wave solutions to a full-dispersion Korteweg–de Vries equation known as the Whitham equation.

Our method also applies to two-dimensional gravity-capillary solitary water waves, the details of which have recently been given for water of finite depth by Groves [8]. There weakly nonlocal analysis leads to the Korteweg–de Vries equation (for strong capillarity) and one-dimensional nonlinear Schrödinger equation (for weak capillarity), and both of these equations have families of explicit solitary-wave solutions. Groves follows the method developed in the present paper, showing how it can be considerably simplified in one spatial dimension, to derive perturbed full-dispersion versions of the Korteweg–de Vries and nonlinear Schrödinger equations. His analysis of the reduced equations is also simpler, since establishing the nondegeneracy of their symmetric solitary-wave solutions amounts to verifying the nondegeneracy of explicit solutions to well-known second-order ordinary differential equations. Note also that while reference [8] is an alternative approach to previously available results, the present paper gives an existence theory for a new type of wave.

Finally, let us remark that Lyapunov–Schmidt reduction in Fourier space, as carried out in Sect. 3 below, has been used by other authors in various forms (e.g. see Pelinovsky and Schneider [18] for a related reduction in a quite different setting).

2. Analyticity

In this section we sketch a proof that the operators K, L given by (1.11) and hence $\mathcal{K}'$ and $\mathcal{L}'$ given by (1.9), (1.10) are analytic at the origin in suitable function spaces (see Corollaries 2.2 and 2.3 below). Full details of the method are given (for different, but related mappings) by Buffoni et al. [4, §2.1].

The boundary-value problem (1.12)–(1.14) can be treated by mapping the fluid domain $\Sigma_\eta = \{(x,y,z)\colon x,z \in \mathbb{R}, -\infty < y < \eta(x,z)\}$ to the lower half-space $\Sigma = \mathbb{R} \times (-\infty,0) \times \mathbb{R}$ using the transformation

$$y' = y - \eta(x,z), \qquad u(x,y',z) = \varphi(x,y,z).$$

The equations are transformed into

$$u_{xx} + u_{yy} + u_{zz} = \partial_x F_1(\eta,u) + \partial_y F_2(\eta,u) + \partial_z F_3(\eta,u), \qquad y<0, \tag{2.1}$$

$$u_y \to 0, \qquad y \to -\infty, \tag{2.2}$$

$$u_y = F_2(\eta,u) + \xi_x, \qquad y=0, \tag{2.3}$$

where we have dropped the primes and defined

$$F_1(\eta,u) = \eta_x u_y, \qquad F_2(\eta,u) = \eta_x u_x + \eta_z u_z - (\eta_x^2+\eta_z^2)u_y, \qquad F_3(\eta,u) = \eta_z u_y,$$

so that

$$K(\eta)\xi = -u_x|_{y=0}, \qquad L(\eta)\xi = -u_z|_{y=0}.$$

We study this boundary-value problem in the function space

$$\mathcal{Z} = \{\eta \in \mathcal{S}'(\mathbb{R}^2) : \|\eta\|_{\mathcal{Z}} := \|\hat{\eta}_1\|_{L^1(\mathbb{R}^2)} + \|\eta_2\|_3 < \infty\}$$

for η and $H^3_\star(\Sigma)$ for u, in which $H^{n+1}_\star(\Sigma)$, $n \in \mathbb{N}$, is the completion of

$$\mathcal{S}(\Sigma) = \{u \in C^\infty(\overline{\Sigma}) : |(x,z)|^m |\partial_x^{\alpha_1}\partial_y^{\alpha_2}\partial_z^{\alpha_3} u| \text{ is bounded for all } m,\alpha_1,\alpha_2,\alpha_3 \in \mathbb{N}_0\}$$

with respect to the norm

$$\|u\|^2_{n+1,\star} := \|u_x\|_n^2 + \|u_y\|_n^2 + \|u_z\|_n^2$$

and $\|\cdot\|_s$ denotes the usual norm for the standard Sobolev space $H^s(\mathbb{R}^2)$ or $H^s(\Sigma)$.

Lemma 2.1. *For each $\xi \in H^{5/2}(\mathbb{R}^2)$ and sufficiently small $\eta \in \mathcal{Z}$ equations (2.1)–(2.3) have a unique solution $u \in H^3_\star(\Sigma)$. Furthermore, the mapping $\eta \mapsto (\xi \mapsto u)$ defines a function $\mathcal{Z} \to \mathcal{L}(H^{5/2}(\mathbb{R}^2), H^3_\star(\Sigma))$ which is analytic at the origin.*

Proof. Suppose that F_1, F_2, $F_3 \in H^2(\Sigma)$, $\xi \in H^{5/2}(\mathbb{R}^2)$ and note that the boundary-value problem

$$u_{xx} + u_{yy} + u_{zz} = \partial_x F_1 + \partial_y F_2 + \partial_z F_3, \qquad y<0,$$

$$u_y \to 0, \qquad y \to -\infty,$$

$$u_y = F_2(\eta,u) + \xi_x, \qquad y=0,$$

has a unique solution $u = S(F_1,F_2,F_3,\xi)$ in $H^3_\star(\Sigma)$ whose gradient is obtained from the explicit formula

$$\begin{aligned} S(F_1,F_2,F_3,\xi) = \mathcal{F}^{-1}\Bigg[&\int_{-\infty}^0 \left(-\frac{\mathrm{i}k_1}{2|k|}\hat{F}_1 - \frac{\mathrm{i}k_3}{2|k|}\hat{F}_3 + \frac{1}{2}\operatorname{sgn}(y-\tilde{y})\hat{F}_2\right) \mathrm{e}^{-|k||y-\tilde{y}|}\,\mathrm{d}\tilde{y} \\ &+ \int_{-\infty}^0 \left(-\frac{\mathrm{i}k_1}{2|k|}\hat{F}_1 - \frac{\mathrm{i}k_3}{2|k|}\hat{F}_3 + \frac{1}{2}\hat{F}_2\right)\mathrm{e}^{|k|(y+\tilde{y})}\,\mathrm{d}\tilde{y} + \frac{\mathrm{i}k_1}{|k|}\hat{\xi}\mathrm{e}^{|k|y}\Bigg] \end{aligned}$$

(with a slight abuse of notation), so that

$$\|S(F_1,F_2,F_3,\xi)\|_{3,\star} \lesssim \|F_1\|_2 + \|F_2\|_2 + \|F_3\|_2 + \|\xi\|_{5/2}.$$

Define

$$T : H^3_\star(\Sigma) \times \mathcal{Z} \times H^{5/2}(\mathbb{R}^2) \to H^3_\star(\Sigma)$$

by

$$T(u,\eta,\xi) = u - S(F_1(\eta,u), F_2(\eta,u), F_3(\eta,u), \xi),$$

so that zeros of $T(\cdot,\eta,\xi)$ correspond to solutions of (2.1)–(2.3). Using the estimates

$$\begin{aligned}\|\eta_x^j w\|_2 &\leq \|\eta_{1x}^j w\|_2 + \|\eta_{2x}^j w\|_2\\ &\lesssim \|\eta_1\|_{3,\infty}^j \|w\|_2 + \|\eta_2\|_3^j \|w\|_2\\ &\lesssim (\|\hat{\eta}_1\|_{L^1(\mathbb{R}^2)} + \|\eta_2\|_3)^j \|w\|_2\\ &= \|\eta\|_{\mathcal{Z}}^j \|w\|_2, \qquad j=1,2,\end{aligned}$$

and similarly

$$\|\eta_z^j w\|_2 \lesssim \|\eta\|_{\mathcal{Z}}^j \|w\|_2, \qquad j=1,2$$

(where we have used the fact that $\mathrm{supp}(\hat{\eta}_1)$ is compact), we find that the mappings $H^3_\star(\Sigma)\times\mathcal{Z}\to H^2(\Sigma)$ given by $(\eta,u)\mapsto F_j(\eta,u)$, $j=1,2,3$, and hence the mapping T, are analytic at the origin. Since $T(0,0,0)=0$ and $\mathrm{d}_1T[0,0,0]=I$ is an isomorphism, the analytic implicit-function theorem yields open neighbourhoods V_1 and V_2 of the origin in respectively $\mathcal{Z}$ and $H^{5/2}(\mathbb{R}^2)$ and an analytic function $v: V_1\times V_2\to H^3_\star(\Sigma)$ such that

$$T(v(\eta,\xi),\eta,\xi)=0.$$

Since v is linear in ξ we can take V_2 to be the entire space $H^{5/2}(\mathbb{R}^2)$. □

Corollary 2.2. *The mappings* $K(\cdot), L(\cdot)\colon \mathcal{Z}\to\mathcal{L}(H^{5/2}(\mathbb{R}^2),H^{3/2}(\mathbb{R}^2))$ *are analytic at the origin.*

In view of Corollary 2.2 we choose M sufficiently small and study the equation

$$\mathcal{K}'(\eta)-2(1-\varepsilon^2)\mathcal{L}'(\eta)=0 \tag{2.4}$$

in the set

$$U=\{\eta\in H^3(\mathbb{R}^2): \|\eta\|_{\mathcal{Z}}<M\},$$

noting that $H^3(\mathbb{R}^2)$ is continuously embedded in $\mathcal{Z}$ and that U is an open neighbourhood of the origin in $H^3(\mathbb{R}^2)$; we proceed accordingly by decomposing $\mathcal{X}=H^3(\mathbb{R}^2)$ into the direct sum of $\mathcal{X}_1=\chi(D)H^3(\mathbb{R}^2)$ and $\mathcal{X}_2=(1-\chi(D))H^3(\mathbb{R}^2)$.

Corollary 2.3. *Equations* (1.9), (1.10) *define functions* $U\to H^1(\mathbb{R}^2)$ *which are analytic at the origin and satisfy* $\mathcal{K}'(0)=\mathcal{L}'(0)=0$.

Proof. The result for $\mathcal{K}'$ follows from (1.9), Corollary 2.2 and the fact that $H^{3/2}(\mathbb{R}^2)$ is an algebra. The result for $\mathcal{L}'$ follows from (1.10) and the observation that $\eta\mapsto\eta_x(1+\eta_x^2+\eta_z^2)^{-1/2}$ and $\eta\mapsto\eta_z(1+\eta_x^2+\eta_z^2)^{-1/2}$ define functions $U\to H^2(\mathbb{R}^2)$ which are analytic at the origin since $H^2(\mathbb{R}^2)$ is an algebra. □

In keeping with Lemma 2.1 and Corollary 2.2 we write

$$u(\eta,\xi)=\sum_{j=0}^\infty u^j(\eta,\xi),$$

where u^j is homogeneous of degree j in η and linear in ξ, and

$$K(\eta)=\sum_{j=0}^\infty K_j(\eta),\quad L(\eta)=\sum_{j=0}^\infty L_j(\eta),\quad \mathcal{K}'(\eta):=\sum_{j=1}^\infty \mathcal{K}_j'(\eta),\quad \mathcal{L}'(\eta):=\sum_{j=1}^\infty \mathcal{L}_j'(\eta).$$

where $K_j(\eta)$, $L_j(\eta)$ and $\mathcal{K}_j'(\eta)$, $\mathcal{L}_j'(\eta)$ are homogeneous of degree j in η. A straightforward calculation shows that

$$u^0(\xi)=\mathcal{F}^{-1}\left[\frac{\mathrm{i}k_1}{|k|}\mathrm{e}^{|k|y}\hat{\xi}\right]$$

and hence that K_0 and L_0 are Fourier multipliers, namely

$$K_0\xi = \mathcal{F}^{-1}\left[\frac{k_1^2}{|k|}\hat{\xi}\right], \qquad L_0\xi = \mathcal{F}^{-1}\left[\frac{k_1k_3}{|k|}\hat{\xi}\right]$$

(we have omitted the argument η on the left-hand sides of these equations).

The following lemma gives expressions for the first few terms in the Taylor expansions of $\mathcal{K}'(\eta)$ and $\mathcal{L}'(\eta)$ at the origin; it is proved by expanding (1.9), (1.10) and examining the boundary-value problems for $u^1(\eta,\eta)$ and $u^2(\eta,\eta)$ to derive the formulae

$$K_1(\eta)\eta = -(\eta\eta_x)_x - K_0(\eta K_0\eta) - L_0(\eta L_0\eta), \tag{2.5}$$

$$L_1(\eta)\eta = -(\eta\eta_x)_z - L_0(\eta K_0\eta) - M_0(\eta L_0\eta) \tag{2.6}$$

with a similar formula for $K_2(\eta_1)\eta_1$ (see Buffoni *et al.* [3, pp. 1032–1033] for details in a similar setting; the restriction to η_1 is necessary to allow the use of higher-order derivatives in these expressions).

Lemma 2.4. (i) *The identities*

$$\begin{aligned}
\mathcal{K}_1'(\eta) &= \eta - \eta_{xx} - \eta_{zz},\\
\mathcal{K}_2'(\eta) &= 0,\\
\mathcal{K}_3'(\eta) &= \frac{1}{2}((\eta_x^2+\eta_z^2)\eta_x)_x + \frac{1}{2}((\eta_x^2+\eta_z^2)\eta_z)_z
\end{aligned} \tag{2.7}$$

hold for each $\eta \in H^3(\mathbb{R}^2)$.

(ii) *The identities*

$$\begin{aligned}
\mathcal{L}_1'(\eta) &= K_0\eta,\\
\mathcal{L}_2'(\eta) &= \tfrac{1}{2}\left(\eta_x^2 - (K_0\eta)^2 - (L_0\eta)^2 - 2(\eta_x\eta)_x - 2K_0(\eta K_0\eta) - 2L_0(\eta L_0\eta)\right)
\end{aligned}$$

hold for each $\eta \in H^3(\mathbb{R}^2)$.

(iii) *The identity*

$$\begin{aligned}
\mathcal{L}_3'(\eta_1) = {} & K_0\eta_1\, K_0(\eta_1K_0\eta_1) + K_0\eta_1\, L_0(\eta_1L_0\eta_1) + L_0\eta_1\, L_0(\eta_1K_0\eta_1) + L_0\eta_1\, M_0(\eta_1L_0\eta_1)\\
& + K_0(\eta_1K_0(\eta_1K_0\eta_1)) + K_0(\eta_1L_0(\eta_1L_0\eta_1)) + L_0(\eta_1L_0(\eta_1K_0\eta_1))\\
& + L_0(\eta_1M_0(\eta_1L_0\eta_1)) + \eta_1(K_0\eta_1)\eta_{1xx} + \tfrac{1}{2}K_0(\eta_1^2\eta_{1xx}) + \tfrac{1}{2}(\eta_1^2K_0\eta_1)_{xx}\\
& + \eta_1(L_0\eta_1)\eta_{1xz} + \tfrac{1}{2}L_0(\eta_1^2\eta_{1xz}) + \tfrac{1}{2}(\eta_1^2L_0\eta_1)_{xz},
\end{aligned}$$

where

$$M_0\xi = \mathcal{F}^{-1}\left[\frac{k_3^2}{|k|}\hat{\xi}\right],$$

holds for each $\eta_1 \in \mathcal{X}_1$ and more generally for any function η_1 whose Fourier transform has compact support.

Finally, we present estimates for the cubic and higher-order parts of $\mathcal{K}'(\eta)$ and $\mathcal{L}'(\eta)$. The results for $\mathcal{L}'(\eta)$ are established by substituting

$$K(\eta) = \sum_{j=0}^{2} K_j(\eta) + K_{\mathrm{c}}(\eta), \qquad L(\eta) = \sum_{j=0}^{2} L_j(\eta) + L_{\mathrm{c}}(\eta)$$

into (1.10) and estimating the resulting formulae for $\mathcal{L}_{\mathrm{c}}(\eta)$ and $\mathcal{L}_{\mathrm{r}}(\eta)$ using the rules

$$\|K_j(\eta)\eta\|_{3/2} \lesssim \|\eta\|_{\mathcal{Z}}^j\|\eta\|_{5/2}, \qquad \|K_{\mathrm{c}}(\eta)\eta\|_{3/2} \lesssim \|\eta\|_{\mathcal{Z}}^3\|\eta\|_{5/2}$$

(with corresponding estimates for $L_j(\eta)(\eta)$, $L_{\mathrm{c}}(\eta)(\eta)$ and derivatives). Since this method yields only

$$\|(K_1(\eta)\eta)^2\|_1,\ \|(L_1(\eta)\eta)^2\|_1 \lesssim \|\eta\|_{\mathcal{Z}}^2\|\eta\|_3^2$$

we do not include the fourth-order terms $-\frac{1}{2}(K_1(\eta)\eta)^2$, $-\frac{1}{2}(L_1(\eta)\eta)^2$ in $\mathcal{L}_{\mathrm{r}}'(\eta)$ and treat them separately later (see in particular Proposition 4.8).

Lemma 2.5. (i) *The quantities*

$$\mathcal{K}_{\mathrm{c}}'(\eta) := \sum_{j=3}^{\infty} \mathcal{K}_j'(\eta), \qquad \mathcal{L}_{\mathrm{c}}'(\eta) := \sum_{j=3}^{\infty} \mathcal{L}_j'(\eta)$$

satisfy the estimates

$$\|\mathcal{K}_{\mathrm{c}}'(\eta)\|_1 \lesssim \|\eta\|_{\mathcal{Z}}^2 \|\eta\|_3, \qquad \|\mathrm{d}\mathcal{K}_{\mathrm{c}}'[\eta](v)\|_1 \lesssim \|\eta\|_{\mathcal{Z}}^2 \|v\|_3 + \|\eta\|_{\mathcal{Z}} \|\eta\|_3 \|v\|_{\mathcal{Z}},$$
$$\|\mathcal{L}_{\mathrm{c}}'(\eta)\|_1 \lesssim \|\eta\|_{\mathcal{Z}}^2 \|\eta\|_3, \qquad \|\mathrm{d}\mathcal{L}_{\mathrm{c}}'[\eta](v)\|_1 \lesssim \|\eta\|_{\mathcal{Z}}^2 \|v\|_3 + \|\eta\|_{\mathcal{Z}} \|\eta\|_3 \|v\|_{\mathcal{Z}}$$

for each $\eta \in U$ and $v \in H^2(\mathbb{R})$.

(ii) *The quantities*

$$\mathcal{K}_{\mathrm{r}}'(\eta) := \sum_{j=4}^{\infty} \mathcal{K}_j'(\eta), \qquad \mathcal{L}_{\mathrm{r}}'(\eta) := \sum_{j=4}^{\infty} \mathcal{L}_j'(\eta) + \tfrac{1}{2}(K_1(\eta)\eta)^2 + \tfrac{1}{2}(L_1(\eta)\eta)^2$$

satisfy the estimates

$$\|\mathcal{K}_{\mathrm{r}}'(\eta)\|_1 \lesssim \|\eta\|_{\mathcal{Z}}^4 \|\eta\|_3, \qquad \|\mathrm{d}\mathcal{K}_{\mathrm{r}}'[\eta](v)\|_1 \lesssim \|\eta\|_{\mathcal{Z}}^4 \|v\|_3 + \|\eta\|_{\mathcal{Z}}^3 \|\eta\|_3 \|v\|_{\mathcal{Z}},$$
$$\|\mathcal{L}_{\mathrm{r}}'(\eta)\|_1 \lesssim \|\eta\|_{\mathcal{Z}}^3 \|\eta\|_3, \qquad \|\mathrm{d}\mathcal{L}_{\mathrm{r}}'[\eta](v)\|_1 \lesssim \|\eta\|_{\mathcal{Z}}^3 \|v\|_3 + \|\eta\|_{\mathcal{Z}}^2 \|\eta\|_3 \|v\|_{\mathcal{Z}}$$

for each $\eta \in U$ and $v \in H^2(\mathbb{R})$.

3. Reduction

Observe that $\eta \in U$ satisfies (2.4) if and only if

$$\eta_1 - \eta_{1xx} - \eta_{1zz} - 2K_0\eta_1 + 2\varepsilon^2 K_0 \eta_1 + \chi(D)\mathcal{N}(\eta_1+\eta_2) = 0, \tag{3.1}$$

$$\eta_2 - \eta_{2xx} - \eta_{2zz} - 2K_0\eta_2 + 2\varepsilon^2 K_0 \eta_2 + (1-\chi(D))\mathcal{N}(\eta_1+\eta_2) = 0, \tag{3.2}$$

in which

$$\mathcal{N}(\eta) = \mathcal{K}_{\mathrm{c}}'(\eta) - 2(1-\varepsilon^2)\big(\mathcal{L}_2'(\eta) + \mathcal{L}_{\mathrm{c}}'(\eta)\big).$$

The nonlinearity in (3.1) is at leading order cubic in η_1 because $\chi(D)\mathcal{L}_2'(\eta_1)$ vanishes; we therefore write it as

$$\eta_1 - \eta_{1xx} - \eta_{1zz} - 2K_0\eta_1 + 2\varepsilon^2 K_0\eta_1 + \chi(D)\left(\mathcal{N}(\eta_1+\eta_2) + 2(1-\varepsilon^2)\mathcal{L}_2'(\eta_1)\right) = 0 \tag{3.3}$$

and make the corresponding adjustment to (3.2), that is 'replacing' its nonlinearity with

$$(1-\chi(D))\left(\mathcal{N}(\eta_1+\eta_2) + 2(1-\varepsilon^2)\mathcal{L}_2'(\eta_1)\right),$$

by writing

$$\eta_2 = F(\eta_1) + \eta_3, \qquad F(\eta_1) := 2(1-\varepsilon^2)\mathcal{F}^{-1}\left[\frac{1-\chi(k)}{g(k)}\mathcal{F}[\mathcal{L}_2'(\eta_1)]\right]$$

(with the requirement that $\eta_1 + F(\eta_1) + \eta_3 \in U$). Equation (3.2) may thus be cast in the form

$$\eta_3 = -\mathcal{F}^{-1}\left[\frac{1-\chi(k)}{g(k)}\mathcal{F}\Big[2(1-\varepsilon^2)\mathcal{L}_2'(\eta_1) + \mathcal{N}(\eta_1 + F(\eta_1) + \eta_3) + 2\varepsilon^2 K_0(F(\eta_1)+\eta_3)\Big]\right], \tag{3.4}$$

where

$$g(k) = 1 + |k|^2 - 2\frac{k_1^2}{|k|} \geq 0$$

with equality if and only if $k = \pm(1,0)$. The following result follows from the fact that $g(k) \gtrsim 1 + |k|^2$ for $|k| - 1 \geq \delta$.

Proposition 3.1. *The mapping*

$$f \mapsto \mathcal{F}^{-1}\left[\frac{1-\chi(k)}{g(k)}\hat{f}\right]$$

defines a bounded linear operator $H^1(\mathbb{R}^2) \to H^3(\mathbb{R}^2)$.

We proceed by solving (3.4) for η_3 as a function of η_1 using a straightforward extension of the standard Banach fixed-point theorem.

Theorem 3.2. *Let* $\mathcal{X}_1$, $\mathcal{X}_2$ *be Banach spaces,* X_1, X_2 *be closed, convex sets in, respectively,* $\mathcal{X}_1$, $\mathcal{X}_2$ *containing the origin and* $\mathcal{G}\colon X_1 \times X_2 \to \mathcal{X}_2$ *be a smooth mapping. Suppose there exists a continuous mapping* $r\colon X_1 \to [0,\infty)$ *such that*

$$\|\mathcal{G}(x_1,0)\| \leq \tfrac{1}{2}r, \quad \|\mathrm{d}_2\mathcal{G}[x_1,x_2]\| \leq \tfrac{1}{3}$$

for each $x_2 \in \overline{B}_r(0) \subseteq X_2$ *and each* $x_1 \in X_1$.

Under these hypotheses there exists for each $x_1 \in X_1$ *a unique solution* $x_2 = x_2(x_1)$ *of the fixed-point equation* $x_2 = \mathcal{G}(x_1,x_2)$ *satisfying* $x_2(x_1) \in \overline{B}_r(0)$. *Moreover* $x_2(x_1)$ *is a smooth function of* $x_1 \in X_1$ *with*

$$\|\mathrm{d}x_2[x_1]\| \leq 2\|\mathrm{d}_1\mathcal{G}[x_1,x_2(x_1)]\|.$$

We apply Theorem 3.2 to equation (3.4) with $\mathcal{X}_1 = \chi(D)H^3(\mathbb{R}^2)$, $\mathcal{X}_2 = (1-\chi(D))H^3(\mathbb{R}^2)$, equipping $\mathcal{X}_1$ with the scaled norm

$$|\!|\!|\eta|\!|\!| := \left(\int_{\mathbb{R}^2}\left(1+\varepsilon^{-2}((|k_1|-1)^2+k_3^2)\right)|\hat{\eta}(k)|^2\,\mathrm{d}k_1\,\mathrm{d}k_3\right)^{1/2}$$

and $\mathcal{X}_2$ with the usual norm for $H^3(\mathbb{R}^2)$, taking

$$X_1 = \{\eta_1 \in \mathcal{X}_1\colon |\!|\!|\eta_1|\!|\!| \leq R_1\}, \qquad X_3 = \{\eta_3 \in \mathcal{X}_2\colon \|\eta_3\|_3 \leq R_3\}$$

and defining $\mathcal{G}$ as the right-hand side of (3.4). (We have written X_3 in place of X_2 for notational consistency.) The calculation

$$\begin{aligned}\int_{\mathbb{R}^2}|\hat{\eta}_1(k)|\,\mathrm{d}k_1\,\mathrm{d}k_3 &= \int_{\mathbb{R}^2}\frac{(1+\varepsilon^{-2}((|k_1|-1)^2+k_3^2))^{1/2}}{(1+\varepsilon^{-2}((|k_1|-1)^2+k_3^2))^{1/2}}|\hat{\eta}_1(k)|\,\mathrm{d}k_1\,\mathrm{d}k_3\\ &\leq 2|\!|\!|\eta|\!|\!|\left(\int_{B_\delta(1,0)}\frac{1}{1+\varepsilon^{-2}((k_1-1)^2+k_3^2)}\,\mathrm{d}k_1\,\mathrm{d}k_3\right)^{1/2}\\ &= 2\sqrt{\pi}\varepsilon(\log(1+\delta^2\varepsilon^{-2}))^{1/2}|\!|\!|\eta|\!|\!|\end{aligned}$$

shows that

$$\|\hat{\eta}_1\|_{L^1(\mathbb{R}^2)} \lesssim \varepsilon^\theta|\!|\!|\eta_1|\!|\!|, \qquad \eta_1 \in \mathcal{X}_1, \tag{3.5}$$

for each fixed $\theta \in (0,1)$. In most of our theory the value of the constant θ is irrelevant; however it will later be given the concrete value $\theta = 5/6$. One can therefore ensure that $\|\hat{\eta}_1\|_{L^1(\mathbb{R}^2)} < M/2$ for all $\eta_1 \in X_1$ for any (large) value of R_1, while the value of R_3 is limited by the condition that $\|F(\eta_1)+\eta_3\|_3 < M/2$ for all $\eta_1 \in X_1$ and $\eta_3 \in X_3$, so that $\eta_1 + F(\eta_1) + \eta_3 \in U$ (Corollary 3.4 below shows that $\|F(\eta_1)\|_3 = O(\varepsilon^\theta)$ uniformly over $\eta_1 \in X_1$).

We proceed by estimating each term in the formula for $\mathcal{G}$, using the inequalities

$$\|\eta\|_\infty \lesssim \|\eta\|_{\mathcal{Z}}, \quad \|\eta\|_{\mathcal{Z}} \lesssim \varepsilon^\theta|\!|\!|\eta_1|\!|\!| + \|\eta_3\|_3, \quad \|\eta\|_3 \lesssim |\!|\!|\eta_1|\!|\!| + \|\eta_3\|_3$$

and making extensive use of the fact that $\mathrm{supp}(\hat{\eta}_1)$ is contained in the fixed bounded set B, so that for example

$$\|\eta_1\|_n \lesssim \|\eta_1\|_0, \qquad \|\eta_1\|_{n,\infty} \lesssim \varepsilon^\theta|\!|\!|\eta_1|\!|\!|$$

for each $n \in \mathbb{N}_0$.

To estimate $F(\eta_1)$ we write $\mathcal{L}_2'(\eta) = m(\eta,\eta)$ with

$$m(u,v) = \tfrac{1}{2}\left(u_x v_x - (K_0 u)(K_0 v) - (L_0 u)(L_0 v)\right) + \tfrac{1}{2}\left(-(u_x v + u v_x)_x - K_0(uK_0 v + vK_0 u) - L_0(uL_0 v + vL_0 u)\right) \qquad (3.6)$$

(see Lemma 2.4(ii)), and observe that

$$\mathrm{d}\mathcal{L}_2'[\eta](v) = 2m(\eta, v).$$

The proof of the following proposition is given (for a related mapping) by Buffoni et al. [4, Proposition 5].

Proposition 3.3. *The estimate*

$$\|m(u,v)\|_1 \lesssim \|u\|_{\mathcal{Z}} \|v\|_3$$

holds for each u*,* $v \in H^3(\mathbb{R}^2)$*.*

Corollary 3.4. *The estimates*

$$\|F(\eta_1)\|_3 \lesssim \varepsilon^\theta |||\eta_1|||^2, \quad \|\mathrm{d}F[\eta_1]\|_{\mathcal{L}(\mathcal{X}_1,\mathcal{X}_2)} \lesssim \varepsilon^\theta |||\eta_1|||$$

hold for each $\eta_1 \in X_1$*.*

Remark 3.5. Noting that

$$K_0 F(\eta_1) = 2(1-\varepsilon^2)\mathcal{F}^{-1}\left[\frac{1-\chi(k)}{g(k)} \frac{k_1^2}{|k|} \mathcal{F}[\mathcal{L}_2'(\eta_1)]\right]$$

and that $\mathrm{supp}(\mathcal{F}[\mathcal{L}_2'(\eta_1)])$ is compact, one finds that $K_0 F(\eta_1)$ satisfies the same estimates as $F(\eta_1)$.

Lemma 3.6. *The quantity*

$$\mathcal{N}_1(\eta_1,\eta_3) = \mathcal{L}_2'(\eta_1 + F(\eta_1) + \eta_3) - \mathcal{L}_2'(\eta_1)$$

satisfies the estimates

(i) $\|\mathcal{N}_1(\eta_1,\eta_3)\|_1 \lesssim \varepsilon^{2\theta}|||\eta_1|||^3 + \varepsilon^\theta |||\eta_1|||^2 \|\eta_3\|_3 + \varepsilon^\theta |||\eta_1||| \|\eta_3\|_3 + \|\eta_3\|_3^2$,
(ii) $\|\mathrm{d}_1\mathcal{N}_1[\eta_1,\eta_3]\|_{\mathcal{L}(\mathcal{X}_1,H^1(\mathbb{R}^2))} \lesssim \varepsilon^{2\theta}|||\eta_1|||^2 + \varepsilon^\theta |||\eta_1||| \|\eta_3\|_3 + \varepsilon^\theta \|\eta_3\|_3$,
(iii) $\|\mathrm{d}_2\mathcal{N}_1[\eta_1,\eta_3]\|_{\mathcal{L}(\mathcal{X}_2,H^1(\mathbb{R}^2))} \lesssim \varepsilon^\theta |||\eta_1||| + \|\eta_3\|_3$

for each $\eta_1 \in X_1$ *and* $\eta_3 \in X_3$*.*

Proof. We estimate

$$\mathcal{N}_1(\eta_1,\eta_3) = 2m(\eta_1, F(\eta_1)+\eta_3) + m(F(\eta_1)+\eta_3, F(\eta_1)+\eta_3)$$

by combining Proposition 3.3 with Corollary 3.4 using the chain rule. □

Lemma 3.7. *The quantity*

$$\mathcal{N}_2(\eta_1,\eta_3) = \mathcal{K}_{\mathrm{c}}'(\eta_1 + F(\eta_1) + \eta_3) - 2(1-\varepsilon^2)\mathcal{L}_{\mathrm{c}}'(\eta_1 + F(\eta_1) + \eta_3),$$

satisfies the estimates

(i) $\|\mathcal{N}_2(\eta_1,\eta_3)\|_1 \lesssim (\varepsilon^\theta |||\eta_1||| + \|\eta_3\|_3)^2 (|||\eta_1||| + \|\eta_3\|_3)$,
(ii) $\|\mathrm{d}_1\mathcal{N}_2[\eta_1,\eta_3]\|_{\mathcal{L}(\mathcal{X}_1,H^1(\mathbb{R}^2))} \lesssim (\varepsilon^\theta |||\eta_1||| + \|\eta_3\|_3)^2$,
(iii) $\|\mathrm{d}_2\mathcal{N}_2[\eta_1,\eta_3]\|_{\mathcal{L}(\mathcal{X}_2,H^1(\mathbb{R}^2))} \lesssim (\varepsilon^\theta |||\eta_1||| + \|\eta_3\|_3)(|||\eta_1||| + \|\eta_3\|_3)$

for each $\eta_1 \in X_1$ *and* $\eta_3 \in X_3$*.*

Proof. We compute the derivatives of $\mathcal{N}_2$ using the chain rule and estimate these expressions using the linearity of the derivative, Lemma 2.5(i) and Corollary 3.4. □

Altogether the above results establish the following estimates for $\mathcal{G}$ (see Proposition 3.1, Remark 3.5 and Lemmata 3.6, 3.7).

Lemma 3.8. *The function* $\mathcal{G} : X_1 \times X_3 \to \mathcal{X}_2$ *satisfies the estimates*

(i) $\|\mathcal{G}(\eta_1,\eta_3)\|_3 \lesssim (\varepsilon^\theta |||\eta_1||| + \|\eta_3\|_3)^2(1 + |||\eta_1||| + \|\eta_3\|_3) + \varepsilon^2\|\eta_3\|_3$,

(ii) $\|\mathrm{d}_1\mathcal{G}[\eta_1,\eta_3]\|_{\mathcal{L}(\mathcal{X}_1,\mathcal{X}_2)} \lesssim (\varepsilon^\theta |||\eta_1||| + \|\eta_3\|_3)(\varepsilon^\theta + \varepsilon^\theta |||\eta_1||| + \|\eta_3\|_3)$,

(iii) $\|\mathrm{d}_2\mathcal{G}[\eta_1,\eta_3]\|_{\mathcal{L}(\mathcal{X}_2)} \lesssim (\varepsilon^\theta |||\eta_1||| + \|\eta_3\|_3)(1 + |||\eta_1||| + \|\eta_3\|_3) + \varepsilon^2$

for each $\eta_1 \in X_1$ *and* $\eta_3 \in X_3$.

Theorem 3.9. *Equation* (3.4) *has a unique solution* $\eta_3 \in X_3$ *which depends smoothly upon* $\eta_1 \in X_1$ *and satisfies the estimates*

$$\|\eta_3(\eta_1)\|_3 \lesssim \varepsilon^{2\theta}|||\eta_1|||^2, \qquad \|\mathrm{d}\eta_3[\eta_1]\|_{\mathcal{L}(\mathcal{X}_1,\mathcal{X}_2)} \lesssim \varepsilon^{2\theta}|||\eta_1|||.$$

Proof. Choosing R_3 and ε sufficiently small, $\sigma > 0$ sufficiently large and setting $r(\eta_1) = \sigma\varepsilon^{2\theta}|||\eta_1|||^2$, one finds that

$$\|\mathcal{G}(\eta_1, 0)\|_3 \le \tfrac{1}{2}r(\eta_1), \qquad \|\mathrm{d}_2\mathcal{G}[\eta_1,\eta_3]\|_{\mathcal{L}(\mathcal{X}_2)} \lesssim \varepsilon^\theta$$

for $\eta_1 \in X_1$ and $\eta_3 \in \overline{B}_{r(\eta_1)}(0) \subset X_3$ (Lemma 3.8(i), (iii)). According to Theorem 3.2 equation (3.4) has a unique solution η_3 in $\overline{B}_{r(\eta_1)}(0) \subset X_3$ depending smoothly upon $\eta_1 \in X_1$ whose derivative is estimated using Lemma 3.8(ii). □

The reduced equation

$$\begin{aligned}\eta_1 - \eta_{1xx} - \eta_{1zz} - 2K_0\eta_1 \\ + 2\varepsilon^2K_0\eta_1 + \chi(D)\big(\mathcal{N}(\eta_1 + F(\eta_1) + \eta_3(\eta_2)) + 2(1-\varepsilon^2)\mathcal{L}_2'(\eta_1)\big) = 0 \end{aligned} \tag{3.7}$$

for $\eta_1 \in X_1$ is obtained by substituting $\eta_2 = \eta_1 + F(\eta_1) + \eta_3(\eta_1)$ into (3.3). Observe that (3.7) is invariant under the reflections $\eta_1(x,z) \mapsto \eta_1(-x,z)$ and $\eta_1(x,z) \mapsto \eta_1(x,-z)$; a familiar argument shows that they are inherited from the corresponding invariance of (3.3), (3.4) under $\eta_1(x,z) \mapsto \eta_1(-x,z)$, $\eta_3(x,z) \mapsto \eta_3(-x,z)$ and $\eta_1(x,z) \mapsto \eta_1(x,-z)$, $\eta_3(x,z) \mapsto \eta_3(x,-z)$ when applying Theorem 3.2.

4. Derivation of the reduced equation

Writing

$$\eta_1 = \eta_1^+ + \eta_1^-,$$

where $\eta_1^+ = \chi^+(D)\eta_1$, $\eta_1^- = \chi^-(D)\eta_1$ and χ^+, χ^- are the indicator functions of respectively $B_\delta(1,0)$ and $B_\delta(-1,0)$ in (3.7), we find that η_1^+ satisfies the equation

$$\begin{aligned}\eta_1^+ - \eta_{1xx}^+ - \eta_{1zz}^+ - 2K_0\eta_1^+ \\ + 2\varepsilon^2K_0\eta_1^+ + \chi^+(D)\big(\mathcal{N}(\eta_1 + F(\eta_1) + \eta_3(\eta_2)) + 2(1-\varepsilon^2)\mathcal{L}_2'(\eta_1)\big) = 0\end{aligned} \tag{4.1}$$

(and η_1^- satisfies its complex conjugate). The next step is to calculate the cubic terms in equation (4.1) explicitly; we employ the following notation for the higher-order terms.

Definition 4.1.

(i) The symbol $\underline{O}(\varepsilon^\gamma|||\eta_1|||^r)$ denotes a smooth function $N : X_1 \to H^1(\mathbb{R}^2)$ with

$$\|N(\eta_1)\|_1 \lesssim \varepsilon^\gamma|||\eta_1|||^r, \qquad \|\mathrm{d}N[\eta_1]\|_{\mathcal{L}(\mathcal{X}_1, H^1(\mathbb{R}^2))} \lesssim \varepsilon^\gamma|||\eta_1|||^{r-1}$$

for each $\eta_1 \in X_1$ (where $\gamma \ge 0$, $r \ge 1$). Furthermore

$$\underline{O}_0(\varepsilon^\gamma|||\eta_1|||^r) := \chi_0(D)\underline{O}(\varepsilon^\gamma|||\eta_1|||^r), \qquad \underline{O}_+(\varepsilon^\gamma|||\eta_1|||^r) := \chi^+(D)\underline{O}(\varepsilon^\gamma|||\eta_1|||^r),$$

where χ_0, χ^+ are the indicator functions of respectively $B_\delta(0,0)$ and $B_\delta(1,0)$.

(ii) The symbol $\underline{O}_n^{\varepsilon}(\|u\|_1^r)$ denotes $\chi_0(\varepsilon D)N(u)$, where N is a smooth function $B_R(0) \subseteq \chi_0(\varepsilon D)H^1(\mathbb{R}^2) \to H^n(\mathbb{R}^2)$ or $B_R(0) \subseteq H^1(\mathbb{R}^2) \to H^n(\mathbb{R}^2)$ with

$$\|N(u)\|_n \lesssim \|u\|_1^r, \qquad \|\mathrm{d}N[u]\|_{\mathcal{L}(H^1(\mathbb{R}^2),H^n(\mathbb{R}^2))} \lesssim \|u\|_1^{r-1}$$

for each $u \in B_R(0)$ (with $r \geq 1$, $n \geq 0$).

A Fourier multiplier $m(D)$ may be approximated by $m(\omega,0)$ when acting upon a function whose Fourier transform is supported near the point $(\omega,0)$. The proof of the following lemma is given by Buffoni *et al.* [4, Lemma 11] (in a slightly different context).

Lemma 4.2. *The estimates*

(i) $\partial_x\eta_1^{\pm} = \pm\mathrm{i}\eta_1^{\pm} + \underline{O}(\varepsilon|\!|\!|\eta_1|\!|\!|)$,
(ii) $\partial_x^2\eta_1^{\pm} = -\eta_1^{\pm} + \underline{O}(\varepsilon|\!|\!|\eta_1|\!|\!|)$,
(iii) $\partial_z\eta_1^{\pm} = \underline{O}(\varepsilon|\!|\!|\eta_1|\!|\!|)$,
(iv) $K_0\eta_1^{\pm} = \eta_1^{\pm} + \underline{O}(\varepsilon|\!|\!|\eta_1|\!|\!|)$,
(v) $L_0\eta_1^{\pm} = \underline{O}(\varepsilon|\!|\!|\eta_1|\!|\!|)$,
(vi) $K_0((\eta_1^{\pm})^2) = 2(\eta_1^{\pm})^2 + \underline{O}(\varepsilon^{1+\theta}|\!|\!|\eta_1|\!|\!|^2)$,
(vii) $L_0((\eta_1^{\pm})^2) = \underline{O}(\varepsilon^{1+\theta}|\!|\!|\eta_1|\!|\!|^2)$,
(viii) $K_0(\eta_1^{+}\eta_1^{-}) = \underline{O}(\varepsilon^{1+\theta}|\!|\!|\eta_1|\!|\!|^2)$,
(ix) $L_0(\eta_1^{+}\eta_1^{-}) = \underline{O}(\varepsilon^{1+\theta}|\!|\!|\eta_1|\!|\!|^2)$,
(x) $\mathcal{F}^{-1}[g(k)^{-1}\mathcal{F}[(\eta_1^{\pm})^2]] = (\eta_1^{\pm})^2 + \underline{O}(\varepsilon^{1+\theta}|\!|\!|\eta_1|\!|\!|^2)$,
(xi) $K_0(\eta_1^{-}(\eta_1^{+})^2) = \eta_1^{-}(\eta_1^{+})^2 + \underline{O}(\varepsilon^{1+2\theta}|\!|\!|\eta_1|\!|\!|^3)$

hold for each $\eta_1 \in X_1$.

We proceed by approximating each term in the nonlinearity on the right-hand side of (4.1) according to the rules given in Lemma 4.2.

Proposition 4.3. *The estimate*

$$F(\eta_1) = -2\left((\eta_1^{+})^2 + (\eta_1^{-})^2\right) + F_{\mathrm{r}}(\eta_1), \qquad F_{\mathrm{r}}(\eta_1) = \underline{O}(\varepsilon^{1+\theta}|\!|\!|\eta_1|\!|\!|^2)$$

holds for each $\eta_1 \in X_1$.

Proof. Using the expansions given in Lemma 4.2, we find that

$$\mathcal{L}_2'(\eta_1) = m(\eta_1,\eta_1) = -\left((\eta_1^{+})^2 + (\eta_1^{-})^2\right) + \underline{O}(\varepsilon^{1+\theta}|\!|\!|\eta_1|\!|\!|^2).$$

It follows that

$$\mathcal{F}^{-1}\left[\frac{1-\chi(k)}{g(k)}\mathcal{F}[\mathcal{L}_2'(\eta_1)]\right] = -\left((\eta_1^{+})^2 + (\eta_1^{-})^2\right) + \underline{O}(\varepsilon^{1+\theta}|\!|\!|\eta_1|\!|\!|^2)$$

because of Lemma 4.2(x) and the fact that

$$\mathcal{F}^{-1}\left[\frac{1-\chi(k)}{g(k)}\mathcal{F}[\underline{O}(\varepsilon^{1+\theta}|\!|\!|\eta_1|\!|\!|^2)]\right] = \underline{O}(\varepsilon^{1+\theta}|\!|\!|\eta_1|\!|\!|^2)$$

(because $(1-\chi(k))g(k)^{-1}$ is bounded). We conclude that

$$F(\eta_1) = 2(1-\varepsilon^2)\mathcal{F}^{-1}\left[\frac{1-\chi(k)}{g(k)}\mathcal{F}[\mathcal{L}_2'(\eta_1)]\right] = -2\left((\eta_1^{+})^2 + (\eta_1^{-})^2\right) + \underline{O}(\varepsilon^{1+\theta}|\!|\!|\eta_1|\!|\!|^2).$$ □

Remark 4.4. The remainder term $F_{\mathrm{r}}(\eta_1)$ in the formula for $F(\eta_1)$ given in Proposition 4.3 satisfies

$$\|F_{\mathrm{r}}(\eta_1)\|_n \lesssim \varepsilon^{1+\theta}|\!|\!|\eta_1|\!|\!|^2, \qquad \|\mathrm{d}F_{\mathrm{r}}[\eta_1]\|_{\mathcal{L}(\mathcal{X}_1,H^n(\mathbb{R}^2))} \lesssim \varepsilon^{1+\theta}|\!|\!|\eta_1|\!|\!|$$

for all $n \in \mathbb{N}_0$ since its Fourier transform is supported in the region $B+B$.

Proposition 4.5. *The estimate*

$$\chi^{+}(D)\mathcal{N}_1(\eta_1,\eta_3(\eta_1)) = 4\chi^{+}(D)\left(\eta_1^{-}(\eta_1^{+})^2\right) + \underline{O}_{+}(\varepsilon^{3\theta}|\!|\!|\eta_1|\!|\!|^3)$$

holds for each $\eta_1 \in X_1$.

Proof. Observe that

$$\begin{aligned}\chi^+(D)\mathcal{N}_1(\eta_1,\eta_3(\eta_1)) &= \chi^+(D)\big(2m(\eta_1,F(\eta_1)+\eta_3)+m(F(\eta_1)+\eta_3,F(\eta_1)+\eta_3)\big)\\ &= 2\chi^+(D)m(\eta_1,F(\eta_1))+\underline{O}(\varepsilon^{3\theta}|||\eta_1|||^3),\end{aligned}$$

in which we have used the calculations

$$m(\eta_1,\eta_3)=\underline{O}(\varepsilon^{3\theta}|||\eta_1|||^3), \qquad m(F(\eta_1),\eta_3)=\underline{O}(\varepsilon^{3\theta}|||\eta_1|||^4)$$

(see Proposition 3.3, Corollary 3.4 and Theorem 3.9) and

$$m(F(\eta_1),F(\eta_1))=\underline{O}(\varepsilon^{3\theta}|||\eta_1|||^4)$$

(because of (3.6) and Proposition 4.3). Observing that

$$m(\eta_1,F_{\rm r}(\eta_1))=\underline{O}(\varepsilon^{1+2\theta}|||\eta_1|||^3)$$

(see Proposition 3.3 and Remark 4.4), we find that

$$\chi^+(D)m(\eta_1,F(\eta_1))=-2\chi^+(D)m(\eta_1^-,(\eta_1^+)^2)+\underline{O}_+(\varepsilon^{3\theta}|||\eta_1|||^3),$$

and it follows from (3.6) and Lemma 4.2 that

$$m(\eta_1^-,(\eta_1^+)^2)=-\eta_1^-(\eta_1^+)^2+\underline{O}(\varepsilon^{3\theta}|||\eta_1|||^3).\square$$

□

Proposition 4.6. *The estimates*

$$\begin{aligned}\chi^+(D)\mathcal{K}_3'(\eta_1+F(\eta_1)+\eta_3(\eta_1)) &= -\tfrac{3}{2}\chi^+(D)\big(\eta_1^-(\eta_1^+)^2\big)+\underline{O}_+(\varepsilon^{3\theta}|||\eta_1|||^3),\\ \chi^+(D)\mathcal{L}_3'(\eta_1+F(\eta_1)+\eta_3(\eta_1)) &= -2\chi^+(D)\big(\eta_1^-(\eta_1^+)^2\big)+\underline{O}_+(\varepsilon^{3\theta}|||\eta_1|||^3)\end{aligned}$$

hold for each $\eta_1\in X_1$.

Proof. Using the estimates for $F(\eta_1)$ and $\eta_3(\eta_1)$ given in Corollary 3.4 and Theorem 3.9, we find that

$$\mathcal{K}_3'(\eta_1+F(\eta_1)+\eta_3(\eta_1))=\mathcal{K}_3'(\eta_1)+\underline{O}(\varepsilon^{3\theta}|||\eta_1|||^4)$$

and

$$\chi^+(D)\mathcal{K}_3'(\eta_1)=-\tfrac{3}{2}\chi^+(D)\big(\eta_1^-(\eta_1^+)^2\big)+\underline{O}_+(\varepsilon^{3\theta}|||\eta_1|||^3))$$

(because of equation (2.7)). It similarly follows from the formula

$$\mathcal{L}_3'(\eta)=-K_0\eta K_1(\eta)\eta-L_0\eta L_1(\eta)\eta-\eta_x^2K_0\eta-\eta_x\eta_zL_0\eta+K_2(\eta)\eta$$

and the fact that $K_2(\eta)=m_2(\eta,\eta)$, where m_2 is a bounded, symmetric bilinear mapping $\mathcal{Z}\times\mathcal{Z}\to\mathcal{L}(H^{5/2}(\mathbb{R}^2),H^{3/2}(\mathbb{R}^2))$, that

$$\mathcal{L}_3'(\eta_1+F(\eta_1)+\eta_3(\eta_1))=\mathcal{L}_3'(F(\eta_1)+\eta_1)+\underline{O}(\varepsilon^{3\theta}|||\eta_1|||^4);$$

using Lemma 2.4(iii) twice yields

$$\mathcal{L}_3'(F(\eta_1)+\eta_1)=\mathcal{L}_3'(\eta_1)+\underline{O}(\varepsilon^{3\theta}|||\eta_1|||^3)$$

and

$$\chi^+(D)\mathcal{L}_3'(\eta_1)=-2\chi^+(D)\big(\eta_1^-(\eta_1^+)^2\big)+\underline{O}_+(\varepsilon^{3\theta}|||\eta_1|||^3). \qquad \square$$

Proposition 4.7. *The estimates*

$$\begin{aligned}\mathcal{K}_{\rm r}'(\eta_1+F(\eta_1)+\eta_3(\eta_1)) &= \underline{O}(\varepsilon^{4\theta}|||\eta_1|||^5),\\ \mathcal{L}_{\rm r}'(\eta_1+F(\eta_1)+\eta_3(\eta_1)) &= \underline{O}(\varepsilon^{3\theta}|||\eta_1|||^4)\end{aligned}$$

hold for each $\eta_1\in X_1$.

Proof. This result follows from Proposition 2.5(ii), Corollary 3.4 and Theorem 3.9. □

Proposition 4.8. *The estimates*

$$-\tfrac{1}{2}\chi^+(D)\big(K_1(\eta_1+F(\eta_1)+\eta_3(\eta_1))(\eta_1+F(\eta_1)+\eta_3(\eta_1))\big)^2=\underline{O}_+(\varepsilon^{3\theta}|||\eta_1|||^4),$$
$$-\tfrac{1}{2}\chi^+(D)\big(L_1(\eta_1+F(\eta_1)+\eta_3(\eta_1))(\eta_1+F(\eta_1)+\eta_3(\eta_1))\big)^2=\underline{O}_+(\varepsilon^{3\theta}|||\eta_1|||^4)$$

hold for each $\eta_1\in X_1$.

Proof. It follows from Corollary 3.4 and Theorem 3.9 that

$$-\tfrac{1}{2}\big(K_1(\eta_1+F(\eta_1)+\eta_3(\eta_1))(\eta_1+F(\eta_1)+\eta_3(\eta_1))\big)^2=-\tfrac{1}{2}(K_1(\eta_1)\eta_1)^2+\underline{O}(\varepsilon^{3\theta}|||\eta_1|||^4),$$

and furthermore

$$-\tfrac{1}{2}\chi^+(D)(K_1(\eta_1)\eta_1)^2=-\tfrac{1}{2}\chi^+(D)\big((\eta_1\eta_{1x})_x+K_0(\eta_1K_0\eta_1)+L_0(\eta_1L_0\eta_1)\big)^2=0$$

because of equation (2.5). The second estimate is derived in the same fashion (with equation (2.6)). □

Corollary 4.9. *The estimate*

$$\chi^+(D)\mathcal{N}_2(\eta_1,\eta_3(\eta_1))=\tfrac{5}{2}\chi^+(D)\big(\eta_1^-(\eta_1^+)^2\big)+\underline{O}_+(\varepsilon^{3\theta}|||\eta_1|||^3)$$

holds for each $\eta_1\in X_1$.

The reduced equation for η_1 is therefore

$$\eta_1^+-\eta_{1xx}^+-\eta_{1zz}^+-2K_0\eta_1^++2\varepsilon^2K_0\eta_1^+-\tfrac{11}{2}\chi^+(D)\big(|\eta_1^+|^2\eta_1^+\big)+\underline{O}_+(\varepsilon^{3\theta}|||\eta_1|||^3)=0,$$

which can be further simplified to

$$\eta_1^+-\eta_{1xx}^+-\eta_{1zz}^+-2K_0\eta_1^++2\varepsilon^2\eta_1^+-\tfrac{11}{2}\chi^+(D)\big(|\eta_1^+|^2\eta_1^+\big)+\underline{O}_+(\varepsilon^{3\theta}|||\eta_1|||)=0$$

by an application of Lemma 4.2(iv). Finally, we introduce the nonlinear Schrödinger scaling

$$\eta_1^+(x,z)=\tfrac{1}{2}\varepsilon\zeta(\varepsilon x,\varepsilon z)\mathrm{e}^{\mathrm{i}x},$$

so that $\zeta\in B_R(0)\subseteq\chi_0(\varepsilon D)H^1(\mathbb{R}^2)$ solves the *perturbed full-dispersion nonlinear Schrödinger equation*

$$\varepsilon^{-2}g(e+\varepsilon D)\zeta+2\zeta-\tfrac{11}{8}\chi_0(\varepsilon D)(|\zeta|^2\zeta)+\varepsilon^{3\theta-2}\underline{O}_0^\varepsilon(\|\zeta\|_1)=0, \tag{4.2}$$

where $R=R_1/\sqrt{2}$ and $e=(1,0)$ (note that $|||\eta_1|||^2=\|\zeta\|_1^2$ and the change of variables from (x,z) to $\varepsilon(x,z)$ introduces a further factor of ε in the remainder term). The invariance of the reduced equation under $\eta_1(x,z)\mapsto\eta_1(\overline{-x,-z})$ and $\eta_1(x,z)\mapsto\eta_1(x,-z)$ is inherited by (4.2), which is invariant under the reflections $\zeta(x,z)\mapsto\overline{\zeta(-x,z)}$ and $\zeta(x,z)\mapsto\zeta(x,-z)$.

Remark 4.10. In the formal limit $\varepsilon=0$ equation (4.2) reduces to the nonlinear Schrödinger equation

$$-\tfrac{1}{2}\zeta_{xx}-\zeta_{zz}+\zeta-\tfrac{11}{16}|\zeta|^2\zeta=0. \tag{4.3}$$

5. Solution of the reduced equation

In this section we complete our existence theory by proving the following theorem.

Theorem 5.1. *For each sufficiently small value of* $\varepsilon>0$ *equation* (4.2) *has two small-amplitude solutions* $\zeta_\varepsilon^\pm$ *in* $\chi_0(\varepsilon D)H^1(\mathbb{R}^2)$ *which satisfy* $\zeta_\varepsilon^\pm(x,z)=\overline{\zeta_\varepsilon^\pm(-x,z)}$, $\zeta_\varepsilon^\pm(x,z)=\zeta_\varepsilon^\pm(x,-z)$ *and* $\|\zeta_\varepsilon^\pm-(\pm\zeta_0)\|_1\lesssim\varepsilon^{1/2}$, *where* $\zeta_0\in\mathcal{S}(\mathbb{R}^2)$ *is the unique symmetric, positive (real) solution of the nonlinear Schrödinger equation* (4.3).

The first step is a result which allows us to 'replace' the nonlocal operator in equation (4.2) by a differential operator.

Proposition 5.2. *The inequality*

$$\left|\frac{\varepsilon^2}{2\varepsilon^2+g(e+\varepsilon k)}-\frac{1}{2+k_1^2+2k_3^2}\right| \lesssim \frac{\varepsilon|k|^3}{(1+|k|^2)^2}$$

holds uniformly over $|k|<\delta/\varepsilon$.

Proof. Clearly

$$\left|\frac{\varepsilon^2}{2\varepsilon^2+g(e+\varepsilon k)}-\frac{1}{2+k_1^2+2k_3^2}\right| = \frac{|g(e+\varepsilon k)-\varepsilon^2(k_1^2+2k_3^2)|}{(2\varepsilon^2+g(e+\varepsilon k))(2+k_1^2+2k_3^2)},$$

while

$$g(e+s)-s_1^2-2s_2^2 \lesssim |s|^3, \qquad |s|\leq\delta,$$

and

$$g(e+s) \gtrsim |s|^2, \qquad s\in\mathbb{R}^2.$$

It follows that

$$\left|\frac{\varepsilon^2}{2\varepsilon^2+g(e+\varepsilon k)}-\frac{1}{2+k_1^2+2k_3^2}\right| \lesssim \frac{\varepsilon|k|^3}{(1+|k|^2)^2}, \qquad |k|<\delta/\varepsilon. \qquad \square$$

Using this proposition, one can write equation (4.2) as

$$\zeta + H_\varepsilon(\zeta)=0, \tag{5.1}$$

where

$$H_\varepsilon(\zeta) = -\tfrac{11}{16}\left(1-\tfrac{1}{2}\partial_x^2-\partial_z^2\right)^{-1}\chi_0(\varepsilon D)\big(|\zeta|^2\zeta\big)+\varepsilon^{1/2}\underline{O}_1^\varepsilon(\|\zeta\|_1)$$

and we have chosen the concrete value $\theta=5/6$, so that $\varepsilon^{3\theta-2}=\varepsilon^{1/2}$. It is convenient to study equation (5.1) in the fixed function space $H^1(\mathbb{R}^2)$ by replacing it by

$$\zeta + \tilde{H}_\varepsilon(\zeta)=0, \tag{5.2}$$

where $\tilde{H}_\varepsilon(\zeta)=H_\varepsilon(\chi_0(\varepsilon D)\zeta)$ (the solution sets of (5.1) and (5.2) evidently coincide). Equation (5.2) is solved using the following variant of the implicit-function theorem.

Theorem 5.3. *Suppose that $\mathcal{X}$ is a Banach space, X_0 and Λ_0 are open neighbourhoods of respectively x_0 in $\mathcal{X}$ and the origin in $\mathbb{R}^n$ and $\mathcal{H}: X_0\times\Lambda_0\to\mathcal{X}$ is a mapping which is differentiable with respect to its first argument. Suppose further that $\mathcal{H}(x_0,0)=0$, $\mathrm{d}_1\mathcal{H}[x_0,0]:\mathcal{X}\to\mathcal{X}$ is an isomorphism,*

$$\lim_{x\to x_0}\|\mathrm{d}_1\mathcal{H}[x,0]-\mathrm{d}_1\mathcal{H}[x_0,0]\|_{\mathcal{L}(\mathcal{X})}=0$$

and

$$\lim_{\lambda\to 0}\|\mathcal{H}(x,\lambda)-\mathcal{H}(x,0)\|_{\mathcal{X}}=0, \quad \lim_{\lambda\to 0}\|\mathrm{d}_1\mathcal{H}[x,\lambda]-\mathrm{d}_1\mathcal{H}[x,0]\|_{\mathcal{L}(\mathcal{X})}=0$$

uniformly over $x\in X_0$.

Under these hypotheses there exist open neighbourhoods X and Λ of respectively x_0 in $\mathcal{X}$ and Λ of the origin in $\mathbb{R}^n$ (with $X\subseteq X_0$, $\Lambda\subseteq\Lambda_0$) and a uniquely determined mapping $h:\Lambda\to X$ such that

(i) $\lim_{\lambda\to 0}h(\lambda)=h(0)=x_0$,
(ii) $\mathcal{H}(h(\lambda),\lambda)=0$ *for all* $\lambda\in\Lambda$,
(iii) $x=h(\lambda)$ *whenever* $(x,\lambda)\in X\times\Lambda$ *satisfies* $\mathcal{H}(x,\lambda)=0$.

Furthermore, the estimate $\|\mathcal{H}(x,\lambda)-\mathcal{H}(x,0)\|_{\mathcal{X}}\lesssim|\lambda|^\alpha$ (with $\alpha>0$) for all $\lambda\in\Lambda_0$ and $x\in X_0$ implies that $\|h(\lambda)-h(0)\|_{\mathcal{X}}\lesssim|\lambda|^\alpha$ for all $\lambda\in\Lambda$.

We proceed by employing Theorem 5.3 with

$$\mathcal{X} = H^1_{\mathrm{e}}(\mathbb{R}^2,\mathbb{C}) = \{\zeta \in H^1(\mathbb{R}^2,\mathbb{C}) : \zeta(x,z) = \overline{\zeta(-x,z)},\ \zeta(x,z) = \zeta(x,-z)\}$$

and $X = B_R(0)$, $\Lambda_0 = (-\varepsilon_0,\varepsilon_0)$, where R and ε_0 are chosen respectively sufficiently large (to accommodate the requirement that $\zeta_0 \in X$) and sufficiently small. Note that negative values of ε have been included so that we can define

$$\mathcal{H}(\zeta,\varepsilon) := \zeta + \tilde{H}_{|\varepsilon|}(\zeta)$$

for ε in the full neighbourhood Λ_0 of the origin in $\mathbb{R}$. We also henceforth explicitly indicate whether functions are real- or complex-valued.

Observe that

$$\begin{aligned}
&\mathcal{H}(\zeta,\varepsilon) - \mathcal{H}(\zeta,0)\\
&\quad = -\tfrac{11}{16}\left(1-\tfrac{1}{2}\partial_x^2-\partial_z^2\right)^{-1}\Big(\chi_0(|\varepsilon|D)\Big(|\chi_0(|\varepsilon|D)\zeta|^2\chi_0(|\varepsilon|D)\zeta\Big) - |\zeta|^2\zeta\Big) + |\varepsilon|^{1/2}\underline{O}_1^{|\varepsilon|}(\|\zeta\|_1)\\
&\quad = -\tfrac{11}{16}\left(1-\tfrac{1}{2}\partial_x^2-\partial_z^2\right)^{-1}\Big(\chi_0(|\varepsilon|D)\Big(|\chi_0(|\varepsilon|D)\zeta|^2\big(\chi_0(|\varepsilon|D)-I\big)\zeta + |\zeta|^2\big(\chi_0(|\varepsilon|D)-I\big)\zeta\\
&\qquad\qquad + \zeta\chi_0(|\varepsilon|D)\zeta\big(\chi_0(|\varepsilon|D)-I\big)\overline{\zeta}\Big)\\
&\qquad\qquad + \Big(\chi_0(|\varepsilon|D)-I\Big)|\zeta|^2\zeta\Big) + |\varepsilon|^{1/2}\underline{O}_1^{|\varepsilon|}(\|\zeta\|_1).
\end{aligned}$$

Noting that

$$\|\chi_0(|\varepsilon|D)-I\|_{\mathcal{L}(H^1(\mathbb{R}^2,\mathbb{C}),H^{1/2}(\mathbb{R}^2,\mathbb{C}))} \lesssim |\varepsilon|^{1/2}$$

because

$$\begin{aligned}
\|\chi_0(|\varepsilon|D)u-u\|_{1/2}^2 &= \int_{|k|>\frac{\delta}{|\varepsilon|}} (1+|k|^2)^{1/2}|\hat{u}|^2\,\mathrm{d}k\\
&\le \sup_{|k|>\frac{\delta}{|\varepsilon|}} (1+|k|^2)^{-1/2}\int_{|k|>\frac{\delta}{|\varepsilon|}} (1+|k|^2)|\hat{u}|^2\,\mathrm{d}k\\
&\le \frac{1}{\big(1+\frac{\delta^2}{|\varepsilon|^2}\big)^{1/2}}\|u\|_1^2,
\end{aligned}$$

and similarly

$$\|\chi_0(|\varepsilon|D)-I\|_{\mathcal{L}(H^{1/2}(\mathbb{R}^2,\mathbb{C}),L^2(\mathbb{R}^2,\mathbb{C}))} \lesssim |\varepsilon|^{1/2},$$

and that pointwise multiplication defines bounded trilinear mappings $H^1(\mathbb{R}^2,\mathbb{C})^3 \to H^{1/2}(\mathbb{R}^2,\mathbb{C})$ and $H^1(\mathbb{R}^2,\mathbb{C})^2 \times H^{1/2}(\mathbb{R}^2,\mathbb{C}) \to L^2(\mathbb{R}^2,\mathbb{C})$ (see Hörmander [10, Theorem 8.3.1]), we find that

$$\|\mathcal{H}(\zeta,\varepsilon)-\mathcal{H}(\zeta,0)\|_1 \lesssim |\varepsilon|^{1/2}$$

uniformly over $\zeta \in B_R(0)$. Here we have also used the estimate $\|\chi_0(|\varepsilon|D)u\|_s \le \|u\|_s$ for all $u \in H^s(\mathbb{R}^2,\mathbb{C})$ and and the fact that $\left(1-\frac{1}{2}\partial_x^2-\partial_z^2\right)^{-1}$ maps $L^2(\mathbb{R}^2,\mathbb{C})$ continuously into $H^1(\mathbb{R}^2,\mathbb{C})$. A similar calculation shows that

$$\|\mathrm{d}_1\mathcal{H}[\zeta,\varepsilon]-\mathrm{d}_1\mathcal{H}[\zeta,0]\|_{\mathcal{L}(H^1(\mathbb{R}^2,\mathbb{C}))} \lesssim |\varepsilon|^{1/2}$$

uniformly over $\zeta \in B_R(0)$.

Furthermore, the equation

$$\mathcal{H}(\zeta,0) = \zeta - \tfrac{11}{16}\left(1-\tfrac{1}{2}\partial_x^2-\partial_z^2\right)^{-1}|\zeta|^2\zeta = 0 \tag{5.3}$$

has a unique symmetric, positive (real) solution $\zeta_0 \in \mathcal{S}(\mathbb{R}^2, \mathbb{C})$ (see Sulem and Sulem [21, §4.2] and the references therein). We show that $\mathrm{d}_1\mathcal{H}[\pm\zeta_0, 0]$ is an isomorphism by means of real coordinates $\zeta_1 = \operatorname{Re}\zeta$ and $\zeta_2 = \operatorname{Im}\zeta$, in terms of which

$$\mathrm{d}_1\mathcal{H}[\pm\zeta_0, 0](\zeta_1 + \mathrm{i}\zeta_2) = \mathcal{H}_1(\zeta_1) + \mathrm{i}\mathcal{H}_2(\zeta_2),$$

where $\mathcal{H}_1 : H^1_\mathrm{e}(\mathbb{R}^2, \mathbb{R}) \to H^1_\mathrm{e}(\mathbb{R}^2, \mathbb{R})$ and $\mathcal{H}_2 : H^1_\mathrm{o}(\mathbb{R}^2, \mathbb{R}) \to H^1_\mathrm{o}(\mathbb{R}^2, \mathbb{R})$ are defined by

$$\mathcal{H}_1(\zeta_1) = \zeta_1 - \tfrac{33}{16}\left(1 - \tfrac{1}{2}\partial_x^2 - \partial_z^2\right)^{-1}\zeta_0^2\zeta_1, \qquad \mathcal{H}_2(\zeta_2) = \zeta_2 - \tfrac{11}{16}\left(1 - \tfrac{1}{2}\partial_x^2 - \partial_z^2\right)^{-1}\zeta_0^2\zeta_2$$

with

$$H^n_\mathrm{e}(\mathbb{R}^2, \mathbb{R}) = \{\zeta_1 \in H^n(\mathbb{R}^2, \mathbb{R}) : \zeta_1(x, z) = \zeta_1(-x, z),\ \zeta_1(x, z) = \zeta_1(x, -z)\},$$
$$H^n_\mathrm{o}(\mathbb{R}^2, \mathbb{R}) = \{\zeta_2 \in H^n(\mathbb{R}^2, \mathbb{R}) : \zeta_2(x, z) = -\zeta_2(-x, z),\ \zeta_2(x, z) = \zeta_1(x, -z)\}$$

for $n \in \mathbb{N}_0$. The formulae

$$\zeta_1 \mapsto \tfrac{33}{16}\left(1 - \tfrac{1}{2}\partial_x^2 - \partial_z^2\right)^{-1}\zeta_0^2\zeta_1, \qquad \zeta_2 \mapsto \tfrac{11}{16}\left(1 - \tfrac{1}{2}\partial_x^2 - \partial_z^2\right)^{-1}\zeta_0^2\zeta_2$$

define compact operators $H^1(\mathbb{R}^2, \mathbb{R}) \to H^1(\mathbb{R}^2, \mathbb{R})$, $H^1_\mathrm{e}(\mathbb{R}^2, \mathbb{R}) \to H^1_\mathrm{e}(\mathbb{R}^2, \mathbb{R})$ and $H^1_\mathrm{o}(\mathbb{R}^2, \mathbb{R}) \to H^1_\mathrm{o}(\mathbb{R}^2, \mathbb{R})$, so that H_1, H_2 are Fredholm with index 0. Writing

$$T_1\zeta_1 = \zeta_1 - \tfrac{1}{2}\zeta_{1xx} - \zeta_{1zz} - \tfrac{33}{16}\zeta_0^2\zeta_1, \qquad T_2\zeta_2 = \zeta_2 - \tfrac{1}{2}\zeta_{2xx} - \zeta_{2zz} - \tfrac{11}{16}\zeta_0^2\zeta_2,$$

we find that the kernels of $\mathcal{H}_1$ and $\mathcal{H}_2$ coincide with respectively the kernels of the linear operators $T_1 : H^2_\mathrm{e}(\mathbb{R}^2, \mathbb{R}) \subseteq L^2_\mathrm{e}(\mathbb{R}^2, \mathbb{R}) \to L^2_\mathrm{e}(\mathbb{R}^2, \mathbb{R})$ and $T_2 : H^2_\mathrm{o}(\mathbb{R}^2, \mathbb{R}) \subseteq L^2_\mathrm{o}(\mathbb{R}^2, \mathbb{R}) \to L^2_\mathrm{o}(\mathbb{R}^2, \mathbb{R})$. It is however known that the kernels of T_1, $T_2 : H^2(\mathbb{R}^2, \mathbb{R}) \subseteq L^2(\mathbb{R}^2, \mathbb{R}) \to L^2(\mathbb{R}^2, \mathbb{R})$ are respectively $\langle \zeta_{0x}, \zeta_{0z} \rangle$ and $\langle \zeta_0 \rangle$ (see Chang *et al.* [5]). The kernels of $\mathcal{H}_1$, $\mathcal{H}_2$ are therefore trivial, so that $\mathcal{H}_1$, $\mathcal{H}_2$ and hence $\mathrm{d}_1\mathcal{H}[\pm\zeta_0, 0]$ are isomorphisms.

It remains to confirm that the formulae

$$\eta = \eta_1 + F(\eta_1) + \eta_3(\eta_1), \quad \eta_1 = \eta_1^+ + \overline{\eta_1^+}, \quad \eta_1^+(x, z) = \tfrac{1}{2}\zeta_\varepsilon^\pm(\varepsilon x, \varepsilon z)\mathrm{e}^{\mathrm{i}x}$$

lead to the estimate

$$\eta(x, z) = \pm\varepsilon\zeta_0(\varepsilon x, \varepsilon z)\cos x + o(\varepsilon)$$

uniformly over $(x, z) \in \mathbb{R}^2$. The key is to show that

$$\|\zeta_\varepsilon^+ - \zeta_0\|_\infty \lesssim \varepsilon^\Delta$$

for any $\Delta \in (0, 1/2)$; here we choose the concrete value $\Delta = 1/4$. This result follows from the calculation

$$\begin{aligned}\|\zeta_\varepsilon^+ - \zeta_0\|_\infty &\lesssim \|\zeta_\varepsilon^+ - \zeta_0\|_{5/4} \\ &= \|(1 + |k|^2)^{5/8}(\hat{\zeta}_\varepsilon^+ - \hat{\zeta}_0)\|_{L^2(|k|<\delta/\varepsilon)} + \|(1 + |k|^2)^{5/8}\hat{\zeta}_0\|_{L^2(|k|>\delta/\varepsilon)}\end{aligned}$$

(because the support of $\hat{\zeta}_\varepsilon$ lies in $\overline{B}_{\delta/\varepsilon}(0)$) and

$$\begin{aligned}\|(1 + |k|^2)^{5/8}(\hat{\zeta}_\varepsilon^+ - \hat{\zeta}_0)\|_{L^2(|k|<\delta/\varepsilon)} &\lesssim \varepsilon^{-1/4}\|(1 + |k|^2)^{1/2}(\hat{\zeta}_\varepsilon^+ - \hat{\zeta}_0)\|_{L^2(|k|<\delta/\varepsilon)} \\ &\leq \varepsilon^{-1/4}\|(1 + |k|^2)^{1/2}(\hat{\zeta}_\varepsilon^+ - \hat{\zeta}_0)\|_0 \\ &= \varepsilon^{-1/4}\|\zeta_\varepsilon^+ - \zeta_0\|_1, \\ &\lesssim \varepsilon^{1/4},\end{aligned}$$

$$\begin{aligned}\|(1 + |k|^2)^{5/8}\hat{\zeta}_0^+\|^2_{L^2(|k|>\delta/\varepsilon)} &= \int_{|k|>\frac{\delta}{\epsilon}} (1 + |k|^2)^{5/4}|\hat{\zeta}_0|^2 \\ &\lesssim \varepsilon\end{aligned}$$

(because $\hat{\zeta}_0 \in \mathcal{S}(\mathbb{R}^2)$, so that in particular $|\hat{\zeta}_0(|k|)|^2 \lesssim (1 + |k|^2)^{-11/4}$). It follows that

$$\begin{aligned}\eta_1^+(x,z) &= \varepsilon\zeta_0(\varepsilon x,\varepsilon z)\mathrm{e}^{\mathrm{i}x} + \tfrac{1}{2}\varepsilon(\zeta_\varepsilon^+ - \zeta_0)(\varepsilon x,\varepsilon z)\mathrm{e}^{\mathrm{i}x}\\ &= \varepsilon\zeta_0(\varepsilon x,\varepsilon z)\mathrm{e}^{\mathrm{i}x} + O(\varepsilon^{5/4}),\end{aligned}$$

uniformly in (x,z). (These estimates remain valid when ζ_ε^+ and ζ_0 are replaced by respectively ζ_ε^- and $-\zeta_0$.)

Furthermore

$$\|\eta_3(\eta_1)\|_\infty \lesssim \|\eta_3(\eta_1)\|_3 \lesssim \varepsilon^{10/6}|||\eta_1|||^2 \lesssim \varepsilon^{10/6}$$

by Theorem 3.9 (recall that we have chosen $\theta = 5/6$), while

$$\|F(\eta_1)\|_\infty = O(\varepsilon^{11/6})$$

because

$$F(\eta_1) = -2\left((\eta_1^+)^2 + \left(\overline{\eta_1^+}\right)^2\right) + F_{\mathrm{r}}(\eta_1),$$

where

$$\|F_{\mathrm{r}}(\eta_1)\|_\infty \lesssim \|F_{\mathrm{r}}(\eta_1)\|_3 \lesssim \|F_{\mathrm{r}}(\eta_1)\|_1 \lesssim \varepsilon^{11/6}|||\eta_1|||^2 \lesssim \varepsilon^{11/6}$$

(see Proposition 4.3; the second estimate follows by the fact that the support of $\mathcal{F}[F_{\mathrm{r}}(\eta_1)]$ is bounded independently of ε).

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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B. Buffoni
École Polytechnique Fédérale de Lausanne (EPFL)
Institut de mathématiques
Station 8
1015 Lausanne
Switzerland

E. Wahlén
Centre for Mathematical Sciences
Lund University
PO Box 11822100 Lund
Sweden

M. D. Groves
Fachrichtung Mathematik
Universität des Saarlandes
Postfach 151150
66041 Saarbrücken
Germany
e-mail: groves@math.uni-sb.de

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