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Symplectic Partitioned Runge-Kutta Methods for High-Order Approximation in Linear-Quadratic Optimal Control of Hamiltonian Systems

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Symplectic partitioned Runge-Kutta (SPRK) methods are known to be a good choice in forward simulations of Hamiltonian systems due to their structure-preserving properties. Recent works study the application of SPRK methods to nonlinear and linear-quadratic optimal control problems howing various advantages of these methods compared to standard non-symplectic integration schemes. Now, our focus is on extending the comparison to SPRK and RK methods of higher orders. For linear-quadratic optimal control problems, we consider the discrete-time Riccati feedback as well as the feedforward control implementation. For applications in which computational power or computation time is limited, low sampling rates are of particular interest. Hence we study this case for the n-fold harmonic oscillator.

1 Introduction

We consider the linear-quadratic (LQ) optimal control problem for a linear time-varying (LTV) Hamiltonian system, i.e. the system matrix A is Hamiltonian and the state x consists of configurations q_i and momenta p_i:

$$\min_u \mathcal{J}(u) = \frac{1}{2}\int_0^T \left\|x(t) - x^{\mathrm{ref}}(t)\right\|_Q^2 + \left\|u(t)\right\|_R^2 \,\mathrm{d}t + \frac{1}{2}\left\|x(T) - x^{\mathrm{ref}}(T)\right\|_{P_T}^2$$

$$\text{w.r.t. } \dot{x}(t) = A(t)x(t) + B(t)u(t), \qquad x(0) = x^0, \qquad x(t) = (q_1, \dots, q_n, p_1, \dots, p_n)^{\top}(t).$$

For this problem, the Ricatti equations can be derived by using the Hamilton function of the problem and applying Pontryagins maximum principle. For its discrete-time counterpart, the Lagrangian is derived and the Karush-Kuhn-Tucker conditions are applied. A sketch of both derivations is shown in Figure 1, where the costate is denoted as θ and the state-costate vector as $z := [x, \theta]^1$. The complete derivation of the discrete-time case for SPRK methods, including the undefined quantities, can be found in [1].

<table>
<tr><td>continuous-time</td><td>discrete-time</td></tr>
<tr><td>$\dot{z} = \overbrace{\begin{pmatrix} A & 0 \\ -Q & -A^{\top} \end{pmatrix}}^{=:\phi_z^{\mathrm{CT}}} z + \underbrace{\begin{pmatrix} B \\ 0 \end{pmatrix}}_{=:\phi_u^{\mathrm{CT}}} u + \underbrace{\begin{pmatrix} 0 \\ Q \end{pmatrix}}_{=:\phi_{x^{\mathrm{ref}}}^{\mathrm{CT}}} x^{\mathrm{ref}}$</td><td>$\begin{aligned} z_{k+1} = &\overbrace{\left(\begin{array}{c|c} I + h\check{H}_b A_s F_1 & 0 \\ \hline -h\check{H}_b\left(Q_s + A_s^{\top} L_2\right) F_1 & I - h\check{H}_b A_s^{\top} L_1 \end{array}\right)}^{=:\phi_z} z_k \\ &+ \underbrace{\left(\begin{array}{c} h\check{H}_b\left(A_s F_2 + B_s\right) \\ \hline -h\check{H}_b\left(Q_s + A_s^{\top} L_2\right) F_2 \end{array}\right)}_{=:\phi_U} U_k + \underbrace{\left(\begin{array}{c} 0 \\ \hline h\check{H}_b\left(Q_s + A_s^{\top} L_2\right) \end{array}\right)}_{=:\phi_{X^{\mathrm{ref}}}} X_k^{\mathrm{ref}} \end{aligned}$</td></tr>
<tr><td>$u = u^* = -R^{-1}B^{\top}\theta$</td><td>$U_k = U_k^* = C_{z^+} z_{k+1} + C_z z_k + C_{X^{\mathrm{ref}}} X_k^{\mathrm{ref}}, \qquad u_k^* = -R^{-1}B^{\top}\theta_k$</td></tr>
<tr><td>$\dot{z} = \overbrace{\begin{pmatrix} A & -BR^{-1}B^{\top} \\ -Q & -A^{\top} \end{pmatrix}}^{=:\Phi^{\mathrm{CT}}} z + \underbrace{\begin{pmatrix} 0 \\ Q \end{pmatrix}}_{=:O^{\mathrm{CT}}} x^{\mathrm{ref}}$</td><td>$\begin{aligned} z_{k+1} = &\overbrace{\left(I - \phi_U C_{z^+}\right)^{-1} \cdot \left(\phi_z + \phi_U C_z\right)}^{=:\Phi} z_k \\ &+ \underbrace{\left(I - \phi_U C_{z^+}\right)^{-1} \cdot \left(\phi_U C_{X^{\mathrm{ref}}} + \phi_{X^{\mathrm{ref}}}\right)}_{=:O} X_k^{\mathrm{ref}} \end{aligned}$</td></tr>
<tr><td>ansatz: $\theta = Px + r$</td><td>ansatz: $\theta_k = P_k x_k + r_k$ and $\alpha_k = \left[\alpha_k^1, \alpha_k^2\right] := O X_k^{\mathrm{ref}}$</td></tr>
<tr><td>$\dot{P} = -\left(PA + A^{\top}P + Q - PBR^{-1}B^{\top}P\right), \quad P(T) = P_T,$
$\dot{r} = -\left(A^{\top}r - Qx^{\mathrm{ref}} - PBR^{-1}B^{\top}r\right), \quad r(T) = -P_T x^{\mathrm{ref}}(T)$</td><td>$P_k = \left(\Phi^{22} - P_{k+1}\Phi^{12}\right)^{-1}\left(-\Phi^{21} + P_{k+1}\Phi^{11}\right), \quad P_N = P_T,$
$r_k = \left(\Phi^{22} - P_{k+1}\Phi^{12}\right)^{-1}\left(r_{k+1} - \alpha_k^2 + P_{k+1}\alpha_k^1\right), \quad r_N = -P_N x_N^{\mathrm{ref}}$</td></tr>
</table>

Fig. 1: Comparison of the derivation of the solution for the continuous-time and discrete-time case.

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[1] The vertical concatenation of the two matrices X, Y with consistent column dimensions is denoted with $[X, Y] := (X^{\top}, Y^{\top})^{\top}$.

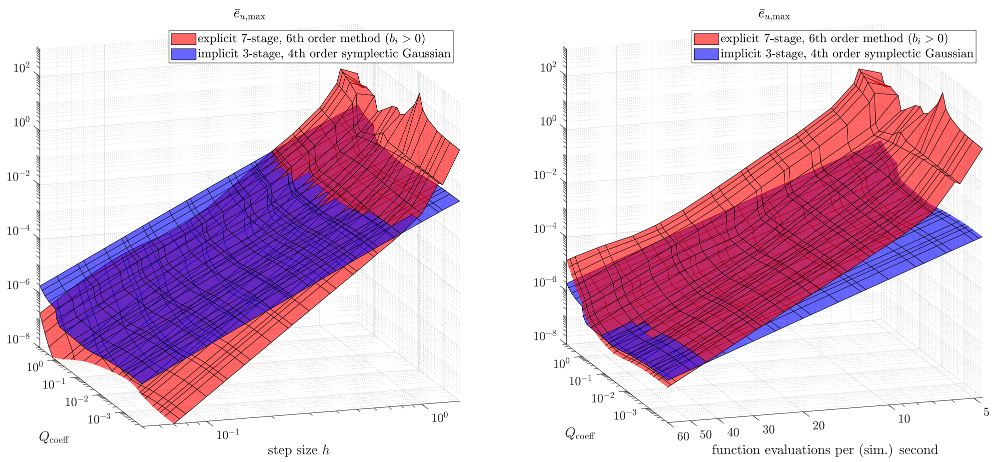
2 Numerical Analysis of the Discrete-Time System Dynamics

In this section, we analyse the convergence of an implicit 3-stage, 4th order SPRK Gaussian method (cf. references in [1]) and an explicit 7-stage, 6th order method, given in [2], with $b_i > 0$ for all $i = 1, \dots, 7$. The non-zero weights are a prerequisite for constructing the adjoint method characterized by the adjoint Butcher tableau $(\bar{\mathsf{A}}, \bar{\mathsf{b}} = \mathsf{b}, \bar{\mathsf{c}} = \mathsf{c})$, where $\bar{\mathsf{A}}_{i,j} = (\mathsf{b}_i \mathsf{b}_j - \mathsf{b}_j \mathsf{A}_{j,i})/\mathsf{b}_i$ and $(\mathsf{A}, \mathsf{b}, \mathsf{c})$ as the original tableau. Following Butcher's simplifying assumption C(2) [3], one usually sets $b_2 = 0$. Thus, the chosen explicit 6th-order method is exceptional and, to the best of the authors' knowledge, there has not been found an explicit method of higher order satisfying $b_2 \neq 0$.

Each simulation of the equations in the right column of Figure 1 yields sequences of states, costates, and inputs that are compared with a reference solution which has been obtained numerically but with high accuracy. Our numerical analysis is based on a Monte-Carlo simulation. Therefore n-fold harmonic oscillators with different size and random physical parameters (resulting from the random number generator seed σ), are generated. For these systems the origin is to be stabilized. For each Monte-Carlo run depending on the system (n, σ), chosen method r, step size h and state weighting matrix $Q = Q_{\text{coeff}} \cdot \tilde{Q}$, $Q_{\text{coeff}} \in \mathbb{R}$, we compute the maximal error and average them over all systems used

$$e_{y,\max}(n, \sigma, r, h, Q_{\text{coeff}}) = \max_{k=0,\dots,N} \left\| y_k - y_{\text{bvp}}(k \cdot h) \right\|_\infty \quad \Rightarrow \quad \bar{e}_{y,\max}(r, h, Q_{\text{coeff}}) = (\hat{n} \cdot \hat{\sigma})^{-1} \cdot \sum_{n=n_1}^{n_{\hat{n}}} \sum_{\sigma=\sigma_1}^{\sigma_{\hat{\sigma}}} e_{y,\max}(\dots),$$

where $y \in \{x, \theta, u\}$. These averaged errors can be plotted logarithmically for each method r . The results are shown exemplarily for the input u in Figure 2. In Figure 2(a), it can be seen that for a fixed step size the explicit method outperforms the SPRK



(a) Plotted over step size h. (b) Plotted over fct. evals. per (sim.) second.

Fig. 2: Errors w.r.t. the optimal input.

method when the maximal error is smaller than 10^{-2}. The situation significantly changes when the function evaluations per simulation second are fixed, see Figure 2(b). Now using the SPRK method is advantageous. The same qualitative behaviour can be observed for the quantities x and θ and, also, the trajectory tracking case.

3 Conclusion

We compared a 4th-order SPRK to an explicit 6th-order method for the discretization of a linear-quadratic optimal control problem using the discrete Riccati equations derived in [1]. As it is well known from numerical error analysis for numerical integration, the higher-order method has smaller error norms for $h \to 0$. However, we showed that it can still be advantageous to use the lower order method: As depicted in Figure 2, the lower-order symplectic method outperforms the other method a) for large step sizes and b) when the number of function evaluations play a crucial role.

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