Strong Subgraph Connectivity of Digraphs: A Survey

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Abstract

In this survey we overview known results on the strong subgraph k-connectivity and strong subgraph k-arc-connectivity of digraphs. After an introductory section, the paper is divided into four sections: basic results, algorithms and complexity, sharp bounds for strong subgraph k-(arc-)connectivity, minimally strong subgraph (k, ℓ)-(arc-) connected digraphs. This survey contains several conjectures and open problems for further study.

Keywords: Strong subgraph k-connectivity; Strong subgraph k-arc-connectivity; Digraph packing; Directed q-linkage; Directed weak q-linkage; Semicomplete digraphs; Eulerian digraphs; Symmetric digraphs; Generalized k-connectivity; Generalized k-edge-connectivity.

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1 Introduction

The generalized k-connectivity $\kappa_k(G)$ of a graph $G = (V, E)$ was introduced by Hager [17] in 1985 ($2 \leq k \leq |V|$). For a graph $G = (V, E)$ and a set $S \subseteq V$ of at least two vertices, an *S-Steiner tree* or, simply, an *S-tree* is a subgraph T of G which is a tree with $S \subseteq V(T)$. Two S-trees T_1 and T_2 are said to be *edge-disjoint* if $E(T_1) \cap E(T_2) = \emptyset$. Two edge-disjoint S-trees T_1 and T_2 are said to be *internally disjoint* if $V(T_1) \cap V(T_2) = S$. The *generalized local connectivity* $\kappa_S(G)$ is the maximum number of internally disjoint S-trees in G. For an integer k with $2 \leq k \leq n$, the *generalized k-connectivity* is defined as

$$\kappa_k(G) = \min\{\kappa_S(G) \mid S \subseteq V(G), |S| = k\}.$$

Observe that $\kappa_2(G) = \kappa(G)$. Li, Mao and Sun [20] introduced the following concept of generalized k-edge-connectivity. The *generalized local edge-connectivity* $\lambda_S(G)$ is the maximum number of edge-disjoint S-trees in G.

For an integer k with $2 \leq k \leq n$, the *generalized k-edge-connectivity* is defined as

$$\lambda_k(G) = \min\{\lambda_S(G) \mid S \subseteq V(G), |S| = k\}.$$

Observe that $\lambda_2(G) = \lambda(G)$. Generalized connectivity of graphs has become an established area in graph theory, see a monograph [19] by Li and Mao on generalized connectivity of undirected graphs.

To extend generalized k-connectivity to directed graphs, Sun, Gutin, Yeo and Zhang [25] observed that in the definition of $\kappa_S(G)$, one can replace "an S-tree" by "a connected subgraph of G containing S." Therefore, Sun et al. [25] defined *strong subgraph k-connectivity* by replacing "connected" with "strongly connected" (or, simply, "strong") as follows. Let $D = (V, A)$ be a digraph of order n, S a subset of V of size k and $2 \leq k \leq n$. A subdigraph H of D is called an *S-strong subgraph* if H is strong and $S \subseteq V(H)$. Two S-strong subgraphs D_1 and D_2 are said to be *arc-disjoint* if $A(D_1) \cap A(D_2) = \emptyset$. Two arc-disjoint S-strong subgraphs D_1 and D_2 are said to be *internally disjoint* if $V(D_1) \cap V(D_2) = S$. Let $\kappa_S(D)$ be the maximum number of internally disjoint S-strong subgraphs in D. The *strong subgraph k-connectivity* of D is defined as

$$\kappa_k(D) = \min\{\kappa_S(D) \mid S \subseteq V, |S| = k\}.$$

By definition, $\kappa_k(D) = 0$ if D is not strong.

As a natural counterpart of the strong subgraph k-connectivity, Sun and Gutin [23] introduced the concept of strong subgraph k-arc-connectivity. Let $D = (V(D), A(D))$ be a digraph of order n, $S \subseteq V$ a k-subset of $V(D)$ and $2 \leq k \leq n$. Let $\lambda_S(D)$ be the maximum number of arc-disjoint S-strong digraphs in D. The *strong subgraph k-arc-connectivity* of D is defined as

$$\lambda_k(D) = \min\{\lambda_S(D) \mid S \subseteq V(D), |S| = k\}.$$

By definition, $\lambda_k(D) = 0$ if D is not strong.

The strong subgraph k-(arc-)connectivity is not only a natural extension of the concept of generalized k-(edge-)connectivity, but also relates to important problems in graph theory. For $k = 2$, $\kappa_2(\overleftrightarrow{G}) = \kappa(G)$ [25] and $\lambda_2(\overleftrightarrow{G}) = \lambda(G)$ [23]. Hence, $\kappa_k(D)$ and $\lambda_k(D)$ could be seen as generalizations of connectivity and edge-connectivity of undirected graphs, respectively. For $k = n$, $\kappa_n(D) = \lambda_n(D)$ is the maximum number of arc-disjoint spanning strong subgraphs of D. Moreover, since $\kappa_S(D)$ and $\lambda_S(D)$ are the number of internally disjoint and arc-disjoint strong subgraphs containing a given set S, respectively, these parameters are relevant to the subdigraph packing problem, see [5–9, 13, 24].

Some basic results are introduced in Section 2. In Section 3, we sum up the results on algorithms and computational complexity for $\kappa_S(D)$, $\kappa_k(D)$, $\lambda_S(D)$ and $\lambda_k(D)$. We present many upper and lower bounds for the parameters $\kappa_k(D)$ and $\lambda_k(D)$ in Section 4. Finally, in Section 5, results on minimally strong subgraph (k, ℓ)-(arc-)connected digraphs are surveyed.

Additional Terminology and Notation. For a digraph D, its *reverse* $D^{\rm rev}$ is a digraph with same vertex set and such that $xy \in A(D^{\rm rev})$ if and only if $yx \in A(D)$. A digraph D is *symmetric* if $D^{\rm rev} = D$. In other words, a symmetric digraph D can be obtained from its underlying undirected graph G by replacing each edge of G with the corresponding arcs of both directions, that is, $D = \overleftrightarrow{G}$. A 2-cycle xyx of a strong digraph D is called a *bridge* if $D - \{xy, yx\}$ is disconnected. Thus, a bridge corresponds to a bridge in the underlying undirected graph of D. An *orientation* of a digraph D is a digraph obtained from D by deleting an arc in each 2-cycle of D. A digraph D is *semicomplete* if for every distinct $x, y \in V(D)$ at least one of the arcs xy, yx is in D. We refer the readers to [3,4,11] for graph theoretical notation and terminology not given here.

2 Basic Results

The following propositions can be easily verified using definitions of $\lambda_k(D)$ and $\kappa_k(D)$.

Proposition 2.1 *[23, 25] Let D be a digraph of order n, and let $k \geq 2$ be an integer. Then*

$$\lambda_{k+1}(D) \leq \lambda_k(D) \text{ for every } k \leq n-1 \tag{1}$$

$$\kappa_k(D') \leq \kappa_k(D), \lambda_k(D') \leq \lambda_k(D) \text{ where } D' \text{ is a spanning subdigraph of } D \tag{2}$$

$$\kappa_k(D) \leq \lambda_k(D) \leq \min\{\delta^+(D), \delta^-(D)\} \tag{3}$$

By Tillson's decomposition theorem [30], we can determine the exact values for $\kappa_k(\overleftrightarrow{K}_n)$ and $\lambda_k(\overleftrightarrow{K}_n)$.

Proposition 2.2 *[25] For $2 \leq k \leq n$, we have*

$$\kappa_k(\overleftrightarrow{K}_n) = \begin{cases} n-1, & \text{if } k \notin \{4,6\}; \\ n-2, & \text{otherwise.} \end{cases}$$

Proposition 2.3 *[23] For $2 \leq k \leq n$, we have*

$$\lambda_k(\overleftrightarrow{K}_n) = \begin{cases} n-1, & \text{if } k \notin \{4,6\}, \text{ or, } k \in \{4,6\} \text{ and } k < n; \\ n-2, & \text{if } k = n \in \{4,6\}. \end{cases}$$

Proposition 2.4 *[23] For every fixed $k \geq 2$, a digraph D is strong if and only if $\lambda_k(D) \geq 1$.*

Sun and Zhang determined the precise value for the strong subgraph k-arc-connectivity of a complete bipartite digraph.

Proposition 2.5 *[28] For two positive integers a and b with $a \leq b$, we have*

$$\lambda_k(\overleftrightarrow{K}_{a,b}) = a$$

for $2 \leq k \leq a+b$.

3 Algorithms and Complexity

3.1 Results for $\kappa_S(D)$ and $\kappa_k(D)$

3.1.1 General digraphs

For a fixed $k \geq 2$, it is easy to decide whether $\kappa_k(D) \geq 1$ for a digraph D: it holds if and only if D is strong. Unfortunately, deciding whether $\kappa_S(D) \geq 2$ is already NP-complete for $S \subseteq V(D)$ with $|S| = k$, where $k \geq 2$ is a fixed integer.

The well-known Directed q-Linkage problem was proved to be NP-complete even for the case that $q = 2$ [16]. The problem is formulated as follows: for a fixed integer $q \geq 2$, given a digraph D and a (terminal) sequence $((s_1, t_1), \ldots, (s_q, t_q))$ of distinct vertices of D, decide whether D has q vertex-disjoint paths $P_1, \ldots, P_q$, where P_i starts at s_i and ends at t_i for all $i \in [q]$.

By using the reduction from the Directed q-Linkage problem, we can prove the following intractability result.

Theorem 3.1 *[25] Let $k \geq 2$ and $\ell \geq 2$ be fixed integers. Let D be a digraph and $S \subseteq V(D)$ with $|S| = k$. The problem of deciding whether $\kappa_S(D) \geq \ell$ is NP-complete.*

In the above theorem, Sun et al. obtained the complexity result of the parameter $\kappa_S(D)$ for an arbitrary digraph D. For $\kappa_k(D)$, they made the following conjecture.

Conjecture 1 *[25] It is NP-complete to decide for fixed integers $k \geq 2$ and $\ell \geq 2$ and a given digraph D whether $\kappa_k(D) \geq \ell$.*

3.1.2 Semicomplete digraphs

Recently, Chudnovsky, Scott and Seymour [14] proved the following powerful result.

Theorem 3.2 *[14] Let D be a digraph and let q and c be fixed positive integers. Given a partition of the vertices of D into c sets each inducing a semicomplete digraph and a terminal sequence $((s_1, t_1), \ldots, (s_q, t_q))$ of distinct vertices of D, the* Directed q-Linkage *for D and $((s_1, t_1), \ldots, (s_q, t_q))$ can be solved in polynomial time.*

The following nontrivial lemma can be deduced from Theorem 3.2.

Lemma 3.3 *[25] Let k and ℓ be fixed positive integers. Let D be a digraph and let $X_1, X_2, \ldots, X_\ell$ be ℓ vertex disjoint subsets of $V(D)$, such that $|X_i| \leq k$ for all $i \in [\ell]$. Let $X = \cup_{i=1}^{\ell} X_i$ and assume that for every $v \in V(D) \setminus X$ and every $w \in V(D)$, there is an arc from v to w or an arc from w to v. Then we can in polynomial time decide if there exist vertex disjoint subsets $Z_1, Z_2, \ldots, Z_\ell$ of $V(D)$, such that $X_i \subseteq Z_i$ and $D[Z_i]$ is strongly connected for each $i \in [\ell]$.*

Using Lemma 3.3, Sun, Gutin, Yeo and Zhang proved the following result for semicomplete digraphs.

Theorem 3.4 *[25] For any fixed integers $k, \ell \geq 2$, we can decide whether $\kappa_k(D) \geq \ell$ for a semicomplete digraph D in polynomial time.*

3.1.3 Eulerian digraphs

Sun and Yeo determined the complexity of the directed 2-linkage problem on Eulerian digraphs.

Theorem 3.5 *[27] The* Directed 2-Linkage *problem restricted to Eulerian digraphs is NP-complete.*

Using Theorem 3.5, Sun and Zhang proved the following:

Theorem 3.6 *[28] Let $k, \ell \geq 2$ be fixed. For any Eulerian digraph D and $S \subseteq V(D)$ with $|S| = k$, deciding whether $\kappa_S(D) \geq \ell$ is NP-complete.*

3.1.4 Symmetric digraphs

Now we turn our attention to symmetric digraphs. We start with the following structural result.

Theorem 3.7 *[25] For every graph G we have $\kappa_2(\overleftrightarrow{G}) = \kappa(G)$.*

Theorem 3.7 immediatly implies the following positive result, which follows from the fact that $\kappa(G)$ can be computed in polynomial time.

Corollary 3.8 *[25] For a graph G, $\kappa_2(\overleftrightarrow{G})$ can be computed in polynomial time.*

Theorem 3.7 states that $\kappa_k(\overleftrightarrow{G}) = \kappa_k(G)$ when $k = 2$. However when $k \geq 3$, then $\kappa_k(\overleftrightarrow{G})$ is not always equal to $\kappa_k(G)$, as can be seen from $\kappa_3(\overleftrightarrow{K_3}) = 2 \neq 1 = \kappa_3(K_3)$. Chen, Li, Liu and Mao [12] introduced the following problem, which turned out to be NP-complete.

CLLM Problem: Given a tripartite graph $G = (V, E)$ with a 3-partition $(\overline{U}, \overline{V}, \overline{W})$ such that $|\overline{U}| = |\overline{V}| = |\overline{W}| = q$, decide whether there is a partition of V into q disjoint 3-sets $V_1, \dots, V_q$ such that for every $V_i = \{v_{i_1}, v_{i_2}, v_{i_3}\}$ $v_{i_1} \in \overline{U}, v_{i_2} \in \overline{V}, v_{i_3} \in \overline{W}$ and $G[V_i]$ is connected.

Lemma 3.9 *[12] The CLLM Problem is NP-complete.*

Now restricted to symmetric digraphs D, for any fixed integer $k \geq 3$, by Lemma 3.9, the problem of deciding whether $\kappa_S(D) \geq \ell$ $(\ell \geq 1)$ is NP-complete for $S \subseteq V(D)$ with $|S| = k$.

Theorem 3.10 *[25] For any fixed integer $k \geq 3$, given a symmetric digraph D, a k-subset S of $V(D)$ and an integer ℓ $(\ell \geq 1)$, deciding whether $\kappa_S(D) \geq \ell$, is NP-complete.*

The last theorem assumes that k is fixed but ℓ is a part of input. When ℓ is fixed but k is a part of input, the problem is still NP-complete.

Theorem 3.11 *[28] For any fixed integer $\ell \geq 2$, given a symmetric digraph D, a k-subset S of $V(D)$ and an integer k ($k \geq 2$), deciding whether $\kappa_S(D) \geq \ell$, is NP-complete.*

Now for the remaining case that both k and ℓ are fixed, the problem of deciding whether $\kappa_S(D) \geq \ell$ for a symmetric digraph D, is polynomial-time solvable. We will start with the following technical lemma.

Lemma 3.12 *[25] Let $k, \ell \geq 2$ be fixed. Let G be a graph and let $S \subseteq V(G)$ be an independent set in G with $|S| = k$. For $i \in [\ell]$, let D_i be any set of arcs with both end-vertices in S. Let a forest F_i in G be called (S, D_i)-*acceptable *if the digraph $\overleftrightarrow{F_i} + D_i$ is strong and contains S. In polynomial time, we can decide whether there exists edge-disjoint forests $F_1, F_2, \ldots, F_\ell$ such that F_i is (S, D_i)-acceptable for all $i \in [\ell]$ and $V(F_i) \cap V(F_j) \subseteq S$ for all $1 \leq i < j \leq \ell$.*

Now we can prove the following result by Lemma 3.12:

Theorem 3.13 *[25] Let $k, \ell \geq 2$ be fixed. For any symmetric digraph D and $S \subseteq V(D)$ with $|S| = k$ we can in polynomial time decide whether $\kappa_S(D) \geq \ell$.*

3.1.5 Open problems

The Directed q-Linkage problem is polynomial-time solvable for planar digraphs [21] and digraphs of bounded directed treewidth [18]. However, it seems that we cannot use the approach in proving Theorem 3.4 directly as the structure of minimum-size strong subgraphs in these two classes of digraphs is more complicated than in semicomplete digraphs. Certainly, we cannot exclude the possibility that computing strong subgraph k-connectivity in planar digraphs and/or in digraphs of bounded directed treewidth is NP-complete.

Problem 3.14 *[25] What is the complexity of deciding whether $\kappa_k(D) \geq \ell$ for fixed integers $k \geq 2$, and $\ell \geq 2$ and a given planar digraph D?*

Problem 3.15 *[25] What is the complexity of deciding whether $\kappa_k(D) \geq \ell$ for fixed integers $k \geq 2$, and $\ell \geq 2$ and a digraph D of bounded directed treewidth?*

It would be interesting to identify large classes of digraphs for which the $\kappa_k(D) \geq \ell$ problem can be decided in polynomial time.

3.2 Results for $\lambda_S(D)$ and $\lambda_k(D)$

3.2.1 General digraphs

Yeo proved that it is an NP-complete problem to decide whether a 2-regular digraph has two arc-disjoint hamiltonian cycles (see, e.g., Theorem 6.6 in [8]). (A digraph is 2-regular if the out-degree and in-degree of every vertex equals 2.) Thus, the problem of deciding whether $\lambda_n(D) \geq 2$ is NP-complete, where n is the order of D. Sun and Gutin [23] extended this result in Theorem 3.16.

Let D be a digraph and let $s_1, s_2, \ldots, s_q, t_1, t_2, \ldots, t_q$ be a collection of not necessarily distinct vertices of D. A *weak q-linkage* from $(s_1, s_2, \ldots, s_q)$ to $(t_1, t_2, \ldots, t_q)$ is a collection of q arc-disjoint paths $P_1, \ldots, P_q$ such that P_i is an (s_i, t_i)-path for each $i \in [q]$. A digraph $D = (V, A)$ is *weakly q-linked* if it contains a weak q-linkage from $(s_1, s_2, \ldots, s_q)$ to $(t_1, t_2, \ldots, t_q)$ for every choice of (not necessarily distinct) vertices $s_1, \ldots, s_q, t_1, \ldots, t_q$. The Directed Weak q-Linkage problem is the following. Given a digraph $D = (V, A)$ and distinct vertices $x_1, x_2, \ldots, x_q, y_1, y_2, \ldots, y_q$; decide whether D contains q arc-disjoint paths $P_1, \ldots, P_q$ such that P_i is an (x_i, y_i)-path. The problem is well-known to be NP-complete already for $q = 2$ [16]. By using the reduction from the Directed Weak q-Linkage problem, we can prove the following intractability result.

Theorem 3.16 *[23] Let $k \geq 2$ and $\ell \geq 2$ be fixed integers. Let D be a digraph and $S \subseteq V(D)$ with $|S| = k$. The problem of deciding whether $\lambda_S(D) \geq \ell$ is NP-complete.*

3.2.2 Semicomplete digraphs, semicomplete compositions and symmetric digraphs

Bang-Jensen and Yeo [8] conjectured the following:

Conjecture 2 *For every $\lambda \geq 2$ there is a finite set $\mathcal{S}_\lambda$ of digraphs such that λ-arc-strong semicomplete digraph D contains λ arc-disjoint spanning strong subgraphs unless $D \in \mathcal{S}_\lambda$.*

Bang-Jensen and Yeo [8] proved the conjecture for $\lambda = 2$ by showing that $|\mathcal{S}_2| = 1$ and describing the unique digraph S_4 of $\mathcal{S}_2$ of order 4. This result and Theorem 4.4 imply the following:

Theorem 3.17 *[23] For a semicomplete digraph D, of order n and an integer k such that $2 \leq k \leq n$, $\lambda_k(D) \geq 2$ if and only if D is 2-arc-strong and the following does not hold: $D \cong S_4$ and $k = 4$.*

Now consider a much larger class of digraphs called semicomplete compositions. Let T be a digraph with t vertices $u_1, \ldots, u_t$ and let $H_1, \ldots, H_t$ be digraphs such that H_i has vertices $u_{i,j_i},\ 1 \leq j_i \leq n_i$. Then the composition $Q = T[H_1, \ldots, H_t]$ is a digraph with vertex set $\bigcup_{i=1}^{t} V(H_i) = \{u_{i,j_i} \mid 1 \leq i \leq t, 1 \leq j_i \leq n_i\}$ and arc set

$$\left(\bigcup_{i=1}^{t} A(H_i)\right) \bigcup \left(\bigcup_{u_i u_p \in A(T)} \{u_{i,j_i} u_{p,q_p} \mid 1 \leq j_i \leq n_i, 1 \leq q_p \leq n_p\}\right).$$

When T is semicomplete, Q is called a *semicomplete composition.* To see that semicomplete compositions form a much larger class of digraphs than semicomplete digraphs, note that the Hamiltonian cycle problem is polynomial time solvable for the latter but NP-complete for the former [1] (see also [5]).

The next result is a characterization of Bang-Jensen, Gutin and Yeo [5], which we reformulate using the notion of $\lambda_n(D)$. In this theorem, we will use the following notation: $\overline{K}_2$ and $\overline{K}_3$ denote digraphs with no arcs of order 2 and 3, respectively, $\overrightarrow{P}_2$ a directed path with two vertices, and $\overrightarrow{C}_3$ a directed cycle with three vertices.

Theorem 3.18 *Let D be a semicomplete composition with $n \geq 2$ vertices. Then $\lambda_n(D) \geq 2$ if and only if D is 2-arc-strong and D is not isomorphic to one of the following four digraphs: S_4, $\overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_2]$, $\overrightarrow{C}_3[\overrightarrow{P}_2, \overline{K}_2, \overline{K}_2]$, $\overrightarrow{C}_3[\overline{K}_2, \overline{K}_2, \overline{K}_3]$.*

It is easy to see that the following holds.

Corollary 3.19 *Let D be a semicomplete composition with $n \geq 2$ vertices. In polynomial time, we can decide whether $\lambda_n(D) \geq 2$.*

Now we turn our attention to symmetric digraphs. We start from characterizing symmetric digraphs D with $\lambda_k(D) \geq 2$, an analog of Theorem 3.17. To prove it we need the following result of Boesch and Tindell [10] translated from the language of mixed graphs to that of digraphs.

Theorem 3.20 *A strong digraph D has a strong orientation if and only if D has no bridge.*

Here is the characterization by Sun and Gutin.

Theorem 3.21 *[23] For a strong symmetric digraph D of order n and an integer k such that $2 \leq k \leq n$, $\lambda_k(D) \geq 2$ if and only if D has no bridge.*

Theorems 3.17 and 3.21 imply the following complexity result, which we believe to be extendable from $\ell = 2$ to any natural $\ell \geq 2$.

Corollary 3.22 *[23] The problem of deciding whether $\lambda_k(D) \geq 2$ is polynomial-time solvable if D is either semicomplete or symmetric digraph of order n and $2 \leq k \leq n$.*

Sun and Gutin gave a lower bound on $\lambda_k(D)$ for symmetric digraphs D.

Theorem 3.23 *[23] For every graph G, we have*

$$\lambda_k(\overleftrightarrow{G}) \geq \lambda_k(G).$$

Moreover, this bound is sharp. In particular, we have $\lambda_2(\overleftrightarrow{G}) = \lambda_2(G)$.

Theorem 3.23 immediately implies the next result, which follows from the fact that $\lambda(G)$ can be computed in polynomial time.

Corollary 3.24 *[23] For a symmetric digraph D, $\lambda_2(D)$ can be computed in polynomial time.*

3.2.3 Open problems

Corollaries 3.22 and 3.24 shed some light on the complexity of deciding, for fixed $k, \ell \geq 2$, whether $\lambda_k(D) \geq \ell$ for semicomplete and symmetric digraphs D. However, it is unclear what is the complexity above for every fixed $k, \ell \geq 2$. If Conjecture 2 is correct, then the $\lambda_k(D) \geq \ell$ problem can be solved in polynomial time for semicomplete digraphs. However, Conjecture 2 seems to be very difficult. It was proved in [25] that for fixed $k, \ell \geq 2$ the problem of deciding whether $\kappa_k(D) \geq \ell$ is polynomial-time solvable for both semicomplete and symmetric digraphs, but it appears that the approaches to prove the two results cannot be used for $\lambda_k(D)$. Some well-known results such as the fact that the hamiltonicity problem is NP-complete for undirected 3-regular graphs, indicate that the $\lambda_k(D) \geq \ell$ problem for symmetric digraphs may be NP-complete, too.

Problem 3.25 *[23] What is the complexity of deciding whether $\lambda_k(D) \geq \ell$ for fixed integers $k \geq 2$ and $\ell \geq 2$, and a semicomplete digraph D?*

Problem 3.26 *[23] What is the complexity of deciding whether $\lambda_k(D) \geq \ell$ for fixed integers $k \geq 2$ and $\ell \geq 2$, and a symmetric digraph D?*

It would be interesting to identify large classes of digraphs for which the $\lambda_k(D) \geq \ell$ problem can be decided in polynomial time.

3.3 Inapproximability results

The famous problem of Steiner Tree Packing is to find a largest collection of edge-disjoint S-Steiner trees in a given undirected graph G. Besides this classical version, researchers have also studied some variations or extensions. For example, Cheriyan and Salavatipour [13], and Sun and Yeo [27] studied the directed Steiner tree packing problem.

Sun and Zhang [28] introduced the following two types of strong subgraph packing problems in digraphs which are analogs of Steiner Tree Packing problem and are related to $\kappa_S(D)$ and $\lambda_S(D)$. The input of Arc-disjoint Strong Subgraph Packing (ASSP) consists of a digraph D and a subset of vertices $S \subseteq V(D)$, and the goal is to find a largest collection of arc-disjoint S-strong subgraphs. Similarly, the input of Internally-disjoint Strong Subgraph Packing (ISSP) consists of a digraph D and a subset of vertices $S \subseteq V(D)$, and the goal is to find a largest collection of internally disjoint S-strong subgraphs.

In the Set Cover Packing problem, the input consists of a bipartite graph $G = (C \cup B, E)$, and the goal is to find a largest collection of pairwise disjoint set covers of B, where a *set cover* of B is a subset $S \subseteq C$ such that each vertex of B has a neighbor in S. Feige et al. [15] proved the following inapproximability result on the Set Cover Packing problem.

Theorem 3.27 *[15] Unless P=NP, there is no $o(\log n)$-approximation algorithm for* Set Cover Packing*, where n is the order of G.*

Sun and Zhang obtained two inapproximability results on ISSP and ASSP by reductions from the Set Cover Packing problem.

Theorem 3.28 *[28] The following assertions hold:*
(i) Unless P=NP, there is no $o(\log n)$*-approximation algorithm for* ISSP*, even restricted to the case that* D *is a symmetric digraph and* S *is independent in* D*, where* n *is the order of* D*.*
(ii) Unless P=NP, there is no $o(\log n)$*-approximation algorithm for* ASSP*, even restricted to the case that* S *is independent in* D*, where* n *is the order of* D*.*

4 Bounds for Strong Subgraph k-(Arc-)Connectivity

4.1 Results for $\kappa_k(D)$

By Propositions 2.1 and 2.2, Sun, Gutin, Yeo and Zhang obtained a sharp lower bound and a sharp upper bound for $\kappa_k(D)$, where $2 \leq k \leq n$.

Theorem 4.1 *[25] Let* $2 \leq k \leq n$*. For a strong digraph* D *of order* n*, we have*

$$1 \leq \kappa_k(D) \leq n-1.$$

Moreover, both bounds are sharp, and the upper bound holds if and only if $D \cong \overleftrightarrow{K}_n$, $2 \leq k \leq n$ *and* $k \notin \{4,6\}$*.*

Sun and Gutin gave the following sharp upper bound for $\kappa_k(D)$ which improves (3) of Proposition 2.1.

Theorem 4.2 *[23] For* $k \in \{2,\ldots,n\}$ *and* $n \geq \kappa(D)+k$*, we have*

$$\kappa_k(D) \leq \kappa(D).$$

Moreover, the bound is sharp.

4.2 Results for $\lambda_k(D)$

By Propositions 2.1 and 2.2, Sun and Gutin obtained a sharp lower bound and a sharp upper bound for $\lambda_k(D)$, where $2 \leq k \leq n$.

Theorem 4.3 *[23] Let* $2 \leq k \leq n$*. For a strong digraph* D *of order* n*, we have*

$$1 \leq \lambda_k(D) \leq n-1.$$

Moreover, both bounds are sharp, and the upper bound holds if and only if $D \cong \overleftrightarrow{K}_n$*, where* $k \notin \{4,6\}$*, or,* $k \in \{4,6\}$ *and* $k < n$*.*

They also gave the following sharp upper bound for $\lambda_k(D)$ which improves (3) of Proposition 2.1.

Theorem 4.4 *[23] For* $2 \leq k \leq n$*, we have*

$$\lambda_k(D) \leq \lambda(D).$$

Moreover, the bound is sharp.

Shiloach [22] proved the following:

Theorem 4.5 *[22] A digraph D is weakly k-linked if and only if D is k-arc-strong.*

Using Shiloach's Theorem, Sun and Gutin [23] proved the following lower bound for $\lambda_k(D)$. Such a bound does not hold for $\kappa_k(D)$ since it was shown in [25] using Thomassen's result in [29] that for every ℓ there are digraphs D with $\kappa(D) = \ell$ and $\kappa_2(D) = 1$.

Proposition 4.6 *[23] Let $k \leq \ell = \lambda(D)$. We have $\lambda_k(D) \geq \lfloor \ell/k \rfloor$.*

For a digraph $D = (V(D), A(D))$, the *complement digraph*, denoted by D^c, is a digraph with vertex set $V(D^c) = V(D)$ such that $xy \in A(D^c)$ if and only if $xy \notin A(D)$.

Given a graph parameter $f(G)$, the Nordhaus-Gaddum Problem is to determine sharp bounds for (1) $f(G) + f(G^c)$ and (2) $f(G)f(G^c)$, and characterize the extremal graphs. The Nordhaus-Gaddum type relations have received wide attention; see a survey paper [2] by Aouchiche and Hansen. By using Proposition 2.4, the following Theorem 4.7 concerning such type of a problem for the parameter λ_k can be obtained.

Theorem 4.7 *[23] For a digraph D with order n, the following assertions holds:*
(i) $0 \leq \lambda_k(D) + \lambda_k(D^c) \leq n - 1$. Moreover, both bounds are sharp. In particular, the lower bound holds if and only if $\lambda(D) = \lambda(D^c) = 0$.
(ii) $0 \leq \lambda_k(D)\lambda_k(D^c) \leq (\frac{n-1}{2})^2$. Moreover, both bounds are sharp. In particular, the lower bound holds if and only if $\lambda(D) = 0$ or $\lambda(D^c) = 0$.

5 Minimally Strong Subgraph (k, ℓ)-(Arc-)Connected Digraphs

5.1 Results for minimally strong subgraph (k, ℓ)-connected digraphs

A digraph $D = (V(D), A(D))$ is called *minimally strong subgraph (k, ℓ)-connected* if $\kappa_k(D) \geq \ell$ but for any arc $e \in A(D)$, $\kappa_k(D - e) \leq \ell - 1$ [23]. By the definition of $\kappa_k(D)$ and Theorem 4.1, we know $2 \leq k \leq n, 1 \leq \ell \leq n-1$. Let $\mathfrak{F}(n, k, \ell)$ be the set of all minimally strong subgraph (k, ℓ)-connected digraphs with order n. We define

$$F(n, k, \ell) = \max\{|A(D)| \mid D \in \mathfrak{F}(n, k, \ell)\}$$

and

$$f(n, k, \ell) = \min\{|A(D)| \mid D \in \mathfrak{F}(n, k, \ell)\}.$$

We further define

$$Ex(n, k, \ell) = \{D \mid D \in \mathfrak{F}(n, k, \ell), |A(D)| = F(n, k, \ell)\}$$

and

$$ex(n,k,\ell) = \{D \mid D \in \mathfrak{F}(n,k,\ell), |A(D)| = f(n,k,\ell)\}.$$

By the definition of a minimally strong subgraph (k,ℓ)-connected digraph, we can get the following observation.

Proposition 5.1 *[23] A digraph D is minimally strong subgraph (k,ℓ)-connected if and only if $\kappa_k(D) = \ell$ and $\kappa_k(D-e) = \ell - 1$ for any arc $e \in A(D)$.*

A digraph D is *minimally strong* if D is strong but $D-e$ is not for every arc e of D.

Proposition 5.2 *[23] The following assertions hold:*
(i) A digraph D is minimally strong subgraph $(k,1)$-connected if and only if D is minimally strong digraph;
(ii) For $k \neq 4, 6$, a digraph D is minimally strong subgraph $(k, n-1)$-connected if and only if $D \cong \overleftrightarrow{K}_n$.

The following result characterizes minimally strong subgraph $(2, n-2)$-connected digraphs.

Theorem 5.3 *[23] A digraph D is minimally strong subgraph $(2, n-2)$-connected if and only if D is a digraph obtained from the complete digraph $\overleftrightarrow{K}_n$ by deleting an arc set M such that $\overleftrightarrow{K}_n[M]$ is a directed 3-cycle or a union of $\lfloor n/2 \rfloor$ vertex-disjoint directed 2-cycles. In particular, we have $f(n,2,n-2) = n(n-1) - 2\lfloor n/2 \rfloor$, $F(n,2,n-2) = n(n-1) - 3$.*

Note that Theorem 5.3 implies that $Ex(n,2,n-2) = \{\overleftrightarrow{K_n} - M\}$ where M is an arc set such that $\overleftrightarrow{K}_n[M]$ is a directed 3-cycle, and $ex(n,2,n-1) = \{\overleftrightarrow{K_n} - M\}$ where M is an arc set such that $\overleftrightarrow{K}_n[M]$ is a union of $\lfloor n/2 \rfloor$ vertex-disjoint directed 2-cycles.

The following result concerns a sharp lower bound for the parameter $f(n,k,\ell)$.

Theorem 5.4 *[23] For $2 \leq k \leq n$, we have*

$$f(n,k,\ell) \geq n\ell.$$

Moreover, the following assertions hold:
(i) If $\ell = 1$, then $f(n,k,\ell) = n$; (ii) If $2 \leq \ell \leq n-1$, then $f(n,n,\ell) = n\ell$ for $k = n \notin \{4,6\}$; (iii) If n is even and $\ell = n-2$, then $f(n,2,\ell) = n\ell$.

To prove two upper bounds on the number of arcs in a minimally strong subgraph (k,ℓ)-connected digraph, Sun and Gutin used the following result, see e.g. [3].

Theorem 5.5 *Every strong digraph D on n vertices has a strong spanning subgraph H with at most $2n-2$ arcs and equality holds only if H is a symmetric digraph whose underlying undirected graph is a tree.*

Proposition 5.6 *[23] We have* (*i*) $F(n,n,\ell) \leq 2\ell(n-1)$; (*ii*) *For every* k ($2 \leq k \leq n$), $F(n,k,1) = 2(n-1)$ *and* $Ex(n,k,1)$ *consists of symmetric digraphs whose underlying undirected graphs are trees.*

The minimally strong subgraph $(2, n-2)$-connected digraphs was characterized in Theorem 5.3. As a simple consequence of the characterization, we can determine the values of $f(n,2,n-2)$ and $F(n,2,n-2)$. It would be interesting to determine $f(n,k,n-2)$ and $F(n,k,n-2)$ for every value of $k \geq 3$ since obtaining characterizations of all $(k, n-2)$-connected digraphs for $k \geq 3$ seems a very difficult problem.

Problem 5.7 *[23] Determine* $f(n,k,n-2)$ *and* $F(n,k,n-2)$ *for every value of* $k \geq 3$.

It would also be interesting to find a sharp upper bound for $F(n,k,\ell)$ for all $k \geq 2$ and $\ell \geq 2$.

Problem 5.8 *[23] Find a sharp upper bound for* $F(n,k,\ell)$ *for all* $k \geq 2$ *and* $\ell \geq 2$.

5.2 Results for minimally strong subgraph (k,ℓ)-arc-connected digraphs

A digraph $D = (V(D), A(D))$ is called *minimally strong subgraph* (k,ℓ)-*arc-connected* if $\lambda_k(D) \geq \ell$ but for any arc $e \in A(D)$, $\lambda_k(D-e) \leq \ell - 1$. By the definition of $\lambda_k(D)$ and Theorem 4.3, we know $2 \leq k \leq n, 1 \leq \ell \leq n-1$. Let $\mathfrak{G}(n,k,\ell)$ be the set of all minimally strong subgraph (k,ℓ)-arc-connected digraphs with order n. We define

$$G(n,k,\ell) = \max\{|A(D)| \mid D \in \mathfrak{G}(n,k,\ell)\}$$

and

$$g(n,k,\ell) = \min\{|A(D)| \mid D \in \mathfrak{G}(n,k,\ell)\}.$$

We further define

$$Ex'(n,k,\ell) = \{D \mid D \in \mathfrak{G}(n,k,\ell), |A(D)| = G(n,k,\ell)\}$$

and

$$ex'(n,k,\ell) = \{D \mid D \in \mathfrak{G}(n,k,\ell), |A(D)| = g(n,k,\ell)\}.$$

Sun and Gutin [23] gave the following characterizations.

Proposition 5.9 *[23] The following assertions hold:*
(*i*) *A digraph* D *is minimally strong subgraph* $(k,1)$-*arc-connected if and only if* D *is minimally strong digraph;*
(*ii*) *Let* $2 \leq k \leq n$. *If* $k \notin \{4,6\}$, *or,* $k \in \{4,6\}$ *and* $k < n$, *then a digraph* D *is minimally strong subgraph* $(k, n-1)$-*arc-connected if and only if* $D \cong \overleftrightarrow{K}_n$.

Theorem 5.10 *[23] A digraph* D *is minimally strong subgraph* $(2, n-2)$-*arc-connected if and only if* D *is a digraph obtained from the complete digraph* $\overleftrightarrow{K}_n$ *by deleting an arc set* M *such that* $\overleftrightarrow{K}_n[M]$ *is a union of vertex-disjoint cycles which cover all but at most one vertex of* $\overleftrightarrow{K}_n$.

Sun and Jin characterized the minimally strong subgraph $(3, n-2)$-arc-connected digraphs.

Theorem 5.11 *[26] A digraph D is minimally strong subgraph $(3, n-2)$-arc-connected if and only if D is a digraph obtained from the complete digraph $\overleftrightarrow{K}_n$ by deleting an arc set M such that $\overleftrightarrow{K}_n[M]$ is a union of vertex-disjoint cycles which cover all but at most one vertex of $\overleftrightarrow{K}_n$.*

Theorems 5.10 and 5.11 imply that the following assertions hold: (i) For $k \in \{2, 3\}$, $Ex'(n, k, n-2) = \{\overleftrightarrow{K_n} - M\}$ where M is an arc set such that $\overleftrightarrow{K}_n[M]$ is a union of vertex-disjoint cycles which cover all but exactly one vertex of $\overleftrightarrow{K}_n$. (ii) For $k \in \{2, 3\}$, $ex'(n, k, n-2) = \{\overleftrightarrow{K_n} - M\}$ where M is an arc set such that $\overleftrightarrow{K}_n[M]$ is a union of vertex-disjoint cycles which cover all vertices of $\overleftrightarrow{K}_n$.

Sun and Jin completely determined the precise value for $g(n, k, \ell)$. Note that $(n, k, \ell) \notin \{(4, 4, 3), (6, 6, 5)\}$ by Theorem 4.3 and the definition of $g(n, k, \ell)$.

Theorem 5.12 *[26] For any triple (n, k, ℓ) with $2 \leq k \leq n, 1 \leq \ell \leq n-1$ such that $(n, k, \ell) \notin \{(4, 4, 3), (6, 6, 5)\}$, we have*

$$g(n, k, \ell) = n\ell.$$

Some results for $G(n, k, \ell)$ were obtained as well.

Proposition 5.13 *[26] We have (i) $G(n, n, \ell) \leq 2\ell(n-1)$; (ii) For every k $(2 \leq k \leq n)$, $G(n, k, 1) = 2(n-1)$ and $Ex'(n, k, 1)$ consists of symmetric digraphs whose underlying undirected graphs are trees; (iii) $G(n, k, n-2) = (n-1)^2$ for $k \in \{2, 3\}$.*

Note that the precise values of $g(n, k, \ell)$ for each pair of k and ℓ and the precise values of $G(n, k, n-2)$ for $k \in \{2, 3\}$ were determined. Hence, similar to problems 5.7 and 5.8, the following problems are also interesting.

Problem 5.14 *[26] Determine $G(n, k, n-2)$ for every value of $k \geq 2$.*

Problem 5.15 *[26] Find a sharp upper bound for $G(n, k, \ell)$ for all $k \geq 2$ and $\ell \geq 2$.*

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