Numerical investigation of nonlinear deflections of an infinite beam on nonlinear and discontinuous elastic foundation

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Abstract

The analysis of static deflections of an infinite beam resting on a nonlinear and discontinuous foundation is not trivial. We apply a recently proposed iterative nonlinear procedure to the analysis. Mathematical models of the elastic foundation are incorporated into the governing nonlinear fourth-order differential equation of the system and then, the differential equation is transformed into an equivalent nonlinear integral equation using Green's functions. Numerical solutions of the integral equation clearly demonstrate herein that our nonlinear iterative numerical method is simple and straightforward for approximate solutions of the static deflection of an infinite beam on a *nonlinear* elastic foundation. Iterative numerical solutions converge fast to corresponding analytic solutions. However, numerical errors are observed in a narrow neighborhood of material discontinuities of foundations.

Keywords: Infinite Beam; Discontinuous Nonlinear Elastic Foundation; Green's Function; Iterative method

1. Introduction

The purpose of this paper is to obtain nonlinear deflections of an infinite beam resting on a discontinuous nonlinear elastic foundation in a numerically stable and efficient way. For many years, the problem has attracted much interests in the fields of mechanical or civil engineering for its wide and practical applications; the analysis of the deflection plays an important role in the design of highways, rail tracks and long pipelines, floating offshore airports, and many other narrow and long structures lying on foundations. Hence, a number of studies have been carried out in order to analyze the nonlinear deflection of a thin body more realistically involving elastic foundations mathematically.

Winkler (1867a, b), in the early stage, proposed the widely known linear elastic foundation model in which the relation between the deflection of a beam and applied force is linear. There have been many studies in the past on the identification of an infinite beam on uniform elastic foundation with linear and nonlinear elasticity (Beaufait & Hoadley, 1980; David, 1988; Failla & Santini, 2007; Hetenyi, 1950; Ho & Chen, 1998; Jang et al., 2011a; Kuo & Lee, 1994; Ma et al., 2009; Sharma & DasGupta, 1975; Timoshenko, 1926; Tsiatas, 2010). Furthermore, some studies have included the effect of curvature as well as displacement of the foundation layer (Pasternak, 1954; Filonenko-Borodich, 1940; Vlasov & Leontev, 1966).

More specifically, Guo & Weitsman (2002) considered non-uniform elastic foundations; they presented an analytical solution for simply supported beams employing a Green's function formulation and elementary numerical techniques. Their foundation model is the simple Winkler model with a spatially varying foundation. Many other studies have considered the nonlinearity of the problem, but their methods are strongly dependent on small perturbation parameters with a limited working range. So they cannot deal with a highly nonlinear system with a more realistic *non-uniform (or discontinuous) elastic* foundation. Moreover, the formulation and the mesh generation in their methods could be time consuming.

Recent research by Choi and Jang (2012) shows analytically the existence and uniqueness of the static deflection of an infinite beam on a *non-uniform and nonlinear* elastic foundation. Moreover, they propose a new approach based on Green's function, which can handle even nonlinear elastic foundations. Hence, in this paper, we apply the Green's function-based method to general problems with non-uniform and nonlinear reaction forces from elastic foundations. The method solves deflection problems iteratively based on the Banach fixed-point theorem (Banach, 1922). Based on the theorem, Jang et al. (2011a) proposed a new iterative method for the analysis of an infinite beam on nonlinear & uniform elastic foundation. Jang & Sung (2012) suggest a general approach to a variable cross-section beam on nonlinear elastic one. A new semi analytical approach to the identification of large deflection of Bernoulli-Euler-v. Karman beam was also proposed (Jang, 2013a). Nonlinear elastic foundation using an one-way spring model (Park et al. 2013) and variable flexural and axial rigidities Bernoulli-Euler-v. Karman beam were also considered (Jang, 2014). Finally, iterative methods also have been successfully applied to problems in ocean engineering such as nonlinear water waves (Jang & Kwon, 2005; Jang et al., 2006; Jang et al., 2007a; Jang & Kwon, 2007b, Jang et al. 2010a) and the simultaneous detection of the nonlinear restoring and nonlinear damping of a nonlinear oscillation (Jang, 2009; Jang et al. 2010b, 2011b, 2011c, 2013b)

In numerical experiments section, the nonlinear deflection of an infinite beam on a non-uniform elastic foundation is successfully obtained, and they show the method is simple and straightforward. They show that an accurate solution can be achieved within a few iterations (2~15) or it converges to the nonlinear solution relatively fast in comparison with other similar studies (Beaufait and Hoadley, 1980; Mahaidi et al. 1990). Finally, we point out that the method does not require a small parameter.

2. Nonlinear integral equation formulation

The problem at hand is the nonlinear deflection of an infinite beam that is resting on a foundation composed of inhomogeneous materials, as depicted in Figure 1. It shows that the discontinuous elastic foundation consists of more than two parts with different spring force $f\left[u(x),x\right]$. The weight of the beam is balanced out with the restoring force and the beam is straight without in-plane loading. In practice, lateral or longitudinal loading and combinations of them can be observed and hence, the bending moment and consequently the governing equation changes accordingly. However, in this paper, such loading condition and coupling between them are not considered, for mathematical simplicity. Instead, we focus on tackling nonlinear restoring forces and the presence of their discontinuities. Here, we describe a procedure to formulate a nonlinear integral equation from the Euler beam equation, a 4th order differential equation that governs beam deflections. According to the classical Euler beam theory, the vertical deflection $u(x)$ satisfies the differential equation as follows:

$$EI\frac{d^4u}{dx^4} = p(x)\,, \tag{1}$$

where the net load distribution $p(x)$ is defined as

$$p(x) = w(x) - f(u,x)\,. \tag{2}$$

In Eq. (2), the upward nonlinear spring force $f(u,x)$ depends both on the beam deflection u and on the position x, and $w(x)$ denotes the loading applied downward. The flexural rigidity of the beam is denoted by EI where E and I are Young's modulus and the mass moment of inertia, respectively.

Substituting Eq. (2) into Eq. (1) yields

$$EI\frac{d^4u}{dx^4} + f(u,x) = w(x) \tag{3}$$

where the boundary conditions are

$$u\,,\ u'\,,\ u''\,,\ \mathrm{u}^{(3)} \text{ and } u^{(4)} \to 0 \text{ as } |x| \to \infty\,. \tag{4}$$

Then Eq. (3) with boundary conditions (4) constructs a well-defined boundary value problem.

We now attempt to seek a nonlinear integral equation, which is equivalent to the nonlinear differential equation (1). We start with a simple modification as

$$EI\frac{d^4u}{dx^4} + ku + N(u,x) = w(x)\,, \tag{5}$$

where the nonlinear spring force is split into two parts:

$$f(u,x) = ku + N(u,x) \tag{6}$$

The first tem on the right, ku, is the linear portion of the reaction force $f(u,x)$ with the linear spring constant k, while $N(u,x)$ is the remaining part, i.e., nonlinear portion of the reaction force $f(u,x)$.

Rearranging the nonlinear term $N(u,x)$, we denote the new forcing term by $\Phi(u,x)$ for the simplicity of our notation as in

$$\frac{d^4u}{dx^4} + ku = \Phi(u,x), \text{ where } \Phi(u,x) \equiv w(x) - N(u,x). \tag{7}$$

The modified differential equation (7) is a starting point to the formulation of a nonlinear integral equation equivalent to the original equation (1). To that end, we first note that the linear solution of Eq. (1), which corresponds to a case with $N(u,x) = 0$ in Eq. (5), was derived by Timoshenko (1926), Kenney (1954), Saito & Murakami (1969), Fryba (1957), they used the Fourier and Laplace transforms to obtain a closed-form solution

$$u(x) = \int_{-\infty}^{\infty} G(x,\xi) w(\xi) d\xi, \tag{8}$$

where G is the following Green's function :

$$G(x,\xi) = \frac{\alpha}{2k} \exp\left(-\frac{\alpha|\xi - x|}{\sqrt{2}}\right) \sin\left(\frac{\alpha|\xi - x|}{\sqrt{2}} + \frac{\pi}{4}\right), \tag{9}$$

with parameter $\alpha = \sqrt[4]{k/EI}$. A localized loading condition was assumed in the derivation of

Eq. (8): u, u', u'', $u^{(3)}$ and $u^{(4)}$ all tend towards zero as $|x| \to \infty$. Green's function is the key to development of the method applied in this paper which plays a crucial role in obtaining the solution of linear differential equations equivalent to the 4th order differential equation in Eq. (7). Here, an infinite differentiability of solution regularity is concerned with the kernel of Green's function via Riemann integration. We extend the relation (8) to a nonlinear case in which the forcing term at the right hand side includes the solution itself. Such extension by a substitution of the Green's function (9) into the relation (8) results in the following nonlinear relation for the case of $N(u,x) \neq 0$:

$$u(x) = \int_{-\infty}^{\infty} G(x,\xi)\Phi(u(\xi),\xi)d\xi \qquad (10)$$

Substituting Eq. (7) into Eq. (10) reveals the following nonlinear Fredholm integral equation for u (Choi & Jang 2012):

$$u(x) = \int_{-\infty}^{\infty} G(x,\xi)w(\xi)d\xi - \int_{-\infty}^{\infty} G(x,\xi)N(u(\xi),\xi)d\xi \qquad (11)$$

The first integral $\int_{-\infty}^{\infty} G(x,\xi)w(x)d\xi$ in Eq. (11) amounts to the linear deflection of an infinite beam on a linear elastic foundation with an artificial linear spring constant k, which is uniform in x. The second term $-\int_{-\infty}^{\infty} G(x,\xi)N(u(\xi),\xi)d\xi$ corresponds to the nonlinear deflection of the beam.

3. Numerical Method

3.1 Iterative procedure for the solution

On the basis of the derived nonlinear integral equation (11), we apply a nonlinear solution procedure. We first define $\pi(x)$ and $\lambda(u)$ for notational simplicity as follows:

$$\pi(x) \equiv \int_{-\infty}^{\infty} G(x,\xi)w(\xi)d\xi \tag{12}$$

and

$$\lambda(u) \equiv -\int_{-\infty}^{\infty} G(x,\xi)N\left[u(\xi),\ \xi\right]d\xi . \tag{13}$$

Then, Eq. (11) is written in a simple form:

$$u(x) = \pi(x) + \lambda(u) \tag{14}$$

Now, the fixed-point iteration is applied to solve Eq. (11) or Eq. (14) as follows (Jang et al. 2011a) ,

$$\mathbf{u}_{n+1}(x) = \boldsymbol{\pi}(x) + \boldsymbol{\lambda}\left[\mathbf{u}_n(x)\right]. \tag{15}$$

The fixed-point iteration is a simple way to get an approximate solution and its convergence was studied by Choi & Jang (2012).

We need to restrict the physical domain $\mathbb{R} = (-\infty,\ \infty)$ to a feasible, computational one, because we cannot handle the infinite domain in a computer without an introduction of a proper nonlinear mapping between the physical domain and a finite computational one. Therefore, we conveniently take the domain $\mathrm{D} = [-R,\ \mathrm{R}]$ for a sufficiently large positive number R, which results in a good approximation of boundary conditions (4). Hence, the integral operator, Eq. (12) and Eq. (13) are computed on the truncated domain as

$$\pi(x) = \int_{-R}^{R} G(x,\xi) w(\xi) d\xi \tag{16}$$

and

$$\lambda(u) = -\int_{-R}^{R} G(x,\xi) N\left[u(\xi),\xi\right] d\xi \,, \tag{17}$$

respectively.

3.2 Discretization of nonlinear integral equation

A numerical integration rule being employed, the recursive equation (15) is discretized to

$$\mathbf{u}_{n+1}(x_i) = \boldsymbol{\pi}(x_i) + \boldsymbol{\lambda}\left[\mathbf{u}_n(x_i)\right], \quad i = 0,\ 1,\ \cdots,\ N \tag{18}$$

where

$$\boldsymbol{\pi}(x_i) = \sum_{j=1}^{N} w_{ij}\, G(x_i,\xi_j) w(\xi_j), \quad i = 0,\ 1,\ \cdots,\ N \tag{19}$$

and

$$\boldsymbol{\lambda}[u_n(x_i)] = -\sum_{j=1}^{N} w_{ij}\, G(x_i,\xi_j) N\left[u_n(\xi_j)\right], \quad i = 0,\ 1,\ \cdots,\ N\,. \tag{20}$$

Quadrature weights are denoted by w_{ij}. When the infinite limits in integrals (12) and (13) are replaced by a large value R, the number N for the summation in Eqs. (19)-(20) represents the number of grid points in the interval $(-R,\ R)$.

4. Numerical experiments

In this section, we examine the applied iterative method through computation to confirm the validity and efficiency. In order to give physically more realistic loading conditions, rectangle-type loadings are used for the simulations. As in the previous sections, an infinite beam is assumed to be on discontinuous and elastic foundations. Without loss of generality, a discontinuity in the nonlinear spring constant of the foundations is placed at $x = 0$.

4.1 Validity of the procedure

In order to demonstrate the accuracy of the applied procedure, we compute analytically a loading which results in our desired deflections $u(x)$ as follows:

$$u(x) = e^{-x^2/s}, \tag{21}$$

where s is a constant which determines how fast the exponential function decays; the smaller s, the faster $u(x)$ decays. We also assume a simple form of the nonlinear spring force $f(u,x)$ of a discontinuous elastic foundation as

$$f(u,x) = \begin{cases} k_1 u + \varepsilon u^p & \text{for } x < 0 \\ k_2 u + \varepsilon u^p & \text{for } x \geq 0 \end{cases}. \tag{22}$$

Firstly, the linear spring coefficients k_1 and k_2 are chosen as the same constant to investigate the validity of the applied method ($k_1 = k_2 = 2$). Other principal properties of the beam are $EI = 1$, $s = 100$ and $\varepsilon = 1/4$. In this section, we use 600 equally spaced grid points (or

$\Delta x = 0.1$) between $-30 \le x \le 30$. Eq. (3) gives an analytical form of the applied loading $w(x)$ for the given foundation model in Eq. (22):

$$w(x) = EI \cdot u^{(4)}(x) + k \cdot u(x) + \varepsilon \cdot u^3(x)$$

$$= \left(\varepsilon + k + \frac{12}{s^2} - \frac{48x^2}{s^3} + \frac{16x^4}{s^4} \right) \cdot e^{-x^2/s} \tag{23}$$

Deflection $u(x)$ in Eq. (21), reaction force $f(u,x)$ in Eq. (22), and the loading $w(x)$ in Eq. (23) satisfy the governing equation (3). Hence, selecting two of them determines the remaining one. We specify an easy $u(x)$ and $f(u,x)$ and then a complicated $w(x)$ in Eq. (23) is obtained. Using the nonlinear iterative procedure of Eq. (18), the recovered numerical solutions converge to the exact solutions and the error reaches a steady state within 20 iterations as shown in Figure 2. The $error(n)$ at the nth iteration step is defined as

$$error(n) = \frac{\|u - u_n\|_2}{\|u\|_2}, \text{ where } \|z\|_2 \equiv \left\{ \sum_{i=1}^{N} |z_i|^2 \right\}^{1/2}. \tag{24}$$

Secondly, the same numerical simulation is performed with discontinuous spring constants $k_1 = 1$ and $k_2 = 2$. The discontinuous loading and converged solution are plotted in Figure 3. Small errors are observed near the discontinuity showing undershoot and overshoot on the left ($k_1 = 1$) and right ($k_2 = 2$), respectively. The result is not contradictory to the previous analytic results presented in Choi & Jang (2012) since the differentiability assumption is not kept in this model. Hence, the contraction property of the integral operator formulation is not guaranteed in our example.

We take another deflection $u(x)=\sin x\cdot e^{-x^2/s}$ as the exact solution to a problem with a nonlinear spring model $f(u,x)=ku(x)+\gamma u^3(x)$. Using Eq. (3), an analytical expression of the corresponding loading is obtained as:

$$\begin{aligned}
w(x) &= EI\cdot u^{(4)}(x)+k\cdot u(x)+\gamma\cdot u^3(x)\\
&=\left(1+k+\frac{12}{s}+\frac{12}{s^2}-\frac{24x^2}{s^2}-\frac{48x^2}{s^3}+\frac{16x^4}{s^4}\right)\cdot e^{-x^2/s}\sin x\\
&\quad+\left(\frac{8x}{s}+\frac{48x}{s^2}-\frac{32x^3}{s^3}\right)\cdot e^{-x^2/s}\cos x\\
&\quad+\gamma\cdot e^{-3x^2/s}\sin^3 x
\end{aligned} \tag{25}$$

Figure 4 shows the iterative solutions and the relative errors ($k_1=k_2=1$).

In this section, we assume analytic solutions and obtain loadings via the governing equation Eq. (3) corresponding to the nonlinear spring force of a nonlinear elastic foundation. Using the nonlinear iteration in Eq. (18), we show that the recovered numerical solution converges to the assumed exact solution, which demonstrates the accuracy and the validity of the nonlinear iterative method. When the foundation is discontinuous, however, the proposed method does not converge to the exact solution in a small region around a point of discontinuity. Since the error is localized near the material discontinuity, numerical solutions of our proposed method are still meaningful.

4.2 Dependence on loading

We now investigate the convergence of the solution in response to the loadings. The applied loading conditions are varied to observe their effects on the deflection of an infinite beam which is resting on the same discontinuous elastic foundation of quadratic form in Eq. (22); $k_1=1$, $k_2=2$, $\varepsilon=1/6$, $p=3$ and $c=1.0$, 1.5 and 2.0 as listed in Table 1.

Since a complicated loading can be approximated by a linear combination of piecewise constant function, we use the following rectangle-type loading $w_{\text{rectangle}}(x)$ which is constant in a finite interval as

$$w_{\text{rectangle}} = \begin{cases} c, & -5 \le x \le 5 \\ 0, & \text{otherwise} \end{cases}. \tag{26}$$

In this study, we truncate the infinite spatial domain to $\mathrm{D} = [-20, 20]$, i.e., $R = 20$ and the number of the sub-interval is 400. Of course, this domain satisfies the boundary condition of Eq. (4). For the numerical quadrature, the Simpson's rule is employed. Figure 5 shows the nonlinear deflections using the iterative procedure in Eq. (18). We confirm that the deflection of the beam gets larger when the magnitude of the applied load increases while the convergence rate gets slower because of the increase of nonlinearity which is proportional to the load intensity.

4. 3 Mathematical models of nonlinear foundations

Here we examine the convergence of the iteration using three mathematically simple models of discontinuous nonlinear elastic foundation. We test our method with simple forms such as polynomials, trigonometric functions, and exponential functions whose combinations can easily represent more complicated and realistic nonlinear foundations.

Model 1: We use the discontinuous nonlinear elastic foundation model that consists of linear and nonlinear polynomials as introduced in Eq. (22). The nonlinearity in this model does not depend on the spatial variable x explicitly but implicitly through the non-uniform displacement $u(x)$.

Model 2: The nonlinear spring force has both a linear and a sinusoidal term,

$$f(u,x)=\begin{cases} k_1 u+\varepsilon u^p \cos x & \text{for } x<0 \\ k_2 u+\varepsilon u^p \cos x & \text{for } x\geq 0 \end{cases}, \tag{27}$$

where k_1 and k_2 are linear spring coefficients and the $\varepsilon u^p \cos x$ is nonlinear spring force with a coefficient $\varepsilon > 0$. The nonlinear term $\varepsilon u^p \cos x$ has explicit dependency on the spatial variable x and the displacement u. In this, nonlinearity can reduce the reaction force because $\cos x$ can be negative, which may not be physical. However, we just want to point out that it is positive around the discontinuity $x = 0$ and the model is purely mathematical, designed for the demonstration of our method.

Model 3: The final model is as follows:

$$f(u,x)=\begin{cases} k_1 u+\varepsilon u^p e^{-x^2/\sigma} & \text{for } x<0 \\ k_2 u+\varepsilon u^p e^{-x^2/\sigma} & \text{for } x\geq 0 \end{cases}, \tag{28}$$

where the linear spring constants k_1 and k_2. The mathematical form of nonlinear elastic foundation is the linear and the combination of linear and exponential forms. The nonlinearity in this model is concentrated around $x = 0$ and the parameter σ controls how far the nonlinearity extends.

The numerical solutions are found for the above three different types of discontinuous elastic foundation in order to demonstrate the capability of the applied method. We also focus on the behavior of the solution as the discontinuity in the linear term at the origin, i.e., $x = 0$, grows.

4.3.1 Model 1

To find the validity of the solution, we model a system of an infinite beam on the discontinuous elastic foundation of model 1 in Eq. (22) with the tabulated properties of Table 2. Figure 6 demonstrates the comparisons of the iterative solutions and shows the convergence behaviors of the iterative solutions according to the loading conditions as listed in Table 2.

In Figure 6, it is noticeable for the iterative solution to overshoot in the first step and then undershoot in the second. From the third step, it is already close to the final state. The relative error norm shows that the iterative procedure converges very quickly within 10 iterations for test cases in Table 2. When the coefficient k_2 increases from 1 to 3, the symmetry around the origin breaks down, which agrees with our intuition. The displacements on the positive x - axis decrease as the reaction forces from the foundation increase. The differences between polynomial order 2 and 3 are not noticeable because the nonlinear reaction term does not make much difference in the range of displacements.

4.3.2 Model 2

Using Eq. (27), the linear plus sinusoidal form of the elastic foundation is applied as $c=1$, $k_1=1$, $\varepsilon=1/6$, $p=3$ and the parameter $k_2=1$, 2, and 3. All test cases are listed in Table 3 and we look at the effect of discontinuity in the linear term. As observed in the previous model 1, introduction of discontinuity term breaks the symmetry in the displacements. As the coefficient k_2 in the positive x - axis increases, the displacements get smaller, which agrees to our intuition. The numerical solution converges very fast within 10 iterations and changes beyond 5 iterations are almost negligible as shown in Figure 7. The iterative solution does not show overshoot and undershoot over the first few iterations, which is also observed in the cases of Model 1.

4.3.3 Model 3

Using Eq. (28), the linear plus exponential form of the elastic foundation is investigated: $c=1$, $k_1=1$, $\varepsilon=1/4$, $p=3$, $\sigma=4$ and the parameter $k_2=1$, 2 and 3 which are summarized in Table 4. The iterates at different steps are shown in Figure 8. The trends of changes in the displacements are similar to that of Model 1 when the discontinuity is introduced to the foundation model.

Numerical simulations with above three models clearly show that our methods can handle nonlinear elastic foundations in the simple mathematical forms, which are not continuous at a discrete point. More complicated foundations and loading can be easily expressed with combinations of the simple forms. Hence, numerical solutions are obtained similarly.

5. Concluding remarks

The highly nonlinear problem of the Euler beam sitting on a nonlinear elastic foundation with discontinuities is solved numerically. Various numerical simulations are performed with the discontinuous nonlinear spring forces of the elastic foundations. For the application of the procedure, the nonlinear differential equation is transformed to the equivalent nonlinear integral one and then the integral operator is recognized. Our current study shows the feasibility of the iterative procedure as well as its convergence. Accuracy of numerical solutions is confirmed with analytical solutions. Convergence to accurate solutions is achieved with 10-20 iterations in all simulations. Regardless of loading conditions, accurate numerical solutions are obtained. The effect of a material discontinuity on the nonlinear deflections is localized numerical errors near the material discontinuity, deserving further investigation.

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Table 1 3 cases of loading conditions for the elastic foundations with cubic nonlinear term

Cases	c	k_1	k_2	ε	p
a	1.0	1	2	1/6	3
b	1.5	1	2	1/6	3
c	2.0	1	2	1/6	3

Table 2 Test cases of model 1 with the elastic foundations of the monomial form

Cases	c	k_1	k_2	ε	p
a	1	1	1	1/4	2
b	1	1	2	1/4	2
c	1	1	3	1/4	2
d	1	1	1	1/4	3
e	1	1	2	1/4	3
f	1	1	3	1/4	3

Table 3 Test cases of model 2 with the elastic foundations of the sinusoidal form

Cases	c	k_1	k_2	ε	p
a	1	1	1	1/6	3
b	1	1	2	1/6	3
c	1	1	3	1/6	3

Table 4 Test cases of model 3 with the elastic foundations of the exponential form

Cases	c	k_1	k_2	ε	p	σ
a	1	1	1	1/4	3	4
b	1	1	2	1/4	3	4
c	1	1	3	1/4	3	4

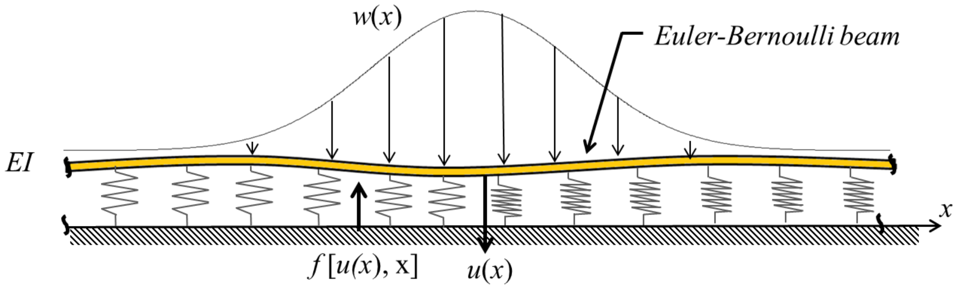


Figure 1 An infinite beam on a discontinuous nonlinear elastic foundation

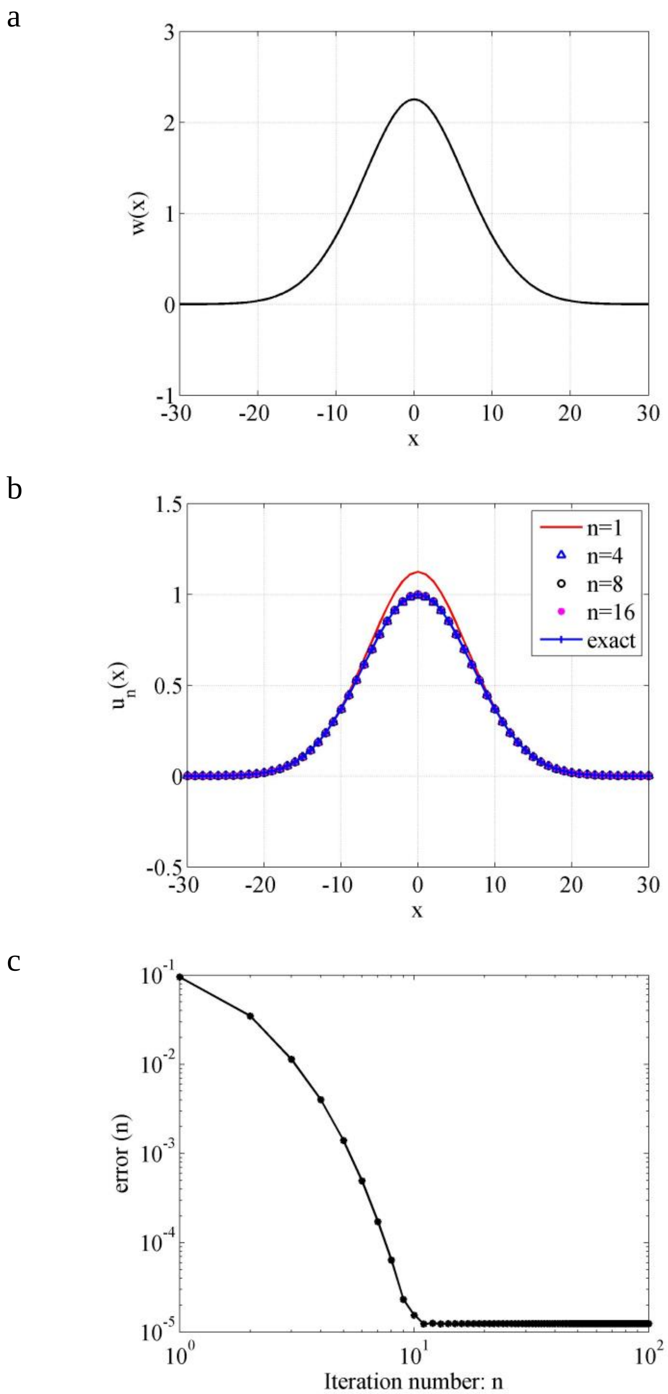


Figure 2 Validity test: (a) localized loading, (b) iterative solutions, and (c) relative errors.

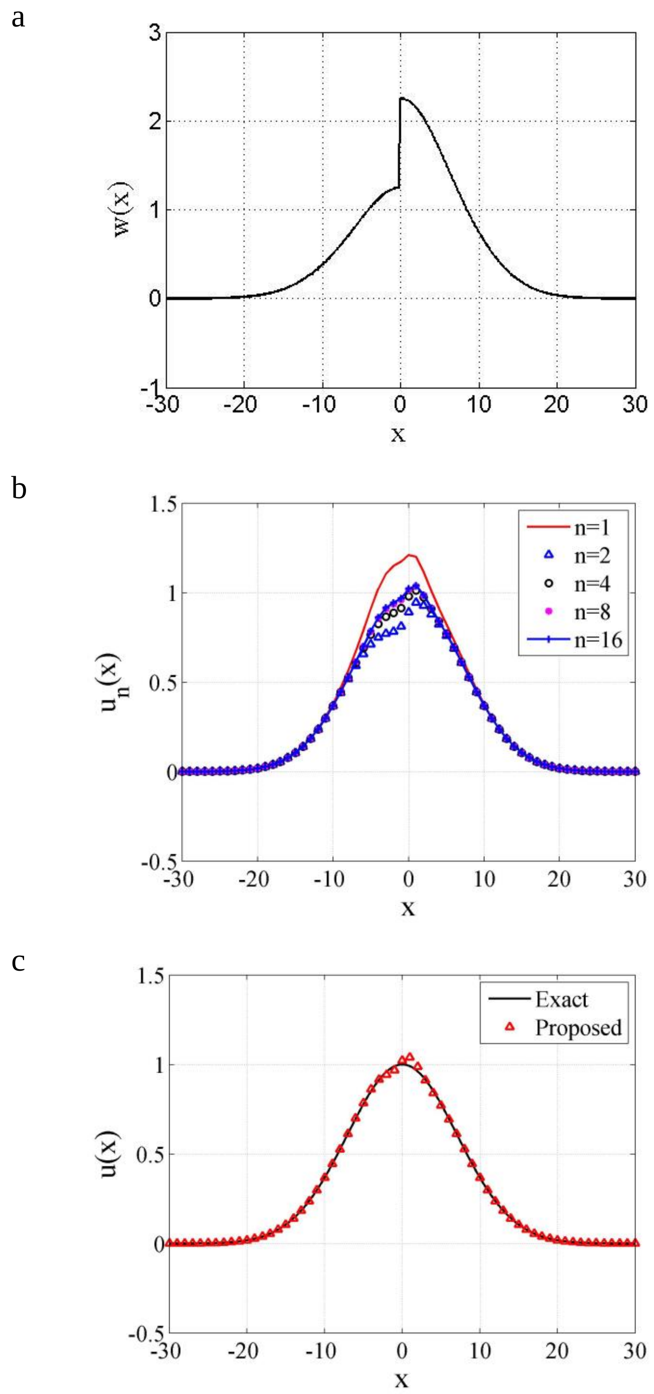


Figure 3 Accuracy test with discontinuity in the nonlinear foundation: (a) discontinuous applied loading due to the material discontinuity, (b) iterative solutions, and (c) comparison of exact solution and converged solution.

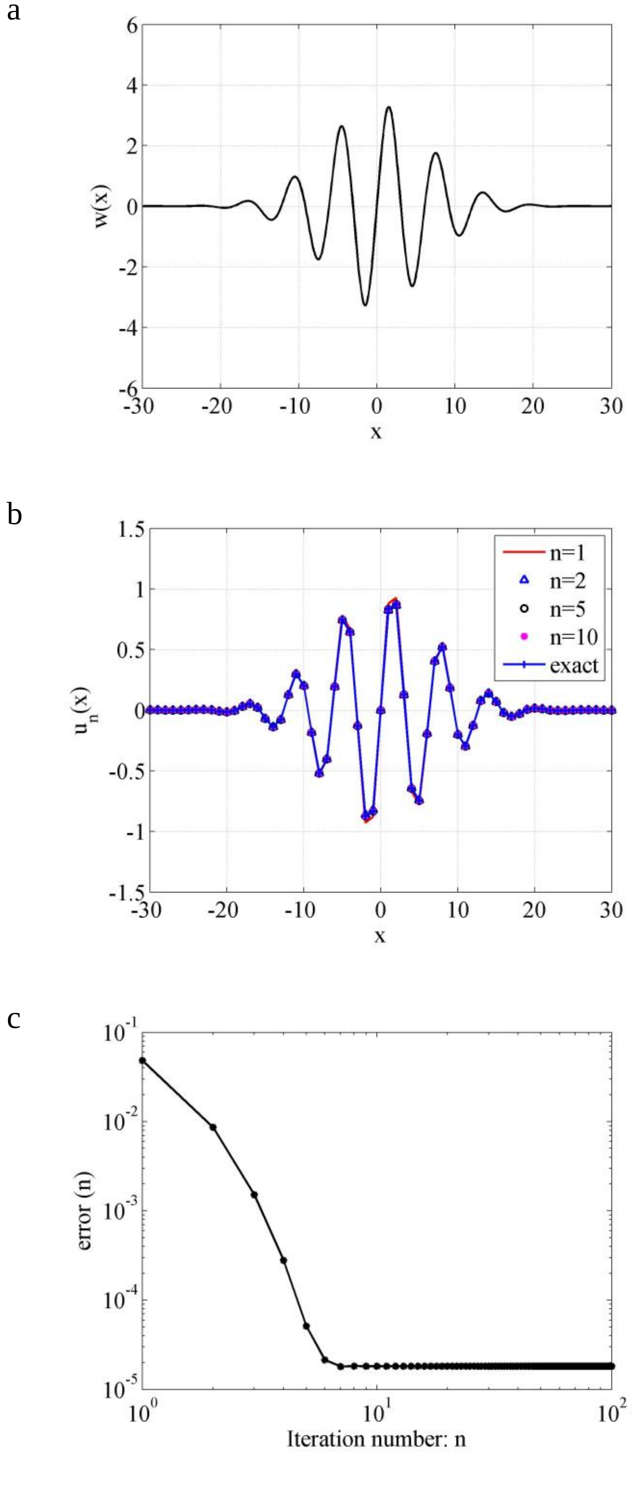
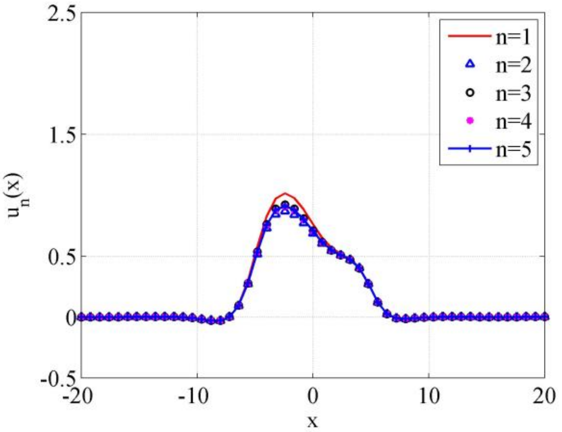
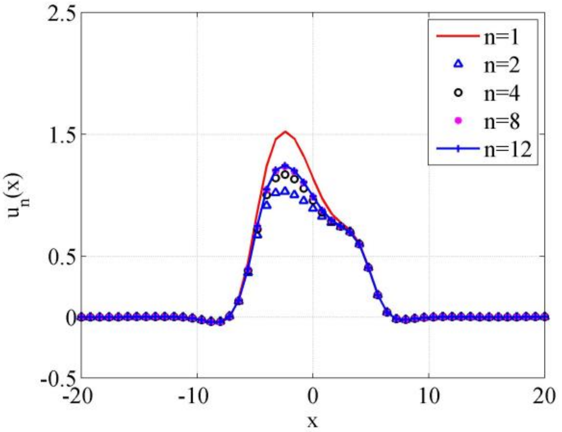


Figure 4 Validity test with uniform nonlinear elastic foundation: (a) applied loading, (b) iterative solutions, and (c) relative errors

a



b



c

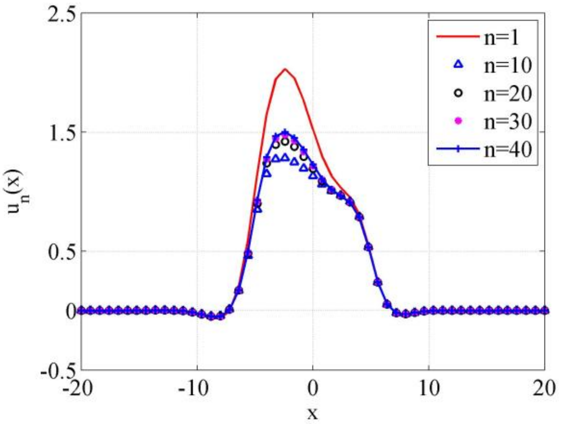
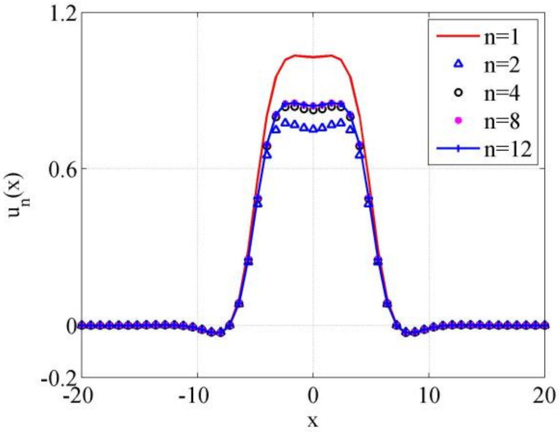
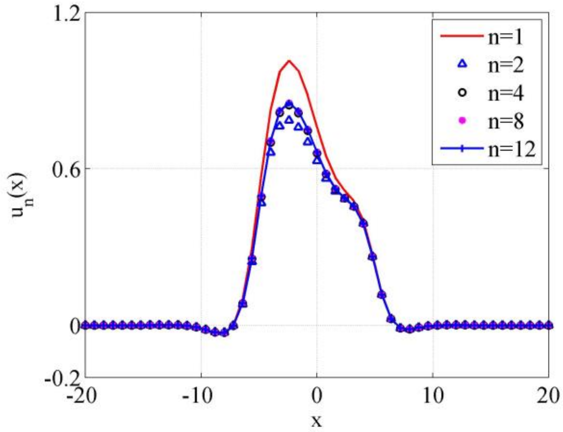


Figure 5 Behaviours of convergence according to 3 different loading conditions $c = 1.0$, 1.5 and 2.0 with other parameters in Table 1.

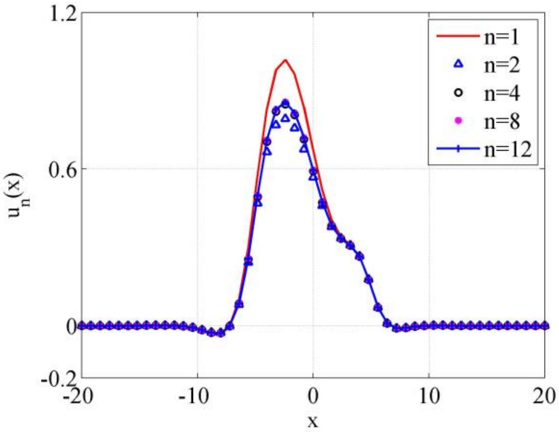
(a) $k_1 = k_2 = 1$



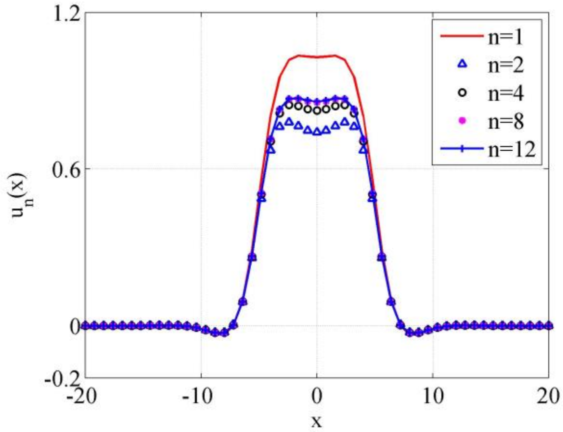
(b) $k_1 = 1,\ k_2 = 2$



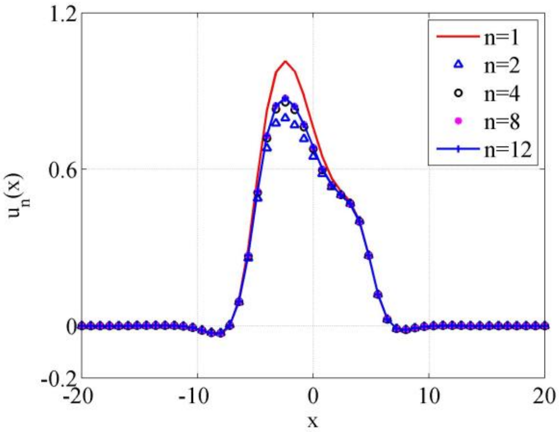
(c) $k_1 = 1,\ k_2 = 3$



(d) $k_1 = k_2 = 1$



(e) $k_1 = 1,\ k_2 = 2$



(f) $k_1 = 1,\ k_2 = 3$

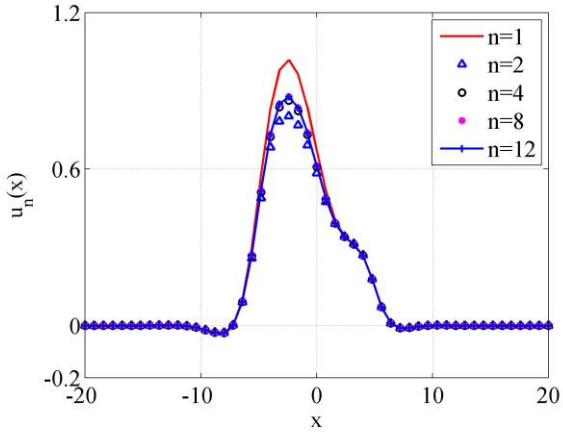


Figure 6 Convergence behaviours of cases in Table 2

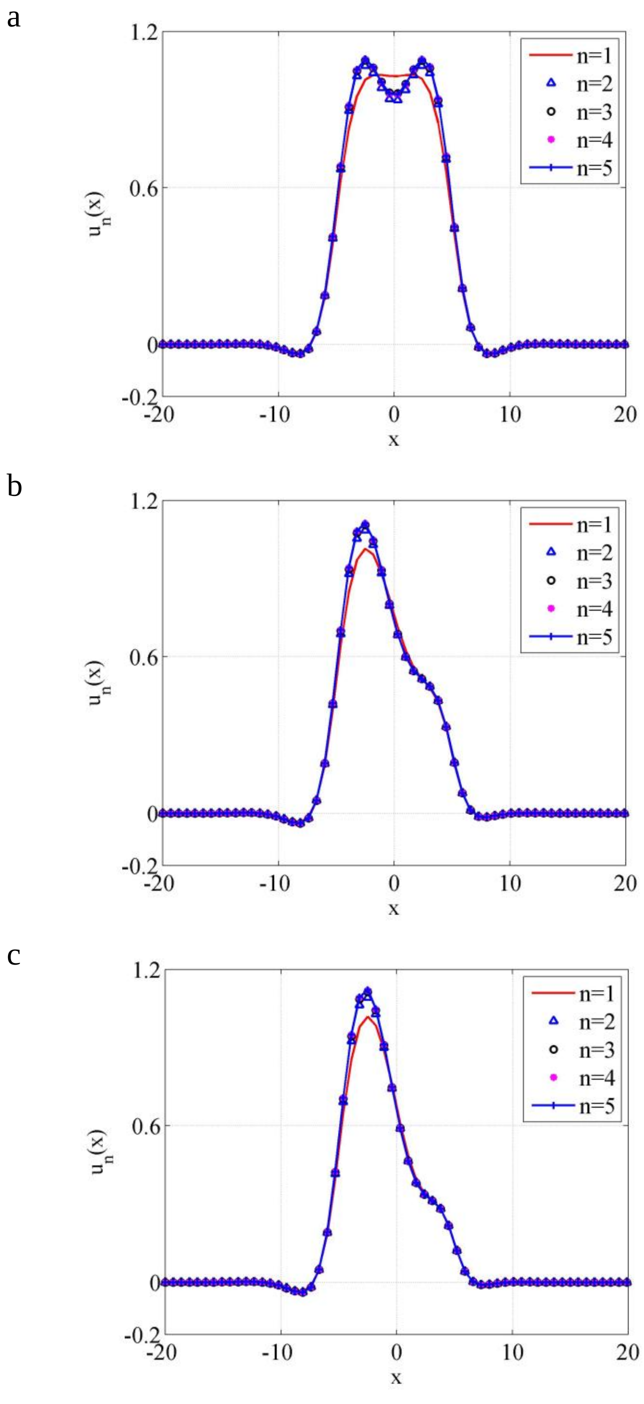


Figure 7 Convergence behaviours of cases in Table 3: (a) $k_1 = k_2 = 1$ (b) $k_1 = 1,\ k_2 = 2$ (c) $k_1 = 1,\ k_2 = 3$

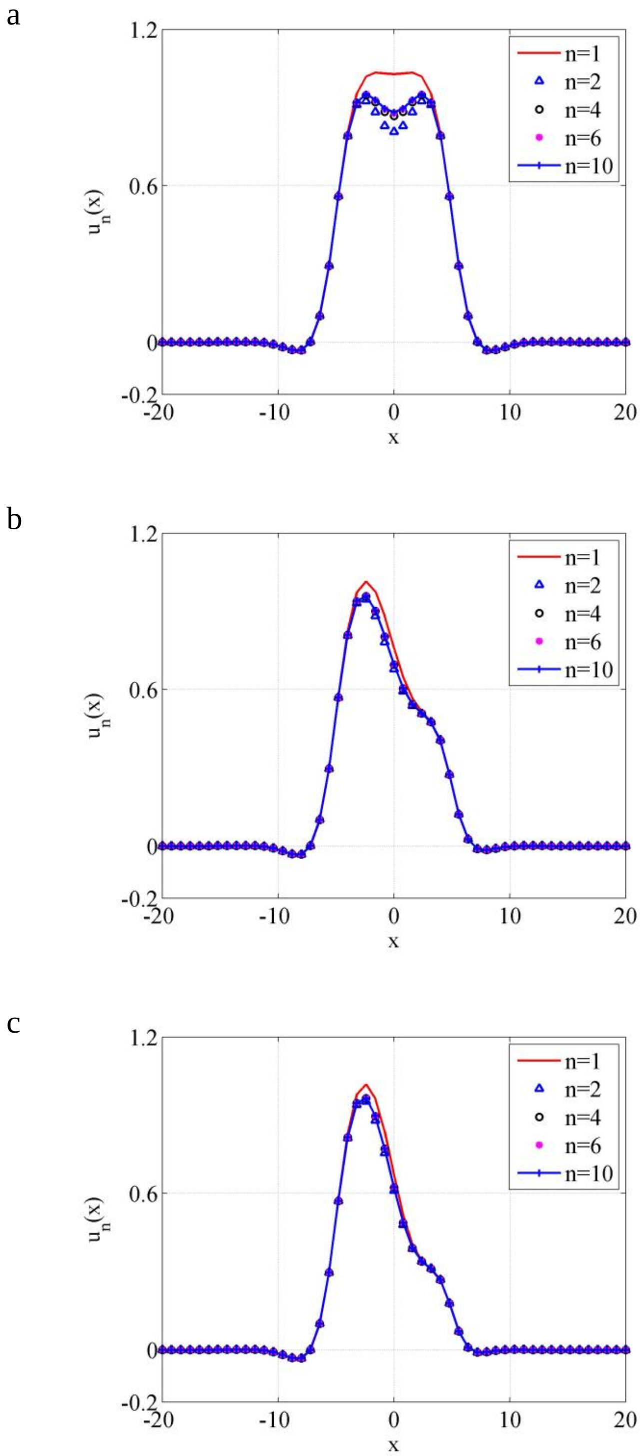


Figure 8 Convergence behaviours of cases in Table 4: (a) $k_1 = k_2 = 1$ (b) $k_1 = 1,\ k_2 = 2$ (c) $k_1 = 1,\ k_2 = 3$