Local interactions of the quantised electromagnetic field

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Declaration

The candidate confirms that the work submitted is his own, except where work which has formed part of jointly authored publications has been included. The contribution of the candidate and the other authors to this work has been explicitly indicated below. The candidate confirms that appropriate credit has been given within the thesis where reference has been made to the work of others.

Some of the work in Chapters 4 and 5 is based on the publication Southall *et al.* (2021), which was carried out in collaboration with Almut Beige, Daniel Hodgson and Robert Purdy.

Some of the work in Chapter 4 is based on the submitted article Hodgson *et al.* (2021), which was carried out in collaboration with Almut Beige, Daniel Hodgson and Robert Purdy.

Life is not a problem to be solved, but a reality to be experienced.

- Søren Kierkegaard

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Abstract

Photons, i.e. the basic energy quanta of monochromatic waves, are highly non-localised and occupy all available space in one dimension. This non-local property can complicate the modelling of the quantised electromagnetic field in the presence of optical elements that are local objects. Therefore, this thesis takes an alternative approach and shows that a local second quantisation of the electromagnetic (EM) field is possible but requires an extension of conventional quantum theory. For light propagating in one dimension, we obtain highly localised bosonic Fock operators, which we do by doubling the usual photon Hilbert space with some photonic modes evolving according to the standard Schrödinger equation and others evolving according to the complex conjugated Schrödinger equation. We also view the quantised EM field as a biorthogonal system. However, we view it as a biorthogonal system where the intersection of the Hilbert space and its dual Hilbert space is non-zero. To the best of our knowledge, this is the first time such a construction of the EM field has been made. These highly localised bosonic Fock operators provide natural building blocks of wave packets of light and enable us to construct locally acting interaction Hamiltonians for two-sided semi-transparent mirrors. Using these Hamiltonians, we produce appropriate classical dynamics of the electric field near a mirror. The question of how to model local transformations of the EM field is a hot topic, as physicists often measure interactions between the EM field and local optical devices in experiments. Therefore, we expect our results to find large appeal across both the quantum optics and non-Hermitian communities. We finish by discussing possible future avenues of research.

Abbreviations

c	Speed of light
$\hbar$	Reduced Plank constant
ϵ_0	Permittivity in free space
μ_0	Permeability in free space
π	Pi
EM	Electromagnetic
H.c.	Hermitian conjugate

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Part I

Background

Chapter 1

Introduction

1.1 Motivation

What do photons and small children have in common? We cannot get them to stay in one location. Further still, it is often challenging to assign a location to a photon in the first place. Because photons are the quanta of light, this apparent lack of locality can make it difficult to model the interactions of light with local objects, such as mirrors, at the quantum level. This research therefore aims to construct a theory that allows us to model local interactions of the quantised electromagnetic (EM) field. This chapter will introduce our study by discussing the background and context, the research problem, our aims and questions, the significance, and the limitations[1].

When one reads the phrase "location of an object", what comes to mind is likely asking how we can assign this object a property to describe where it is in some reference frame. For example, a point's coordinate in 3-D Euclidean space, the coordinate of an electron in space-time, or the location of a football on a football pitch. When we say "local transformation of an object", what comes to mind is likely a process involving changing the property that describes the object's location. Using the examples above, this could be the rotation of a point about the origin in Euclidean space, the boost of an electron along the x-axis in space-time, or the kicking of a football on a football pitch. Of course, in this

[1]Disclaimer: Our research cannot be used to help small children stay in one place.

thesis we want to model local interactions of the EM field. We therefore describe what we want a local transformation of the EM field to look like.

In classical electrodynamics, we often characterise light by its local properties such as local amplitudes, direction of propagation and polarisation. Its fundamental equations of motion – Maxwell's equations – are local differential equations. Practically, we assume that the classical EM field comprises a continuum of local field excitations. In contrast, quantum electrodynamics routinely decomposes the EM field into monochromatic waves, which are called monochromatic photons and are highly non-local. A key reason for this decomposition is that these monochromatic waves have well-defined energies and so can be helpful for modelling interactions between the EM field and objects where energy conservation is concerned. For example, the emission of a photon when an excited atom drops to its ground state (Gerry & Knight, 2004). However, such a non-local approach can result in more complicated equations of motion than strictly necessary. For example, the Green's functions of macroscopic quantum electrodynamics correlate an observer's position with all spatial positions and photon frequencies (Buhmann, 2013; Philbin, 2010; Scheel & Buhmann, 2009)[1]. Therefore, this thesis takes an alternative approach and quantises the EM field in position space. As in classical electrodynamics, our equations of motion only depend on local properties. Hence, we expect them to find many applications, for example, in modelling systems involving local light-matter interactions or featuring ultrabroadband photonic wave-packets (Javid *et al.*, 2021; Nasr *et al.*, 2008; Okano *et al.*, 2015; Tanaka *et al.*, 2012).

In quantum physics, we usually represent physical quantities, such as the energy of a system, by mathematical objects known as observables. These observables are a certain class of mathematical objects called operators. This representation applies to the electric field at a point $x \in \mathbb{R}$ in 1-dimensional space, which we represent by the operator $\mathbf{E}(x)$.

Furthermore, we can often represent observables by the sum of operators known as creation and annihilation operators. Suppose $a^a(k)$ and $a^c(k)$[2], where

[1] That's not to say Green's functions aren't useful; they have been found to be advantageous in the quantisation of the EM field within certain dielectrics (Gruner & Welsch, 1996).

[2] The reason for this non-standard notation will become clear shortly.

k is a continuous variable, are two operators that satisfy the commutation relation

$$[a^a(k), a^c(k')] = \delta(k - k')\,. \tag{1.1}$$

Then we can call $a^c(k)$ a bosonic creation operator and $a^a(k)$ its corresponding bosonic annihilation operator. We explain the properties of these operators more in the main sections of this thesis, but for now, what is important to know is that Eq. (1.1) is zero if $k \neq k'$ and non-zero if $k = k'$.

For a given quantum system, an operator known as the Hamiltonian, $H(t)$, governs time evolution. Hamiltonians that are Hermitian satisfy the equation

$$H(t) = H^\dagger(t)\,, \tag{1.2}$$

where the $\dagger$ symbol denotes the Hermitian conjugate, as this leads to unitary dynamics, which we discuss more in the main text. Whats important to know now is that unitary dynamics are important for maintaining a probabilistic interpretation of quantum theory. The Hermitian conjugate of an operator is closely related to the system's inner product, $\langle , \rangle$. The inner product is a property of the Hilbert space, a mathematical space we use to model our system. In conventional quantum physics, the Hermitian conjugate of an operator is equal to its complex conjugate transpose and, to be consistent, we reserve the $\dagger$ notation only for the conventional inner product, unless we explicitly state otherwise. In what we call the Heisenberg picture, all time-dependence goes into the observables. If the time dependence of an observable, $A(t)$, is due only to a system's Hermitian Hamiltonian, its time derivative is

$$\frac{\mathrm{d}}{\mathrm{d}t}A(t) = -\frac{i}{\hbar}[A(t), H(t)]\,. \tag{1.3}$$

This equation means that solving the dynamics of operators will involve computing the commutator between these operators and the systems' Hamiltonian.

Light is bosonic and this has been shown in experiments (English *et al.*, 2010; Hong *et al.*, 1987). Suppose, then, that we can construct the electric field operator, $\mathbf{E}(x)$, out of a sum of bosonic creation and annihilation operators that take the position coordinate x as their continuous variable so that

$$\mathbf{E}(x) = \sum_{i=0}^{N} b_i a_i^a(x) + c_i a_i^c(x)\,, \tag{1.4}$$

where b_i and c_i are constants, and N is a positive integer. If we can do this we can then construct Hermitian Hamiltonians out of annihilation/creation operators corresponding to the creation/annihilation operators in the electric field operator to utilise the commutation relation Eq. (1.1) and obtain transformations that only act at certain locations.

Unfortunately, in the standard description the electric field operator in 1-dimension is proportional to (Bennett *et al.*, 2015)

$$\mathbf{E}(x) \propto \sum_{s,\lambda} a_{s\lambda}(x) + a^{\dagger}_{s\lambda}(x)\,, \tag{1.5}$$

where

$$[a_{s\lambda}(x), a^{\dagger}_{s\lambda}(x')] \neq \delta(x - x')\,. \tag{1.6}$$

Here s denotes the direction of propagation, in either the positive or negative direction, and λ denotes the polarisation, which we can label either horizontal, H, or vertical, V. Here, the $\dagger$ superscript corresponds to the Hermitian conjugate in conventional quantum physics. We describe the electric field operator in much more detail in the following sections. However, for now, we point out that the above commutation relation means that $a_{s\lambda}(x)$ is not the annihilation operator of its Hermitian conjugate, $a^{\dagger}_{s\lambda}(x)$, which makes it difficult to construct conventionally Hermitian Hamiltonians that alter the EM field only at specific locations. To construct such Hamiltonians, we require two steps. First, we will need to construct the creation/annihilation operators corresponding to the annihilation/creation operators in Eq. (1.5), $a_{s\lambda}(x)$ and $a^{\dagger}_{s\lambda}(x)$. We will achieve this by introducing monochromatic photon modes that evolve according to the complex conjugate Heisenberg equation, which, as we will see, is equivalent to introducing negative frequency modes in free space[1]. Second, we then need to identify a system where the creation and annihilation operators corresponding to $a_{s\lambda}(x)$ and $a^{\dagger}_{s\lambda}(x)$ are also their respective Hermitian conjugates, and that they evolve with appropriate dynamics. We will achieve this by modelling the EM field as a biorthogonal system (Brody, 2013). Biorthogonal quantum mechanics is a form of non-Hermitian physics where a dual Hilbert space is used in addition to

[1] We *do not* localise monochromatic photons. However, we use them to construct localised states.

the original Hilbert space for calculations on a system, and we review the details of such a system in the next chapter. However, we find that to achieve appropriate dynamics for the EM field observables, we have to model the EM field as a biorthogonal system that has a non-zero intersection between its Hilbert space and dual Hilbert space. To the best of our knowledge, this is the first time such a construction of the EM field has been made.

The construction of such local creation/annihilation operators is easier said than done, however. For example, there are certain no go theorems such as the Hegerfeldt (1998a) and Malament (1996) theorems that put constraints on locality. The Hegerfeldt (1998a) theorem, in particular, asserts that even if a state is initially localised to some region V, at a time $t > 0$ the probability to detect it anywhere outside the region V will be non-zero if the Hamiltonian is positive and evolves according to the Schrödinger equation (Hegerfeldt, 2001). The Schrödinger equation governs how states evolve in the Schrödinger picture where all time-dependence from the Hamiltonian goes into the states.

Furthermore, they are not an entirely new concept, either (Bialynicki-Birula, 1996; Chan *et al.*, 2002; Cook, 1982a; Hawton & Debierre, 2017; Raymer & Walmsley, 2020; Sipe, 1995; Smith & Raymer, 2007). For example, Smith & Raymer (2007) introduced modes of the electric field that can be local under an inner product that is adjusted from the conventional one. However, their modes fit the conditions for Hegerfeldt's theorem to be applied so that their modes do not remain localised in time. Also, adjusting the inner product may change the normalisation of certain states. For example, a property of monochromatic photons is that they are pair-wise orthonormal to each other with respect to the standard inner product. In this thesis, we wish to preserve this property as it allows us to construct states that behave classically. Hawton & Debierre (2017) used negative frequencies to overcome the Hegerfeldt theorem and biorthogonal quantum physics (Brody, 2013) to construct local states of the EM field. However, we show that the concept of negative frequencies can only describe certain local interactions. We also show we need a biorthogonal system with a nonzero intersection between its Hilbert space and dual Hilbert space to calculate EM expectation values near a semi-transparent mirror. Hawton & Debierre (2017) also uses a time-dependent inner product in what is called the interaction picture. It has

often been debated if an interaction picture can exist (Earman & Fraser, 2005). Discussing the existence of this picture is not the purpose of this thesis, but, in any case, it would be helpful to have an inner product that does not require us to be in a particular picture or be time-dependent. Fermi (1932) also used negative frequencies to overcome a causality problem between the interaction of two atoms via radiation. There, he claimed that the negative terms added were negligible, but we now know that they were vital for his proof (Hegerfeldt, 1994).

As photons are chargeless, they are identical to their antiparticles, anti-photons (Bialynicki-Birula, 1996). This is perhaps why photons with negative frequencies, or that evolve according the complex conjugate Schrödinger equation, are often neglected. Like Hawton & Debierre (2017) did with negative frequencies, we show that photons with complex conjugated dynamics can be used to overcome the Hegerfeldt (1998a); Malament (1996) locality no-go theorems in quantum field theory.

Therefore, this thesis will aim to construct the EM field observables out of bosonic operators such that we do not violate key no-go theorems, we do not require a time-dependent inner product and can operate in any picture. Having local bosonic operators also provide natural building blocks of wave packets of light as they help us to construct states, $|1_{s\lambda}(x)\rangle$, such that

$$\langle 1_{s\lambda}(x)|1_{s\lambda}(x')\rangle = \delta(x-x')\,. \tag{1.7}$$

Therefore, they provide natural building blocks of wave packets of light that are localised within a certain region.

To verify its utility, we shall use it to model a physical scenario where the EM field is interacting with two-sided semi-transparent mirror. We focus on two-sided semitransparent mirrors as this topic has already attracted a lot of interest in the literature and they are devices that are often used in optical cavities (Agarwal, 1975; Carniglia & Mandel, 1971; Collett & Gardiner, 1984; Creatore & Andreani, 2008; Dalton *et al.*, 1999; Dawson *et al.*, 2020, 2021; Dilley *et al.*, 2012; Furtak-Wells *et al.*, 2018; Gardiner & Collett, 1985; Glauber & Lewenstein, 1991; Huttner & Barnett, 1992; Knöll *et al.*, 1987; Meschede *et al.*, 1990; Wang *et al.*, 2021). In addition to using classical Green's functions (Buhmann, 2013; Gruner & Welsch, 1996; Philbin, 2010; Scheel & Buhmann, 2009), it is possible

to describe semi-transparent mirrors by restricting the Hilbert space of the EM field onto a subset of so-called triplet modes (Carniglia & Mandel, 1971). These consist of incident, reflected and transmitted waves and can be used to reproduce the well-known classical dynamics of field expectation values for light approaching a semi-transparent mirror from one side. However, they did not give a locally acting Hamiltonian for the mirror and cannot describe situations in which wave packets approach a mirror surface from both sides without resulting in the prediction of nonphysical interference effects (Żakowicz, 1995). Some authors therefore prefer phenomenological approaches such as the input-output formalism (Collett & Gardiner, 1984; Dilley *et al.*, 2012; Gardiner & Collett, 1985) or a quantum mirror image detector method that maps light scattering by semi-transparent mirrors onto analogous free-space scenarios (Dawson *et al.*, 2020, 2021; Furtak-Wells *et al.*, 2018). Although these models describe well the experiments that they have been designed for, they have not been derived from basic principles.

The mirror image method of classical electrodynamics simply describes light scattering by replacing any wave packet which comes in contact with the scattering object, at least partially, by its mirror image (Jackson, 1975). For semi-transparent mirrors, the mirror image is a wave packet with reduced field amplitudes which travels in the opposite direction and seems to emerge from the other side. In this thesis, we take a similar approach. We quantise the EM field in terms of local bosonic operators starting from the standard description of the quantised EM field. Afterwards, we construct locally-acting mirror Hamiltonians and show that these reproduce well-known classical dynamics. For example, they can cause a complete conversion of incoming into outgoing wave packets without altering the dynamics of outgoing wave packets. To relate further to classical dynamics, we show how these Hamiltonians reproduce the classical electric field expectation values, using coherent states, near a semi-transparent mirror. We achieve both of these with exactly solvable systems. Since most quantum systems have a Hamiltonian, the same should apply to optical elements.

The research we present will contribute to the body of knowledge on local interactions of the EM field, shed new light on how we can model experimental scenarios, and describe how to utilise a peculiar non-Hermitian construction.

Therefore, we expect our results to find large appeal across both the quantum optics and non-Hermitian communities.

We have kept the research in this thesis to light propagation in one spatial dimension, but work on generalising to three spatial dimensions is ongoing.

1.2 Thesis outline

Chapter 2 reviews some properties of quantum physics. Chapter 3 reviews the quantised EM field in free space. In chapter 4, we describe how to model the free space EM field as a biorthogonal system, including how to define appropriate dynamical equations and an appropriate inner product. This also includes how to construct the EM field operators using local bosonic modes. Chapter 5 describes how to solve certain bosonic systems and model light scattering by a two-sided semi-transparent mirror. We conclude this thesis in chapter 6.

Chapter 2

Quantum theory

This chapter reviews some of the basic foundations of quantum theory. We begin by taking a whistle-stop tour of quantum theory's development, starting with quantum mechanics in the early 1900s and its mathematical formulation. Next, we review the postulates of quantum theory, the uncertainty principle, and the quantum harmonic oscillator and its coherent states. Last, we review some aspects of biorthogonal quantum (Brody, 2016) and pseudo-Hermitian (Mostafazadeh, 2010) physics. The theory we review in this section will give us the tools required for our results later in this thesis.

2.1 Basic fundamentals of quantum theory

2.1.1 In the beginning

Around 120 years ago, quantum theory was born through the collection of ideas from many outstanding scientists. It is often believed, and commonly still taught in schools, that Max Planck first set quantum theory in motion in 1900. It was during this year that he first presented his work on black-body radiation using the concept of energy quanta, which leads to avoiding the so-called ultraviolet catastrophe in classical physics (Klein, 1962).

However, whether it was Planck that truly founded quantum theory is hard to say, as it has been claimed that his idea of quantum theory is a lot different from how we see it today. Around the same time, others had similar, perhaps influenced ideas, most notably Bohr and Einstein (Kragh, 2000). As for who was the true founder of quantum theory, if such a founder exists, who can say?[1] What is objectively clear, however, is that many scientists and philosophers have contributed to the field and have made it what it is today — an extremely useful theory to understand our physical universe (Dirac, 1925; Schrödinger, 1926).

We do not claim to hold the truth of what the exact interpretation of quantum theory should be. However, in typical fashion, in this thesis we interpret quantum theory as a probabilistic theory, in which we have a well-defined set of rules, or formalism, to follow in performing our calculations. We then use postulates to correspond to how this theory describes physical reality with calculations (Born, 1969; Faye, 2008). People may have subjective interpretations, but (ideally) we want objective calculations[2]. This interpretation is not the only interpretation of quantum theory. For example, it could be said that it is a kinematic theory in which physical states are wave functions (Carroll, 2020; Crease, 2019; Wilce, 2010). However, some say that states in quantum theory are just a construct of the observer and not an objective state in reality because measurements alone cannot determine the state (Hartle, 1968). For example, if we measure the position and momentum of a classical state, we have only one state that corresponds

[1] I certainly cannot answer that.

[2] Two parties may disagree on blue paint being suitable for a kitchen wall, but they can both agree that the paint is blue (relativity not taken into account).

to our measurement outcome. In contrast, if we take the same measurements on a quantum state, we can have many states that could give us the same measurement outcomes. We can still deduce what quantum state we have in a system — we just need to know how it was prepared.

We will not dwell on the philosophical pondering on what quantum theory is — this a theoretical physics thesis, not a philosophy one — in what follows, we shall review the formalities of quantum theory we need for the calculations in this thesis. In a fashion that the reader should find very familiar and similar to that found in many quantum textbooks (Griffiths & Schroeter, 2018; Nielsen & Chuang, 2011).

2.1.2 The Hilbert space

In quantum physics, we often use a mathematical structure called a Hilbert space to represent a system. The elements of the Hilbert space are known as state vectors and, for a given Hilbert space, these state vectors represent possible states of the system. We will refer to state vectors simply as states.

A Hilbert space, $\mathcal{H}$, is an inner product space so that it associates with a pair of states, $|\psi\rangle, |\phi\rangle \in \mathcal{H}$, a complex number via an inner product, which is denoted by $\langle |\psi\rangle, |\phi\rangle\rangle$. This inner product satisfies the following properties:

1. The inner product of a pair of states is equal to the complex conjugate of the swapped states. This means $\langle |\psi\rangle, |\phi\rangle\rangle = \langle |\phi\rangle, |\psi\rangle\rangle^*$ where the asterisk denotes complex conjugation.

2. The inner product is linear in its first argument. This means $\langle a|\psi\rangle_1 + b|\psi\rangle_2, |\phi\rangle\rangle = a\langle |\psi\rangle, |\phi\rangle_1\rangle + b\langle |\psi\rangle_2, |\phi\rangle\rangle$ where $a, b \in \mathbb{C}$.

3. The inner product is positive definite. This means $\langle |\psi\rangle, |\psi\rangle\rangle > 0$ if $|\psi\rangle \neq 0$ and $\langle |\psi\rangle, |\psi\rangle\rangle = 0$ if $|\psi\rangle = 0$.

We define the norm of a state by $\| |\psi\rangle \| = \sqrt{\langle |\psi\rangle, |\psi\rangle\rangle}$. We define operators acting on our Hilbert space by A where $A : \mathcal{H} \to \mathcal{H}$. We denote the adjoint of an operator by $A^\dagger$ and it satisfies $\langle |\psi\rangle, A|\phi\rangle\rangle = \langle A^\dagger|\psi\rangle, |\phi\rangle\rangle$. We call an operator Hermitian if it satisfies $A^\dagger = A$. Additionally, if the an operator satisfies $A = A^\dagger$

and the domain of A is equal to the domain of $A^\dagger$ then it is self-adjoint (Ballentine, 2014).

As we consider only closed systems in this thesis, we assert that our Hilbert spaces only contain normalised, $|\psi\rangle$, states so that $\| |\psi\rangle \| = 1$, or $\langle |1(x)\rangle , |1(x')\rangle \rangle = \delta(x - x')$ where x is a continuous variable (more on this later). This allows us to obtain a probabilistic interpretation from our theory according to the Born (1926) rule.

Perhaps the most commonly seen inner product used in quantum theory is $\langle |\psi\rangle , |\phi\rangle \rangle = \langle\phi| \, |\psi\rangle$ where $\langle\phi|$ is the complex conjugate transpose of $|\phi\rangle$. We refer to this inner product as the conventional inner product. For simplicity, $\langle\phi| \, |\psi\rangle$ is often denoted as $\langle\phi|\psi\rangle$, and we shall do the same in this thesis. However, quantum theory is not limited to just the conventional inner product, and many more have been applied by adjusting this conventional inner product; For example, the PT inner product used by Bender & Boettcher (1998). In particular, we can adjust the conventional inner product by introducing a metric operator, g, in the inner product so that

$$\langle |\psi\rangle , |\phi\rangle \rangle = \langle\phi| \, g \, |\psi\rangle \ , \tag{2.1}$$

where $g = I_d$ gives the conventional inner product. Interestingly, this metric g is not physically observable (Brody, 2016), which opens the door to many physical observables that are not conventionally Hermitian — as long as they are Hermitian with respect to a suitable inner product, they will produce real expectation values! For many more examples, see Mostafazadeh (2010), but of relevant use to us here are inner products seen in biorthogonal quantum mechanics. We will delve into this later, but for now we describe the postulates of quantum mechanics.

2.1.3 Postulates of quantum mechanics

In finite dimensional systems, a key difference between classical and quantum probability theory is that in quantum theory there exists continuous and reversible transformations between the pure states of a system (Hardy, 2001). Here a state is a pure state if it is equal to a state in the system's Hilbert space as described earlier, so that we do not need a density operator to describe it (Fano, 1957). In quantum information and two dimensions, we can view pure states as points

on the unit Bloch sphere (Nielsen & Chuang, 2011). The Bloch sphere gives a geometrical interpretation of all the possible pure states of a two-level system[1], or one qubit, where we label the two levels $|0\rangle$ and $|1\rangle$ in Fig. 2.1. We can represent any point on this sphere by the state $|\psi\rangle = \cos(\theta/2)\,|0\rangle + e^{i\phi}\sin(\theta/2)\,|1\rangle$[2], where $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$. We give the Bloch sphere example as it illustrates nicely the continuous transformation property between pure states that can arise in quantum theory but not in classical theory. In classical information theory, the only possible states of a two-level system would be $|0\rangle$ or $|1\rangle$. Whereas in the quantum case, we continuously transform from $|0\rangle$ to $|1\rangle$, which we can observe by continuously moving our point on the Bloch sphere at $|0\rangle$ to $|1\rangle$. During this process, the probability of measuring the state to be $|0\rangle$ decreases, whereas the probability of measuring the state to be $|1\rangle$ increases.

We now present the postulates of quantum theory that we use to model our universe probabilistically.

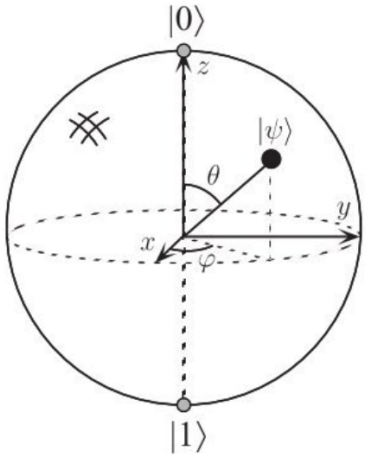

Figure 2.1: The Bloch sphere (Nielsen & Chuang, 2011)

Postulate 1

A state vector $|\psi\rangle$ represents the state of a system. This state vector belongs to the system's Hilbert space $\mathcal{H}$, which contains all possible equivalence classes of normalised state vectors of the system. Here two normalised state vectors, $|\psi\rangle$ and $|\phi\rangle$, are equivalent if $|\psi\rangle = c\,|\phi\rangle$ for some complex number c where $|c| = 1$. We will refer to state vectors simply as states.

[1] For a n-level system we need the complex projective space $\mathbb{CP}^{n-1}$ of complex dimension $n-1$.

[2] More accurately this should be $|\psi\rangle = e^{i\Theta}(\cos(\theta/2)\,|0\rangle + e^{i\phi}\sin(\theta/2)\,|1\rangle)$, but as overall phase factors produce no observable effect we leave out the $e^{i\Theta}$.

Postulate 2

Self-adjoint operators represent physically observable quantities, e.g. position, and we call these operators observables. The normalised eigenstates of self-adjoint operators provide an orthonormal basis for the Hilbert space. This means that if $|a_i\rangle$ denotes the set of eigenstates of a self-adjoint operator A, where $\langle |a_i\rangle , |a_j\rangle\rangle = \delta_{ij}$, then we can write any state, $|\psi\rangle$ in the form $|\psi\rangle = \sum_i c_i |a_i\rangle$ where $c_i = \langle |\psi\rangle , |a_i\rangle\rangle$.

Postulate 3

When we measure an observable A, the result is an eigenvalue of the corresponding observable. However, if we were to carry out this measurement on an infinite number of identical states, $|\psi\rangle$, the average value of all these measurements is

$$\langle A\rangle_{|\psi\rangle} = \langle A|\psi\rangle , |\psi\rangle\rangle , \tag{2.2}$$

which we call the expectation value. As our universe is real, all expectation values of physical observables should be real. Therefore, we require $\langle A\rangle_{|\psi\rangle} = \langle A\rangle^*_{|\psi\rangle}$. Because A is Hermitian, we have

$$\begin{aligned} \langle A|\psi\rangle , |\psi\rangle\rangle^* &= \langle |\psi\rangle , A|\psi\rangle\rangle \quad \text{using property 1 of the Hilbert space} \\ &= \langle A|\psi\rangle , |\psi\rangle\rangle \quad \text{since } A = A^\dagger \end{aligned} \tag{2.3}$$

and so we conclude that $\langle A\rangle_{|\psi\rangle}$ is a real number, as it should be. This also implies that the eigenvalues of A are real. Setting $|\psi\rangle = |a_i\rangle$ we find

$$a_i = \langle A|a_i\rangle , |a_i\rangle\rangle = \langle |a_i\rangle , A|a_i\rangle\rangle = a_i^* . \tag{2.4}$$

However, this does not mean the converse that only Hermitian operators have real eigenvalues (Bender & Boettcher, 1998).

Postulate 4

For an observable, A, and (normalised) state, $|\psi\rangle$, the probability density of measuring the eigenvalue a_i of A is $P(a_i) = |\langle |a_i\rangle , |\psi\rangle\rangle|^2$ where $|a_i\rangle$ is the corresponding eigenstate of a_i. In this thesis, our observables will only have non-degenerate

spectra. We define the uncertainty of the observable to be equal to the standard deviation of the observable

$$\Delta A_{|\psi\rangle} = \sqrt{\langle A^2 \rangle_{|\psi\rangle} - \langle A \rangle^2_{|\psi\rangle}} \,. \tag{2.5}$$

A measurement of A will only ever produce one of its eigenvalues. Therefore, if the uncertainty is nonzero, $|\psi\rangle$ will be a superposition of more than one of A's eigenstates. If the uncertainty is zero, then $|\psi\rangle$ is an eigenstate of A. We often refer to the uncertainty of a measurement as its fluctuations.

Postulate 5

Immediately after measurement of an observable, the state collapses to the corresponding eigenstate of the eigenvalue measured. This is sometimes referred to as the collapse of the wave function. It also ensures that repeating the same measurement on the *same* state gives the same result.

Postulate 6

States evolve according to the Schrödinger equation, which is

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} |\psi(t)\rangle = H(t) |\psi(t)\rangle \,, \tag{2.6}$$

where $H(t)$ is the Hamiltonian of the system — the operator that corresponds to the total energy of the system — and is potentially time-dependent. The solution to this equation is

$$\begin{aligned} |\psi(t)\rangle &= U(t) |\psi(0)\rangle \,, \\ U(t) &= \exp\left(-\frac{i}{\hbar} \int_0^t \mathrm{d}t' \, H(t') \right) \,, \end{aligned} \tag{2.7}$$

where $|\psi(0)\rangle$ is the initial state of the system at time $t = 0$. In closed systems, we want to conserve total probability during time evolution. Via the Born rule, this means we want the absolute inner product between states to be invariant under time evolution so that

$$|\langle U(t) |\psi\rangle \, , U(t) |\phi\rangle \rangle| = |\langle |\psi\rangle \, , |\phi\rangle \rangle| \,. \tag{2.8}$$

Wigner's theorem states that $U(t)$ can either be unitary or antiunitary (Ballentine, 2014). If $U(t)$ is unitary, we have

$$\langle U(t)\,|\psi\rangle\,, U(t)\,|\phi\rangle\rangle = \langle|\psi\rangle\,, |\phi\rangle\rangle. \tag{2.9}$$

and $U^{-1}(t) = U^{\dagger}(t)$. If $U(t)$ is antiunitary, we have

$$\langle U(t)\,|\psi\rangle\,, U(t)\,|\phi\rangle\rangle = \langle|\psi\rangle\,, |\phi\rangle\rangle^{*}\,. \tag{2.10}$$

By Eq. (2.7) time evolution will be unitary only if $H^{\dagger}(t) = H(t)$. So again, we see the utility of using Hermitian operators. Even if a non-Hermitian Hamiltonian has real eigenvalues, the inner product will still need to be adjusted for it to generate unitary dynamics.

This concludes the postulates of quantum mechanics. Later in the thesis, we shall use them to calculate how light evolves near a semi-transparent mirror. To do so, we need an appropriate state, observable, and mirror system Hamiltonian to calculate expectation values, which we describe later.

2.1.4 The uncertainty principle

In quantum theory, the position operator x and momentum operator p along the x direction are canonical conjugate quantities and so obey the canonical commutation relation[1]

$$[x, p] = i\hbar\,. \tag{2.11}$$

This equation leads to the following relationship between the standard deviations of the position and momentum operators (Robertson, 1929)

$$\Delta x \Delta p \geq \frac{\hbar}{2}\,, \tag{2.12}$$

which is known as the uncertainty principle — it dictates how much spread in momentum and position a quantum state can have. To see this, assume $|x\rangle$ is an eigenstate of the position operator. By the canonical commutation relation, $|x\rangle$ cannot also be an eigenstate of the momentum operator. However, as p is self-adjoint, $|x\rangle$ is a superposition of momentum eigenstates. Therefore, $|x\rangle$ has

[1] The commutator between two operators A and B is $[A, B] = AB - BA$

a spread in momentum with its uncertainty dictated by the uncertainty principle (in this case Δp cannot be a finite number to satisfy Eq. (2.12), as $\Delta x = 0$, so we must have maximum uncertainty in momentum). This is not a problem: it is a fundamental aspect of quantum physics. We mentioned earlier that our Hilbert space, $\mathcal{H}$, only contains states that are normalised to unity, however, the position and momentum eigenstates are not normalised to unity. This is because the position and momentum are continuous variables and so the inner product between their respective eigenstates is

$$\langle |x\rangle , |x'\rangle \rangle = \delta(x - x') , \qquad \langle |p\rangle , |p'\rangle \rangle = \delta(p - p') , \tag{2.13}$$

where the $\delta(y)$ function has the property that

$$\int_{-\epsilon}^{\epsilon} \delta(y) \mathrm{d}y = 1 , \tag{2.14}$$

for $\epsilon > 0$. The $\delta(y)$ function blows up to infinity at $y = 0$; therefore, the inner products $\langle |x\rangle , |x'\rangle \rangle$ and $\langle |p\rangle , |p'\rangle \rangle$ blow up to infinity when $x = x'$ and $p = p'$, respectively. Clearly, the position and momentum eigenstates do not belong in our Hilbert space. Fortunately, one can define an extended space, $\Omega^{\times}$, that contains the position and momentum eigenstates, in addition to $\mathcal{H}$, and a subspace of $\mathcal{H}$, Ω, such that for any $|\psi\rangle \in \Omega$, $|\phi\rangle \in \Omega^{\times}$ we have that $\langle |\phi\rangle , |\psi\rangle \rangle$ is finite. The corresponding triplet

$$\Omega \subset \mathcal{H} \subset \Omega^{\times} , \tag{2.15}$$

is called a rigged Hilbert space and these spaces have found many uses in quantum theory (Ballentine, 2014; de la Madrid, 2005).

Later in the thesis we will calculate how the EM field changes in the presence of a mirror in a fully quantum setting. However, we will have to make sure that the produced expectation values evolve as in classical physics. To reproduce these dynamics, we will use quantum states with the most classical properties and therefore use states that minimise the uncertainty relation so that $\Delta x \Delta p = \frac{\hbar}{2}$.

2.1.5 The quantum harmonic oscillator

Ubiquitous in quantum physics is the quantum harmonic oscillator, and this is especially so for the quantised EM field. The EM field itself is a collection of

quantised harmonic oscillators (Gerry & Knight, 2004). We, therefore, briefly review the undamped quantum harmonic oscillator in one dimension, using conventional Hermitian physics, since this is its simplest case and will allow us to introduce concepts such as creation/annihilation operators, which will be very useful later.

The Hamiltonian of this system is

$$H = \frac{p^2}{2m} + \frac{1}{2}Kx^2\,, \tag{2.16}$$

where p, x and m are the particle's momentum, displacement from the origin — the origin being the equilibrium point where the particle experiences no restoring force — and mass, respectively. The constant K is the positive restoring force constant. However, as this is a quantum harmonic oscillator we promote x and p to operators that must obey the canonical commutation relation $[x, p] = i\hbar$. Because p and x are real observables, we must have $p = p^\dagger$ and $x = x^\dagger$. Setting $\omega = \sqrt{K/m}$ and

$$\begin{aligned} a &= \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{i}{m\omega}p\right)\,, \\ a^\dagger &= \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{i}{m\omega}p\right)\,, \end{aligned} \tag{2.17}$$

we find the Hamiltonian and the commutator between a and $a^\dagger$ is equal to

$$\begin{aligned} H &= \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)\,, \\ [a, a^\dagger] &= 1\,. \end{aligned} \tag{2.18}$$

To see why this representation is useful, we define $|n\rangle$ to be the eigenstates of H with corresponding eigenvalues E_n. We use the commutators

$$\begin{aligned} [H, a^\dagger] &= \hbar\omega a^\dagger\,, \\ [H, a] &= -\hbar\omega a\,, \end{aligned} \tag{2.19}$$

to calculate

$$\begin{aligned} Ha^\dagger\,|n\rangle &= (E_n + \hbar\omega)a^\dagger\,|n\rangle\,, \\ Ha\,|n\rangle &= (E_n - \hbar\omega)a\,|n\rangle\,. \end{aligned} \tag{2.20}$$

We see that if $|n\rangle$ is an eigenstate of H then $a^\dagger |n\rangle$ is also an eigenstate of H except with eigenvalue $E_n + \hbar\omega$. Similarly, $a|n\rangle$ is also an eigenstate of H except with eigenvalue $E_n - \hbar\omega$. Therefore, we refer to $a^\dagger$ and a as creation and annihilation operators, respectively.

For a given $|n\rangle$, we can construct other eigenstates by acting integer products of either creation or annihilation operators on this state. In particular, since H is a positive operator, there must be a state, $|0\rangle$, that has the property $a|0\rangle = 0$. Otherwise, there would be a $m \in \mathbb{Z}$ such that $H(a^m|n\rangle) = E(a^m|n\rangle)$ where $E < 0$. We call $|0\rangle$ the vacuum state. From Eq. (2.16) we see that this state has energy $\frac{1}{2}\hbar\omega$, and we refer to this energy as the vacuum energy.

We can now construct our eigenstates by acting integer multiples of $a^\dagger$ on the vacuum and define

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle\,. \tag{2.21}$$

to be the orthonormal eigenstates of H. It will be useful to define the number operator

$$N = a^\dagger a, \tag{2.22}$$

which has the useful property $N|n\rangle = n|n\rangle$. In other words, its eigenvalues are the numbers of "energy quanta" in the Hamiltonian's eigenstates. With this it is easy to see that the eigenstates of H satisfy $H|n\rangle = \hbar\omega(n + \frac{1}{2})|n\rangle$.

Lastly, the Hamiltonian of a collection of N uncoupled harmonic oscillators is

$$H = \sum_{i=1}^{\mathrm{N}} \hbar\omega_i \left(a_i^\dagger a_i + \frac{1}{2}\right)\,, \tag{2.23}$$

where

$$[a_i, a_j^\dagger] = \delta_{ij}, \quad [a_i, a_j] = 0, \quad [x_i, p_j] = i\hbar\delta_{ij}\,, \tag{2.24}$$

and $\omega_i = \sqrt{K_i/m}$, where K_i is the restoring force constant of the i^{th} oscillator. Therefore, we refer to the ith creation/annihilation operator pair as a creation/annihilation operator pair for the ω_i mode. The operators x_i and p_i are position and momentum operators, respectively, of the i^{th} oscillator.

Interestingly, although neither a nor $a^\dagger$ are Hermitian, and so are not observables, their sum and product are Hermitian and are therefore observables. This

is in contrast to x and p where both are Hermitian, but their product is not. For example, the commutator $[x, p] = i\hbar$ is not an observable.

We introduce the so-called wave function of a state

$$\psi(x) = \langle x|\psi\rangle \ , \tag{2.25}$$

where

$$\int_{-\infty}^{\infty} \mathrm{d}x|\psi(x)|^2 = 1 \, , \tag{2.26}$$

because the probability to find the state anywhere along the x-axis must be equal to 1. The vacuum state wave function is given by $\psi_-(x) = \exp\left(-\frac{1}{4}\omega x^2\right)$ (Bender & Turbiner, 1993).

Suppose we make the substitution $\omega \to -\omega$. The Hamiltonian remains the same yet the eigenvalues are now negative and so the ground state wave function is not normalisable. How can this be? To address this, it was pointed out in Bender & Turbiner (1993) that in order to replace ω by $-\omega$ *and* obtain the correct analytic continuation one must also extend x into the complex plane and let $x = ir$ where $r \in \mathbb{R}$. This is because if one simply negates ω then the normalised ground state wave function is now the different solution $\psi_+(x) = \exp\left(\frac{1}{4}\omega x^2\right)$ where $\omega < 0$, see Fig. 2.2, and so we must rotate x onto the complex axis to obtain a correct analytic continuation.

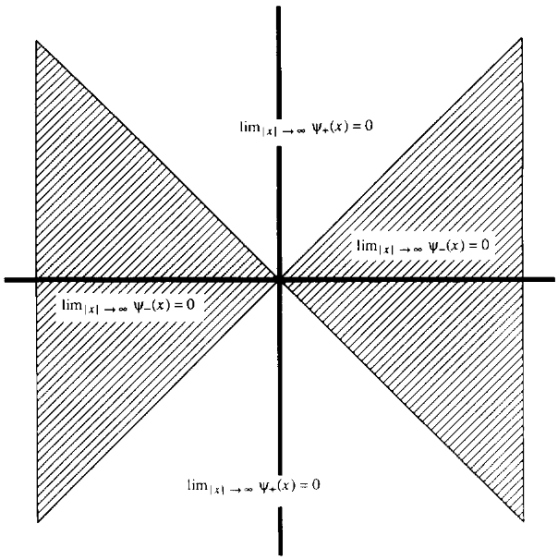

Figure 2.2: Above is the complex x-plane. Shaded regions show where $\psi_-(x) = \exp\left(-\frac{1}{4}\omega x^2\right)$ is normalised and the un-shaded regions show where $\psi_+(x) = \exp\left(\frac{1}{4}\omega x^2\right)$ is normalised (Bender & Turbiner, 1993).

We point out that we could repeat the above analysis with the substitution

$$i \to -i \, , \tag{2.27}$$

without changing any expectation values of observable quantities. This is because any physical expectation value or probability will be a real quantity, C. Complex conjugating C will leave it unchanged; therefore, complex conjugating the operator and states of an expectation value will leave the value unchanged. With this substitution, we have

$$\begin{aligned} [x,p] = i\hbar \quad &\rightarrow \quad -i\hbar = [x^*, p^*]\,, \\ [a, a^\dagger] = 1 \quad &\rightarrow \quad 1 = [a^*, a^{\dagger *}]\,. \end{aligned} \tag{2.28}$$

From the first line, we point out that just because an operator is Hermitian, it does not mean that it is the complex conjugate of itself.

Coherent states

We now ask what states of the single quantised harmonic oscillator minimise the uncertainty principle. To do so, we rearrange Eq. (2.17) to find

$$\begin{aligned} x &= \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right)\,, \\ p &= -i\sqrt{\frac{\hbar m\omega}{2}}\left(a - a^\dagger\right), \end{aligned} \tag{2.29}$$

which leads to

$$\begin{aligned} x^2 &= \frac{\hbar}{2m\omega}(aa + a^\dagger a^\dagger + 2a^\dagger a + 1)\,, \\ p^2 &= -\frac{\hbar m\omega}{2}(aa + a^\dagger a^\dagger - (2a^\dagger a + 1))\,. \end{aligned} \tag{2.30}$$

Therefore, we can guess that states that will minimise the uncertainty principle between position and momentum are the eigenstates of the annihilation operator. To see this, if we assume $a\,|\alpha\rangle = \alpha\,|\alpha\rangle$, where $\alpha \in \mathbb{C}$, we can calculate

$$\begin{aligned} \langle\alpha|x^2|\alpha\rangle - \langle\alpha|x|\alpha\rangle^2 &= \frac{\hbar}{2m\omega}\,, \\ \langle\alpha|p^2|\alpha\rangle - \langle\alpha|p|\alpha\rangle^2 &= \frac{\hbar m\omega}{2}\,. \end{aligned} \tag{2.31}$$

Then, using Eq. (2.5), we find

$$\Delta x \Delta p = \frac{\hbar}{2}\,, \tag{2.32}$$

which is the minimum value the uncertainty relation in Eq.(2.12) can be.

We define the following state

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger} |0\rangle \ , \tag{2.33}$$

where $\alpha = |\alpha|^{i\theta}$ is a complex number. We call these states coherent states. For these states we have $a|\alpha\rangle = \alpha|\alpha\rangle$ so that they are indeed eigenstates of the annihilation operator with eigenvalue α. To see this, suppose that two operators A and B have commutator c such that $[A, c] = [B, c] = 0$. This means

$$[A, e^{\alpha B}] = \alpha c e^{\alpha B} \, , \tag{2.34}$$

Setting $A = a$ and $B = a^\dagger$ we can now calculate

$$a|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} a e^{\alpha a^\dagger} |0\rangle = e^{-\frac{|\alpha|^2}{2}} \alpha e^{\alpha a^\dagger} |0\rangle = \alpha |\alpha\rangle \ . \tag{2.35}$$

Furthermore, the inner product between two coherent states, $|\alpha\rangle$ and $|\beta\rangle$, is

$$\langle\beta|\alpha\rangle = \exp\left(-\frac{1}{2}\left(|\beta|^2 + |\alpha|^2 - 2\alpha\beta^*\right)\right) \ . \tag{2.36}$$

So, although they are normalised, they are not orthogonal to each other. We expected this as coherent states are not eigenstates of a Hermitian operator. Nevertheless, we have

$$\frac{1}{\pi}\int \mathrm{d}\alpha^2 \, |\alpha\rangle\langle\alpha| = \sum_{n=0}^{\infty} |n\rangle\langle n| = \mathbb{1} \, , \tag{2.37}$$

which means coherent states provide an over-complete basis.

Calculating the uncertainty in position and momentum we find

$$\begin{aligned} \Delta x_{|\alpha\rangle} &= \sqrt{\frac{\hbar}{2m\omega}} \, , \\ \Delta p_{|\alpha\rangle} &= \sqrt{\frac{\hbar m\omega}{2}} \, , \end{aligned} \tag{2.38}$$

so that $\Delta x \Delta p = \frac{\hbar}{2}$ for coherent states. Thus they have minimum uncertainty for position and momentum.

Calculating the evolution of coherent states, we find

$$U(t)|\alpha\rangle = |e^{-i\omega t}\alpha\rangle = |\alpha(t)\rangle \ . \tag{2.39}$$

where $\alpha(t) = e^{-i\omega t}\alpha$. Therefore, under time evolution coherent states remain coherent states and so keep their minimum uncertainty in position and momentum. Calculating the position and momentum expectation values themselves we find

$$\begin{aligned} \langle x\rangle_{|\alpha(t)\rangle} &= \sqrt{\frac{2\hbar}{m\omega}}|\omega|\cos(\theta-\omega t)\,, \\ \langle p\rangle_{|\alpha(t)\rangle} &= \sqrt{2\hbar m\omega}|\alpha|\sin(\theta-\omega t)\,, \end{aligned} \tag{2.40}$$

which oscillate sinusoidal like a classical system. Because of this classical sinusoidal oscillation property, coherent states will appeal to us when we ask what states produce classical-like EM field expectation values in free space.

This concludes our discussion of the quantised harmonic oscillator; we will use the introduced concepts throughout this thesis.

2.2 Biorthogonal quantum mechanics

Systems possessing curious non-Hermitian structure, with respect to the conventional inner product, have gained much interest in recent years (Bender & Boettcher, 1998; Mostafazadeh, 2010). We have seen this in quantum optics, where certain systems have been modelled using biorthogonal quantum physics (El-Ganainy *et al.*, 2018; Hawton & Debierre, 2017; Smith & Raymer, 2007). Later in this thesis, we will equip our EM field Hilbert space with an inner product using the help of biorthogonal quantum mechanics. In this subsection, we therefore review some properties of biorthogonal quantum mechanics, which are closely related to pseudo-Hermitian Hamiltonians. We refer to non-Hermitian operators as operators that are non-Hermitian with respect to the conventional inner product[1], and reserve the dagger notation, †, to denote the adjoint of an operator with respect to the conventional inner product. For clarity, we work in an N-dimensional Hilbert space but will discuss the extension to infinite-dimensional Hilbert spaces.

Suppose a set of N linearly independent states, $\{|\alpha_n\rangle\}_{n=1}^{n=N}$, spans a Hilbert space, $\mathcal{H}$, which aren't necessarily orthonormal with respect to the conventional

[1] This does not mean they are not Hermitian with respect to a different inner product!

inner product. It is possible to obtain a set of states $\{|\beta_n\rangle\}_{n=1}^{n=N}$ such that $\langle\beta_i|\alpha_j\rangle = \delta_{ij}$ Brody (2013, 2016). For a state

$$|\psi\rangle = \sum_n a_n |\alpha_n\rangle \ , \tag{2.41}$$

its associated state, $|\tilde{\psi}\rangle \in \mathcal{H}^*$, is defined by[1]

$$|\tilde{\psi}\rangle = \sum_n a_n |\beta_n\rangle \ , \tag{2.42}$$

and the inner product by

$$\langle |\psi_1\rangle , |\psi_2\rangle \rangle^{bio} = \langle \tilde{\psi}_2 | \psi_1 \rangle \ . \tag{2.43}$$

Operators that are Hermitian with respect to this inner product are of the form

$$A = \sum_{n,m} a_{nm} |\alpha_n\rangle \langle\beta_m| \ , \tag{2.44}$$

where $a^*_{nm} = a_{mn}$.The set $\{|\beta_n\rangle\}_{n=1}^{n=N}$ is called the biorthonormal basis associated with $\{|\alpha_n\rangle\}_{n=1}^{n=N}$ and $\{|\alpha_n\rangle , |\beta_n\rangle\}_{n=1}^{n=N}$ is a biorthonormal system. For the case where $\{|\alpha_n\rangle\}_{n=1}^{n=N}$ is already orthonormal, selecting $|\beta_n\rangle = |\alpha_n\rangle$ reduces the system to conventional Hermitian physics.

Biorthogonal quantum physics is related to pseudo-Hermitian physics. A Hamiltonian is pseudo-Hermitian if it is of the form

$$H^\dagger = \eta H \eta^{-1} \, , \tag{2.45}$$

where $\eta = \eta^\dagger$. This H is Hermitian with respect to the inner product

$$\langle |\psi_1\rangle , |\psi_2\rangle \rangle^{\eta} = \langle \psi_2 | \, \eta \, | \psi_1 \rangle \ . \tag{2.46}$$

We see this by calculating

$$\langle H |\psi_1\rangle , |\psi_2\rangle \rangle^{\eta} = \langle |\psi_1\rangle , H |\psi_2\rangle \rangle^{\eta} \ . \tag{2.47}$$

[1]Here an asterisk is using to denote the dual Hilbert space as this is how it is usually denoted. In this thesis, when we use an asterisk on anything other than a Hilbert space it denotes complex conjugation.

Because H is Hermitian with respect to this η inner product, its set of (normalised) eigenstates, $\{|\phi_n\rangle\}_{n=1}^{n=N}$, is an orthonormal basis for the whole Hilbert space and its eigenvalues are real. The eigenstates of H can be used to construct an identity operator

$$I_d = \sum_n |\phi_n\rangle \langle\phi_n| \eta \,, \tag{2.48}$$

in the sense that when it acts on a state it leaves the state unchanged. To see this, we first point out that because $\{|\phi_n\rangle\}_{n=1}^{n=N}$ is an orthornomal basis for the Hilbert space we can represent any state in this Hilbert space as $|\psi\rangle = \sum_n a_n |\phi_n\rangle$. With this we calculate

$$\begin{aligned} I_d |\psi\rangle &= \sum_n |\phi_n\rangle \langle\phi_n| \eta \sum_m a_m |\phi_m\rangle \\ &= \sum_n \sum_m a_m |\phi_n\rangle \langle\phi_n| \eta |\phi_m\rangle \\ &= \sum_n \sum_m a_m |\phi_n\rangle \delta_{nm} \\ &= \sum_n a_n |\phi_n\rangle \\ &= |\psi\rangle \ . \end{aligned} \tag{2.49}$$

Now, if we define

$$|\gamma_n\rangle = \eta |\phi_n\rangle \ , \tag{2.50}$$

we see that $\{|\phi_n\rangle, |\gamma_n\rangle\}_{n=1}^{n=N}$ is a biorthonormal system. So the identity operator can also be represented as

$$I_d = \sum_n |\phi_n\rangle \langle\gamma_n| \ . \tag{2.51}$$

These $|\gamma_n\rangle$ states are the eigenstates of $H^\dagger$ as

$$H^\dagger |\gamma_n\rangle = a_n |\gamma_n\rangle \ , \tag{2.52}$$

where a_n is the eigenvalue of H with corresponding eigenstate $|\phi_n\rangle$. These states are orthonormal with respect to the η^{-1} inner product defined by

$$\langle |\psi_1\rangle, |\psi_2\rangle \rangle^{\eta^{-1}} = \langle\psi_2| \eta^{-1} |\psi_1\rangle \ . \tag{2.53}$$

So, if $\{|\phi_n\rangle, |\gamma_n\rangle\}_{n=1}^{n=N}$ is a biorthonormal system equipped with the η inner product, then $\{|\gamma_n\rangle, |\eta_n\rangle\}_{n=1}^{n=N}$ is a biorthonormal system equipped with the η^{-1} inner product.

However, the $|\gamma_n\rangle$ states are not in general orthonormal with respect to the η inner product defined above. To see this, simply compute

$$\langle |\gamma_n\rangle, |\gamma_m\rangle\rangle^{\eta} = \langle \phi_m | \eta^3 | \phi_n \rangle \neq \delta_{nm} \,. \tag{2.54}$$

This is true for general η. For example, we can set η and η^{-1} (Mostafazadeh, 2010)

$$\begin{aligned} \eta &= \sum_n |\gamma_n\rangle \langle \gamma_n| \,, \\ \eta^{-1} &= \sum_n |\phi_n\rangle \langle \phi_n| \,. \end{aligned} \tag{2.55}$$

However, they are orthonormal for the case $\eta = \mathbb{1}$. In this case, the η inner product Eq. (2.46) reduces to the conventional inner product of quantum mechanics and $|\phi_n\rangle = |\psi_n\rangle$.

Eq. (2.54) is not a problem if the only states in the Hilbert space, $\mathcal{H}$, are states of the form in Eq. (2.41) that are normalised with respect to the η inner product in Eq. (2.46) and the observables are of the form in Eq. (2.44). In this case, one can select the η inner product for the Hilbert space and find that the biorthogonal QM is indistinguishable to conventional Hermitian QM (Brody, 2016). Likewise, if the Hilbert space only contains states that belong to $\mathcal{H}^*$, where the states are all normalised with respect to the η^{-1} inner product, and the observables are Hermitian with respect to the η^{-1} inner product in Eq. (2.53), one can simply select the η^{-1} inner product for the Hilbert space. Again, this is indistinguishable to conventional Hermitian physics.

2.2.1 Infinite dimensional systems

We kept the preceding discussion to N-dimensional systems for clarity. However, when we quantise the EM field in free space, we have an infinite-dimensional system. It is possible to carry over our analysis of N-dimensional systems into infinite-dimensional systems; however, one needs to be careful with how η behaves (Kretschmer & Szymanowski, 2004; Mostafazadeh, 2013) and also that basis states are actually orthonormal (Brody, 2013). We show that we do not run into these potential issues with the EM field. In any case, at the forefront of our minds is making sure we can use the theory we present to make physical predictions, see Fig. 2.3. In the next section, we quantise the EM field in free space in a very standard way. We do this as the standard way of quantising the EM field has been very successful in predicting phenomena (Gerry & Knight, 2004). So we use it as a starting point to help make sure our theory is consistent with what is already known.

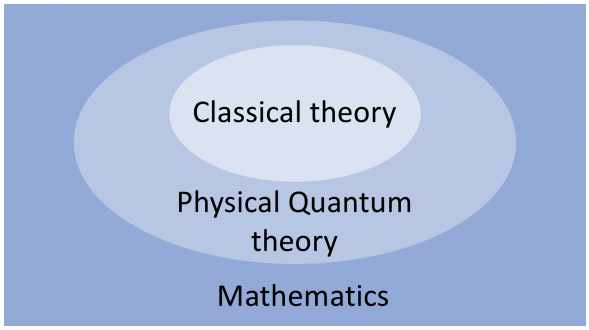

Figure 2.3: If something is nonphysical with part of the mathematics, it might be that that part doesn't belong in our physical quantum theory.

2.3 Summary

- We gave a brief whistle-stop tour of the beginnings of quantum theory, to help introduce concepts of quantum theory.
- We reviewed the Hilbert space structure of quantum theory, and how we can use postulates to relate this theory to reality.
- We reviewed the uncertainty principle, and how it will be used later in this thesis.

- We reviewed a simple quantum harmonic oscillator. In particular, we looked at coherent states of this system and the position and momentum expectation values of these states oscillate sinusoidally and have minimum uncertainty in these expectation values.
- We reviewed standard biorthogonal quantum physics, and how it relates to pseudo-Hermitian physics.

Chapter 3

The standard description of the quantised EM field

This chapter describes the quantised EM field in free space with the help of a fairly recent paper by Bennett *et al.* (2015). Here, we use a bottom-up approach that starts with the system's energy eigenstates, monochromatic photons. We then review states of the EM field that exhibit the most classical behaviour and call these states the coherent states of the EM field. These states will be important later when we ask how to produce classical dynamics of the EM field near a mirror from a fully quantum setting.

3.1 The quantised EM field in free space

This section looks at the quantised EM field in one-dimensional free space. By free space we mean a homogeneous, non-dispersive, non-absorbing medium with permittivity ε_0 and permeability μ_0. Here, the speed of light is equal to

$$c = 1/\sqrt{\varepsilon_0 \mu_0}\,. \tag{3.1}$$

The most fundamental equations in EM theory are Maxwell's equations that relate the electric and magnetic field vectors. In free space they are

$$\begin{aligned} \nabla \times \mathbf{E}(\mathbf{x},t) &= -\frac{\partial \mathbf{B}(\mathbf{x},t)}{\partial t}\,, \\ \nabla \times \mathbf{B}(\mathbf{x},t) &= \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}(\mathbf{x},t)}{\partial t}\,, \\ \nabla \cdot \mathbf{E}(\mathbf{x},t) &= 0\,, \\ \nabla \cdot \mathbf{B}(\mathbf{x},t) &= 0\,, \end{aligned} \tag{3.2}$$

where $\mathbf{x} \in \mathbb{R}^3$ (Jackson, 1975). The classical energy of the EM field is the Hamiltonian

$$H_{\text{eng}} = \int \mathrm{d}\mathbf{V} \left[\varepsilon_0 \mathbf{E}(\mathbf{x},t)^2 + \frac{1}{\mu_0} \mathbf{B}(\mathbf{x},t)^2 \right]\,, \tag{3.3}$$

where V denotes the volume that contains the EM field. In the following, we have a closer look at this system's Hilbert space, Hamiltonian and the local EM field operators $\mathbf{E}(x)$ and $\mathbf{B}(x)$. We take a bottom-up approach by starting with defining the excitations of this system, defining the EM field operators in terms of these excitations and then showing that they satisfy the free space Maxwell equations[1]. We then describe the states whose EM field expectation values oscillate like a classical sinusoidal wave.

Using Eqs. (3.2) and the vector identity

$$\nabla \times \nabla \times \mathbf{V} = \nabla(\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V}\,, \tag{3.4}$$

[1] We are not aiming to give a comprehensive derivation of the quantised EM field here, just one that works and agrees with standard results. In doing so we take inspiration from Bennett *et al.* (2015).

we see that

$$\begin{aligned} \frac{\partial^2}{\partial t^2}\mathbf{E}(\mathbf{x},t) &= c^2\nabla^2\mathbf{E}(\mathbf{x},t)\,, \\ \frac{\partial^2}{\partial t^2}\mathbf{B}(\mathbf{x},t) &= c^2\nabla^2\mathbf{B}(\mathbf{x},t)\,, \end{aligned} \tag{3.5}$$

and so the E and B fields separately obey the wave equation. Suppose we are only interested in waves that propagate along one axis in an Euclidean coordinate system. To model this, we denote a position vector here by $(x, y, z) \in \mathbb{R}^3$ and label the unit vectors along each axis by $\mathbf{e}_i$ for $i = x, y, z$, respectively. These unit vectors satisfy $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$. Without loss of generality, we choose the x-axis to be the propagation axis so that the EM fields depend on x and t. This means the EM wave equations (3.5) reduce to

$$\begin{aligned} \frac{\partial^2}{\partial t^2}\mathbf{E}(x,t) &= c^2\frac{\partial^2}{\partial x^2}\mathbf{E}(x,t)\,, \\ \frac{\partial^2}{\partial t^2}\mathbf{B}(x,t) &= c^2\frac{\partial^2}{\partial x^2}\mathbf{B}(x,t)\,. \end{aligned} \tag{3.6}$$

This is useful as the solution to the equation

$$\frac{\partial^2}{\partial t^2}f = c^2\frac{\partial^2}{\partial x^2}f\,, \tag{3.7}$$

is

$$f = Af(x - ct) + Bf(x + ct)\,, \tag{3.8}$$

where A and B are constants. Therefore, we can assign to the EM fields two directions of propagation: in the positive x direction, $\mathbf{E}(x,t) = \mathbf{E}(x - ct, 0)$, or in the negative x direction, $\mathbf{E}(x,t) = \mathbf{E}(x + ct, 0)$. The EM field operators will therefore need to account for these two directions of propagation. Lastly, we have two orthornomal polarisations for the E field propagating along the x-axis and without loss of generality we choose them to be $\mathbf{e}_y$ and $\mathbf{e}_z$. In one spatial dimension, Eq. (3.3) reduces to

$$H_{\text{eng}} = \frac{A}{2}\int_{-\infty}^{\infty}\mathrm{d}x\,\left[\varepsilon_0\mathbf{E}(x,t)^2 + \frac{1}{\mu_0}\mathbf{B}(x,t)^2\right]\,, \tag{3.9}$$

where A is the area around the x-axis that contains the EM field. This is an integral over the whole x-axis, and is invariant under spatial translations of the

x-axis, so we assume that, when quantised, eigenstates of this Hamiltonian will not be x-dependent, i.e. fully nonlocal, and so can have complete certainty in momentum, which is characterised by their wave number k and direction of propagation. The two possible directions of propagation being either in the positive x direction or in the negative x direction. Unlike quantising in a box of finite length where k takes discrete values related to the length of the box, we expect k to take a continuum of values in the range $(0, \infty)$. The x values in Eq. (3.9) are in the range $(-\infty, \infty)$. If we wanted k to characterise the direction of propagation, it would also take in values in the range $(-\infty, \infty)$, so that the sign of k would direction of propagation. However, later in this thesis, we introduce negative frequencies, and it is our preference to associate negative frequencies with negative wave numbers. To denote the direction of propagation, we will use a parameter s, which we go into more detail shortly.

To describe our quantised EM field, we know from experiments that light can have bosonic properties (Hong *et al.*, 1987). It will therefore be convenient to represent states using a bosonic Fock basis (Fock, 1932; Lancaster & Blundell, 2014) as this allows us to construct observables using the same operators used in describing states, thus simplifying calculations. So we want to describe a single excitation of this system by a creation operator acting on the normalised vacuum state of the system

$$|1_{s\lambda}(k)\rangle \quad = \quad b^{\dagger}_{s\lambda}(k)\,|0\rangle\,, \quad \langle 0|0\rangle = 1\,, \tag{3.10}$$

and we call these states single monochromatic photon states or just single photon states. We denote photon annihilation operators by $b_{s\lambda}(k)$, which annihilate the vacuum state above. We have chosen b instead of a to prepare for following sections. The commutator between the creation and annihilation operators needs to take into account the two polarisations of the system, $\lambda = H, V$, the two directions of propagation, $s = \pm 1$, and of course the distinct wave number that an excitation can have $k \in (0, \infty)$. This notation is slightly different from standard notation where k can be negative but also denotes direction of propagation. We have done this to prepare for later sections. To be sure, momentum is a vector. If we had to account for photons propagating in 3-d then using the $s = \pm 1$ notation would not be possible. It is only possible here as the photons we are using only

propagate along the x-axis in either the positive x direction or the negative x direction. The photons satisfy the Fock bosonic commutation relations

$$\left[b_{s\lambda}(k), b^\dagger_{s'\lambda'}(k')\right] = \delta_{ss'}\,\delta_{\lambda,\lambda'}\,\delta(k-k')\,,\ \left[b_{s\lambda}(k), b_{s'\lambda'}(k')\right] = \left[b^\dagger_{s\lambda}(k), b^\dagger_{s'\lambda'}(k')\right] = 0\,. \tag{3.11}$$

If we choose our inner product so that $b_{s\lambda}(k)$ is the hermitian conjugate of $b^\dagger_{s\lambda}(k)$ then we find

$$\begin{aligned}\langle 1_{s\lambda}(k)|1_{s'\lambda'}(k')\rangle &= \langle 0|\left[b_{s\lambda}(k), b^\dagger_{s'\lambda'}(k')\right]|0\rangle \\ &= \delta_{ss'}\,\delta_{\lambda,\lambda'}\,\delta(k-k')\,,\end{aligned} \tag{3.12}$$

so the single photon states are indeed orthonormal in the continuous variable sense[1]. To be sure, these excitations are not normalised states — take the overlap of two identical states and it is equal to $\delta(0)$. However, we can still use them to construct normalised states by taking appropriate integrals over them (Eisaman *et al.*, 2011). For example, if we define

$$|\psi\rangle = \int_0^\infty \mathrm{d}k f(k) b^\dagger_{1H}(k)\,|0\rangle\ , \tag{3.13}$$

where

$$\int_0^\infty \mathrm{d}k |f(k)|^2 = 1\,, \tag{3.14}$$

then

$$\langle\psi|\psi\rangle = 1\,. \tag{3.15}$$

We now want the EM field Hilbert space of the quantised EM field to contain the normalised vacuum state $|0\rangle$ and all superpositions of normalised states we can get by applying the creation operators $b^\dagger_{s\lambda}(k)$ multiple times to the vacuum state. Therefore, it includes the states

$$|n_{s\lambda}(k)\rangle = \frac{1}{\sqrt{n!}}\left(b^\dagger_{s\lambda}(k)\right)^n|0\rangle\ , \tag{3.16}$$

[1]Regarding the creation/annihilation operator commutator, when going from a discrete number of modes to a continuum of modes we can roughly think of it as taking $[b_i, b^\dagger_j] = \delta_{ij} \to [b(x), b^\dagger(x')] = \delta(x-x')$ for some continuous variable x, as well as turning any summations over the discrete set into an integral over the variable x.

which contain exactly n photons in the (k, s, λ) mode. We call these states number states. A general basis state of our system is a tensor product state of the form

$$\prod_{i=0}^{N} \prod_{s=\pm 1} \prod_{\lambda=H,V} |n_{s\lambda}(k_i)\rangle \ , \tag{3.17}$$

for some N, which could be infinity, and where $|n_{s\lambda}(k_i)\rangle$ denotes a number state in the (k_i, s, λ) mode. We call this state a Fock state generated by the $b^\dagger_{s\lambda}(k)$ creation operators. Next, experiments have shown that the electromagnetic field's energy increases by $\hbar ck$ whenever a photon of wavenumber k is added (Bennett *et al.*, 2015). Combining this with the photon number states, the Hamiltonian of the quantised electromagnetic field is such that

$$H_{\text{eng}} |n_{s\lambda}(k)\rangle = (\hbar ck n_{s\lambda}(k) + H_{\text{ZPE}}) \ |n_{s\lambda}(k)\rangle \, , \tag{3.18}$$

where H_{ZPE} is the zero point energy term, so that $H_{\text{eng}} |0\rangle = H_{\text{ZPE}} |0\rangle$. Using the fact that this Hamiltonian has the same energy level structure as a harmonic oscillator, we can also use the creation and annihilation operators to write the above Hamiltonian in the form

$$H_{\text{eng}} = \sum_{s,\lambda} \int_0^\infty \mathrm{d}k \, \hbar ck \, b^\dagger_{s\lambda}(k) b_{s\lambda}(k) + H_{ZPE} \, . \tag{3.19}$$

We have described what we want our Hilbert space and Hamiltonian to be, but can we construct EM field operators that satisfy Maxwell's equations where the time evolution is generated by the Hamiltonian in Eq. 3.19? The answer is: yes, we can! Furthermore, we can do it all using the conventional inner product of quantum physics, which makes life easier. We mentioned that the Hamiltonian would generate the dynamics of the EM field operators, but in section 2.1.3 we described how states evolve. We therefore describe the Heisenberg picture where it is the operators that evolve and states remain stationary.

Working with the conventional inner product, with an observable, A, Hamiltonian $H(t)$ and state $|\psi\rangle$, the expectation value is equal to

$$\langle A \rangle_{|\psi\rangle} = \langle \psi(t)| \, A \, |\psi(t)\rangle = \langle \psi| \, U(t)^\dagger A U(t) \, |\psi\rangle = \langle \psi| \, A(t) \, |\psi\rangle \ , \tag{3.20}$$

where $A(t) = U(t)^\dagger A U(t)$. So we see that if we allow states to remain stationary and operators to evolve as described above, we get the same expectation value.

The Heisenberg equation describes the dynamics of operators in the Heisenberg picture and, using the above equation, it is

$$\frac{\mathrm{d}}{\mathrm{d}t}A(t) \;=\; -\frac{\mathrm{i}}{\hbar}\left[A(t), H(t)\right], \tag{3.21}$$

if the original observable A has no explicit time-dependence.

Now, if we define our EM field observables, $\mathbf{E}(x)$ and $\mathbf{B}(x)$ by (Bennett *et al.*, 2015)

$$\begin{aligned}
\mathbf{E}^{(b)}(x) &= \sum_{s,\lambda}\int_0^\infty \mathrm{d}k \sqrt{\frac{\hbar c k}{4\pi\varepsilon_0 A}}\, \mathrm{e}^{i(skx+\phi)}\, b_{s\lambda}(k)\, \mathbf{e}_\lambda + H.c.\,, \\
\mathbf{B}^{(b)}(x) &= \sum_{s,\lambda}\int_0^\infty \mathrm{d}k \,\frac{s}{c}\sqrt{\frac{\hbar c k}{4\pi\varepsilon_0 A}}\, \mathrm{e}^{i(skx+\phi)}\, b_{s\lambda}(k)\, \mathbf{e}_x \times \mathbf{e}_\lambda + H.c.\,,
\end{aligned} \tag{3.22}$$

where $\phi \in [0, 2\pi)$ is a free parameter and we have added the (b) superscript for convenience, and then substitute these field operators into (3.9) we have

$$H_{\mathrm{eng}} = \sum_{s,\lambda}\int_0^\infty \mathrm{d}k\, \hbar c k\, b^\dagger_{s\lambda}(k) b_{s\lambda}(k) + \sum_{s,\lambda}\int_0^\infty \mathrm{d}k \frac{\hbar c k}{2}\delta(0)\,. \tag{3.23}$$

This Hamiltonian is equal to the Hamiltonian in Eq. (3.19) where the second term is the zero-point energy term and so we denote

$$H_{\mathrm{ZPE}} = \sum_{s,\lambda}\int_0^\infty \mathrm{d}k \frac{\hbar c k}{2}\delta(0)\,. \tag{3.24}$$

The zero-point energy term is infinite; however, it does not contribute to dynamics in the Heisenberg equation. We, therefore, remove this term when this Hamiltonian is used to calculate dynamics and denote

$$H_{\mathrm{free}} = \sum_{s,\lambda}\int_0^\infty \mathrm{d}k\, \hbar c k\, b^\dagger_{s\lambda}(k) b_{s\lambda}(k)\,, \tag{3.25}$$

because this Hamiltonian generates dynamics in free space.

Furthermore, we see that Eq. (3.23) looks a lot like the Hamiltonian of a set of N harmonic oscillators Eq. (2.23)! The expectation being that we now have a continuum of oscillators, rather than a discrete set, with the summation replaced with an integral. For example, in the N case we had $\omega_i \in \{\omega_j\}_{j=0}^N$, whereas in

the continuum case we have $\omega_k \in \{ck : 0 < k < \infty\}$. This suggests we can define the hermitian operators

$$x_{s\lambda}(k) = \sqrt{\frac{\hbar}{2ck}} \left(b_{s\lambda}(k) + b^\dagger_{s\lambda}(k) \right) ,$$
$$p_{s\lambda}(k) = -i\sqrt{\frac{\hbar ck}{2}} \left(b_{s\lambda}(k) - b^\dagger_{s\lambda}(k) \right) , \tag{3.26}$$

where

$$[x_{s\lambda}(k), p_{s\lambda}(k')] = i\hbar\delta(k - k') , \tag{3.27}$$

so that we can write Eq. (3.23) in the form

$$H_{\text{eng}} = \sum_{s,\lambda} \int_0^\infty \mathrm{d}k \, \frac{1}{2} \left((ck)^2 x^2_{s\lambda}(k) + p^2_{s\lambda}(k) \right) . \tag{3.28}$$

We call $x_{s\lambda}(k)$ and $p_{s\lambda}(k)$ the canonical position and momentum operators, respectively, of the s, λ, k mode.

It is easy to check that the photon number states $|n_{s\lambda}(k)\rangle$ are eigenstates of the energy observable H_{eng} with their eigenvalues given by $n\hbar ck + H_{\text{ZPE}}$. These states evolve in the Schrödinger picture as $|n_{s\lambda}(k,t)\rangle = e^{-inckt} |n_{s\lambda}(k)\rangle$. Creation and annihilation operators evolve in the Heisenberg picture as

$$\begin{aligned} b_{s\lambda}(k) &= e^{-ickt} b_{s\lambda}(k) \\ b^\dagger_{s\lambda}(k) &= e^{ickt} b_{s\lambda}(k) . \end{aligned} \tag{3.29}$$

Using the above equations, the EM field operators Eq. (3.22) evolve, in the Heisenberg picture, as

$$\begin{aligned} \mathbf{E}^{(b)}(x,t) &= \sum_{s,\lambda} \int_0^\infty \mathrm{d}k \sqrt{\frac{\hbar ck}{4\pi\varepsilon_0 A}} \, \mathrm{e}^{i(sk(x-sct)+\phi)} \, b_{s\lambda}(k) \, \mathbf{e}_\lambda + H.c. , \\ \mathbf{B}^{(b)}(x,t) &= \sum_{s,\lambda} \int_0^\infty \mathrm{d}k \, \frac{s}{c} \sqrt{\frac{\hbar ck}{4\pi\varepsilon_0 A}} \, \mathrm{e}^{i(sk(x-sct)+\phi)} \, b_{s\lambda}(k) \, \mathbf{e}_x \times \mathbf{e}_\lambda + H.c. \end{aligned} \tag{3.30}$$

These satisfy Maxwell's equations and $\mathbf{E}(x,t) = \mathbf{E}(x - sct, 0)$ where $s = \pm 1$, which is exactly what we were looking for.

3.2 Coherent states of the EM field

Clearly, we have

$$\langle n_{s\lambda}(k')|\,\mathbf{E}^{(b)}(x,t)\,|n_{s\lambda}(k)\rangle = 0\,, \tag{3.31}$$

which means the number states have zero electric field expectation value. A classical mode of the EM field with a well-defined frequency travels with a sinusoidal shape. Therefore, we construct states such that their E field expectation value will behave like these classical mode. Building on the coherent states of a single harmonic oscillator we described in section 2.1.5, we define a single mode coherent state of the EM field by

$$|\alpha_{s\lambda}(k)\rangle = e^{-\frac{|\alpha|^2}{2}} e^{\alpha b^\dagger_{s\lambda}(k)}\,|0\rangle\ , \tag{3.32}$$

where $\alpha = |\alpha|e^{i\theta}$ is a complex number. These states have the properties

$$\begin{aligned} b_{s'\lambda'}(k')\,|\alpha_{s\lambda}(k)\rangle &= \delta_{ss'}\delta_{\lambda\lambda'}\delta(k-k')\alpha\,|\alpha_{s\lambda}(k)\rangle\ , \\ \langle\alpha_{s'\lambda'}(k')|\alpha_{s\lambda}(k)\rangle &= \delta_{ss'}\delta_{\lambda\lambda'}\delta(k-k')\,. \end{aligned} \tag{3.33}$$

Using these properties, we find

$$\langle\alpha_{s\lambda}(k')|\mathbf{E}^{(b)}(x,t)|\alpha_{s\lambda}(k)\rangle = \delta(k-k')A_{mp}|\alpha|\sqrt{k}\cos(k(x-sct)+\phi+\theta)\mathbf{e}_\lambda\,, \tag{3.34}$$

where

$$A_{mp} = \sqrt{\frac{\hbar c}{4\pi\varepsilon_0 A}}\,, \tag{3.35}$$

which oscillates like a classical travelling wave.

Similar to the number states, these cannot correspond to physical states, because $||\,|\alpha_{s\lambda}(k)\rangle\,|| \neq 1$, but we can still use them to construct normalised states whose EM field expectation values will oscillate like classical waves. For example, if we define the state

$$|\psi_{\alpha,f}\rangle = \int_0^\infty \mathrm{d}k\, f(k)\,|\alpha_{s\lambda}(k)\rangle\ , \tag{3.36}$$

where

$$\int_0^\infty \mathrm{d}k\,|f(k)|^2 = 1\,, \tag{3.37}$$

then $|| \, |\psi_{\alpha,f}\rangle \, || = 1$ and

$$\langle \mathbf{E}^{(b)}(x,t)\rangle_{|\psi_{\alpha,f}\rangle} = 2A_{mp}\int_0^\infty \mathrm{d}k\, |f(k)|^2|\alpha|\sqrt{k}\cos\left(k(x-sct)+\phi+\theta\right)\mathbf{e}_\lambda\,, \quad (3.38)$$

which is just an integral over a collection of frequencies k with $|f(k)|^2$ acting like a weight function.

To be sure, we cannot quite call these coherent wave packets classical like yet as we still need to make sure it has minimum electric field fluctuations. In other words, we need to check that the electric field fluctuations for these $|\psi_{\alpha,f}\rangle$ states, $\Delta E_{|\psi_{\alpha,f}\rangle}$, is the same as the vacuum fluctuations, $\Delta E_{|0\rangle}$. However, we have $\Delta E_{|\psi_{\alpha,f}\rangle} \neq \Delta E_{|0\rangle}$. This makes sense because the $|\psi_{\alpha,f}\rangle$ states are not eigenstates of $\mathbf{E}^+(x,t)$, where

$$\mathbf{E}^+(x,t) = \sum_{s,\lambda}\int_0^\infty \mathrm{d}k\,\sqrt{\frac{\hbar ck}{4\pi\varepsilon_0 A}}\,\mathrm{e}^{i(sk(x-sct)+\phi)}\,b_{s\lambda}(k)\,\mathbf{e}_\lambda\,. \quad (3.39)$$

Therefore, to find our most classical EM field states, we do not just want a superposition of single-mode coherent states, but a coherent superposition of single-mode coherent states. We define the state

$$|\alpha^k_{s\lambda}\rangle = \exp\left(-\frac{1}{2}\int_0^\infty \mathrm{d}k|\alpha^k|^2\right)\exp\left(\int_0^\infty \mathrm{d}k\,\alpha^k b^\dagger_{s\lambda}(k)\right)|0\rangle\,, \quad (3.40)$$

where $\alpha^k = |\alpha^k|e^{\theta^k}$ maps each k onto a complex number such that the above integrals are well-defined (Zhang, 1999). These states satisfy $\langle\alpha^k_{s\lambda}|\alpha^k_{s\lambda}\rangle = 1$ as

$$\langle\beta^k_{s\lambda}|\alpha^k_{s\lambda}\rangle = \exp\left(-\frac{1}{2}\int_0^\infty \mathrm{d}k|\beta^k|^2 - \frac{1}{2}\int_0^\infty \mathrm{d}k|\alpha^k|^2 + \int_0^\infty \mathrm{d}k\beta^{k*}\alpha^k\right)\,, \quad (3.41)$$

and are eigenstates of $\mathbf{E}^+(x,t)$. To see this, we can compute directly

$$\mathbf{E}^+(x,t)\,|\alpha^k_{s\lambda}\rangle = A_{mp}\int_0^\infty \mathrm{d}k'\sqrt{k'}\alpha^{k'}e^{i(sk'(x-sct)+\phi)}\mathbf{e}_\lambda\,|\alpha^k_{s\lambda}\rangle\,, \quad (3.42)$$

where we have applied the commutator identity

$$[A,B^n] = nB^{n-1}[A,B] \quad \text{if} \quad [B,[A,B]] = 0\,, \quad (3.43)$$

to find

$$\left[b_{s'\lambda'}(k'),\left(\int_0^\infty \mathrm{d}k\alpha^k b^\dagger_{s\lambda}(k)\right)^n\right] = n\left(\int_0^\infty \mathrm{d}k\alpha^k b^\dagger_{s\lambda}(k)\right)^{n-1}\alpha^{k'}\delta_{ss'}\delta_{\lambda\lambda'}\,. \quad (3.44)$$

The E field expectation value of these states is therefore

$$\langle \mathbf{E}(x,t)\rangle_{|\alpha^k_{s\lambda}\rangle} = 2A_{mp}\int_0^\infty \mathrm{d}k\sqrt{k}|\alpha^k|\cos(k(x-sct)+\phi+\theta^k)\mathbf{e}_\lambda\,, \tag{3.45}$$

which looks a lot like Eq. (3.38); they are in fact equal if we set $\alpha^k = |f(k)|^2\alpha$. However, unlike the $|\psi_{\alpha,f}\rangle$ states, we have $\Delta E_{|\alpha^k_{s\lambda}\rangle} = \Delta E_{|0\rangle}$. We therefore call the $|\alpha^k_{s\lambda}\rangle$ states: coherent states of the EM field. We will come across these coherent states later in the paper when we describe how a mirror device locally changes the EM field.

3.3 Summary

- We reviewed a quantisation of the free space EM field propagating in free space. In this case, the EM field Eq. (3.30) total energy observables in Eq. (3.30) and Eq. (3.28), respectively, are represented by Hermitian operators. We reviewed the energy eigenstates of this system – monochromatic photons, which are fully nonlocal and states of well-defined momentum.

- We reviewed coherent states of the EM field and some of their properties. Of particular use is that their electric field expectation values have minimum uncertainty and oscillate like a classical sinusoidal mode.

Part II

New results

Chapter 4

The EM field as a biorthogonal system

Until now, we have considered the EM field a conventional quantum system. States evolve according to the usual Schrödinger equation, and the inner product used is the conventional one. In this chapter, we describe how to model the EM field using local bosonic operators, using biorthogonal physics (Brody, 2013) and complex conjugated time evolution operators. We discuss locality theorems in QFT and how they relate to our theory. Local modes of the EM field have been discussed before (Bialynicki-Birula, 1996; Hawton & Debierre, 2017; Smith & Raymer, 2007). What is new is that our system's inner product provides a unitary transformation between the local bosonic operators and monochromatic photon operators, and our local bosonic operators comprise operators that evolve according to the Heisenberg equation and complex conjugated Heisenberg equation. What is also new is that the biorthogonal system we shall use to model the EM field contains states that belong to the intersection of the Hilbert space and dual Hilbert space. We discuss the implications of this, and we describe how to compute the dynamics of our system.

4.1 The position space representation

In classical electrodynamics, we often employ a local description of the EM field when modelling experiments. Similarly, this thesis establishes a local description of electrodynamics but for the quantised EM field. To do so, we first have a closer look at the local EM field operators in Eq. (3.22). We can write these operators in the more compact form

$$\begin{aligned} \mathbf{E}(x) &= \sum_{s,\lambda} \sqrt{\frac{\hbar c}{2\varepsilon A}}\, a_{s\lambda}(x)\, \mathbf{e}_\lambda + H.c.\,, \\ \mathbf{B}(x) &= \sum_{s,\lambda} \frac{s}{c} \sqrt{\frac{\hbar c}{2\varepsilon A}}\, a_{s\lambda}(x)\, \mathbf{e}_x \times \mathbf{e}_\lambda + H.c.\,, \end{aligned} \tag{4.1}$$

where

$$a_{s\lambda}(x) = \int_0^\infty \mathrm{d}k \sqrt{\frac{k}{2\pi}}\, \mathrm{e}^{iskx}\, b_{s\lambda}(k)\,. \tag{4.2}$$

This operator contains all the position dependence of the EM field operators and so is the obvious candidate for a locally acting annihilation operator of the EM field. A pair of local creation and annihilation operators of the EM field is desirable as it will allow us to construct Hamiltonians that locally alter the EM field. In the Heisenberg picture and using Eq.(3.25), the time dependence of these operators is

$$a_{s\lambda}(x,t) = U^\dagger_{\text{free}}(t) a_{s\lambda}(x) U_{\text{free}}(t) = a_{s\lambda}(x - sct, 0)\,. \tag{4.3}$$

This equation means that the above operator can correspond to wave packets that move at the speed of light in the s direction.

Unfortunately, there is a problem. The operators $a_{s\lambda}(x)$ and $a^\dagger_{s\lambda}(x)$ do not obey bosonic commutator relations. To see this, suppose

$$|1_{s\lambda}(x)\rangle = a^\dagger_{s\lambda}(x)\,|0\rangle\,, \tag{4.4}$$

is the state of a single field excitation associated with the $a_{s\lambda}(x)$ operator. One can show that

$$\langle 1_{s\lambda}(x)|1_{s'\lambda'}(x')\rangle = \langle 0|\big[a_{s\lambda}(x), a^\dagger_{s'\lambda'}(x')\big]|0\rangle\,, \tag{4.5}$$

where

$$\left[a_{s\lambda}(x), a^{\dagger}_{s'\lambda'}(x')\right] = \delta_{ss'}\,\delta_{\lambda\lambda'}\int_0^{\infty} \mathrm{d}k\,\frac{k}{2\pi}\,\mathrm{e}^{\mathrm{i}sk(x-x')}\,. \tag{4.6}$$

Therefore, single excitation states in Eq. (4.4) are not pairwise orthogonal. Therefore, the $a_{s\lambda}(x)$ and the $a^{\dagger}_{s\lambda}(x)$ are not the bosonic annihilation and creation operators of highly localised field excitations we want them to be and so we need to adjust our theory.

4.2 A description of the quantised EM field in position space using local bosonic operators

In this section, we construct annihilation operators of highly localised excitations. To do so, we identify operators, $\tilde{a}_{s\lambda}(x)$, such that

$$[\tilde{a}_{s\lambda}(x), a^{\dagger}_{s'\lambda'}(x')] = \delta_{ss'}\,\delta_{\lambda\lambda'}\,\delta(x-x')\,. \tag{4.7}$$

We use a tilde here to emphasise that $\tilde{a}_{s\lambda}(x) \neq a_{s\lambda}(x)$. To achieve this identification, we use Eq. (4.6) as a starting point so that, when $s = s'$ and $\lambda = \lambda'$, we want Eq. (4.7) to be the delta function

$$\delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k\,e^{ik(x-x')}\,. \tag{4.8}$$

Comparing the two equations, we see there are effectively two things we need to do: decrease the lower k integral limit from 0 to $-\infty$, and remove the k factor inside the integral of Eq. (4.6). Getting an $\tilde{a}_{s\lambda}(x)$ that does this requires doubling the degrees of freedom of the EM field to get our desired k range.

4.2.1 Doubling the degrees of freedom of the quantised EM field

A closer look at the EM field operators shows we can reproduce any E or B expectation value satisfying Maxwell's equations by complex conjugating both the

operator and state within that expectation value. For example, if $\langle\psi(t)|\mathbf{E}(x)|\psi(t)\rangle$ is real then it is equal to its complex conjugate

$$\begin{aligned}\langle\psi(t)|\mathbf{E}(x)|\psi(t)\rangle &= (\langle\psi(t)|\mathbf{E}(x)|\psi(t)\rangle)^* \\ &= \langle\psi(t)|^*\mathbf{E}^*(x)|\psi(t)\rangle^* \ . \end{aligned} \tag{4.9}$$

Therefore, it is tempting to think that these complex conjugate states, $|\psi(t)\rangle^* = U^*(t)\,|\psi(0)\rangle^*$, and operators, $\mathbf{E}^*(x)$, are redundant as they seem to give us no new, or changing of, information that we extract from a system. Here we show that they have a use: they can extend our k range from 0 to $-\infty$ in Eq. (4.6), which will allow us to construct Hamiltonians that locally alter the EM operators at specific locations. Furthermore, one often associates complex conjugation of states or operators with parity-time, or PT, physics (Barnett *et al.*, 2000; Vaccaro, 2011, 2016). Time reversal in quantum physics is, after all, an anti-unitary operator. For example, a classical E-wave, $\cos(kx - \omega t + \phi)$, is indistinguishable if we make the substitutions $x \to -x$, $t \to -t$ and $\phi \to -\phi$ as $\cos(-kx + \omega t - \phi) = \cos(kx - \omega t + \phi)$. However, we do not dwell on the philosophical implications or origins of these complex conjugate states, as this is not the purpose of this thesis – this thesis shows how we can use them. In particular, how we can use them to construct bosonic Fock operators that have local commutation relations.

More concretely, we now define additional bosonic photon annihilation and creation operators, $c_{s\lambda}(k)$ and $c^\dagger_{s\lambda}(k)$, that commute with the original photon operators. So that $[c_{s\lambda}(k), c^\dagger_{s'\lambda'}(k')] = \delta_{ss'}\delta_{\lambda\lambda'}\delta(k-k')$, where s and λ still denote direction of propagation and polarisation, respectively. However, in contrast to the original photon states, these states evolve according to the complex conjugate Schrödinger equation. If a state, $|\psi\rangle$, evolves according to the complex conjugate Schrödinger equation then its time evolution is governed by the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = \frac{i}{\hbar}H^*\,|\psi(t)\rangle \ . \tag{4.10}$$

where H^* is the complex conjugate of the systems Hamiltonian. Using H^* has important consequences when we model systems that are more complex than free space, where $H^*_{\text{free}} = H_{\text{free}}$. We will see examples of such Hamiltonians in the

next section. We refer to these photons that evolve according to the complex conjugate Schödinger equation as c photons.

Using this notation, we introduce additional terms to the free space Hamiltonian

$$H^{(c)}_{\text{free}} = \sum_{s,\lambda} \int_0^\infty \mathrm{d}k\, \hbar ck\, c^\dagger_{s\lambda}(k) c_{s\lambda}(k)\,, \tag{4.11}$$

and EM field observables

$$\begin{aligned} \mathbf{E}^{(c)}(x) &= \sum_{s,\lambda} \int_0^\infty \mathrm{d}k \sqrt{\frac{\hbar ck}{4\pi\varepsilon A}}\, \mathrm{e}^{-\mathrm{i}(skx+\phi)}\, c_{s\lambda}(k)\, \mathbf{e}_\lambda + H.c.\,, \\ \mathbf{B}^{(c)}(x) &= \sum_{s,\lambda} \int_0^\infty \mathrm{d}k \frac{s}{c} \sqrt{\frac{\hbar ck}{4\pi\varepsilon A}}\, \mathrm{e}^{-\mathrm{i}(skx+\phi)}\, c_{s\lambda}(k)\, \mathbf{e}_x \times \mathbf{e}_\lambda + H.c.\,. \end{aligned} \tag{4.12}$$

An observable A, which acts only on the c photon states, in the Heisenberg picture now satisfies a complex conjugate Heisenberg equation,

$$\begin{aligned} A(t) &= U^{\dagger *}(t) A U^*(t)\,, \\ \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t} A(t) &= \frac{\mathrm{i}}{\hbar}\left[A(t), H^*\right], \end{aligned} \tag{4.13}$$

if A has no implicit time-dependence so that

$$\frac{\partial}{\partial t} A = 0\,. \tag{4.14}$$

Hence, in contrast to the dynamics of the b photons in Eq. (3.29), the c operators evolve such that

$$\begin{aligned} c_{s\lambda}(k,t) &= \mathrm{e}^{\mathrm{i}ckt}\, c_{s\lambda}(k,0)\,, \\ c^\dagger_{s\lambda}(k,t) &= \mathrm{e}^{-\mathrm{i}ckt}\, c^\dagger_{s\lambda}(k,0)\,, \end{aligned} \tag{4.15}$$

in the Heisenberg picture.

We set the free space Hamiltonian and EM field operators equal to the sum of the b and c terms

$$\begin{aligned} H_{\text{free}} &= H^{(b)}_{\text{free}} + H^{(c)}_{\text{free}}\,, \\ \mathbf{E}(x) &= \mathbf{E}^{(b)}(x) + \mathbf{E}^{(c)}(x)\,, \\ \mathbf{B}(x) &= \mathbf{B}^{(b)}(x) + \mathbf{B}^{(c)}(x)\,. \end{aligned} \tag{4.16}$$

We now have an over-complete description of real EM field expectation values in the sense that for every b photon state we can find a c photon state such that they both produce the same EM field expectation values. However, this over-completeness allows us to construct annihilation operators for highly localised field excitations that remain local under time evolution. In the following, we denote

$$\begin{aligned} |1_{s\lambda}(k)\rangle_b &= b^\dagger_{s\lambda}(k)\,|0\rangle\ , \\ |1_{s\lambda}(k)\rangle_c &= c^\dagger_{s\lambda}(k)\,|0\rangle\ , \end{aligned} \tag{4.17}$$

and we shall use the b, c subscripts on a state or operator to denote the same operator except with its c or b photon operators set to zero, respectively. For example, if $|\psi\rangle = |1_{s\lambda}(k)\rangle_b + |1_{s\lambda}(k)\rangle_c$ then $|\psi\rangle_b = |1_{s\lambda}(k)\rangle_b$. Likewise, $H_{free,b} = H^{(b)}_{\text{free}}$ and $H_{\text{free,c}} = H^{(c)}_{\text{free}}$. We have essentially taken a vector sum of two copies of the total Hilbert space, the difference between the two copies is that they have different time evolution equations.

As we now have two time evolution equations, we define the time evolution operator of a system with Hamiltonian H to be

$$\overline{U}(t) = \exp\left(-\frac{i}{\hbar}\int_0^t \mathrm{d}t(H_b - H_c^*)\right). \tag{4.18}$$

By using the Baker-Campbell-Hausdorff formula (Gerry & Knight, 2004), we also have

$$\overline{U}(t) = U_b(t)U_c^*(t)\,. \tag{4.19}$$

Therefore, in the Schrödinger picture the time derivative of a state, $|\psi(t)\rangle$, that is initially $|\psi\rangle$ is

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle &= \frac{\mathrm{d}}{\mathrm{d}t}\overline{U}(t)\,|\psi\rangle \\ &= -\frac{i}{\hbar}H_b\,|\psi(t)\rangle + \frac{i}{\hbar}H_c^*\,|\psi(t)\rangle \end{aligned} \tag{4.20}$$

Therefore, the time derivative of an initial operator with no explicit time

dependence operator, A, in the Heisenberg picture is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}A(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\overline{U}^{\dagger}(t)A\overline{U}(t)\right)\\ &= \overline{U}^{\dagger}(t)\left(-\frac{i}{\hbar}\left[A,H_b\right]\right)\overline{U}(t)+\overline{U}^{\dagger}(t)\left(\frac{i}{\hbar}\left[A,H_c^*\right]\right)\overline{U}(t)\\ &= -\frac{i}{\hbar}\left[A(t),H_b\right]+\frac{i}{\hbar}\left[A(t),H_c^*\right]. \end{aligned} \tag{4.21}$$

From this we conclude that the time derivative of such an operator is 0 if

$$\left[A,H_b\right]-\left[A,H_c^*\right]=0\,. \tag{4.22}$$

4.2.2 Local bosonic operators of the EM field

In this section we adjust $a_{s\lambda}^{\dagger}(x)$ and $a_{s\lambda}(x)$ so that they form bosonic operators. Taking the electric field observable $\mathbf{E}(x)$ in Eq. (4.16), we write it in the form

$$\begin{aligned}\mathbf{E}(x) &= \sum_{s,\lambda}\sqrt{\frac{\hbar c}{2\varepsilon A}}\,a_{s\lambda}(x)\,\mathbf{e}_{\lambda}+H.c.\,,\\ \mathbf{B}(x) &= \sum_{s,\lambda}\frac{s}{c}\sqrt{\frac{\hbar c}{2\varepsilon A}}\,a_{s\lambda}(x)\,\mathbf{e}_x\times\mathbf{e}_{\lambda}+H.c.\,,\end{aligned} \tag{4.23}$$

which looks a lot like Eq. (4.1) except we now have

$$a_{s\lambda}^{\dagger}(x)=\int_0^{\infty}\mathrm{d}k\sqrt{\frac{|k|}{2\pi}}\left[\mathrm{e}^{-\mathrm{i}skx}\,b_{s\lambda}^{\dagger}(k)+\mathrm{e}^{\mathrm{i}skx}\,c_{s\lambda}^{\dagger}(k)\right], \tag{4.24}$$

which we call a local bosonic creation operator.

We can also write the electric field operator as

$$\mathbf{E}(x)=\sum_{s,\lambda}\mathrm{E}_{s\lambda}(x)\mathbf{e}_{\lambda}\,, \tag{4.25}$$

where

$$\mathrm{E}_{s\lambda}(x)=\sqrt{\frac{\hbar c}{2\varepsilon A}}\,a_{s\lambda}(x)+H.c. \tag{4.26}$$

This representation will be useful for later. However, to justify this name we still need an operator that commutes with the local bosonic operator to a δ function –

this will be its corresponding annihilation operator. We call this operator $a^{\rm bio}_{s\lambda}(x)$ and define it by

$$a^{bio}_{s\lambda}(x) = \int_0^\infty \mathrm{d}k \sqrt{\frac{1}{2\pi|k|}} \left[\mathrm{e}^{\mathrm{i}skx}\, b_{s\lambda}(k) + \mathrm{e}^{-\mathrm{i}skx}\, c_{s\lambda}(k)\right] . \tag{4.27}$$

Clearly, we have

$$\left[a^{bio}_{s\lambda}(x), a^{\dagger}_{s'\lambda'}(x')\right] \;=\; \delta_{ss'}\,\delta_{\lambda\lambda'}\,\delta(x-x')\,, \tag{4.28}$$

which is the desired bosonic commutation relation. Therefore, $a^{\dagger}_{s\lambda}(x)$ and $a^{bio}_{s\lambda}(x)$ are a local bosonic creation and annihilation operator pair. We have added the absolute symbol for convenience as it will be useful for comparing to the negative frequency case later, it does not affect the local commutator above. Computing the time evolution of these local operators we find

$$\begin{aligned} \overline{U}^{\dagger}(t) a^{\dagger}_{s\lambda}(x) \overline{U}(t) &= a^{\dagger}_{s\lambda}(x - sct)\,, \\ \overline{U}^{\dagger}(t) a_{s\lambda}(x) \overline{U}(t) &= a_{s\lambda}(x - sct)\,. \end{aligned} \tag{4.29}$$

However, different from conventional quantum theory, the creation operator $a^{\dagger}_{s\lambda}(x)$ is no longer the Hermitian conjugate with respect to the usual inner product of its corresponding annihilation operator. In other words, $a_{s\lambda}(x,t)$ is not the annihilation operator corresponding to the creation operator $a^{\dagger}_{s\lambda}(x)$. By taking the Hermitian conjugate of Eq. (4.28), we get a second pair of operators, $a_{s\lambda}(x)$ and $a^{\rm bio\,\dagger}_{s\lambda}(x)$, that also have a local bosonic commutator relation

$$\left[a_{s\lambda}(x), a^{\rm bio\,\dagger}_{s'\lambda'}(x')\right] \;=\; \delta_{ss'}\,\delta_{\lambda\lambda'}\,\delta(x-x')\,. \tag{4.30}$$

In the following, we refer to the second pair as the bio-local operators. So a single mode EM field operator contains two local bosonic operators where each operator belongs to a different creation/annihilation operator pair. We therefore define the bio local states

$$|1_{s\lambda}(x)\rangle^{bio} \;=\; a^{bio\,\dagger}_{s\lambda}(x)\,|0\rangle\;, \tag{4.31}$$

as the EM field operators contain the annihilation operator corresponding to this excitation.

We can represent the monochromatic photons in terms of the local bosonic operators and bio local bosonic operators via the transformations

$$\begin{aligned}
b^\dagger_{s\lambda}(k) &= \int_{-\infty}^{\infty} \mathrm{d}x\, \frac{1}{\sqrt{2\pi|k|}}\, \mathrm{e}^{iskx}\, a^\dagger_{s\lambda}(x) = \int_{-\infty}^{\infty} \mathrm{d}x\, \sqrt{\frac{|k|}{2\pi}}\, \mathrm{e}^{\mathrm{i}skx}\, a^{\dagger bio}_{s\lambda}(x)\,, \\
b_{s\lambda}(k) &= \int_{-\infty}^{\infty} \mathrm{d}x\, \sqrt{\frac{|k|}{2\pi}}\, \mathrm{e}^{-\mathrm{i}skx}\, a^{\mathrm{bio}}_{s\lambda}(x) = \int_{-\infty}^{\infty} \mathrm{d}x\, \frac{1}{\sqrt{2\pi|k|}}\, \mathrm{e}^{-\mathrm{i}skx}\, a_{s\lambda}(x)\,. \\
c^\dagger_{s\lambda}(k) &= \int_{-\infty}^{\infty} \mathrm{d}x\, \frac{1}{\sqrt{2\pi|k|}}\, \mathrm{e}^{iskx}\, a^\dagger_{s\lambda}(-x) = \int_{-\infty}^{\infty} \mathrm{d}x\, \sqrt{\frac{|k|}{2\pi}}\, \mathrm{e}^{iskx}\, a^{\dagger bio}_{s\lambda}(-x)\,, \\
c_{s\lambda}(k) &= \int_{-\infty}^{\infty} \mathrm{d}x\, \sqrt{\frac{|k|}{2\pi}}\, \mathrm{e}^{-iskx}\, a^{bio}_{s\lambda}(-x) = \int_{-\infty}^{\infty} \mathrm{d}x\, \frac{1}{\sqrt{2\pi|k|}}\, \mathrm{e}^{-iskx}\, a_{s\lambda}(-x)\,.
\end{aligned} \tag{4.32}$$

Photons, local states and bio local states, share the same counting operator

$$\begin{aligned}
N &= \int_0^{\infty} \mathrm{d}k\, (a^\dagger_{s\lambda}(k) a_{s\lambda}(k) + b^\dagger_{s\lambda}(k) b_{s\lambda}(k)) \\
&= \int_{-\infty}^{\infty} \mathrm{d}x\, a^{\dagger\, bio}_{s\lambda}(x) a_{s\lambda}(x) \\
&= \int_{-\infty}^{\infty} \mathrm{d}x\, a^\dagger_{s\lambda}(x) a^{bio}_{s\lambda}(x)\,.
\end{aligned} \tag{4.33}$$

With this number operator it is easy to see that the following state

$$|n_{s\lambda}(x)\rangle = \frac{1}{\sqrt{n!}} \left(a^\dagger_{s\lambda}(x)\right)^n |0\rangle\ , \tag{4.34}$$

contains n local bosons in the s, λ, x mode, since $N\,|n_{s\lambda}(x)\rangle = n\,|n_{s\lambda}(x)\rangle$. Therefore, we call N the boson number operator of the EM field, as opposed to just the photon number operator.

We can represent the local bosonic operators in terms of their bio operators, and vice-versa, by

$$\begin{aligned}
a_{s\lambda}(x) &= \int_{\infty}^{\infty} \mathrm{d}x' \int_0^{\infty} \mathrm{d}k \frac{|k|}{\pi} \cos(k(x-x'))\, a^{\mathrm{bio}}_{s\lambda}(x')\,, \\
a^{\mathrm{bio}}_{s\lambda}(x) &= \int_{\infty}^{\infty} \mathrm{d}x' \int_0^{\infty} \mathrm{d}k \frac{1}{\pi|k|} \cos(k(x-x'))\, a_{s\lambda}(x')\,.
\end{aligned} \tag{4.35}$$

We can use these representations to write the free space free space Hamiltonian in Eq. (4.16) in terms of the local and bio-local operators

$$\begin{aligned} H_{\text{free}} &= \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}x' \int_{0}^{\infty} \mathrm{d}k \frac{\hbar c k}{\pi} \cos(sk(x-x')) a^{\dagger}_{s\lambda}(x) a^{bio}_{s\lambda}(x') \\ &= \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}x' \int_{0}^{\infty} \mathrm{d}k \frac{\hbar c k}{\pi} \cos(sk(x-x')) a^{\dagger\, bio}_{s\lambda}(x) a_{s\lambda}(x') \,. \end{aligned} \tag{4.36}$$

We only write the above forms of the Hamiltonian to be complete, the original one written in terms of monochromatic photon operators works fine for our purposes.

For the energy Hamiltonian, we take the classical expression for energy of the electromagnetic field in 1-d (Jackson, 1975)

$$H_{\text{eng}} = \frac{A}{2} \int_{-\infty}^{\infty} \mathrm{d}x \left[\varepsilon_0 \mathbf{E}(x,t)^2 + \frac{1}{\mu_0} \mathbf{B}(x,t)^2 \right] , \tag{4.37}$$

and substitute into it our field observables from Eq. (4.23). We find that the energy observable is now

$$\begin{aligned} H_{\text{eng}} &= \sum_{s,\,\lambda} \int_{-\infty}^{\infty} \mathrm{d}x \, \hbar c \xi^{\dagger}_{s\lambda}(x) \xi_{s\lambda}(x) \\ &= \sum_{s,\,\lambda} \int_{0}^{\infty} \mathrm{d}k \, \hbar c \frac{k}{2} \left(b^{\dagger}_{s\lambda}(k) b_{s\lambda}(k) + c^{\dagger}_{s\lambda}(k) c_{s\lambda}(k) + 2 b^{\dagger}_{s\lambda}(k) c^{\dagger}_{s\lambda}(k) + \text{H.c.} \right) , \end{aligned} \tag{4.38}$$

where

$$\xi_{s\lambda}(x) = a_{s\lambda}(x) + a^{\dagger}_{s\lambda}(x) \,. \tag{4.39}$$

Since this is a positive operator, both b and c photons have only positive energy expectation values. We need the additional interference terms between the b and c modes as both modes can produce real EM fields on the x-axis. The interference between these real fields will, of course, affect the total energy. As a state can evolve according to one of two dynamical equations, we are not thinking of the energy observable as the operator that generates dynamics, since both types of states will contribute to the systems' total energy. However, when we consider either only the b photons or only the c photons, the free space Hamiltonian and energy observable are the same as we only have one dynamical equation of motion. Nevertheless, H_{eng} is still the operator that gives energy expectation values.

Using Eq. (3.26), we define

$$\begin{aligned} x^b_{s\lambda}(k) &= \sqrt{\frac{\hbar}{2ck}}\left(b_{s\lambda}(k)+b^\dagger_{s\lambda}(k)\right), \\ p^b_{s\lambda}(k) &= -i\sqrt{\frac{\hbar ck}{2}}\left(b_{s\lambda}(k)-b^\dagger_{s\lambda}(k)\right), \\ x^c_{s\lambda}(k) &= \sqrt{\frac{\hbar}{2ck}}\left(c_{s\lambda}(k)+c^\dagger_{s\lambda}(k)\right), \\ p^c_{s\lambda}(k) &= i\sqrt{\frac{\hbar ck}{2}}\left(c_{s\lambda}(k)-c^\dagger_{s\lambda}(k)\right). \end{aligned} \tag{4.40}$$

Here we have assumed that $[x^c_{s\lambda}(k), p^c_{s\lambda}(k')]$ is the complex conjugate of $[x^b_{s\lambda}(k), p^b_{s\lambda}(k')]$ so that

$$\begin{aligned} [x^b_{s\lambda}(k), p^b_{s\lambda}(k')] &= i\hbar\delta(k-k'), \\ [x^c_{s\lambda}(k), p^c_{s\lambda}(k')] &= -i\hbar\delta(k-k'). \end{aligned} \tag{4.41}$$

Using the above operators, we can write the energy Hamiltonian in the form

$$\begin{aligned} H_{\text{eng}} = \sum_{s,\lambda}\int_0^\infty \mathrm{d}k\,\frac{1}{2}\Big((ck)^2 x^b_{s\lambda}(k)^2 + p^b_{s\lambda}(k)^2 + (ck)^2 x^c_{s\lambda}(k)^2 + p^c_{s\lambda}(k)^2\Big) \\ + \Big((ck)^2 x^b_{s\lambda}(k)x^c_{s\lambda}(k) + p^b_{s\lambda}(k)p^c_{s\lambda}(k)\Big). \end{aligned} \tag{4.42}$$

The first line is the sum of all the independent harmonic oscillator Hamiltonians corresponding to the b and c photons, see Eq. (3.28), and the second line corresponds to the interference between a b oscillator with wave number k and a c oscillator with wave number k, for all k.

Computing the time evolution of the energy operator with respect to the free space Hamiltonian, we find

$$\overline{U}^\dagger(t) H_{\text{eng}} \overline{U}(t) = H_{\text{eng}}, \tag{4.43}$$

because

$$[H_{\text{eng}}, H_{\text{free}\,b}] - [H_{\text{eng}}, H^*_{\text{free}\,c}] = 0. \tag{4.44}$$

The energy observable is therefore invariant under time translations[1] and so the

[1] It would be interesting to derive the Lagrangian of this system apply Noether's theorem (Peskin & Schroeder, 1995) to deduce conserved quantities. However, this is beyond the scope of this thesis.

energy expectation values of states are conserved because

$$\langle\psi|\overline{U}^{\dagger}(t)H_{\text{eng}}\overline{U}(t)|\psi\rangle = \langle\psi|H_{\text{eng}}|\psi\rangle \ . \tag{4.45}$$

We point out that $H_{\text{free}} = H^{*}_{\text{free}}$. This means that instead of using complex conjugate dynamical equations, we could have introduced negative frequencies instead for this case. We will discuss negative frequencies in more detail later in this thesis.

So, for each creation operator $a_{s\lambda}(x)$ and annihilation operator $a^{\dagger}_{s\lambda}(x)$ within the EM field observable we have identified a corresponding annihilation and creation operator, respectively. However, under the standard inner product, the inner product between two local states, $|1_{s\lambda}(x)\rangle$ and $|1_{s'\lambda'}(x')\rangle$, is

$$\langle 1_{s'\lambda'}(x')|1_{s\lambda}(x)\rangle = \delta_{ss'}\delta_{\lambda\lambda'}\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}k|k|e^{ik(x-x')} \neq \delta_{ss'}\delta_{\lambda\lambda'}\delta(x-x')\,. \tag{4.46}$$

Not only can this equation be non-zero when $x \neq x'$, but due the the $|k|$ term in the integral it also does not have the right units if the overlap between a local state at x and a local state at x' is to be considered as a probability density. However, we could still construct states that are an integral over these local states such that their overlap is a unit-less number. For example, the state

$$|\psi\rangle = \int_{-\infty}^{\infty}\mathrm{d}x\, f(x)\,|1_{s\lambda}(x)\rangle \ , \tag{4.47}$$

can be normalised to the unit-less number 1 if $f(x)$ has units $m^{\frac{1}{2}}$.

Furthermore, in contrast to the local operators in Eq. (4.2), there exists one-to-one transformations between the momentum space Fock operators and the position space Fock operators. However, these transformations are non-unitary with respect to the standard inner product. Therefore, although the $|1_{s\lambda}(k)\rangle$ states form an orthonormal basis, the $|1_{s\lambda}(x)\rangle$ states will not form an orthonormal basis, and so we need to adjust our inner product to make these local states pairwise orthogonal.

We make this adjustment by identifying the quantised EM field as a biorthogonal system (Brody, 2013; Mostafazadeh, 2010). This identification will generalise our Hilbert space's scalar product to manage exact calculations, i.e. compute expectation values, with systems that locally alter EM fields. However, we still want

to include monochromatic modes in our Hilbert space, so we need to make sure that $|1_{s\lambda}(k)\rangle$ states remain pairwise orthogonal with respect to the generalised inner product. So we want an inner product where the transformation between the local bosonic states and monochromatic photon states is unitary. We describe this in detail in the next section. However, before we do, we describe another way to obtain local bosons of the EM field.

Using negative frequencies and a scaling operator $\mathcal{R}$ to construct local modes of the EM field

Earlier, we alluded to using negative frequencies monochromatic photons instead of monochromatic photons that evolve with respect to a complex conjugate Schrödinger equation. In this scenario, states only evolve according to the conventional Schrödinger equation, but we introduce monochromatic photon states that have negative eigenvalues so that k in $b_{s\lambda}(k)$ can be a value in $(-\infty, \infty)$. We refer to those $b_{s\lambda}(k)$ modes where $k \in (-\infty, 0)$ as negative frequency modes. When we take these negative frequencies into account, the free space Hamiltonian is

$$H_{\text{free}}^{\text{neg}} = \sum_{s,\lambda} \int_{-\infty}^{\infty} \mathrm{d}k \, \hbar c k \, b_{s\lambda}^{\dagger}(k) b_{s\lambda}(k) \,, \tag{4.48}$$

where the lower integral limit has been extended from 0 to $-\infty$.

Suppose instead we defined the EM field operators by (Cook, 1982a,b; Hodgson *et al.*, 2021)

$$\begin{aligned} \mathbf{E}(x) &= \sum_{s,\lambda} \sqrt{\frac{\hbar c}{2\varepsilon A}} \, \mathcal{R}(A_{s\lambda}(x)) \, \mathbf{e}_{\lambda} + H.c. \,, \\ \mathbf{B}(x) &= \sum_{s,\lambda} \frac{s}{c} \sqrt{\frac{\hbar c}{2\varepsilon A}} \, \mathcal{R}(A_{s\lambda}(x)) \, \mathbf{e}_x \times \mathbf{e}_{\lambda} + H.c. \,, \end{aligned} \tag{4.49}$$

where

$$A_{s\lambda}(x) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}k \, \mathrm{e}^{\mathrm{i}skx} \, b_{s\lambda}(k) \,. \tag{4.50}$$

and $\mathcal{R}$ is a super operator such that

$$\mathcal{R}(b_{s\lambda}(k)) = \sqrt{|k|} b_{s\lambda}(k) \,. \tag{4.51}$$

We could just show the square root term when needed instead of using the $\mathcal{R}$ operator, however, we have included it here to be consistent with Hodgson *et al.* (2021). We therefore see that the EM fields in this case

$$\begin{aligned}\mathbf{E}(x) &= \sum_{s,\lambda}\int_{-\infty}^{\infty} \mathrm{d}k \sqrt{\frac{\hbar c k}{4\pi\varepsilon_0 A}}\, \mathrm{e}^{\mathrm{i}skx}\, b_{s\lambda}(k)\, \mathbf{e}_\lambda + H.c.\,, \\ \mathbf{B}(x) &= \sum_{s,\lambda}\int_{-\infty}^{\infty} \mathrm{d}k \frac{s}{c} \sqrt{\frac{\hbar c k}{4\pi\varepsilon_0 A}}\, \mathrm{e}^{\mathrm{i}skx}\, b_{s\lambda}(k)\, \mathbf{e}_x \times \mathbf{e}_\lambda + H.c.\,, \end{aligned} \tag{4.52}$$

so that they are equal to the usual EM field operators except with the lower integral limit extended from 0 to $-\infty$ (Bennett *et al.*, 2015). This was the route taken in Hodgson *et al.* (2021).

Therefore, in the context of negative frequencies, the energy operator is

$$H_{\mathrm{eng}}^{\mathrm{neg}} = \sum_{s,\,\lambda}\int_{-\infty}^{\infty} \mathrm{d}k\, \hbar c \frac{|k|}{2}\left(b_{s\lambda}^\dagger(k) b_{s\lambda}(k) + b_{s\lambda}^\dagger(k) b_{s\lambda}^\dagger(-k) + \mathrm{H.c.}\right), \tag{4.53}$$

which we found by substituting EM field operators in Eq. (4.49) into Eq. (3.9). We see that negative frequency photons still have positive energies.

Here we have

$$[A_{s\lambda}(x), A_{s'\lambda'}^\dagger(x')] = \delta_{ss'}\delta_{\lambda\lambda'}\delta(x-x')\,, \tag{4.54}$$

so that the creation operator corresponding to $A_{s\lambda}(x)$ is its Hermitian conjugate. Therefore, when the option is available, it might be more desirable to work with the $A_{s\lambda}(x)$ as opposed to the $\mathcal{R}(A_{s\lambda}(x))$ operators, as it could lead to simpler calculations. For example, when we examine the effect of the time evolution operator generated by the free space Hamiltonian in Eq. 4.48 we find

$$U_{\mathrm{free}}^\dagger(t,0)\mathcal{R}(A_{s\lambda}(x))U_{\mathrm{free}}(t,0) = \mathcal{R}\left(U_{\mathrm{free}}(t,0)A_{s\lambda}(x)U_{\mathrm{free}}(t,0)\right)\,, \tag{4.55}$$

where

$$U_{\mathrm{free}}(t,0) = \exp\left(-\frac{i}{\hbar}H_{\mathrm{free}}^{\mathrm{neg}}\right)\,. \tag{4.56}$$

In fact, Eq. (4.55) holds for any transformation that couples $b_{s\lambda}(k)$ modes to $b_{s'\lambda'}(k)$ or $b_{s'\lambda'}(-k)$ modes (Hodgson *et al.*, 2021). However, any transformation that does this has to be wholly delocalised in position space. An example of this is a Hamiltonian that couples a k mode to a k mode. Additionally, scattering

operators of locally acting Hamiltonians can also simply couple k to k (Southall *et al.*, 2021). In these scenarios, it might be straightforward to use the $a_{s\lambda}(x)$ operators, as opposed to the $A_{s\lambda}(x)$, operators, as we can apply the $\mathcal{R}$ super operator at the end of the calculation rather than during it.

Unfortunately, Eq. (4.55) does not apply when $U(t,0)$ is a transformation with local properties, i.e. when it does not couple $b_{s\lambda}(k)$ modes to an $b_{s'\lambda'}(k)$ or $b_{s'\lambda'}(-k)$ mode. For example, a time evolution operator corresponding to a Hamiltonian with a locally acting term. Perhaps this is because the $A_{s\lambda}(x)$ operators relate to the $\mathcal{R}(A_{s\lambda}(x))$ operators via the equation

$$A_{s\lambda}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}x' \int_{-\infty}^{\infty} \mathrm{d}k \, \frac{1}{\sqrt{|k|}} e^{isk(x-x')} \mathcal{R}(A_{s\lambda}(x')) \, . \tag{4.57}$$

Therefore, the $A_{s\lambda}(x)$ and $\mathcal{R}(A_{s\lambda}(x))$ operators are not locally connected in the sense that $A_{s\lambda}(x_0)$ is equal to a superposition of the $\mathcal{R}(A_{s\lambda}(x))$ operators at places other than x_0. This relationship means if we have a transformation that alters the $a_{s\lambda}(x)$ operators only at a position x_0, this transformation will alter the $A_{s\lambda}(x)$ operators at multiple places. In other words, a transformation that couples $A_{s\lambda}(x_1)$ only to $A_{s\lambda}(x_2)$ won't necessarily couple $\mathcal{R}(A_{s\lambda}(x_1))$ only to $\mathcal{R}(A_{s\lambda}(x_2))$.

A novel result in Southall *et al.* (2021) was using the $A_{s\lambda}(x)$ to construct locally acting mirror Hamiltonians and states that reproduce the dynamics of the classical mirror image method. However, the results in Southall *et al.* (2021) was not enough to reproduce the dynamics of the classical mirror image method for the EM field operators themselves. We discuss the details of such Hamiltonians in the next chapter and how they relate to Hamiltonians that reproduce the dynamics of the classical mirror image method for the EM field operators.

To finish this section, we point out that we could identify an operator $\tilde{A}^{\dagger}_{s\lambda}(x)$ such that

$$[\mathcal{R}(A_{s\lambda}(x)), \tilde{A}^{\dagger}_{s\lambda}(x)] = \delta_{ss'}\delta_{\lambda\lambda'}\delta(x-x') \, , \tag{4.58}$$

to help us obtain local transformations of the EM field using negative frequencies. However, later in this thesis, we explain why using the concept of negative frequencies is more restrictive than using modes that evolve using complex conjugate equations.

4.2.3 The inner product

Defining a suitable inner product on our Hilbert is of vital importance. Quantum theory is, after all, an abstract probability space that we can use as a tool to predict experimental outcomes. If we cannot extract probabilities from the theory we present in this thesis, it is not very helpful. We construct an orthonormal basis out of the local states $|1_{s\lambda}(x)\rangle$ so that if we have a local state at x, then we have a 0 probability of measuring it to be at $x' \neq x$. This orthogonal property will make it easier for us to construct Hamiltonians that act on the EM field only at specific locations.

Adjusting the inner product from the conventional one is something that many have done in the context of local quantum theories of light (Hawton & Debierre, 2017; Smith & Raymer, 2007). In particular, Bialynicki-Birula (1996) used an inner product for photon wave functions in coordinate representation. This is fine for states that are normalised with respect to this adjusted inner product, but in our case we need to be careful so that we preserve the normalisation of monochromatic photons. Preserving the normalisation of monochromatic photons is something we wish to retain as they can be used, for example, to construct coherent states of the EM field.

We take the standard inner product as a starting point, where we define the inner product between two states $|\psi\rangle, |\phi\rangle$ by

$$\langle |\psi\rangle, |\phi\rangle \rangle = \langle \phi|\psi\rangle \ . \tag{4.59}$$

Clearly, $\langle |1_{s\lambda}(x)\rangle, |1_{s\lambda}(x')\rangle \rangle \neq \delta(x - x')$. To address this, we point out that for every $|1_{s\lambda}(x)\rangle$ there is a $|1_{s\lambda}(x')\rangle^{\text{bio}}$ such that

$$\langle |1_{s\lambda}(x')\rangle, |1_{s\lambda}(x)\rangle^{\text{bio}} \rangle = \delta(x - x') \, . \tag{4.60}$$

Therefore, it would be useful to identify our Hilbert space as a biorthogonal system where the $|1_{s\lambda}(x')\rangle^{\text{bio}}$ are the biorthogonal basis states to the $|1_{s\lambda}(x)\rangle$ states; as we have local creation operators in the EM field observables. Likewise, we have that for every $|1_{s\lambda}(x)\rangle^{\text{bio}}$ there is a $|1_{s\lambda}(x')\rangle$ such that

$$\langle |1_{s\lambda}(x')\rangle^{\text{bio}}, |1_{s\lambda}(x)\rangle \rangle = \delta(x - x') \, . \tag{4.61}$$

Therefore, it would be useful to identify our Hilbert space as a biorthogonal system where the $|1_{s\lambda}(x')\rangle^{\text{bio}}$ are the biorthogonal basis states to the $|1_{s\lambda}(x)\rangle$ states; as we have bio-local annihilation operators in the EM field observables.

From section 2.2, we know that pseudo-Hermitian Hamiltonians are related to biorthogonal systems and satisfy the condition

$$H^\dagger = \eta^{-1} H \eta \,. \tag{4.62}$$

Considering only single excitation states[1], we define η and η^{-1} operators by (Mostafazadeh, 2010)

$$\begin{aligned} \eta &= \sum_{s,\lambda} \int_{-\infty}^{\infty} \mathrm{d}x \, |1_{s\lambda}(x)\rangle^{\text{bio}} \, \langle 1_{s\lambda}(x)|^{\text{bio}} \,, \\ \eta^{-1} &= \sum_{s,\lambda} \int_{-\infty}^{\infty} \mathrm{d}x \, |1_{s\lambda}(x)\rangle \, \langle 1_{s\lambda}(x)| \,. \end{aligned} \tag{4.63}$$

Using Eq. (4.24) and Eq. (4.27), we can write the above η and η^{-1} operators in the forms

$$\begin{aligned} \eta &= \sum_{s,\lambda} \int_0^{\infty} \mathrm{d}k \frac{1}{|k|} \left(|1_{s\lambda}(k)\rangle_b \, \langle 1_{s\lambda}(k)|_b + |1_{s\lambda}(k)\rangle_c \, \langle 1_{s\lambda}(k)|_c \right) \,, \\ \eta^{-1} &= \sum_{s,\lambda} \int_0^{\infty} \mathrm{d}k |k| \left(|1_{s\lambda}(k)\rangle_b \, \langle 1_{s\lambda}(k)|_b + |1_{s\lambda}(k)\rangle_c \, \langle 1_{s\lambda}(k)|_c \right) \,. \end{aligned} \tag{4.64}$$

From this we can see why η^{-1} is the inverse of η as we have

$$\eta\eta^{-1} = \sum_{s,\lambda} \int_0^{\infty} \mathrm{d}k \left(|1_{s\lambda}(k)\rangle_b \, \langle 1_{s\lambda}(k)|_b + |1_{s\lambda}(k)\rangle_c \, \langle 1_{s\lambda}(k)|_c \right) \,, \tag{4.65}$$

which acts as an identity for single excitation states. With these η operators, we can define an inner product so that the local states are pairwise orthonormal. For example, if we define an η inner product by

$$\langle |\psi\rangle \,, |\phi\rangle \rangle^{\eta} = \langle \psi | \, \eta \, |\phi\rangle \,, \tag{4.66}$$

[1] A primary goal of this thesis is to describe how to alter the free space EM field operators locally, and these operators are just linear equations of the local creation operators and bio-local annihilation operators. For simplicity, we therefore consider single excitation states.

then

$$\langle |1_{s\lambda}(x')\rangle , |1_{s\lambda}(x)\rangle\rangle^{\eta} = \delta(x - x') . \tag{4.67}$$

From Eq. (4.64) we can see that η is a positive operator as when it acts on a state it scales each $|1_{s\lambda}(k)\rangle_b$ and $|1_{s\lambda}(k)\rangle_c$ by a positive $|k|^{-1}$ factor. If, in addition, $\sqrt{\eta}$ and $\sqrt{\eta^{-1}}$ are bounded, this η inner product is related to a Hermitian inner product by

$$\langle |\psi\rangle , |\phi\rangle\rangle^{\eta} = \langle \sqrt{\eta}\,|\psi\rangle , \sqrt{\eta}\,|\phi\rangle\rangle \ . \tag{4.68}$$

However, if we decide to use the conventional Hermitian inner product we will have to use Hermitian Hamiltonains and so make it very difficult to construct Hamiltonians that locally act on the $|1_{s\lambda}(kx)\rangle$ states.

Furthermore, the η inner product does not work if we apply it to photon states as

$$\langle |1_{s\lambda}(k')\rangle , |1_{s\lambda}(k)\rangle\rangle^{\eta} \neq \delta(k - k') , \tag{4.69}$$

for the same reason that the local states under the standard inner product are not orthonormal – the transformation relating the photon and local states is non-unitary (with respect to either the η or standard inner product!). This η inner product will therefore change expectation values of Hermitian observables with respect to photon states. Not good. Further still, if we define an η^{-1} inner product by

$$\langle |\psi\rangle , |\phi\rangle\rangle^{\eta^{-1}} = \langle \psi|\,\eta^{-1}\,|\phi\rangle . \tag{4.70}$$

then we find that the bio-local states are orthonormal with respect to this η^{-1} inner product, but the local states and monochromatic photon states are not.

Let us pause for a moment. We said earlier that we are interested in locally transforming the EM field operators. It is tempting to think that if we use these transformed operators to calculate expectation values with respect to photonic states, we only need the standard inner product because this calculation will not involve any initial local states or bio local states from Eq. (4.31). However, in the next section, we will use non-Hermitian[1] Hamiltonians to locally alter the

[1]The Hamiltonians that we later define will still be Hermitian with respect to the generalised inner product and so will generate unitary dynamics with this inner product.

EM field operators. This use of non-Hermitian Hamiltonians means that in the Schrödinger picture photon states evolving according to these Hamiltonians will not remain normalised under the standard inner product as the time evolution operator acting on it will be non-unitary with respect to the standard inner product. We cannot use any of the η inner products either, as using these will mean that the photon state is not initially normalised. We want the local states and monochromatic states to be excitations of the same vacuum state of the EM field's Hilbert space – by definition we must have only one inner product. Therefore, we shall define a generalised inner product, $\langle , \rangle^{\text{bio}}$, such that

$$\begin{aligned} \left\langle |1_{s\lambda}(x)\rangle \, , |1_{s\lambda}(x')\rangle \right\rangle^{\text{bio}} &= \delta(x-x') \, , \\ \left\langle |1_{s\lambda}(x)\rangle^{\text{bio}} \, , |1_{s\lambda}(x')\rangle^{\text{bio}} \right\rangle^{\text{bio}} &= \delta(x-x') \, , \\ \left\langle |1_{s\lambda}(k)\rangle \, , |1_{s\lambda}(k')\rangle \right\rangle^{\text{bio}} &= \delta(k-k') \, . \end{aligned} \tag{4.71}$$

We cast our minds back to the definition of the local creation operator Eq. (4.24) and replace the $\sqrt{|k|}$ term with a $f(k)$ to define the operators

$$\begin{aligned} A^{\dagger}_{s\lambda}(x) &= \int_0^{\infty} \mathrm{d}k \frac{f(k)}{\sqrt{2\pi}} \left[\mathrm{e}^{-\mathrm{i}skx} \, b^{\dagger}_{s\lambda}(k) + \mathrm{e}^{\mathrm{i}skx} \, c^{\dagger}_{s\lambda}(k) \right] \, , \\ A^{\dagger\,\text{bio}}_{s\lambda}(x) &= \int_0^{\infty} \mathrm{d}k \frac{1}{f(k)\sqrt{2\pi}} \left[\mathrm{e}^{-\mathrm{i}skx} \, b^{\dagger}_{s\lambda}(k) + \mathrm{e}^{\mathrm{i}skx} \, c^{\dagger}_{s\lambda}(k) \right] \, , \end{aligned} \tag{4.72}$$

where $f(k)$ is such that the equations

$$\begin{aligned} b^{\dagger}_{s\lambda}(k) &= \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{1}{f(k)\sqrt{2\pi}} \, e^{iskx} \, A^{\dagger}_{s\lambda}(x) \, , \\ c^{\dagger}_{s\lambda}(k) &= \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{1}{f(k)\sqrt{2\pi}} \, e^{-iskx} \, A^{\dagger}_{s\lambda}(x) \, , \end{aligned} \tag{4.73}$$

are well-defined. We refer to $f(k)$ as the Fourier weight function.

We define a function, S, that, for a given $f(k)$, takes any operator or state and sends any Fourier weight term to its reciprocal. This means that if an operator or state has Fourier weight terms contained within, $A(f(k))$ or $|\psi(f(k))\rangle$, then

$$\begin{aligned} S\left(A\left(f(k)\right)\right) &= A\left(\frac{1}{f(k)}\right) \, , \\ S\left(|\psi\left(f(k)\right)\rangle\right) &= \left|\psi\left(\frac{1}{f(k)}\right)\right\rangle \, . \end{aligned} \tag{4.74}$$

Clearly, we have that $S^{-1} = S$. Applying this to $A^\dagger_{s\lambda}(x)$, $b^\dagger_{s\lambda}(k)$, $c^\dagger_{s\lambda}(k)$, $A^\dagger_{s\lambda}(x)\left|0\right\rangle$, $\left|1_{s\lambda}(k)\right\rangle_b$ and $\left|1_{s\lambda}(k)\right\rangle_c$ we find

$$\begin{aligned} S\left(A^\dagger_{s\lambda}(x)\right) &= A^{\dagger\,\mathrm{bio}}_{s\lambda}(x)\,, & S\left(A^\dagger_{s\lambda}(x)\left|0\right\rangle\right) &= A^{\dagger\,\mathrm{bio}}_{s\lambda}(x)\left|0\right\rangle\,, \\ S\left(b^\dagger_{s\lambda}(k)\right) &= b^\dagger_{s\lambda}(k)\,, & S\left(\left|1_{s\lambda}(k)\right\rangle_b\right) &= \left|1_{s\lambda}(k)\right\rangle_b\,, \\ S\left(c^\dagger_{s\lambda}(k)\right) &= c^\dagger_{s\lambda}(k)\,, & S\left(\left|1_{s\lambda}(k)\right\rangle_c\right) &= \left|1_{s\lambda}(k)\right\rangle_c\,. \end{aligned} \tag{4.75}$$

With this operator we henceforth define the bio of an operator by $A^{\mathrm{bio}} = S(A)$, and the bio of a state by $\left|\psi\right\rangle^{\mathrm{bio}} = S(\left|\psi\right\rangle)$. Suppose, for a given $f(k)$, we define an inner product by

$$\left\langle\left|\psi\right\rangle, \left|\phi\right\rangle\right\rangle^* = S\left(\left\langle\phi\right|\right)\left|\psi\right\rangle. \tag{4.76}$$

Under this inner product, we have

$$\begin{aligned} \left\langle\left|1_{s\lambda}(k')\right\rangle, \left|1_{s\lambda}(k)\right\rangle\right\rangle^* &= \delta(k-k'), \\ \left\langle A^\dagger_{s\lambda}(x)\left|0\right\rangle, A^\dagger_{s\lambda}(x)\left|0\right\rangle\right\rangle^* &= \delta(x-x'), \\ \left\langle A^{\dagger\,\mathrm{bio}}_{s\lambda}(x)\left|0\right\rangle, A^{\dagger\,\mathrm{bio}}_{s\lambda}(x)\left|0\right\rangle\right\rangle^* &= \delta(x-x')\,. \end{aligned} \tag{4.77}$$

We define the inner product where $f(k) = \sqrt{|k|}$ by[1]

$$\left\langle\left|\psi\right\rangle, \left|\phi\right\rangle\right\rangle^{\mathrm{bio}} = S\left(\left\langle\phi\right|\right)\left|\psi\right\rangle. \tag{4.78}$$

In the following, we refer to the above scalar product as the biorthogonal, or bio, or generalised inner product to distinguish it from the conventional one. The generalised inner product includes the standard inner product as a special case. For example, $\left|1_{s\lambda}(k)\right\rangle^{\mathrm{bio}} = \left|1_{s\lambda}(k)\right\rangle$ as $a^\dagger_{s\lambda}(k)$ is the Hermitian conjugate of its annihilation operator. Therefore, if we are only interested in using photonic states, photonic Hamiltonians, and Hermitian observables, there is no need to use the generalised inner product since it is equivalent to the standard inner product. The bio of $\left|1_{s\lambda}(x)\right\rangle$ is $\left|1_{s\lambda}(x)\right\rangle^{\mathrm{bio}}$, hence we used the bio superscript to denote bio

[1] We used a general $f(k)$ for motivation and for the purpose of this thesis we only need a S operator that works for $f(k) = \sqrt{|k|}$.

local states earlier. Using the generalised inner product, we have

$$\begin{aligned} \langle |1_{s\lambda}(k')\rangle , |1_{s\lambda}(k)\rangle \rangle^{bio} &= \delta(k-k') , \\ \langle |1_{s\lambda}(x')\rangle , |1_{s\lambda}(x)\rangle \rangle^{bio} &= \delta(x-x') , \\ \langle |1_{s\lambda}(x')\rangle^{bio} , |1_{s\lambda}(x)\rangle^{bio} \rangle^{bio} &= \delta(x-x') , \end{aligned} \tag{4.79}$$

so each of the photon, local and bio local states are separately pairwise orthonormal. We can conclude that the one-to-one transformation between the photon and local operators is indeed unitary with respect to the generalised inner product. But not with respect to the standard or η inner products. We shall therefore use this generalised inner product for the Hilbert space we use to model local interactions of the EM field.

Before we use these operators to compute expectation values, we still need to define how they evolve with respect to time. States contained in $\mathcal{H}$ evolve according a time evolution operator that is generated by H, states contained in $\mathcal{H}^*$ evolve according to a time evolution operator that is generated by $H^\dagger$ Brody (2013). In conventional Hermitian physics, all states evolve according to one Hamiltonian. Suppose our total Hilbert space of states is

$$\mathcal{H}^T = \mathcal{H} \oplus \mathcal{H}^* . \tag{4.80}$$

It is interesting to ask how do we compute calculations with a system whose states may belong to $\mathcal{H} \cap \mathcal{H}^* \neq \varnothing$, and, if so, how do we compute calculations on such a system? We address this in the next section where we look at time evolution of expectation values. It turns out that we need to use both H and $H^\dagger$ in calculating the dynamics of expectation values.

To finish this subsection, we point out that if it were not for the $\sqrt{|k|}$ term in the EM field operators, there would be no need to invoke biorthogonal quantum physics. This term seems to behave as a "Fourier weight" term in the momentum space representation of the $a^\dagger_{s\lambda}(x)$'s. Hodgson *et al.* (2021) and Southall *et al.* (2021) studied the same operators in detail except with complex conjugate dynamical equations replaced by negative frequencies and the Fourier weight term set to one. These certainly have their uses; however, we want to transform the EM operators locally, so we need to include the $\sqrt{|k|}$ as the EM field operators

include the $\sqrt{|k|}$. Also, unlike the free space Hamiltonian, we will use Hamiltonians that are not the complex conjugate of themselves, but are bio Hermitian, so that $H \neq H^*$ and $H^{\dagger\,\mathrm{bio}} = H$. We therefore need the complex conjugate dynamical equations.

4.2.4 The dynamics of the biorthogonal EM field

This section's primary aim is to describe our biorthogonal system's dynamics, which differ from a conventional quantum system and a conventional biorthogonal system. This will give us the tools required to model more complex scenarios than free space where we have Hamiltonians that locally alter the EM field operators. However, we recover the conventional dynamics as a particular case.

In this thesis we use Hamiltonians, that are potentially time-dependent $H(t)$, that we construct using creation/annihilation operators. In order for our time evolution operators generated by these Hamiltonians, $U(t)$ as described earlier, to be unitary we must make sure that when acting on a quantum state, $|\psi\rangle$, it will preserve the state's normalisation, i.e. ${}^{bio}\langle\psi(t)|\psi(t)\rangle = {}^{bio}\langle\psi|\psi\rangle$. This means that we require

$$\begin{aligned} {}^{bio}\langle\psi|\,\overline{U}^{\dagger\,bio}(t)\overline{U}(t)\,|\psi\rangle &= 1 \\ \Rightarrow \overline{U}^{\dagger\,bio}(t)\overline{U}(t) &= \mathrm{I_d}\,, \end{aligned} \tag{4.81}$$

which is true if

$$H^{\dagger\,bio} = H\,. \tag{4.82}$$

We therefore work with Hamiltonians of this form, as they will generate unitary dynamics. Taking the Hermitian conjugate of each side of this equation gives us $H^{bio} = H^\dagger$. So we see that the bio of a bio Hermitian Hamiltonian is equal to its Hermitian conjugate. The free space Hamiltonian Eq. (4.16) is bio Hermitian and Hermitian; thus, it generates unitary dynamics, with respect to both the standard inner product and generalised inner product.

The dynamics of state vectors

In biorthogonal quantum physics, if a state evolves using a Hamiltonain, H, its associated state evolves using $H^\dagger$ (Brody, 2013). Therefore, in our case states in $\mathcal{H}$ evolve using H and states in $\mathcal{H}^*$ evolve using H^{bio}.

$$\begin{aligned} |\psi(t)\rangle &= \overline{U}(t)\,|\psi(0)\rangle\ , \\ |\psi(t)\rangle^{bio} &= \overline{U}^{bio}(t)\,|\psi(0)\rangle^{bio}\,. \end{aligned} \tag{4.83}$$

If the Hamiltonian is not Hermitian, then $\overline{U}(t) \neq \overline{U}^{bio}(t)$. It is therefore important to differentiate what form of the Hamiltonian generates time dynamics for a particular state. We say that if a state evolves with respect to H, it belongs to the Hilbert space of states, $\mathcal{H}$. Whereas, if a state evolves with respect to H^{bio}, it belongs to the bio Hilbert space of states, $\mathcal{H}^*$.

For a state with at most one excitation in a mode[1], $|\phi\rangle$, we say that it belongs to $\mathcal{H}$ if it is normalised with respect to to the η inner product Eq. (4.66); and it belongs to $\mathcal{H}^*$ if it is normalised with respect to the η^{-1} inner product as described by Eq. (4.70). For example, the states

$$\begin{aligned} |\psi\rangle &= \int_{-\infty}^{\infty} \mathrm{d}x\, f(x)\,|1_{s\lambda}(x)\rangle\ , \\ |\phi\rangle &= \int_{-\infty}^{\infty} \mathrm{d}x\, g(x)\,|1_{s\lambda}(x)\rangle^{bio}\ , \end{aligned} \tag{4.84}$$

are normalised with respect to the η and η^{-1} inner product, respectively, so long as

$$\int_{-\infty}^{\infty} \mathrm{d}x\,|f(x)|^2 = \int_{-\infty}^{\infty} \mathrm{d}x\,|g(x)|^2 = 1\,. \tag{4.85}$$

Clearly, we have $|\phi\rangle \in \mathcal{H} \iff |\phi\rangle^{bio} \in \mathcal{H}^*$. If a general state remains unchanged by the operator S, so that $S(|\psi\rangle) = |\psi\rangle)$, then we say it belongs to the intersection of $\mathcal{H}$ and $\mathcal{H}^*$, $\mathcal{H}\cap\mathcal{H}^*$. For example, the monochromatic photon states. We explain why shortly.

With this understanding, a state in $\mathcal{H}$ evolves using H, whereas a state in $\mathcal{H}^*$ evolves using H^{bio}. It turns out that, when a state and Hermitian operator

[1]The difficultly in generalising this to a general tensor product state with N excitations is defining a suitable N. However, this is fine for our purposes as we are interested locally transforming the EM field operators, which are single excitation operators.

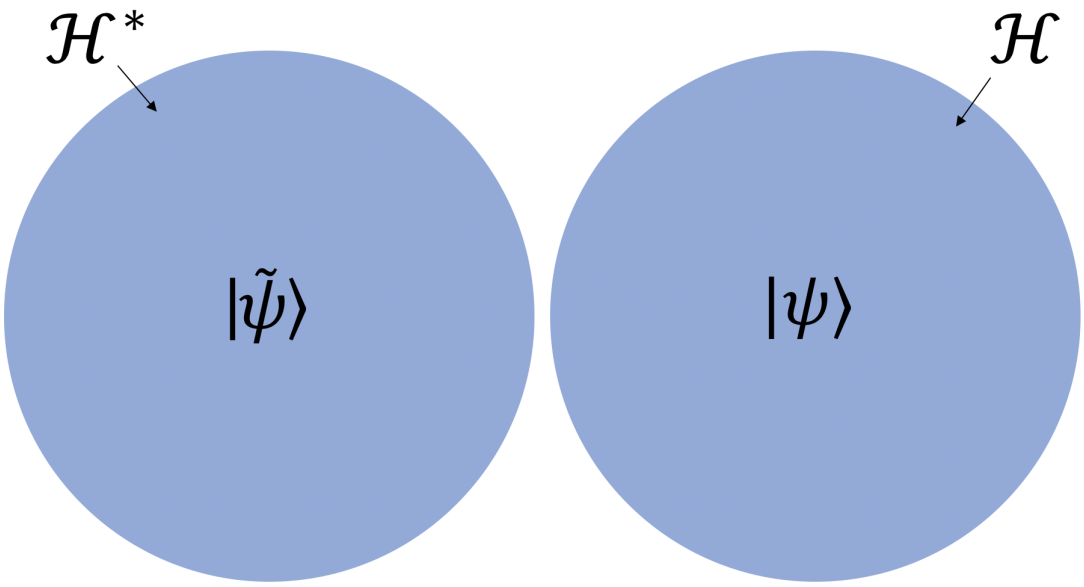

Figure 4.1: A schematic diagram of a typical biorthogonal system. If a state $|\psi\rangle$ evolves with respect to H, where $H = H^{\dagger bio}$, then it belongs to the Hilbert space $\mathcal{H}$ of states and its bio state $|\psi\rangle^{bio}$ evolves with respect to H^{bio} and belongs to the bio Hilbert space $\mathcal{H}^*$.

are equal to their bio state and bio operator, $|\psi\rangle^{bio} = |\psi\rangle$ and $A^{bio} = A$, then calculating the corresponding expectation value using H is indistinguishable to using H^{bio}. We refer to states and operators of this form as photonic. We see this in the following theorem:

Theorem 4.2.1. *If a Hamiltonian is bio Hermitian but not Hermitian, so that for its corresponding time evolution operator $\overline{U}(t) \neq \overline{U}^{bio}(t)$, whereas an operator, A, is both Hermitian and bio Hermitian, $A = A^\dagger = A^{bio}$, and a state, $|\psi\rangle$, is equal to its bio state, so $|\psi\rangle = |\psi\rangle^{bio}$, then $\langle A\,|\psi(t)\rangle\,, |\psi(t)\rangle\rangle^{bio} = \langle A|\psi(t)\rangle^{bio}, |\psi(t)\rangle^{bio}\rangle^{bio}$ where $|\psi(t)\rangle = \overline{\mathrm{U}}(\mathrm{t})\,|\psi\rangle$.*

Proof.

$$\begin{aligned}
\langle A\,|\psi(t)\rangle\,, |\psi(t)\rangle\rangle^{bio} &= \langle\psi|\,\overline{U}^{\dagger\,bio}(t)A\overline{U}(t)\,|\psi\rangle \\
&= (\langle\psi|\,\overline{U}^{\dagger\,\mathrm{bio}}(t)A\overline{U}(t)\,|\psi\rangle)^\dagger \\
&= \langle\psi|\,\overline{U}^\dagger(t)A\overline{U}^{bio}(t)\,|\psi\rangle \\
&= \langle A\,|\psi(t)\rangle^{bio}, |\psi(t)\rangle^{bio}\rangle^{bio}. \qquad (4.86)
\end{aligned}$$

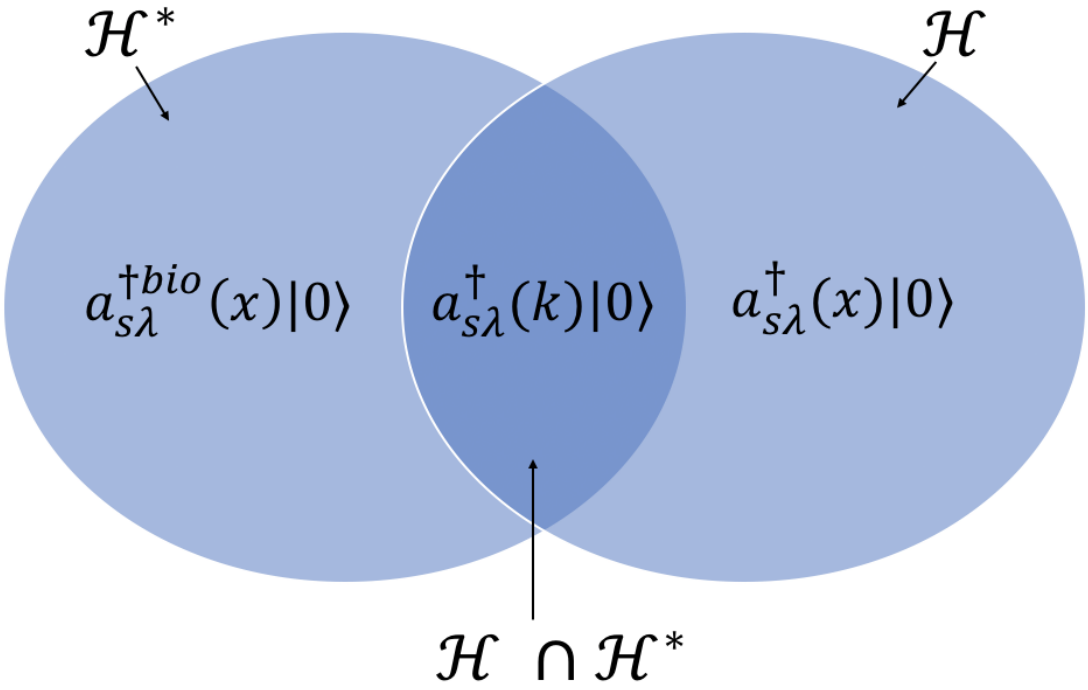

Figure 4.2: A schematic diagram of our EM Hilbert space. In contrast to the previous figure, we have here that the intersection between the Hilbert space of states and its bio space of states is non-empty and contains photonic excitations. Shown above are the single excitation building blocks of $\mathcal{H}^*$, $\mathcal{H} \cap \mathcal{H}^*$ and $\mathcal{H}$.

In the second line we have used the property that A has real expectation values being bio Hermitian and with this value being a scalar it is therefore equal to its Hermitian conjugate. In the third line we have used the property that A is Hermitian. ■

This theorem is why we say $\mathcal{H} \cap \mathcal{H}^*$ contains states satisfying $|\psi\rangle^{bio} = |\psi\rangle$.

When we take into account that c photons evolve to the complex conjugate Schrödinger equation, we find that the time evolution of local states is

$$\begin{aligned} |1_{s\lambda}(x,t)\rangle &= U_b(t)\,|1_{s\lambda}(x)\rangle_b + U_c^*(t)\,|1_{s\lambda}(x)\rangle_c\ , \\ |1_{s\lambda}(x,t)\rangle^{bio} &= U_b^{bio}(t)\,|1_{s\lambda}(x)\rangle_b + U_c^{*bio}(t)\,|1_{s\lambda}(x)\rangle_c\ . \end{aligned} \tag{4.87}$$

We will need to take this into account if we work with Hamiltonians, H, such that $H^* \neq H$ or $H^\dagger \neq H$.

To understand other cases, i.e. where A is bio Hermitian but not Hermitian, we have to look at how operators evolve.

The dynamics of operators

To work out what dynamical equation states evolve according to, we looked at what subspace contained them. Similarly, to determine what dynamical equation operators evolve according to, we look at what subspace they act on. This is perhaps easier to see in our quantised EM field, where we construct our states by acting creation operators on the vacuum state. For example, as $a^\dagger_{s\lambda}(x)\left|0\right\rangle$ evolves using H in the Schrödinger picture, then its corresponding creation and annihilation operators, $a^\dagger_{s\lambda}(x)$ and $a^{bio}_{s\lambda}(x)$, should evolve using H in the Heisenberg picture. Likewise, as $a^{\dagger bio}_{s\lambda}(x)\left|0\right\rangle$ evolves using H^{bio} in the Schrödinger picture, its corresponding creation and annihilation operators should evolve using H^{bio} in the Heisenberg picture.

Suppose we consider an operator of the form

$$A = \sum_i A_i \,, \tag{4.88}$$

where each A_i either maps states from the Hilbert space to itself, from the bio Hilbert space to itself, or from the intersection of the two spaces to itself, i.e. $A_i : \mathcal{H} \to \mathcal{H}$, $A_i : \mathcal{H}^* \to \mathcal{H}^*$ or $A_i : \mathcal{H}^* \cap \mathcal{H} \to \mathcal{H}^* \cap \mathcal{H}$, respectively. As always, we assert that expectation values in the Heisenberg picture must be equal to expectation values in the Schrödinger picture and so an expectation value $\langle A\rangle_{|\psi(t)\rangle}$ will only be valid if $|\psi(t)\rangle$ and A_i evolve using the same form of the Hamiltonian. With this we can define how to calculate time dependent expectation values $\langle A\rangle_{|\psi(t)\rangle}$ with system Hamiltonian H and initial photonic state $|\psi(0)\rangle$.

If A_i maps states from $\mathcal{H}$ to $\mathcal{H}$ we use H to generate its dynamics, which gives the Heisenberg equation

$$\frac{\mathrm{d}}{\mathrm{d}t}A_i(t) = -\frac{i}{\hbar}\left[A_i(t), H_b\right] + \frac{i}{\hbar}\left[A_i(t), H_c^*\right]. \tag{4.89}$$

Whereas, if it maps states from $\mathcal{H}^*$ to $\mathcal{H}^*$ we use H^{bio} to generate its dynamics, which gives the Heisenberg equation

$$\frac{\mathrm{d}}{\mathrm{d}t}A_i(t) = -\frac{i}{\hbar}\left[A_i(t), H_b^{bio}\right] + \frac{i}{\hbar}\left[A_i(t), H_c^{bio\,*}\right]. \tag{4.90}$$

This means that the local Fock operators, $a^{bio}_{s\lambda}(x)$ and $a^{\dagger}_{s\lambda}(x)$, evolve according to Eq. (4.89), and the bio local Fock operators, $a_{s\lambda}(x)$ and $a^{\dagger bio}_{s\lambda}(x)$, evolve according to Eq. (4.90). If A_i maps states from $\mathcal{H}^* \cap \mathcal{H}$ to $\mathcal{H}^* \cap \mathcal{H}$ then from the theorem above, you can use either if the state it acts on also evolves using the same form of the Hamiltonian.

Furthermore, what this also means is that for certain expectation values we may also need to use H and H^{bio} in our calculations. For example, suppose we have a photonic state, $|\psi\rangle$, and the Hermitian operator $A = a_{s\lambda}(x) + a^{\dagger}_{s\lambda}(x)$. Here we have

$$\begin{aligned} \langle A \rangle_{|\psi(t)\rangle} &= \langle \psi(t)|a_{s\lambda}(x)|\psi(t)\rangle + \langle \psi(t)|a^{\dagger}_{s\lambda}(x)|\psi(t)\rangle \\ &= \langle \psi|\overline{U}^{\dagger}(t,0)a_{s\lambda}(x)\overline{U}^{bio}(t,0)|\psi\rangle + \langle \psi|\overline{U}^{\dagger bio}(t,0)a^{\dagger}_{s\lambda}(x)\overline{U}(t,0)|\psi\rangle \,. \end{aligned}$$

What this means is that Hermitian, but not bio Hermitian, operators can give real expectation values with respect to photonic states. For example, the quantity

$$\overline{U}^{\dagger bio}(t,0)a^{\dagger}_{s\lambda}(x)\overline{U}(t,0) + \overline{U}^{\dagger}(t,0)a_{s\lambda}(x)\overline{U}^{bio}(t,0)\,, \tag{4.91}$$

is Hermitian for all t. This only works for photonic states and does not apply to local or bio local states as in that case the Heisenberg and Schrödinger pictures would not be consistent.

Suppose we have an operator

$$B = \prod_i B_i\,, \tag{4.92}$$

where each B_i is of the form described by Eq. (4.88). We say that the time dependence of this B operator is

$$B(t) = \prod_i B_i(t)\,. \tag{4.93}$$

To the best of our knowledge, there is nothing in the literature concerning mathematical structures of the form in Fig. 4.2, where we have a biorthogonal system with a non-zero intersection between its Hilbert space and bio or dual Hilbert space so that we need both H and $H^{\dagger}$ to calculate the dynamics of expectation values. To be sure, just because there is nothing in the literature

does not mean that such a system can exist. Conversely, it also does not mean that it cannot exist. In any case, in the next chapter, we show that if one follows the rules set out in this chapter, one can use the local and bio local bosonic Fock operators to construct mirror Hamiltonians that affect the EM field only at strictly local points.

4.3 Addressing the locality no-go theorems in QFT

One can find various detailed discussions of particle localisation in quantum field theory, particularly on the non-localisation of monochromatic photons, throughout the literature (Bialynicki-Birula, 1996; Halvorson & Clifton, 2002; Keller, 2005; Knight, 1961; Lamb, 1995; Newton & Wigner, 1949). We *are not* trying to localise monochromatic photons. What we have done is constructed two pairs of local bosonic Fock operators

$$\begin{aligned}\left[a^{bio}_{s\lambda}(x),\, a^{\dagger}_{s\lambda}(x)\right] &= \delta(x-x')\,,\\ \left[a_{s\lambda}(x),\, a^{\dagger\, bio}_{s\lambda}(x)\right] &= \delta(x-x')\,.\end{aligned} \tag{4.94}$$

As far as we know, Titulaer & Glauber (1966) were the first to study local operators of the EM field, and these operators have since received much attention in the literature (Bialynicki-Birula, 1996; Chan *et al.*, 2002; Sipe, 1995; Smith & Raymer, 2007). For example, Smith & Raymer (2007) introduced dual operators and adjusted their Hilbert space's inner product so that their wave-packet modes are orthonormal. However, their Hamiltonian that governs dynamics has its eigenvalues bounded from below, so their wave packets cannot remain strictly local throughout all time (Hegerfeldt, 1974, 1998a,b), which is what we want. They also construct their wave-packet modes out of photonic modes that propagate in all directions in a 3-D system and so do not have the property Eq. (4.3). These wave-packet modes certainly have their merits, however, and for a recent review see Raymer & Walmsley (2020).

Using the above commutators, the inner product between a local state at x and a local state at x', evolving in free space is

$$\begin{aligned}\left\langle \overline{U}_{\text{free}} \left|1_{s\lambda}(x')\right\rangle, \overline{U}_{\text{free}} \left|1_{s\lambda}(x)\right\rangle \right\rangle^{bio} &= \left\langle \left|1_{s\lambda}(x'-sct)\right\rangle, \left|1_{s\lambda}(x-sct)\right\rangle \right\rangle^{bio} \\ &= \delta(x-x'), \end{aligned} \tag{4.95}$$

so we see that states remain local throughout time evolution in the sense that they remain pairwise orthogonal. Therefore, we comment that these operators do not violate the following QFT locality no-go theorems: Knight's theorem, Malament's theorem, and Hegerfeldt's theorem (Hegerfeldt, 1998a; Knight, 1961; Malament, 1996). Knight's theorem asserts that a state, $|\psi\rangle$ cannot be strictly local if it contains a finite number of the system's harmonic oscillator Hamiltonian eigenstates, as described by de Bièvre (2006); Hawton & Debierre (2017). We do not have that here as $a^{\dagger}_{s\lambda}(x)\left|0\right\rangle$ currently contains all $b^{\dagger}_{s\lambda}(k)$ and $c^{\dagger}_{s\lambda}(k)$ in the range $k \in (0, \infty)$. A condition for both Malament's and Hegerfeldt's theorem is that all states evolve according to the Schrödinger equation and the system's Hamiltonian eigenvalues are bounded from below. Our local states comprise photon states that evolve according to the Schrödinger equation *and* states that evolve according to the complex conjugate Schrödinger equation and so do not satisfy the Hegerfeldt or Malament theorems conditions.

4.4 Summary

- We represented the EM field in terms of local bosonic operators, which we did by introducing the c photon operators that evolve according to the complex conjugate Heisenberg equation. Introduction of the c photons ensured that we did not violate the Hegerfeldt and Malament theorems.

- Due to Eq. (4.38), both the b and c photons have positive energy expectation values.

- We generalised the inner product on our Hilbert space with the help of biorthogonal quantum physics, so that the photon, local and bio-local Fock states are now all pairwise orthonormal, respectively. This means that the

transformations between these states in Eq. (4.32) is now unitary. This is a new result and is important so that we can construct locally acting mirror Hamiltonians with exactly solvable dynamics, as we will see in the next section.

- We described how different states and operators evolved using H, H^{bio}, or either. In particular, from theorem 4.2.1 a photonic state may evolve using H or H^{bio}, the operator $a^{\dagger}_{s\lambda}(x)$ evolves using H and the operator $a_{s\lambda}(x)$ evolves using H^{bio}. This new result has implications when we look at how the electric field expectation value of a EM field coherent state evolves near locally acting mirror Hamiltonian.

Chapter 5

Modelling light scattering experiments with local bosonic operators

This chapter begins by looking at exactly solvable bosonic systems where the Hamiltonians of interest are coupling Hamiltonians between different bosonic modes. Building on these Hamiltonians, we construct locally acting Hamiltonians that couple between different local bosonic modes and exactly solve the dynamics of a local operator in the presence of such a Hamiltonian. Following this, we use an interaction picture to show how we can examine the scattering dynamics of monochromatic photons. Next, we construct mirror systems that reproduce the classical behaviour of light in free space near a mirror. Mirrors are a popular topic in the literature as they are often used in optical experiments (Agarwal, 1975; Carniglia & Mandel, 1971; Dawson *et al.*, 2020). What is new in our approach is that we construct locally acting mirror Hamiltonians with exactly solvable dynamics that distinguishes between incoming and outgoing wave-packets. Finally, we show that the electric field expectation values of EM field coherent states in the presence of a mirror reproduce the appropriate dynamics of a classical sinusoidal electric wave near a mirror.

5.1 Exactly solvable bosonic quantum systems

In this section, we reverse engineer the question: "How does a Hamiltonian, $H(t)$, evolve an operator A?" and ask instead "If an operator evolves as $A(t)$, then what must the systems Hamiltonian be?". This question is helpful to ask since we often know what we want our outcome to be in physics or that we know from classical physics how expectation values should evolve. For example, for a wave packet travelling towards a mirror, we might want the outcome to be a wave packet travelling away from the mirror. In this case, if we can deduce the Hamiltonian from the outcome, we can construct that Hamiltonian and apply it to our system.

Suppose we want a coupling Hamiltonian for N modes that gives the following operator transformation:

$$A^\dagger(t) = \frac{1}{(\sum_{j=1}^{N} f_j(t)^2)^{\frac{1}{2}}} \sum_{i=1}^{N} e^{i\phi_i} f_i(t) a_i^\dagger \,. \tag{5.1}$$

Here the $f(t)_i$'s are any real differentiable functions, there is no T such that $f_i(T)$=0 for all i, and $[a_i, a_j^\dagger] = \delta_{ij}$. We have $[A(t), A^\dagger(t)] = 1$. The same $A^\dagger(t)$ can be given by different $f_i(t)$'s, i.e. the $f_i(t)$'s are not unique. For example, comparing the 2-mode case $f_1(t) = \cos(t)$, $f_2(t) = \sin(t)$ with $f_1(t) = 1$, $f_2(t) = \tan(t)$, both with $\phi_1 = \phi_2 = 0$, we find that both lead to

$$A^\dagger(t) = \cos(t) a_1^\dagger + \sin(t) a_2^\dagger \,. \tag{5.2}$$

The Hamiltonian that satisfies Heisenberg's equation of motion here is

$$H(t) = \frac{i\hbar}{\sum_{j=1}^{N} (f_j(t)^2)} \sum_{k=1}^{N} \sum_{m>k}^{N} \left(e^{i(\phi_k - \phi_m)} (f_k(t) \dot{f}_m(t) - \dot{f}_k(t) f_m(t)) a_k^\dagger a_m \right) + H.c. \,, \tag{5.3}$$

where the dot notation denotes the derivative with respect to time. This is so, because in this case we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} A^\dagger(t) &= \frac{1}{\left(\sum_{j=1}^{N} f_j(t)^2\right)^{\frac{3}{2}}} \sum_{i=1}^{N} \sum_{k=1}^{N} f_k(t) \left(\dot{f}_i(t) f_k(t) - f_i(t) \dot{f}_k(t) \right) e^{i\phi_i} a_i^\dagger \\ &= -\frac{i}{\hbar} [A^\dagger(t), H(t)] \,. \end{aligned} \tag{5.4}$$

Here we are applying a time-dependent coupling between each mode so that the maximum number of terms in (5.3) is $N(N+1)/2$. The above transformation is a novel result.

Suppose we want to know how the initial operator $A^\dagger(0) = a_1^\dagger$ evolves under the Hamiltonian

$$H(t) = i\hbar g(t)(e^{i\phi}a_1^\dagger a_2 - e^{-i\phi}a_2^\dagger a_1)\,, \tag{5.5}$$

where $g(t)$ is *any* integrable and continuous function. One can then show that the operator

$$A^\dagger(t) = \cos\left(\int_0^t g(t')dt' + c_1\right) a_1^\dagger + e^{-i\phi}\sin\left(\int_0^t g(t')dt' + c_1\right) a_2^\dagger\,, \tag{5.6}$$

where $c_1 = \arctan(b_2/b_1)$, satisfies Heisenberg's equation. Suppose $g(t)$ has discontinuities, but is still integrable. Let $\{t_i^d\}$, where $t_i < t_{i+1}$, be the set of the times at which it is discontinuous. Then the relevant $A^\dagger(t)$ is

$$A^\dagger(t) = \cos\left(G(t)\right) a_1^\dagger + e^{-i\phi}\sin\left(G(t)\right) a_2^\dagger\,. \tag{5.7}$$

Here $G(t)$ is

$$G(t) = \begin{cases} \int_0^t g(t')dt' + c_0, & \text{if } 0 < t \leq t_1^d \\ \int_{t_j^d}^t g(t')dt' + c_j, & \text{if } t_j^d < t \leq t_{j+1}^d \end{cases} \tag{5.8}$$

with $c_j = G(t_j^d)$ and $t_{j+1}^d = \infty$ if it does not exist.

5.2 Modelling scattering experiments with locally acting interaction Hamiltonians

We now construct a system that locally transforms the free space E and B field operators from left moving to right moving, anywhere on the x-axis. To do so, we take the free space Hamiltonian and add a locally acting static Hamiltonian, $H_{\text{loc}}(x)$. While the free space Hamiltonian term propagates wave packets at the speed of light, the interaction Hamiltonian will couple local modes at x, with $s = 1$, to local modes at $-x$, with $s = -1$ where λ remains the same, subject to a local potential $\Omega(x)$[1]. When the interaction term represents a mirror at $x = 0$,

[1]Instead of coupling x to $-x$ we could couple x to $-x + r$ where r is a real constant. However, it is unnecessary for this thesis.

the local potential will be non-zero only when x is close to 0. We define $H_{\text{loc}}(x)$ by

$$H_{\text{loc}}(x) = \sum_{\lambda} \int_{-\infty}^{\infty} \mathrm{d}x\, i\hbar\Omega(x) \left[e^{i\phi}\, a^{\dagger}_{-1\lambda}(-x) a^{\text{bio}}_{1\lambda}(x) - e^{-i\phi}\, a^{\dagger}_{1\lambda}(x) a^{\text{bio}}_{-1\lambda}(-x) \right] . \tag{5.9}$$

Here $\Omega(x)$ is a locally integrable function, ϕ is a free parameter that determines what phase factor modes accumulates during this interaction.

This Hamiltonian is bio Hermitian, but not Hermitian. Its Hermitian conjugate, with $\phi = -\frac{\pi}{2}$ for simplicity, is

$$H^{\dagger}_{\text{loc}}(x) = \sum_{\lambda} \int_{-\infty}^{\infty} \mathrm{d}x\, \hbar\Omega(x) \left[a^{\dagger\,\text{bio}}_{1\lambda}(x)\, a_{-1\lambda}(-x) - a^{\dagger\,\text{bio}}_{-1\lambda}(-x) a_{1\lambda}(x) \right] . \tag{5.10}$$

The local mode operators have become bio mode operators and vice-versa. To show that Eq. (5.10) is not equal to Eq. (5.9), we calculate the commutator of each with $b_{-1}(k_1)$, followed by the commutator of each result with $b^{\dagger}_1(k_2)$ where $k_1 \neq k_2$. The two results are not the same. Using Eq. (4.24), we find

$$\left[[b_{-1}(k_1), H_{\text{loc}}(x)]\, , b^{\dagger}_1(k_2) \right] = \frac{1}{2\pi} \frac{\sqrt{|k_1|}}{\sqrt{|k_2|}} \int_{-\infty}^{\infty} \mathrm{d}x \Omega(x) e^{-ix(k_1 - k_2)} . \tag{5.11}$$

We repeat this except with $H_{\text{loc}}(x)$ replaced by $H^{\dagger}_{\text{int}}(x)$ to find

$$\left[\left[b_{-1}(k_1), H^{\dagger}_{\text{loc}}(x) \right] , b^{\dagger}_1(k_2) \right] = \frac{1}{2\pi} \frac{\sqrt{|k_2|}}{\sqrt{|k_1|}} \int_{-\infty}^{\infty} \mathrm{d}x \Omega(x) e^{-ix(k_1 - k_2)} . \tag{5.12}$$

Therefore, the equality $H^{\dagger}_{\text{loc}}(x) = H_{\text{int}}(x)$ requires

$$|k_1| = |k_2| , \tag{5.13}$$

which gives us a contradiction as $k_1 \neq k_2$. So we can confidently say that we have a non-Hermitian Hamiltonian that gives us exact dynamics here.

However, as Eq. (5.9) is bio-Hermitian, it generates unitary dynamics with respect to the generalised inner product. Calculating the dynamics of local creation operators under the above Hamiltonian is straightforward since the true Hermitian conjugate of $a^{\dagger}_{s\lambda}(x)$ with respect to the generalised inner product is

$a^{\rm bio}_{s\lambda}(x)$ and not $a_{s\lambda}(x)$. It is the former that is in the above Hamiltonian and not the latter.

The same Hamiltonian except where † denotes the true Hermitian conjugate with respect to the generalised inner product, so that the Hermitian conjugate of an operator A is $A^\dagger$ such that $\langle |\psi\rangle, A^\dagger |\phi\rangle\rangle^{\rm bio} = \langle A|\psi\rangle, |\phi\rangle\rangle^{\rm bio}$, is

$$H_{\rm loc}(x) = \sum_\lambda \int_{-\infty}^{\infty} {\rm d}x\, i\hbar\Omega(x) \left[e^{i\phi}\, a^\dagger_{-1\lambda}(-x) a_{1\lambda}(x) - e^{-i\phi}\, a^\dagger_{1\lambda}(x) a_{-1\lambda}(-x) \right] . \tag{5.14}$$

So, with respect to the generalised inner product, we have

$$[a_{s\lambda}(x), a^\dagger_{s'\lambda'}(x')] = \delta_{ss'}\delta_{\lambda\lambda'}\delta(x-x') . \tag{5.15}$$

However, to be consistent we reserve † to denote the conventional Hermitian adjoint.

Our total Hamiltonian is the sum of the free space Hamiltonian and the $H_{\rm loc}(x)$ Hamiltonian above:

$$H = H_{\rm free} + H_{\rm int}(x) . \tag{5.16}$$

We can calculate the dynamics of local state initially at x_0, polarisation λ and propagating in the $s = 1$ direction, $|1_{1\lambda}(x_0)\rangle$, with respect to this Hamiltonian. Using Eq. (4.20), the full time dependent solution, with no approximations, is

$$\begin{aligned} \overline{U}(t)\,|1_{1\lambda}(x_0)\rangle &= \cos\left(\Theta(x_0,t)\right)|1_{1\lambda}(x_0+ct)\rangle \\ &\quad +\sin\left(\Theta(x_0,t)\right)|1_{-1\lambda}(-(x_0+ct),\phi)\rangle , \end{aligned} \tag{5.17}$$

where

$$\Theta(x_0,t) = \int_0^t \Omega(x_0+ct'){\rm d}t' , \tag{5.18}$$

and

$$|1_{s\lambda}(x,\phi)\rangle = \int_0^\infty {\rm d}k \sqrt{\frac{|k|}{2\pi}}\, e^{-iskx} e^{i\phi}\, b^\dagger_{s\lambda}(k)\,|0\rangle + \int_0^\infty {\rm d}k \sqrt{\frac{|k|}{2\pi}}\, e^{iskx} e^{-i\phi}\, c^\dagger_{s\lambda}(k)\,|0\rangle . \tag{5.19}$$

The bio of this state is

$$\begin{aligned} \overline{U}^{\rm bio}(t)\,|1_{1\lambda}(x_0)\rangle^{\rm bio} &= \cos\left(\Theta(x_0,t)\right)|1_{1\lambda}(x_0+ct)\rangle^{\rm bio} \\ &\quad +\sin\left(\Theta(x_0,t)\right)|1_{-1\lambda}(-(x_0+ct),\phi)\rangle^{\rm bio} . \end{aligned} \tag{5.20}$$

The ϕ in the second line represents the $e^{i\phi}$ and $e^{-i\phi}$ phases that the b and c modes have accumulated respectively. Therefore, the interaction above transfers the same real phase to real E and B field expectation values produced separately by b and c photon coherent states. Intuitively, this makes sense as the c photons are the "complex conjugate" b photons. The same solution can be obtained using negative frequencies but there ϕ has to be restricted to 0 or π as $e^{i\pi} = e^{-i\pi}$. We explain why shortly.

As we have solved the Schrödinger equation for local states in the presence of a local coupling Hamiltonian, we have an exact description of their dynamics in this system. Dependent on what the coupling $\Omega(x)$ is we can faithfully transfer a $s = 1$ local state, initially at $x = x_0$, into a $s = -1$ state at a time t if $\int_0^t \Omega(x_0 + ct')\mathrm{d}t' = \pi/2 + n\pi$ where $n \in \mathbb{Z}$. We utilise this in the next section.

Last, we solved the dynamics of single excitation local and bio local states as an *example* to motivate the strictly local properties of the local coupling Hamiltonian in Eq. (5.9). Even if we have no local states in an EM field calculation, we still have a local operator, $a^\dagger_{s\lambda}(x)$, and a bio local operator, $a^{\mathrm{bio}}_{s\lambda}(x)$, in the EM field operators. If we were to construct wave packets using the local modes, such as $|\psi\rangle = A^\dagger_{s\lambda}|0\rangle$ where

$$A^\dagger_{s\lambda} = \int_{-\infty}^{\infty} \mathrm{d}x f(x) a^\dagger_{s\lambda}(x)\,, \tag{5.21}$$

we immediately know how it transforms in the presence of Eq. (5.9) since we know how each $a^\dagger_{s\lambda}(x)$ transforms. For example,

$$|\psi(t)\rangle = \int_{-\infty}^{\infty} \mathrm{d}x f(x) |1_{1\lambda}(x,t)\rangle\,, \tag{5.22}$$

where $|1_{1\lambda}(x,t)\rangle$ is given by Eq. (5.17).

5.2.1 The interaction on photons and scattering operator

We can also solve exact dynamics for an initial single monochromatic photon state, $|\psi(0)\rangle = |1_{1\lambda}(k_0)\rangle_a$. As the monochromatic photon is a fully non-local state, but the Hamiltonian is locally acting, this transformation will couple k

modes to k' where $k \neq k'$. Using equation (4.32), we have

$$\begin{aligned} |\psi(t)\rangle &= \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{1}{\sqrt{2\pi|k|}} \, e^{ikx} \overline{U}(t) \, |1_{1\lambda}(x_0)\rangle \\ &= \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{1}{\sqrt{2\pi|k|}} \, e^{ikx} \Big(\cos\left(\Theta(x_0,t)\right) |1_{1\lambda}(x_0+ct)\rangle \\ &\quad + \sin\left(\Theta(x_0,t)\right) |1_{-1\lambda}(-(x_0+ct),\phi)\rangle \Big) \,. \end{aligned} \tag{5.23}$$

However, as mentioned previously, photons can evolve with respect to H or H^{bio} – which one it is will depend on the other operators within the states of the inner product that you are calculating. As shown in theorem 4.2.1, if all these operators are photonic then for expectation values it does not matter.

From equation (5.23), it is clear that photons undergoing a transformation due to the local Hamiltonian are now not photonic as $|\psi(t)\rangle \neq |\psi(t)\rangle^{\text{bio}}$. However, we ask the question "Can we make the whole of the monochromatic photon experience the local transformation, so that the overall transformation is non-local?". To answer this question we go into an interaction picture with respect to the free space Hamiltonian.

The interaction picture

Our mirror system Hamiltonian is of the form

$$H = H_{\text{free}} + H_{\text{loc}}(x) \,. \tag{5.24}$$

We define the state vector in the interaction picture as

$$|\psi_{\text{I}}(t)\rangle = \overline{U}^{\dagger\,\text{bio}}_{\text{free}}(t) \, |\psi(t)\rangle \,, \tag{5.25}$$

where $|\psi_{\text{I}}(t)\rangle$ and $|\psi(t)\rangle$ denote the state vectors of wave packets in the interaction and in the Schrödinger picture, respectively. Using this definition, one can show that $|\psi_I(t)\rangle$ evolves with the usual Schrödinger equation with respect to the potentially time-dependent interaction Hamiltonian, $H_{\text{I}}(x,t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}|\psi_{\text{I}}(t)\rangle = -\frac{i}{\hbar} H_{\text{I}\,b}(x,t) \, |\psi_{\text{I}}(t)\rangle_b + \frac{i}{\hbar} H^*_{\text{I}\,c}(x,t) \, |\psi_{\text{I}}(t)\rangle_c \,. \tag{5.26}$$

where

$$H_{\rm I}(x,t) \;=\; \overline{U}^{\dagger\,\rm bio}_{\rm free}(t)\, H_{\rm loc}(x)\, \overline{U}_{\rm free}(t)\,. \tag{5.27}$$

The solution to this equation, where the state is initially equal to $|\psi_{\rm I}(0)\rangle$, is

$$|\psi_{\rm I}(t)\rangle = \overline{U}_{\rm I}(t)|\psi_{\rm I}(0)\rangle\,, \tag{5.28}$$

where $\overline{U}_{\rm I}(t)$ is the interaction picture time evolution operator

$$\overline{U}_{\rm I}(t) = T\left[\exp\left(-\frac{i}{\hbar}\int_0^t {\rm d}t' H_{{\rm I}\,b}(x,t')\right)\times\exp\left(-\frac{i}{\hbar}\int_0^t {\rm d}t' H^*_{{\rm I}\,c}(x,t')\right)\right]\,, \tag{5.29}$$

and T indicates that the exponential is time ordered. This is because in general an interaction Hamiltonian does not commute with itself at different times, i.e. $[H_{\rm I}(x,t), H_{\rm I}(x,t')] \neq 0$. However, in this thesis we have $[H_{\rm I}(x,t), H_{\rm I}(x,t')] = 0$ and so for simplicity will drop the time ordering symbol.

In this thesis, we wish to represent states by operators acting on the vacuum state, for example $|\psi_{\rm I}(t)\rangle = A^\dagger_I(t)\,|0\rangle$, so we need to expand on equation (5.28) where $|\psi_{\rm I}(0)\rangle = A^\dagger_{\rm I}(0)\,|0\rangle$. We find

$$\begin{aligned} |\psi_{\rm I}(t)\rangle &= \overline{U}_{\rm I}(t)|\psi_{\rm I}(0)\rangle \\ &= A^\dagger_{\rm I}(t)\,|0\rangle\,, \end{aligned} \tag{5.30}$$

where $A^\dagger_{\rm I}(t) = \overline{U}_{\rm I}(t)A^\dagger_{\rm I}(0)\overline{U}^{\dagger\,\rm bio}_{\rm I}(t)$. Here we have used the fact that $1 = \overline{U}^{\dagger\,\rm bio}_{\rm I}(t)\overline{U}_{\rm I}(t)$ and that $H_{\rm I}(x,t)$ annihilates the vaccum. From this we also find

$$\begin{aligned} \frac{\rm d}{{\rm d}t}\,|\psi_{\rm I}(t)\rangle &= \frac{{\rm d}A^\dagger_{\rm I}(t)}{{\rm d}t}\,|0\rangle \\ &= \left(\frac{i}{\hbar}[A^\dagger_{\rm I}(t), H_{I,b}(x,t)] - \frac{i}{\hbar}[A^\dagger_{\rm I}(t), H^*_{{\rm I}\,c}(x,t)]\right)|0\rangle\,. \end{aligned} \tag{5.31}$$

To go back into the Schrödinger picture we use

$$\begin{aligned} |\psi(t)\rangle &= \overline{U}_{\rm free}(t)\,|\psi_{\rm I}(t)\rangle \\ &= A^\dagger_{\rm S}(t)\,|0\rangle\,, \end{aligned} \tag{5.32}$$

where $A^\dagger(t) = \overline{U}_{\rm free}(t)A^\dagger_{\rm I}(t)\overline{U}^{\dagger\rm bio}_{\rm free}(t)$.

Monochromatic photon transformation in the interaction picture

Using equation (5.27) to take $H_{\text{loc}}(x)$ in (5.9) into the interaction picture with respect to the free space Hamiltonian H_{free} yields

$$H_{\text{I}}(x,t) = \sum_{\lambda} \int_{-\infty}^{\infty} \mathrm{d}x \hbar\Omega(x+ct)\, i(e^{i\phi} a^{\dagger}_{-1\lambda}(-x) a^{\text{bio}}_{1\lambda}(x) - e^{-i\phi} a^{\dagger}_{1\lambda}(x) a^{\text{bio}}_{-1\lambda}(-x))\,. \tag{5.33}$$

This looks a lot like the $H_{\text{loc}}(x)$ in Eq. (5.9), except now the potential $\Omega(x)$ is moving at the speed of light.

The interaction picture scattering operator is

$$S_{\text{I}} = \exp\left[-\frac{i}{\hbar}\int_{-\infty}^{\infty} \mathrm{d}t H_{\text{I}}(x,t)_b\right] \times \exp\left[\frac{i}{\hbar}\int_{-\infty}^{\infty} \mathrm{d}t {H^*}_{\text{I}}(x,t)_c\right]. \tag{5.34}$$

The first part on the R.H.S. is the familiar scattering operator. However, we have the addition of the second term because the c monochromatic photons evolve with respect to a different dynamical equation to the b monochromatic photons. We calculate the interaction picture scattering operator, and not the Schrödinger picture scattering operator, as in this picture we can visualise the $t = \pm\infty$ integral limits in (5.34) moving the local potential along the whole x-axis. We keep the scattering operator name by convention. However, it is not a scattering operator in the usual sense when calculating its effect on monochromatic photons. The usual sense being that if we go back and forward far enough in time, the Hamiltonian will have "turned off". As the photons exist across the whole x-axis, this will never happen. Indeed, we only get a non-local transformation by strictly using the limits $t = \pm\infty$. However, if our initial states were local states, we could think of the scattering operator in the usual sense.

Substituting (5.33) into (5.34) we find

$$S_{\text{I}} = \exp\left[\sum_{\lambda}\int_{0}^{\infty} \mathrm{d}k\, \Omega_x (e^{i\phi} b^{\dagger}_{-1\lambda}(k) b_{1\lambda}(k) + e^{-i\phi} c^{\dagger}_{-1\lambda}(k) c_{1\lambda}(k) - H.c.)\right], \tag{5.35}$$

where $\Omega_x = \frac{1}{c}\int_{-\infty}^{\infty} \mathrm{d}x\, \Omega(x)$, i.e. it is a constant for a given $\Omega(x)$ that is integrable across the whole x-axis. To calculate the effects of this scattering operator, we introduce an effective Hamiltonian, H_{eff}, and time interval, $(0, t_{\text{eff}})$, such that its corresponding time evolution operator is equal to the scattering operator above.

However, we stress that the only information we can infer from this effective Hamiltonian is the effect of the scattering operator and not any exact dynamics.

For simplicity, we choose our effective Hamiltonian to be

$$H_{\text{eff}} = \sum_{\lambda} i\hbar \int_0^{\infty} \mathrm{d}k\, \Omega_x((e^{i\phi} b^{\dagger}_{-1\lambda}(k) b_{1\lambda}(k) - H.c.) + (e^{-i\phi} c^{\dagger}_{-1\lambda}(k) c_{1\lambda}(k) - H.c.))\,, \tag{5.36}$$

and the effective time interval to be $(0, 1)$. Clearly, the time evolution operator corresponding to this Hamiltonian is equal to the scattering operator above, $U_{\text{eff}}(0,1) = S_{\text{I}}$ where $U_{\text{eff}}(0,1)$ is generated by the standard Schrödinger equation. This means we also only need to use one equation of motion for this effective Hamiltonian, as the crux is $U_{\text{eff}}(0,1) = S_{\text{I}}$ here.

We point out that this effective Hamiltonian is Hermitian, whereas the locally acting Hamiltonian that gives exact dynamics is not Hermitian (but is bio Hermitian). Indeed, infinitely many effective Hermitian Hamiltonians give the same scattering operator and infinitely many effective bio Hermitian, but not Hermitian, Hamiltonians that give the same scattering operator. For example, instead of using $H_{\text{I}}(x,t)$ in the scattering operator, we could have easily just used $H^{\text{bio}}_{\text{I}}(x,t)$ as they give the same scattering transformation. This is good because the goal here is to provide a non-local transformation between photons, so we need the scattering operator to be unitary with resepct to the conventional inner product.

Suppose we have initial states $|1_{1\lambda}(k)\rangle_b$ and $|1_{1\lambda}(k)\rangle_c$. The effect of the above scattering operator above on these states is

$$\begin{aligned} S_I\,|1_{1\lambda}(k)\rangle_b &= U_{\text{eff}}(0,1) b^{\dagger}_{1\lambda}(k) U^{\dagger}_{\text{eff}}(0,1)\,|0\rangle \\ &= \cos(\Omega_x)\,|1_{1\lambda}(k)\rangle + e^{i\phi}\sin(\Omega_x)\,|1_{-1\lambda}(k)\rangle\,. \\ S_{\text{I}}\,|1_{1\lambda}(k)\rangle_c &= U_{\text{eff}}(0,1) c^{\dagger}_{1\lambda}(k) U^{\dagger}_{\text{eff}}(0,1)\,|0\rangle \\ &= \cos(\Omega_x)\,|1_{1\lambda}(k)\rangle + e^{-i\phi}\sin(\Omega_x)\,|1_{-1\lambda}(k)\rangle\,. \end{aligned} \tag{5.37}$$

From this we see that the $s = 1$ photon will have made a full transfer to an $s = -1$ photon if $\Omega_x = \frac{\pi}{2} + n\pi$ where $n \in \mathbb{Z}$.

However, in this case the scattering operator does not distinguish between incoming and outgoing local states. If we want this distinction, and to calculate

exact dynamics, we calculate the effect of the time evolution operator generated by the Hamiltonian in Eq. (5.16).

Finally, as the effective Hamiltonian commutes with the energy observable Eq. (4.38), i.e. $[H_{\text{eff}}, E_{\text{eng}}] = 0$, and we have that this interaction Hamiltonian conserves energy.

5.3 Using the concept of negative frequencies only is not enough

Suppose we take the same locally acting Hamiltonian in Eq. (5.9) except with the operators, $A_{s\lambda}$ and $A^{\dagger}_{s\lambda}(x)$, defined as in Eq. (4.50) in section 4.2.2

$$H^{\text{neg}}_{\text{loc}}(x) = \sum_{\lambda} \int_{-\infty}^{\infty} \mathrm{d}x \, i\hbar\Omega(x) \left[e^{i\phi} \, A^{\dagger}_{-1\lambda}(-x) A_{1\lambda}(x) - e^{-i\phi} \, A^{\dagger}_{1\lambda}(x) A_{-1\lambda}(-x) \right] . \tag{5.38}$$

Then, using Eq. (4.48), we define the total Hamiltonian to be

$$H = H^{\text{neg}}_{\text{free}} + H^{\text{neg}}_{\text{loc}}(x) \, . \tag{5.39}$$

The time evolution of the state $A^{\dagger}_{1\lambda}(x_0) \left|0\right\rangle$ under the above Hamiltonian with $\phi = 0$ is

$$\begin{aligned} U(t) A^{\dagger}_{1\lambda}(x_0) \left|0\right\rangle \quad &= \quad \cos\left(\Theta(x_0, t)\right) A^{\dagger}_{1\lambda}(x + ct) \left|0\right\rangle \\ &\quad + \sin\left(\Theta(x_0, t)\right) A^{\dagger}_{-1\lambda}(-(x + ct)) \left|0\right\rangle \, . \end{aligned} \tag{5.40}$$

We state this as this was a key result in Southall *et al.* (2021), as we had reproduced the classical mirror image effect for the $A^{\dagger}_{1\lambda}(x_0) \left|0\right\rangle$ states with exactly solvable dynamics. This novel result did not require biorthogonal quantum mechanics and used the conventional Schrödinger equation.

Suppose we then take this Hamiltonian into the interaction picture with respect to the free space Hamiltonian in Eq. (4.48), it becomes

$$H_{\text{I}}(x, t) = \sum_{\lambda} \int_{-\infty}^{\infty} \mathrm{d}x \hbar\Omega(x + ct) \, i(e^{i\phi} A^{\dagger}_{-1\lambda}(-x) A_{1\lambda}(x) - e^{-i\phi} A^{\dagger}_{1\lambda}(x) A_{-1\lambda}(-x)) \, . \tag{5.41}$$

When we calculate the scattering operator corresponding to this interaction Hamiltonian we find

$$S_{\rm I} = \exp\left[\sum_{\lambda}\int_{-\infty}^{\infty} {\rm d}k\, \Omega_x (e^{i\phi} b^{\dagger}_{-1\lambda}(k) b_{1\lambda}(k) - H.c.)\right]. \tag{5.42}$$

This scattering operator is equal to the effective time evolution operator $U(0,1)$ with the effective Hamiltonian

$$H_{\rm eff} = \sum_{\lambda} i\hbar \int_{-\infty}^{\infty} {\rm d}k\, \Omega_x (e^{i\phi} b^{\dagger}_{-1\lambda}(k) b_{1\lambda}(k) - H.c.). \tag{5.43}$$

Look familiar? It should, we followed the same method we used to calculate the effective Hamiltonian in Eq. (5.36), which was the effective Hamiltonian we used that corresponded to the scattering operator of Eq. (5.33).

Now, this effective Hamiltonian still looks like a trivial coupling Hamiltonian that only couples between modes of the same energy and so it is tempting to think that this must preserve energy. When we examine the effect of the scattering operator on a photon which is an energy eigenstate we find

$$\begin{aligned} S_{\rm I}\left|1_{1\lambda}(k)\right\rangle &= U_{\rm eff}(0,1) b^{\dagger}_{1\lambda}(k) U^{\dagger}_{\rm eff}(0,1)\left|0\right\rangle \\ &= \cos(\Omega_x)\left|1_{1\lambda}(k)\right\rangle + e^{i\phi}\sin(\Omega_x)\left|1_{-1\lambda}(k)\right\rangle , \\ S_{\rm I}\left|1_{1\lambda}(-k)\right\rangle &= U_{\rm eff}(0,1) b^{\dagger}_{1\lambda}(-k) U^{\dagger}_{\rm eff}(0,1)\left|0\right\rangle \\ &= \cos(\Omega_x)\left|1_{1\lambda}(-k)\right\rangle + e^{i\phi}\sin(\Omega_x)\left|1_{-1\lambda}(-k)\right\rangle , \end{aligned} \tag{5.44}$$

which looks a lot like the $b_{s\lambda}(k)$ and $c_{s\lambda}(k)$ scattering transformations in Eq. (5.37). The exception being that the negative frequency states accumulate the same phase, $e^{i\phi}$, as the positive frequency states upon reflection, whereas the c photon states accumulate the complex conjugate phase, $e^{-i\phi}$. The former property is problematic as it allows for energy to not be conserved. To show this, we calculate the commutator between the effective Hamiltonian Eq. (5.43) and energy operator Eq. (4.53) because if this is non-zero then

$$U(0,1) H_{\rm eng} U(0,1) \neq H_{\rm eng}\,, \tag{5.45}$$

so that energy is not conserved. Our energy observable now includes the double annihilation and creation terms $b_{s\lambda}(k) b_{s\lambda}(-k) + H.c.$ It is the commutation relationship with these operators that we turn our attention to. Calculating this

directly, and setting $\hbar = c = \Omega_x = 1$ and $r = 0$ for convenience, we find

$$[H_{\text{eng}}\,, H_{\text{eff}}] = 2\cos\left(\phi + \frac{\pi}{2}\right)\int_{-\infty}^{\infty} \mathrm{d}k|k|\, b_{1\lambda}(k)b_{-1\lambda}(-k) + H.c.\,, \tag{5.46}$$

where it is only the double creation and annihilation operator terms in the energy Hamiltonian that contribute to the non-zero terms in the above commutator. This commutator is zero if $\phi = n\pi$ where $n \in \mathbb{Z}$, which corresponds to $e^{i\phi}$ being real. With this choice of $\phi = n\pi$, the c and negative frequency monochromatic photons both gain the same real phase upon reflection. However, in general $\phi \in \mathbb{R}$.

To see this non-energy preserving property in action, we set $\phi = \frac{3\pi}{2}$ and examine the change in electric fields of the single mode coherent states $|\alpha_{1\lambda}(k_0)\rangle$ and $|-\alpha^*_{1\lambda}(-k_0)\rangle$ and the tensor product state $|\alpha_{1\lambda}(k_0)\rangle\,|-\alpha^*_{1\lambda}(-k_0)\rangle$ where $\alpha = |\alpha|e^{i\theta}$. Setting $\frac{c}{2}\sqrt{\frac{\hbar c}{\varepsilon A\pi}}\sqrt{|k_0|}|\alpha| = 1$ for convenience, the free space electric field expectation values of these states are

$$\begin{aligned}
\langle\alpha_{1\lambda}(k_0')|\mathbf{E}(x,t)|\alpha_{1\lambda}(k_0)\rangle &= \delta(k_0 - k_0')\cos(k_0x - k_0ct + \theta)\mathbf{E}_\lambda\,,\\
\langle -\alpha^*_{1\lambda}(-k_0')|\mathbf{E}(x,t)|-\alpha^*_{1\lambda}(-k_0)\rangle &= \delta(k_0 - k_0')\cos(-k_0x + k_0ct - \theta + \pi)\mathbf{E}_\lambda\,,\\
&= -\langle\alpha_{1\lambda}(k_0')|\mathbf{E}(x,t)|\alpha_{1\lambda}(k_0)\rangle\,,
\end{aligned} \tag{5.47}$$

and

$$\langle\alpha_{1\lambda}(k_0')|\,\langle -\alpha^*_{1\lambda}(-k_0')|\,\mathbf{E}(x,t)\,|\alpha_{1\lambda}(k_0)\rangle\,|-\alpha^*_{1\lambda}(-k_0)\rangle = 0\,, \tag{5.48}$$

where the time evolution is due the free space Hamiltonian only so that $\mathbf{E}(x,t) = U^\dagger_{free}(t)\mathbf{E}(x)U(t)$ with H in Eq. (4.48). A similar story holds for the magnetic field also, the magnetic field expectation value of the tensor product state $|\alpha_{1\lambda}(k_0)\rangle\,|-\alpha^*_{1\lambda}(-k_0)\rangle$ is zero etc. This means that the energy expectation value of this state is 0, as it should be. However, setting $\phi = \frac{3\pi}{2}$ in (5.43) and denoting $|\psi\rangle^S = S_{\text{I}}\,|\psi\rangle$ we find that

$$\begin{aligned}
\langle\alpha_{1\lambda}(k_0')|^S\mathbf{E}(x,t)|\alpha_{1\lambda}(k_0)\rangle^S &= \delta(k_0 - k_0')\cos(-k_0x - k_0ct + \theta + \frac{\pi}{2})\mathbf{E}_\lambda\,,\\
\langle -\alpha^*_{1\lambda}(-k_0')|^S\mathbf{E}(x,t)|-\alpha^*_{1\lambda}(-k_0)\rangle^S &= \delta(k_0 - k_0')\cos(k_0x + k_0ct - \theta + \pi + \frac{\pi}{2})\mathbf{E}_\lambda\,,\\
&= \langle\alpha_{1\lambda}(k_0')|^S\mathbf{E}(x,t)|\alpha_{1\lambda}(k_0)\rangle^S\,.
\end{aligned} \tag{5.49}$$

This means that

$$\begin{aligned}
&\langle\alpha_{1\lambda}(k_0')|^S\,\langle -\alpha^*_{1\lambda}(-k_0')|^S\,\mathbf{E}(x,t)\,|\alpha_{1\lambda}(k_0)\rangle^S\,|-\alpha^*_{1\lambda}(-k_0)\rangle^S\\
&\qquad = 2\,\langle\alpha_{1\lambda}(k_0')|^S\mathbf{E}(x,t)|\alpha_{1\lambda}(k_0)\rangle^S \neq 0\,,
\end{aligned} \tag{5.50}$$

and a similar story holds for the magnetic field. We therefore do not have energy conservation. But we should expect this as the energy and scattering operators do not commute with each other. However, we do have energy conservation here if $e^{i\phi} \in \mathbb{R}$. This is why we use the complex conjugate Schrödinger and Heisenberg equations, to account for mirror Hamiltonians where it is possible for reflected or transmitted waves to obtain a phase ϕ that need not be 0 or π.

5.4 The electric field near two-sided semi-transparent mirrors

In this section, we give a concrete application of the theory presented in this thesis. We ask the question; "How does the electric field behave in the presence of a localised mirror?". Suppose that, before introducing the mirror, our E-field is a collection of classical sinusoidal frequencies that are polarised in the λ direction, travelling in the positive x direction, and have the same phase Θ

$$\mathbf{E}(x,t) = \int_0^\infty \mathrm{d}k g(k) \cos(k(x - sct) + \Theta)\mathbf{e}_\lambda \,, \tag{5.51}$$

where $g(k)$ is a real function. To model this scenario, we shall use coherent states of the EM field described in Section 3.2:

$$|\alpha^k_{s\lambda}\rangle_b = \exp\left(-\frac{1}{2}\int_0^\infty \mathrm{d}k|\alpha^k|^2\right)\exp\left(\int_0^\infty \mathrm{d}k\, \alpha^k b^\dagger_{s\lambda}(k)\right)|0\rangle \,, \tag{5.52}$$

because if we set

$$\alpha^k = \frac{g(k)}{2A_{mp}\sqrt{k}}e^{i\Theta} \,, \tag{5.53}$$

we have

$$\langle \mathbf{E}(x,t)\rangle_{|\alpha^k_{s\lambda}\rangle_b} = \int_0^\infty \mathrm{d}k g(k) \cos(k(x - sct) + \Theta)\mathbf{e}_\lambda \,. \tag{5.54}$$

with $\phi = 0$ in the EM field operators. Additionally, this expectation value has minimum uncertainty.

Furthermore, we could also use the c photon coherent state

$$|\alpha^k_{s\lambda}\rangle = \exp\left(-\frac{1}{2}\int_0^\infty \mathrm{d}k|\alpha^k|^2\right)\exp\left(\int_0^\infty \mathrm{d}k\, \alpha^k c^\dagger_{s\lambda}(k)\right)|0\rangle \,, \tag{5.55}$$

because if, in this case, we set

$$\alpha^k = \frac{g(k)}{2A_{mp}\sqrt{k}} e^{-i\Theta}\,, \tag{5.56}$$

with $g(k) = g(-k)$, then we have

$$\langle \mathbf{E}(x,t)\rangle_{|\alpha^k_{s\lambda}\rangle_c} = \int_0^\infty \mathrm{d}k g(k)\cos(k(x-sct)+\Theta)\mathbf{e}_\lambda\,. \tag{5.57}$$

For the mirror Hamiltonian itself, we start with a toy model given by the local Hamiltonian given earlier in equation (5.9) with $\phi = 0$ for simplicity.

$$H_{\mathrm{loc}}(x) = \sum_\lambda \int_{-\infty}^{\infty} \mathrm{d}x\, i\hbar\Omega(x)\left[a^\dagger_{-1\lambda}(-x)a^{\mathrm{bio}}_{1\lambda}(x) - a^\dagger_{1\lambda}(x)a^{\mathrm{bio}}_{-1\lambda}(-x)\right]\,. \tag{5.58}$$

For now there is no need to fix $\Omega(x)$ so we leave it general until later as this helps for explanation purposes. We do not discuss the origin of this Hamiltonian much; for example, if it derives from an atom-field interaction, etc.

We introduce our mirror Hamiltonian so that our total system Hamiltonian is

$$H_{\mathrm{sys}} = H_{\mathrm{free}} + H_{\mathrm{loc}}(x)\,. \tag{5.59}$$

We can use this to reproduce the classical mirror image method, but from a fully quantum foundation. Now, when calculating the E field expectation value of this state under H_{sys} we end up with

$$\langle \mathbf{E}(x,t)\rangle_{|\alpha^k_{s\lambda}\rangle_b} = \sum_{s',\lambda} c\sqrt{\frac{\hbar c}{2\varepsilon A}}(\langle\alpha^k_{s\lambda}|_b\, a_{s'\lambda}(x,t)\,|\alpha^k_{s\lambda}\rangle_b + \langle\alpha^k_{s\lambda}|_b\, a^\dagger_{s'\lambda}(x,t)\,|\alpha^k_{s\lambda}\rangle_b)\mathbf{e}_\lambda\,. \tag{5.60}$$

Using Section 4.2.4, the time dependence of each operator is

$$\begin{aligned} a_{s\lambda}(x,t) &= \overline{U}^\dagger(t)a_{s\lambda}(x)\overline{U}^{\mathrm{bio}}(t) \\ &= \cos\left(\int_0^t \Omega(x-sct')\mathrm{d}t'\right)a_{s\lambda}(x-sct) \\ &\quad -(s)\sin\left(\int_0^t \Omega(x-sct')\mathrm{d}t'\right)a_{-s\lambda}(-(x-sct))\,, \end{aligned} \tag{5.61}$$

and

$$\begin{aligned} a^\dagger_{s\lambda}(x,t) &= \overline{U}^{\dagger\,\mathrm{bio}}(t)a^\dagger_{s\lambda}(x)\overline{U}(t) \\ &= \cos\left(\int_0^t \Omega(x-sct')\mathrm{d}t'\right) a^\dagger_{s\lambda}(x-sct) \\ &\quad -(s)\sin\left(\int_0^t \Omega(x-sct')\mathrm{d}t'\right) a^\dagger_{-s\lambda}(-(x-sct))\,. \end{aligned} \tag{5.62}$$

These transformations ensure that the EM field operators remain Hermitian. However, this operator is not Hermitian with respect to the generalised inner product we use for our Hilbert space. So we have a non-Hermitian operator that gives real expectation values. This only applies because the states we used here belong to $\mathcal{H}\cap\mathcal{H}^*$ as $S(|\alpha^k_{s\lambda}\rangle_b) = |\alpha^k_{s\lambda}\rangle_b$. Combining the above equations with the electric field representation in Eq. (4.25), we see that the time evolution of the electric field operator is

$$\begin{aligned} \mathbf{E}(x,t) &= \sum_{s,\lambda}\Bigg(\cos\left(\int_0^t \Omega(x-sct')\mathrm{d}t'\right) E_{s\lambda}(x-sct,0) \\ &\quad -(s)\sin\left(\int_0^t \Omega(x-sct')\mathrm{d}t'\right) E_{-s\lambda}(-(x-sct),0)\Bigg)\mathbf{e}_\lambda\,. \end{aligned} \tag{5.63}$$

Therefore, a left moving electric field at x can transform into a right moving electric field depending on the initial location x and coupling parameter $\Omega(x)$. In Southall *et al.* (2021), we showed how to apply this transformation to the local operators

$$a_{s\lambda}(x) = \int_{-\infty}^{\infty}\mathrm{d}k\,\frac{1}{\sqrt{2\pi}}e^{iskx}\,b_{s\lambda}(k)\,, \tag{5.64}$$

which are bosonic with respect to the conventional scalar product. However, in this thesis, we have showed how to achieve a mirror transformation for the physically observable electric field itself.

Suppose our initial state is only in the $s=1$ mode, the only non-zero terms will then be from $s=1$ operators in the E field operator. Therefore, equation (5.60) reduces to

$$\begin{aligned} \langle\mathbf{E}(x,t)\rangle_{|\alpha^k_{s\lambda}\rangle_b} &= \Bigg(\cos\left(\int_0^t \Omega(x-ct')\mathrm{d}t'\right)\int_0^\infty \mathrm{d}k\,g(k)\cos(k(x-ct)+\Theta) \\ &\quad +\sin\left(\int_0^t \Omega(x+ct')\mathrm{d}t'\right)\int_0^\infty \mathrm{d}k\,g(k)\cos(k(-x-ct)+\Theta)\Bigg)\mathbf{e}_\lambda\,, \end{aligned} \tag{5.65}$$

with no approximations. We have now solved the electric field value of a coherent state with the system Hamiltonian in Eq. (5.59) at *any* position x, time $t \geq 0$ and locally integrable $\Omega(x)$. In the next section, we set $\Omega(x)$ to reproduce the classical E field's expected dynamics near a mirror. Before we do, however, we make a remark regarding energy conservation.

Substituting the creation and annihilation operators above into the EM field operators

$$\begin{aligned} \mathbf{E}(x,t) &= \sum_{s,\lambda} \sqrt{\frac{\hbar c}{2\varepsilon A}}\, a_{s\lambda}(x,t)\, \mathbf{e}_\lambda + H.c.\,, \\ \mathbf{B}(x,t) &= \sum_{s,\lambda} \frac{s}{c} \sqrt{\frac{\hbar c}{2\varepsilon A}}\, a_{s\lambda}(x,t)\, \mathbf{e}_x \times \mathbf{e}_\lambda + H.c.\,, \end{aligned} \tag{5.66}$$

and then substituting these EM operators in the classical energy expression of the EM field

$$H_{\text{eng}}(t) = \frac{A}{2} \int_{-\infty}^{\infty} \mathrm{d}x \, \left[\varepsilon_0 |\mathbf{E}(x,t)|^2 + \frac{1}{\mu_0} |\mathbf{B}(x,t)|^2 \right] , \tag{5.67}$$

we find

$$H_{\text{eng}}(t) = \sum_{s,\,\lambda} \int_{-\infty}^{\infty} \mathrm{d}x\, \hbar c \left(a_{s\lambda}(x) + a_{s\lambda}^\dagger(x) \right) \left(a_{s\lambda}(x) + a_{s\lambda}^\dagger(x) \right) , \tag{5.68}$$

which is exactly equal to the classical free space energy observable in Eq. (4.38). This shows that the above EM field transformations preserve energy. However, we point out here that

$$\begin{aligned} H_{\text{eng}}(t) &\neq \overline{U}^{\dagger\,\text{bio}}(t) H_{\text{eng}} \overline{U}(t)\,, \\ H_{\text{eng}}(t) &\neq \overline{U}^{\dagger}(t) H_{\text{eng}} \overline{U}^{\text{bio}}(t)\,, \end{aligned} \tag{5.69}$$

due the operator transformations in Eq. (5.61) and Eq. (5.62). In conventional quantum physics where the Hamiltonian generates only one equation of motion this would not be allowed.

5.5 The mirror Hamiltonian

We want our mirror to have the properties that it is only non-zero for a finite, but not point-like, distance around $x = 0$ and that it will fully turn around any incoming E-field but will not affect any outgoing field. Therefore, we choose $\Omega(x)$ such that

$$\Omega(x) = \begin{cases} \frac{\pi}{2} & \text{if } -1/2 \leq x \leq 1/2 \\ 0 & \text{otherwise.} \end{cases} \tag{5.70}$$

We choose this $\Omega(x)$ because

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \Omega(x) = \frac{\pi}{2}\,, \tag{5.71}$$

and we know from the last subsection that coupling parameters with this property fully transfer s local operators into $-s$ local operators. Of course, there are *infinitely* many other mirror type Hamiltonians we can create. We choose the one here as it is simple and works as proof of concept. We set $\Theta = 0$ and $g(k) = \delta(k-1)$ for simplicity. This means

$$\begin{aligned} \langle \mathbf{E}(x,t)\rangle_{|\alpha^k_{s\lambda}\rangle_b} \;=\; & \left(\cos\left(\int_0^t \Omega(x-ct')\mathrm{d}t'\right)\cos(x-ct)\right. \\ & \left. +\sin\left(\int_0^t \Omega(x+ct')\mathrm{d}t'\right)\cos(-x-ct)\right)\mathbf{e}_\lambda\,. \end{aligned} \tag{5.72}$$

Therefore, we see that the E field expectation value is the sum of a left travelling wave and a right travelling wave, where the amplitude of each depends on the coupling parameter $\Omega(x)$.

The following graphs show how the E field expectation value of the coherent state changes in the presence of a mirror, as described above. The plots show equation Eq. (5.72) with $\Omega(x)$ given by Eq. (5.70) for x in the range $(-50, 50)$.

If $x > \pi/2$ then whilst $x - ct > \pi/2$ the E-field is exactly what it would be without the interaction. Once $x - ct < \pi/2$ the amplitude of this wave decreases continuously, because of integral over $\Omega(x - ct)$ in the cos term above, until eventually $x - ct = -\pi/2$ and the amplitude becomes zero as $\int_0^t \Omega(x - ct')\mathrm{d}t' = \pi/2$. The E-field will then remain zero at this location x. For any $t > 0$ at $x > \pi/2$, there is no contribution from a left travelling wave as the sin term in Eq. (5.72) remains zero.

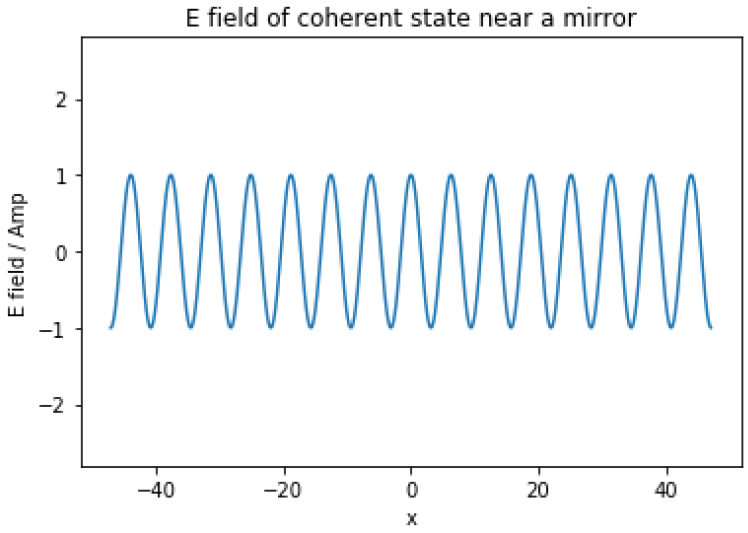

Figure 5.1: The E field at $t = 0$. Here it is just a travelling wave in free space.

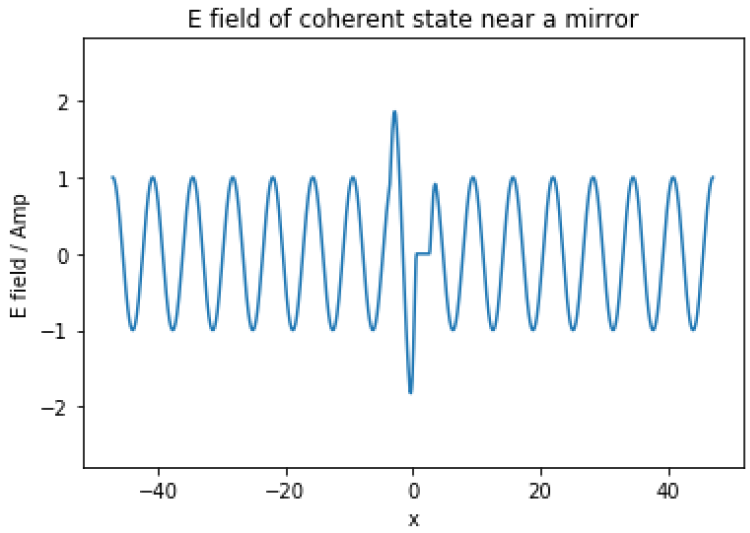

Figure 5.2: The E field at $t = \frac{\pi}{c}$. Here we see a standing wave forming on the L.H.S. of the mirror.

However, suppose that $x < -\pi/2$. Here the amplitude contribution from a right travelling wave does not change as the integral over $\Omega(x - ct')$ in the cos term above remains 0, which is good as any right moving wave here is arriving from where there has been no interaction. However, once $x + ct \geq -\pi/2$, the

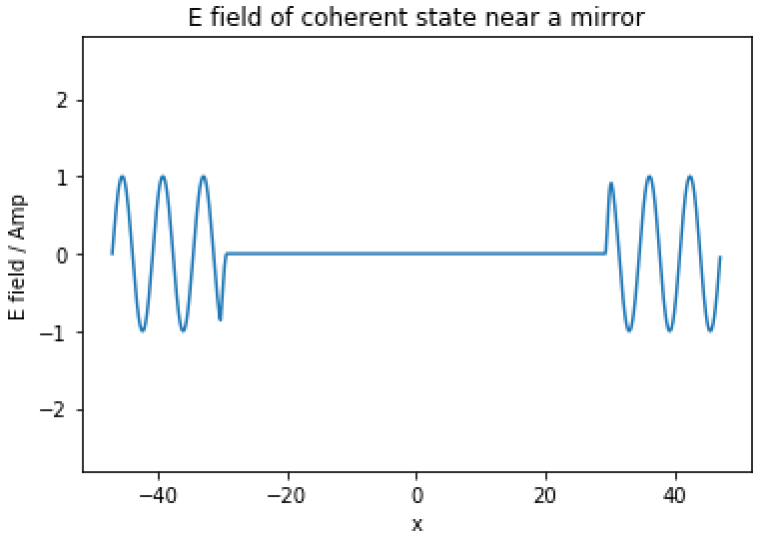

Figure 5.3: The E field at $t = \frac{19\pi}{2c}$. We chose this time so that the standing wave has zero amplitude.

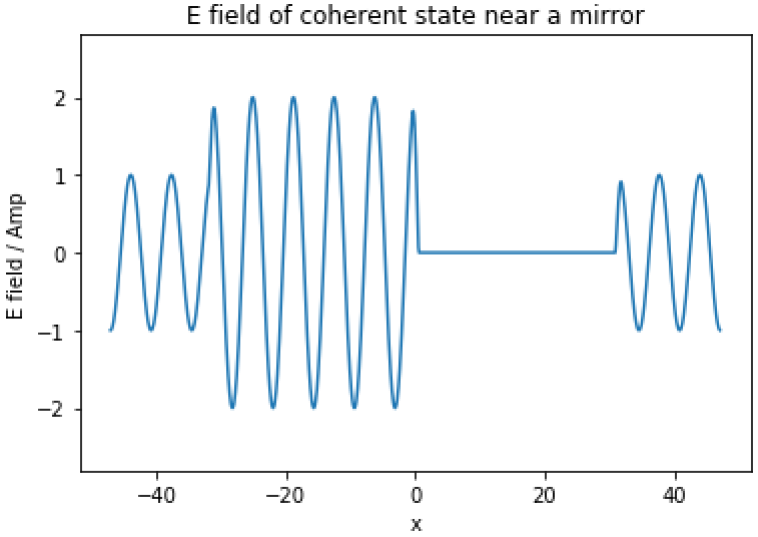

Figure 5.4: The E field at $t = \frac{10\pi}{c}$. We chose this time so that the standing wave has a maximum amplitude.

left travelling wave contribution increases as the sin term monotonically increases from 0 to eventually 1 when $x + ct = \pi/2$ and remains 1 after that. Once this happens, the left and right travelling waves contributions interfere, creating a

standing wave.

We have reproduced a classical sinusoidal electric wave's behaviour in the presence of a mirror from a purely quantum perspective. In doing so, we extended conventional quantum theory and used no approximations nor any assumptions once we understood the correct way to interpret the extended quantum theory. Thus, we have shown how the local quantum theory of light we presented in this thesis can model a system with local boundary conditions that alters the EM field only at specific locations. This is important as a typical situation in an experiment arises where the electric field is altered locally, so a theory that accurately models this is critical.

5.6 Summary

- We derived what Hamiltonian satisfies the Heisenberg equation for the following bosonic mode transformation

$$A^\dagger(t) = \frac{1}{(\sum_{j=1}^{N} f_j(t)^2)^{\frac{1}{2}}} \sum_{i=1}^{N} e^{i\phi_i} f_i(t) a_i^\dagger \,, \tag{5.73}$$

  where $[a_i, a_j^\dagger] = \delta_{ij}$, $[a_i, a_j] = 0$ and the $f(t)_i$'s are any real differentiable functions, there is no T such that $f_i(T)$=0 for all i. The Hamiltonian itself is

$$H(t) = \frac{i\hbar}{\sum_{j=1}^{N} (f_j(t)^2)} \sum_{k=1}^{N} \sum_{m>k}^{N} \left(e^{i(\phi_k - \phi_m)} (f_k(t)\dot{f}_m(t) - \dot{f}_k(t) f_m(t)) a_k^\dagger a_m \right) + H.c. \,, \tag{5.74}$$

  This is a new result. What it means is that if one has N bosonic modes and wants a transformation of the the form in Eq. (5.73), they can readily compute the Hamiltonian.

- We constructed locally acting Hamiltonians of the form

$$H_{\text{loc}}(x) = \sum_{\lambda} \int_{-\infty}^{\infty} \mathrm{d}x \, i\hbar\Omega(x) \left[e^{i\phi} \, a_{-1\lambda}^\dagger(-x) a_{1\lambda}^{\text{bio}}(x) - e^{-i\phi} \, a_{1\lambda}^\dagger(x) a_{-1\lambda}^{\text{bio}}(-x) \right] . \tag{5.75}$$

We then used these Hamiltonians in conjugation with the free space Hamiltonian to construct mirror systems and solved the dynamics of local, bio-local and photon states within these mirror systems. For the local and bio-local states, these Hamiltonians coupled a state to its mirror image state subject to a local potential $\Omega(x)$. The role of the free space Hamiltonian was to then to propagate these local and bio-local states at the speed of light along the x-axis. This is a new result as it is the first time a locally acting mirror Hamiltonian has been derived (Southall *et al.*, 2021).

- We calculated the scattering operator in the interaction picture of these mirror systems and computed its effect on monochromatic photons.

- For our final new result, we showed how these mirror systems alter the electric field expectation values of coherent states and that it reproduces the appropriate classical dynamics. This extended our results in Southall *et al.* (2021) where we constructed states whose dynamics could be solved in the presence of a mirror but the same could not be said for the electric field operator itself.

Chapter 6

Conclusions and outlook

This research aimed to introduce a formalism that allows us to model experiments that alter the EM field only at specific locations. By using complex conjugated Schrödinger and Heisenberg equations, modelling the EM field as a biorthogonal system with a non-zero intersection between its Hilbert space and dual Hilbert space, and then applying these techniques to model the transformation of a classical sinusoidal E wave near a mirror, it can be concluded that we have achieved our aim. More work is needed to discuss the Lorentz transformations of the mirror interactions we presented. However, to start this off we could look at proving the Lorentz invariance of the commutators

$$\begin{aligned} \left[a_{s\lambda}(x), a^{\mathrm{bio}\,\dagger}_{s\lambda}(x')\right] &= \delta(x-x')\,, \\ \left[a^{\mathrm{bio}}_{s\lambda}(x), a^{\dagger}_{s\lambda}(x')\right] &= \delta(x-x')\,. \end{aligned} \tag{6.1}$$

That is, proving

$$\delta(x-x') = \delta(\Delta x - \Delta x')\,, \tag{6.2}$$

where Δ is a Lorentz transformation.

We presented our new results in part II of this thesis. In chapter 4, we constructed local bosonic operators out of the free space EM field operators such that there exists a one-to-one *and* unitary transformation between these operators and monochromatic photon operators. This was a novel result. To achieve this, we modelled the field as a biorthogonal system with a non-zero intersection between its Hilbert space and dual Hilbert space. To the best of our knowledge, before

this research, no such system had been examined before. We also extended the Hilbert space of the EM field to include monochromatic photon states that evolve according to the complex conjugate Schrödinger equation. In chapter 5, we applied the theory from chapter 4 to model light in certain scattering descriptions. In particular, we showed that this approach enables us to construct locally acting mirror Hamiltonians that reproduce the well-known dynamics of wave packets in the presence of two-sided semi-transparent mirrors. When analysing the dynamics of the quantised EM field in the presence of a mirror interface in the interaction picture, we found that the corresponding scattering operator in Eq. (5.35), which maps the states of incoming onto the states of outgoing wave packets, couples monochromatic photons to monochromatic photons of the same frequency and is equal to an effective time evolution operator generated by a conventionally Hermitian Hamiltonian. This showed that describing overall scattering transformations, in this case, does not require an extension of the standard description of photonic wave packets. However, to compute the time evolution of a state at a time $t > 0$, we required extending the standard description to include states that evolve according the complex conjugate Schrödinger equation.

To be sure, until that point, we had only looked at how quantum states had evolved in the presence of a mirror Hamiltonian. Therefore, to give a full flavour of its utility, we computed how the expectation value of a physical observable, the electric field operator, changes in the presence of a mirror Hamiltonian for a coupling potential $\Omega(x)$ as described in the main text. We used a coherent state, commonly referred to as "the most classical" quantum state, to highlight the correspondence to classical physics. We calculated this computation exactly. We found the results agreed with classical physics – any outgoing electric field was not affected by the mirror, and any incoming electric field was reflected and created a standing wave with the incoming electric field on the same side. In the example, we used a fully reflecting mirror, i.e. with a reflective coefficient equal to 1, and a phase accumulation upon reflection of 1 on one side and -1 on the other. However, these values can be adjusted.

In this thesis, we showed how to utilise biorthogonal quantum mechanics as a tool to model locally acting mirror Hamiltonians that reproduce the appropriate dynamics of a classical electric field sinusoidal wave near a mirror. This is a novel

result. To do so, it involved allowing part of the electric field operator to evolve using H and the remaining part to evolve using H^{bio}, as well as stating that the electric field operator can only be used as an observable for the states where $S(|\psi\rangle) = |\psi\rangle$. In other words, we started with the quantised EM field, recognised the utility of biorthogonal quantum mechanics, and so applied it accordingly. However, we do not claim to know that, given a general biorthogonal quantum mechanics system, what are its properties such that we can apply the results in this thesis to it. This is something we would like to work on in the future.

The last chapter showed how using both b and c photons can make it easy for us to construct locally acting Hamiltonians. However, we point out that, when computing the electric field expectation values of a coherent state near the mirror, the states we used were composed only of b photons. This means that when we computed the full time-dependent expectation value, we could have commuted the c photon operators in the state with the b photon operators in the time evolution operator and electric field operator to move them to the vacuum bra/ket where they annihilate the vacuum. Therefore, the electric field expectation value we obtained is equal to the expectation value with the same state, operator and Hamiltonian, except with the c photon operators removed.

Furthermore, we also know that

$$\langle A\,|\psi(t)\rangle\,,|\psi(t)\rangle\rangle = \langle A\,|\psi(t)\rangle\,,|\psi(t)\rangle\rangle^* = \langle A^*\,|\psi(t)\rangle^*\,,|\psi(t)\rangle^*\rangle\ , \tag{6.3}$$

if $\langle A\,|\psi\rangle\,,|\psi\rangle\rangle$ is real. This equation means that for any expectation value comprising only of b photons, we can construct an equivalent expectation value using only c photons. It is, therefore, interesting to ask if the c photons[1] are physically distinguishable from b photons? Or do they just make computations easier and allow us to construct local models? Perhaps the answers to these questions lie in optical homodyne tomography, where Smithey *et al.* (1993) have succeeded in measuring the Wigner distribution and density matrix of certain light modes. For example, coherent states generated by b photons may rotate in the opposite direction to c photons in phase space. Additionally, perhaps it will be possible to interfere b and c photons to create novel effects. Last, if we can construct projectors for both the b photons and c photons, then we could distinguish the two

[1] Or equivalently negative frequency photons in free space.

by applying projective measurements onto the b and c photon subspaces. This of course assumes that the generation of both is possible. Optical phase conjugation (He, 2002; Pepper, 1982) may hold the answers here.

In addition to constructing locally acting Hamiltonians, these operators naturally lend themselves to the modelling of the quantised EM field in in-homogeneous media, are likely to provide new insight into fundamental quantum effects (Maybee *et al.*, 2019) and quantum information processing with photonic wave packets (Wang *et al.*, 2019).

Appendix A

Code

This is program will calculate the E-field expectation value described in chapter 5.5. We wrote in the python programming language.

```
# importing the required modules
import matplotlib.pyplot as plt
import numpy as np

#Calculating Omega(x) integrals
def int_omega_plus(x,t,height):
    if x > 0.5:
        int_omega_plus = 0
    elif x > -0.5:
        if x + t > 0.5:
            int_omega_plus = (height)*(0.5-x)
        else:
            int_omega_plus = (height)*(t)
    else:
        if x + t < -0.5:
            int_omega_plus = 0
        elif x + t < 0.5:
            int_omega_plus = (height)*(x + t + 0.5)
        else:
            int_omega_plus = height
```

```
    return int_omega_plus

def int_omega_minus(x,t,height):
    if x < -0.5:
        int_omega_minus = 0
    elif x < 0.5:
        if x - t < -0.5:
            int_omega_minus = (height)*(x+0.5)
        else:
            int_omega_minus = (height)*(t)
    else:
        if x - t > 0.5:
            int_omega_minus = 0
        elif x - t > -0.5:
            int_omega_minus = (height)*(0.5-(x-t))
        else:
            int_omega_minus = height
    return int_omega_minus

#setting our initial time t=0
t = 0

#setting the height of our potential, pi/2\pm n*pi corresponds to
#full reflection.
#for this potential, the height is equal to it's area under the
#curve, i.e. area= width x height and we have width=1 here.
#This parameter can be adjusted to see different reflective and
#transmissve rates.
height =(np.pi/2)

#creating our while loop, during each iteration it will print
#the <E(x,t)> values to the console, increase t, then wait
#for the users command to continue with the new t value
#To exit the loop will require the users input in the console,
```

```
#i.e. crtl + c
while t > -1:
    #creating our x-axis array
    x = np.arange(-15*np.pi,15*np.pi,0.1)

    #creating our empty omega minus and plus integrals
    int_omega_plus_arr = []
    int_omega_minus_arr = []

    #appending our omega integral values for each x to our arrays
    for x in x:
        int_omega_plus_arr.append(int_omega_plus(x,t, height))
        int_omega_minus_arr.append(int_omega_minus(x,t, height))

    #Taking the cos and sin of each value in the arrays
    int_omega_minus_cos = np.cos(int_omega_minus_arr)
    int_omega_plus_sin = np.sin(int_omega_plus_arr)

    x = np.arange(-15*np.pi,15*np.pi,0.1)
    #creating our right moving wave terms
    y = np.cos(x-t)*int_omega_minus_cos
    #creating our left moving wave terms
    z = np.cos(-x-t)*int_omega_plus_sin
    #adding the two together
    w = y + z

    # Add title and axis names
    plt.title('E field of coherent state near a mirror')
    plt.xlabel('x')
    plt.ylabel('E field / Amp')


    #creating our plots
```

```
#plot of right moving wave
#plt.plot(x,y)

#plot of left moving wave
#plt.plot(x,z)

#plot of the sum.
plt.plot(x,w)
plt.axes().set_aspect(12, 'datalim')
plt.show()
#increasing t, can adjust this for smoother animations.
t += (np.pi/10)
#command to allow user to continue when enter is pressed
#hold down enter to see 'animation'
wait = input("PRESS ENTER TO CONTINUE.")
```

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