Attitude regulation with bounded control in the presence of large disturbances with bounded moving average

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***Abstract*—The attitude regulation problem with bounded control for a class of satellites in the presence of large disturbances, with bounded moving average, is solved using a Lyapunov-like design. The analysis and design approaches are introduced in the case in which the underlying system is an integrator and are then applied to the satellite attitude regulation problem. The performance of the resulting closed-loop systems are studied in detail and it is shown that trajectories are ultimately bounded despite the effect of the persistent disturbance. Simulation results on a model of a small satellite subject to large, but bounded in moving average, disturbances are presented.**

***Index Terms*—Robust attitude regulation, Ultimate boundedness, Bounded control, CubeSats**

I. Introduction

In the past few decades satellites have been widely developed for commercial, communication, military and scientific purposes. These include tasks which play important roles in modern society, such as long-distance signal transmission [1], [2], navigation [3], [4], Earth observation [5] and weather and climate monitoring [6]. To guarantee reliable and repeatable operations, high-accuracy attitude control algorithms have to be implemented and various control methods have been ultilized. Traditional control architectures exploit PD-like controllers [7], PID controllers [8], LQR-based designs [9], [10], sliding mode controllers [11] and back-stepping-based designs [12]. Of these, PD-like, PID and LQR controllers are most widely used in applications because of their simple structure and design process.

A class of satellites, which is becoming very popular because of the modest construction and deployment costs, is that of the so-called CubeSats [13]–[18]. Unfortunately, CubeSats have a limited power budget and, because of their small dimension, are very sensitive to environmental disturbances, such as the Earth gravity gradient, aerodynamic drag, solar radiation pressure [19], [20], and to disturbances resulting from the coupling of the on-board electronics with the Earth magnetic field (often known as the residual dipole torque) [21]. In addition, the size and weight limitations impose strict constraints on the attitude control torque, which is often smaller, in magnitude, than the combined torque generated by the disturbances [22].

CubeSats attitude regulation in the presence of disturbances has been studied in [23] (in which the gravity gradient is regarded as the main external disturbance) and in [24] (in which multiple external disturbances are considered). In both papers, however, no consideration has been given to control bounds or to the fact that the instantaneous amplitude of the external disturbance may exceed the available control torque.

Such considerations are the point of departure for this paper. In particular, we study control systems which are perturbed by (additive) external disturbances, the instantaneous amplitude of which is not constrained to be smaller than the amplitude of the control signal. To illustrate this class of control problems, which to the best of our knowledge has not received attention in the control literature, we study initially the case of a scalar linear system (an integrator) with matched additive disturbance and with bounded input. For this *toy* example we design a state feedback control law which, on the basis of an integral bound on the disturbance, yields ultimately bounded trajectories and stability of a set containing the origin.

This design idea is then extended to solve the attitude control problem for a satellite subject to external torque disturbances, which satisfy an integral bound, in the presence of bounded control. We consider a saturated PD-like control structure, the gains of which are tuned as a function of the available bound on the disturbance, the available control torque, and the desired attitude accuracy. Similarly to the case of the *toy* example, we show that the trajectories of the closed-loop system are ultimately bounded and a residual set around the origin, the size of which depends upon the bound on the disturbances, is stable.

The paper is organised as follows. In Section II we discuss the class of disturbances considered and we present integral bounds which are used to characterize their long term properties, or their properties in a given time window. Relations among these bounds and implications in terms of the

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instantaneous amplitude of the disturbance are also discussed. In Section III a scalar linear system with matched additive disturbance and bounded input is studied as a motivating example. Two state feedback controllers, designed on the basis of a given integral bound on the external disturbance, are presented and the resulting properties of the closed-loop system are discussed. By exploiting the ideas of Section III, Section IV provides a solution to the bounded input attitude regulation problem for a satellite subject to disturbances satisfying an integral bound. In Section V simulation results for the systems studied in Sections III and IV driven by randomly generated disturbances satisfying the considered integral bounds are presented. Section VI contains conclusions and outlooks.

II. The class of admissible disturbances

Disturbances are ubiquitous in practical control problems. While there are several ways in which disturbances can be *measured*, for example using L_p norms, where typically $p = 1,\ 2,\ \infty$, or the notion of power, in this paper we consider bounds in terms of windowed averaged p-norms defined as follows. Differently from traditional L_p norms, the introduced windowed averaged p-norms are useful to "measure" disturbances, the amplitude of which may be very large only for short periods.

Definition 1 *Let $w : \Re^{\geq 0} \to \Re^d$ be a piece-wise continuous vector-valued function of time. Consider a window $[t_1, t_2]$, with $0 \leq t_1 < t_2$. Let*

$$\|w\|_{p,a}^{[t_1,t_2]} = \frac{1}{t_2 - t_1}\left[\int_{t_1}^{t_2}\sum_{i=1}^{d}|w_i(\tau)|^p\ d\tau\right]^{1/p},$$

for p any finite integer, or

$$\|w\|_{\infty,a}^{[t_1,t_2]} = \frac{1}{t_2 - t_1}\max_{i=1,\cdots,d}\ \max_{t\in[t_1,t_2]}|w_i(t)|,$$

for $p = \infty$.

The average $p-$norm of w with moving window of size $T > 0$ is given by

$$\|w\|_{p,\ a}^{T} = \sup_{t\geq 0}\|w\|_{p,a}^{[t,\ t+T]}. \tag{1}$$

Definition 2 *The set of all signals w with finite average p-norm over a moving window of size $T > 0$ is denoted as $L_{p,\ a}^{T}$.*

The class of disturbances characterised in Definition 1 possesses some interesting properties and relations with disturbances satisfying L_p norm bounds, as summarised in the following statement.

Proposition 1 *Let w be a piece-wise continuous vector-valued function.*

(P1) *$L_p \subset L_{p,a}^T$, that is, $w \in L_p$ implies $w \in L_{p,a}^T$, for all $T > 0$.*
(P2) *$w \in L_{p,a}^T$, for some $T > 0$, does not imply $w \in L_p$, that is there exists signals $w \in L_{p,a}^T$ such that $w \notin L_p$.*
(P3) *If $w \in L_{1,a}^T \bigcap L_{\infty,a}^T$, then $\|w\|_{2,a}^T < \left(\|w\|_{\infty,a}^T\|w\|_{1,a}^T\right)^{1/2}$, hence $w \in L_{2,a}^T$.*
(P4) *If w is a periodic signal with period T, then $w \in L_{p,a}^T$.*
(P5) *If w is a power signal and $w \in L_{\infty,a}^T$, then $w \in L_{2,a}^T$.*

Proof: We prove each item individually.

(P1) Let $w \in L_p$, that is

$$\|w\|_p = \left[\int_0^{+\infty}\sum_{i=1}^{d}|w_i(\tau)|^p\ d\tau\right]^{1/p} < \infty,$$

or, equivalently,

$$\|w\|_p^p = \int_0^{+\infty}\sum_{i=1}^{d}|w_i(\tau)|^p\ d\tau < \infty.$$

Note now that, for all $t \geq 0$,

$$\begin{aligned} T(\|w\|_{p,a}^T)^p &= \int_t^{t+T}\sum_{i=1}^{d}|w_i(\tau)|^p\ d\tau \\ &= \int_0^{+\infty}\sum_{i=1}^{d}|w_i(\tau)|^p\ d\tau - \int_0^{t}\sum_{i=1}^{d}|w_i(\tau)|^p\ d\tau \\ &\quad - \int_{t+T}^{+\infty}\sum_{i=1}^{d}|w_i(\tau)|^p\ d\tau \\ &= \|w\|_p^p - \int_0^{t}\sum_{i=1}^{d}|w_i(\tau)|^p\ d\tau \\ &\quad - \int_{t+T}^{+\infty}\sum_{i=1}^{d}|w_i(\tau)|^p\ d\tau \\ &< \infty, \end{aligned}$$

hence we conclude that $\|w\|_{p,a}^T < \infty$, for any $T > 0$, which complete the proof of (P1).

(P2) From the proof of (P1) we have that if $w \in L_{p,a}^T$ then

$$\begin{aligned} T(\|w\|_{p,a}^T)^p = \|w\|_p^p &- \int_0^{t}\sum_{i=1}^{d}|w_i(\tau)|^p\ d\tau \\ &- \int_{t+T}^{+\infty}\sum_{i=1}^{d}|w_i(\tau)|^p\ d\tau \\ &< \infty, \end{aligned}$$

which does not impose any restriction on $\|w\|_p$. In summary, $w \in L_{p,\ a}^T$ for some $T > 0$, does not imply $w \in L_p$, which proves (P2).

(P3) By Definition 1

$$\|w\|_{1,a}^T = \frac{1}{T}\int_t^{t+T}\sum_{i=1}^{d}|w_i(\tau)|d\tau.$$

Moreover, since $w \in L_{1,a}^T \bigcap L_{\infty,a}^T$, then

$$\begin{aligned} \int_t^{t+T}\sum_{i=1}^{d}|w_i(\tau)|^2 d\tau &= \int_t^{t+T}|w(\tau)||w(\tau)|d\tau \\ &\leq T^2(\|w\|_{\infty,a}^T\|w\|_{1,a}^T) \\ &< \infty. \end{aligned}$$

As a result

$$\begin{aligned} \|w\|_{2,a}^T &= \frac{1}{T}\left[\int_t^{t+T}\sum_{i=1}^{d}|w_i(\tau)|^2 d\tau\right]^{1/2} \\ &\leq (\|w\|_{\infty,a}^T\|w\|_{1,a}^T)^2 \\ &< \infty, \end{aligned}$$

which completes the proof of item (P3).

(P4) Since w is a periodic signal with period T then, for all $t \geq 0$ and for all finite p,

$$\int_t^{t+T} \sum_{i=1}^{d} |w_i(\tau)|^p d\tau = C < \infty$$

for some $C > 0$. Note that a similar condition holds for $p = \infty$. Thus, for any $t \geq 0$ and any finite p,

$$\|w\|_{p,a}^T = \frac{1}{T}\left[\int_t^{t+T} \sum_{i=1}^{d} |w_i(\tau)^p| d\tau\right]^{1/p} = \frac{\bar{C}}{T} < \infty,$$

where $\bar{C} = C^{1/p}$. Again, a similar condition holds for $p = \infty$. As a result (P4) holds.

(P5) If w is a (one-sided) power signal, then the limit

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T w(t)^2 dt$$

exists. Now, since $\|w\|_{\infty,a}^T < \infty$, for $0 < T < \infty$, one has

$$\begin{aligned}\frac{1}{T}\int_t^{t+T} w(t)^2 dt &= \frac{1}{T}\int_t^{t+T} \sum_{i=1}^{d} |w_i(t)|^2 \; dt \\ &\leq \|w\|_{\infty,a}^{T2} \frac{1}{T}\int_t^T dt \\ &= \|w\|_{\infty,a}^{T2} \\ &< \infty.\end{aligned}$$

Thus

$$\begin{aligned}\|w\|_{2,a}^T &= \frac{1}{T}\left[\int_t^{t+T} \sum_{i=1}^{d} |w_i(t)|^2 \; dt\right]^{1/2} \\ &= \left(\frac{1}{T}\right)^{1/2}\left[\frac{1}{T}\int_t^{t+T} \sum_{i=1}^{d} |w_i(t)|^2 \; dt\right]^{1/2} \\ &< \left(\frac{1}{T}\right)^{1/2} \infty \\ &< \infty,\end{aligned}$$

which implies that (P5) holds. □

To illustrate the introduced notions, consider two periodic disturbances with period $T = 5s$, as displayed in Figures 1 and 3. Figures 2 and 4 show that the considered disturbances have bounded average norms, and unbounded L_p norms, for any finite p.

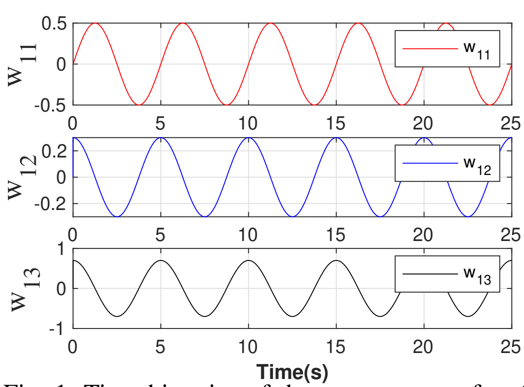

Fig. 1. Time histories of the components of $w_1(t)$.

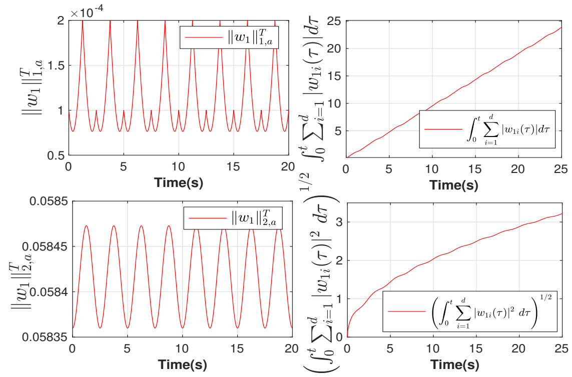

Fig. 2. $L_{p,a}^T$ norms (left), and $\left(\int_0^t \sum_{i=1}^d |w_{1i}(\tau)|^p d\tau\right)^{1/p}$ (right), for $p = 1, 2$.

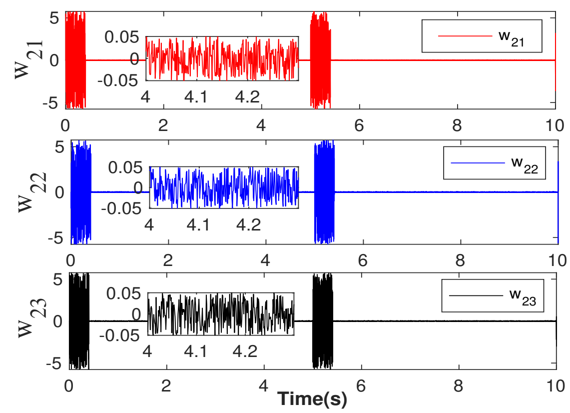

Fig. 3. Time histories of the components of $w_2(t)$.

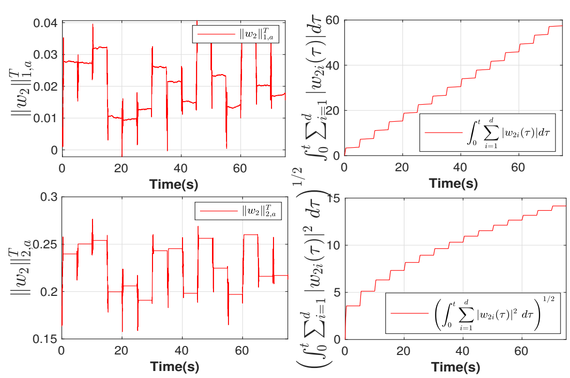

Fig. 4. $L_{p,a}^T$ norms (left), and $\left(\int_0^t \sum_{i=1}^d |w_{2i}(\tau)|^p d\tau\right)^{1/p}$ (right), for $p = 1, 2$.

The selected disturbances are such that $\|w\|_{p,a}^T \leq 1 - \varepsilon < 1$ for $p = 1, 2$ and $\varepsilon \in (0, 1)$, conditions which will be used in Sections III and IV, whereas they have unbounded L_1 and L_2 norms.

III. A PERTURBED INTEGRATOR WITH BOUNDED CONTROL

In this section we introduce the main tools for control design in the presence of *persistent* disturbances by studying the robust stabilization problem for a scalar system described by the equation

$$\dot{x} = u + w, \tag{2}$$

where $x(t) \in I\!\!R$ is the state of the system, $u(t) \in [-1,1]$ refers to the control input and $w(t) \in I\!\!R$ denotes the external disturbance. In addition, the disturbance w is such that

$$\|w\|_{1,a}^T = \frac{1}{T}\int_t^{t+T} |w| d\tau \leq 1 - \varepsilon < 1, \tag{3}$$

for all $t \geq 0$ and for some, not necessarily known, $T > 0$ and some $0 < \varepsilon < 1$.

Proposition 2 *Consider the system (2). Suppose the disturbance satisfies the condition (3). Let*

$$u = -\text{sign}(x), \tag{4}$$

where

$$\text{sign}(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

Then the state of the closed-loop system is ultimately bounded and $\limsup_{t\to\infty} |x(t)| \leq 2T$.

Proof: Consider the function $V(x) = \frac{1}{2}x^2$, which is smooth, positive definite, bounded from below and has a global minimizer at $x = 0$. Along the trajectories of the closed-loop system, to be understood as in [27], one has

$$\dot{V} \leq -|x| + |x||w| = \sqrt{2V}(-1 + |w|). \tag{5}$$

Suppose now that $|x(t)| > \varepsilon T$. Integrating both sides of (5) from t to $t + T$, by condition (3), one has

$$\begin{aligned} \sqrt{V(t+T)} - \sqrt{V(t)} &\leq \frac{\sqrt{2}T}{2}\left[-1 + \frac{1}{T}\int_t^{t+T} |w| d\tau\right] \\ &\leq -\frac{\sqrt{2}}{2}\varepsilon T < 0, \end{aligned} \tag{6}$$

hence

$$|x(t+T)| - |x(t)| \leq -\varepsilon T,$$

that is

$$0 \leq |x(t+T)| \leq |x(t)| - \varepsilon T, \tag{7}$$

which implies that there exists a finite k such that $|x(t + kT)| \leq \varepsilon T$.

Suppose now that $0 \leq |x(t)| \leq \varepsilon T$, for some $t \geq 0$. Note that

$$-1 - |w| \leq \dot{x} = -\text{sign}(x) + w \leq 1 + |w|,$$

hence a direct integration yields

$$-T - \int_t^{t+T} |w(\tau)| d\tau \leq x(t+T) - x(t) \leq T + \int_t^{t+T} |w(\tau)| d\tau,$$

that is

$$\begin{aligned} |x(t+T)| &\leq |x(t)| + T + \int_t^{t+T} |w(\tau)| d\tau \\ &\leq \varepsilon T + T(2 - \varepsilon) \\ &\leq 2T, \end{aligned}$$

We therefore conclude that for any initial condition, the trajectories of the system enter the set $|x(t)| \leq \varepsilon T$ in finite time and are such that $|x(t)| \leq 2T$, afterwards, which proves the claim. □

Remark 1 *The condition expressed in Proposition 1 and in equation (3) reduce, at T tends to zero, to*

$$\lim_{T\to 0} \frac{1}{T}\int_t^{t+T} |w| d\tau = |w(t)| \leq 1 - \varepsilon < 1,$$

which implies that all trajectories converge to zero (in finite time).

Remark 2 *The controller (4) is similar to a bang-bang controller, with the difference that the controller (4) does not solve an optimal control problem: it is developed to deal with control systems affected by disturbances, the instantaneous amplitude of which is not constrained to be smaller than the maximum amplitude of the control signal.*

To show that a similar result holds also for the $L_{2,a}^T$ norm, assume that

$$\|w\|_{2,a}^T = \frac{1}{T}\left[\int_t^{t+T} |w|^2 d\tau\right]^{1/2} \leq 1 - \varepsilon < 1, \tag{8}$$

for some $\varepsilon \in (0,1)$, that is

$$\int_t^{t+T} |w|^2 d\tau \leq (1 - \tilde{\varepsilon})T^2, \tag{9}$$

where $\tilde{\varepsilon} = (2 - \varepsilon)\varepsilon$, $\tilde{\varepsilon} \in (0,1)$, and for some $T > 0$.

Proposition 3 *Consider the system (2). Suppose that the disturbance satisfies the condition (8) and* $T < \frac{1}{1-\tilde{\varepsilon}}$. *Let*

$$u = -\text{sign}(x). \tag{10}$$

Then the state of the closed-loop system is ultimately bounded and $\limsup_{t\to\infty} |x(t)| \leq \frac{5}{2}T$.

Proof: Consider again the function $V(x) = \frac{1}{2}x^2$ and note that

$$\begin{aligned} \dot{V} &\leq -|x| + |x||w| \\ &= -|x| + \sqrt{|x|}(\sqrt{|x|}|w|) \\ &\leq -|x| + \frac{|x|}{2} + \frac{|x||w|^2}{2} \\ &\leq \frac{\sqrt{2V}}{2}\left(-1 + |w|^2\right). \end{aligned} \tag{11}$$

Suppose now that $|x(t)| > \frac{T}{2}$. Integrating both sides of (11) from t to $t + T$ and exploiting condition (9), one has

$$\begin{aligned} \sqrt{V(t+T)} - \sqrt{V(t)} &\leq \frac{\sqrt{2}T}{4}\left[-1 + \frac{1}{T}\int_t^{t+T} |w|^2 d\tau\right] \\ &\leq -\frac{\sqrt{2}T}{4}\left[1 - (1 - \tilde{\varepsilon})T\right]. \end{aligned} \tag{12}$$

Hence, since T is such that $T < \frac{1}{1-\tilde{\varepsilon}}$ then $1 - (1 - \tilde{\varepsilon})T > 0$, and

$$\sqrt{V(t+T)} - \sqrt{V(t)} < -\frac{\sqrt{2}T}{4},$$

and, equivalently,

$$|x(t+T)| - |x(t)| \leq -\frac{T}{2},$$

that is

$$0 \le |x(t+T)| \le |x(t)| - \frac{T}{2}, \tag{13}$$

which implies that there exists a finite k such that $|x(t+kT)| \le \frac{T}{2}$.

Suppose now that $0 \le |x(t)| \le \frac{T}{2}$, for some $t \ge 0$. Note that

$$-1 - \frac{|w|^2+1}{2} \le \dot{x} = -\text{sign}(x) + w \le 1 + \frac{|w|^2+1}{2},$$

hence a direct integration yields

$$\begin{aligned}|x(t+T)| &\le |x(t)| + \frac{1}{2}\left(2T + T + \int_t^{t+T} |w(\tau)|^2 d\tau\right)\\ &\le \frac{T}{2} + \frac{T}{2}\left(3 + (1-\tilde{\varepsilon})T\right).\end{aligned}$$

As a result

$$|x(t+T)| \le T\left(2 + \frac{1}{2}(1-\tilde{\varepsilon})T\right). \tag{14}$$

Recalling that $T < \frac{1}{1-\tilde{\varepsilon}}$, we have that $|x(t+T)| \le \frac{5T}{2}$. We therefore conclude that, for any initial condition, the trajectories of the system enter the set $|x(t)| \le \frac{T}{2}$ in finite time and are such that $|x(t)| \le \frac{5}{2}T$, afterward, which proves the claim. □

In what follows we show that the bound of Proportion 2 can be obtained also with a continuous control law. To this end, consider the control law

$$u(t) = -\text{sat}(Kx), \tag{15}$$

with $\text{sat}(Kx) = \min\{1, K|x|\}\text{sign}(x)$, and $K > 0$.

Corollary 1 *Consider the system (2) in closed-loop wtih the controller (15). Then the state of the closed-loop system is ultimately bounded. In particular,*

i) *if w satisfies the condition (3) one has $\limsup_{t\to\infty} |x(t)| \le 2T$, if $\varepsilon T \ge \frac{1}{K}$, and $\limsup_{t\to\infty} |x(t)| \le \frac{1}{K} + (2-\varepsilon)T$, if $\varepsilon T \le \frac{1}{K}$;*
ii) *if w satisfies the condition (8) one has $\limsup_{t\to\infty} |x(t)| \le \frac{5}{2}T$, if $\frac{2}{K} \le T < \frac{1}{1-\tilde{\varepsilon}}$ and $\limsup_{t\to\infty} |x(t)| \le \frac{1}{K} + 2T$, if $T \le \frac{2}{K} < \frac{1}{1-\tilde{\varepsilon}}$, where $\tilde{\varepsilon}$ is defined in (9).*

The claim of Corollary 1 follows from the proof of Propositions 2 and 3, respectively. Note that in this case, $u(x) = -\text{sign}(x)$ for all $|x| \ge \frac{1}{K}$. The proof of claim i) and ii) are given below.

Proof:

i) Consider the function $V(x) = \frac{1}{2}x^2$, which is smooth, positive definite, bounded from below and has a global minimizer at $x = 0$. Along the trajectories of the closed-loop system one has

$$\dot{V} \le -|x| + |x||w| = \sqrt{2V}(-1 + |w|). \tag{16}$$

Suppose now that $|Kx(t)| > 1$, i.e. $|x(t)| > \frac{1}{K}$. Integrating both sides of (16) from t to $t+T$, by condition (3), one has

$$\begin{aligned}\sqrt{V(t+T)} - \sqrt{V(t)} &\le \frac{\sqrt{2}T}{2}\left[-1 + \frac{1}{T}\int_t^{t+T} |w| d\tau\right]\\ &\le -\frac{\sqrt{2}}{2}\varepsilon T\\ &< 0,\end{aligned} \tag{17}$$

hence

$$|x(t+T)| - |x(t)| \le -\varepsilon T,$$

that is

$$0 \le |x(t+T)| \le |x(t)| - \varepsilon T, \tag{18}$$

which implies that there exists a finite k such that $|x(t+kT)| \le \varepsilon T$.
Suppose now that $0 \le |Kx(t)| \le 1$, i.e. $0 \le |x(t)| \le \frac{1}{K}$, for some $t \ge 0$. Note that

$$-1 - |w| \le \dot{x} = -\text{sign}(x) + w \le 1 + |w|,$$

hence a direct integration yields

$$-T - \int_t^{t+T} |w(\tau)| d\tau \le x(t+T) - x(t) \le T + \int_t^{t+T} |w(\tau)| d\tau,$$

that is

$$\begin{aligned}|x(t+T)| &\le |x(t)| + T + (1-\varepsilon)T\\ &\le \frac{1}{K} + T + (1-\varepsilon)T.\end{aligned} \tag{19}$$

Thus, if T is such that $\varepsilon T \ge \frac{1}{K}$, one has

$$|x(t+T)| \le 2T,$$

which is as discussed in Proposition 2. Similarly, if T is such that $\varepsilon T \le \frac{1}{K}$, then there exists a finite k such that $|x(t+kT)| \le \varepsilon T \le \frac{1}{K}$ and

$$|x(t+T)| \le \frac{1}{K} + (2-\varepsilon)T.$$

We therefore conclude that for any initial condition the trajectories of the system enter the set $|x(t)| \le \varepsilon T$ in finite time and are such that $|x(t)| \le 2T$ afterward, if $\varepsilon T \ge \frac{1}{K}$, and $|x(t)| \le \frac{1}{K} + (2-\varepsilon)T$ afterward, if $\varepsilon T \le \frac{1}{K}$, which proves the claim i).

ii) Consider again the function $V(x) = \frac{1}{2}x^2$ and note that

$$\begin{aligned}\dot{V} &\le -|x| + |x||w|\\ &= -|x| + \sqrt{|x|}(\sqrt{|x|}|w|)\\ &\le -|x| + \frac{|x|}{2} + \frac{|x||w|^2}{2}\\ &\le \frac{|x|}{2}\left(-1 + |w|^2\right)\\ &\le \frac{\sqrt{2V}}{2}\left(-1 + |w|^2\right).\end{aligned} \tag{20}$$

Suppose now that $|Kx(t)| > 1$, i.e. $|x(t)| > \frac{1}{K}$. Integrating both sides of (20) from t to $t+T$ and exploiting condition (9), one has

$$\sqrt{V(t+T)}-\sqrt{V(t)} \leq \frac{\sqrt{2}T}{4}\left[-1+\frac{1}{T}\int_t^{t+T}|w|^2 d\tau\right] \leq -\frac{\sqrt{2}T}{4}\left[1-(1-\tilde{\varepsilon})T\right]. \quad (21)$$

Recalling that T is such that $T < \frac{1}{1-\tilde{\varepsilon}}$, then

$$\sqrt{V(t+T)}-\sqrt{V(t)} < -\frac{\sqrt{2}T}{4},$$

and, equivalently,

$$|x(t+T)|-|x(t)| \leq -\frac{T}{2},$$

that is

$$0 \leq |x(t+T)| \leq |x(t)|-\frac{T}{2}, \quad (22)$$

which implies that there exists a finite k such that $|x(t+kT)| \leq \frac{T}{2}$.

Suppose now that $0 \leq |Kx(t)| \leq 1$, i.e. $0 \leq |x(t)| \leq \frac{1}{K}$, for some $t \geq 0$. Note that

$$-1-\frac{|w|^2+1}{2} \leq \dot{x} = -K|x|\text{sign}(x)+w \leq 1+\frac{|w|^2+1}{2},$$

hence a direct integration yields

$$|x(t+T)| \leq |x(t)|+\frac{1}{2}\left(2T+T+\int_t^{t+T}|w(\tau)|^2 d\tau\right),$$

that is

$$|x(t+T)| \leq \frac{1}{K}+\frac{T}{2}\left(3+(1-\tilde{\varepsilon})T\right). \quad (23)$$

Thus if T is such that $\frac{1}{K} \leq \frac{T}{2} < \frac{1}{2(1-\tilde{\varepsilon})}$, i.e. $\frac{2}{K} \leq T < \frac{1}{1-\tilde{\varepsilon}}$ one has

$$|x(t+T)| \leq \frac{5T}{2},$$

which is as discussed in Proposition 3. Similarly, if T is such that $\frac{T}{2} \leq \frac{1}{K} \leq \frac{1}{2(1-\tilde{\varepsilon})}$ there exists a finite k such that $|x(t+kT)| \leq \frac{T}{2} \leq \frac{1}{K}$ and

$$|x(t+T)| \leq \frac{1}{K}+2T.$$

We therefore conclude that, for any initial condition, the trajectories of the system enter the set $|x(t)| \leq \frac{T}{2}$ in finite time and are such that $|x(t)| \leq \frac{5}{2}T$ afterward, if $\frac{2}{K} \leq T \leq \frac{1}{1-\tilde{\varepsilon}}$, and $|x(t)| \leq \frac{1}{K}+2T$ afterward, if $\frac{T}{2} \leq \frac{1}{K} \leq \frac{1}{2(1-\tilde{\varepsilon})}$, which proves the claim ii).

□

IV. Satellite attitude control system

Consider now the attitude control problem for a fully actuated satellite, described by the equations

$$\begin{aligned} J\dot{\omega}+\omega^{\times}J\omega &= \tau+w, \\ \dot{q} &= \frac{1}{2}\begin{pmatrix} 0 & -\omega^{\mathrm{T}} \\ \omega & -\omega^{\times}\end{pmatrix}\begin{pmatrix} q_0 \\ q_v\end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} -\omega^{\mathrm{T}}q_v \\ -\omega^{\times}q_v+q_0 I_3\omega\end{pmatrix}, \end{aligned} \quad (24)$$

where $\omega = (\omega_1\ \omega_2\ \omega_3)^{\mathrm{T}}$ denotes the angular velocity, and $q = (q_0,\ q_v^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{4\times 1}$, with $q_0 = \cos\frac{\vartheta}{2}$, $q_v = (q_1,\ q_2,\ q_3)^{\mathrm{T}} = e\sin\frac{\vartheta}{2}$ is the quaternion vector such that $q_v^T q_v + q_0^2 = 1$. The variable ϑ represents the rotation angle with respect to the Euler axes and $e = (e_x,\ e_y,\ e_z)^{\mathrm{T}}$ denotes the vector basis of the Euler axes satisfying $\|e\|_1 = 1$. $J \in \mathbb{R}^{3\times 3}$ is the positive definite inertia matrix. $\tau = (\tau_1\ \ \tau_2\ \ \tau_3)^{\mathrm{T}}$ is the input torque and w represents the action of the external disturbance. Finally,

$$\omega^{\times} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

Let now

$$\tau(t) = -\frac{k_1}{2}q_v(t)-k_2\text{sat}\left(J\omega(t)\right), \quad (25)$$

with the saturation function defined as

$$\text{sat}(x) = \frac{x}{\sqrt{1+x^T J^{-1}x}},$$

with $k_1 > 0$ and $k_2 \geq \frac{\alpha}{1-\alpha^2}$, for some $\alpha \in (0,\ 1)$, such that $\frac{k_1}{2}+k_2 \leq |\tau_{max}|$, where τ_{max} is selected on the basis of the specific application.

Proposition 4 *Consider the system (24) with the controller (25). Assume the disturbance satisfies (3), then the angular velocity and quaternion are ultimately bounded, and there exists* $\underline{\zeta} > 0$ *such that* $\limsup_{t\to\infty}\|\omega(t)\| = \|J\|^{-1/2}\underline{\zeta}$.

Proof: Consider the function

$$V(t) = V_1(t)+V_2(t), \quad (26)$$

with $V_1(t) = k_1[q_v^{\mathrm{T}}q_v+(1-q_0)^2]$ and $V_2(t) = \omega^{\mathrm{T}}J\omega$. Recalling now that by definition of $q(t)$, one has $V(t) = 2k_1(1-q_0)+\omega(t)^{\mathrm{T}}J\omega(t)$.

Differentiating along the trajectories of the closed-loop system yields

$$\begin{aligned} \dot{V}(t) &= -2k_1\dot{q}_0+2\omega(t)^{\mathrm{T}}J\dot{\omega}(t) \\ &= k_1\omega(t)^{\mathrm{T}}q_v(t)+2\omega(t)^{\mathrm{T}}J\big[-J^{-1}\big(\omega(t)^{\times}J\omega(t) \\ &\quad +\tau(t)+w(t)\big)\big] \\ &= k_1\omega(t)^{\mathrm{T}}q_v(t)+2\omega(t)^{\mathrm{T}}\big(\tau(t)+w(t)\big). \end{aligned} \quad (27)$$

Recalling that $\sqrt{V_2} \leq \sqrt{V}$, and substituting (25) into (27) yields

$$\begin{aligned} \dot{V}(t) &= -2k_2\omega(t)^{\mathrm{T}}\text{sat}(J\omega(t))+2\omega(t)^{\mathrm{T}}w(t) \\ &\leq -2k_2\frac{\sqrt{V_2(t)}}{\sqrt{1+V_2(t)}}\sqrt{V_2(t)}+2\sqrt{V_2(t)}\|J\|^{-\frac{1}{2}}\|w(t)\|_2 \\ &\leq -2k_2\frac{\sqrt{V_2(t)}}{\sqrt{1+V_2(t)}}\sqrt{V(t)-V_1(t)}+2\sqrt{V(t)}\|J\|^{-\frac{1}{2}}\|w(t)\|_1 \\ &\leq -2k_2\frac{\sqrt{V_2(t)}}{\sqrt{1+V_2(t)}}\left(\frac{1}{\alpha}\sqrt{V(t)}-\alpha\sqrt{V_1(t)}\right) \\ &\quad +2\sqrt{V(t)}\|J\|^{-\frac{1}{2}}\|w(t)\|_1, \end{aligned} \quad (28)$$

for any $\alpha \in (0,1)$. Since $\sqrt{V_1} \leq \sqrt{V}$, we conclude that

$$\dot{V}(t) \leq -2k_2 \frac{\sqrt{V_2(t)}}{\sqrt{1+V_2(t)}} \left(\frac{1}{\alpha} - \alpha\right) \sqrt{V(t)} + 2\sqrt{V(t)}\|J\|^{-\frac{1}{2}}\|w(t)\|_1, \quad (29)$$

hence

$$\frac{\dot{V}(t)}{\sqrt{V(t)}} \leq -\frac{2k_2(1-\alpha^2)}{\alpha} \frac{\sqrt{V_2(t)}}{\sqrt{1+V_2(t)}} + 2\|J\|^{-\frac{1}{2}}\|w(t)\|_1. \quad (30)$$

Let $\xi(t) = \sqrt{V(t)}$, $\zeta(t) = \sqrt{V_2(t)}$, and note that $\dot{\xi}(t) = \frac{1}{2}\frac{\dot{V}(t)}{\sqrt{V(t)}}$, thus equation (30) can be rewritten as:

$$\dot{\xi}(t) \leq -\frac{k_2(1-\alpha^2)}{\alpha} \frac{\zeta(t)}{\sqrt{1+\zeta(t)^2}} + \|J\|^{-\frac{1}{2}}\|w(t)\|_1. \quad (31)$$

Consider now the auxiliary system

$$\dot{\eta}(t) = -\frac{k_2(1-\alpha^2)}{\alpha} \frac{\zeta(t)}{\sqrt{1+\zeta(t)^2}} + \|J\|^{-\frac{1}{2}}\|w(t)\|_1 \quad (32)$$

According to the Comparison Principle [26], if $\eta(0) \geq \xi(0)$, then $\eta(t) \geq \xi(t) \geq 0$ for all $t \geq 0$. Integrating both sides of equation (32) in the time intervals $[0,\ T], [T,\ 2T], \cdots, [(K-1)T,\ T]$ yields

$$\begin{aligned}
\eta(T) - \eta(0) &= -\frac{k_2(1-\alpha^2)}{\alpha} \int_0^T \frac{\zeta(t)}{\sqrt{1+\zeta(t)^2}} dt \\
&\quad + \|J\|^{-\frac{1}{2}} \int_0^T \|w\|_1 dt \\
&\leq -\frac{k_2(1-\alpha^2)}{\alpha} \frac{\underline{\zeta}_1}{\sqrt{1+\overline{\zeta}_1^2}} T \\
&\quad + \|J\|^{-\frac{1}{2}}(1-\varepsilon)T \\
\eta(2T) - \eta(T) &\leq -\frac{k_2(1-\alpha^2)}{\alpha} \frac{\underline{\zeta}_2}{\sqrt{1+\overline{\zeta}_2^2}} T \\
&\quad + \|J\|^{-\frac{1}{2}}(1-\varepsilon)T \\
&\vdots \\
\eta(KT) - \eta((K-1)T) &\leq -\frac{k_2(1-\alpha^2)}{\alpha} \frac{\underline{\zeta}_K}{\sqrt{1+\overline{\zeta}_K^2}} T \\
&\quad + \|J\|^{-\frac{1}{2}}(1-\varepsilon)T,
\end{aligned} \quad (33)$$

where $\underline{\eta}_i = \min_{t\in((i-1)T,iT]}(\eta_i(t))$, $\overline{\eta}_i = \max_{t\in((i-1)T,iT]}(\eta_i(t))$, $\underline{\zeta}_i = \min_{t\in((i-1)T,iT]}(\zeta_i(t))$, $\overline{\zeta}_i = \max_{t\in((i-1)T,iT]}(\zeta_i(t))$, $i = 1, \cdots, K$. Adding the left and right hand sides of the inequalities (33) yields

$$0 \leq \eta(KT) \leq \eta(0) - \frac{k_2(1-\alpha^2)}{\alpha} \frac{K\underline{\zeta}}{\sqrt{1+\overline{\zeta}^2}} T + \|J\|^{-\frac{1}{2}}(1-\varepsilon)KT, \quad (34)$$

in which $\underline{\zeta} = \min(\underline{\zeta}_i)$ and $\overline{\zeta} = \max(\bar{\zeta}_i)$, $i = 1, \cdots, K$. According to the definition of ζ one has

$$\dot{\zeta}(t) = \tfrac{1}{2}V_2(t)^{-\frac{1}{2}}\dot{V}_2(t) = \tfrac{1}{2}\zeta^{-1}(t)\dot{V}_2(t). \quad (35)$$

Suppose now that $\zeta(t^*) = \underline{\zeta}$ at $t = t^*$. Then by integrating both side of (35) from t^* to $t^* + t$ yields

$$\bar{\zeta} = \underline{\zeta} + \frac{V(t+t^*) - V(t^*)}{2} \int_{t^*}^{t^*+t} \zeta(\tau)^{-1} d\tau,\ t \in [0,\ KT]. \quad (36)$$

Define now

$$\psi(\zeta^*, \varepsilon) = \varphi(\zeta^*) - \phi(\varepsilon), \quad (37)$$

where $\varphi(\zeta^*) = \frac{k_2(1-\alpha^2)}{\alpha} \frac{\zeta}{\sqrt{1+\overline{\zeta}^2}}$ and $\phi(\varepsilon) = \|J\|^{-\frac{1}{2}}(1-\varepsilon)$.

Consider now a fixed T and let $K \to +\infty$. Then equation (34) can be rewritten as

$$0 \leq \eta(KT) \leq \eta(0) - \lim_{K\to+\infty} KT\psi(\zeta^*, \varepsilon). \quad (38)$$

Consider now the case in which $\psi(\zeta^*, \varepsilon) \leq 0$ as $K \to +\infty$. Then, by the definition of $\psi(\zeta^*, \varepsilon)$, one has

$$0 \ \leq \ \underline{\zeta} \ \leq \ \frac{\alpha\|J\|^{-\frac{1}{2}}(1-\varepsilon)\sqrt{1+\bar{\zeta}^2}}{k_2(1-\alpha^2)},$$

hence

$$\lim_{K\to+\infty} \eta(KT) \leq \eta(0) - \lim_{K\to+\infty} KT\psi(\zeta^*, \varepsilon) \to +\infty. \quad (39)$$

Then according to the Comparison Principle [26], for any $\eta(0) \geq \xi(0)$ one can obtain

$$+\infty \geq \eta(t) \geq \xi(t) = \sqrt{V(t)} = \sqrt{V_1(t) + V_2(t)},$$

which is a contradiction.

Note now that V_1 is bounded by definition, and $\zeta(t) = \sqrt{V_2(t)} = \sqrt{\omega^{\mathrm{T}} J \omega}$ cannot increase for all $t \in [0, KT]$, since there always exists a time instant t^{**} such that $\mathrm{sat}(J\omega(t^{**})) = 1$. For all t such that $|J\omega(t)| \geq 1$, the equation (31) can be rewritten as

$$\dot{\xi}(t) \leq -\frac{k_2(1-\alpha^2)}{\alpha}\|J\|^{-\frac{1}{2}} + \|J\|^{-\frac{1}{2}}\|w(t)\|_1. \quad (40)$$

Integrating both sides of (40) in the time interval $[t^{**},\ t^{**} + T]$ yields

$$\begin{aligned}
\xi(t^{**} + T) - \xi(t^{**}) &\leq -\frac{k_2(1-\alpha^2)}{\alpha}\|J\|^{-\frac{1}{2}}T + \|J\|^{-\frac{1}{2}}(1-\varepsilon)T \\
&\leq \frac{-k_2(1-\alpha^2) + \alpha}{\alpha}\|J\|^{-\frac{1}{2}}T - \varepsilon^* T,
\end{aligned} \quad (41)$$

where $\varepsilon^* = \|J\|^{-\frac{1}{2}}\varepsilon$, since k_2 is such that

$$k_2 \ \geq \ \frac{\alpha}{1-\alpha^2} \ > \ 0 \quad (42)$$

for some $\alpha \in (0,1)$. Then

$$\sqrt{V(t^{**} + T)} - \sqrt{V(t^{**})} = \xi(t^{**}) - \xi(t^{**}) < -\varepsilon^* T,$$

that is

$$\sqrt{V(t^{**} + T)} \leq \sqrt{V(t^{**})} - \varepsilon^* T, \quad (43)$$

which implies that the condition $\psi(\zeta^*, \varepsilon) \leq 0$ cannot hold for $K \to +\infty$. Hence $\xi(t)$ decrease after reaching its peak value, i.e. $\lim_{t\to+\infty}\xi(t)<+\infty$. Thus, $\psi(\zeta^*,\varepsilon)>0$, $K\to+\infty$.

Suppose now that $\psi(\zeta^*,\varepsilon)>0$, as $K\to+\infty$. Then

$$0\le\eta(KT)\le\eta(0)-\lim_{K\to+\infty}KT\psi(\zeta^*,\varepsilon)<\eta(0), \tag{44}$$

which implies that η decreases and enters a neighbourhood of the origin. Moreover

$$0<\lim_{K\to+\infty}KT\psi(\zeta^*,\varepsilon)<\eta(0) \tag{45}$$

and

$$0<\psi(\zeta^*,\varepsilon)\to\frac{\eta(0)}{KT}\to 0,\ K\to+\infty. \tag{46}$$

Then for $K\to+\infty$ there exists a small scalar $\bar{\varepsilon}>0$ such that $|\psi(\zeta^*,\varepsilon)|<\bar{\varepsilon}$. Hence, the lower bound $\underline{\zeta}$ for ζ is such that

$$\left|\underline{\zeta}-\frac{\alpha\|J\|^{-\frac{1}{2}}(1-\varepsilon)\sqrt{1+\bar{\zeta}^2}}{k_2(1-\alpha^2)}\right|<\varepsilon^{**} \tag{47}$$

with $\varepsilon\in(0,1)$ and $\varepsilon^{**}=\frac{\alpha\bar{\varepsilon}\sqrt{1+\bar{\zeta}^2}}{k_2(1-\alpha^2)}$.

We therefore conclude that, for any initial condition the states of the closed-loop system (24) with the controller (25) are ultimately bounded, and there exists $\underline{\zeta}>0$ such that $\|\omega(t)\|\le\frac{\alpha\|J\|^{-\frac{1}{2}}(1-\varepsilon)\sqrt{1+\bar{\zeta}^2}}{k_2(1-\alpha^2)}$ for some finite t and

$$\limsup_{t\to+\infty}\|\omega(t)\|=\|J\|^{-1/2}\underline{\zeta}$$

with $\underline{\zeta}\in\left(\frac{\alpha\|J\|^{-\frac{1}{2}}(1-\varepsilon)\sqrt{1+\bar{\zeta}^2}}{k_2(1-\alpha^2)}-\varepsilon^{**},\frac{\alpha\|J\|^{-\frac{1}{2}}(1-\varepsilon)\sqrt{1+\bar{\zeta}^2}}{k_2(1-\alpha^2)}+\varepsilon^{**}\right)$.

□

Remark 3 *On the basis of the analysis in Proposition 3, a similar result can be obtained using bounds on the average p-norms of the disturbance w.*

Remark 4 *Considering the natural characteristics of space environmental torques (mainly generated by the Earth gravity gradient, geomagnetic field, aerodynamic drag and solar radiation pressure), one can assume that all these torques are L_p norm bounded [28], [29]. Thus condition (3) is reasonable.*

V. SIMULATION RESULTS

In this section we illustrate the performance of the considered perturbed closed-loop systems.

To begin with consider the perturbed integrator system (2), with the control law $u=-\mathrm{sat}(Kx)$. Let $\varepsilon=0.1$, $K=1$, and

$$w(t)=\begin{cases}\bar{\alpha}, & nT<t<\lambda T+nT,\\ \bar{\beta}, & \lambda T+nT<t<(n+1)T,\end{cases} \tag{48}$$

with $\bar{\alpha}=\frac{1}{3}(2\tilde{\alpha}\,\mathrm{rand}(\cdot)-\tilde{\alpha})$, $\bar{\beta}=\frac{1}{3}(2\tilde{\beta}\,\mathrm{rand}(\cdot)-\tilde{\beta})$, where $\mathrm{rand}(\cdot)$ gives uniformly distributed random numbers in the set $[0,1]$. By choosing $\tilde{\alpha}=10$,

$$\tilde{\beta}=0.9\frac{1-\varepsilon-\lambda\tilde{\alpha}}{1-\lambda}<\frac{1-\varepsilon-\lambda\tilde{\alpha}}{1-\lambda},$$
$$\lambda=0.9\frac{1-\varepsilon}{\tilde{\alpha}}<\frac{1-\varepsilon}{\tilde{\alpha}},$$

one has

$$\frac{1}{T}\int_t^{t+T}|w|d\tau\le\frac{1}{T}(\lambda\tilde{\alpha}T+(1-\lambda)\tilde{\beta}T)\le 1-\varepsilon<1,$$

which satisfies the condition in (3). Figures 5 and 6 show a realization of the selected disturbance and its averaged $L_{1,a}^T$ norm, for $T=1$. Fig. 7 displays time histories of the state of the system (2) in closed loop with the controller (15) for two values of T.

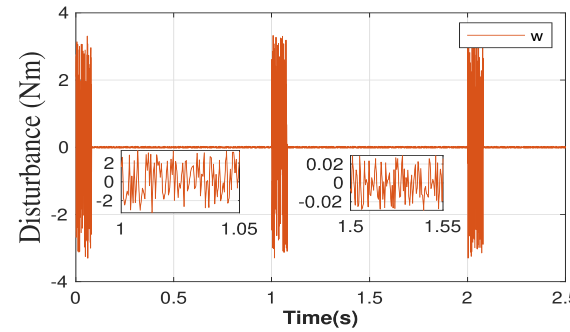

Fig. 5. Realization of the disturbance w, with $T=1$.

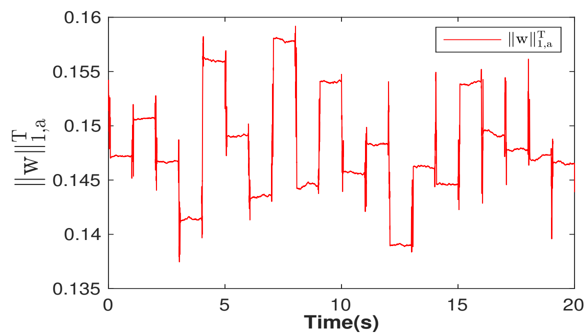

Fig. 6. Time history of $\frac{1}{T}\int_t^{t+T}|w|d\tau$, with $T=1$.

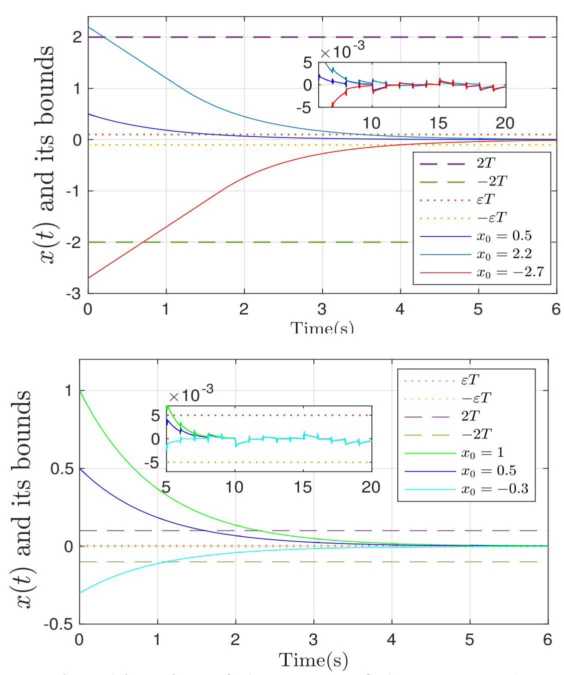

Fig. 7. Time histories of the states of the system (2), with the disturbance w for $T=1$ (top) and $T=0.05$ (bellow), and for various initial conditions.

A. Attitude regulation

Consider now the attitude regulation problem described in Section IV. Let

$$J = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} \mathrm{kg} \cdot \mathrm{m}^2,$$

and

$$w(t) = \begin{cases} \begin{pmatrix} w_{\alpha_1} \\ w_{\alpha_2} \\ w_{\alpha_3} \end{pmatrix}, & nT < t < \lambda T + nT \\ \begin{pmatrix} w_{\beta_1} \\ w_{\beta_2} \\ w_{\beta_3} \end{pmatrix}, & \lambda T + nT < t < (n+1)T \end{cases}$$

with $w_{\alpha_i} = 2\alpha_i \operatorname{rand}(\cdot) - \alpha_i \le |\alpha_i|$, $w_{\beta_j} = 2\beta_j \operatorname{rand}(\cdot) - \beta_j \le |\beta_j|, i, j = 1, 2, 3$.

Note now that

$$\frac{1}{T}\int_t^{t+T} |w| d\tau \le \frac{1}{T}(\lambda \bar{\alpha} T + (1-\lambda)\bar{\beta} T) \le 1 - \varepsilon < 1 \quad (49)$$

with $\varepsilon \in (0,1)$, $\bar{\alpha} = |\alpha_1| + |\alpha_2| + |\alpha_3|$ and $\bar{\beta} = |\beta_1| + |\beta_2| + |\beta_3|$. In what follows we select $\alpha_1 = \alpha_2 = \alpha_3 = \frac{\bar{\alpha}}{3}$, $\beta_1 = \beta_2 = \beta_3 = \frac{\bar{\beta}}{3}$, $\bar{\alpha} = 10\sqrt{3}\mathrm{Nm}$, $T = 1\mathrm{s}$, and

$$\bar{\beta} = 0.9 \cdot \frac{1 - \varepsilon - \lambda \bar{\alpha}}{1 - \lambda} < \frac{1 - \varepsilon - \lambda \bar{\alpha}}{1 - \lambda},$$
$$\lambda = 0.9 \cdot \frac{1 - \varepsilon}{\bar{\alpha}} < \frac{1 - \varepsilon}{\bar{\alpha}}$$

for $\varepsilon = 0.1$. The controller gains k_1 and k_2 are selected such as $\frac{k_1}{2} + k_2 = 1$ with $k_2 \ge \frac{\alpha}{1-\alpha^2}$ and $\alpha \in (0,1)$. The initial conditions are selected as $\omega(0) = [0.05, 0.02, 0.05]^{\mathrm{T}}\mathrm{deg/s}$, $q(0) = [0.83, 0.03, 0.02, 0.02]^{\mathrm{T}}$. Figs. 8 and 9 show a realization of the disturbance and its averaged $L_{1,a}^T$ norm, for $T = 1$.

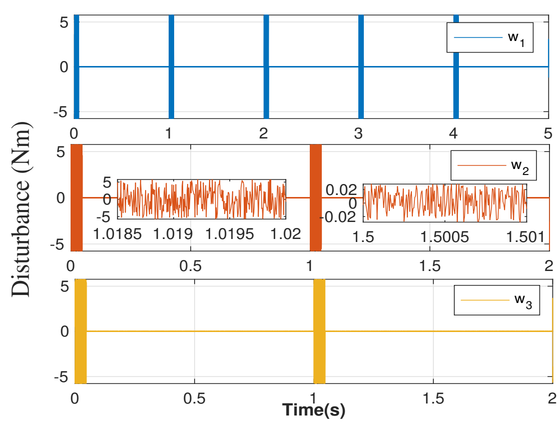

Fig. 8. Realization of the disturbance w for $T = 1\mathrm{s}$.

To assess the performance of the closed loop system let $E_q = \int_0^t (q_v' q_v + (1 - q_0)^2) dt = \int_0^t (1 - q_0) dt$, and note that E_q represents the quaternion energy in the time interval $[0, t)$. Fig. 10 and Fig. 11 display the quaternion energy as a function of the controller parameters.

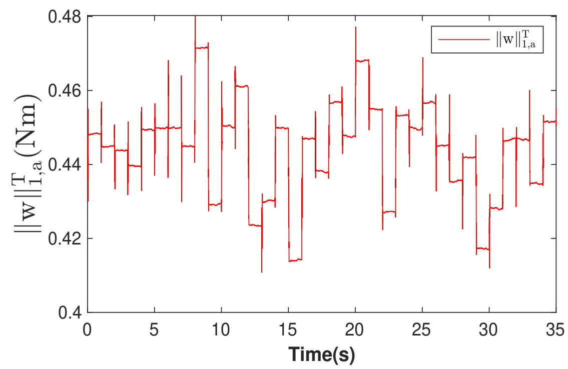

Fig. 9. Time history of $\frac{1}{T}\int_t^{t+T} |w| d\tau$, with $T = 1\mathrm{s}$.

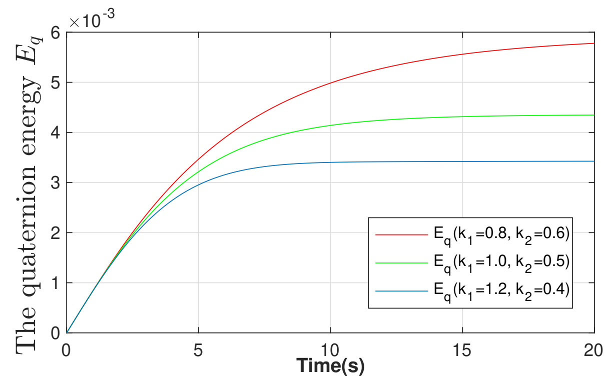

Fig. 10. The quaternion energy for different control parameters and a fixed disturbance.

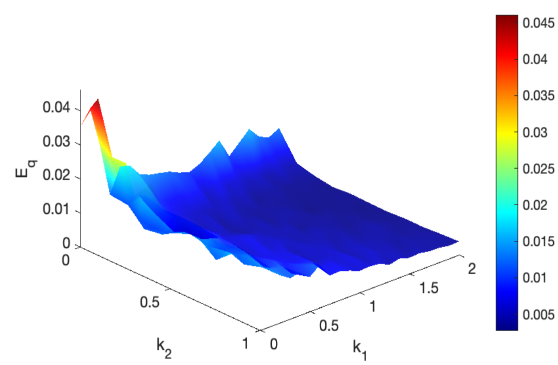

Fig. 11. Mean quaternion energy for different disturbances as a function of k_1 and k_2.

We conclude from Fig. 10 and Fig 11 that a good selection of the control gains is $k_1 = 1.2$ and $k_2 = 0.4$. Moreover, if we select $\alpha = 0.1$ the condition $k_2 \ge \frac{\alpha}{1-\alpha^2}$ is satisfied. For such values of k_i $(i = 1, 2)$ the time histories of the angular velocity w and of the quaternion q are displayed in Figs. 12 and 13. Consistent with the results of Proposition 4, the state of the system is ultimately bounded despite the presence of the persistent disturbance.

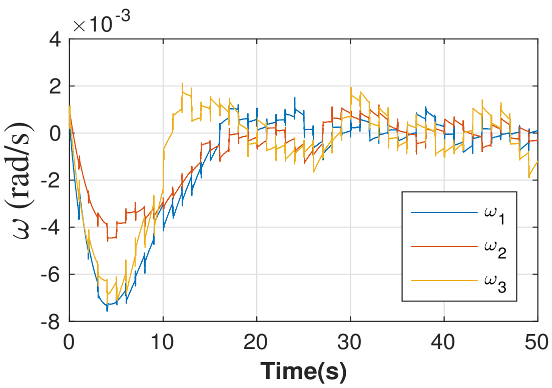

Fig. 12. Time histories of the angular velocity $\boldsymbol{\omega}$.

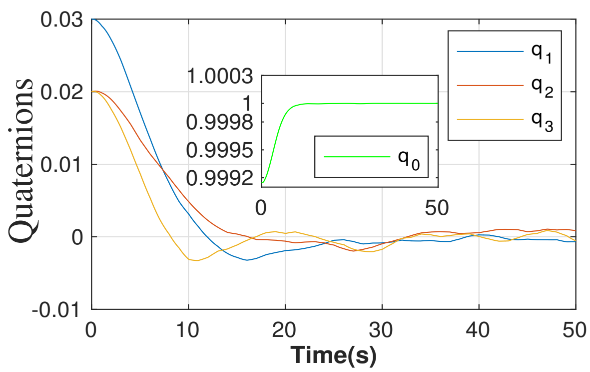

Fig. 13. Time histories of the quaternion q.

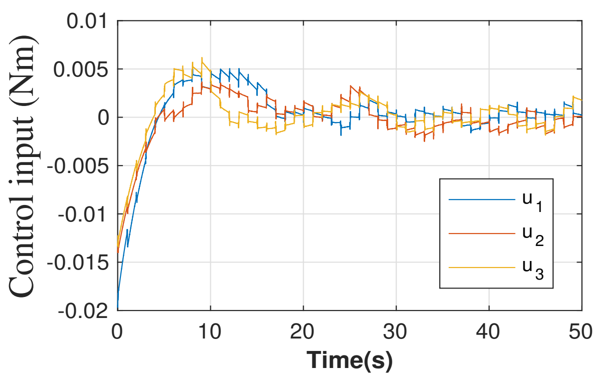

Fig. 14. Time histories of the control input u.

B. COMPASS-1 based simulation

In this section, we demonstrate the performance for the closed-loop system of the CubeSat COMPASS-1 [19]. The COMPASS-1 is designed by students at Aachen University of Applied Sciences in Germany. It has been launched in April 2008 with an orbit inclination and altitude approximately of 98° and 800 km, respectively. Its orbit period is approximate 1.7h. The dimensions of the COMPASS-1 are $10 \times 10 \times 10\ \mathrm{cm}^3$, its weight is less than 1kg and its inertia matrix is [25]

$$J = \begin{pmatrix} 0.00198 & 0 & 0 \\ 0 & 0.0021 & 0 \\ 0 & 0 & 0.00188 \end{pmatrix} \mathrm{kg} \cdot \mathrm{m}^2.$$

According to the data recorded in [19], the worst case disturbance value for low orbit standard CubeSats like COMPASS-1 is 5.79×10^{-7} Nm, including Aerodynamic Drag (1.34×10^{-7} Nm), Solar Pressure (2.62×10^{-7} Nm) and Residual Dipole (4.59×10^{-7} Nm).

Thus, we consider disturbance w to be such that $\|w_{\alpha_i}\| \le 5.79 \times 10^{-7}$Nm. Selecting $\bar{\alpha}$ as in (49), with $\|\bar{\alpha}\| = \sqrt{3}|\alpha_i| = 5.79 \times 10^{-7}$Nm, yields $|w_{\alpha_i}| \le |\alpha_i| = \frac{1}{\sqrt{3}} \times 5.79 \times 10^{-7}$Nm and $|\bar{\alpha}| = 3|\alpha_i| = \sqrt{3} \times 5.79 \times 10^{-7}$Nm. Then

$$\lambda = 0.9 \cdot \frac{(1-\varepsilon) \times 10^{-7}}{\bar{\alpha}} < \frac{(1-\varepsilon) \times 10^{-7}}{\bar{\alpha}},$$
$$\bar{\beta} = 0.9 \cdot \frac{(1-\varepsilon) \times 10^{-7} - \lambda \bar{\alpha}}{1-\lambda} < \frac{(1-\varepsilon) \times 10^{-7} - \lambda \bar{\alpha}}{1-\lambda}$$

for $\varepsilon = 0.1$. As a result, the condition in (3) gives

$$\begin{aligned} \frac{1}{T}\int_t^{t+T} |w| d\tau &\le \frac{1}{T}(\lambda \bar{\alpha} T + (1-\lambda)\bar{\beta} T) \\ &= (1-\varepsilon) \times 10^{-7} \\ &= 0.9 \times 10^{-7}. \end{aligned}$$

For CubeSats like COMPASS-1, the input control which can be provided is limited. Note that, the inertia matrix J of COMPASS-1 is such that $\|J\| = 0.0021$. Selecting $k_1 = 1.2 \times 10^{-5}$ and $k_2 = 0.4 \times 10^{-5}$, as shown in Fig. 15 and Fig. 17, the magnitude of the control torque u is comparable, yet smaller, than the magnitude of the worst-case disturbance. Since the inertia matrix is small, the response is also very slow as is demonstrated in Fig. 18, but approximate attitude regulations are achieved despite the fact that the control torque is smaller in amplitude than the worst-case disturbance. In addition, we can conclude from Fig. 20 that for smaller control amplitude, as shown in Fig. 19, where $k_1 = 1.2 \times 10^{-7}$ and $k_2 = 0.4 \times 10^{-7}$, a reduced precision attitude regulation can be maintained over a large period, which means that the input torque can be reduced for a short period, if only low precision is needed, for energy-saving purpose. Moreover, Figs. 21 and 22 show the time histories of the control input and of the attitude regulation performance for $k_1 = 1.2 \times 10^{-3}$ and $k_2 = 0.4 \times 10^{-3}$. With a larger control amplitude, a faster attitude regulation is achieved.

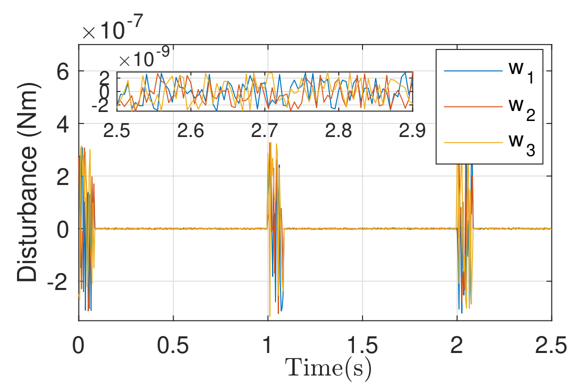

Fig. 15. Realization of the disturbance w, with $T = 1$s.

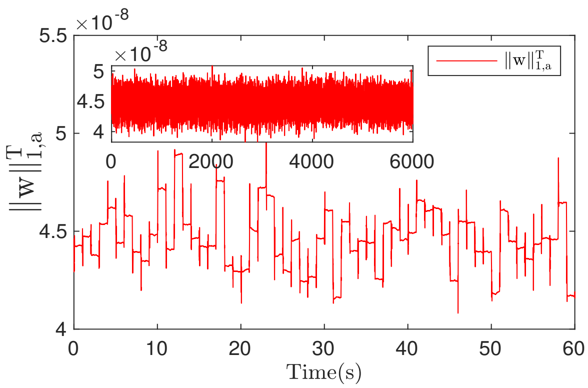

Fig. 16. Time history of $\frac{1}{T}\int_t^{t+T}|w|d\tau$.

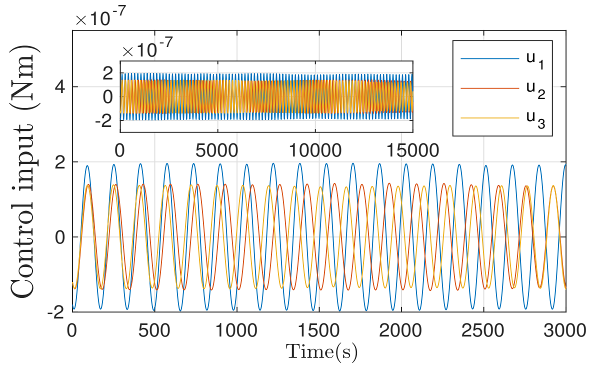

Fig. 17. Time histories of the control input u, with $k_1 = 1.2\times 10^{-5}$ and $k_2 = 0.4\times 10^{-5}$.

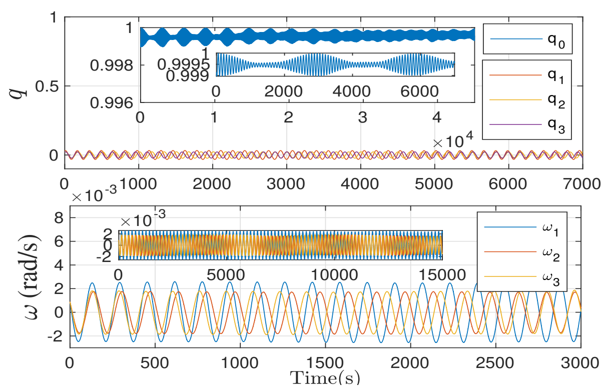

Fig. 18. Time histories of the quaternion q and of the angular velocity ω.

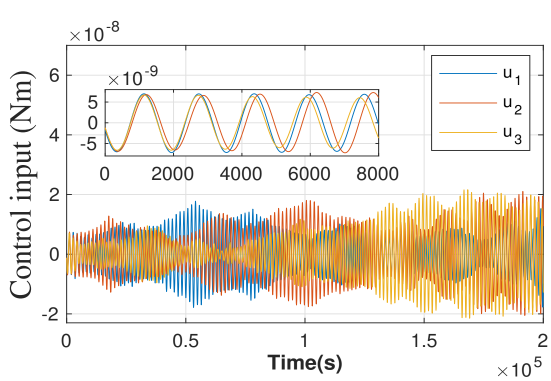

Fig. 19. Time histories of the control input u, with $k_1 = 1.2\times 10^{-7}$ and $k_2 = 0.4\times 10^{-7}$.

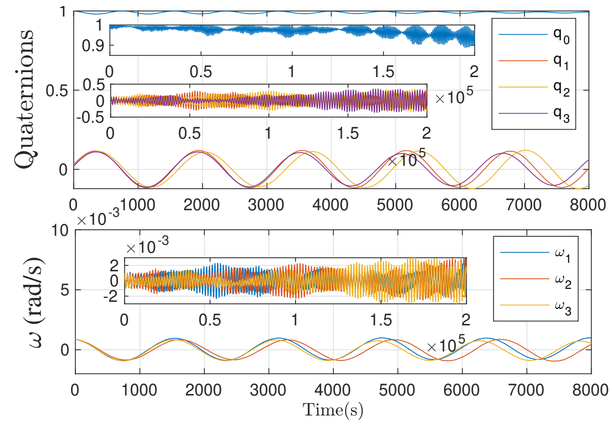

Fig. 20. Time histories of quaternion q and of the angular velocity ω.

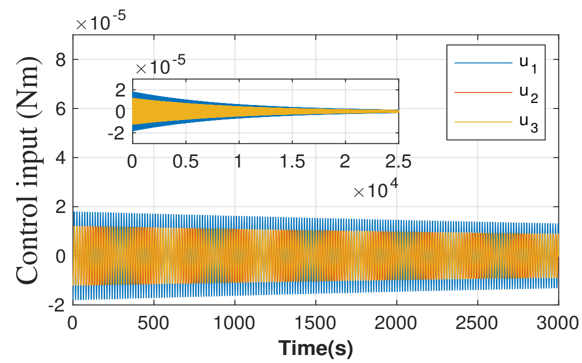

Fig. 21. Time histories of the control input u, with $k_1 = 1.2\times 10^{-3}$ and $k_2 = 0.4\times 10^{-3}$.

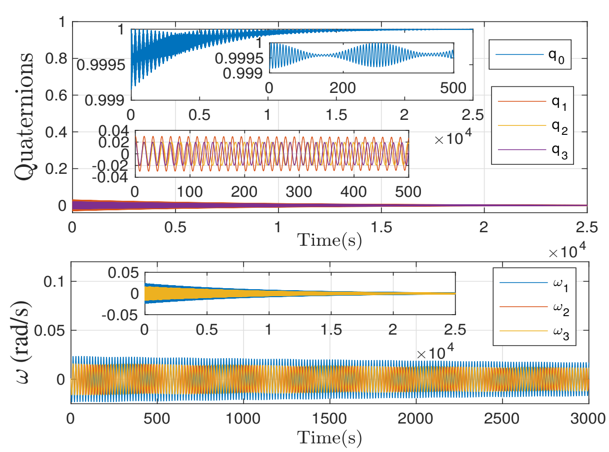

Fig. 22. Time histories of the quaternion q and of the angular velocity ω.

VI. Conclusion

This paper has studied the attitude regulation problem for a class of satellites with bounded control input in the presence of persistent disturbances with bounded windowed norms. A Lyapunov-like analysis, which is firstly introduced in the case in which the underlying system is a disturbed integrator and then extended to the satellite attitude regulation problem, is developed. A detailed analysis of the performance of the resulting closed-loop systems is given and it is shown that the trajectories are ultimately bounded even in the presence of a persistent disturbance. Simulation results on the model of a small satellite subject to large, but bounded in moving average, disturbances are presented.

Future work will extend the proposed analysis tools to a more general classes system.

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