Enhanced dissipation and Taylor dispersion in higher-dimensional parallel shear flows

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ABSTRACT. We consider the evolution of a passive scalar advected by a parallel shear flow in an infinite cylinder with bounded cross section, in arbitrary space dimension. The essential parameters of the problem are the molecular diffusivity ν, which is assumed to be small, and the wave number k in the streamwise direction, which can take arbitrary values. Under generic assumptions on the shear velocity v, we obtain optimal decay estimates for large times, both in the enhanced dissipation regime $\nu \ll |k|$ and in the Taylor dispersion regime $|k| \ll \nu$. Our results can be deduced from resolvent estimates using a quantitative version of the Gearhart-Prüss theorem, or can be established more directly via the hypocoercivity method. Both approaches are implemented in the present example, and their relative efficiency is compared.

1. Introduction

The evolution of a passive scalar advected by a parallel shear flow and undergoing molecular diffusion is an idealized problem which plays an important role in hydrodynamic stability theory. This is perhaps the simplest model demonstrating how advection by an incompressible flow, which has no dissipative effect by itself, can strengthen the action of diffusion and lead to energy dissipation at a much faster rate. The relative importance of advection and diffusion is measured by the Péclet number, which is inversely proportional to the molecular diffusion coefficient ν. We are interested in the regime of large Péclet numbers, where two different phenomena can occur depending on the streamwise wavenumber k. If $\mathrm{Pe}^{-1} \ll |k|L$, where L is a characteristic length of the domain, the lifetime of the Fourier mode with wavenumber k is not proportional to Pe, as in usual diffusion, but typically to $\mathrm{Pe}^{1/3}$ or $\mathrm{Pe}^{1/2}$ depending on the shear velocity. This phenomenon is usually called *accelerated diffusion* or *enhanced dissipation* [3, 11, 34]. In contrast the Fourier modes corresponding to $|k|L \ll \mathrm{Pe}^{-1} \ll 1$ evolve diffusively, with an effective diffusion coefficient that is proportional to Pe and therefore inversely proportional to the molecular viscosity ν. This effect, which is only observed in very long or infinite cylinders, is called *Taylor dispersion* or sometimes *Taylor-Aris dispersion* [2, 35, 36, 40].

From a mathematical point of view, numerous results describing the enhancement of diffusion due to advection by a divergence-free vector field were obtained both in the deterministic and in the stochastic setting, see [6, 13, 14, 16, 17, 20, 21, 30, 33] and the references therein. For the specific case of a parallel flow in a two-dimensional strip, the enhanced dissipation effect for a passive scalar was thoroughly studied in [1, 7, 38], and the estimates derived on that model also play a crucial role in the stability analysis of the shear flow as a stationary solution of the Navier-Stokes equations, see [5, 10, 18, 23, 28, 39]. The corresponding problem in higher-dimensional cylinders did not attract much attention so far, except in particular examples such as the plane Couette flow [8] and the pipe Poiseuille flow [12]. On the other hand, a rigorous justification of Taylor dispersion using self-similar variables and center manifold theory was achieved in [4], see also [31]. It is worth mentioning that, although the enhanced dissipation and the Taylor dispersion have the

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same physical origin, the mathematical techniques used in [4] and [7] are completely different, and rely on distinct assumptions on the shear velocity.

In the present paper, we reopen the study of a passive scalar advected by a parallel flow with a double goal: we aim at investigating the *higher-dimensional case*, which has received less attention so far, with an approach that covers *in a unified way* the enhanced dissipation and the Taylor dispersion regimes. To state our results, we introduce some notation. Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded domain, and $v : \overline{\Omega} \to \mathbb{R}$ be a smooth function. We consider the evolution of a passive scalar in the infinite cylinder $\Sigma = \mathbb{R} \times \Omega \subset \mathbb{R}^{d+1}$ under the action of the shear velocity $u(x,y) = (v(y), 0)^T$. The density $f(x,y,t)$ of the passive scalar satisfies the advection-diffusion equation

$$\partial_t f(x,y,t) + v(y)\partial_x f(x,y,t) \,=\, \nu\Delta f(x,y,t)\,, \quad (x,y) \in \Sigma\,, \quad t > 0\,, \tag{1.1}$$

where $\nu > 0$ is the molecular diffusion coefficient and $\Delta = \partial_x^2 + \Delta_y$ denotes the Laplace operator acting on all variables $(x,y) \in \Sigma$. We supplement (1.1) with homogeneous Neumann conditions at the boundary $\partial\Sigma = \mathbb{R} \times \partial\Omega$. Applying a Galilean transformation if needed, we can assume without loss of generality that the shear velocity v has zero average over Ω. If L denotes the diameter of Ω and U is the maximum of $|v|$ on $\overline{\Omega}$, the Péclet number is defined as

$$\mathrm{Pe} \,=\, \frac{UL}{\nu}\,.$$

We are interested in the long-time behavior of the solutions of (1.1) in the regime where $\mathrm{Pe} \gg 1$. It is convenient to introduce dimensionless variables defined by

$$\tilde{x} \,=\, \frac{x}{L}\,, \qquad \tilde{y} \,=\, \frac{y}{L}\,, \qquad \tilde{t} \,=\, \frac{Ut}{L}\,, \qquad \tilde{v} \,=\, \frac{v}{U}\,.$$

Dropping all tildes for notational simplicity, we arrive at the same equation (1.1) where $L = U = 1$ and $\nu = \mathrm{Pe}^{-1}$ is now a dimensionless parameter.

Since equation (1.1) is invariant under translations in the horizontal direction, it is useful to consider the (partial) Fourier transform formally defined by

$$\hat{f}(k,y,t) \,=\, \int_{\mathbb{R}} f(x,y,t)\,\mathrm{e}^{-ikx}\,\mathrm{d}x\,, \quad k \in \mathbb{R}\,, \quad y \in \Omega\,, \quad t > 0\,. \tag{1.2}$$

This quantity satisfies the evolution equation

$$\partial_t \hat{f}(k,y,t) + ikv(y)\hat{f}(k,y,t) \,=\, \nu\bigl(-k^2 + \Delta_y\bigr)\hat{f}(k,y,t)\,, \quad y \in \Omega\,, \quad t > 0\,, \tag{1.3}$$

where the horizontal wavenumber $k \in \mathbb{R}$ is now a parameter. The horizontal diffusion $-\nu k^2$ in (1.3) plays only a minor role in the regime we consider, and can be conveniently eliminated by the change of dependent variables

$$\hat{f}(k,y,t) \,=\, \mathrm{e}^{-\nu k^2 t} g(k,y,t)\,, \quad k \in \mathbb{R}\,, \quad y \in \Omega\,, \quad t > 0\,. \tag{1.4}$$

This leads to the “hypoelliptic” evolution equation

$$\partial_t g(k,y,t) + ikv(y)g(k,y,t) \,=\, \nu\Delta_y g(k,y,t)\,, \quad y \in \Omega\,, \quad t > 0\,, \tag{1.5}$$

which is the starting point of our analysis. As already mentioned, we suppose that g satisfies the homogeneous Neumann conditions at the boundary, but our results also hold, with a similar proof, if we assume instead that $g = 0$ on $\partial\Omega$. We also suppose that the horizontal wave number k is nonzero, otherwise (1.5) reduces to the usual heat equation in Ω. It is not difficult to verify that, for all initial data $g_0 \in L^2(\Omega)$, equation (1.5) has a unique global solution $g(k,\cdot) \in C^0([0,+\infty), L^2(\Omega))$ such that $g(k,0) = g_0$. Our goal is to estimate the decay rate of that solution in $L^2(\Omega)$ as $t \to +\infty$.

We first consider the situation where the cross section Ω is one-dimensional. Our first main result can be stated as follows.

THEOREM 1.1. *Assume that $d = 1$, $\Omega = (0,1)$, and that $v : [0,1] \to \mathbb{R}$ is a C^m function, for some $m \in \mathbb{N}^*$, whose derivatives up to order m do not vanish simultaneously:*

$$|v'(y)| + |v''(y)| + \cdots + |v^{(m)}(y)| \,>\, 0\,, \quad \textit{for all } y \in [0,1]\,. \tag{1.6}$$

Then there exist positive constants C_1, C_2 such that, for all $\nu > 0$, all $k \neq 0$, and all initial data $g_0 \in L^2(\Omega)$, the solution of (1.5) *satisfies, for all $t \geq 0$,*

$$\|g(k,t)\|_{L^2(\Omega)} \,\leq\, C_1\, \mathrm{e}^{-C_2\lambda_{\nu,k}t}\,\|g_0\|_{L^2(\Omega)}\,, \quad \textit{where}\ \ \lambda_{\nu,k} \,=\, \begin{cases} \nu^{\frac{m}{m+2}}|k|^{\frac{2}{m+2}} & \textit{if}\ \ 0 < \nu \leq |k|\,, \\ \frac{k^2}{\nu} & \textit{if}\ \ 0 < |k| \leq \nu\,. \end{cases} \tag{1.7}$$

The main novel feature of Theorem 1.1 is to exhibit a decay rate $\lambda_{\nu,k}$ which undergoes a *continuous transition* from the enhanced dissipation regime $\nu \ll |k|$ to the Taylor dispersion regime $|k| \ll \nu$. In previous mathematical works, both situations were studied using different methods, making the comparison more difficult. When $\nu \leq |k|$, the expression (1.7) of $\lambda_{\nu,k}$ is certainly not new: it was obtained in [**7**], up to a logarithmic correction of purely technical origin, and it can also be deduced from the general criteria formulated in [**38**], although this requires some work. This instructive formula shows that the long-time behavior of the solutions of (1.5) is determined, in this regime, by the degree of *degeneracy of the critical points* of the shear function v. In the most common cases, the shear flows under consideration are either monotone ($m = 1$) or have nondegenerate critical points ($m = 2$). Accordingly, the lifetime $1/\lambda_{\nu,k}$ of the Fourier mode indexed by k is proportional to $\nu^{-1/3}$ or $\nu^{-1/2}$ when $\nu \ll 1$, and is therefore much shorter than the diffusive time scale ν^{-1}. In the Taylor dispersion regime $|k| \leq \nu$, the decay rate $\lambda_{\nu,k} = k^2/\nu$ has the same dependence upon the Fourier parameter k as the purely diffusive rate νk^2, but $\lambda_{\nu,k} \gg \nu k^2$ when $\nu \ll 1$. So, in all cases, the expression (1.7) of $\lambda_{\nu,k}$ reveals the strong influence of advection on the solutions of (1.5) when the Péclet number ν^{-1} is sufficiently large.

In the higher-dimensional case $d \geq 2$, the situation is similar and we still expect that the decay rate of the solutions of (1.5) is determined, when $\nu \ll |k|$, by the degree of degeneracy of the critical points of v. This is more difficult to prove, however, because the behavior of a function near its critical points can take more diverse forms in higher dimensions. To limit the complexity, we assume here that v is a *Morse function*, which means that v has only a finite number of critical points in Ω, all of which are nondegenerate. For simplicity, we also suppose that v has no critical point on the boundary $\partial\Omega$, although this additional restriction could be dispensed with. Our second main result is:

THEOREM 1.2. *Assume that $v : \overline{\Omega} \to \mathbb{R}$ is a smooth Morse function with no critical point on the boundary $\partial\Omega$. There exist positive constants C_1, C_2 such that, for all $\nu > 0$, all $k \neq 0$, and all initial data $g_0 \in L^2(\Omega)$, the solution of* (1.5) *satisfies estimate* (1.7) *for all $t \geq 0$, where $m = 1$ if v has no critical point in Ω and $m = 2$ if v has at least one critical point in Ω.*

Many classical examples, such as the plane Couette flow or the Poiseuille flow in a cylindrical pipe, are covered by Theorem 1.2, but of course one can imagine more degenerate situations where v is not a Morse function. In Section 2 below, we formulate a general condition on the level sets of v (Assumption 2.2) which ensures that estimate (1.7) holds for all $\nu > 0$ and all $k \neq 0$. We then prove that our assumption holds for a one-dimensional map satisfying (1.6) and for a Morse function in any dimension, but we also give other examples which do not fall into these categories. One may conjecture that estimate (1.7) holds for some $m \in \mathbb{N}^*$ if $v \in C^m(\overline{\Omega})$ and if, for any critical point $\bar{y} \in \overline{\Omega}$, there exists an integer $n \in \{1, \ldots, m\}$ such that the n-th order differential $\mathrm{d}^n v(\bar{y})$ is nondegenerate, but proving that using our techniques requires nontrivial additional work. In a different direction, we also believe that Theorem 1.2 remains true for Morse-Bott functions, whose critical points can form submanifolds of nonzero dimension, but this interesting question is left for a future work.

At this point, it is important to observe that, although Theorems 1.1 and 1.2 treat the enhanced dissipation and the Taylor dispersion regimes in a unified way, the minimal assumptions on v that are needed to obtain the decay estimate (1.7) are very different in both situations. On the one hand, we expect that the expression of $\lambda_{\nu,k}$ is optimal when $\nu \ll |k|$ if v has indeed a critical point where the n-th order differential vanishes for all $n < m$. In particular, there is no enhanced dissipation effect at all if, for instance, v is constant on a nonempty open subset of Ω. In contrast, the following result shows that estimate (1.7) holds in the Taylor dispersion regime whenever the shear velocity v is not identically constant.

THEOREM 1.3. *If $v : \overline{\Omega} \to \mathbb{R}$ is continuous and not identically constant, there exist positive constants C_1, C_2 such that the decay estimate (1.7) holds in the Taylor dispersion regime $0 < |k| \le \nu$.*

The rest of this paper is organized as follows. In Section 2, we derive accurate resolvent estimates for the linear operator $H_{\nu,k} = -\nu\Delta_y + ikv(y)$, which is (up to a sign) the generator of the evolution defined by (1.5). We impose an abstract condition on the shear velocity which, in the enhanced dissipation regime, implies that all level sets of v are "H^1-thin", in a sense that is made precise in Appendix A. We then check the validity of our assumption in concrete situations, for one-dimensional maps satisfying (1.6) and for Morse functions in all space dimensions. Finally, the semigroup estimate (1.7) is obtained from the resolvent bounds using a quantitative version of the Gearhart-Prüss theorem which was recently obtained in [**38**], see also [**25**]. This concludes the proof of Theorems 1.1–1.3.

In Section 3, we give an alternative proof of Theorem 1.2 using the hypocoercivity method of Villani [**37**], which was already used in [**7**] to establish the enhanced dissipation estimate when $d = 1$. This second approach is, in some sense, more direct and more elementary, since it relies on relatively straightforward energy estimates for the solutions of the evolution equation (1.5) in $H^1(\Omega)$. However, to avoid problematic contributions from the boundary, we now have to impose that $g = 0$ on $\partial\Omega$, or alternatively that $\Omega = \mathbb{T}^d$ (the d-dimensional torus). Moreover, to decrease complexity, we restrict ourselves to the Morse case where $m = 1$ or 2, which means that the shear velocity has a finite number of critical points which are all nondegenerate. This implies in particular that the lowest eigenvalue of the semi-classical Hamiltonian $-\nu\Delta_y + |\nabla v|^2$ in $L^2(\Omega)$ is bounded from below by $C\nu^{\frac{m-1}{m}}$ as $\nu \to 0$, and this information is used to derive the differential inequalities that eventually lead to estimate (1.7), up to a logarithmic correction in the enhanced dissipation regime when $m = 2$. As is shown in [**7**], the restriction $m \le 2$ can be removed at the expense of introducing an energy functional with variable coefficients, which is constructed using a suitable partition of unity. In the Taylor dispersion setting, we also recover Theorem 1.3 under slightly more restrictive assumptions on the velocity profile, see Remark 3.10 below for a precise statement.

The efficiency of the methods implemented in this paper is briefly compared in the final Section 4. In Appendix A, we introduce and study a specific notion of "thinness" for arbitrary subsets of $\mathbb{R}^d$, which is closely related to our Assumption 2.2. Finally, a few technical estimates that are used in the proof of Theorem 1.2 are collected in Appendix B.

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2. Resolvent estimates

This section is devoted to the proof of our main results using a first approach, which relies on spectral theory and resolvent estimates. We recall that $\Omega \subset \mathbb{R}^d$ is a smooth bounded domain, and $v : \overline{\Omega} \to \mathbb{R}$ a smooth function. Given any $\nu > 0$ and any $k \neq 0$, the linear evolution equation (1.5) can we written in the abstract form

$$\partial_t g + H_{\nu,k} g \,=\, 0\,, \qquad \text{where} \qquad H_{\nu,k} \,=\, -\nu\Delta_y + ikv(y)\,. \tag{2.1}$$

We consider $H_{\nu,k}$ as a linear operator in the Hilbert space $X = L^2(\Omega)$ with domain

$$D(H) \,=\, \bigl\{ g \in H^2(\Omega)\,;\, \mathcal{N}\cdot\nabla g = 0 \text{ on } \partial\Omega \bigr\}\,,$$

where $\mathcal{N}$ denotes the outward unit normal on the boundary $\partial\Omega$. Being a bounded perturbation of the Neumann Laplacian $-\nu\Delta_y$ in Ω, the operator $H_{\nu,k}$ has compact resolvent, hence purely discrete spectrum [**29**]. Moreover, for all $g \in D(H)$, we have the identities

$$\mathrm{Re}\,\langle H_{\nu,k} g, g\rangle \,=\, \nu\|\nabla g\|^2 \ge 0\,, \qquad \text{and} \qquad \mathrm{Im}\,\langle H_{\nu,k} g, g\rangle \,=\, k\int_\Omega v(y)|g(y)|^2\,\mathrm{d}y\,, \tag{2.2}$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in X and $\|\cdot\|$ the corresponding norm. This implies that the numerical range of $H_{\nu,k}$ is included in the infinite strip

$$S_k \,=\, \bigl\{z\in\mathbb{C}\,;\,\mathrm{Re}(z)\ge 0\,,\ |\mathrm{Im}(z)|\le |k|\|v\|_{L^\infty(\Omega)}\bigr\}\,.$$

The eigenvalues of $H_{\nu,k}$ form a sequence $(\mu_n)_{n\in\mathbb{N}}$ of complex numbers which satisfy $\mu_n\in S_k$ for all $n\in\mathbb{N}$ and $\mathrm{Re}(\mu_n)\to+\infty$ as $n\to\infty$. Furthermore, if we assume that $k\neq 0$ and v is not identically constant, we deduce from the first relation in (2.2) that $\mathrm{Re}(\mu_n)>0$ for all $n\in\mathbb{N}$, so that the imaginary axis in the complex plane is included in the resolvent set of the operator $H_{\nu,k}$. Our goal is to obtain accurate estimates on the resolvent norm $\|(H_{\nu,k}-z)^{-1}\|$ for all $z\in i\mathbb{R}$. In particular, we need a precise lower bound on the pseudospectral abscissa

$$\Psi(\nu,k)\,:=\,\Bigl(\sup_{z\in i\mathbb{R}}\|(H_{\nu,k}-z)^{-1}\|\Bigr)^{-1}\,, \tag{2.3}$$

as a function of the parameters ν and k. This quantity was introduced and studied in [22] for a related problem. It is easy to verify that $\mathrm{Re}(\mu_n)\ge\Psi(\nu,k)$ for all $n\in\mathbb{N}$. More importantly, a recent result due to Dongyi Wei [38], see also Helffer & Sjöstrand [25], shows that the quantity $\Psi(\nu,k)$ entirely controls the decay rate of the semigroup generated by $-H_{\nu,k}$. Indeed, applying [38, Theorem 1.3], we obtain:

PROPOSITION 2.1. *The operator $-H_{\nu,k}$ is the generator of a strongly continuous semigroup in X which satisfies, for all $g\in X$ and all $t\ge 0$,*

$$\|\mathrm{e}^{-tH_{\nu,k}}g\|\,\le\,\mathrm{e}^{\pi/2}\,\mathrm{e}^{-t\Psi(\nu,k)}\|g\|\,. \tag{2.4}$$

So, to obtain the decay estimate (1.7) with $C_1=\mathrm{e}^{\pi/2}$, all we need is to derive a lower bound of the form $\Psi(\nu,k)\ge C_2\lambda_{\nu,k}$ for some positive constant C_2 that is independent of the parameters ν,k.

To do that, we first exploit the assumption that $k\neq 0$ and write any $z\in i\mathbb{R}$ in the form $z=ik\lambda$ with $\lambda\in\mathbb{R}$, so that

$$H_{\nu,k}-z\,=\,H_{\nu,k,\lambda}\,:=\,-\nu\Delta_y+ik\bigl(v(y)-\lambda\bigr)\,. \tag{2.5}$$

It is apparent from (2.5) that the estimates we can hope for depend on the properties of the level sets

$$E_\lambda\,=\,\bigl\{y\in\Omega\,;\,v(y)=\lambda\bigr\}\,,\qquad \lambda\in\mathbb{R}\,.$$

For instance, if E_λ has nonempty interior for some $\lambda\in\mathbb{R}$, then clearly $H_{\nu,k,\lambda}g=-\nu\Delta_y g$ for any function $g\in D(H)$ that is supported in the interior of E_λ. As is easily verified, this implies that $\Psi(\nu,k)\le C\nu$ for some positive constant C, which means that there is no enhanced dissipation effect in such a case. So, to observe a nontrivial influence of the advection term in (1.5), we must assume at least that all levels sets of the function v are "thin" in an appropriate sense.

To formulate our assumption precisely, we introduce the following notation. We give ourselves a positive integer $m\in\mathbb{N}^*$, which will be related to the maximal degree of degeneracy of the critical points of v, as is explained in the introduction. For any $\lambda\in\mathbb{R}$ and any $\delta>0$, we then define the "thickened level set"

$$E^m_{\lambda,\delta}\,=\,\bigl\{y\in\Omega\,;\,|v(y)-\lambda|<\delta^m\bigr\}\,, \tag{2.6}$$

which is the union of the level sets $E_{\lambda'}$ for all $\lambda'\in(\lambda-\delta^m,\lambda+\delta^m)$. We also consider the δ-neighborhood

$$\mathcal{E}^m_{\lambda,\delta}\,=\,\bigl\{y\in\Omega\,;\,\mathrm{dist}(y,E^m_{\lambda,\delta})<\delta\bigr\}\,, \tag{2.7}$$

where "dist" denotes the Euclidean distance in $\mathbb{R}^d$. Our assumption on the function $v:\Omega\to\mathbb{R}$ is:

ASSUMPTION 2.2. There exist a positive integer $m\in\mathbb{N}^*$ and positive real constants C_0,δ_0 such that, for all $\lambda\in\mathbb{R}$ and all $\delta\in(0,\delta_0]$, the following inequality holds for all $g\in H^1(\Omega)$:

$$\int_{\mathcal{E}^m_{\lambda,\delta}}|g(y)|^2\,\mathrm{d}y\,\le\,\frac{1}{2}\int_\Omega|g(y)|^2\,\mathrm{d}y+C_0\delta^2\int_\Omega|\nabla g(y)|^2\,\mathrm{d}y\,. \tag{2.8}$$

REMARK 2.3. The factor $1/2$ in (2.8) can be replaced by any fixed real number $\kappa \in (0,1)$ without altering the definition, see Lemma A.3 below for a similar statement. To avoid introducing yet another parameter, we stick to Assumption 2.2 as it is stated, but it is useful to keep the general case in mind.

It is not easy to characterize precisely the functions v that satisfy Assumption 2.2, but the following observations can be made. First, we emphasize that inequality (2.8) must hold for *all sufficiently small* $\delta > 0$; having it satisfied for just *one* small $\delta > 0$ is infinitely less restrictive, as is shown in Lemma 2.5 below. Next, if we substitute E_λ for $E^m_{\lambda,\delta}$ in the definition (2.7), Assumption 2.2 exactly means that the level sets E_λ are "H^1-thin" according to the definition given in Appendix A. As is shown there, any Lipschitz graph or any submanifold of nonzero codimension is H^1-thin. As a consequence, if $\mathcal{E}^m_{\lambda,\delta}$ is contained in a neighborhood of size $\mathcal{O}(\delta)$ of a Lipschitz graph or a submanifold of nonzero codimension, then inequality (2.8) holds. In contrast, if $\mathcal{E}^m_{\lambda,\delta}$ contains a ball of radius $R(\delta)$ such that $R(\delta)/\delta \to +\infty$ as $\delta \to 0$, then (2.8) fails. So Assumption 2.2 roughly means that, locally, the set $\mathcal{E}^m_{\lambda,\delta}$ is no thicker than $\mathcal{O}(\delta)$ in some direction.

At this point we can explain the role played by the integer $m \in \mathbb{N}$ in definitions (2.6), (2.7). Assume for instance that $0 \in \Omega$ and that $v(y) = |y|^n$ near the origin, where $n \in \mathbb{N}$ and $n \geq 2$. Then for $\delta > 0$ sufficiently small, the thickened level set $E^m_{0,\delta}$ contains the ball of radius $\delta^{m/n}$ centered at the origin, so that the δ-neighborhood $\mathcal{E}^m_{0,\delta}$ contains the ball of radius $\delta + \delta^{m/n}$. As we just saw, for inequality (2.8) to hold, this radius must be $\mathcal{O}(\delta)$ as $\delta \to 0$, which is the case if $m \geq n$. So a necessary condition for Assumption 2.2 to hold is that the integer m be chosen sufficiently large, depending on the degree of degeneracy of the critical points of the function v. For instance, we can take $m = 1$ if v has no critical points, and $m = 2$ if all critical points are nondegenerate, i.e. if v is a Morse function.

The main result of this section is:

PROPOSITION 2.4. *Assume that the shear velocity $v : \Omega \to \mathbb{R}$ satisfies Assumption 2.2 for some positive $m \in \mathbb{N}^*$. Then there exists a constant $C > 0$ such that, for all $\nu > 0$ and all $k \neq 0$,*

$$\Psi(\nu, k) \geq C\, \lambda_{\nu,k}\,, \tag{2.9}$$

where $\Psi(\nu,k)$ is defined in (2.3) and $\lambda_{\nu,k}$ in (1.7).

PROOF. Fix $\nu > 0$, $k \neq 0$, $\lambda \in \mathbb{R}$, and $\delta \in (0, \delta_0)$, where $\delta_0 > 0$ is as in Assumption 2.2. For simplicity we denote $H = H_{\nu,k,\lambda}$, where $H_{\nu,k,\lambda}$ is defined in (2.5). We introduce the localization function

$$\chi(y) = \phi\Big(\frac{1}{\delta}\,\mathrm{sign}\big(v(y) - \lambda\big)\,\mathrm{dist}\big(y, E^m_{\lambda,\delta}\big)\Big)\,, \qquad y \in \Omega\,,$$

where $\phi : \mathbb{R} \to [-1,1]$ is the unique odd function such that $\phi(t) = \min(t,1)$ for $t \geq 0$. We have the following three properties:

i) χ is locally Lipschitz in Ω with $\|\nabla\chi\|_{L^\infty} \leq 1/\delta$;
ii) $\chi(y)\big(v(y) - \lambda\big) \geq 0$ for all $y \in \Omega$;
iii) $\mathcal{E}^m_{\lambda,\delta} = \{y \in \Omega\,;\, |\chi(y)| < 1\}$.

These properties mean that χ is a Lipschitz regularization of the discontinuous function $\mathrm{sign}(v - \lambda)$, such that the transitions between the values -1 and $+1$ occur within the region $\mathcal{E}^m_{\lambda,\delta}$. To prove the Lipschitz continuity, we observe that $E^m_{\lambda,\delta}$ is an open neighborhood of the level set E_λ, so that $\mathrm{dist}(y, E^m_{\lambda,\delta})$ vanishes near the points where $\mathrm{sign}(v(y){-}\lambda)$ is discontinuous. The remaining properties are obvious by construction.

For any $g \in D(H)$ we have, as in (2.2),

$$\mathrm{Re}\langle Hg, g\rangle = \nu\|\nabla g\|^2\,, \qquad \text{hence} \qquad \nu\|\nabla g\|^2 \leq \|Hg\|\,\|g\|\,. \tag{2.10}$$

Moreover, a direct calculation shows that $\mathrm{Im}\langle Hg, \chi g\rangle = \nu\,\mathrm{Im}\langle \nabla g, g\nabla\chi\rangle + k\langle \chi(v-\lambda)g, g\rangle$. Since $|\chi| \leq 1$ and $|\nabla\chi| \leq 1/\delta$ we deduce that

$$|k|\langle \chi(v-\lambda)g, g\rangle \leq \|Hg\|\,\|g\| + \frac{\nu}{\delta}\,\|\nabla g\|\,\|g\| \leq \|Hg\|\,\|g\| + \frac{\nu^{1/2}}{\delta}\,\|Hg\|^{1/2}\,\|g\|^{3/2}\,, \tag{2.11}$$

where the second inequality follows from (2.10). We now decompose

$$\|g\|^2 \,=\, \int_{\Omega\setminus\mathcal{E}} |g(y)|^2\,\mathrm{d}y \,+\, \int_{\mathcal{E}} |g(y)|^2\,\mathrm{d}y\,, \qquad \text{where} \quad \mathcal{E} \,=\, \mathcal{E}^m_{\lambda,\delta}\,, \tag{2.12}$$

and we estimates both terms separately using (2.10), (2.11).

1. If $y \in \Omega\setminus\mathcal{E}$, then $y \notin E^m_{\lambda,\delta}$, hence $|v(y)-\lambda| \ge \delta^m$ by definition. In view of properties ii), iii) above, we even have $\chi(y)\big(v(y)-\lambda\big) \ge \delta^m$, so that

$$\begin{aligned}
\int_{\Omega\setminus\mathcal{E}} |g(y)|^2\,\mathrm{d}y \,&\le\, \frac{1}{\delta^m}\int_{\Omega\setminus\mathcal{E}} \chi(y)\big(v(y)-\lambda\big)|g(y)|^2\,\mathrm{d}y \,\le\, \frac{1}{|k|\delta^m}\,|k|\langle\chi(v-\lambda)g,g\rangle \\
&\le\, \frac{1}{|k|\delta^m}\Big(\|Hg\|\,\|g\| + \frac{\nu^{1/2}}{\delta}\,\|Hg\|^{1/2}\,\|g\|^{3/2}\Big) \\
&\le\, \Big(\frac{1}{|k|\delta^m} + \frac{\nu}{k^2\delta^{2m+2}}\Big)\|Hg\|\,\|g\| + \frac{1}{4}\,\|g\|^2\,,
\end{aligned} \tag{2.13}$$

where in the second line we used (2.11) and in the third line Young's inequality.

2. Since $\mathcal{E} = \mathcal{E}^m_{\lambda,\delta}$, inequality (2.8) gives

$$\int_{\mathcal{E}} |g(y)|^2\,\mathrm{d}y \,\le\, \frac{1}{2}\,\|g\|^2 + C_0\delta^2\|\nabla g\|^2 \,\le\, \frac{1}{2}\,\|g\|^2 + \frac{C_0\delta^2}{\nu}\,\|Hg\|\,\|g\|\,. \tag{2.14}$$

Combining (2.12)–(2.14), we arrive at

$$\frac{1}{4}\,\|g\| \,\le\, \Big(\frac{1}{|k|\delta^m} + \frac{\nu}{k^2\delta^{2m+2}} + \frac{C_0\delta^2}{\nu}\Big)\,\|Hg\|\,. \tag{2.15}$$

We now choose

$$\delta \,=\, \delta_0\Big(\frac{\nu}{|k|}\Big)^{\frac{1}{m+2}} \quad \text{if} \;\; \nu \le |k|\,, \qquad \text{and} \qquad \delta \,=\, \delta_0 \quad \text{if} \;\; \nu \ge |k|\,. \tag{2.16}$$

Then (2.15) shows that

$$\|Hg\| \,\ge\, C\begin{cases} \nu^{\frac{m}{m+2}}|k|^{\frac{2}{m+2}}\|g\| & \text{if} \quad 0 < \nu \le |k|\,, \\ \frac{k^2}{\nu}\|g\| & \text{if} \quad 0 < |k| \le \nu\,, \end{cases}$$

where the constant C depends only on C_0, δ_0, and m. In other words, we have $\|Hg\| \ge C\lambda_{\nu,k}\|g\|$ for all $g \in D(H)$, where $\lambda_{\nu,k}$ is as in (1.7). Since $H = H_{\nu,k,\lambda}$ and

$$\Psi(\nu,k) \,=\, \inf\Big\{\|H_{\nu,k,\lambda}g\|\,;\, \lambda\in\mathbb{R}\,,\; g\in D(H)\,,\; \|g\| = 1\Big\}\,,$$

we obtain the desired inequality (2.9). □

It is clear that the decay estimate (1.7) follows immediately from inequalities (2.4) and (2.9). So, to prove Theorems 1.1–1.3, what remains to be done is verifying the validity of Assumption 2.2. We first investigate under which conditions a continuous function v satisfies inequality (2.8) for some *fixed* $\delta > 0$.

LEMMA 2.5. *If $v : \overline{\Omega} \to \mathbb{R}$ is continuous and not identically constant, then for any sufficiently small $\delta > 0$ there exist constants $C > 0$ and $\kappa \in (0,1)$ such that, for all $\lambda \in \mathbb{R}$ and all $g \in H^1(\Omega)$,*

$$\int_{\mathcal{E}^1_{\lambda,\delta}} |g(y)|^2\,\mathrm{d}y \,\le\, \kappa\int_\Omega |g(y)|^2\,\mathrm{d}y + C\int_\Omega |\nabla g(y)|^2\,\mathrm{d}y\,. \tag{2.17}$$

PROOF. Since v is not constant, we can pick $y_1, y_2 \in \Omega$ such that $v(y_2) > v(y_1)$. We define

$$\lambda_0 \,=\, \frac{v(y_2)+v(y_1)}{2}\,, \qquad \gamma \,=\, \frac{v(y_2)-v(y_1)}{6}\,,$$

so that $v(y_1) = \lambda_0 - 3\gamma$ and $v(y_2) = \lambda_0 + 3\gamma$. We next choose $\delta > 0$ small enough so that

i) $\delta < \gamma$;

ii) $B(y_j, \delta) \subset \Omega$ for $j = 1, 2$, where $B(y_j, \delta)$ is the open ball of radius δ centered at y_j;

iii) for all $y, \tilde{y} \in \Omega$ with $|y - \tilde{y}| < \delta$, one has $|v(y) - v(\tilde{y})| < \gamma$.

Property iii) holds because v is continuous on the compact set $\overline{\Omega}$, hence uniformly continuous in Ω.

Given any $\lambda \in \mathbb{R}$, we claim that

$$B(y_2,\delta) \,\subset\, \Omega \setminus \mathcal{E}^1_{\lambda,\delta} \quad \text{if } \lambda \le \lambda_0\,, \qquad \text{and} \qquad B(y_1,\delta) \,\subset\, \Omega \setminus \mathcal{E}^1_{\lambda,\delta} \quad \text{if } \lambda \ge \lambda_0\,. \tag{2.18}$$

Let us prove the first assertion, the second one being similar. Assume thus that $\lambda \le \lambda_0$. By definition, if $y \in \mathcal{E}^1_{\lambda,\delta}$, there exists $\tilde{y} \in E^1_{\lambda,\delta}$ such that $|y - \tilde{y}| < \delta$, hence by iii) and i) above

$$v(y) \,<\, v(\tilde{y}) + \gamma \,<\, \lambda + \delta + \gamma \,<\, \lambda_0 + 2\gamma\,.$$

On the other hand, for any $y \in B(y_2, \delta)$, one has $v(y) > v(y_2) - \gamma = \lambda_0 + 2\gamma$. It follows that $B(y_2, \delta) \cap \mathcal{E}^1_{\lambda,\delta} = \emptyset$, which is the first assertion in (2.18).

For any $\lambda \in \mathbb{R}$, it follows from (2.18) that $|\mathcal{E}^1_{\lambda,\delta}| \le |\Omega| - |B(\delta)|$, where $|B(\delta)|$ is the measure of a ball of radius δ in $\mathbb{R}^d$. We now take $\rho > 0$ small enough so that

$$\kappa \,:=\, \bigl(1+\rho\bigr)\Bigl(1 - \frac{|B(\delta)|}{|\Omega|}\Bigr) \,<\, 1\,, \qquad \text{hence} \quad |\mathcal{E}^1_{\lambda,\delta}| \,\le\, \frac{\kappa\,|\Omega|}{1+\rho}\,. \tag{2.19}$$

If $g \in H^1(\Omega)$, we decompose $g = \langle g\rangle + \tilde{g}$ where $\langle g\rangle$ is the average of g over Ω. Using Young's inequality in the form $|g|^2 = \bigl|\langle g\rangle + \tilde{g}\bigr|^2 \le (1+\rho)|\langle g\rangle|^2 + \bigl(1+1/\rho\bigr)|\tilde{g}|^2$, we obtain

$$\begin{aligned}
\int_{\mathcal{E}^1_{\lambda,\delta}} |g|^2 \,\mathrm{d}y \,&\le\, (1+\rho)\int_{\mathcal{E}^1_{\lambda,\delta}} |\langle g\rangle|^2 \,\mathrm{d}y + \bigl(1+1/\rho\bigr)\int_{\mathcal{E}^1_{\lambda,\delta}} |\tilde{g}|^2 \,\mathrm{d}y \\
&\le\, (1+\rho)\frac{|\mathcal{E}^1_{\lambda,\delta}|}{|\Omega|}\int_\Omega |\langle g\rangle|^2 \,\mathrm{d}y + \bigl(1+1/\rho\bigr)\int_\Omega |\tilde{g}|^2 \,\mathrm{d}y \\
&\le\, \kappa\int_\Omega |g|^2 \,\mathrm{d}y + \bigl(1+1/\rho\bigr)C_W^2\int_\Omega |\nabla g|^2 \,\mathrm{d}y\,,
\end{aligned}$$

where in the last line we used (2.19), the obvious bound $\|\langle g\rangle\| \le \|g\|$, and the Poincaré-Wirtinger inequality $\|\tilde{g}\| \le C_W\|\nabla\tilde{g}\|$. This gives the desired inequality (2.17). □

PROOF OF THEOREM 1.3. In the Taylor dispersion regime where $|k| \le \nu$, the proof of Proposition 2.4 requires Assumption 2.2 only for a fixed value of the parameter δ, see (2.16). So, if $v : \overline{\Omega} \to \mathbb{R}$ is continuous and not identically constant, we can use instead of (2.8) inequality (2.17), which is given by Lemma 2.5. The only (minor) difference is the factor κ in the right-hand side of (2.17), which may be larger than $1/2$, in which case one should modify (2.13) so that the factor $1/4$ in the last member is replaced by $(1-\kappa)/2$. The rest of the proof is unchanged, and gives the desired inequality (2.9) when $|k| \le \nu$. □

We next investigate the validity of inequality (2.8) for *all sufficiently small* $\delta > 0$, which requires much stronger assumptions on the function v. We begin with the one-dimensional case, which is simpler.

2.1. The one-dimensional case. If $d = 1$, we can take $\Omega = (0, L)$, for some $L > 0$. Given a nonzero integer $m \in \mathbb{N}$, we suppose that $v : [0, L] \to \mathbb{R}$ is a function of class C^m whose derivatives up to order m do not vanish simultaneously:

$$|v'(y)| + |v''(y)| + \cdots + |v^{(m)}(y)| \,>\, 0\,, \quad \text{for all } y \in [0, L]\,. \tag{2.20}$$

Our goal is to show that such a function satisfies Assumption 2.2, for the same value of m.

We recall that $H^1(\Omega) \subset C^0(\overline{\Omega})$ and that the following inequality

$$\|g\|_{L^\infty}^2 \;\le\; \frac{1}{L}\,\|g\|_{L^2}^2 + 2\|g\|_{L^2}\|g'\|_{L^2}$$

holds for any $g \in H^1(\Omega)$. If $E \subset \Omega$ is any measurable set, we thus have

$$\int_E |g|^2\,\mathrm{d}y \;\le\; |E|\Big(\frac{1}{L}\,\|g\|_{L^2}^2 + 2\|g\|_{L^2}\|g'\|_{L^2}\Big) \;\le\; \Big(\frac{1}{4} + \frac{|E|}{L}\Big)\|g\|_{L^2}^2 + 4|E|^2\|g'\|_{L^2}^2\,, \tag{2.21}$$

where in the second step we used Young's inequality. If $|E| \le \delta$ for some $\delta \le L/4$, we see that inequality (2.21) is precisely of the form (2.8). This observation reveals that, in the one-dimensional case, it is sufficient to show that $|\mathcal{E}^m_{\lambda,\delta}| = \mathcal{O}(\delta)$ as $\delta \to 0$, uniformly with respect to $\lambda \in \mathbb{R}$.

We first estimate the length of the thickened level sets $E^m_{\lambda,\delta}$ defined in (2.6).

LEMMA 2.6. *If $v \in C^m([0,L])$ satisfies (2.20), there exist positive constants C_0, δ_0 such that, for all $\lambda \in \mathbb{R}$ and all $\delta \in (0,\delta_0)$, one has $|E^m_{\lambda,\delta}| \le C_0\delta$.*

PROOF. Since $E^m_{\lambda,\delta} = \emptyset$ when $\lambda \notin \mathrm{supp}(v)$ and δ is sufficiently small, we need only prove the result for λ in a compact neighborhood of the support of v. Thus, by a finite covering argument, it is sufficient to prove that, for any $\lambda_0 \in \mathbb{R}$, we have the bound $|E^m_{\lambda,\delta}| \le C\delta$ for all $\lambda \in \mathbb{R}$ sufficiently close to λ_0 and for all $\delta > 0$ small enough. From now on, we fix $\lambda_0 \in \mathbb{R}$ and we consider the level set

$$E_{\lambda_0} \;=\; \Big\{y \in [0,L]\,;\, v(y) = \lambda_0\Big\} \;=\; \big\{y_1,\dots,y_N\big\}\,,$$

which is a finite set since, by (2.20), the zeros of the function $y \to v(y) - \lambda_0$ are isolated. If $E_{\lambda_0} = \emptyset$, then $E_\lambda = \emptyset$ when λ is sufficiently close to λ_0, hence also $E^m_{\lambda,\delta} = \emptyset$ when $\delta > 0$ is small enough. So we need only consider the situation where $N = \mathrm{card}(E_{\lambda_0}) \ge 1$. In that case, a standard continuity argument shows that, for any $\epsilon > 0$, there exists $\eta > 0$ such that, if $|\lambda - \lambda_0| < \eta$ and $0 < \delta < \eta$, any point $y \in E^m_{\lambda,\delta}$ satisfies $\mathrm{dist}(y, E_{\lambda_0}) < \epsilon$. In particular, if $\epsilon > 0$ is sufficiently small, the thickened level set $E^m_{\lambda,\delta}$ is included in the union of the pairwise disjoint intervals $(y_j - \epsilon, y_j + \epsilon)$, where $j = 1\dots N$. This means that it is enough to consider the intersection $E^m_{\lambda,\delta} \cap (y_j - \epsilon, y_j + \epsilon)$ for each $j \in \{1,\dots,N\}$, which reduces the problem to the particular case $N = 1$.

In the rest of the proof, we thus assume that $E_{\lambda_0} = \{y_1\}$ for some $y_1 \in [0,L]$. According to (2.20), there exists $n \in \{1,\dots,m\}$ such that $v^{(j)}(y_1) = 0$ for $j = 1,\dots,n-1$ and $v^{(n)}(y_1) \neq 0$. We distinguish two cases according to the parity of n.

Case 1 : n is odd. For definiteness, we suppose that $y_1 \in (0,L)$ and $v^{(n)}(y_1) > 0$ (the other situations can be treated in the same way). We introduce the function $w(x) = v(y_1 + x) - \lambda_0$, which satisfies $w(0) = 0$. By assumption, there exist an open interval $\mathcal{I} \subset \mathbb{R}$ containing the origin and two positive constants κ_1, κ_2 such that

$$\kappa_1 x^{n-1} \;\le\; w'(x) \;\le\; \kappa_2 x^{n-1}\,, \qquad \text{for all } x \in \mathcal{I}\,.$$

In particular, for any λ in a small neighborhood $\mathcal{V}$ of the origin, the equation $w(x) = \lambda$ has exactly one solution x_λ in $\mathcal{I}$, which satisfies $\lambda x_\lambda \ge 0$ and

$$\frac{n|\lambda|}{\kappa_2} \;\le\; |x_\lambda|^n \;\le\; \frac{n|\lambda|}{\kappa_1}\,, \qquad \text{for all } \lambda \in \mathcal{V}\,. \tag{2.22}$$

If $w(x) \ge \lambda$ for some $x \in \mathcal{I}$ and some $\lambda \in \mathcal{V}$, one has

$$w(x) - \lambda \;=\; w(x) - w(x_\lambda) \;=\; \int_{x_\lambda}^x w'(y)\,\mathrm{d}y \;\ge\; \frac{\kappa_1}{n}\big(x^n - x_\lambda^n\big)\,,$$

and a similar estimate holds when $w(x) \le \lambda$. Reducing the neighborhood $\mathcal{V}$ if necessary and choosing a sufficiently small $\delta_0 > 0$, we conclude that, if $\lambda \in \mathcal{V}$ and $0 < \delta < \delta_0$, then

$$x \in \tilde{E}^n_{\lambda,\delta} \;:=\; \big\{x \in \mathcal{I}\,;\, |w(x) - \lambda| \;<\; \delta^n\big\} \qquad \Longrightarrow \qquad |x^n - x_\lambda^n| \;<\; \frac{n\delta^n}{\kappa_1}\,. \tag{2.23}$$

We now estimate the measure of the set $\tilde{E}^n_{\lambda,\delta}$ in (2.23). Assume for instance that $\lambda \geq 0$ (the other case being similar). If $\lambda \geq \delta^n$, then $x_\lambda \geq (n/\kappa_2)^{1/n}\delta$ by (2.22), hence by (2.23) any $x \in \tilde{E}^n_{\lambda,\delta}$ satisfies

$$|x - x_\lambda| \,<\, \frac{n\delta^n}{\kappa_1}\,\frac{1}{x^{n-1} + \cdots + x_\lambda^{n-1}} \,\leq\, \Bigl(\frac{n}{\kappa_1}\Bigr)\Bigl(\frac{\kappa_2}{n}\Bigr)^{1-1/n}\delta\,. \tag{2.24}$$

If $0 \leq \lambda \leq \delta^n$, then $x_\lambda \leq (n/\kappa_1)^{1/n}\delta$ by (2.22), hence using (2.23) again we find that any $x \in \tilde{E}^n_{\lambda,\delta}$ satisfies

$$|x|^n \,\leq\, |x^n - x_\lambda^n| + x_\lambda^n \,<\, \frac{2n}{\kappa_1}\,\delta^n\,.$$

In all cases we obtain $|\tilde{E}^n_{\lambda,\delta}| \leq C\delta$, where the constant depends only on n, κ_1, κ_2. Returning to the original function v, we conclude that, if $\lambda - \lambda_0 \in \mathcal{V}$ and $0 < \delta < \delta_0$, we have the estimate

$$|E^m_{\lambda,\delta}| \,\leq\, |E^n_{\lambda,\delta}| \,=\, |\tilde{E}^n_{\lambda,\delta}| \,\leq\, C(n,\kappa_1,\kappa_2)\delta\,. \tag{2.25}$$

Case 2: n is even. We again assume that $y_1 \in (0,L)$ and $v^{(n)}(y_1) > 0$. If $w(x) = v(y_1 + x) - \lambda_0$, there exists an open interval $\mathcal{I} \subset \mathbb{R}$ containing the origin such that

$$\kappa_1 x^n \,\leq\, x w'(x) \,\leq\, \kappa_2 x^n\,, \qquad \text{for all } x \in \mathcal{I}\,.$$

If the parameter λ is restricted to a small neighborhood $\mathcal{V}$ of the origin, the equation $w(x) = \lambda$ has no solution in $\mathcal{I}$ when $\lambda < 0$, and exactly two solutions $x_\lambda^\pm \in \mathcal{I}$ when $\lambda > 0$, which satisfy

$$x_\lambda^- < 0 < x_\lambda^+\,, \qquad \text{and} \qquad \frac{n\lambda}{\kappa_2} \,\leq\, \bigl|x_\lambda^\pm\bigr|^n \,\leq\, \frac{n\lambda}{\kappa_1}\,. \tag{2.26}$$

Assume now that $\lambda \in \mathcal{V}$ and $0 < \delta < \delta_0$ for some sufficiently small $\delta_0 > 0$. If $\lambda \geq \delta^n$ and x belongs to the set $\tilde{E}^n_{\lambda,\delta}$ defined in (2.23), then taking $x_\lambda \in \{x_\lambda^-, x_\lambda^+\}$ such that $x x_\lambda \geq 0$ we have, as in (2.23), (2.24),

$$|x^n - x_\lambda^n| \,<\, \frac{n\delta^n}{\kappa_1}\,, \qquad \text{hence} \quad |x - x_\lambda| \,<\, \Bigl(\frac{n}{\kappa_1}\Bigr)\Bigl(\frac{\kappa_2}{n}\Bigr)^{1-1/n}\delta\,.$$

If $\lambda \leq \delta^n$ and $x \in \tilde{E}^n_{\lambda,\delta}$, then $w(x) < \lambda + \delta^n \leq 2\delta^n$, and using the lower bound $w(x) \geq (\kappa_1/n)x^n$ we deduce that $|x| < (2n/\kappa_1)^{1/n}\delta$. In all cases we thus obtain $|\tilde{E}^n_{\lambda,\delta}| \leq C\delta$, and we conclude as in (2.25). ☐

The same estimate also holds for the δ-neighborhoods of the thickened level sets:

LEMMA 2.7. *If $v \in C^m([0,L])$ satisfies (2.20), there exists positive constants C_1, δ_1 such that, for all $\lambda \in \mathbb{R}$ and all $\delta \in (0,\delta_1)$, one has $|\mathcal{E}^m_{\lambda,\delta}| \leq C_1\delta$.*

PROOF. Again it is sufficient to estimate the measure of $\mathcal{E}^m_{\lambda,\delta}$ for λ close to some $\lambda_0 \in \mathbb{R}$, and for $\delta > 0$ sufficiently small. The proof of Lemma 2.6 shows that, under those assumptions, the thickened level set $E^m_{\lambda,\delta}$ is contained in the union of a finite number of intervals, the lengths of which are bounded by $C_0\delta$. Therefore, the δ-neighborhood $\mathcal{E}^m_{\lambda,\delta}$ is contained in a finite union of intervals of lengths $(C_0 + 2)\delta$, which gives the desired conclusion. ☐

PROOF OF THEOREM 1.1. If $d = 1$ and v satisfies (1.6), Assumption 2.2 holds as a consequence of Lemma 2.7 and estimate (2.21). Thus, combining Propositions 2.1 and 2.4, we obtain the desired conclusion. ☐

2.2. The case of Morse functions. Checking Assumption 2.2 in the higher-dimensional case $d \geq 2$ is more difficult, and we only consider here an important example. We assume that $\Omega \subset \mathbb{R}^d$ is a bounded domain with smooth boundary $\partial\Omega$, and that $v : \overline{\Omega} \to \mathbb{R}$ is a smooth *Morse function* with no critical point at the boundary. By this we first mean that v has only a finite number of critical points in Ω, all of which are nondegenerate. Moreover, by Whitney's extension theorem [**27**], v can be extended to a smooth function on $\Omega_0 := \{y \in \mathbb{R}^d\,;\, \mathrm{dist}(y,\Omega) < \epsilon_0\}$ for some $\epsilon_0 > 0$, and we assume that this extension (still denoted by v) has no critical point on $\partial\Omega$.

LEMMA 2.8. *If $v : \overline{\Omega} \to \mathbb{R}$ is a smooth Morse function with no critical point on $\partial\Omega$, then Assumption 2.2 holds with $m = 1$ if v has no critical point in Ω, and with $m = 2$ if v has at least one critical point in Ω.*

PROOF. As in Lemma 2.6, it is sufficient to prove that, for all $\lambda_0 \in \mathbb{R}$, there exists a constant $C > 0$ such that inequality (2.8) holds for all λ sufficiently close to λ_0, all $\delta > 0$ sufficiently small, and all $g \in H^1(\Omega)$. We thus fix $\lambda_0 \in \mathbb{R}$ and we denote by $y_1, \dots, y_N \in \Omega$ the critical points of v that lie on the level set E_{λ_0} (if there are none, we simply set $N = 0$). By the *Morse lemma* [32], for any $j \in \{1, \dots, N\}$ and any sufficiently small $\epsilon > 0$, there exist a neighborhood V_j of y_j in Ω and a smooth diffeomorphism $\phi_j : B(0,\epsilon) \to V_j$ such that $\phi_j(0) = y_j$ and

$$v(\phi_j(x)) = \lambda_0 + |x'|^2 - |x''|^2 , \qquad \text{for all } x = (x', x'') \in B(0,\epsilon) , \tag{2.27}$$

where $x' \in \mathbb{R}^{d_1}$, $x'' \in \mathbb{R}^{d_2}$ with $d_1 + d_2 = d$. In other words, after a smooth change of coordinates, we can assume that v takes the canonical form (2.27) near the critical point y_j.

Similarly, in a neighborhood of any non-critical point $y \in \overline{E}_{\lambda_0} := \{y \in \overline{\Omega}\,;\, v(y) = \lambda_0\}$, we can transform v into an affine function by a change of coordinates. The compact set $\overline{E}_{\lambda_0}$ being covered by these neighborhoods and by the sets V_j for $j = 1, \dots, N$, we can extract a finite subcover. We can thus find M points $y_{N+1}, \dots, y_{N+M} \in \overline{E}_{\lambda_0}$ such that, for any $j \in \{N{+}1, \dots, N{+}M\}$, there exist a neighborhood V_j of y_j in Ω_0 and a smooth diffeomorphism $\phi_j : B(0,\epsilon) \to V_j$ such that $\phi_j(0) = y_j$ and

$$v(\phi_j(x)) = \lambda_0 + x_d , \qquad \text{for all } x = (x_1, \dots, x_d) \in B(0,\epsilon) . \tag{2.28}$$

Moreover we have

$$\overline{E}_{\lambda_0} \subset V := V_1 \cup \dots \cup V_N \cup V_{N+1} \cup \dots \cup V_{N+M} \subset \Omega_0 . \tag{2.29}$$

Our next tool is a smooth partition of unity (χ_j) associated with the open cover (2.29). More precisely, there exists smooth functions $\chi_j : \mathbb{R}^d \to \mathbb{R}$ such that $K_j := \mathrm{supp}(\chi_j) \subset V_j$ for $j = 1, \dots, N+M$, and

$$\sum_{j=1}^{N+M} \chi_j(y)^2 \le 1 \quad \text{for all } y \in \Omega_0 , \qquad \sum_{j=1}^{N+M} \chi_j(y)^2 = 1 \quad \text{for all } y \in E , \tag{2.30}$$

for some open set $E \subset \mathbb{R}^d$ with $\overline{E}_{\lambda_0} \subset E \subset \overline{E} \subset V$. For later use, we observe that $\mathcal{E}^m_{\lambda,\delta} \subset E$ whenever λ is sufficiently close to λ_0 and $\delta > 0$ is sufficiently small.

Now, let $g \in H^1(\Omega)$. Since the boundary $\partial\Omega$ is smooth, we can extend g to a function $\tilde{g} \in H^1(\Omega_0)$ which satisfies

$$\|\tilde{g}\|_{L^2(\Omega_0)} \le 2\|g\|_{L^2(\Omega)} , \qquad \|\tilde{g}\|_{H^1(\Omega_0)} \le C\|g\|_{H^1(\Omega)} , \tag{2.31}$$

for some constant $C > 0$ (independent of g). This extension will allow us to treat the boundary points $y_j \in \partial\Omega$ exactly as the interior points $y_j \in \Omega$. Since $\mathcal{E}^m_{\lambda,\delta} \subset E$ when λ is sufficiently close to λ_0 and $\delta > 0$ is sufficiently small, we can use the partition of unity (2.30) to decompose

$$\int_{\mathcal{E}^m_{\lambda,\delta}} |g(y)|^2 \,\mathrm{d}y = \sum_{j=1}^{N+M} \int_{\mathcal{E}^m_{\lambda,\delta}} |g(y)|^2 \chi_j(y)^2 \,\mathrm{d}y = \sum_{j=1}^{N+M} \int_{\mathcal{E}^m_{\lambda,\delta}} |g_j(y)|^2 \,\mathrm{d}y , \tag{2.32}$$

where $g_j := \tilde{g}\,\chi_j$ is supported in $K_j \subset V_j$ for $j = 1, \dots, N{+}M$.

It remains to estimate the integrals in the right-hand side of (2.32), which can be written in the form

$$I_j := \int_{\mathcal{E}^m_{\lambda,\delta} \cap K_j} |g_j(y)|^2 \,\mathrm{d}y = \int_{\phi_j^{-1}(\mathcal{E}^m_{\lambda,\delta} \cap K_j)} |g_j(\phi_j(x))|^2 \, |J_{\phi_j}(x)| \,\mathrm{d}x , \tag{2.33}$$

where J_{ϕ_j} is the Jacobian determinant of the diffeomorphism ϕ_j. We can find constants $L, \Lambda \ge 1$ such that

$$L^{-1}|x - \tilde{x}| \le |\phi_j(x) - \phi_j(\tilde{x})| \le L|x - \tilde{x}| , \qquad \text{and} \qquad \Lambda^{-1} \le |J_{\phi_j}(x)| \le \Lambda ,$$

for all points $x, \tilde{x} \in B(0,\epsilon)$ and all integers $j \in \{1, \ldots, N{+}M\}$. It is thus straightforward to verify that $\phi_j^{-1}(\mathcal{E}^m_{\lambda,\delta} \cap K_j) \subset \mathcal{E}^m_{\lambda,L\delta,j}$ if $\delta > 0$ is small enough, where

$$\mathcal{E}^m_{\lambda,\delta,j} := \bigl\{x \in B(0,\epsilon)\,;\, \mathrm{dist}(x, E^m_{\lambda,\delta,j}) < \delta\bigr\}\,, \quad E^m_{\lambda,\delta,j} := \bigl\{x \in B(0,\epsilon)\,;\, \bigl|v(\phi_j(x)) - \lambda\bigr| < \delta^m\bigr\}\,.$$

Note that the sets $E^m_{\lambda,\delta,j}, \mathcal{E}^m_{\lambda,\delta,j}$ are defined in terms of the canonical forms $v \circ \phi_j$ exactly as the sets (2.6), (2.7) were defined in terms of the original function v. We now distinguish two cases:

1: *The critical points*. If $N \ge 1$ and $j \in \{1, \ldots, N\}$, we necessarily have $m = 2$. In view of the canonical form (2.27), we apply Lemma B.2 if $d_1 d_2 = 0$ (the case of a local extremum) or Lemma B.3 if $d_1 d_2 > 0$ (the case of a saddle point). Denoting $h_j = g_j \circ \phi_j$, we thus obtain

$$I_j \,\le\, \Lambda \int_{\mathcal{E}^m_{\lambda,L\delta,j}} |g_j(\phi_j(x))|^2 \,\mathrm{d}x \,\le\, C\delta \|h_j\| \|\nabla h_j\| \,\le\, \kappa \|h_j\|^2 + C\delta^2 \|\nabla h_j\|^2\,, \tag{2.34}$$

where the constant $\kappa > 0$ can be taken arbitrarily small. Returning to the original variable y we arrive at

$$\begin{aligned} \|h_j\|^2_{L^2} \,&=\, \int_{B(0,\epsilon)} |g_j(\phi_j(x))|^2 \,\mathrm{d}x \,\le\, \Lambda \int_{V_j} |g_j(y)|^2 \,\mathrm{d}y \,=\, \Lambda \int_{V_j} |\tilde{g}_j(y)|^2 \chi_j(y)^2 \,\mathrm{d}y \\ \|\nabla h_j\|^2_{L^2} \,&\le\, C \int_{V_j} |\nabla g_j(y)|^2 \,\mathrm{d}y \,\le\, C \int_{V_j} \Bigl(|\nabla \tilde{g}(y)|^2 \chi_j(y)^2 + |\tilde{g}(y)|^2 |\nabla \chi_j(y)|^2\Bigr) \,\mathrm{d}y\,. \end{aligned} \tag{2.35}$$

2: *The non-critical points*. If $j \in \{N{+}1, \ldots, N{+}M\}$ and $m = 1$ or 2, then due to the simple canonical form (2.28) we have the inclusions $\mathcal{E}^m_{\lambda,L\delta,j} \subset \mathcal{E}^1_{\lambda,L\delta,j} \subset E^1_{\lambda,2L\delta,j}$. Therefore, applying Lemma A.5 we obtain the same estimate (2.34) for the quantity I_j, and (2.35) is unchanged too. Note that, if $y_j \in \partial\Omega$, the open set V_j is not entirely contained in Ω, so that we need to use the extension $\tilde{g}$ instead of g in the right-hand side of (2.35).

Summarizing, we deduce from (2.30), (2.31), and (2.32)–(2.35) that

$$\begin{aligned} \int_{\mathcal{E}^m_{\lambda,\delta}} |g(y)|^2 \,\mathrm{d}y \,=\, \sum_{j=1}^{N+M} I_j \,&\le\, \kappa\Lambda \int_{\Omega_0} |\tilde{g}(y)|^2 \,\mathrm{d}y + C\delta^2 \int_{\Omega_0} \Bigl(|\nabla \tilde{g}(y)|^2 + |\tilde{g}(y)|^2\Bigr) \,\mathrm{d}y \\ &\le\, \frac{1}{2} \int_{\Omega} |g(y)|^2 \,\mathrm{d}y + C\delta^2 \int_{\Omega} |\nabla g(y)|^2 \,\mathrm{d}y\,, \end{aligned}$$

provided $\kappa\Lambda < 1/8$ and $\delta > 0$ is sufficiently. This gives the desired estimate (2.8). ☐

PROOF OF THEOREM 1.2. If v is Morse function, Assumption 2.2 holds with $m = 1$ or 2 in view of Lemma 2.8, and the desired conclusion follows from Propositions 2.1 and 2.4. ☐

2.3. Additional examples. Lemmas 2.7 and 2.8 give general conditions that imply the validity of Assumption 2.2, but our approach has a much broader scope and allows us to treat many flows which do not fall into these categories. We give two simple examples here, which indicate in which direction the assumptions of Theorem 1.2 could be weakened.

EXAMPLE 2.9. Assume that $\Omega = B(0,1) \subset \mathbb{R}^d$ and that $v(y) = 1 - |y|^m$, for some $m \in \mathbb{N}^*$. Then Assumption 2.2 is verified for that value of m, so that estimate (1.7) holds for all $\nu > 0$ and all $k \neq 0$.

To see this, we observe that, if $\lambda \in \mathbb{R}$ and $\delta > 0$, we have $\mathcal{E}^m_{\lambda,\delta} \subset \bigl\{y \in \Omega\,;\, R_1 \le |y| < R_2\bigr\}$, where

$$R_1 \,=\, \bigl(1 - \lambda - \delta^m\bigr)_+^{1/m} - \delta\,, \qquad R_2 \,=\, \bigl(1 - \lambda + \delta^m\bigr)_+^{1/m} + \delta\,.$$

An easy calculation shows that $R_2 - R_1 \le 4\delta$. So, if $g \in H^1(\Omega)$ and if $\tilde{g} \in H^1(\mathbb{R}^d)$ is an extension of g to $\Omega_0 = \mathbb{R}^d$ satisfying (2.31), we can apply Lemma B.1 in Appendix B to obtain the estimate

$$\int_{\mathcal{E}^m_{\lambda,\delta}} |g(y)|^2 \,\mathrm{d}y \,\le\, \int_{\{R_1 \le |y| < R_2\}} |\tilde{g}(y)|^2 \,\mathrm{d}y \,\le\, 8\delta \,\|\tilde{g}\|_{L^2} \,\|\nabla \tilde{g}\|_{L^2}\,,$$

from which (2.8) easily follows using (2.31) and Young's inequality. Note that the value $m = 1$ is allowed, in which case the function v is not smooth. In fact, we can allow m to be an arbitrary positive real number in that example, but the constant C_0 in (2.8) diverges in the singular limit $m \to 0$.

EXAMPLE 2.10. Assume that $\Omega = B(0,1) \times B(0,1) \subset \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ and that $v(y,z) = 1 - |y|^m$ for all $(y,z) \in \Omega$. If $d_1 \ge 1$ we have the same conclusions as in Example 2.9.

We can assume that $d_2 \ge 1$ too, otherwise we recover Example 2.9. In the case $m = 2$, which is already interesting, the function v is Morse-Bott: its critical points form a submanifold $S = \{(0,z)\,;\, |z| < 1\}$, and the second differential of v is nondegenerate in the directions that are transverse to S. This example can be treated in the same way as the previous one. We have $\mathcal{E}^m_{\lambda,\delta} \subset \bigl\{(y,z) \in \Omega\,;\, R_1 \le |y| < R_2\bigr\}$, with R_1, R_2 as above, so using Lemma B.1 and Fubini's theorem we obtain

$$\int_{\mathcal{E}^m_{\lambda,\delta}} |g(y,z)|^2 \,\mathrm{d}y\,\mathrm{d}z \,\le\, \int_{\mathbb{R}^{d_2}} \int_{\{R_1 \le |y| < R_2\}} |\tilde g(y,z)|^2 \,\mathrm{d}y\,\mathrm{d}z \,\le\, 8\delta\, \|\tilde g\|_{L^2}\, \|\nabla \tilde g\|_{L^2}\,,$$

and estimate (2.8) follows by the same argument. Note that the domain Ω is not smooth in that example.

3. Energy estimates

In this section we give an alternative proof of a slight variation of Theorems 1.2–1.3, using a direct energy method usually referred to as hypocoercivity [37].To avoid a few technicalities related to boundaries, we restrict ourselves to the following two cases:

1. $\Omega = \mathbb{T}^d$, the d-dimensional periodic box;
2. $\Omega \subset \mathbb{R}^d$ a smooth bounded domain, with *homogeneous Dirichlet boundary conditions* for g.

The result involves an energy functional of the form

$$\Phi = \frac{1}{2}\left[\|g\|^2 + \alpha\|\nabla g\|^2 + 2\beta \mathrm{Re}\langle ikg\nabla v, \nabla g\rangle + \gamma k^2 \|g\nabla v\|^2\right], \tag{3.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $X = L^2(\Omega)$ and $\|\cdot\|$ the associated norm. The parameters α, β, γ depend on ν, k in a different way according to whether we consider the enhanced dissipation regime or the Taylor dispersion regime. We prove the following result:

THEOREM 3.1. *Assume that $v : \overline{\Omega} \to \mathbb{R}$ is a smooth Morse function with no critical point on the boundary $\partial\Omega$. There exist positive constants β_0, C_3 such that, for all $\nu > 0$, all $k \ne 0$, and all initial data $g_0 \in H^1_0(\Omega)$, the solution of* (1.5) *satisfies, for all $t \ge 0$,*

$$\Phi(t) \,\le\, \mathrm{e}^{-C_3\lambda_{\nu,k}t}\,\Phi(0)\,, \quad \text{where } \lambda_{\nu,k} \,=\, \begin{cases} \nu^{\frac{m}{m+2}}|k|^{\frac{2}{m+2}} & \text{if } 0 < \nu \le \beta_0|k|\,, \\ \frac{k^2}{\nu} & \text{if } 0 < \beta_0|k| \le \nu\,. \end{cases} \tag{3.2}$$

Here $m = 1$ if v has no critical point in Ω, and $m = 2$ if v has at least one critical point in Ω.

REMARK 3.2. If $\Omega = \mathbb{T}^d$, it is understood that $\overline{\Omega} = \mathbb{T}^d$, $\partial\Omega = \emptyset$, and $H^1_0(\Omega) = H^1(\mathbb{T}^d)$. In that case, any Morse function v has critical points in Ω, so that we necessarily have $m = 2$.

The decay rate $\lambda_{\nu,k}$ in (3.2) is essentially the same as the one in (1.7), except for the threshold parameter β_0 which will be taken smaller than 1 in the proof of Theorem 3.1. However, the main difference with the arguments developed in Section 2 is the use of the H^1-type energy (3.1). A simple argument allows us to translate estimate (3.2) on the energy functional Φ into a semigroup bound of the form (1.7), as is stated below.

COROLLARY 3.3. *Under the assumptions of Theorem 3.1, there exist positive constants β_0, C_1, C_2 such that, for all $\nu > 0$, all $k \ne 0$, and all initial data $g_0 \in L^2(\Omega)$, the solution of* (1.5) *satisfies the estimate*

$$\|g(k,t)\| \,\le\, C_1\left(1 + \frac{|k|}{\nu}\right)^{\frac{m-1}{m+2}} \mathrm{e}^{-C_2\lambda_{\nu,k}t}\,\|g_0\|\,, \tag{3.3}$$

for all $t \ge 0$, where $\lambda_{\nu,k}$ is given by (3.2).

REMARK 3.4. In the enhanced dissipation regime, when $m \geq 2$, the prefactor appearing in (3.3) implies a logarithmic correction on the decay rate $\lambda_{\nu,k}$, as already noticed in [7]. Such a correction is not present when $m = 1$, namely when there are no critical points. Also, in the Taylor dispersion case, the prefactor is harmless since $1 \leq 1 + \frac{|k|}{\nu} \leq 1 + \beta_0^{-1}$.

In the case of homogeneous Dirichlet boundary conditions, the decay estimate (3.3) is not interesting in the Taylor dispersion regime, because it is in fact weaker than what can be deduced from the simple energy balance (3.4) in view of the Poincaré inequality.

The rest of this section is devoted to proving Theorem 3.1.

3.1. Energy identities. We start the discussion with some energy identities that will be used to build the hypocoercivity functional Φ in (3.1). Below, we indicate by Δ and ∇ the Laplacian and the gradient with respect to the space variable $y \in \Omega$.

LEMMA 3.5. *Let g solve (1.5) either in the torus $\mathbb{T}^d$, or in a bounded domain $\Omega \subset \mathbb{R}^d$ with homogeneous Dirichlet boundary conditions. Then we have the following balances:*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|g\|^2 + \nu\|\nabla g\|^2 \;=\; 0\,, \tag{3.4}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla g\|^2 + \nu\|\Delta g\|^2 \;=\; -\mathrm{Re}\langle ikg\nabla v, \nabla g\rangle\,, \tag{3.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Re}\langle ikg\nabla v, \nabla g\rangle + k^2\|g\nabla v\|^2 \;=\; -2\nu\mathrm{Re}\langle ik\nabla v\cdot\nabla g, \Delta g\rangle - \nu\mathrm{Re}\langle ikg\Delta v, \Delta g\rangle\,, \tag{3.6}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|g\nabla v\|^2 + \nu\||\nabla v|\nabla g\|^2 \;=\; -2\nu\mathrm{Re}\langle gD^2v\nabla v, \nabla g\rangle\,. \tag{3.7}$$

PROOF. All the identities are established by direct computation, using integration by parts. There are no boundary terms if $\Omega = \mathbb{T}^d$, and in the other case the contributions from the boundary vanish thanks to the homogeneous Dirichlet conditions. The L^2 balance (3.4) follows directly by testing (1.5) with g and using the antisymmetry property $\mathrm{Re}\langle ivg, g\rangle = 0$. Testing (1.5) with $-\Delta g$ we also obtain (3.5) by a simple integration by parts. Turning to (3.6), we use (1.5) to compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Re}\langle ikg\nabla v, \nabla g\rangle \;=\; \nu\mathrm{Re}\left[\langle ik(\nabla v)\Delta g, \nabla g\rangle + \langle ikg\nabla v, \nabla\Delta g\rangle\right] + k^2\mathrm{Re}\left[\langle vg\nabla v, \nabla g\rangle - \langle g\nabla v, \nabla(vg)\rangle\right].$$

We treat the ν term integrating by parts as

$$\mathrm{Re}\left[\langle i(\nabla v)\Delta g, \nabla g\rangle + \langle ig\nabla v, \nabla\Delta g\rangle\right] \;=\; -2\mathrm{Re}\langle i\nabla v\cdot\nabla g, \Delta g\rangle - \mathrm{Re}\langle ig\Delta v, \Delta g\rangle\,,$$

while for the second term we compute

$$\langle vg\nabla v, \nabla g\rangle - \langle g\nabla v, \nabla(vg)\rangle \;=\; -\|g\nabla v\|^2\,,$$

and (3.6) follows. For (3.7), thanks to antisymmetry we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|g\nabla v\|^2 \;&=\; \nu\mathrm{Re}\langle g\nabla v, (\nabla v)\Delta g\rangle - \mathrm{Re}\langle g\nabla v, ikvg\nabla v\rangle \\ &=\; -\nu\||\nabla v|\nabla g\|^2 - 2\nu\mathrm{Re}\langle gD^2v\nabla v, \nabla g\rangle\,.\end{aligned}$$

This concludes the proof. □

REMARK 3.6. The relations (3.4), (3.5), and (3.7) remain valid if g satisfies the homogeneous Neumann conditions on $\partial\Omega$, but to obtain (3.6) one has to assume in addition that the normal derivative of the shear velocity v vanishes identically on the boundary. This additional hypothesis is not very natural, as it is not satisfied in many classical examples, such as the cylindrical Poiseuille flow. Moreover, it conflicts with our forthcoming assumption that v has no critical points on $\partial\Omega$, see Proposition 3.7 below. For these reasons, we prefer assuming in this section that $g = 0$ on $\partial\Omega$, or alternatively that $\Omega = \mathbb{T}^d$.

For each $k \in \mathbb{R}$, the (frequency-localized) energy functional Φ in (3.1) depends on the positive coefficients α, β, γ, to be chosen depending on k and ν. For the moment, we assume that

$$\frac{\beta^2}{\alpha\gamma} \le \frac{1}{16}, \tag{3.8}$$

a condition that guarantees the coercivity of Φ. Indeed, since

$$2\beta|k||\langle g\nabla v, \nabla g\rangle| \le 2\beta|k|\|g\nabla v\|\|\nabla g\| \le \frac{\alpha}{4}\|\nabla g\|^2 + \frac{4\beta^2 k^2}{\alpha}\|g\nabla v\|^2 \le \frac{\alpha}{4}\|\nabla g\|^2 + \frac{\gamma k^2}{4}\|g\nabla v\|^2,$$

we obtain

$$\frac{1}{8}\left[4\|g\|^2 + 3\alpha\|\nabla g\|^2 + 3\gamma k^2\|g\nabla v\|^2\right] \le \Phi \le \frac{1}{8}\left[4\|g\|^2 + 5\alpha\|\nabla g\|^2 + 5\gamma k^2\|g\nabla v\|^2\right]. \tag{3.9}$$

Moreover, Lemma 3.5 readily implies that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Phi + \nu\|\nabla g\|^2 + \alpha\nu\|\Delta g\|^2 + \beta k^2\|g\nabla v\|^2 + \gamma\nu k^2\||\nabla v|\nabla g\|^2 \\ = -\alpha\mathrm{Re}\langle ikg\nabla v, \nabla g\rangle - 2\beta\nu\mathrm{Re}\langle ik\nabla v\cdot\nabla g, \Delta g\rangle \\ - \beta\nu\mathrm{Re}\langle ikg\Delta v, \Delta g\rangle - 2\gamma\nu k^2\langle gD^2 v\nabla v, \nabla g\rangle.\end{aligned} \tag{3.10}$$

In addition to (3.8), let us assume from now on that

$$\frac{\alpha^2}{\nu} \le \beta. \tag{3.11}$$

In this way, the first term on the right-hand side of (3.10) can be estimated as

$$\alpha|\langle ikg\nabla v, \nabla g\rangle| \le \alpha|k|\|g\nabla v\|\|\nabla g\| \le \frac{\nu}{2}\|\nabla g\|^2 + \frac{\alpha^2 k^2}{2\nu}\|g\nabla v\|^2 \le \frac{\nu}{2}\|\nabla g\|^2 + \frac{\beta k^2}{2}\|g\nabla v\|^2. \tag{3.12}$$

Moreover, thanks to (3.8) we have

$$2\beta\nu|\langle ik\nabla v\cdot\nabla g, \Delta g\rangle| \le 2\beta\nu|k|\||\nabla v|\nabla g\|\|\Delta g\| \le \frac{\alpha\nu}{4}\|\Delta g\|^2 + \frac{\gamma\nu k^2}{4}\||\nabla v|\nabla g\|^2, \tag{3.13}$$

and

$$\beta\nu|\langle ikg\Delta v, \Delta g\rangle| \le \beta\nu|k|\|g\Delta v\|\|\Delta g\| \le \frac{\alpha\nu}{4}\|\Delta g\|^2 + \frac{\gamma\nu k^2}{16}\|g\Delta v\|^2. \tag{3.14}$$

Finally, we have

$$2\gamma\nu k^2|\langle gD^2 v\nabla v, \nabla g\rangle| \le \frac{\gamma\nu k^2}{4}\||\nabla v|\nabla g\|^2 + 4\gamma\nu k^2\|gD^2 v\|^2. \tag{3.15}$$

Combining (3.10) and (3.12)–(3.15), we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi + \frac{\nu}{2}\|\nabla g\|^2 + \frac{\alpha\nu}{2}\|\Delta g\|^2 + \frac{\beta k^2}{2}\|g\nabla v\|^2 + \frac{\gamma\nu k^2}{2}\||\nabla v|\nabla g\|^2 \le 5\gamma\nu k^2\|gD^2 v\|^2. \tag{3.16}$$

In order to obtain a differential inequality for the functional Φ, it remains to bound the remainder term in the right-hand side of (3.16), and to show that Φ itself can be controlled using the positive terms in the left-hand side.

3.2. A semi-classical estimate. One of the key elements of the proof via hypocoercivity is the following inequality, which we state in a fairly general way.

PROPOSITION 3.7. *Let $w : \Omega \to \mathbb{R}$ be a smooth function such that*

(H1) $w \in C^1(\overline{\Omega})$, and $\nabla w \neq 0$ on $\partial\Omega$;

(H2) *w has a finite number of critical points $y_1, \dots, y_N$ in Ω. For $1 \le j \le N$ there exist a radius $R_j \in (0,1)$, an integer $m_j \ge 2$, and a constant $C \ge 1$ such that*

$$\frac{1}{C}|y-y_j|^{m_j-1} \,\le\, |\nabla w(y)| \,\le\, C|y-y_j|^{m_j-1}, \qquad \forall\, y \in B(y_j,R_j)\,, \tag{3.17}$$

where $B(y_j,R_j)$ denotes the ball of radius R_j centered at y_j.

Then there exists a constant $C_{sp} \ge 1$ such that, for all $\sigma \in (0,1]$ and all $\varphi \in H^1(\Omega)$, the following inequality holds

$$\sigma^{\frac{m-1}{m}}\|\varphi\|^2 \,\le\, C_{sp}\left[\sigma\|\nabla\varphi\|^2 + \|\varphi\nabla w\|^2\right]\,, \tag{3.18}$$

where $m = \max\{m_1,\dots,m_N\}$ if $N \ge 1$ and $m = 1$ if w has no critical point in Ω.

Estimate (3.18) means that the ground state of the Schrödinger operator $-\sigma\Delta + |\nabla w|^2$ in $L^2(\Omega)$ is bounded from below by $C_{sp}^{-1}\,\sigma^{\frac{m-1}{m}}$ in the semi-classical limit $\sigma \to 0$. There is of course an abundant literature on spectral bounds for semi-classical Schrödinger operators. The background material can be found in the excellent references [**24**, **26**], where estimate (3.18) is established at least in some special cases. For the sake of completeness, we give below a short proof of (3.18) under our general assumptions.

The proof of Proposition 3.7 relies on the following lemma.

LEMMA 3.8. *Let $R > 0$ and denote by $B_R \subset \mathbb{R}^d$ the ball of radius R centered at 0. If $\varphi \in H^1_0(B_R)$ and $\ell \in \mathbb{N}$, then*

$$\|\varphi\|_{L^2(B_R)} \,\le\, C\|\nabla\varphi\|_{L^2(B_R)}^{\frac{\ell}{\ell+1}}\||y|^\ell\varphi\|_{L^2(B_R)}^{\frac{1}{\ell+1}}\,, \tag{3.19}$$

where the constant $C > 0$ depends only on d and ℓ.

PROOF. There is nothing to prove if $\ell = 0$, so we assume henceforth that $\ell \ge 1$. Passing to polar coordinates and decomposing φ in spherical harmonics as in the proof of Lemma B.1 below, we see that it is sufficient to prove (3.19) in the particular case where φ is *radially symmetric*. Under that assumption, we can integrate by parts and obtain

$$\|\varphi\|^2 \,=\, A_d\int_0^R \varphi(r)^2 r^{d-1}\,\mathrm{d}r \,=\, -\frac{2A_d}{d}\int_0^R r\varphi(r)\varphi'(r)r^{d-1}\,\mathrm{d}r\,,$$

where $A_d = 2\pi^{d/2}\Gamma(d/2)^{-1}$ is the area of the unit sphere in $\mathbb{R}^d$. By Hölder's inequality we then find

$$A_d\int_0^R r|\varphi(r)||\varphi'(r)|r^{d-1}\,\mathrm{d}r \,\le\, \|\varphi'\|\|r^\ell\varphi\|^{\frac{1}{\ell}}\|\varphi\|^{1-\frac{1}{\ell}}\,,$$

hence

$$\|\varphi\|^2 \,\le\, \frac{2}{d}\|\varphi'\|\|r^\ell\varphi\|^{\frac{1}{\ell}}\|\varphi\|^{1-\frac{1}{\ell}}\,.$$

This gives the desired inequality (3.19) for a radially symmetric φ, and the general case follows. □

We are now ready to prove the semi-classical estimate (3.18).

PROOF OF PROPOSITION 3.7. For definiteness we consider the case of a bounded domain $\Omega \subset \mathbb{R}^d$; the argument is similar in the periodic case. In assumption (H2), we suppose without loss of generality that the balls $B(y_j,R_j)$ are pairwise disjoint. For each $j \in \{1,\dots,N\}$, let χ_j be a smooth cut-off function such that $\mathrm{supp}(\chi_j) \subset B(y_j,R_j)$ and $\chi_j = 1$ on $B(y_j,R_j/2)$. We can also assume that there exists a smooth function χ_0 such that

$$\chi_0(y)^2 + \sum_{j=1}^N \chi_j(y)^2 \,=\, 1\,, \qquad \text{for all } y \in \mathbb{R}^d\,,$$

so that the family $\{\chi_j^2\}$ for $j = 0,\dots,N$ is a smooth partition of unity.

Fix $\varphi \in H^1(\Omega)$. For any $j \in \{1, \dots, N\}$ we have $\varphi\chi_j \in H^1_0(B(y_j, R_j))$, so using assumption (3.17) and Lemma 3.8 with $\ell = m_j - 1$, we obtain

$$\|\varphi\chi_j\|_{L^2(B(y_j,R_j))} \le C\|\nabla(\varphi\chi_j)\|^{\frac{m_j-1}{m_j}}_{L^2(B(y_j,R_j))}\|\varphi\chi_j\nabla w\|^{\frac{1}{m_j}}_{L^2(B(y_j,R_j))}, \qquad j \ge 1.$$

Equivalently, by Young's inequality, there exists a constant $c_1 > 0$ such that

$$\sigma^{\frac{m_j-1}{m_j}}\|\varphi\chi_j\|^2 \le c_1\left[\sigma\|\nabla(\varphi\chi_j)\|^2 + \|\varphi\chi_j\nabla w\|^2\right], \qquad j \ge 1. \tag{3.20}$$

We remark that inequality (3.20) is also satisfied when $j = 0$, with $m_0 = 1$, since by construction the function $|\nabla w|$ is bounded away from zero on the support of $\varphi\chi_0$. Taking this observation into account, we can compute

$$\begin{aligned}
\sigma\|\nabla\varphi\|^2 + \|\varphi\nabla w\|^2 &= \sum_{j\ge 0}\sigma\|\chi_j\nabla\varphi\|^2 + \|\varphi\chi_j\nabla v\|^2 \\
&= \sum_{j\ge 0}\sigma\|\nabla(\varphi\chi_j) - \varphi\nabla\chi_j\|^2 + \|\varphi\chi_j\nabla v\|^2 \\
&= \sum_{j\ge 0}\sigma\|\nabla(\varphi\chi_j)\|^2 + \|\varphi\chi_j\nabla v\|^2 - 2\sigma\langle\nabla(\varphi\chi_j), \varphi\nabla\chi_j\rangle + \sigma\|\varphi\nabla\chi_j\|^2 \\
&\ge \frac{1}{2}\sum_{j\ge 0}\left[\sigma\|\nabla(\varphi\chi_j)\|^2 + \|\varphi\chi_j\nabla v\|^2\right] - c_2\sigma\|\varphi\|^2 \\
&\ge \frac{1}{2c_1}\sum_{j\ge 0}\sigma^{\frac{m_j-1}{m_j}}\|\varphi\chi_j\|^2 - c_2\sigma\|\varphi\|^2 \\
&= \frac{1}{2c_1}\sigma^{\frac{m-1}{m}}\|\varphi\|^2 - c_2\sigma\|\varphi\|^2 \ge \frac{1}{4c_1}\sigma^{\frac{m-1}{m}}\|\varphi\|^2,
\end{aligned}$$

provided $\sigma \in (0, \sigma_0]$ for some σ_0 small enough. A simple rescaling of σ then implies (3.18), hence concluding the proof. □

3.3. Velocity profiles with simple critical points. If v is a Morse function, namely if all critical points of v are nondegenerate, the strategy is to estimate the right-hand side of (3.16) using the smoothness of v, and then to apply inequality (3.18) with $m = 2$ (or $m = 1$ if v has no critical point). For the sake of clarity, we concentrate here on the harder case $m = 2$.

PROOF OF THEOREM 3.1, CASE $m = 2$. Our starting point is inequality (3.16). Since $v \in C^2(\overline{\Omega})$, there exists a positive constant $c_v = c(\|v\|_{W^{2,\infty}})$ such that $\|D^2 v\|_{L^\infty} \le c_v/5$, hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi + \frac{\nu}{2}\|\nabla g\|^2 + \frac{\beta k^2}{2}\|g\nabla v\|^2 \le c_v\gamma\nu k^2\|g\|^2.$$

We need to show that the right-hand side can be absorbed in the left-hand side thanks to inequality (3.18) and to a suitable choice of the parameters α, β, γ, in compliance with (3.8) and (3.11). Assuming for the moment that such a choice can be made, so that

$$c_v\gamma\nu k^2\|g\|^2 \le \frac{\nu}{4}\|\nabla g\|^2 + \frac{\beta k^2}{4}\|g\nabla v\|^2, \tag{3.21}$$

we then find

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi + \frac{\nu}{4}\|\nabla g\|^2 + \frac{\beta k^2}{4}\|g\nabla v\|^2 \le 0.$$

Using (3.21) once more, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi + \frac{c_v\gamma\nu k^2}{2}\|g\|^2 + \frac{\nu}{8}\|\nabla g\|^2 + \frac{\beta k^2}{8}\|g\nabla v\|^2 \le 0,$$

or equivalently

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi + \frac{c_v\gamma\nu k^2}{8}\left[4\|g\|^2 + \frac{1}{5c_v\alpha\gamma k^2}5\alpha\|\nabla g\|^2 + \frac{\beta}{5c_v\gamma^2\nu k^2}5\gamma k^2\|g\nabla v\|^2\right] \le 0\,. \tag{3.22}$$

In order to fulfill (3.8) and (3.11) with an equality, we make the choices

$$\alpha^2 = \beta\nu\,, \qquad \gamma = \frac{16\beta^{3/2}}{\nu^{1/2}}\,, \tag{3.23}$$

and rewrite (3.22) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi + 2c_v\beta^{3/2}\nu^{1/2}k^2\left[4\|g\|^2 + \frac{1}{80c_v\beta^2k^2}5\alpha\|\nabla g\|^2 + \frac{1}{1280c_v\beta^2k^2}5\gamma k^2\|g\nabla v\|^2\right] \le 0\,. \tag{3.24}$$

We now consider two complementary regimes, verify (3.21) and close a proper Gronwall estimate for Φ.

$\diamond$ *Enhanced dissipation.* For some $\beta_0 \in (0,1)$ to be fixed and independent of ν, k, we take

$$\beta = \frac{\beta_0}{|k|}\,, \quad \text{and we assume} \quad \frac{\nu}{|k|} \le \beta_0\,. \tag{3.25}$$

To verify (3.21), we use inequality (3.18) with $w = v$, $m = 2$, and

$$\sigma = \frac{\nu}{\beta k^2} \le 1\,.$$

We thus obtain

$$\beta^{1/2}\nu^{1/2}|k|\|g\|^2 \le C_{sp}\left[\nu\|\nabla g\|^2 + \beta k^2\|g\nabla v\|^2\right]\,. \tag{3.26}$$

It follows that inequality (3.21) is verified provided

$$c_v\gamma\nu k^2 \le \frac{\beta^{1/2}\nu^{1/2}|k|}{4C_{sp}}\,.$$

In view of (3.23) and (3.25), this is equivalent to $\beta_0 \le (64\,c_vC_{sp})^{-1}$, which is simply requiring β_0 to be small enough. Going back to (3.24) and possibly reducing β_0 further so that $1280\,c_v\beta_0^2 \le 1$, we find from the coercivity of Φ in (3.9) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi + 16\,c_v\beta_0^{3/2}\nu^{1/2}|k|^{1/2}\Phi \le 0\,, \tag{3.27}$$

which immediately implies (3.2).

$\diamond$ *Taylor dispersion.* If $\nu|k|^{-1} \ge \beta_0$, we take

$$\beta = \frac{\beta_1}{\nu}\,, \tag{3.28}$$

for some $\beta_1 \in (0, \beta_0^2]$ to be fixed and independent of ν, k. Now, choosing $\sigma = 1$ in (3.18) and using the assumption that $\beta_1 \le \beta_0^2$, we find

$$\beta k^2\|g\|^2 \le C_{sp}\left[\nu\|\nabla g\|^2 + \beta k^2\|g\nabla v\|^2\right]\,.$$

It follows that inequality (3.21) is verified provided

$$c_v\gamma\nu k^2 \le \frac{\beta k^2}{4C_{sp}}\,.$$

From (3.23) and (3.28), this is equivalent to $\beta_1^{1/2} \le (64\,c_vC_{sp})^{-1}$. Going back to (3.24), we possibly reduce β_1 further so that $1280\,c_v\beta_1^2 \le \beta_0^2$. From the coercivity of Φ in (3.9), we then find

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi + 16\,c_v\beta_1^{3/2}\frac{k^2}{\nu}\Phi \le 0\,.$$

and the proof is now complete. $\square$

REMARK 3.9. In the simpler case $m = 1$, the only difference is the scaling of α, β, γ with respect to ν, k, in the enhanced dissipation regime. Specifically, the choice of β in (3.25) has to be changed to

$$\beta = \beta_0 \frac{\nu^{1/3}}{|k|^{4/3}}. \tag{3.29}$$

In fact, an even simpler proof can be carried out, without the use of the term multiplied by γ in (3.1). See [15] for a proof in the one dimensional case.

REMARK 3.10. In the Taylor dispersion regime, we really only used that v is twice continuously differentiable and not identically constant. Indeed, all we need is that inequality (3.18) holds when $\sigma = 1$, and this does not require any particular structure. Therefore, the above argument also gives an alternative proof of Theorem 1.3 under slightly more restrictive regularity assumptions on v.

It remains to give a proof of the semigroup estimate (3.3). The argument follows essentially that of [18, 19], and we give the details here for completeness.

PROOF OF COROLLARY 3.3. We first consider the enhanced dissipation regime. Let $T_{\nu,k} = 1/\lambda_{\nu,k}$ be the relevant time scale. From the energy balance (3.4) and the mean-value theorem, we may find a time $t_0 \in (0, T_{\nu,k})$ such that

$$2\|\nabla g(t_0)\|^2 \leq \frac{\lambda_{\nu,k}}{\nu}\|g_0\|^2 = \left(\frac{|k|}{\nu}\right)^{\frac{2}{m+2}} \|g_0\|^2.$$

In turn, from the definition of α, β, γ in (3.23) and (3.25) (or in (3.29) for $m = 1$), the above inequality can be rewritten as

$$\alpha\|\nabla g(t_0)\|^2 \leq \frac{\beta_0^{1/2}}{2}\|g_0\|^2. \tag{3.30}$$

Since ∇v is bounded on Ω and $t \mapsto \|g(t)\|^2$ is decreasing by (3.4), we infer from (3.9) that

$$\Phi(t_0) \leq \frac{1}{8}\left[4\|g(t_0)\|^2 + 5\alpha\|\nabla g(t_0)\|^2 + 5\gamma k^2\|g(t_0)\nabla v\|^2\right] \leq K_0(1 + \gamma k^2)\|g_0\|^2, \tag{3.31}$$

for some constant $K_0 > 0$ which is independent on ν, k. Hence, for any $t \geq T_{\nu,k}$, the differential inequality (3.27) implies that

$$\frac{1}{2}\|g(t)\|^2 \leq \Phi(t) \leq \mathrm{e}^{-16c_v\beta_0^{3/2}\lambda_{\nu,k}(t-t_0)}\Phi(t_0) \leq K_0(1 + \gamma k^2)\,\mathrm{e}^{16c_v\beta_0^{3/2}}\mathrm{e}^{-16c_v\beta_0^{3/2}\lambda_{\nu,k}t}\|g_0\|^2. \tag{3.32}$$

In terms of powers of ν and k, we have that

$$\beta \sim \frac{\nu^{\frac{2-m}{m+2}}}{|k|^{\frac{4}{m+2}}}, \qquad \gamma \sim \frac{\nu^{\frac{2(1-m)}{m+2}}}{|k|^{\frac{6}{m+2}}} \qquad \Longrightarrow \qquad \gamma k^2 \sim \left(\frac{|k|}{\nu}\right)^{\frac{2(m-1)}{m+2}},$$

so that (3.32) gives the semigroup estimate (3.3) for $t \geq T_{\nu,k}$. Since inequality (3.3) is trivially satisfied when $t < T_{\nu,k}$, the proof is complete in the enhanced dissipation regime.

In the Taylor dispersion regime, the argument is analogous, and in fact more elementary due to the simple scaling of α, β, γ. Indeed, since $\alpha \sim 1$ and $\gamma \sim \nu^{-2}$, we only have to replace (3.30) and (3.31) by

$$\alpha\|\nabla g(t_0)\|^2 \leq \frac{\beta_1^{1/2}}{2}\left(\frac{k}{\nu}\right)^2\|g_0\|^2, \qquad \Phi(t_0) \leq K_0\left(1 + \left(\frac{k}{\nu}\right)^2\right)\|g_0\|^2, \tag{3.33}$$

respectively. The rest of the argument is exactly the same as before, and leads to (3.3). $\square$

4. Conclusions

The results of this paper emphasize the link between enhanced dissipation and Taylor dispersion in parallel shear flows, thereby demonstrating that these phenomena, which have a common origin, can be analyzed using the same mathematical tools. In the Taylor dispersion regime, an optimal decay estimate is obtained under very general assumptions on the shear velocity v, see Theorem 1.3, but our approach does not give the classical formula for the effective diffusion constant in the asymptotic regime where $|k| \ll \nu$. That formula can be established using homogenization theory [30, 33], see also [4] for a rigorous proof based on center manifold theory. In the enhanced dissipation regime, which requires more precise assumptions, we obtain (to our knowledge) the first general result concerning the higher-dimensional case, see Theorem 1.2. For simplicity, we suppose that the shear velocity is a Morse function, but it is clear that our methods can be extended to more general situations, some examples of which are given in Section 2.3. When the cross-section of the domain is one-dimensional, we only need to suppose that the critical points of v are non-degenerate in the sense of (1.6), and we recover the main conclusions of the previous works [7, 22].

A notable feature of our analysis is to provide for our main results two different proofs, which are based either on L^2 resolvent estimates (Section 2) or on H^1 energy estimates (Section 3). Both methods have their own advantages and drawbacks, and it is worth drawing a little summary at this point.

i) The first approach, based on resolvent estimates for the generator of the linear evolution equation (1.5), is very general. It can be used even if the shear velocity is not smooth, see [38], and it is relatively insensitive to the choice of the boundary conditions. Thanks to the semigroup bounds recently obtained in [25, 38], it gives optimal decay estimates for the solutions of (1.5) in $L^2(\Omega)$, without the logarithmic corrections originating from the hypocoercivity method [7]. It relies entirely on standard techniques for the analysis of linear partial differential operators, and can therefore be applied to higher-order dissipative operators, involving for instance the bilaplacian. However, when the cross-section of our domain has dimension $d \geq 2$, Assumption 2.2 on the level sets of the shear velocity v is not easy to verify, and this is why we restrict ourselves to the relatively simple case of Morse functions.

ii) The second approach, inspired from Villani's work on hypocoercivity [37], has the advantage of dealing directly with the evolution equation, which makes it potentially applicable to nonlinear problems as well (see [9]), although this possibility has not been widely explored so far. It is based on rather elementary H^1 energy estimates, which however impose some restrictions concerning the boundary conditions. When the shear velocity v is a Morse function, it essentially relies on the standard semi-classical estimate (3.18), which is certainly easier to prove than (2.8) in the higher-dimensional case. But if v has degenerate critical points, the coefficients α, β, γ in the functional (3.1) have to be replaced by y-dependent functions, which makes the calculations more complicated. As a final remark, the estimates given by the hypocoercivity method are naturally expressed in terms of the H^1-type functional Φ, and logarithmic corrections may appear when translating them into ordinary L^2 estimates for the evolution equation (1.5).

Appendix A. On H^1-thin sets

Assumption 2.2 in Section 2 is closely related to a notion of "thinness" for subsets of $\mathbb{R}^d$ which seems quite natural, although we were not able to locate it in the literature. In this section, we introduce this notion and discuss a few elementary properties.

DEFINITION A.1. A set $E \subset \mathbb{R}^d$ is H^1*-thin* if there exist positive constants C and δ_0 such that, for all $\delta \in (0, \delta_0)$ and all $g \in H^1(\mathbb{R}^d)$, the following inequality holds:

$$\int_{E_\delta} g(x)^2 \,\mathrm{d}x \;\leq\; \frac{1}{2}\int_{\mathbb{R}^d} g(x)^2 \,\mathrm{d}x + C\delta^2 \int_{\mathbb{R}^d} |\nabla g(x)|^2 \,\mathrm{d}x\,, \tag{A.1}$$

where $E_\delta = \bigl\{x \in \mathbb{R}^d\,;\, \mathrm{dist}(x, E) < \delta\bigr\}$.

REMARK A.2. In particular, if we take $g \in H_0^1(E_\delta)$ in (A.1), we see that Poincaré's inequality holds in E_δ with constant $\sqrt{2C}\delta$, for all sufficiently small $\delta > 0$. It is not clear if this property is sufficient to characterize H^1-thin sets.

We first observe that the factor $1/2$ in (A.1) can be replaced by an arbitrary real number $\kappa \in (0,1)$ without altering the definition.

LEMMA A.3. *Fix any $\kappa \in (0,1)$. A set $E \subset \mathbb{R}^d$ is H^1-thin if and only if there exist positive constants C and δ_0 such that, for all $\delta \in (0,\delta_0)$ and all $g \in H^1(\mathbb{R}^d)$,*

$$\int_{E_\delta} g(x)^2 \,\mathrm{d}x \,\le\, \kappa \int_{\mathbb{R}^d} g(x)^2 \,\mathrm{d}x + C\delta^2 \int_{\mathbb{R}^d} |\nabla g(x)|^2 \,\mathrm{d}x\,. \tag{A.2}$$

PROOF. Increasing the value of κ obviously makes inequality (A.2) weaker. To prove Lemma A.3, we have to show that it is possible to *decrease* the value of κ in (A.2), at the expense of modifying the constants C and δ_0. To see that, assume that (A.2) holds for some $\kappa \in (0,1)$, and take $g \in H^1(\mathbb{R}^d)$. For any $N \in \mathbb{N}^*$, we have

$$\int_{E_{(N+1)\delta}} g(x)^2 \,\mathrm{d}x \,\le\, \kappa \int_{\mathbb{R}^d} g(x)^2 \,\mathrm{d}x + C(N{+}1)^2\delta^2 \int_{\mathbb{R}^d} |\nabla g(x)|^2 \,\mathrm{d}x\,, \tag{A.3}$$

provided $0 < \delta < \delta_0/(N+1)$. Define $f \in H^1(\mathbb{R}^d)$ by $f = \chi g$, where

$$\chi(x) \,=\, \phi\Big(\frac{\mathrm{dist}(x,E) - \delta}{N\delta}\Big)\,, \qquad \phi(t) \,=\, \begin{cases} 1 & \text{if } t \le 0 \\ 1-t & \text{if } 0 \le t \le 1 \\ 0 & \text{if } t \ge 1 \end{cases}.$$

Note that f vanishes outside $E_{(N+1)\delta}$, and coincides with g on E_δ. Applying (A.2) to f, we thus find

$$\begin{aligned} \int_{E_\delta} g(x)^2 \,\mathrm{d}x \,&=\, \int_{E_\delta} f(x)^2 \,\mathrm{d}x \,\le\, \kappa \int_{\mathbb{R}^d} f(x)^2 \,\mathrm{d}x + C\delta^2 \int_{\mathbb{R}^d} |\nabla f(x)|^2 \,\mathrm{d}x \\ &\le\, \kappa \int_{E_{(N+1)\delta}} g(x)^2 \,\mathrm{d}x + 2C\delta^2 \int_{E_{(N+1)\delta}} \big(|\nabla g|^2 + |\nabla\chi|^2 g^2\big) \,\mathrm{d}x\,. \end{aligned} \tag{A.4}$$

Since $|\nabla\chi| \le 1/(N\delta)$, we deduce from (A.3), (A.4) that

$$\int_{E_\delta} g(x)^2 \,\mathrm{d}x \,\le\, \Big(\kappa^2 + \frac{2C}{N^2}\Big) \int_{\mathbb{R}^d} g(x)^2 \,\mathrm{d}x + C(N)\delta^2 \int_{\mathbb{R}^d} |\nabla g(x)|^2 \,\mathrm{d}x\,,$$

for some constant $C(N)$ independent of δ. If we take N large enough, the coefficient in front of the first integral in the right-hand side can be made as close as we wish to $\kappa^2 < \kappa$. Repeating the argument, we see that this coefficient can be made arbitrarily small. □

It is clear from the definition that, if $E \subset \mathbb{R}^d$ is H^1-thin, then any subset $F \subset E$ is H^1-thin a fortiori. Also, using Lemma A.3, it is easy to verify that a finite union of H^1-thin sets is H^1-thin. Indeed, if E, F are arbitrarily measurable subsets of $\mathbb{R}^d$ we have, for all $g \in H^1(\mathbb{R}^d)$ and all $\delta > 0$,

$$\int_{(E\cup F)_\delta} g^2 \,\mathrm{d}x \,=\, \int_{E_\delta \cup F_\delta} g^2 \,\mathrm{d}x \,\le\, \int_{E_\delta} g^2 \,\mathrm{d}x + \int_{F_\delta} g^2 \,\mathrm{d}x\,.$$

If E, F are H^1-thin, both integrals in the right-hand side can be estimated as in (A.2) with $\kappa = 1/4$, which yields inequality (A.1) for $E \cup F$. A less immediate property is stated in the following lemma.

LEMMA A.4. *If $E \subset \mathbb{R}^d$ is measurable and H^1-thin, then E has zero Lebesgue measure.*

PROOF. We can assume without loss of generality that E is bounded, so that $|E| < \infty$. Given any $\epsilon > 0$ we define $g_\epsilon = \mathbf{1}_E * \chi_\epsilon$, where $\mathbf{1}_E$ is the characteristic function of E and χ_ϵ is a standard approximation of unity:

$$\chi_\epsilon(x) \,=\, \frac{1}{\epsilon^d}\,\chi\Big(\frac{x}{\epsilon}\Big)\,, \qquad \chi \in C_c^\infty(\mathbb{R}^d)\,, \qquad \int_{\mathbb{R}^d} \chi(x)\,\mathrm{d}x \,=\, 1\,.$$

Since E is H^1-thin and $g_\epsilon \in H^1(\mathbb{R}^d)$, inequality (A.1) shows that, for any small $\delta > 0$,

$$\int_E g_\epsilon(x)^2 \,\mathrm{d}x \;\le\; \int_{E_\delta} g_\epsilon(x)^2 \,\mathrm{d}x \;\le\; \frac{1}{2}\int_{\mathbb{R}^d} g_\epsilon(x)^2 \,\mathrm{d}x + C\delta^2 \int_{\mathbb{R}^d} |\nabla g_\epsilon(x)|^2 \,\mathrm{d}x\,.$$

Therefore, taking the limit $\delta \to 0$, we obtain

$$\int_E g_\epsilon(x)^2 \,\mathrm{d}x \;\le\; \frac{1}{2}\int_{\mathbb{R}^d} g_\epsilon(x)^2 \,\mathrm{d}x\,. \tag{A.5}$$

Now, in the limit $\epsilon \to 0$, we have $g_\epsilon \to \mathbf{1}_E$ in $L^2(\mathbb{R}^d)$, so that both integrals in (A.5) converge to the same value $|E|$. We thus obtain the inequality $|E| \le |E|/2$, which implies that $|E| = 0$. $\square$

The following lemma is useful to construct concrete examples of H^1-thin sets.

LEMMA A.5. *The graph of any Lipschitz function $h : \mathbb{R}^{d-1} \to \mathbb{R}$ is H^1-thin in $\mathbb{R}^d$.*

PROOF. Let $E = \bigl\{(x, h(x)) \in \mathbb{R}^d\,;\, x \in \mathbb{R}^{d-1}\bigr\}$ be the graph of h, and let M be a Lipschitz constant of h. We first observe that, for any $\delta > 0$, we have the inclusion

$$E_\delta \,\subset\, \Gamma_\delta \,:=\, \bigl\{(x,y) \in \mathbb{R}^d\,;\, x \in \mathbb{R}^{d-1}\,,\; |y - h(x)| < N\delta\bigr\}\,, \tag{A.6}$$

where $N = (1+M^2)^{1/2}$. Indeed, for all $x_1, x_2 \in \mathbb{R}^{d-1}$ and all $z \in \mathbb{R}$, we have

$$\begin{aligned}\bigl|(x_1, h(x_1)+z) - (x_2, h(x_2))\bigr|^2 \;&=\; |x_1 - x_2|^2 + |z + h(x_1) - h(x_2)|^2 \\ &\ge\; |x_1 - x_2|^2 + \bigl(|z| - M|x_1 - x_2|\bigr)_+^2 \;\ge\; \frac{z^2}{1+M^2}\,,\end{aligned}$$

where the last inequality is obvious if $|z| \le M|x_1 - x_2|$, and can be obtained by minimizing the function $a \mapsto a^2 + (|z| - Ma)^2$ in the converse case. Taking the infimum over $x_2 \in \mathbb{R}^{d-1}$, we obtain the estimate

$$\mathrm{dist}\Bigl((x_1, h(x_1)+z)\,,\, E\Bigr) \;\ge\; \frac{|z|}{\sqrt{1+M^2}}\,, \qquad \forall\, x_1 \in \mathbb{R}^{d-1}\,,\; \forall\, z \in \mathbb{R}\,,$$

which in turn implies (A.6). Now, if $g \in C_c^1(\mathbb{R}^d)$, we have for any $x \in \mathbb{R}^{d-1}$:

$$\int_{h(x)-N\delta}^{h(x)+N\delta} g(x,y)^2 \,\mathrm{d}y \;\le\; 2N\delta \sup_{y\in\mathbb{R}} g(x,y)^2 \;\le\; 2N\delta \biggl(\int_{\mathbb{R}} g(x,y)^2 \,\mathrm{d}y\biggr)^{1/2} \biggl(\int_{\mathbb{R}} \partial_y g(x,y)^2 \,\mathrm{d}y\biggr)^{1/2}\,,$$

where we used the bound $\|f\|_{L^\infty}^2 \le \|f\|_{L^2}\|f'\|_{L^2}$ which holds for all $f \in H^1(\mathbb{R})$. Integrating both sides over $x \in \mathbb{R}^{d-1}$ and using Schwarz's inequality together with (A.6), we arrive at

$$\int_{E_\delta} g(x)^2 \,\mathrm{d}x\,\mathrm{d}y \;\le\; \int_{\Gamma_\delta} g(x)^2 \,\mathrm{d}x\,\mathrm{d}y \;\le\; 2N\delta\, \|g\|_{L^2} \|\nabla g\|_{L^2}\,.$$

By density, this bound remains valid for all $g \in H^1(\mathbb{R}^d)$, and (A.1) then follows from Young's inequality. $\square$

It is clear from Definition A.1 that the family of H^1-thin sets is invariant under the action of the Euclidean group in $\mathbb{R}^d$. It is also easy to verify that H^1-thin sets are stable under dilations. Combining these observations with Lemma A.5, we conclude that any submanifold S of $\mathbb{R}^d$ of nonzero codimension is H^1-thin. More generally, any m-rectifiable set $E \subset \mathbb{R}^d$ with $m \le d-1$ is H^1-thin.

Appendix B. Geometric lemmas

In this section we collect some basic estimates for levels sets of Morse functions near critical points, which are used in Section 2.2. We assume henceforth that the space dimension d is at least 2. Our starting point is:

LEMMA B.1. *For all $g \in H^1(\mathbb{R}^d)$ and all $R_2 \ge R_1 \ge 0$, we have*

$$\int_{R_1 \le |x| \le R_2} g(x)^2 \,\mathrm{d}x \;\le\; 2(R_2 - R_1)\, \|g\|_{L^2}\, \|\nabla g\|_{L^2}\,. \tag{B.1}$$

PROOF. We first prove (B.1) in the particular case where $g \in C^1_c(\mathbb{R}^d)$ and g is *radially symmetric*. Under those assumptions, we can integrate by parts and obtain, for any $r > 0$,

$$-\int_r^\infty 2g(s)g'(s)s^{d-1}\,\mathrm{d}s \;=\; g(r)^2 r^{d-1} + (d-1)\int_r^\infty g(s)^2 s^{d-2}\,\mathrm{d}s\,.$$

Using Schwarz's inequality, we deduce

$$g(r)^2 r^{d-1} + (d-1)\int_r^\infty g(s)^2 s^{d-2}\,\mathrm{d}s \;\le\; 2\Big(\int_r^\infty g(s)^2 s^{d-1}\,\mathrm{d}s\Big)^{1/2}\Big(\int_r^\infty g'(s)^2 s^{d-1}\,\mathrm{d}s\Big)^{1/2}.$$

In particular, we have

$$A_d\, g(r)^2\, r^{d-1} \;\le\; 2\,\|g\|_{L^2}\,\|\nabla g\|_{L^2}\,, \qquad \forall\, r > 0\,,$$

where $A_d = 2\pi^{d/2}\Gamma(d/2)^{-1}$ is the area of the unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$. Integrating both sides over the interval $[R_1, R_2]$, we obtain the desired inequality (B.1).

For a general function $g \in C^1_c(\mathbb{R}^d)$, we introduce polar coordinates $x = r\omega$ and use the decomposition

$$g(r\omega) \;=\; \sum_{n\in\mathbb{N}} g_n(r)\, Y_n(\omega)\,, \qquad r \in \mathbb{R}_+\,, \quad \omega \in \mathbb{S}^{d-1}\,,$$

where the spherical harmonics $Y_n(\omega)$ are eigenfunctions of the Laplace-Beltrami operator on $\mathbb{S}^{d-1}$, and are normalized so that the family $(Y_n)_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{S}^{d-1})$. Using Parseval's identity and the previous step, we deduce that

$$\begin{aligned}\int_{R_1\le|x|\le R_2} g(x)^2\,\mathrm{d}x \;&=\; \sum_{n\in\mathbb{N}}\int_{R_1\le|x|\le R_2} g_n(|x|)^2\,\mathrm{d}x \;\le\; 2(R_2-R_1)\sum_{n\in\mathbb{N}}\|g_n\|_{L^2}\|g_n'\|_{L^2}\\ &\le\; 2(R_2-R_1)\Big(\sum_{n\in\mathbb{N}}\|g_n\|_{L^2}^2\Big)^{1/2}\Big(\sum_{n\in\mathbb{N}}\|g_n'\|_{L^2}^2\Big)^{1/2} \;\le\; 2(R_2-R_1)\,\|g\|_{L^2}\,\|\nabla g\|_{L^2}\,,\end{aligned}$$

because $\|g\|_{L^2}^2 = \sum_{n\in\mathbb{N}}\|g_n\|_{L^2}^2$ and $\|\nabla g\|_{L^2}^2 \ge \sum_{n\in\mathbb{N}}\|g_n'\|_{L^2}^2$. This proves (B.1) for all $g \in C^1_c(\mathbb{R}^d)$, and the general case follows by density. □

Now, let $v : \mathbb{R}^d \to \mathbb{R}$ be a smooth function and $m \in \mathbb{N}^*$ a nonzero integer. In analogy with (2.6), (2.7), we define, for all $\lambda \in \mathbb{R}$ and all $\delta > 0$,

$$E^m_{\lambda,\delta} \;=\; \big\{x \in \mathbb{R}^d\,;\, |v(x)-\lambda| < \delta^m\big\}\,, \qquad \mathcal{E}^m_{\lambda,\delta} \;=\; \big\{x \in \mathbb{R}^d\,;\, \mathrm{dist}(x, E^m_{\lambda,\delta}) < \delta\big\}\,. \tag{B.2}$$

LEMMA B.2. *Assume that $v(x) = |x|^2$ for all $x \in \mathbb{R}^d$. Then for any $\lambda \in \mathbb{R}$, any $\delta > 0$, and any $g \in H^1(\mathbb{R}^d)$, we have*

$$\int_{E^2_{\lambda,\delta}} g(x)^2\,\mathrm{d}x \;\le\; 2\sqrt{2}\,\delta\,\|g\|_{L^2}\,\|\nabla g\|_{L^2}\,, \qquad \int_{\mathcal{E}^2_{\lambda,\delta}} g(x)^2\,\mathrm{d}x \;\le\; 2(1{+}\sqrt{3})\,\delta\,\|g\|_{L^2}\,\|\nabla g\|_{L^2}\,. \tag{B.3}$$

PROOF. Since $v(x) = |x|^2$, the definition (B.2) implies that $E^2_{\lambda,\delta} \subset \{x \in \mathbb{R}^d\,;\, R_1 \le |x| < R_2\}$ where $R_1 = (\lambda-\delta^2)_+^{1/2}$ and $R_2 = (\lambda+\delta^2)_+^{1/2}$. Considering three cases according to whether $\lambda \le -\delta^2$,

$\lambda \in (-\delta^2, \delta^2)$, or $\lambda \ge \delta^2$, it is straightforward to verify that $R_2 - R_1 \le \sqrt{2}\,\delta$ in all situations, hence (B.1) gives the first inequality in (B.3). The same argument applies to $\mathcal{E}^2_{\lambda,\delta}$ if we define

$$R_1 \,=\, \bigl((\lambda-\delta^2)_+^{1/2} - \delta\bigr)_+\,, \qquad R_2 \,=\, (\lambda+\delta^2)_+^{1/2} + \delta\,.$$

Again considering all possible cases, we find that $R_2 - R_1 \le (1+\sqrt{3})\delta$, and the second inequality in (B.3) follows in the same way. □

LEMMA B.3. *Assume that $d = d_1 + d_2$ with $d_1, d_2 \ge 1$, and that $v(x) = |y|^2 - |z|^2$ for all $x = (y,z) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Then for any $\lambda \in \mathbb{R}$, any $\delta > 0$, and any $g \in H^1(\mathbb{R}^d)$, we have*

$$\int_{E^2_{\lambda,\delta}} g(x)^2 \,\mathrm{d}x \,\le\, 2\sqrt{2}\,\delta\,\|g\|_{L^2}\,\|\nabla g\|_{L^2}\,, \qquad \int_{\mathcal{E}^2_{\lambda,\delta}} g(x)^2 \,\mathrm{d}x \,\le\, 4(2+\sqrt{2})\,\delta\,\|g\|_{L^2}\,\|\nabla g\|_{L^2}\,. \tag{B.4}$$

PROOF. We have by definition

$$E^2_{\lambda,\delta} \,=\, \bigl\{(y,z) \in \mathbb{R}^d\,;\, |z|^2 + \lambda - \delta^2 < |y|^2 < |z|^2 + \lambda + \delta^2\bigr\}\,.$$

It follows that $E^2_{\lambda,\delta} \subset \{(y,z); R_1(z) \le |y| < R_2(z)\}$ where

$$R_2(z) \,=\, \bigl(|z|^2 + \lambda + \delta^2\bigr)_+^{1/2}\,, \qquad R_1(z) \,=\, \bigl(|z|^2 + \lambda - \delta^2\bigr)_+^{1/2}\,.$$

As before we have $R_2(z) - R_1(z) \le \sqrt{2}\,\delta$. Thus applying Lemma B.1 and Fubini's theorem, we obtain

$$\begin{aligned}
\int_{E^2_{\lambda,\delta}} g(y,z)^2 \,\mathrm{d}y\,\mathrm{d}z \,&\le\, \int_{\mathbb{R}^{d_2}}\biggl(\int_{R_1(z)\le|y|\le R_2(z)} g(y,z)^2\,\mathrm{d}y\biggr)\mathrm{d}z \\
&\le\, 2\sqrt{2}\,\delta \int_{\mathbb{R}^{d_2}}\biggl(\int_{\mathbb{R}^{d_1}} g(y,z)^2\,\mathrm{d}y\biggr)^{1/2}\biggl(\int_{\mathbb{R}^{d_1}} |\nabla_y g(y,z)|^2\,\mathrm{d}y\biggr)^{1/2}\mathrm{d}z \\
&\le\, 2\sqrt{2}\,\delta\,\|g\|_{L^2}\,\|\nabla g\|_{L^2}\,,
\end{aligned}$$

which is the first inequality in (B.4).

The proof of the second inequality is slightly more complicated. If $(y,z) \in \mathcal{E}^2_{\lambda,\delta}$, then by definition there exists $(\tilde y, \tilde z) \in E^2_{\lambda,\delta}$ such that $|y-\tilde y|^2 + |z - \tilde z|^2 < \delta^2$. Let $\mu = |\tilde y|^2 - |\tilde z|^2 \in (\lambda - \delta^2, \lambda + \delta^2)$. If $\mu \ge 0$ we have $|\tilde y| = \sqrt{\mu + |\tilde z|^2}$, hence

$$\begin{aligned}
\Bigl||y| - \sqrt{\mu + |z|^2}\Bigr| \,&\le\, \Bigl||y| - |\tilde y|\Bigr| + \Bigl||\tilde y| - \sqrt{\mu + |\tilde z|^2}\Bigr| + \Bigl|\sqrt{\mu+|\tilde z|^2} - \sqrt{\mu + |z|^2}\Bigr| \\
&\le\, |y - \tilde y| + |z - \tilde z| \,<\, \sqrt{2}\,\delta\,.
\end{aligned}$$

A similar argument shows that $\bigl||z| - \sqrt{|\mu| + |y|^2}\bigr| < \sqrt{2}\,\delta$ if $\mu \le 0$. Thus $\mathcal{E}^2_{\lambda,\delta} \subset F_{\lambda,\delta} \cup G_{\lambda,\delta}$ where

$$\begin{aligned}
F_{\lambda,\delta} \,&=\, \Bigl\{(y,z)\,;\, \bigl||y| - \sqrt{\mu + |z|^2}\bigr| < \sqrt{2}\,\delta \text{ for some } \mu \ge 0 \text{ with } |\mu - \lambda| < \delta^2\Bigr\}\,, \\
G_{\lambda,\delta} \,&=\, \Bigl\{(y,z)\,;\, \bigl||z| - \sqrt{|\mu| + |y|^2}\bigr| < \sqrt{2}\,\delta \text{ for some } \mu \le 0 \text{ with } |\mu - \lambda| < \delta^2\Bigr\}\,.
\end{aligned}$$

We now distinguish three cases.

Case 1 : $\lambda \ge \delta^2$. Then $G_{\lambda,\delta} = \emptyset$ and $F_{\lambda,\delta} \subset \{(y,z); R_1(z) \le |y| < R_2(z)\}$ where

$$R_1(z) \,=\, \bigl((\lambda - \delta^2 + |z|^2)^{1/2} - \sqrt{2}\,\delta\bigr)_+\,, \qquad R_2(z) \,=\, (\lambda + \delta^2 + |z|^2)^{1/2} + \sqrt{2}\,\delta\,.$$

It is easy to verify that $R_2(z) - R_1(z) \le (2+\sqrt{2})\delta$, hence proceeding as above we find

$$\int_{\mathcal{E}^2_{\lambda,\delta}} g(x)^2\,\mathrm{d}x \,\le\, \int_{F_{\lambda,\delta}} g(x)^2\,\mathrm{d}x \,\le\, 2(2+\sqrt{2})\,\delta\,\|g\|_{L^2}\,\|\nabla g\|_{L^2}\,.$$

Case 2 : $\lambda \le -\delta^2$. Then $F_{\lambda,\delta} = \emptyset$ and a similar argument shows that

$$\int_{\mathcal{E}^2_{\lambda,\delta}} g(x)^2 \,\mathrm{d}x \;\le\; \int_{G_{\lambda,\delta}} g(x)^2 \,\mathrm{d}x \;\le\; 2(2+\sqrt{2})\,\delta\, \|g\|_{L^2}\, \|\nabla g\|_{L^2}\,.$$

Case 3 : $-\delta^2 < \lambda < \delta^2$. Here both sets $F_{\lambda,\delta}, G_{\lambda,\delta}$ are nonempty, and must be considered. We first observe that $F_{\lambda,\delta} \subset \{(y,z);\, R_1(z) \le |y| < R_2(z)\}$ where

$$R_1(z) \;=\; \big((\lambda-\delta^2+|z|^2)_+^{1/2} - \sqrt{2}\,\delta\big)_+\,, \qquad R_2(z) \;=\; (\lambda+\delta^2+|z|^2)^{1/2} + \sqrt{2}\,\delta\,.$$

One verifies that $R_2(z) - R_1(z) \le (2+\sqrt{2})\,\delta$, and it follows that

$$\int_{F_{\lambda,\delta}} g(x)^2 \,\mathrm{d}x \;\le\; 2(2+\sqrt{2})\,\delta\, \|g\|_{L^2}\, \|\nabla g\|_{L^2}\,.$$

A similar argument gives the same estimate for the integral over $G_{\lambda,\delta}$, and since $\mathcal{E}^2_{\lambda,\delta} \subset F_{\lambda,\delta} \cup G_{\lambda,\delta}$ we arrive at the second estimate in (B.4) in all cases. □

References

[1] D. Albritton, R. Beekie, and M. Novack, *Enhanced dissipation and Hörmander's hypoellipticity*, arXiv e-prints (May 2021), available at 2105.12308.

[2] R Aris, *On the dispersion of a solute in a fluid flowing through a tube*, Proc. Roy. Soc. London A **235** (1956), 67–77.

[3] K. Bajer, A. P. Bassom, and A. D. Gilbert, *Accelerated diffusion in the centre of a vortex*, J. Fluid Mech. **437** (2001), 395–411.

[4] M. Beck, O. Chaudhary, and C. E. Wayne, *Rigorous justification of Taylor dispersion via center manifolds and hypocoercivity*, Arch. Ration. Mech. Anal. **235** (2020), no. 2, 1105–1149.

[5] M. Beck and C. E. Wayne, *Metastability and rapid convergence to quasi-stationary bar states for the two-dimensional Navier-Stokes equations*, Proc. Roy. Soc. Edinburgh Sect. A **143** (2013), no. 5, 905–927.

[6] J. Bedrossian, A. Blumenthal, and S. Punshon-Smith, *Almost-sure enhanced dissipation and uniform-in-diffusivity exponential mixing for advection-diffusion by stochastic Navier-Stokes*, Probab. Theory Related Fields **179** (2021), no. 3-4, 777–834.

[7] J. Bedrossian and M. Coti Zelati, *Enhanced dissipation, hypoellipticity, and anomalous small noise inviscid limits in shear flows*, Arch. Ration. Mech. Anal. **224** (2017), no. 3, 1161–1204.

[8] J. Bedrossian, P. Germain, and N. Masmoudi, *On the stability threshold for the 3D Couette flow in Sobolev regularity*, Ann. of Math. (2) **185** (2017), no. 2, 541–608.

[9] J. Bedrossian and S. He, *Suppression of blow-up in Patlak-Keller-Segel via shear flows*, SIAM J. Math. Anal. **49** (2017), no. 6, 4722–4766.

[10] J. Bedrossian, V. Vicol, and F. Wang, *The Sobolev stability threshold for 2D shear flows near Couette*, J. Nonlinear Sci. **28** (2018), no. 6, 2051–2075.

[11] R Camassa, R McLaughlin, and C Viotti, *Analysis of passive scalar advection in parallel shear flows: Sorting of modes at intermediate time scales*, Phys. Fluids **22** (2010), 117103.

[12] Q. Chen, D. Wei, and Z. Zhang, *Linear stability of pipe Poiseuille flow at high Reynolds number regime*, arXiv e-prints (Oct. 2019), available at 1910.14245.

[13] M. Colombo, M. Coti Zelati, and K. Widmayer, *Mixing and diffusion for rough shear flows*, Ars Inveniendi Analytica (2021), Paper No. 2, 21pp.

[14] P. Constantin, A. Kiselev, L. Ryzhik, and A. Zlatoš, *Diffusion and mixing in fluid flow*, Ann. of Math. (2) **168** (2008), no. 2, 643–674.

[15] M. Coti Zelati, *Stable mixing estimates in the infinite Péclet number limit*, J. Funct. Anal. **279** (2020), no. 4, 108562, 25.

[16] M. Coti Zelati, M. G. Delgadino, and T. M. Elgindi, *On the relation between enhanced dissipation timescales and mixing rates*, Comm. Pure Appl. Math. **73** (2020), no. 6, 1205–1244.

[17] M. Coti Zelati and T. D. Drivas, *A stochastic approach to enhanced diffusion*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **22** (2021), 811–834.

[18] M. Coti Zelati, T. M. Elgindi, and K. Widmayer, *Enhanced dissipation in the Navier-Stokes equations near the Poiseuille flow*, Comm. Math. Phys. **378** (2020), no. 2, 987–1010.

[19] M. Coti Zelati and G. A. Pavliotis, *Homogenization and hypocoercivity for Fokker-Planck equations driven by weakly compressible shear flows*, IMA J. Appl. Math. **85** (2020), no. 6, 951–979.

[20] A. Fannjiang and G. Papanicolaou, *Convection enhanced diffusion for periodic flows*, SIAM J. Appl. Math. **54** (1994), no. 2, 333–408.

[21] A. Fannjiang and G. Papanicolaou, *Convection-enhanced diffusion for random flows*, J. Statist. Phys. **88** (1997), no. 5-6, 1033–1076.

[22] I. Gallagher, T. Gallay, and F. Nier, *Spectral asymptotics for large skew-symmetric perturbations of the harmonic oscillator*, Int. Math. Res. Not. IMRN **12** (2009), 2147–2199.

[23] E. Grenier, T. T. Nguyen, F. Rousset, and A. Soffer, *Linear inviscid damping and enhanced viscous dissipation of shear flows by using the conjugate operator method*, J. Funct. Anal. **278** (2020), no. 3, 108339, 27.

[24] B. Helffer and J. Sjöstrand, *Puits multiples en mécanique semi-classique. VI. Cas des puits sous-variétés*, Ann. Inst. H. Poincaré Phys. Théor. **46** (1987), no. 4, 353–372.

[25] B. Helffer and J. Sjöstrand, *Improving semigroup bounds with resolvent estimates*, Integral Equations Operator Theory **93** (2021), no. 3, Paper No. 36, 41.

[26] B. Helffer and F. Nier, *Hypoelliptic estimates and spectral theory for Fokker-Planck operators and Witten Laplacians*, Lecture Notes in Mathematics, vol. 1862, Springer-Verlag, Berlin, 2005.

[27] L. Hörmander, *The analysis of linear partial differential operators. I*, Classics in Mathematics, Springer-Verlag, Berlin, 2003.

[28] S. Ibrahim, Y. Maekawa, and N. Masmoudi, *On pseudospectral bound for non-selfadjoint operators and its application to stability of Kolmogorov flows*, Ann. PDE **5** (2019), no. 2, Paper No. 14, 84.

[29] T. Kato, *Perturbation theory for linear operators*, Classics in Mathematics, Springer-Verlag, Berlin, 1995.

[30] A. J. Majda and P. R. Kramer, *Simplified models for turbulent diffusion: theory, numerical modelling, and physical phenomena*, Phys. Rep. **314** (1999), no. 4-5, 237–574.

[31] G. N. Mercer and A. J. Roberts, *A centre manifold description of contaminant dispersion in channels with varying flow properties*, SIAM J. Appl. Math. **50** (1990), no. 6, 1547–1565.

[32] J. Milnor, *Morse theory*, Annals of Mathematics Studies, No. 51, Princeton University Press, Princeton, N.J., 1963.

[33] G. A. Pavliotis and A. M. Stuart, *Multiscale methods*, Texts in Applied Mathematics, vol. 53, Springer, New York, 2008.

[34] P. B. Rhines and W. R. Young, *How rapidly is a passive scalar mixed within closed streamlines?*, Journal of Fluid Mechanics **133** (1983), 133–145.

[35] G. I. Taylor, *Dispersion of soluble matter in solvent flowing slowly through a tube*, Proc. Roy. Soc. London A **219** (1953), 186–203.

[36] G. I. Taylor, *Dispersion of matter in turbulent flow through a tube*, Proc. Roy. Soc. London A **223** (1954), 446–468.

[37] C. Villani, *Hypocoercivity*, Mem. Am. Math. Soc. **202** (2009), iv+141 pp.

[38] D. Wei, *Diffusion and mixing in fluid flow via the resolvent estimate*, Sci. China Math. **64** (2021), no. 3, 507–518.

[39] D. Wei, Z. Zhang, and W. Zhao, *Linear inviscid damping and enhanced dissipation for the Kolmogorov flow*, Adv. Math. **362** (2020), 106963, 103.

[40] W. R. Young and S. Jones, *Shear dispersion*, Physics of Fluids A: Fluid Dynamics **3** (1991), no. 5, 1087–1101.

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