Federated Learning via Inexact ADMM

Shenglong Zhou and Geoffrey Ye Li ITP Lab, Department of EEE Imperial College London, United Kingdom Emails:{shenglong.zhou, geoffrey.li}@imperial.ac.uk

Abstract: One of the crucial issues in federated learning is how to develop efficient optimization algorithms. Most of the current ones require full devices participation and/or impose strong assumptions for convergence. Different from the widely-used gradient descent-based algorithms, this paper develops an inexact alternating direction method of multipliers (ADMM), which is both computation and communication-efficient, capable of combating the stragglers' effect, and convergent under mild conditions.

Keywords: Partial device participation, inexact ADMM, communication and computationefficiency, global convergence

1 Introduction

Federated learning (FL), originated from [1, 2], gains its popularity recently due to its ability to address extensive applications, such as vehicular communications [3, 4, 5, 6], digital health [7], and mobile edge and over-the-air computing [8, 9, 10, 11], but is facing many challenges. One of them is how to develop efficient optimization algorithms for different purposes, such as saving communication resources, accelerating the learning process, coping with the stragglers' effect, preserving data privacy, and so on. We refer to some nice surveys [12, 13, 14] for more challenges.

1.1 Prior arts

It is known that FL trains a model across multiple decentralized edge devices (or clients) holding local data, without exchanging them. The training is usually coordinated by a central server that chooses which devices to participate in the training and collects parameters from them for averaging periodically. Based on this, FL algorithms can be categorized into two groups: full and partial device participation.

a) Full device participation. There is an impressive body of work on developing algorithms based on full device participation. The stochastic gradient descent (SGD) is one of the most extensively used schemes [15, 16, 17, 18, 19]. As gradients are constructed using randomly chosen data, SGD often assumes the local data is identically and independently distributed (i.i.d.) to establish the convergence theory, which is unrealistic for FL settings where data distributions are usually heterogeneous. A parallel line of research aims to investigate non-stochastic gradient descent-based algorithms, namely, all data is used to construct the gradient. Thanks to this, these algorithms do not need assumptions on data distributions but still impose restrictive assumptions on the learning models [20, 21, 22, 23, 24].

In addition to gradient descent-based frameworks, algorithms from primal-dual perspective have also drawn much attention. Typical representative is the popular ADMM. It has two versions: exact and inexact ADMM. The former requests clients to update their parameters through solving sub-problems exactly, which hence incurs expensive computational cost [25, 26, 27, 28, 29, 30, 31]. Therefore, inexact ADMM is an alternative to reduce the computational complexity for clients [32, 33, 34, 35] as they can update their parameters via solving sub-problems approximately.

We would like to point out that most of these algorithms not only require full device participation but also impose relatively strong assumptions on the model to establish the convergence properties. Common assumptions include the gradient Lipschitz continuity (also known as L-smoothness), strong smoothness, convexity, or strong convexity.

b) Partial device participation. As stated in [36], full device participation in the training at each step suffers from the so-called "stragglers' effect" (which means everyone waits for the slowest) in real-world applications. Thus it is more realistic for FL algorithms to select partial devices to join in the training at each step. Motivated by this, FedAvg, a state-of-the-art algorithm for FL, has been proposed in [37] and its convergence has been established under assumptions of strongly convexity and smoothness in [36]. Recently, FedProx has been developed in [38] to tackle the statistical heterogeneity. We note that both algorithms were designed from the primal perspective. Hence, it is interesting to see the performance of a primal-dual algorithm (e.g., ADMM) for FL using a partial device participation strategy.

1.2 Our contributions

The main contribution of this paper is to develop an inexact ADMM-based FL algorithm (FedADMM, see Algorithm 1) with the following advantages.

a) Communication and computation efficiency. The framework states the global averaging occurs only at certain steps (e.g., at step k that is a multiple of a pre-defined integer k_0). This means the communication rounds (CR) can be affected by setting a proper k_0 . It is shown that the larger k_0 the fewer CR for the algorithm to converge, see Figure 2. In addition to the communication efficiency, FedADMM suggests selected clients solve their sub-problems approximately with a flexible accuracy. In this regard, clients can relax the accuracy to reduce the computational complexity of solving their sub-problems.

b) Avoiding the stragglers' effect. In FedADMM, at each round of communication, the sever randomly divides all clients into two groups. One group adopts the inexact ADMM to update their parameters, while parameters in the second group remain unchanged, which means the server can put stragglers into the second group to avoid their impact on the training.

c) Convergence under mild conditions. We have proven that FedADMM converges to a stationary point, see Definition 4.1, of the learning optimization in (2.3) only under two mild conditions: gradient Lipschitz continuity and the coerciveness of the objective function, see Theorem 4.2. If we further assume the convexity, then it can achieve the optimal parameter, see Corollary 4.1.

d) *High numerical performance*. The numerical comparisons with two state-of-the-art algorithms have demonstrated that FedADMM can learn the parameter using the fewest CR and consuming the shortest computational time.

1.3 Organization

The paper is organized as follows. In the next section, we summarize all notations used in this paper and introduce ADMM and FL mathematically. In Section 3, we present algorithm FedADMM, followed by highlighting its advantages. We then establish its global convergence in Section 4 and conduct some numerical comparison in Section 5. Concluding remarks are given in the last section.

2 Preliminaries

This section presents the notation that will be employed throughout this paper, followed by introducing the framework of ADMM and FL mathematically.

2.1 Notation

We use plain, bold, and capital letters to present scalars, vectors, and matrices, respectively, e.g., k and σ are scalars, \mathbf{w} and π are vectors, W and Π are matrices. Let $\lfloor t \rfloor$ be the largest integer smaller than t + 1 (e.g., $\lfloor 1.1 \rfloor = \lfloor 2 \rfloor = 2$). Denote $[m] := \{1, 2, \dots, m\}$ with ':=' meaning define and \mathbb{R}^n the *n*-dimensional Euclidean space equipped with inner product $\langle \cdot, \cdot \rangle$ defined by $\langle \mathbf{w}, \mathbf{z} \rangle := \sum_i w_i z_i$. The 2-norm is written as $\|\cdot\|$, i.e., $\|\mathbf{w}\|^2 = \langle \mathbf{w}, \mathbf{w} \rangle$. Function f is said to be gradient Lipschitz continuous with a constant r > 0 if

$$\|\nabla f(\mathbf{w}) - \nabla f(\mathbf{z})\| \le r \|\mathbf{w} - \mathbf{z}\|$$
(2.1)

for any two vectors \mathbf{w} and \mathbf{z} , where $\nabla f(\mathbf{w})$ is the gradient of f with respect to \mathbf{w} . Hereafter, for two groups of vectors \mathbf{w}_i and π_i in \mathbb{R}^n , we denote

$$W:=(\mathbf{w}_1,\mathbf{w}_2,\cdots,\mathbf{w}_m), \quad \Pi:=(\boldsymbol{\pi}_1,\boldsymbol{\pi}_2,\cdots,\boldsymbol{\pi}_m).$$

Similar rules are also applied for W^k, W^*, W^∞ and Π^k, Π^*, Π^∞ . Here k, * and ∞ mean the iteration number, optimality and accumulation, e.g., see Corollary 4.1.

2.2 ADMM

We refer to the earliest work [39] and a nice book [40] for more details of ADMM and briefly introduce it as follows: Given an optimization problem,

$$\min_{\mathbf{w}\in\mathbb{R}^n,\mathbf{z}\in\mathbb{R}^q} f(\mathbf{w}) + g(\mathbf{z}), \text{ s.t. } A\mathbf{w} + B\mathbf{z} - \mathbf{b} = 0,$$

where $A \in \mathbb{R}^{p \times n}$, $B \in \mathbb{R}^{p \times q}$, and $\mathbf{b} \in \mathbb{R}^{p}$, its corresponding augmented Lagrange function is

$$\mathcal{L}(\mathbf{w}, \mathbf{z}, \boldsymbol{\pi}) := f(\mathbf{w}) + g(\mathbf{z}) + \langle A\mathbf{w} + B\mathbf{z} - \mathbf{b}, \boldsymbol{\pi} \rangle + \frac{\sigma}{2} \|A\mathbf{w} + B\mathbf{z} - \mathbf{b}\|^2,$$

where $\boldsymbol{\pi}$ is the Lagrange multiplier and σ is a given positive constant. Then starting with an initial point ($\mathbf{w}^0, \mathbf{z}^0, \boldsymbol{\pi}^0$), ADMM performs the following steps iteratively,

$$\begin{cases} \mathbf{w}^{k+1} = \operatorname{argmin}_{\mathbf{w} \in \mathbb{R}^n} \mathcal{L}(\mathbf{w}, \mathbf{z}^k, \boldsymbol{\pi}^k), \\ \mathbf{z}^{k+1} = \operatorname{argmin}_{\mathbf{z} \in \mathbb{R}^q} \mathcal{L}(\mathbf{w}^{k+1}, \mathbf{z}, \boldsymbol{\pi}^k), \\ \boldsymbol{\pi}^{k+1} = \boldsymbol{\pi}^k + \sigma(A\mathbf{w}^{k+1} + B\mathbf{z}^{k+1} - \mathbf{b}). \end{cases}$$
(2.2)

2.3 Federated learning

Given m local clients (or devices) with datasets $\{\mathcal{D}_1, \mathcal{D}_2, \cdots, \mathcal{D}_m\}$, each client has the total loss

$$f_i(\mathbf{w}) := \frac{1}{d_i} \sum_{\mathbf{x} \in \mathcal{D}_i} \ell_i(\mathbf{w}; \mathbf{x}),$$

where $\ell_i(\cdot; \mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ is a continuous loss function and bounded from below, d_i is the cardinality of \mathcal{D}_i , and $\mathbf{w} \in \mathbb{R}^n$ is the parameter to be learned. The overall loss function can be defined by

$$f(\mathbf{w}) := \sum_{i=1}^{m} \alpha_i f_i(\mathbf{w}),$$

where α_i is a positive weight satisfying $\sum_{i=1}^{m} \alpha_i = 1$. The task of FL is to learning an optimal parameter \mathbf{w}^* that minimizes the overall loss, namely,

$$\mathbf{w}^* := \underset{\mathbf{w} \in \mathbb{R}^n}{\operatorname{argmin}} f(\mathbf{w}). \tag{2.3}$$

Since f_i is supposed to be bounded from below, we have

$$f^* := f(\mathbf{w}^*) > -\infty. \tag{2.4}$$

3 FL via Inexact ADMM

By introducing auxiliary variables, $\mathbf{w}_i = \mathbf{w}$, problem (2.3) can be rewritten as the following form,

$$\min_{\mathbf{w},W} \quad \sum_{i=1}^{m} \alpha_i f_i(\mathbf{w}_i), \quad \text{s.t.} \quad \mathbf{w}_i = \mathbf{w}, \ i \in [m].$$
(3.1)

Throughout the paper, we shall focus on the above optimization problem instead of problem (2.3) as they are equivalent to each other. For simplicity, we also denote

$$F(W) := \sum_{i=1}^{m} \alpha_i f_i(\mathbf{w}_i). \tag{3.2}$$

Clearly, $F(\mathbf{w}, \mathbf{w}, \cdots, \mathbf{w}) = f(\mathbf{w})$.

3.1 Algorithmic design

To implement ADMM for (3.1), the corresponding augmented Lagrange function is defined by,

$$\mathcal{L}(\mathbf{w}, W, \Pi) := \sum_{i=1}^{m} (\underbrace{\alpha_i f_i(\mathbf{w}_i) + \langle \mathbf{w}_i - \mathbf{w}, \pi_i \rangle + \sigma_i / 2 \| \mathbf{w}_i - \mathbf{w} \|^2}_{=:L(\mathbf{w}, \mathbf{w}_i, \pi_i)}),$$
(3.9)

where Π is the Lagrange multiplier, and $\sigma_i > 0, i \in [m]$. Similar to (2.2), we have the framework of ADMM for problem (3.1). That is, for an initial point (\mathbf{w}^0, W^0, Π^0) and any $k \ge 0$, perform the following updates iteratively,

$$\begin{cases} \mathbf{w}^{k+1} = \operatorname{argmin}_{\mathbf{w}} \mathcal{L}(\mathbf{w}, W^k, \Pi^k) \\ = \frac{1}{\sigma} \sum_{i=1}^m (\sigma_i \mathbf{w}_i^k + \boldsymbol{\pi}_i^k), \\ \mathbf{w}_i^{k+1} = \operatorname{argmin}_{\mathbf{w}_i} L(\mathbf{w}^{k+1}, \mathbf{w}_i, \boldsymbol{\pi}^k), \quad i \in [m], \\ \boldsymbol{\pi}_i^{k+1} = \boldsymbol{\pi}_i^k + \sigma_i(\mathbf{w}_i^{k+1} - \mathbf{w}^{k+1}), \quad i \in [m], \end{cases}$$
(3.10)

where $\sigma := \sum_{i=1}^{m} \sigma_i$. Based on the framework of ADMM, we present our algorithm in Algorithm 1 and highlight its advantages in the sequel.

Algorithm 1: FL via inexact ADMM (FedADMM)

Initialize an integer $k_0 > 0$ and $\Omega^0 = [m]$. Set k = 0. Denote $\tau_k := \lfloor k/k_0 \rfloor$ and $\mathbf{g}_i^k := \alpha_i \nabla f_i(\mathbf{w}_i^k)$. All clients $i \in [m]$ initialize $\epsilon_i^0, \sigma_i > 0, \nu_i \in [1/2, 1), \mathbf{w}_i^0, \ \mathbf{\pi}_i^0 = -\mathbf{g}_i^0, \ \mathbf{z}_i^0 = \sigma_i \mathbf{w}_i^0 + \mathbf{\pi}_i^0$ and send the sever σ_i to calculate $\sigma = \sum_{i=1}^m \sigma_i$.

for
$$k = 0, 1, 2, \cdots$$
 do

if $k \in \mathcal{K} := \{0, k_0, 2k_0, 3k_0, \dots\}$ then

Weights upload: (Communication occurs)

Clients in Ω^{τ_k} send their parameters $\{\mathbf{z}_i^k : i \in \Omega^{\tau_k}\}$ to the server.

Global averaging:

The server calculates average parameter $\mathbf{w}^{\tau_{k+1}}$ by

$$\mathbf{w}^{\tau_{k+1}} = \frac{1}{\sigma} \sum_{i=1}^{m} \mathbf{z}_i^k. \tag{3.3}$$

Weights feedback: (Communication occurs)

The server randomly selects clients in [m] to form a subset $\Omega^{\tau_{k+1}}$ and broadcasts them parameter $\mathbf{w}^{\tau_{k+1}}$.

 \mathbf{end}

for every $i \in \Omega^{\tau_{k+1}}$ do

Local update: Client i updates its parameters as follows:

$$\epsilon_i^{k+1} \leq \nu_i \epsilon_i^k, \tag{3.4}$$

Find
$$\mathbf{w}_i^{k+1}$$
 such that $\|\boldsymbol{g}_i^{k+1} + \boldsymbol{\pi}_i^k + \sigma_i(\mathbf{w}_i^{k+1} - \mathbf{w}^{\tau_{k+1}})\|^2 \le \epsilon_i^{k+1}$

$$(3.5)$$

by solving
$$\min_{\mathbf{w}_i} L(\mathbf{w}^{\tau_{k+1}}, \mathbf{w}_i, \boldsymbol{\pi}^k)$$
,

$$\boldsymbol{\pi}_{i}^{k+1} = \boldsymbol{\pi}_{i}^{k} + \sigma_{i}(\mathbf{w}_{i}^{k+1} - \mathbf{w}^{\tau_{k+1}}), \tag{3.6}$$

$$\mathbf{z}_{i}^{k+1} = \sigma_{i} \mathbf{w}_{i}^{k+1} + \boldsymbol{\pi}_{i}^{k+1}.$$
(3.7)

end

for every $i \notin \Omega^{\tau_{k+1}}$ do

<u>Local invariance</u>: Client i keeps their parameters by

$$(\epsilon_i^{k+1}, \mathbf{w}_i^{k+1}, \boldsymbol{\pi}_i^{k+1}, \mathbf{z}_i^{k+1}) = (\epsilon_i^k, \mathbf{w}_i^k, \boldsymbol{\pi}_i^k, \mathbf{z}_i^k).$$
(3.8)

 \mathbf{end}

end

3.2 Communication efficiency

The framework of ADMM in (3.10) indicates that the global averaging (i.e., the first sub-problem) and local updates (i.e., the last two sub-problems) are repeated at every step. In FL settings, this manifests that local clients and the central server have to communicate at every step. However, frequent communications would come at a huge price like long time for learning and large amounts of communication resources (e.g., power and bandwidth).

Therefore, in Algorithm 1, we allow a portion of clients (i.e., clients in $\Omega^{\tau_{k+1}}$) to update their pa-

rameters a few times (i.e., k_0 times) and then upload them to the central server. In other words, the central server collects parameters from local clients only when $k \in \mathcal{K} = \{0, k_0, 2k_0, 3k_0, \cdots\}$. Here, choosing a proper k_0 can reduce CR significantly so as to save resources. It is worth mentioning that such an idea has been extensively used in [15, 41, 16, 17, 18, 19].

3.3 Local invariance

It is worth mentioning that clients outside Ω^{τ_k} do not need to do anything at steps $k, k+1, \dots, k+k_0-1$. We use (3.8) for the purpose of notational convenience when conducting convergence analysis. Moreover, (3.8) also allows the server to record the previous uploaded parameters from clients outside Ω^{τ_k} . Precisely, for any $i \notin \Omega^{\tau_k}$, at step $k \in \mathcal{K}$, let k_i be the largest integer in [0, k) such that $i \in \Omega^{\tau_{k_i}}$. In other words, k_i is the last time that i was selected. Hence, (3.8) implies $\mathbf{z}_i^t \equiv \mathbf{z}_i^{k_i}, \forall t = k_i, k_i + 1, \dots, k$ and thus

$$\mathbf{w}^{\tau_{k+1}} = \frac{1}{\sigma} (\sum_{i \in \Omega^{\tau_k}} \mathbf{z}_i^k + \sum_{i \notin \Omega^{\tau_k}} \mathbf{z}_i^{k_i}) = \frac{1}{\sigma} \sum_{i=1}^m \mathbf{z}_i^k.$$

This means the sever uses the previously uploaded parameters (i.e., $\mathbf{z}_i^{k_i}$) from clients outside Ω^{τ_k} and the currently uploaded parameters (i.e., \mathbf{z}_i^k) from clients in Ω^{τ_k} .

3.4 Fast computation

We emphasize that \mathbf{w}_i^{k+1} in (3.5) is well defined. In fact,

$$\alpha_i \nabla f_i(\mathbf{v}_i^*) + \boldsymbol{\pi}_i^k + \sigma_i(\mathbf{v}_i^* - \mathbf{w}^{\tau_{k+1}}) = 0, \qquad (3.11)$$

where \mathbf{v}_i^* is any optimal solution to $\min_{\mathbf{w}_i} L(\mathbf{w}^{\tau_{k+1}}, \mathbf{w}_i, \pi^k)$. This means there always exists a point satisfying the condition in (3.5). We note that $\min_{\mathbf{w}_i} L(\mathbf{w}^{\tau_{k+1}}, \mathbf{w}_i, \pi^k)$ is an unconstrained optimization problem, so many solvers can be used to solve it. However, we are interested in algorithms that can find \mathbf{w}_i^{k+1} quickly. Particularly, we initialize $\mathbf{v}_i^0 = \mathbf{w}^{\tau_{k+1}}$, and perform the following steps, for $\ell = 0, 1, 2, \cdots, \kappa$,

$$\mathbf{v}_{i}^{\ell+1} = \operatorname{argmin}_{\mathbf{w}_{i}} \langle \mathbf{w}_{i} - \mathbf{w}^{\tau_{k+1}}, \boldsymbol{\pi}_{i}^{k} \rangle + \frac{\sigma_{i}}{2} \| \mathbf{w}_{i} - \mathbf{w}^{\tau_{k+1}} \|^{2} + \alpha_{i} (f_{i}(\mathbf{v}_{i}^{\ell}) + \langle \nabla f_{i}(\mathbf{v}_{i}^{\ell}), \mathbf{w}_{i} - \mathbf{v}_{i}^{\ell} \rangle + \frac{r_{i}}{2} \| \mathbf{w}_{i} - \mathbf{v}_{i}^{\ell} \|^{2})$$

$$= \frac{1}{\alpha_{i}r_{i} + \sigma_{i}} (\alpha_{i}r_{i}\mathbf{v}_{i}^{\ell} + \sigma_{i}\mathbf{w}^{\tau_{k+1}} - (\alpha_{i}\nabla f_{i}(\mathbf{v}_{i}^{\ell}) + \boldsymbol{\pi}_{i}^{k})), \qquad (3.12)$$

where $r_i > 0$ which can be the Lipschitz continuous constant if f_i is Lipschitz continuous and κ is a given maximum number of steps to update \mathbf{v}_i^{ℓ} . The following theorem states that using (3.12) to find $\mathbf{w}_i^{k+1} = \mathbf{v}_i^{\kappa+1}$ can guarantee the condition in (3.5) within a small number of iterations κ .

Theorem 3.1. Suppose that every $f_i, i \in [m]$ is gradient Lipschitz continuous with $r_i > 0$ and Hessian matrix $\nabla^2 f_i \succeq -s_i I$ with $s_i \ge 0$. By setting $\sigma_i \ge \alpha_i s_i + \rho \alpha_i r_i/2$ with $\rho > 1$, client $i \in [m]$ can find $\mathbf{w}_i^{k+1} = \mathbf{v}_i^{\kappa+1}$ such that (3.5) through (3.12) with at most κ steps, where

$$\kappa = \log_{\varrho} \left[\frac{2(\alpha_i^2 r_i^2 + \sigma_i^2) \| \mathbf{w}^{\tau_{k+1}} - \mathbf{v}_i^* \|^2}{\epsilon_i^{k+1}} \right] - 1.$$
(3.13)

Here, $\nabla^2 f_i \succeq -s_i I$ means $\nabla^2 f_i + s_i I \succeq 0$, a positive semi-definite matrix. Plentiful non-convex and all convex functions satisfy this condition. For convex functions, we could choose $s_i = 0$. The above theorem implies that clients can set a slightly large accuracy ϵ_i^{k+1} to ensure a small κ in (3.13) for the sake of fastening their computations. Therefore, the computational cost can be saved in comparison with solving the second sub-problem of (3.10) exactly.

3.5 Coping with straggler's effect

The framework enables FedADMM to deal with the straggler's effect as partial clients are selected for the training at every step. This strategy is similar to FedAvg [37, 36] and FedProx [38]. Their algorithmic frameworks are summarized in Algorithm 2, where FedAvg corresponds to the case of $B = \infty$ in its original version. For FedProx, we use the gradient decent-based scheme (3.17) to solve the sub-problem with a proximal term.

Algorithm 2: FedAvg and FedProx.

Initialize an integer $k_0, \gamma, \mu > 0$ and $\Omega^0 = [m]$. Set k = 0. All clients $i \in [m]$ initialize $\mathbf{w}_i^0 = 0$. for $k = 0, 1, 2, \cdots$ do if $k \in \mathcal{K} := \{0, k_0, 2k_0, 3k_0, \dots\}$ then Weights upload: (Communication occurs) Clients in Ω^{τ_k} send $\{\mathbf{w}_i^k : i \in \Omega^{\tau_k}\}$ to the server. *Global averaging:* The server averages $\mathbf{w}^{\tau_{k+1}}$ by [FedAvg] $\mathbf{w}^{\tau_{k+1}} = \frac{1}{m} \sum_{i=1}^{m} \mathbf{w}_i^k.$ (3.14)[FedProx] $\mathbf{w}^{\tau_{k+1}} = \frac{1}{|\Omega^{\tau_k}|} \sum_{i \in \Omega^{\tau_k}} \mathbf{w}_i^k.$ (3.15)Weights feedback: (Communication occurs) The server randomly selects clients to form $\Omega^{\tau_{k+1}}$ and broadcasts them $\mathbf{w}^{\tau_{k+1}}$. end for every $i \in \Omega^{\tau_{k+1}}$ do *Local update:* Client *i* updates its parameter by $[\text{FedAvg}] \quad \mathbf{w}_i^{k+1} = \begin{cases} \mathbf{w}^{\tau_{k+1}} - \frac{\gamma}{m} \nabla f_i(\mathbf{w}^{\tau_{k+1}}), & k \in \mathcal{K}, \\ \mathbf{w}_i^k - \frac{\gamma}{m} \nabla f_i(\mathbf{w}_i^k), & k \notin \mathcal{K}. \end{cases}$ (3.16)[FedProx] $\mathbf{w}_i^{k+1} = \mathbf{w}_i^k + \mu(\mathbf{w}_i^k - \mathbf{w}_{k+1}^{\tau}) - \frac{\gamma}{m} \nabla f_i(\mathbf{w}_i^k).$ (3.17)end for every $i \notin \Omega^{\tau_{k+1}}$ do <u>Local invariance</u>: Client *i* keeps $\mathbf{w}_i^{k+1} = \mathbf{w}_i^k$. end

end

There are some difference among FedAvg, FedProx and our algorithm. First of all, the global averaging for FedProx is taken on the selected clients in Ω^{τ_k} by (3.15), while FedADMM and FedAvg assemble parameters of all clients, see (3.3) and (3.14). In addition, FedAvg and FedProx average

parameters \mathbf{w}_i^k directly. However, FedADMM aggregates \mathbf{z}_i^k , which are the combinations of primal variables \mathbf{w}_i^k and dual variables π_i^k . To this end, it is more secured to protect clients' data when communicating with the server.

4 Convergence analysis

This section aims to establish the global convergence for FedADMM, before which we define the optimality conditions of problem (3.1) and (2.3) as follows.

4.1 Stationary point

Definition 4.1. A point $(\mathbf{w}^*, W^*, \Pi^*)$ is a stationary point of problem (3.1) if it satisfies

$$\begin{cases} \alpha_i \nabla f_i(\mathbf{w}_i^*) + \boldsymbol{\pi}_i^* = 0, & i \in [m], \\ \mathbf{w}_i^* - \mathbf{w}^* = 0, & i \in [m], \\ \sum_{i=1}^m \boldsymbol{\pi}_i^* = 0. \end{cases}$$
(4.1)

It is not difficult to prove that any locally optimal solution to problem (3.1) must satisfy (4.1). If f_i is convex for every $i \in [m]$, then a point is a globally optimal solution if and only if it satisfies condition (4.1). Moreover, a stationary point (\mathbf{w}^*, W^*, Π^*) of problem (3.1) indicates

$$\nabla f(\mathbf{w}^*) = \sum_{i=1}^m \alpha_i \nabla f_i(\mathbf{w}^*) = -\sum_{i=1}^m \pi_i^* = 0.$$
(4.2)

That is, \mathbf{w}^* is also a stationary point of problem (2.3).

4.2 Some assumptions

Assumption 4.1. Every $f_i, i \in [m]$ is gradient Lipschitz continuous with a constant $r_i > 0$.

Assumption 4.2. Function f is coercive. That is, $f(\mathbf{w}) \to +\infty$ when $\|\mathbf{w}\| \to +\infty$.

Scheme 4.1. The sever randomly selects Ω^{τ} that satisfies

$$\Omega^{\tau+1} \cup \Omega^{\tau+2} \cup \dots \cup \Omega^{\tau+s_0} = [m], \quad \forall \tau = 0, s_0, 2s_0, \dots$$

where s_0 is a pre-defined postive integer.

Such a scheme indicates that for each group of s_0 sets $\{\Omega^{\tau+1}, \Omega^{\tau+2}, \dots, \Omega^{\tau+s_0}\}$, all clients should be chosen at least once. In other words, for any client $i \in [m]$, the maximum gap between its two consecutive selections is no more than $2s_0$, namely,

$$\max \left\{ \begin{aligned} u - v : & i \in \Omega^v, i \in \Omega^u, \\ u - v : & i \notin \Omega^\tau, \tau = v + 1, \cdots, u - 1 \end{aligned} \right\} \le 2s_0.$$

$$(4.3)$$

Remark 4.1. Scheme 4.1 can be satisfied with a high probability. In fact, if $\Omega^1, \Omega^2, \cdots$ are selected independent and indices in Ω^t are uniformly sampled from [m] without replacement, then the probability of client i being selected in $\{\Omega^{\tau+1}, \Omega^{\tau+2}, \cdots, \Omega^{\tau+s_0}\}$ is

$$p_i := 1 - \mathbb{P}(i \notin \Omega^{\tau+1}, i \notin \Omega^{\tau+2}, \cdots, i \notin \Omega^{\tau+s_0})$$

= $1 - \mathbb{P}(i \notin \Omega^{\tau+1}) \mathbb{P}(i \notin \Omega^{\tau+2}) \cdots \mathbb{P}(i \notin \Omega^{\tau+s_0})$
= $1 - (1 - \frac{|\Omega^{\tau+1}|}{m})(1 - \frac{|\Omega^{\tau+2}|}{m}) \cdots (1 - \frac{|\Omega^{\tau+s_0}|}{m}),$

which tends to 1. For example, $p_i = 1 - 10^{-5}$ if $s_0 = 5$ and $|\Omega^{\tau}| = 0.9m$ for any $\tau \ge 1$.

4.3 Global convergence

Now with the help of Assumptions 4.1 and 4.2, our first result shows the following decreasing property of the sequence associated with $\mathcal{L}^k := \mathcal{L}(\mathbf{w}^{\tau_k}, W^k, \Pi^k)$.

Lemma 4.1. Under Assumption 4.1, it holds that

$$\widetilde{\mathcal{L}}^k - \widetilde{\mathcal{L}}^{k+1} \ge \frac{\sigma}{2} \|\mathbf{w}^{\tau_{k+1}} - \mathbf{w}^{\tau_k}\|^2 + \frac{c}{4} \sum_{i=1}^m \|\mathbf{w}_i^{k+1} - \mathbf{w}_i^k\|^2,$$

where $\widetilde{\mathcal{L}}^k$ and c are defined as

$$\widetilde{\mathcal{L}}^{k} := \mathcal{L}^{k} + \sum_{i=1}^{m} \left(\frac{\nu_{i}}{(1-\nu_{i})\alpha_{i}r_{i}} + \frac{10(1+\nu_{i})}{\sigma_{i}(1-\nu_{i})} \right) \epsilon_{i}^{k},$$

$$c := \min_{i \in [m]} \frac{(\sigma_{i} + \alpha_{i}r_{i})(2\sigma_{i} - 5\alpha_{i}r_{i})}{\sigma_{i}}.$$

$$(4.4)$$

Using this result enables to prove that whole sequences of three objectives $f(\mathbf{w}^{\tau_k})$, $F(W^k)$, and \mathcal{L}^k converge.

Theorem 4.1. Suppose that Assumptions 4.1 and 4.2 hold. Every client $i \in [m]$ chooses $\sigma_i > 5\alpha_i r_i/2$ and the sever selects Ω^{τ_k} as Scheme 4.1. Then the following results hold.

- a) Sequence $\{(\mathbf{w}^{\tau_k}, W^k, \Pi^k)\}$ is bounded.
- b) Three sequences $\{\mathcal{L}^k\}$, $\{F(W^k)\}$, and $\{f(\mathbf{w}^{\tau_k})\}$ converge to the same value, namely,

$$\lim_{k \to \infty} \mathcal{L}^k = \lim_{k \to \infty} F(W^k) = \lim_{k \to \infty} f(\mathbf{w}^{\tau_k}).$$
(4.5)

c) $\nabla F(W^k)$ and $\nabla f(\mathbf{w}^{\tau_k})$ eventually vanish, namely,

$$\lim_{k \to \infty} \nabla F(W^k) = \lim_{k \to \infty} \nabla f(\mathbf{w}^{\tau_k}) = 0.$$
(4.6)

Theorem 4.1 states that the objective function values of sequence $\{(\mathbf{w}^{\tau_k}, W^k, \Pi^k)\}$ converge. In the following theorem, we would like to see the convergence performance of the sequence itself.

Theorem 4.2. Suppose that Assumptions 4.1 and 4.2 hold. Every client $i \in [m]$ chooses $\sigma_i > 5\alpha_i r_i/2$ and the sever selects Ω^{τ_k} as Scheme 4.1. Then the following results hold.

- a) Any accumulating point $(\mathbf{w}^{\infty}, W^{\infty}, \Pi^{\infty})$ of sequence $\{(\mathbf{w}^{\tau_k}, W^k, \Pi^k)\}$ is a stationary point of (3.1), where \mathbf{w}^{∞} is a stationary point of (2.3).
- b) If further assuming that \mathbf{w}^{∞} is isolated, then the whole sequence converges to $(\mathbf{w}^{\infty}, W^{\infty}, \Pi^{\infty})$.

It is worth mentioning that the establishments of Theorems 4.1 and 4.2 do not rely on the choices of Ω^{τ_k} explicitly due to Scheme 4.1. If the sever generates Ω^{τ_k} randomly rather than using Scheme 4.1, then the above two theorems are valid with a high probability.

In addition, since no convexity of f_i or f is imposed, the sequence is guaranteed to converge to the stationary point of problems (3.1) and (2.3). In other words, if we assume the convexity of f, then the sequence is capable of converging to the optimal solution to problems (3.1) and (2.3), which is stated by the following corollary. **Corollary 4.1.** Suppose that Assumptions 4.1 and 4.2 hold and f is convex. Every client $i \in [m]$ chooses $\sigma_i > 5\alpha_i r_i/2$ and the sever selects Ω^{τ_k} as Scheme 4.1. Then the following results hold.

a) Three sequences converge to the optimal function value of (2.3), namely,

$$\lim_{k \to \infty} \mathcal{L}^k = \lim_{k \to \infty} F(W^k) = \lim_{k \to \infty} f(\mathbf{w}^{\tau_k}) = f^*.$$
(4.7)

- b) Any accumulating point $(\mathbf{w}^{\infty}, W^{\infty}, \Pi^{\infty})$ of sequence $\{(\mathbf{w}^{\tau_k}, W^k, \Pi^k)\}$ is an optimal solution to (3.1), where \mathbf{w}^{∞} is an optimal solution to (2.3).
- c) If f is strongly convex, then whole sequence converges the unique optimal solution $(\mathbf{w}^*, W^*, \Pi^*)$ to (3.1), where \mathbf{w}^* is the unique optimal solution to (2.3).

Remark 4.2. Regarding assumptions in Corollary 4.1, f being strongly convex does not require that every $f_i, i \in [m]$ is strongly convex. If one of f_i s is strongly convex and the remaining is convex, then $f = \sum_{i=1}^{m} \alpha_i f_i$ is strongly convex. Moreover, the strongly convexity suffices to the coerciveness of f. Therefore, under the strongly convexity, Assumption 4.2 can be exempted.

5 Numerical Experiments

This section conducts some numerical experiments to demonstrate the performance of FedADMM. All numerical experiments are implemented through MATLAB (R2019a) on a laptop with 32GB memory and 2.3Ghz CPU.

5.1 Testing example

Example 5.1 (Linear regression with non-i.i.d. data). For this problem, local clients have their objective functions as

$$f_i(\mathbf{w}) = \sum_{t=1}^{d_i} \frac{1}{2d_i} (\langle \mathbf{a}_i^t, \mathbf{w} \rangle - b_i^t)^2,$$

where $\mathbf{a}_i^t \in \mathbb{R}^n$ and $b_i^t \in \mathbb{R}$ are the t-th sample for client i. We first pick m integers d_1, \dots, d_m randomly from [50, 150] and denote $d := d_1 + \dots + d_m$. Then we generate $\lfloor d/3 \rfloor$ samples (\mathbf{a}, b) from the standard normal distribution, $\lfloor d/3 \rfloor$ samples from the Student's t distribution with degree 5, and $d - 2\lfloor d/3 \rfloor$ samples from the uniform distribution in [-5, 5]. Now we shuffle all samples and divide them into m parts with sizes $d_1 \dots, d_m$ for m clients. In the regard, each client has non-i.i.d. data. For simplicity, we fix n = 100 and alter $m \in \{32, 64, 96, 128, 160\}$.

Table 1: Descriptions of two real datasets.

Data	Datasets	Source	n	d
qot	Qsar oral toxicity	uci	1024	8992
sct	Santander customer transaction	kaggle	200	200000

Example 5.2 (Logistic regression). For this problem, local clients have their objective functions as

$$f_i(\mathbf{w}) = \frac{1}{d_i} \sum_{t=1}^{d_i} (\ln(1 + e^{\langle \mathbf{a}_i^t, \mathbf{w} \rangle}) - b_i^t \langle \mathbf{a}_i^t, \mathbf{w} \rangle) + \frac{\lambda}{2} \|\mathbf{w}\|^2,$$

where $\mathbf{a}_i^t \in \mathbb{R}^n, b_i^t \in \{0, 1\}$, and $\lambda > 0$ is a penalty parameter (e.g., $\lambda = 0.001$ in our numerical experiments). We use two real datasets described in Table 1 to generate (\mathbf{a}, b) . Again, we randomly split d samples into m groups for m clients.

5.2 Implementations

We fix $\alpha_i = 1/m, i \in [m]$ in model (2.3) and initialize $\mathbf{w}_i^0 = \boldsymbol{\pi}_i^0 = 0$. Parameters are set as follows: $\sigma_i = 0.2r_i/m$, where r_i is the gradient Lipschitz continuous constant for f_i , $\epsilon_i^0 = k_0^2$, and $\nu_i = 0.95$ for all $i \in [m]$. We terminate our algorithm if $k \ge 10^4$ or solution \mathbf{w}^{τ_k} satisfies

$$\operatorname{Error} := \|\nabla f(\mathbf{w}^{\tau_k})\|^2 \le 5n10^{-4}/d.$$
(5.1)

In the subsequent experiments, $\Omega^1, \Omega^2, \cdots$ are selected independently with $|\Omega^{\tau}| = \rho m$ for any $\tau \ge 1$, where $\rho \in (0, 1]$. Indices in each Ω^{τ} are uniformly sampled from [m] without replacement.

5.3 Benchmark algorithms

We will compare our proposed method with FedAvg [37] and FedProx [38] presented in Algorithm 2. Their starting points are initialized as $\mathbf{w}_i^0 = 0$ and the learning rate is set as $\gamma = \gamma_k(a) := a/\log_2(k+1)$ with a = 0.005 for Example 5.1 and a = 0.5d/m for Example 5.2. To ensure relatively fair comparisons, we terminate them and FedADMM if condition (5.1) is met or CR are over 1000.

5.4 Numerical comparisons

In this part, we conduct some simulation to demonstrate the performance of three algorithms by reporting the following factors: objective $f(\mathbf{w}^{\tau_k})$, CR, and computational time (in second).



Figure 1: Objective v.s. CR.

a) Effect of k_0 . By fixing m = 64 and $\rho = 0.5$ meaning half clients chosen for the training (i.e., $|\Omega^{\tau}| = 0.5m$), we apply three algorithms into solving two examples under $k_0 = 1, 5, 10$ and report the results in Figure 1. One can observe that (i) with the increase of CR, all algorithms eventually achieve the same objective function values (i.e., the optimal function values), (ii) for each algorithm, the larger k_0 the fewer CR needed to converge, and (iii) FedADMM outperforms FedProx which behaves better than FedAvg.

Next, we generate 20 instances for each example solved by one algorithm with a fixed $k_0 \in [20]$ and present the average results in Figure 2, where the following comments can be declared: (i) for Example 5.1, FedAvg and FedProx reach the maximum of CR and their computational time is



rising along with k_0 increasing, (ii) for Example 5.2, CR and time are descending when k_0 is getting bigger, (iii) for both examples, FedADMM performs the best in terms of using the fewest CR and running the fastest.

b) Effect of Ω^{τ} . To see this, we fix $m = 64, k_0 = 10$ and alter $\rho \in [0.1, 1]$. Since $|\Omega^{\tau}| = \rho m$, the large value of ρ means more clients are selected for the training. Similarly, we generate 20 instances for each example solved by one algorithm with a fixed ρ and report the average results in Figure



3. We can conclude that (i) for Example 5.1, FedAvg and FedProx reach the maximum allowed number of CR, and their computational time is rising along with ρ increasing, (ii) for Example 5.2, there are declining trends for CR generated by FedAvg and FedADMM. By contrast, CR from FedProx stabilizes at a certain level with the changing of ρ . This is probably because of averaging scheme (3.15) that only uses selected clients. (iii) Once again, FedADMM outperforms the other two algorithms for both examples.

c) Effect of m. To see this, we fix $\rho = 0.5, k_0 = 10$ and alter $m \in \{32, 64, 96, 128, 160\}$. The average results over 20 instances are presented in Figure 4, where for both examples, CR and computational time from FedAvg and FedProx increase when m gets bigger. However, CR for FedADMM does not fluctuate significantly. Apparently, our algorithm performs better than the other two algorithms.

6 Conclusion

We developed an inexact ADMM-based FL algorithm, FedADMM. The periodic global averaging allows it to reduce CR so as to save communication resources. Solving sub-problems inexactly alleviates clients' computational burdens significantly, thereby accelerating the learning process. Partial device participation in the algorithm eliminates the stragglers' effect. Those merits show the strong potential of FedADMM for real-world applications like vehicular communications, mobile edge and over-the-air computing.

References

[1] J. Konečný, B. McMahan, and D. Ramage, "Federated optimization: Distributed optimization beyond the datacenter," *arXiv preprint arXiv:1511.03575*, 2015.

- [2] J. Konečný, H. B. McMahan, D. Ramage, and P. Richtárik, "Federated optimization: Distributed machine learning for on-device intelligence," arXiv preprint arXiv:1610.02527, 2016.
- [3] S. Samarakoon, M. Bennis, W. Saad, and M. Debbah, "Distributed federated learning for ultra-reliable low-latency vehicular communications," *IEEE Trans. Commun.*, vol. 68, no. 2, pp. 1146–1159, 2019.
- [4] S. R. Pokhrel, "Federated learning meets blockchain at 6g edge: A drone-assisted networking for disaster response," in *DroneCom*, 2020, pp. 49–54.
- [5] A. M. Elbir, B. Soner, and S. Coleri, "Federated learning in vehicular networks," *arXiv preprint arXiv:2006.01412*, 2020.
- [6] J. Posner, L. Tseng, M. Aloqaily, and Y. Jararweh, "Federated learning in vehicular networks: opportunities and solutions," *IEEE Netw.*, vol. 35, no. 2, pp. 152–159, 2021.
- [7] N. Rieke, J. Hancox, W. Li, F. Milletari, H. R. Roth, S. Albarqouni, S. Bakas, M. N. Galtier,
 B. A. Landman, K. Maier-Hein *et al.*, "The future of digital health with federated learning," *NPJ Digit. Med.*, vol. 3, no. 1, pp. 1–7, 2020.
- [8] Y. Mao, C. You, J. Zhang, K. Huang, and K. B. Letaief, "A survey on mobile edge computing: The communication perspective," *IEEE Commun. Surv. Tutor.*, vol. 19, no. 4, pp. 2322–2358, 2017.
- [9] K. Yang, T. Jiang, Y. Shi, and Z. Ding, "Federated learning via over-the-air computation," *IEEE Trans. Wirel. Commun.*, vol. 19, no. 3, pp. 2022–2035, 2020.
- [10] S. Zhou and G. Y. Li, "Communication-efficient ADMM-based federated learning," arXiv preprint arXiv:2110.15318, 2021.
- [11] H. Ye, L. Liang, and G. Y. Li, "Decentralized federated learning with unreliable communications," *IEEE J. Sel. Top. Signal Process.*, 2022.
- [12] P. Kairouz, H. B. McMahan, B. Avent, A. Bellet, M. Bennis, A. N. Bhagoji, K. Bonawitz, Z. Charles, G. Cormode, R. Cummings *et al.*, "Advances and open problems in federated learning," *Found. Trends Mach. Learn.*, vol. 14, no. 1-2, pp. 1–210, 2019.
- [13] T. Li, A. K. Sahu, A. Talwalkar, and V. Smith, "Federated learning: Challenges, methods, and future directions," *IEEE Signal Process. Mag.*, vol. 37, no. 3, pp. 50–60, 2020.
- [14] Z. Qin, G. Y. Li, and H. Ye, "Federated learning and wireless communications," *IEEE Wirel. Commun.*, 2021.
- [15] S. Zhang, A. Choromanska, and Y. LeCun, "Deep learning with elastic averaging SGD," in *NeurIPS*, vol. 1. Cambridge, MA, USA: MIT Press, 2015, p. 685–693.
- [16] S. U. Stich, "Local SGD converges fast and communicates little," in *ICLR*, 2019.
- [17] H. Yu, S. Yang, and S. Zhu, "Parallel restarted SGD with faster convergence and less communication: Demystifying why model averaging works for deep learning," AAAI, vol. 33, no. 1, pp. 5693–5700, 2019.

- [18] T. Lin, S. U. Stich, K. K. Patel, and M. Jaggi, "Don't use large mini-batches, use local SGD," in *ICLR*, 2020.
- [19] J. Wang and G. Joshi, "Cooperative SGD: A unified framework for the design and analysis of local-update SGD algorithms," J. Mach. Learn. Res., vol. 22, no. 213, pp. 1–50, 2021.
- [20] V. Smith, S. Forte, M. Chenxin, M. Takáč, M. I. Jordan, and M. Jaggi, "Cocoa: A general framework for communication-efficient distributed optimization," J. Mach. Learn. Res., vol. 18, p. 230, 2018.
- [21] T. Chen, G. Giannakis, T. Sun, and W. Yin, "Lag: Lazily aggregated gradient for communication-efficient distributed learning," Adv. Neural Inf. Process. Syst., vol. 2018-December, pp. 5050–5060, 2018.
- [22] S. Wang, T. Tuor, T. Salonidis, K. K. Leung, C. Makaya, T. He, and K. Chan, "Adaptive federated learning in resource constrained edge computing systems," *IEEE J. Sel. Areas Commun.*, vol. 37, no. 6, pp. 1205–1221, 2019.
- [23] Y. Liu, Y. Sun, and W. Yin, "Decentralized learning with lazy and approximate dual gradients," *IEEE Trans. Signal Process.*, vol. 69, pp. 1362–1377, 2021.
- [24] Q. Tong, G. Liang, T. Zhu, and J. Bi, "Federated nonconvex sparse learning," arXiv preprint arXiv:2101.00052, 2020.
- [25] T. Zhang and Q. Zhu, "Dynamic differential privacy for ADMM-based distributed classification learning," *IEEE Trans. Inf. Forensics Secur.*, vol. 12, no. 1, pp. 172–187, 2016.
- [26] Q. Li, B. Kailkhura, R. Goldhahn, P. Ray, and P. K. Varshney, "Robust federated learning using ADMM in the presence of data falsifying byzantines," arXiv preprint arXiv:1710.05241, 2017.
- [27] X. Zhang, M. M. Khalili, and M. Liu, "Improving the privacy and accuracy of ADMM-based distributed algorithms," in *PMLR*, 2018, pp. 5796–5805.
- [28] Y. Guo and Y. Gong, "Practical collaborative learning for crowdsensing in the internet of things with differential privacy," in CNS. IEEE, 2018, pp. 1–9.
- [29] X. Zhang, M. M. Khalili, and M. Liu, "Recycled ADMM: Improve privacy and accuracy with less computation in distributed algorithms," in *Allerton*. IEEE, 2018, pp. 959–965.
- [30] Z. Huang, R. Hu, Y. Guo, E. Chan-Tin, and Y. Gong, "DP-ADMM: ADMM-based distributed learning with differential privacy," *IEEE Trans. Inf. Forensics Secur.*, vol. 15, pp. 1002–1012, 2019.
- [31] A. Elgabli, J. Park, S. Ahmed, and M. Bennis, "L-FGADMM: Layer-wise federated group ADMM for communication efficient decentralized deep learning," in *WCNC*, 2020, pp. 1–6.
- [32] J. Ding, S. M. Errapotu, H. Zhang, Y. Gong, M. Pan, and Z. Han, "Stochastic ADMM based distributed machine learning with differential privacy," in *SecureComm.* Springer, 2019, pp. 257–277.

- [33] S. Yue, J. Ren, J. Xin, S. Lin, and J. Zhang, "Inexact-ADMM based federated meta-learning for fast and continual edge learning," in ACM Mobihoc, 2021, pp. 91–100.
- [34] M. Ryu and K. Kim, "Differentially private federated learning via inexact ADMM with multiple local updates," arXiv preprint arXiv:2202.09409, 2022.
- [35] X. Zhang, M. Hong, S. Dhople, W. Yin, and Y. Liu, "Fedpd: A federated learning framework with optimal rates and adaptivity to non-iid data," arXiv preprint arXiv:2005.11418, 2020.
- [36] X. Li, K. Huang, W. Yang, S. Wang, and Z. Zhang, "On the convergence of FedAvg on non-iid data," in *ICLR*, 2020.
- [37] B. McMahan, E. Moore, D. Ramage, S. Hampson, and B. A. y Arcas, "Communication-efficient learning of deep networks from decentralized data," in *PMLR*, 2017, pp. 1273–1282.
- [38] T. Li, A. K. Sahu, M. Zaheer, M. Sanjabi, A. Talwalkar, and V. Smith, "Federated optimization in heterogeneous networks," *Proceedings of Machine Learning and Systems*, vol. 2, pp. 429–450, 2020.
- [39] D. Gabay and B. Mercier, "A dual algorithm for the solution of nonlinear variational problems via finite element approximation," *Comput. Math. with Appl.*, vol. 2, no. 1, pp. 17–40, 1976.
- [40] S. Boyd, N. Parikh, and E. Chu, *Distributed optimization and statistical learning via the alternating direction method of multipliers*. Now Publishers Inc, 2011.
- [41] S. Zheng, Q. Meng, T. Wang, W. Chen, N. Yu, Z.-M. Ma, and T.-Y. Liu, "Asynchronous stochastic gradient descent with delay compensation," in *ICML*, vol. 7, 2017, p. 4120–4129.
- [42] J. Moré and D. Sorensen, "Computing a trust region step," SIAM J. Sci. Statist. Comput., vol. 4, no. 3, pp. 553–572, 1983.

7 Appendix: Proofs of all theorems

For any $\mathbf{w}_1, \mathbf{w}_2$, and $\mathbf{w}_i \in {\mathbf{w}_1, \mathbf{w}_2}$, it follows that $\mathbf{w}_2 + t(\mathbf{w}_1 - \mathbf{w}_2) - \mathbf{w}_i = (t - 1)(\mathbf{w}_1 - \mathbf{w}_2)$ or $= t(\mathbf{w}_1 - \mathbf{w}_2)$. If function is gradient Lipschitz continuous with constant r, then the Mean Value Theorem suffices to

$$f(\mathbf{w}_{1}) - f(\mathbf{w}_{2}) - \langle \nabla f(\mathbf{w}_{i}), \mathbf{w}_{1} - \mathbf{w}_{2} \rangle$$

$$= \int_{0}^{1} \langle \nabla f(\mathbf{w}_{2} + t(\mathbf{w}_{1} - \mathbf{w}_{2})) - \nabla f(\mathbf{w}_{i}), \mathbf{w}_{1} - \mathbf{w}_{2} \rangle dt$$

$$\leq \int_{0}^{1} r \|\mathbf{w}_{2} + t(\mathbf{w}_{1} - \mathbf{w}_{2}) - \mathbf{w}_{i}\| \|\mathbf{w}_{1} - \mathbf{w}_{2}\| dt$$

$$= \frac{r}{2} \|\mathbf{w}_{1} - \mathbf{w}_{2}\|^{2}.$$
(7.1)

If $\nabla^2 f \succeq -sI$, then the Mean Value Theorem brings out

$$\langle \nabla f(\mathbf{w}_1) - \nabla f(\mathbf{w}_2), \mathbf{w}_1 - \mathbf{w}_2 \rangle$$

= $\int_0^1 \langle \nabla^2 f(\mathbf{w}_2 + t(\mathbf{w}_1 - \mathbf{w}_2))(\mathbf{w}_1 - \mathbf{w}_2), \mathbf{w}_1 - \mathbf{w}_2 \rangle \rangle dt$ (7.2)
 $\geq -s \|\mathbf{w}_1 - \mathbf{w}_2\|^2.$

For any vectors \mathbf{w}_i , matrix $H \succeq 0$, and t > 0, we have

$$2\langle \mathbf{w}_{1}, \mathbf{w}_{2} \rangle \leq t \| \mathbf{w}_{1} \|^{2} + (1/t) \| \mathbf{w}_{2} \|^{2},$$

$$\| \mathbf{w}_{1} + \mathbf{w}_{2} \|^{2} \leq (1+t) \| \mathbf{w}_{1} \|^{2} + (1+1/t) \| \mathbf{w}_{2} \|^{2},$$

$$\| \sum_{i=1}^{m} \mathbf{w}_{i} \|^{2} \leq m \sum_{i=1}^{m} \| \mathbf{w}_{i} \|^{2}.$$

(7.3)

For notational simplicity, hereafter, we denote

$$\Delta \mathbf{w}^{\tau_{k+1}} := \mathbf{w}^{\tau_{k+1}} - \mathbf{w}^{\tau_k} \quad \Delta \overline{\mathbf{w}}_i^{k+1} := \mathbf{w}_i^{k+1} - \mathbf{w}^{\tau_{k+1}},$$

$$\Delta \mathbf{w}_i^{k+1} := \mathbf{w}_i^{k+1} - \mathbf{w}_i^k, \quad \Delta \pi_i^{k+1} := \pi_i^{k+1} - \mathbf{w}_i^k,$$

$$\Delta \epsilon_i^{k+1} := \epsilon_i^{k+1} - \epsilon_i^k,$$

$$(7.4)$$

and let $\mathbf{w}^k \to \mathbf{w}$ stand for $\lim_{k\to\infty} \mathbf{w}^k = \mathbf{w}$.

7.1 Proof of Theorem 3.1

Proof. Since $\mathbf{v}_i^{\ell+1}$ is a solution to problem (3.12), it satisfies the following optimality condition,

$$\alpha_i \nabla f_i(\mathbf{v}_i^{\ell}) + \boldsymbol{\pi}_i^k + \sigma_i(\mathbf{v}_i^{\ell+1} - \mathbf{w}^{\tau_{k+1}}) + \alpha_i r_i(\mathbf{v}_i^{\ell+1} - \mathbf{v}_i^{\ell}) = 0,$$

which subtracting (3.11) gives rise to

$$\begin{aligned} -\sigma_i(\mathbf{v}_i^{\ell+1} - \mathbf{v}_i^*) - \alpha_i r_i(\mathbf{v}_i^{\ell+1} - \mathbf{v}_i^\ell) &= \alpha_i(\nabla f_i(\mathbf{v}_i^\ell) - \nabla f_i(\mathbf{v}_i^*)) \\ &= \alpha_i(\nabla f_i(\mathbf{v}_i^\ell) - \nabla f_i(\mathbf{v}_i^{\ell+1}) + \nabla f_i(\mathbf{v}_i^{\ell+1}) - \nabla f_i(\mathbf{v}_i^*)). \end{aligned}$$

Using the condition allows us to obtain

$$\begin{aligned} -\alpha_{i}s_{i}\|\mathbf{v}_{i}^{\ell+1}-\mathbf{v}_{i}^{*}\|^{2} &\leq \langle \mathbf{v}_{i}^{\ell+1}-\mathbf{v}_{i}^{*}, \alpha_{i}(\nabla f_{i}(\mathbf{v}_{i}^{\ell+1})-\nabla f_{i}(\mathbf{v}_{i}^{*})) \rangle \\ &= \langle \mathbf{v}_{i}^{\ell+1}-\mathbf{v}_{i}^{*}, -\sigma_{i}(\mathbf{v}_{i}^{\ell+1}-\mathbf{v}_{i}^{*}) - \alpha_{i}r_{i}(\mathbf{v}_{i}^{\ell+1}-\mathbf{v}_{i}^{\ell})) \rangle \\ &+ \langle \mathbf{v}_{i}^{\ell+1}-\mathbf{v}_{i}^{*}, \alpha_{i}(\nabla f_{i}(\mathbf{v}_{i}^{\ell+1})-\nabla f_{i}(\mathbf{v}_{i}^{\ell})) \rangle \\ &\leq -(\sigma_{i}+\alpha_{i}r_{i})\|\mathbf{v}_{i}^{\ell+1}-\mathbf{v}_{i}^{*}\|^{2} - \alpha_{i}r_{i}\langle \mathbf{v}_{i}^{\ell+1}-\mathbf{v}_{i}^{*}, \mathbf{v}_{i}^{*}-\mathbf{v}_{i}^{\ell}\rangle \\ &+ \frac{\alpha_{i}r_{i}}{2}(\|\mathbf{v}_{i}^{\ell+1}-\mathbf{v}_{i}^{*}\|^{2}+\|\mathbf{v}_{i}^{\ell+1}-\mathbf{v}_{i}^{\ell}\|^{2}) \\ &= -\sigma_{i}\|\mathbf{v}_{i}^{\ell+1}-\mathbf{v}_{i}^{*}\|^{2} - \frac{\alpha_{i}r_{i}}{2}\|\mathbf{v}_{i}^{\ell}-\mathbf{v}_{i}^{*}\|^{2} \\ &\leq -(\alpha_{i}s_{i}+\frac{\rho\alpha_{i}r_{i}}{2})\|\mathbf{v}_{i}^{\ell+1}-\mathbf{v}_{i}^{*}\|^{2} - \frac{\alpha_{i}r_{i}}{2}\|\mathbf{v}_{i}^{\ell}-\mathbf{v}_{i}^{*}\|^{2}, \end{aligned}$$

which immediately results in

$$\|\mathbf{v}_{i}^{\kappa+1} - \mathbf{v}_{i}^{*}\|^{2} \leq \frac{1}{\varrho} \|\mathbf{v}_{i}^{\kappa} - \mathbf{v}_{i}^{*}\|^{2} \leq \frac{1}{\varrho^{2}} \|\mathbf{v}_{i}^{\kappa-1} - \mathbf{v}_{i}^{*}\|^{2} \leq \dots \leq \frac{1}{\varrho^{\kappa+1}} \|\mathbf{v}_{i}^{0} - \mathbf{v}_{i}^{*}\|^{2}.$$

Now letting $\mathbf{w}_i^{k+1} = \mathbf{v}_i^{\kappa+1}$, we verify the condition in (3.5) by

$$\begin{split} \|\boldsymbol{g}_{i}^{k+1} + \boldsymbol{\pi}_{i}^{k} + \sigma_{i}(\mathbf{w}_{i}^{k+1} - \mathbf{w}^{\tau_{k+1}})\|^{2} \\ \stackrel{(3.11)}{=} \|\boldsymbol{g}_{i}^{k+1} - \alpha_{i}\nabla f_{i}(\mathbf{v}_{i}^{*}) + \sigma_{i}(\mathbf{w}_{i}^{k+1} - \mathbf{v}_{i}^{*})\|^{2} \\ \leq 2(\alpha_{i}^{2}r_{i}^{2} + \sigma_{i}^{2})\|\mathbf{w}_{i}^{k+1} - \mathbf{v}_{i}^{*}\|^{2} \\ \leq 2(\alpha_{i}^{2}r_{i}^{2} + \sigma_{i}^{2})/\varrho^{\kappa+1}\|\mathbf{v}_{i}^{0} - \mathbf{v}_{i}^{*}\|^{2} \leq \epsilon_{i}^{k+1}, \end{split}$$

which by $\mathbf{v}_i^0 = \mathbf{w}^{\tau_{k+1}}$ implies that

$$\kappa = \log_{\varrho} \left\lfloor \frac{2(\alpha_i^2 r_i^2 + \sigma_i^2) \|\mathbf{w}^{\tau_{k+1}} - \mathbf{v}_i^*\|^2}{\epsilon_i^{k+1}} \right\rfloor - 1.$$

The whole proof is finished.

7.2 Key lemma

Lemma 7.1. The following statements are valid. a) For any $k \in \mathcal{K}$,

$$\sum_{i=1}^{m} (\sigma_i (\mathbf{w}_i^k - \mathbf{w}^{\tau_{k+1}}) + \pi_i^k) = 0.$$
(7.5)

b) For any $k \ge 0$ and any $i \in [m]$,

$$\varphi_i^{k+1} := g_i^{k+1} + \pi_i^{k+1} \quad and \quad \|\varphi_i^{k+1}\|^2 \le \epsilon_i^{k+1}.$$
(7.6)

c) Under Assumption 4.1, for any $k \ge 0$ and any $i \in [m]$,

$$\| \triangle \boldsymbol{\pi}_{i}^{k+1} \|^{2} \leq \frac{5\alpha_{i}^{2}r_{i}^{2}}{4} \| \triangle \mathbf{w}_{i}^{k+1} \|^{2} - \frac{10(1+\nu_{i})}{1-\nu_{i}} \triangle \epsilon_{i}^{k+1}.$$
(7.7)

d) Under Scheme 4.1, for any $i \in [m]$,

$$\epsilon_i^{k+1} \to 0. \tag{7.8}$$

Proof. a) For any $i \in [m]$ and at (k + 1)th iteration, let k_i be the largest integer in [-1, k] such that $i \in \Omega^{\tau_{k_i+1}}$. This implies that client i is not selected in all $\Omega^{\tau_{k_i+2}}, \Omega^{\tau_{k_i+3}} \cdots, \Omega^{\tau_{k+1}}$, which by (3.8) yields

$$(\epsilon_i^{\ell+1}, \mathbf{w}_i^{\ell+1}, \pi_i^{\ell+1}, \mathbf{z}_i^{\ell+1}) \equiv (\epsilon_i^{k_i+1}, \mathbf{w}_i^{k_i+1}, \pi_i^{k_i+1}, \mathbf{z}_i^{k_i+1}), \quad \forall \ell = k_i, k_i + 1, \cdots, k.$$
(7.9)

For any client $i \in \Omega^{\tau_{k+1}}$, we have (3.7). For any client $i \notin \Omega^{\tau_{k+1}}$, if $k_i \ge 0$, then $(\mathbf{w}_i^{k_i+1}, \boldsymbol{\pi}_i^{k_i+1}, \mathbf{z}_i^{k_i+1})$ also satisfies (3.7) due to $i \in \Omega^{\tau_{k_i+1}}$, which by condition (7.9) implies that $(\mathbf{w}_i^{k+1}, \boldsymbol{\pi}_i^{k+1}, \mathbf{z}_i^{k+1})$ satisfies (3.7). If $k_i = -1$, this means that is client *i* has never been selected. Then by (7.9) and our initialization, we have

$$\mathbf{z}_i^{k+1} = \mathbf{z}_i^{k_i+1} = \mathbf{z}_i^0 = \sigma_i \mathbf{w}_i^0 + \boldsymbol{\pi}_i^0 = \sigma_i \mathbf{w}_i^{k+1} + \boldsymbol{\pi}_i^{k+1}.$$

Hence, (3.7) is still valid. Overall, we can conclude that (3.7) holds for every $i \in [m]$ and $k \ge 0$. Now, for any $k \in \mathcal{K}$,

$$\sum_{i=1}^{m} (\sigma_i (\mathbf{w}_i^k - \mathbf{w}^{\tau_{k+1}}) + \boldsymbol{\pi}_i^k) \stackrel{(3.7)}{=} \sum_{i=1}^{m} \sigma_i (\mathbf{z}_i^k - \mathbf{w}^{\tau_{k+1}}) \stackrel{(3.3)}{=} 0.$$

b) For any $i \in \Omega^{\tau_{k+1}}$, solution \mathbf{w}_i^{k+1} in (3.5) satisfies

$$\varphi_{i}^{k+1} = g_{i}^{k+1} + \pi_{i}^{k+1} \stackrel{(3.6)}{=} g_{i}^{k+1} + \pi_{i}^{k} + \sigma_{i} \triangle \overline{\mathbf{w}}_{i}^{k+1}, \\ \|\varphi_{i}^{k+1}\|^{2} \stackrel{(3.4)}{\leq} \epsilon_{i}^{k+1}.$$
(7.10)

For any $i \notin \Omega^{\tau_{k+1}}$, we have

$$\varphi_i^{k_i+1} = g_i^{k_i+1} + \pi_i^{k_i+1}, \ \|\varphi_i^{k_i+1}\|^2 \le \epsilon_i^{k_i+1}$$

due to $i \in \Omega^{\tau_{k_i+1}}$. This together with (7.9) implies that (7.6) is still true. So, (7.6) holds for any $i \in [m]$ and any $k \ge 0$.

c) For any $i \in \Omega^{\tau_{k+1}}$, it follows from (7.6) and the gradient Lipschitz continuity of f_i that

$$\begin{split} \| \triangle \boldsymbol{\pi}_{i}^{k+1} \|^{2} &\leq \| \boldsymbol{g}_{i}^{k+1} - \boldsymbol{g}_{i}^{k} + \boldsymbol{\varphi}_{i}^{k+1} - \boldsymbol{\varphi}_{i}^{k} \|^{2} \\ &\leq \frac{5\alpha_{i}^{2}r_{i}^{2}}{4} \| \triangle \mathbf{w}_{i}^{k+1} \|^{2} + 5 \| \boldsymbol{\varphi}_{i}^{k+1} - \boldsymbol{\varphi}_{i}^{k} \|^{2} \\ &\leq \frac{5\alpha_{i}^{2}r_{i}^{2}}{4} \| \triangle \mathbf{w}_{i}^{k+1} \|^{2} + 10(\epsilon_{i}^{k+1} + \epsilon_{i}^{k}) \\ &\leq \frac{5\alpha_{i}^{2}r_{i}^{2}}{4} \| \triangle \mathbf{w}_{i}^{k+1} \|^{2} + \frac{10(1+\nu_{i})}{1-\nu_{i}} (\epsilon_{i}^{k} - \epsilon_{i}^{k+1}). \end{split}$$

For any $i \notin \Omega^{\tau_{k+1}}$, we have $\pi_i^{k+1} = 0$ by (3.8), and thus the above condition is still valid.

d) For sufficiently large $k \in \mathcal{K}$, any client *i* has been selected at least $k/(s_0k_0)$ times due to (4.3). This means that (3.4) (e.g., $\epsilon_i^{k+1} = \epsilon_i^k/2$) occurs at least $k/(s_0k_0)$ times, leading to $\epsilon_i^{k+1} \to 0$. \Box

7.3 Proof of Lemma 4.1

Proof. We estimate gap $(\mathcal{L}^{k+1} - \mathcal{L}^k)$ by decomposing it as

$$\mathcal{L}^{k+1} - \mathcal{L}^k =: p_1^k + p_2^k + p_3^k, \tag{7.11}$$

with

$$p_{1}^{k} := \mathcal{L}(\mathbf{w}^{\tau_{k+1}}, W^{k}, \Pi^{k}) - \mathcal{L}^{k},$$

$$p_{2}^{k} := \mathcal{L}(\mathbf{w}^{\tau_{k+1}}, W^{k+1}, \Pi^{k}) - \mathcal{L}(\mathbf{w}^{\tau_{k+1}}, W^{k}, \Pi^{k}),$$

$$p_{3}^{k} := \mathcal{L}^{k+1} - \mathcal{L}(\mathbf{w}^{\tau_{k+1}}, W^{k+1}, \Pi^{k}).$$
(7.12)

Estimate p_1^k . If $k \notin \mathcal{K}$, then $\mathbf{w}^{\tau_{k+1}} = \mathbf{w}^{\tau_k}$, thereby leading to

$$p_1^k = \mathcal{L}(\mathbf{w}^{\tau_{k+1}}, W^k, \Pi^k) - \mathcal{L}^k = 0 = -\frac{\sigma}{2} \| \triangle \mathbf{w}^{\tau_{k+1}} \|^2.$$
(7.13)

If $k \in \mathcal{K}$, then we have (7.5) and

$$\frac{\sigma_i}{2} \|\mathbf{w}_i^k - \mathbf{w}^{\tau_{k+1}}\|^2 - \frac{\sigma_i}{2} \|\Delta \overline{\mathbf{w}}_i^k\|^2
= \langle \Delta \mathbf{w}^{\tau_{k+1}}, -\sigma_i(\mathbf{w}_i^k - \mathbf{w}^{\tau_{k+1}}) \rangle - \frac{\sigma_i}{2} \|\Delta \mathbf{w}^{\tau_{k+1}}\|^2.$$
(7.14)

These facts also allow us to derive that

$$p_{1}^{k} \stackrel{(3.9)}{=} \sum_{i=1}^{m} \langle \Delta \mathbf{w}^{\tau_{k+1}}, -\boldsymbol{\pi}_{i}^{k} \rangle + \sum_{i=1}^{m} \langle \frac{\sigma_{i}}{2} \| \mathbf{w}_{i}^{k} - \mathbf{w}^{\tau_{k+1}} \|^{2} - \frac{\sigma_{i}}{2} \| \mathbf{w}_{i}^{k} - \mathbf{w}^{\tau_{k}} \|^{2})$$

$$\stackrel{(7.14)}{=} \sum_{i=1}^{m} \langle \Delta \mathbf{w}^{\tau_{k+1}}, -\boldsymbol{\pi}_{i}^{k} - \sigma_{i} (\mathbf{w}_{i}^{k} - \mathbf{w}^{\tau_{k+1}}) \rangle - \sum_{i=1}^{m} \frac{\sigma_{i}}{2} \| \Delta \mathbf{w}^{\tau_{k+1}} \|^{2}$$

$$\stackrel{(7.15)}{=} -\frac{\sigma}{2} \| \Delta \mathbf{w}^{\tau_{k+1}} \|^{2}.$$

Estimate p_2^k . We consider two cases: $i \notin \Omega^{\tau_{k+1}}$ and $i \in \Omega^{\tau_{k+1}}$. For the former, $\mathbf{w}_i^{k+1} = \mathbf{w}_i^k$ from (3.8) suffices to

$$u_i^k := L(\mathbf{w}^{\tau_{k+1}}, \mathbf{w}_i^{k+1}, \boldsymbol{\pi}_i^k) - L(\mathbf{w}^{\tau_{k+1}}, \mathbf{w}_i^k, \boldsymbol{\pi}_i^k) = 0.$$

For the latter, it follows from (7.3) that

$$\frac{\sigma_i}{2} \| \triangle \overline{\mathbf{w}}_i^{k+1} \|^2 - \frac{\sigma_i}{2} \| \mathbf{w}_i^k - \mathbf{w}^{\tau_{k+1}} \|^2 = \langle \triangle \mathbf{w}_i^{k+1}, \sigma_i \triangle \overline{\mathbf{w}}_i^{k+1} \rangle - \frac{\sigma_i}{2} \| \triangle \mathbf{w}_i^{k+1} \|^2.$$
(7.16)

Now we have the following chain of inequalities,

$$\begin{split} u_{i}^{k} &\stackrel{(3.9)}{=} & \alpha_{i}f_{i}(\mathbf{w}_{i}^{k+1}) - \alpha_{i}f_{i}(\mathbf{w}_{i}^{k}) + \langle \bigtriangleup \mathbf{w}_{i}^{k+1}, \pi_{i}^{k} \rangle + \frac{\sigma_{i}}{2} \|\bigtriangleup \overline{\mathbf{w}}_{i}^{k+1}\|^{2} - \frac{\sigma_{i}}{2} \|\mathbf{w}_{i}^{k} - \mathbf{w}^{\tau_{k+1}}\|^{2} \\ \stackrel{(7.16),(3.6)}{=} & \alpha_{i}f_{i}(\mathbf{w}_{i}^{k+1}) - \alpha_{i}f_{i}(\mathbf{w}_{i}^{k}) + \langle \bigtriangleup \mathbf{w}_{i}^{k+1}, \pi_{i}^{k+1} \rangle - \frac{\sigma_{i}}{2} \|\bigtriangleup \mathbf{w}_{i}^{k+1}\|^{2} \\ \stackrel{(7.1)}{\leq} & \frac{\alpha_{i}r_{i} - \sigma_{i}}{2} \|\bigtriangleup \mathbf{w}_{i}^{k+1}\|^{2} + \langle \bigtriangleup \mathbf{w}_{i}^{k+1}, g_{i}^{k+1} + \pi_{i}^{k+1} \rangle \\ \stackrel{(7.10)}{=} & \frac{\alpha_{i}r_{i} - \sigma_{i}}{2} \|\bigtriangleup \mathbf{w}_{i}^{k+1}\|^{2} + \langle \bigtriangleup \mathbf{w}_{i}^{k+1}, \varphi_{i}^{k+1} \rangle \\ \stackrel{(7.3)}{\leq} & \frac{3\alpha_{i}r_{i} - 2\sigma_{i}}{4} \|\bigtriangleup \mathbf{w}_{i}^{k+1}\|^{2} + \frac{1}{\alpha_{i}r_{i}} \|\varphi_{i}^{k+1}\|^{2} \\ \stackrel{(7.6)}{\leq} & \frac{3\alpha_{i}r_{i} - 2\sigma_{i}}{4} \|\bigtriangleup \mathbf{w}_{i}^{k+1}\|^{2} + \frac{\epsilon_{i}^{k+1}}{\alpha_{i}r_{i}} \\ \stackrel{(3.4)}{\leq} & \frac{3\alpha_{i}r_{i} - 2\sigma_{i}}{4} \|\bigtriangleup \mathbf{w}_{i}^{k+1}\|^{2} - \frac{\nu_{i}\bigtriangleup \epsilon_{i}^{k+1}}{(1 - \nu_{i})\alpha_{i}r_{i}}. \end{split}$$

Therefore, for both cases, we obtain

$$p_{2}^{k} = \sum_{i=1}^{m} u_{i}^{k} \leq \sum_{i=1}^{m} \left(\frac{3\alpha_{i}r_{i} - 2\sigma_{i}}{4} \| \Delta \mathbf{w}_{i}^{k+1} \|^{2} - \frac{\nu_{i} \Delta \epsilon_{i}^{k+1}}{(1 - \nu_{i})\alpha_{i}r_{i}} \right).$$
(7.17)

Estimate p_3^k . We consider two cases: $i \notin \Omega^{\tau_{k+1}}$ and $i \in \Omega^{\tau_{k+1}}$. For the former, $\pi_i^{k+1} = \pi_i^k$ from (3.8) suffices to

$$v_i^k := L(\mathbf{w}^{\tau_{k+1}}, \mathbf{w}_i^{k+1}, \pi_i^k) - L(\mathbf{w}^{\tau_{k+1}}, \mathbf{w}_i^{k+1}, \pi_i^k) = 0.$$

For the latter, it is easy to see that

$$v_i^k \stackrel{(3.9)}{=} \langle \triangle \overline{\mathbf{w}}_i^{k+1}, \triangle \pi_i^{k+1} \rangle \stackrel{(3.6)}{=} \frac{1}{\sigma_i} \| \triangle \pi_i^{k+1} \|^2 \stackrel{(7.7)}{\leq} \frac{5\alpha_i^2 r_i^2}{4\sigma_i} \| \triangle \mathbf{w}_i^{k+1} \|^2 - \frac{10(1+\nu_i)}{\sigma_i(1-\nu_i)} \triangle \epsilon_i^{k+1}.$$

Therefore, for both cases

$$p_{3}^{k} = \sum_{i=1}^{m} v_{i}^{k} \leq \sum_{i=1}^{m} \left(\frac{5\alpha_{i}^{2}r_{i}^{2}}{4\sigma_{i}} \| \Delta \mathbf{w}_{i}^{k+1} \|^{2} - \frac{10(1+\nu_{i})}{\sigma_{i}(1-\nu_{i})} \Delta \epsilon_{i}^{k+1} \right).$$
(7.18)

Overall, combining (7.11), (7.15), (7.17) and (7.18), we obtain

$$\mathcal{L}^{k+1} - \mathcal{L}^{k} = p_{1}^{k} + p_{2}^{k} + p_{3}^{k} \leq -\frac{\sigma}{2} \| \triangle \mathbf{w}^{\tau_{k+1}} \|^{2} - \sum_{i=1}^{m} \left(\frac{\nu_{i}}{(1-\nu_{i})\alpha_{i}r_{i}} + \frac{10(1+\nu_{i})}{\sigma_{i}(1-\nu_{i})} \right) \triangle \epsilon_{i}^{k+1} - \sum_{i=1}^{m} \left(\frac{2\sigma_{i} - 3\alpha_{i}r_{i}}{4} - \frac{5\alpha_{i}^{2}r_{i}^{2}}{4\sigma_{i}} \right) \| \triangle \mathbf{w}_{i}^{k+1} \|^{2},$$

which displays the result.

Lemma 7.2. Suppose that Assumptions 4.1 and 4.2 hold. Every client $i \in [m]$ chooses $\sigma_i > 5\alpha_i r_i/2$ and the sever selects Ω^{τ_k} as Scheme 4.1. Then the following results hold.

- a) Sequence $\{\widetilde{\mathcal{L}}^k\}$ is non-increasing.
- b) $\widetilde{\mathcal{L}}^k \ge f(\mathbf{w}^{\tau_k}) \ge f^* > -\infty$ for any $k \ge 1$.
- c) The limits of all the following terms are zero, namely,

$$(\epsilon_i^{k+1}, \Delta \mathbf{w}^{\tau_{k+1}}, \ \Delta \overline{\mathbf{w}}_i^{k+1}, \ \Delta \mathbf{w}_i^{k+1}, \ \Delta \pi_i^{k+1}) \to 0.$$
(7.19)

Proof. a) It follows from $\sigma_i > 5\alpha_i r_i/2$ that c > 0, which by Lemma 4.1 can conclude the conclusion.

b) The gradient Lipschitz continuity of f_i implies

$$\alpha_{i}f_{i}(\mathbf{w}^{\tau_{k}}) - \alpha_{i}f_{i}(\mathbf{w}_{i}^{k}) \stackrel{(7.1)}{\leq} \langle \Delta \overline{\mathbf{w}}_{i}^{k}, \boldsymbol{g}_{i}^{k} \rangle + \frac{r_{i}\alpha_{i}}{2} \| \Delta \overline{\mathbf{w}}_{i}^{k} \|^{2}$$

$$\stackrel{(7.6)}{=} \langle \Delta \overline{\mathbf{w}}_{i}^{k}, \boldsymbol{\pi}_{i}^{k} + \boldsymbol{\varphi}_{i}^{k} \rangle + \frac{r_{i}\alpha_{i}}{2} \| \Delta \overline{\mathbf{w}}_{i}^{k} \|^{2}$$

$$\stackrel{(7.3)}{\leq} \langle \Delta \overline{\mathbf{w}}_{i}^{k}, \boldsymbol{\pi}_{i}^{k} \rangle + \frac{3r_{i}\alpha_{i}}{4} \| \Delta \overline{\mathbf{w}}_{i}^{k} \|^{2} + \frac{1}{\alpha_{i}r_{i}} \| \boldsymbol{\varphi}_{i}^{k} \|^{2}$$

$$\stackrel{(7.6)}{\leq} \langle \Delta \overline{\mathbf{w}}_{i}^{k}, \boldsymbol{\pi}_{i}^{k} \rangle + \frac{3r_{i}\alpha_{i}}{4} \| \Delta \overline{\mathbf{w}}_{i}^{k} \|^{2} + \frac{\epsilon_{i}^{k}}{\alpha_{i}r_{i}}.$$

The above relation and $\nu_i \ge 1/2$ gives rise to

$$\begin{aligned} \widetilde{\mathcal{L}}^{k} \stackrel{(4.4)}{\geq} & \sum_{i=1}^{m} (L(\mathbf{w}^{\tau_{k}}, \mathbf{w}_{i}^{k}, \pi_{i}^{k}) + \frac{\nu_{i}\epsilon_{i}^{k}}{(1-\nu_{i})\alpha_{i}r_{i}}) \\ \stackrel{(3.9)}{\geq} & \sum_{i=1}^{m} (\alpha_{i}f_{i}(\mathbf{w}_{i}^{k}) + \langle \bigtriangleup \overline{\mathbf{w}}_{i}^{k}, \pi_{i}^{k} \rangle + \frac{\sigma_{i}}{2} \|\bigtriangleup \overline{\mathbf{w}}_{i}^{k}\|^{2} + \frac{\epsilon_{i}^{k}}{\alpha_{i}r_{i}}) \\ & \geq & \sum_{i=1}^{m} (\alpha_{i}f_{i}(\mathbf{w}^{\tau_{k}}) + \frac{2\sigma_{i}-3r_{i}\alpha_{i}}{4} \|\bigtriangleup \overline{\mathbf{w}}_{i}^{k}\|^{2}) \\ & \geq & \sum_{i=1}^{m} \alpha_{i}f_{i}(\mathbf{w}^{\tau_{k}}) = f(\mathbf{w}^{\tau_{k}}) \geq f^{*} \stackrel{(2.4)}{>} -\infty. \end{aligned}$$

c) Using Lemma 4.1 and $\widetilde{\mathcal{L}}^k > -\infty$ enables to show that

$$\sum_{k\geq 0} \left(\frac{\sigma}{2} \| \Delta \mathbf{w}^{\tau_{k+1}} \|^2 + \sum_{i=1}^m \frac{c}{2} \| \Delta \mathbf{w}_i^{k+1} \|^2 \right)$$

$$\leq \sum_{k\geq 0} \left(\widetilde{\mathcal{L}}^k - \widetilde{\mathcal{L}}^{k+1}\right) = \widetilde{\mathcal{L}}^0 - f^* < +\infty.$$
(7.20)

The above condition means $\|\triangle \mathbf{w}^{\tau_{k+1}}\| \to 0$ and $\|\triangle \mathbf{w}_i^{k+1}\| \to 0$, which by (7.7) and (7.8) derives $\|\triangle \pi_i^{k+1}\| \to 0$. Finally, for $i \in \Omega^{\tau_{k+1}}$, it follows from (3.6) that $\|\sigma_i \triangle \overline{\mathbf{w}}_i^{k+1}\| = \|\triangle \pi_i^{k+1}\| \to 0$. For $i \notin \Omega^{\tau_{k+1}}$, then $i \in \Omega^{\tau_{k+1}}$, where k_i is defined the same as that in the proof of Lemma 7.1 a). Based on (4.3), we have $\tau_{k+1} - \tau_{k_i+1} \leq 2s_0$. As a consequence,

$$\begin{split} \| \triangle \overline{\mathbf{w}}_{i}^{k+1} \|^{2} \stackrel{(7.9)}{=} \| \mathbf{w}_{i}^{k_{i}+1} - \mathbf{w}^{\tau_{k+1}} \|^{2} \\ &\leq 2 \| \triangle \overline{\mathbf{w}}_{i}^{k_{i}+1} \|^{2} + 2 \| \mathbf{w}^{\tau_{k_{i}+1}} - \mathbf{w}^{\tau_{k+1}} \|^{2} \\ \stackrel{(3.6)}{=} \frac{2}{\sigma_{i}^{2}} \| \triangle \pi_{i}^{k_{i}+1} \|^{2} + 2 \| \sum_{\tau=\tau_{k_{i}+1}}^{\tau_{k+1}-1} (\mathbf{w}^{\tau} - \mathbf{w}^{\tau+1}) \|^{2} \\ \stackrel{(7.3)}{\leq} \frac{2}{\sigma_{i}^{2}} \| \triangle \pi_{i}^{k_{i}+1} \|^{2} + 4s_{0} \sum_{\tau=\tau_{k_{i}+1}}^{\tau_{k+1}-1} \| \mathbf{w}^{\tau} - \mathbf{w}^{\tau+1} \|^{2} \rightarrow 0, \end{split}$$
(7.21)

where the last relationship is due to $\triangle \pi_i^k \rightarrow \text{and } \triangle \mathbf{w}^{\tau_{k+1}} \rightarrow 0$. The whole proof is finished. \Box

7.4 Proof of Theorem 4.1

Proof. a) By Lemma 7.2 that $\widetilde{\mathcal{L}}^1 \geq f(\mathbf{w}^{\tau_{k+1}})$ and f being coercive, we can claim the boundedness of sequence $\{\mathbf{w}_i^{k+1}\}$ immediately. This calls forth the boundedness of sequence $\{\mathbf{w}_i^{k+1}\}$ as $\Delta \overline{\mathbf{w}}_i^{k+1} \to 0$ from (7.19), thereby delivering

$$\begin{aligned} \|\boldsymbol{\pi}_{i}^{k+1}\| &\stackrel{(7.6)}{=} & \|\boldsymbol{\varphi}_{i}^{k+1} - \boldsymbol{g}_{i}^{k+1}\| \\ &\leq & \|\boldsymbol{\varphi}_{i}^{k+1}\| + \|\boldsymbol{g}_{i}^{k+1} - \boldsymbol{g}_{i}^{0}\| + \|\boldsymbol{g}_{i}^{0}\| \\ &\stackrel{(7.6),(2.1)}{\leq} \sqrt{\epsilon_{i}^{k+1}} + \alpha_{i}r_{i}\|\mathbf{w}_{i}^{k+1} - \mathbf{w}_{i}^{0}\| + \|\boldsymbol{g}_{i}^{0}\| < +\infty. \end{aligned}$$

This shows the boundedness of $\{\pi_i^{k+1}\}$. Overall, sequence $\{(\mathbf{w}^{\tau_{k+1}}, W^{k+1}, \Pi^{k+1})\}$ is bounded.

b) It follows from Lemma 7.2 that $\{\widetilde{\mathcal{L}}^k\}$ is non-increasing and bounded from below. Therefore, the whole sequence $\{\widetilde{\mathcal{L}}^k\}$ converges and $\widetilde{\mathcal{L}}^{k+1} \to \mathcal{L}^{k+1}$ due to $\epsilon_i^{k+1} \to 0$ in (7.19). Again by (7.19) and the boundedness of sequence $\{\pi_i^{k+1}\}$, we can prove that

$$\mathcal{L}^{k+1} - F(W^{k+1}) \stackrel{(3.9)}{=} \sum_{i=1}^{m} (\langle \bigtriangleup \overline{\mathbf{w}}_i^{k+1}, \pi_i^{k+1} \rangle + \frac{\sigma_i}{2} \| \bigtriangleup \overline{\mathbf{w}}_i^{k+1} \|^2) \to 0.$$
(7.22)

It follows from Mean Value Theory that

$$f_i(\mathbf{w}_i^{k+1}) = f_i(\mathbf{w}^{\tau_{k+1}}) + \langle \triangle \overline{\mathbf{w}}^{k+1}, \nabla f_i(\mathbf{w}_t) \rangle,$$

where $\mathbf{w}_t := (1-t)\mathbf{w}^{\tau_{k+1}} + t\mathbf{w}_i^{k+1}$ for some $t \in (0,1)$. Since $\{\mathbf{w}^{\tau_{k+1}}, \mathbf{w}_i^{k+1}\}$ is bounded, so is \mathbf{w}_t . This calls forth $f_i(\mathbf{w}_i^{k+1}) - f_i(\mathbf{w}^{\tau_{k+1}}) \to 0$ due to $\Delta \overline{\mathbf{w}}^{k+1} \to 0$. Using this condition obtains

$$\mathcal{L}^{k+1} - f(\mathbf{w}^{\tau_{k+1}}) = \sum_{i=1}^{m} (\alpha_i f_i(\mathbf{w}_i^{k+1}) - \alpha_i f_i(\mathbf{w}^{\tau_{k+1}}) + \langle \triangle \overline{\mathbf{w}}^{k+1}, \boldsymbol{\pi}_i^{k+1} \rangle + \frac{\sigma_i}{2} \| \triangle \overline{\mathbf{w}}^{k+1} \|^2) \to 0$$

b) Let $\ell := (\tau_{k+1} - 1)k_0 \in \mathcal{K}$, then for any k, it has

$$\ell + 1 = (\tau_{k+1} - 1)k_0 + 1 \le k + 1 \le \tau_{k+1}k_0,$$

$$\tau_{\ell+1} = \lfloor (\ell+1)/k_0 \rfloor = \lfloor \tau_{k+1} - 1 - 1/k_0 \rfloor = \tau_{k+1}.$$
(7.23)

Since $\ell \in \mathcal{K}$, we have

$$\sum_{i=1}^{m} \boldsymbol{\pi}_{i}^{\ell+1} \stackrel{(7.19)}{\rightarrow} \sum_{i=1}^{m} (\sigma_{i}(\bigtriangleup \overline{\mathbf{w}}_{i}^{\ell+1} - \bigtriangleup \mathbf{w}_{i}^{\ell+1}) - \bigtriangleup \boldsymbol{\pi}_{i}^{\ell+1} + \boldsymbol{\pi}_{i}^{\ell+1})$$
$$= \sum_{i=1}^{m} (\sigma_{i}(\mathbf{w}_{i}^{\ell} - \mathbf{w}^{\tau_{\ell+1}}) + \boldsymbol{\pi}_{i}^{\ell}) \stackrel{(7.5)}{=} 0.$$

We note that sequence $\{\epsilon_i^{k+1}\}$ is non-increasing and thus obtain $\epsilon_i^{\ell+1} \leq \epsilon_i^{k+1}$ from (7.23), thereby rendering that

$$\begin{split} \|\boldsymbol{\pi}_{i}^{k+1} - \boldsymbol{\pi}_{i}^{\ell+1}\|^{2} &\stackrel{(7.6)}{=} & \|\boldsymbol{\varphi}_{i}^{k+1} - \boldsymbol{\varphi}_{i}^{\ell+1} - \boldsymbol{g}_{i}^{k+1} + \boldsymbol{g}_{i}^{\ell+1}\|^{2} \\ &\stackrel{(7.6),(2.1)}{\leq} & 3\epsilon_{i}^{k+1} + 3\epsilon_{i}^{\ell+1} + 3\alpha_{i}^{2}r_{i}^{2}\|\mathbf{w}_{i}^{k+1} - \mathbf{w}_{i}^{\ell+1}\|^{2} \\ &\stackrel{(7.23)}{=} & 6\epsilon_{i}^{k+1} + 3\alpha_{i}^{2}r_{i}^{2}\|\mathbf{w}_{i}^{k+1} - \mathbf{w}^{\tau_{k+1}} + \mathbf{w}^{\tau_{\ell+1}} - \mathbf{w}_{i}^{\ell+1}\|^{2} \\ &\leq & 6\epsilon_{i}^{k+1} + 6\alpha_{i}^{2}r_{i}^{2}(\|\Delta\overline{\mathbf{w}}_{i}^{k+1}\|^{2} + \|\Delta\overline{\mathbf{w}}_{i}^{\ell+1}\|^{2}) \\ &\stackrel{(7.19)}{\to} & 0. \end{split}$$

Using the above two conditions immediately derives that

$$\sum_{i=1}^{m} \pi_i^{k+1} \to 0.$$
 (7.24)

Taking the limit on both sides of (7.6) gives us

$$\nabla F(W^{k+1}) = \sum_{i=1}^{m} \boldsymbol{g}_{i}^{k+1} \stackrel{(7.24)}{\to} \sum_{i=1}^{m} (\boldsymbol{g}_{i}^{k+1} + \boldsymbol{\pi}_{i}^{k+1})$$

$$\stackrel{(7.6)}{=} \sum_{i=1}^{m} \boldsymbol{\varphi}_{i}^{k+1} \stackrel{(7.7)}{\to} 0,$$
(7.25)

which together with $\Delta \overline{\mathbf{w}}_i^{k+1} \to 0$ and the gradient Lipschitz continuity yields that $\nabla f(\mathbf{w}^{\tau_{k+1}}) = \sum_{i=1}^m \alpha_i \nabla f_i(\mathbf{w}^{\tau_{k+1}}) \to 0$. This completes the whole proof.

7.5 Proof of Theorem 4.2

Proof. a) Let $(\mathbf{w}^{\infty}, W^{\infty}, \Pi^{\infty})$ be any accumulating point of the sequence, it follows from (7.6) and (7.7) that

$$0 = \alpha_i \nabla f_i(\mathbf{w}_i^\infty) + \boldsymbol{\pi}_i^\infty.$$

By $(\mathbf{w}_i^{k+1} - \mathbf{w}^{\tau_{k+1}}) \to 0$ and (7.24), we have

$$0 = \mathbf{w}_i^{\infty} - \mathbf{w}^{\infty}, \quad 0 = \sum_{i=1}^m \boldsymbol{\pi}_i^{\infty}.$$

Therefore, recalling (4.1), $(\mathbf{w}^{\infty}, W^{\infty}, \Pi^{\infty})$ is a stationary point of problem (3.1) and \mathbf{w}^{∞} is a stationary point of problem (2.3).

b) It follows from [42, Lemma 4.10], $\triangle \mathbf{w}^{\tau_{k+1}} \to 0$ and \mathbf{w}^{∞} being isolated that the whole sequence, $\{\mathbf{w}^{\tau_{k+1}}\}$ converges to \mathbf{w}^{∞} , which by $\triangle \overline{\mathbf{w}}_i^{k+1} \to 0$ implies that $\{W^{k+1}\}$ also converges to W^{∞} . Finally, this together with (7.6) and (7.7) results in the convergence of $\{\Pi^{k+1}\}$.

7.6 Proof of Corollary 4.1

Proof. a) The convexity of f and the optimality of \mathbf{w}^* yields

$$f(\mathbf{w}^{\tau_k}) \ge f(\mathbf{w}^*) \ge f(\mathbf{w}^{\tau_k}) + \langle \nabla f(\mathbf{w}^{\tau_k}), \mathbf{w}^* - \mathbf{w}^{\tau_k} \rangle.$$
(7.26)

Theorem 4.1 ii) states that

$$\lim_{k \to \infty} \nabla F(W^k) = \lim_{k \to \infty} \nabla f(\mathbf{w}^{\tau_k}) = 0.$$

Using this and the boundedness of $\{\mathbf{w}^{\tau_k}\}$ from Theorem 4.2, we take the limit of both sides of (7.26) to derive that $f(\mathbf{w}^{\tau_k}) \to f(\mathbf{w}^*)$, which recalling Theorem 4.1 i) yields (4.7).

b) The conclusion follows from Theorem 4.2 ii) and the fact that the stationary points are equivalent to optimal solutions if f is convex.

c) The strong convexity of f means that there is a positive constance ν such that

$$f(\mathbf{w}^{\tau_k}) - f(\mathbf{w}^*) \ge \langle \nabla f(\mathbf{w}^*), \mathbf{w}^{\tau_k} - \mathbf{w}^* \rangle + \frac{\nu}{2} \|\mathbf{w}^{\tau_k} - \mathbf{w}^*\|^2 = \frac{\nu}{2} \|\mathbf{w}^{\tau_k} - \mathbf{w}^*\|^2,$$

where the equality is due to (4.2). Taking limit of both sides of the above inequality shows $\mathbf{w}^{\tau_k} \to \mathbf{w}^*$ since $f(\mathbf{w}^{\tau_k}) \to f(\mathbf{w}^*)$. This together with (7.19) yields $\mathbf{w}_i^k \to \mathbf{w}^*$. Finally, $\pi_i^k \to \pi_i^*$ because of

$$\begin{aligned} \|\boldsymbol{\pi}_{i}^{k} - \boldsymbol{\pi}_{i}^{*}\|^{2} \stackrel{(4.1),(7.6)}{=} \|\boldsymbol{\varphi}_{i}^{k} + \boldsymbol{g}_{i}^{k} - \alpha_{i} \nabla f_{i}(\mathbf{w}^{*})\|^{2} \\ \stackrel{(2.1)}{\leq} 2\alpha_{i}^{2} r_{i}^{2} \|\mathbf{w}_{i}^{k} - \mathbf{w}^{*}\|^{2} + 2\epsilon_{i}^{k} \to 0, \end{aligned}$$

displaying the desired result.