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UNIQUENESS AND REGULARITY FOR THE NAVIER--STOKES--CAHN--HILLIARD SYSTEM *

ANDREA GIORGINI[†], ALAIN MIRANVILLE[‡], AND ROGER TEMAM[§]

Abstract. The motion of two contiguous incompressible and viscous fluids is described within the diffuse interface theory by the so-called Model H. The system consists of the Navier--Stokes equations, which are coupled with the Cahn--Hilliard equation associated to the Ginzburg--Landau free energy with physically relevant logarithmic potential. This model is studied in bounded smooth domains in $\mathbb{R}^d$, $d = 2$, and $d = 3$ and is supplemented with a no-slip condition for the velocity, homogeneous Neumann boundary conditions for the order parameter and the chemical potential, and suitable initial conditions. We study uniqueness and regularity of weak and strong solutions. In a two-dimensional domain, we show the uniqueness of weak solutions and the existence and uniqueness of global strong solutions originating from an initial velocity $u_0 \in \mathbf{V}_\sigma$, namely, $u_0 \in \mathbf{H}_0^1(\Omega)$ such that $\operatorname{div} u_0 = 0$. In addition, we prove further regularity properties and the validity of the instantaneous separation property. In a three-dimensional domain we show the existence and uniqueness of local strong solutions with initial velocity $u_0 \in \mathbf{V}_\sigma$.

Key words. Navier--Stokes equations, Cahn--Hilliard equation, logarithmic potential, uniqueness, strong solutions

AMS subject classifications. 35Q35, 35K61, 76D03

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1. Introduction. In the diffuse interface theory, the motion of two incompressible and viscous fluids and the evolution of the interface that separates them are described by the Model H. The domain Ω of $\mathbb{R}^d$, $d = 2$ or $d = 3$, is filled with a mixture of two fluids with the same density; the concentrations of the fluids are φ_i, $i = 1,2$, where $\varphi_i \in [0,1]$ and $\varphi_1 + \varphi_2 = 1$. The physics of the Model H is such that the interface between the two fluids is assumed to be a narrow region with finite thickness. The concentrations are uniform (equal to 0 or 1) in subregions of Ω and vary steeply but continuously across the thin interface layer. This formulation allows large interface deformations and topological changes of the interfaces in the mixture. After the seminal work [57] on critical points of single and binary fluids, a detailed derivation of the Model H was proposed in [53] and [78] for the flow driven by capillarity forces. The model is based on the balance of mass and momentum

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of Indiana University. \dagger Department of Mathematics \& Institute for Scientific Computing and Applied Mathematics,
Indiana
University, Bloomington, IN 47405 (agiorgin@iu.edu).

\ddagger Laboratoire de Math\'ematiques et Applications, Universit\'e de Poitiers, UMR CNRS 7348, Boulevard Marie et Pierre Curie - T\'el\'eport 2, F-86962, Chasseneuil Futuroscope Cedex, France, and School of Mathematical Sciences, Xiamen University, Xiamen, Fujian, China, and Fudan University (Fudan Fellow), Shanghai, China (alain.miranville@math.univ-poitiers.fr).

\S Institute for Scientific Computing and Applied Mathematics, Indiana University, Bloomington, IN 47405 (temam@indiana.edu).

that are combined with constitutive laws compatible with a version of the second law of thermodynamics. Model H has been employed in several numerical studies for concrete applications. Relevant examples are interface stretching during mixing [25], thermocapillary flows [62], droplet formation and collision, moving contact lines, and large-deformation flows [60, 68]. For a review on these topics we refer the reader to

[10] and the references therein. Further generalizations of the Model H have been dis cussed for fluid mixtures with different densities in [8, 11, 18, 33, 69], and for contact angle problems and ternary fluids in [19, 65] and the references therein.

Assuming that density differences are negligible, we consider two state variables: the volume-averaged fluid velocity $u = u(x,t)$ and the difference of the fluids con -

centrations (order parameter) $\varphi = \varphi(x,t)$, equal to $\varphi_1 - \varphi_2$ in the notation above, where $x \in \Omega \subset \mathbb{R}^d$, $d = 2$ or $d = 3$, Ω being a bounded domain with smooth bound ary $\partial \Omega$, and t the time. The evolution of the two state variables is governed by the Navier--Stokes--Cahn--Hilliard (NSCH) system, which reads in dimensionless form:

$$\mathrm{div}(\nu(\varphi)Du) + \nabla \pi = \mu \nabla \varphi,$$

$$
(1.1)\quad \begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \\ \operatorname{div} \boldsymbol{u} = 0, \\ \partial_t\varphi + \boldsymbol{u}\cdot\nabla\varphi = \Delta\mu, \\ \mu = -\Delta\varphi + \Psi^\prime(\varphi), \end{cases} \quad \text{in } \Omega\times(0,T),
$$

subject to the boundary and initial conditions

$$
(1.2)\quad \begin{cases} \boldsymbol{u} = \mathbf{0}, \quad \partial_{\mathbf{n}}\mu = \partial_{\mathbf{n}}\varphi = 0 & \text{on } \partial\Omega\times(0,T), \\ \boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0, \quad \varphi(\cdot,0) = \varphi_0 & \text{in } \Omega. \end{cases}
$$

Here $\mathbf{n}$ is the unit outward normal vector to the boundary $\partial\Omega$, $Du = \frac{1}{2}\bigl(\nabla u + (\nabla u)^t\bigr)$ is the symmetric gradient, $\pi = \pi(x,t)$ is the pressure, and $\mu = \mu(x,t)$ is the socalled chemical potential. The potential Ψ is the physically relevant homogeneous free energy density introduced in [22] and defined as

$$
(1.3)\quad \Psi(z) = \frac{\theta}{2}\Big((1+z)\log(1+z) + (1-z)\log(1-z)\Big) - \frac{\theta_0}{2}z^2 \quad \forall z \in [-1,1],
$$

where θ and θ_0 are related to the absolute temperature of the mixture and the crit- ical temperature, respectively. These two constant parameters satisfy the physical relations $0 < \theta < \theta_0$. This condition implies the double-well form to the potential (1.3). The mathematical analysis of (1.1)--(1.2) may lead to a solution φ with arbitrary values in $\mathbb{R}$ whatever the potential Ψ, but we have to keep in mind that, by its very definition, $-1 \leq \varphi \leq 1$ (± 1 represent the pure concentrations), and we call these *physical* solutions. Now, assuming that ν_1 and ν_2 are the viscosities of the two homogeneous fluids, the viscosity of the mixture is modeled by the concentration dependent term $\nu = \nu(\varphi)$. In the unmatched viscosity case ($\nu_1 \not= \nu_2$), a typical form for

ν is the linear combination (see, e.g., [65] and Remark 2.1 below):

$$\nu(z) = \nu_1 \frac{1+z}{2} + \nu_2 \frac{1-z}{2} \quad \forall z \in [-1,1] \tag{1.4}$$

The particular case $\nu_1 = \nu_2$ is called matched viscosity case, and ν is a positive constant.

In the literature, the NSCH system has been widely studied by considering regular approximations of the logarithmic potential (1.3). Typical examples are polynomial- like functions, such as $\Psi_0(z) = \frac{\kappa}{4}(z^2 - \beta^2)^2$, where $\kappa > 0$ is related to θ and θ_0, and $\pm\beta$ are the two minima of Ψ. In the matched viscosity case, the mathematical analysis of problem (1.1)--(1.2) with regular potentials is now well established, at least for classical boundary conditions. We refer the reader to [17, 15, 41, 44, 43, 48] (see also [16, 23, 46] for the analysis of similar systems). In the unmatched viscosity case, the author in [17] proved the global existence of weak solutions and the existence and uniqueness of strong solutions (global if $d = 2$, local if $d = 3$). Concerning the longtime behavior, the existence of the trajectory attractor is showed in [44], while the convergence to equilibrium is established in [85] for periodic boundary conditions. However, in the case of polynomial potentials, it is worth recalling that it is not possible to guarantee the existence of *physical* solutions, that is, solutions for which $-1 \leq \varphi(x,t) \leq 1$, for almost every $x \in \Omega$ and $t > 0$.

On the other hand, few results are available for the original Model H with logarithmic potential (1.3). The NSCH system with unmatched viscosities and logarithmic potential has been only studied in [2], where existence of global weak (*physical*) solutions and existence and uniqueness of strong solutions (global if $d = 2$, local if $d = 3$) are shown (see [2, Theorem 1 and 2]). In particular, in two dimensions, assuming $u_0 \in \mathbf{V}_2^{1+r}(\Omega)$ for $r > 0$, where $\mathbf{V}_2^{1+r}(\Omega) = (\mathbf{V}_\sigma, \mathbf{W}_\sigma)_{r,2}$ is an interpolation space, and $\mathbf{V}_\sigma$ and $\mathbf{W}_\sigma$ are defined below in section 2, and assuming a natural higher-order condition on φ_0 (cf. Theorem 4.1 below), the corresponding strong solution (u, φ) is global in time and unique. In three dimensions, the local existence and uniqueness of strong solutions is achieved provided that the initial velocity u_0 belongs to $\mathbf{V}_2^{1+r}(\Omega)$ with $r > \frac{1}{2}$. The restriction on the initial velocity in $\mathbf{V}_2^{1+r}$ ($r > 0$ if $d = 2$ and

$r > \frac{1}{2}$ if $d = 3$) is due to the uniqueness result [2, Proposition 1], which requires that $u \in L^\infty(0,T;\mathbf{W}^{1,q}(\Omega))$, with $q > 2$ if $d = 2$ and $q = 3$ if $d = 3$, being not true for classical strong solutions of the Navier--Stokes equations for an initial velocity $u_0 \in \mathbf{V}_\sigma$. In addition, the author in [2] shows that any weak solution is more regular on the interval $[T,\infty)$, for some $T > 0$ which is not explicitly estimated. It satisfies the so-called *asymptotic* separation property (see [2, Lemma 12]), namely,

$$\exists \delta > 0, \exists T > 0 : \|\varphi(t)\|_{L^\infty(\Omega)} \leq 1 - \delta \quad \forall t \geq T. \tag{1.5}$$

This is a key property in order to show that any single trajectory converges to an equilibrium [2, Theorem 3]. We also mention the results in [6, 45, 71], where the global existence of weak solutions to similar systems has been established. In [6] the author considers a version of the NSCH system for non-Newtonian fluids, in [45] the authors study the NSCH system with boundary conditions that account for a moving contact line slip velocity, whereas in [71] the authors consider the NSCH-Oono system. For the sake of completeness, we refer the interested reader to [3, 1, 4, 5, 7] for the analysis of the NSCH system with different densities. Finally, we mention among many references [12, 13, 27, 28, 26, 31, 32, 37, 38, 39, 47, 51, 52, 54, 55, 56, 61, 64, 63, 58, 68, 74, 75, 59, 76, 79, 83, 84] for the numerical analysis, in particular stability and convergence analysis, numerical simulations, and control problems of the NSCH system. At this stage we note that to date some important issues are still unsolved, such as the uniqueness of weak solutions of the NSCH in dimension two as well as the uniqueness of strong solutions with initial velocity in $\mathbf{V}_\sigma$ in both two and three dimensions. It is not even known whether such properties hold in the simpler case with matched viscosities. Besides, uniqueness of weak solutions in dimension two is an open question even for the NSCH system with regular potential and unmatched viscosities.

The aim of this work is to answer positively to the above mentioned open questions. Our main results for the NSCH system with unmatched viscosities are the following:

1. If $d = 2$, we show the uniqueness of weak (*physical*) solutions.
2. If $d = 2$, we prove the global existence and uniqueness of strong solutions when $u_0 \in \mathbf{V}_\sigma$.

3. If $d = 2$, we show that any (weak or strong) solution becomes instantaneously more regular (that is, on $[\tau, \infty)$ for any $\tau > 0$), and it satisfies the *instantaneous* separation property, namely,

$$\forall \tau > 0, \exists \delta = \delta(\tau) > 0 : \|\varphi(t)\|_{L^\infty(\Omega)} \leq 1 - \delta \quad \forall t \geq \tau. \tag{1.6}$$

4. If $d = 3$, we prove the local existence and uniqueness of strong solutions when $u_0 \in \mathbf{V}_\sigma$.

We observe that the technique here employed to prove the uniqueness of weak solutions in dimension two can be applied to show the same result for the following two cases: logarithmic potential and matched viscosities as well as regular potentials and unmatched viscosities (see Remark 3.2 and 3.3). It is worth mentioning that our method not only entails the uniqueness of weak solutions in dimension two but a continuous dependence estimate on the initial data with a time-dependent exponent.

The mathematical analysis presented in this paper may be employed to investigate other diffuse interface models with logarithmic potential (1.3), also in connection with the study of optimal control problems and the analysis of numerical schemes. Among several models, we mention those systems that involve different laws for the velocity field, such as the Hele--Shaw and Brinkman approximations [29, 50] or regularized family of the Navier--Stokes equations [46] (see also [23]). It would be interesting as well to analyze modified equations of the Cahn--Hilliard type [19, 49, 70, 71] or the Allen--Cahn equation (see, e.g., [42]). A further important issue would be to extend the analysis to the nonisothermal version of the Model H introduced in [34, 35] and to the Model H with mass transfer and chemotaxis presented in [66].

Plan of the paper. In section 2 we introduce the functions spaces, the main assumptions of the paper, and we report a result of existence of weak solutions. In section 3 we discuss the uniqueness of weak solutions in two dimensions. Section 4 is devoted to analysis of strong solutions, the instantaneous regularization of weak solutions, and the separation property in space dimension two. Section 5 is devoted to the study of strong solutions in space dimension three. We report in Appendixes A and B some mathematical tools regarding the Neumann and Stokes problems.

2. Preliminaries.

2.1. Notation and functions spaces. Let X be a (real) Banach or Hilbert space with norm denoted by $\| \cdot \|_X$. The boldface letter $\mathbf{X}$ stands for the vectorial space X^d (d is the spatial dimension), which consists of vector-valued functions u with all components belonging to X, with norm $\| \cdot \|_{\mathbf{X}}$. Let Ω be a bounded domain in $\mathbb{R}^d$, where $d = 2$ or $d = 3$, with smooth boundary $\partial \Omega$. We denote by $W^{k,p}(\Omega)$, $k \in \mathbb{N}$, the Sobolev space of functions in $L^p(\Omega)$ with distributional derivatives of order less than or equal to k in $L^p(\Omega)$ and by $\| \cdot \|_{W^{k,p}(\Omega)}$ its norm. For $k \in \mathbb{N}$, the Hilbert space $W^{k,2}(\Omega)$ is denoted by $H^k(\Omega)$ with norm $\| \cdot \|_{H^k(\Omega)}$. We denote by $H^1_0(\Omega)$ the closure of $C^\infty_0(\Omega)$ in $H^1(\Omega)$ and by $H^{-1}(\Omega)$ its dual space. We define $H = L^2(\Omega)$. Its inner product and norm are denoted by $(\cdot,\cdot)$ and $\| \cdot \|$, respectively. We set $V = H^1(\Omega)$ with norm $\| \cdot \|_V$, and we denote its dual space by V' with norm $\| \cdot \|_{V'}$. The symbol $\langle \cdot,\cdot \rangle$ will stand for the duality product between V and V'. We denote by $\overline{u}$ the average of u over Ω; that is, $\overline{u} = |\Omega|^{-1}\langle u,1\rangle$ for all $u \in V'$. By the generalized Poincaré inequality (see [80, Chapter II, section 1.4]), we recall that $u \to (\|\nabla u\|^2 + |\overline{u}|^2)^{\frac{1}{2}}$ is a norm on V equivalent to the natural one. We recall the following Gagliardo–Nirenberg and Agmon inequalities (see, e.g., [81])

$$\|u\|_{L^4(\Omega)} \leq C\|u\|^{\frac{1}{2}}\|u\|_V^{\frac{1}{2}} \quad \forall u \in V \quad \text{if } d = 2, \tag{2.1}$$

$$\|u\|_{L^3(\Omega)} \leq C\|u\|^{\frac{1}{2}}\|u\|_V^{\frac{1}{2}} \quad \forall u \in V \quad \text{if } d = 3, \tag{2.2}$$

$$\|u\|_{L^\infty(\Omega)} \leq C\|u\|^{\frac{1}{2}}\|u\|_{H^2(\Omega)}^{\frac{1}{2}} \quad \forall u \in H^2(\Omega) \quad \text{if } d = 2, \tag{2.3}$$

$$\|\nabla u\|_{\mathbf{L}^4(\Omega)} \leq C\|u\|_{L^\infty(\Omega)}^{\frac{1}{2}}\|u\|_{H^2(\Omega)}^{\frac{1}{2}} \quad \forall u \in H^2(\Omega) \quad \text{if } d = 2,3, \tag{2.4}$$

and the Brezis–Gallouet inequality (see [21])

$$\|u\|_{L^\infty(\Omega)} \leq C\|u\|_V \left[\log\left(\mathrm{e} + \frac{\|u\|_{H^2(\Omega)}}{\|u\|_V}\right)\right]^{\frac{1}{2}} \quad \forall u \in H^2(\Omega) \quad \text{if } d = 2. \tag{2.5}$$

We now introduce the Hilbert space of solenoidal vector-valued functions. We denote by $\mathcal{C}^{\infty}_{0,\sigma}(\Omega)$ the space of divergence free vector fields in $\mathcal{C}^{\infty}_{0}(\Omega)$. We define $\mathbf{H}_\sigma$ and $\mathbf{V}_\sigma$ as the closure of $\mathcal{C}^{\infty}_{0,\sigma}(\Omega)$ with respect to the $\mathbf{H}$ and $\mathbf{H}^1_0(\Omega)$ norms, respectively. We also use $(\cdot,\cdot)$ and $\|\cdot\|$ for the norm and the inner product in $\mathbf{H}_\sigma$. The space $\mathbf{V}_\sigma$ is endowed with the inner product and norm $(\boldsymbol{u},\boldsymbol{v})_{\mathbf{V}_\sigma} = (\nabla \boldsymbol{u}, \nabla \boldsymbol{v})$ and $\|\boldsymbol{u}\|_{\mathbf{V}_\sigma} = \|\nabla \boldsymbol{u}\|$, respectively. We denote by $\mathbf{V}_\sigma'$ its dual space. We recall that Korn's inequality entails

$$\|\nabla \boldsymbol{u}\| \leq \sqrt{2}\|D\boldsymbol{u}\| \leq \sqrt{2}\|\nabla \boldsymbol{u}\| \quad \forall\, \boldsymbol{u} \in \mathbf{V}_\sigma,$$

where $D\boldsymbol{u} = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^t)$. In turn, the above inequality gives that $\boldsymbol{u} \rightarrow \|D\boldsymbol{u}\|$ is a norm on $\mathbf{V}_\sigma$ equivalent to the initial norm. We consider the Hilbert space $\mathbf{W}_\sigma = \mathbf{H}^2(\Omega) \cap \mathbf{V}_\sigma$ with inner product and norm $(\boldsymbol{u},\boldsymbol{v})_{\mathbf{W}_\sigma} = (\mathbf{A}\boldsymbol{u},\mathbf{A}\boldsymbol{v})$ and $\|\boldsymbol{u}\|_{\mathbf{W}_\sigma} = \|\mathbf{A}\boldsymbol{u}\|$, where $\mathbf{A}$ is the Stokes operator (see Appendix B for the definition and some properties). We recall that there exists $C > 0$ such that

$$\|\boldsymbol{u}\|_{H^2(\Omega)} \leq C\|\boldsymbol{u}\|_{\mathbf{W}_\sigma} \quad \forall\, \boldsymbol{u} \in \mathbf{W}_\sigma. \tag{2.6}$$

Finally, we introduce the trilinear continuous form on $\mathbf{H}^1_0(\Omega)$

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = \int_\Omega (\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{w}\,\mathrm{d}x = \sum_{i,j=1}^{2}\int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j\,\mathrm{d}x, \quad \forall\, \boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \mathbf{H}^1_0(\Omega),$$

satisfying the relation $b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v}) = 0$ for all $\boldsymbol{u} \in \mathbf{V}_\sigma$ and $\boldsymbol{v} \in \mathbf{H}^1(\Omega)$.

2.2. Main assumptions. We require that the viscosity $\nu \in \mathcal{C}^2(\mathbb{R})$ satisfies

$$0 < 2\nu_\ast \leq \nu(z) \leq \nu^\ast \quad \forall\, z \in \mathbb{R}, \tag{2.7}$$

for some positive values $\nu_\ast, \nu^\ast$. The singular potential Ψ belongs to the class of functions $\mathcal{C}([-1,1]) \cap \mathcal{C}^3(-1,1)$ and has the form

$$\Psi(z) = F(z) - \frac{\theta_0}{2} z^2 \quad \forall\, z \in [-1,1] \tag{2.8}$$

with

$$\lim_{z\to -1} F'(z) = -\infty, \quad \lim_{z\to 1} F'(z) = +\infty, \quad F''(z) \geq \theta > 0, \tag{2.9}$$

and

$$\theta_0 - \theta = \alpha > 0. \tag{2.10}$$

We define $F(z) = +\infty$ for any $z \notin [-1,1]$. We assume without loss of generality that is convex and $F(0) = 0$. In addition, we require that F

$$F''(z) \leq C e^{C|F'(z)|} \quad \forall\, z \in (-1,1) \tag{2.11}$$

for some positive constant C. Also, we assume that there exists $\gamma \in (0,1)$ such that F'' is nondecreasing in $[1-\gamma, 1)$ and nonincreasing in $(-1, -1+\gamma]$.

Remark 2.1. The above assumptions are satisfied and motivated by the logarithmic potential (1.3). In that case, Ψ is extended by continuity at $z = \pm 1$. Notice also that the viscosity function (1.4) can be easily extended on the whole $\mathbb{R}$ in such way to comply (2.7). Moreover, other physically relevant profiles can be considered (up to a suitable extension), such as (see, e.g., [52, 36])

$$\nu(z) = \frac{\nu_1 \nu_2}{\nu_1\left(\frac{1-z}{2}\right) + \nu_2\left(\frac{1+z}{2}\right)}, \quad \text{or} \quad \nu(z) = \nu_1 e^{\left(\log\left(\frac{\nu_2}{\nu_1}\right)\left(\frac{1-z}{2}\right)\right)} \quad \forall\, z \in [-1,1],$$

where ν_1 and ν_2 are the constant viscosities of the two fluids.

General agreement. Throughout the paper, the symbol C denotes a positive constant which may be estimated in terms of Ω and of the parameters of the system (see "Main assumptions"). Any further dependence will be explicitly pointed out when necessary. In particular, the notation $C = C(\kappa_1,\dots,\kappa_n)$ denotes a positive constant which explicitly depends on the quantities κ_i, $i = 1,\dots,n$.

2.3. Existence of weak solutions. Let us introduce the notion of weak solution.

Definition 2.2. *Let* $T > 0$ *and* $d = 2,3$. *Given* $u_0 \in \mathbf{H}_\sigma$, $\varphi_0 \in V \cap L^\infty(\Omega)$ *with* $\|\varphi_0\|_{L^\infty(\Omega)} \leq 1$ *and* $|\overline{\varphi_0}| < 1$, *a pair* (u,φ) *is a weak solution to* (1.1)–(1.2) *on* $[0,T]$ *if*

$$u \in L^\infty(0,T;\mathbf{H}_\sigma) \cap L^2(0,T;\mathbf{V}_\sigma),\ \partial_t u \in L^{\frac{4}{d}}(0,T;\mathbf{V}'_\sigma),$$

$$\varphi \in L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)) \cap H^1(0,T;V'),$$

$$\varphi \in L^\infty(\Omega \times (0,T)), \quad \text{with} \quad |\varphi(x,t)| < 1 \quad \text{a.e. } (x,t) \in \Omega \times (0,T),$$

and satisfies

$$\langle \partial_t u, v\rangle + b(u,u,v) + (\nu(\varphi)Du, Dv) = (\mu\nabla\varphi, v) \quad \forall v \in \mathbf{V}_\sigma, \tag{2.12}$$

$$\langle \partial_t \varphi, v\rangle + (u\cdot\nabla\varphi, v) + (\nabla\mu, \nabla v) = 0 \quad \forall v \in V, \tag{2.13}$$

for almost every $t \in (0,T)$, *where* $\mu \in L^2(0,T;V)$ *is given by* $\mu = -\Delta\varphi + \Psi'(\varphi)$. *Moreover,* $\partial_n\varphi = 0$ *a.e. on* $\partial\Omega \times (0,T)$, $u(\cdot,0) = u_0$, *and* $\varphi(\cdot,0) = \varphi_0$ *in* Ω.

Remark 2.3. Notice that (2.12) is equivalent to

$\langle \partial_t u, v\rangle - (u \otimes u, \nabla v) + (\nu(\varphi)Du, Dv) = (\nabla \varphi \otimes \nabla \varphi, \nabla v) \quad \forall v \in \mathbf{V}_\sigma,$
for almost every $t \in (0,T)$, where $(v \otimes w)_{ij} = v_i w_j$, $i,j = 1, 2$, in light of the equalities

(2.14) $$(u \cdot \nabla)u = \mathrm{div}(u \otimes u)$$

and

$$\mu\nabla\varphi = \nabla\Big(\frac{1}{2}|\nabla\varphi|^2 + \Psi(\varphi)\Big) - \mathrm{div}\,(\nabla\varphi \otimes \nabla\varphi).$$

The following existence result of weak solutions has been proven in [2, Theorem 1] (see also [71]).

Theorem 2.4. *Let* $d = 2,3$. *Assume that* $u_0 \in \mathbf{H}_\sigma$, $\varphi_0 \in V \cap L^\infty(\Omega)$ *with* $\|\varphi_0\|_{L^\infty(\Omega)} \leq 1$ *and* $|\overline{\varphi_0}| < 1$. *Then, for any* $T > 0$, *there exists a weak solution* (u,φ) *to* (1.1)–(1.2) *on* $[0,T]$ *in the sense of Definition* 2.2 *such that*

(2.15) $u \in \mathcal{C}([0,T],\mathbf{H}_\sigma)$, *if* $d = 2$, $u \in \mathcal{C}_w([0,T],\mathbf{H}_\sigma)$ *if* $d = 3$,

(2.16) $\varphi \in \mathcal{C}([0,T],V) \cap L^4(0,T;H^2(\Omega)) \cap L^2(0,T;W^{2,p}(\Omega))$,

where $2 \leq p < \infty$ *is arbitrary if* $d = 2$ *and* $p = 6$ *if* $d = 3$. *Moreover, given the energy of the system*

(2.17) $$\mathcal{E}(\boldsymbol{u},\varphi) = \frac{1}{2}\|\boldsymbol{u}\|^2 + \frac{1}{2}\|\nabla\varphi\|^2 + \int_\Omega \Psi(\varphi)\,dx,$$

any weak solution satisfies the energy inequality

(2.18) $$\mathcal{E}(\boldsymbol{u}(t),\varphi(t)) + \int_\tau^t \Big(\|\sqrt{\nu(\varphi(s))}D\boldsymbol{u}(s)\|^2 + \|\nabla\mu(s)\|^2\Big)\,ds \leq \mathcal{E}(\boldsymbol{u}(\tau),\varphi(\tau))$$

for almost every $0 \leq \tau < T$, *including* $\tau = 0$, *and every* $t \in [\tau,T]$. *If* $d = 2$, *then* (2.18) *holds with equality for every* $0 \leq \tau < t \leq T$.

Remark 2.5. We observe that any admissible initial condition in Theorem 2.4 is such that $\Psi(\varphi_0) \in L^1(\Omega)$, so that $\mathcal{E}(u_0,\varphi_0) < \infty$. However, due to $|\overline{\varphi_0}| < 1$, φ_0 cannot be a pure concentration, i.e., $\varphi_0 \equiv 1$ or $\varphi_0 \equiv -1$.

Remark 2.6. The regularity $\varphi \in L^4(0,T;H^2(\Omega))$ is not proved in [2, 71], but it has been recently shown in [50]. Given a weak solution (u,φ), it can be inferred from Theorem A.2 in Appendix A with $f = \mu + \theta_0\varphi \in L^2(0,T;V)$ and $u = \varphi \in L^\infty(0,T;V)$ (cf. also (4.23) below).

3. Uniqueness of weak solutions in two dimensions. In this section we prove the uniqueness of weak solutions for the two-dimensional NSCH system with unmatched viscosities. The key idea is to derive a differential inequality involving norms (for the difference of two solutions) weaker than the natural ones given by the energy of the system (cf. (2.17)). We take full advantage of the regularity properties of the Neumann and Stokes operators which allow us to recover coercive terms. In such a way, we are able to handle the Korteweg force (i.e., the term $\mu\nabla\varphi$) in the Navier-Stokes equations and the convective terms. This technique will be also employed to show the uniqueness of strong solutions if $d = 3$.

Theorem 3.1. *Let* $d = 2$. *Given* (u_0,φ_0) *be such that* $u_0 \in \mathbf{H}_\sigma$, $\varphi_0 \in V$, $\|\varphi_0\|_{L^\infty(\Omega)} \leq 1$, *and* $|\overline{\varphi_0}| < 1$, *the weak solution to* (1.1)–(1.2) *on* $[0,T]$ *with initial datum* (u_0,φ_0) *is unique.*

Proof. Let (u_1,φ_1) and (u_2,φ_2) be two weak solutions to (1.1)–(1.2) on $[0,T]$ with the same initial datum (u_0,φ_0). We define $u = u_1 - u_2$ and $\varphi = \varphi_1 - \varphi_2$. According to Remark 2.3, u and φ solve

$$\begin{aligned}
&\langle \partial_t u, v\rangle - (u_1 \otimes u, \nabla v) - (u \otimes u_2, \nabla v) + (\nu(\varphi_1) Du, \nabla v) \\
&\quad + ((\nu(\varphi_1) - \nu(\varphi_2)) Du_2, \nabla v) = (\nabla\varphi_1 \otimes \nabla\varphi, \nabla v) \\
&\quad + (\nabla\varphi \otimes \nabla\varphi_2, \nabla v) \quad \forall v \in \mathbf{V}_\sigma,
\end{aligned} \tag{3.1}$$

$$\langle \partial_t \varphi, v\rangle + (u_1 \cdot \nabla \varphi, v) + (u \cdot \nabla \varphi_2, v) + (\nabla \mu, \nabla v) = 0 \qquad \forall v \in V, \tag{3.2}$$

where $\mu = -\Delta \varphi + \Psi'(\varphi_1) - \Psi'(\varphi_2)$. Taking $v = 1$ in (3.2) and observing that the integrals over Ω of $u_1 \cdot \nabla \varphi$ and $u \cdot \nabla \varphi_2$ vanish, we have $\varphi(t) = \varphi(0) = 0$ for all $t \in [0,T]$. We rewrite (3.2) as

$$\langle \partial_t \varphi, v\rangle - (\varphi u_1, \nabla v) - (\varphi_2 u, \nabla v) + (\nabla \mu, \nabla v) = 0 \qquad \forall v \in V, \tag{3.3}$$

and we recall the following estimates (cf. (2.15)–(2.16))

$$\| u_i(t)\| \leq C_0, \quad \| \varphi_i(t)\|_V \leq C_0, \quad \| \varphi_i(t)\|_{L^\infty(\Omega)} \leq 1 \quad \forall t \in [0,T], \ i = 1,2, \tag{3.4}$$

where the positive constant C_0 depends on $\mathcal{E}(\boldsymbol{u}_0, \varphi_0)$. Now, taking $v = A_0^{-1}\varphi$ in (3.3) (see Appendix A for the definition of A_0) and using (A.3), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\varphi\|_\ast^2 + (\mu, \varphi) = \mathcal{I}_1 + \mathcal{I}_2,$$

where $\|\varphi\|_* = \|\nabla A_0^{-1}\varphi\|$ and

$$\mathcal{I}_1 = (\varphi u_1, \nabla A_0^{-1} \varphi), \quad \mathcal{I}_2 = (\varphi_2 u, \nabla A_0^{-1} \varphi). \tag{3.5}$$

By the assumptions on Ψ, we have

$$\begin{aligned}(\mu, \varphi) &= \|\nabla \varphi\|^2 + (\Psi'(\varphi_1) - \Psi'(\varphi_2), \varphi) \\ &\geq \|\nabla \varphi\|^2 - \alpha \|\varphi\|^2,\end{aligned}$$

where α is defined in (2.10). By definition of A_0^{-1}, we get

$$\begin{aligned}\alpha\|\varphi\|^2 &= \alpha(\nabla A_0^{-1}\varphi, \nabla\varphi) \\ &\leq \frac{1}{2}\|\nabla\varphi\|^2 + \frac{\alpha^2}{2}\|\varphi\|_*^2,\end{aligned} \tag{3.6}$$

and we end up with

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|_*^2 + \frac{1}{2}\|\nabla\varphi\|^2 \le \frac{\alpha^2}{2}\|\varphi\|_*^2 + \mathcal{I}_1 + \mathcal{I}_2. \tag{3.7}$$

Taking $v = \mathbf{A}^{-1}u$ in (3.1) (see Appendix B for the definition of $\mathbf{A}$), we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|_\sharp^2 + (\nu(\varphi_1) D\boldsymbol{u}, \nabla\mathbf{A}^{-1}\boldsymbol{u}) = \mathcal{I}_3 + \mathcal{I}_4 + \mathcal{I}_5, \tag{3.8}$$

where $\|\boldsymbol{u}\|_\sharp = \|\nabla\mathbf{A}^{-1}\boldsymbol{u}\|$ and

$$\mathcal{I}_3 = -((\nu(\varphi_1)-\nu(\varphi_2))D\boldsymbol{u}_2, \nabla\mathbf{A}^{-1}\boldsymbol{u}),$$
$$\mathcal{I}_4 = (\boldsymbol{u}_1\otimes\boldsymbol{u}, \nabla\mathbf{A}^{-1}\boldsymbol{u}) + (\boldsymbol{u}\otimes\boldsymbol{u}_2, \nabla\mathbf{A}^{-1}\boldsymbol{u}),$$
$$\mathcal{I}_5 = (\nabla\varphi_1\otimes\nabla\varphi, \nabla\mathbf{A}^{-1}\boldsymbol{u}) + (\nabla\varphi\otimes\nabla\varphi_2, \nabla\mathbf{A}^{-1}\boldsymbol{u})$$

Recalling that $\mathrm{div}({}^t(\nabla v)) = \nabla(\mathrm{div} v)$ and $\mathbf{A}^{-1}u \in L^2(0,T;D(\mathbf{A}))$, and integrating by parts, we obtain

$$\begin{aligned}(\nu(\varphi_1)D\boldsymbol{u}, \nabla\mathbf{A}^{-1}\boldsymbol{u}) &= (\nabla\boldsymbol{u}, \nu(\varphi_1)D\mathbf{A}^{-1}\boldsymbol{u})\\ &= -(u, \mathrm{div}(\nu(\varphi_1)D\mathbf{A}^{-1}\boldsymbol{u}))\\ &= -(\boldsymbol{u}, \nu'(\varphi_1)D\mathbf{A}^{-1}\boldsymbol{u}\nabla\varphi_1) - \frac{1}{2}(\boldsymbol{u}, \nu(\varphi_1)\Delta\mathbf{A}^{-1}u).\end{aligned} \tag{3.9}$$

By the properties of the Stokes operator (cf. Appendix B), there exists $p \in L^2(0,T;V)$ such that $-\Delta\mathbf{A}^{-1}u + \nabla p = u$ a.e. in $\Omega \times (0,T)$. By (B.5) and (B.7), we have

$$\|p\| \le C\|\nabla\mathbf{A}^{-1}\boldsymbol{u}\|^{\frac{1}{2}}\|\boldsymbol{u}\|^{\frac{1}{2}}, \quad \|p\|_V \le C\|\boldsymbol{u}\|. \tag{3.10}$$

Therefore, we are led to

$$\begin{aligned}-\frac{1}{2}(\boldsymbol{u}, \nu(\varphi_1)\Delta\mathbf{A}^{-1}\boldsymbol{u}) &= \frac{1}{2}(\nu(\varphi_1)\boldsymbol{u}, \boldsymbol{u}) - \frac{1}{2}(\nu(\varphi_1)\boldsymbol{u}, \nabla p)\\ &\ge \nu_*\|\boldsymbol{u}\|^2 + \frac{1}{2}(\nu'(\varphi_1)\nabla\varphi_1\cdot\boldsymbol{u}, p).\end{aligned} \tag{3.11}$$

Here we have used $\mathrm{div} u = 0$. We now set

$$\mathcal{H}(t) = \frac{1}{2}\|\boldsymbol{u}(t)\|_\sharp^2 + \frac{1}{2}\|\varphi(t)\|_*^2,$$

and

$$\mathcal{I}_6 = (\boldsymbol{u}, \nu'(\varphi_1) D\mathbf{A}^{-1}\boldsymbol{u}\nabla\varphi_1), \quad \mathcal{I}_7 = -\frac{1}{2}(\nu'(\varphi_1)\nabla\varphi_1 \cdot \boldsymbol{u}, p).$$

Summing (3.7) and (3.8), in light of (3.9) and (3.11), we arrive at

$$\text{(3.12)} \qquad \frac{d}{dt}\mathcal{H} + \frac{\nu_*}{2}\|\boldsymbol{u}\|^2 + \frac{1}{2}\|\nabla\varphi\|^2 \leq \alpha_2\mathcal{H} + \sum_{k=1}^{7}\mathcal{I}_k,$$

where $\mathcal{I}_1$ and $\mathcal{I}_2$ are defined in (3.5). We proceed by estimating all the remainder terms on the right-hand side of (3.12). Hereafter the positive constant C_i, $i \in \mathbb{N}$, depends on ν_*, ν', Ω, C_0, and the constants that appear in the mentioned embedding results and interpolation inequalities. By the embedding $V \hookrightarrow L^6(\Omega)$, the Poincaré inequality, and the uniform bound (3.4), we have

$$\begin{aligned}\mathcal{I}_1 &\leq \|\varphi\|_{L^6(\Omega)}\|\boldsymbol{u}_1\|_{L^3(\Omega)}\|\varphi\|_* \\ &\leq \frac{1}{8}\|\nabla\varphi\|^2 + C_1\|\boldsymbol{u}_1\|_{\mathbf{L}^3(\Omega)}^2\|\varphi\|_*^2,\end{aligned}$$

and

$$\begin{aligned}\mathcal{I}_2 &\leq \|\varphi_2\|_{L^\infty(\Omega)}\|\boldsymbol{u}\|\|\varphi\|_* \\ &\leq \frac{\nu_*}{8}\|\boldsymbol{u}\|^2 + C_2\|\varphi\|_*^2.\end{aligned}$$

By (2.1), (2.6), and (3.4), we get

$$\begin{aligned}\mathcal{I}_4 &\leq \Big(\|\boldsymbol{u}_1\|_{\mathbf{L}^4(\Omega)} + \|\boldsymbol{u}_2\|_{\mathbf{L}^4(\Omega)}\Big)\|\boldsymbol{u}\|\|\nabla\mathbf{A}^{-1}\boldsymbol{u}\|_{\mathbf{L}^4(\Omega)} \\ &\leq C\Big(\|\boldsymbol{u}_1\|^{\frac12}\|\boldsymbol{u}_1\|_{\mathbf{V}_\sigma}^{\frac12} + \|\boldsymbol{u}_2\|^{\frac12}\|\boldsymbol{u}_2\|_{\mathbf{V}_\sigma}^{\frac12}\Big)\|\boldsymbol{u}\|_\sharp^{\frac12}\|\boldsymbol{u}\|^{\frac32} \\ &\leq \frac{\nu_*}{8}\|\boldsymbol{u}\|^2 + C_3\Big(\|\boldsymbol{u}_1\|_{\mathbf{V}_\sigma}^2 + \|\boldsymbol{u}_2\|_{\mathbf{V}_\sigma}^2\Big)\|\boldsymbol{u}\|_\sharp^2,\end{aligned}$$

and

$$\begin{aligned}\mathcal{I}_5 &\leq \Big(\|\nabla\varphi_1\|_{\mathbf{L}^\infty(\Omega)} + \|\nabla\varphi_2\|_{\mathbf{L}^\infty(\Omega)}\Big)\|\nabla\varphi\|\|\nabla\mathbf{A}^{-1}\boldsymbol{u}\|\\ &\leq \frac{1}{8}\|\nabla\varphi\|^2 + C_4\Big(\|\nabla\varphi_1\|^2_{\mathbf{L}^\infty(\Omega)} + \|\nabla\varphi_2\|^2_{\mathbf{L}^\infty(\Omega)}\Big)\|\boldsymbol{u}\|_\sharp^2,\end{aligned}$$

Being ν' globally bounded, by using (2.4) and the estimates for the pressure (3.10), we find

$$\begin{aligned}\mathcal{I}_6 &\leq C\|\boldsymbol{u}\|\|D\mathbf{A}^{-1}\boldsymbol{u}\|\|\nabla\varphi_1\|_{\mathbf{L}^\infty(\Omega)}\\ &\leq \frac{\nu_*}{8}\|\boldsymbol{u}\|^2 + C_5\|\nabla\varphi_1\|^2_{\mathbf{L}^\infty(\Omega)}\|\boldsymbol{u}\|_\sharp^2,\end{aligned}$$

and

$$\begin{aligned}\mathcal{I}_7 &\leq C\|\nabla\varphi_1\|_{\mathbf{L}^4(\Omega)}\|\boldsymbol{u}\|\|p\|_{L^4(\Omega)}\\ &\leq C\|\varphi_1\|^{\frac12}_{L^\infty(\Omega)}\|\varphi_1\|^{\frac12}_{H^2(\Omega)}\|\boldsymbol{u}\|\|p\|^{\frac12}\|p\|_V^{\frac12}\\ &\leq C\|\varphi_1\|^{\frac12}_{H^2(\Omega)}\|\nabla\mathbf{A}^{-1}\boldsymbol{u}\|^{\frac14}\|\boldsymbol{u}\|^{\frac74}\\ &\leq \frac{\nu_*}{8}\|\boldsymbol{u}\|^2 + C_6\|\varphi_1\|^4_{H^2(\Omega)}\|\boldsymbol{u}\|_\sharp^2.\end{aligned}$$

Finally, regarding $\mathcal{I}_3$, by using (2.5), we obtain

$$\begin{aligned}\mathcal{I}_3 &= \Big(\int_0^1 \nu'\big(s\varphi_1 + (1-s)\varphi_2\big)\,\mathrm{d}s\,\varphi D\boldsymbol{u}_2, \nabla\mathbf{A}^{-1}\boldsymbol{u}\Big)\\ &\leq C\|D\boldsymbol{u}_2\|\|\varphi\|_{L^\infty(\Omega)}\|\nabla\mathbf{A}^{-1}\boldsymbol{u}\|\\ &\leq C_7\|\boldsymbol{u}_2\|_{\mathbf{V}_\sigma}\|\nabla\varphi\|\Big[\log\Big(\mathrm{e} + \frac{\|\varphi\|_{H^2(\Omega)}}{\|\nabla\varphi\|}\Big)\Big]^{\frac12}\|\boldsymbol{u}\|_\sharp.\end{aligned}$$

Note that, when $\varphi \equiv 0$, the logarithmic term on the right-hand side is assumed to be zero. Collecting the above estimates, we find the differential inequality

(3.13)
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H} + \frac{\nu_*}{2}\|\boldsymbol{u}\|^2 + \frac14\|\nabla\varphi\|^2 \leq \mathcal{Y}_1\mathcal{H} + C_7\|\boldsymbol{u}_2\|_{\mathbf{V}_\sigma}\Big[\mathcal{H}\|\nabla\varphi\|^2\log\Big(\mathrm{e} + \frac{\|\varphi\|_{H^2(\Omega)}}{\|\nabla\varphi\|}\Big)\Big]^{\frac12},$$

where

$$\begin{aligned}\mathcal{Y}_1(t) = C_8\Big(&1 + \|\boldsymbol{u}_1(t)\|^2_{\mathbf{L}^3(\Omega)} + \|\boldsymbol{u}_1(t)\|^2_{\mathbf{V}_\sigma} + \|\boldsymbol{u}_2(t)\|^2_{\mathbf{V}_\sigma}\\ &+ \|\nabla\varphi_1(t)\|^2_{\mathbf{L}^\infty(\Omega)} + \|\nabla\varphi_2(t)\|^2_{\mathbf{L}^\infty(\Omega)} + \|\varphi_1(t)\|^4_{H^2(\Omega)}\Big).\end{aligned}$$

Thanks to Theorem 2.4 and the Sobolev embedding $W^{2,3}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$, valid in space dimension two, we deduce that $\mathcal{Y}_1$ belongs to $L^1(0,T)$. In addition, recalling from (3.4) that $\|\nabla\varphi\| \leq C_0$, we have

$$\log\Big(\mathrm{e} + \frac{\|\varphi\|_{H^2(\Omega)}}{\|\nabla\varphi\|}\Big) \leq \log\Big(\frac{C_8\big(\|\nabla\varphi\| + \|\varphi\|_{H^2(\Omega)}\big)}{\|\nabla\varphi\|^2}\Big).$$

Therefore, denoting

$$\mathcal{G}(t) = \frac14\|\nabla\varphi(t)\|^2, \quad \mathcal{Y}_2(t) = C_7\|\boldsymbol{u}_2(t)\|_{\mathbf{V}_\sigma}, \quad \mathcal{S}(t) = \frac{C_8}{4}\Big(\|\nabla\varphi(t)\| + \|\varphi(t)\|_{H^2(\Omega)}\Big),$$

we rewrite the differential inequality (3.13) as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H} + \mathcal{G} \leq \mathcal{Y}_1\mathcal{H} + \mathcal{Y}_2\Big[\mathcal{H}\mathcal{G}\log\Big(\frac{\mathcal{S}}{\mathcal{G}}\Big)\Big]^{\frac{1}{2}}. \tag{3.14}$$

Note that $\frac{\mathcal{S}}{\mathcal{G}} \geq 1$ for the choice of C_8. Since $\mathcal{Y}_2 \in L^2(0,T)$, $\mathcal{S} \in L^1(0,T)$, and $\mathcal{H}(0) = 0$, we can apply [67, Lemma 2.2] to conclude that $\mathcal{H}(t) = 0$ for all $t \in [0,T]$, which implies the uniqueness of weak solutions. ∎

Remark 3.2. An immediate consequence of the argument performed in the proof of Theorem 3.1 is the uniqueness of weak solutions to the NSCH system in dimension two with matched viscosities (i.e., $\nu(s) = 1$). In that particular case, let us consider (u_1,φ_1) and (u_2,φ_2) are two weak solutions to (1.1)–(1.2) on $[0,T]$ with initial data (u_{01},φ_{01}) and (u_{02},φ_{02}), respectively, where (u_{0i},φ_{0i}), $i = 1,2$, comply the assumptions of Theorem 2.4 and $\overline{\varphi_{01}} = \overline{\varphi_{02}}$. Then, following line by line the above proof and observing that $\mathcal{I}_3 = 0$, we end up with the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H} \leq \mathcal{Y}_1\mathcal{H},$$

where $\mathcal{H}$ and $\mathcal{Y}_1$ are defined above. Hence, we can infer from the Gronwall lemma the following continuous dependence estimate:

$$\|u_1(t) - u_2(t)\|_{\mathbf{V}_\sigma'} + \|\varphi_1(t) - \varphi_2(t)\|_{V_0'} \leq C\|u_{01} - u_{02}\|_{\mathbf{V}_\sigma'} + C\|\varphi_{01} - \varphi_{02}\|_{V_0'}$$

for all $t \in [0,T]$. Here, C is a positive constant depending on T and $\mathcal{E}(u_{0i},\varphi_{0i})$, $i = 1,2$, but is independent of the specific form of the initial data.

Remark 3.3. The proof of Theorem 3.1 also allows us to deduce the uniqueness of weak solutions to problem (1.1)–(1.2) with unmatched viscosities and regular potential (cf. Ψ_0 in "Introduction"). The only changes in the proof arise from the different regularity of weak solutions. Indeed, the global bound in L^∞ is not known in this case, but any weak solution satisfies $\varphi \in L^2(0,T;H^3(\Omega))$ and $\varphi \in L^\infty(0,T;V)$ (see [17, 43]). Thus, the two terms which need a different control are $\mathcal{I}_2$ and $\mathcal{I}_7$. Nonetheless, they can be simply estimated in the following way

$$\mathcal{I}_2 \leq \frac{\nu_*}{8}\|\boldsymbol{u}\|^2 + C_2\|\varphi_2\|_{L^\infty(\Omega)}^2\|\varphi\|_*^2,$$

and, by using (2.1),

$$\begin{aligned}
\mathcal{I}_7 &\leq C\|\nabla\varphi_1\|_{\mathbf{L}^4(\Omega)}\|\boldsymbol{u}\|\|p\|_{L^4(\Omega)} \\
&\leq C\|\nabla\varphi_1\|^{\frac{1}{2}}\|\varphi_1\|_{H^2}^{\frac{1}{2}}\|\boldsymbol{u}\|\|p\|^{\frac{1}{2}}\|p\|_V^{\frac{1}{2}} \\
&\leq C\|\varphi_1\|_{H^2(\Omega)}^{\frac{1}{2}}\|\nabla\mathbf{A}^{-1}\boldsymbol{u}\|^{\frac{1}{4}}\|\boldsymbol{u}\|^{\frac{7}{4}} \\
&\leq \frac{\nu_*}{4}\|\boldsymbol{u}\|^2 + C_6\|\varphi_1\|_{H^2(\Omega)}^4\|\boldsymbol{u}\|_\sharp^2.
\end{aligned}$$

Since by interpolation $\varphi_i \in L^2(0,T;L^\infty(\Omega)) \cap L^4(0,T;H^2(\Omega))$, $i = 1,2$, it is easily seen that we end up with a differential equation having the same form of (3.14).

Remark 3.4. In the three dimensional case, the above proof does not allow us to deduce even a weak-strong uniqueness property, which is classical with the Navier--Stokes equations; that is, the weak solution is unique when a strong solution exists. In this case, this is due to the form of $\mathcal{I}_4$ involving both u_1 and u_2. Hence, we only expect a (conditional) uniqueness result provided that both solutions u_1 and u_2 are more regular than Definition 2.2 (at least u_1, u_2 satisfy the classical condition in [81, Remark 3.81]).

We conclude this section with a continuous dependence estimate in dual space norms with a time-dependent double exponential growth.

Proposition 3.5. *Let $d = 2$. Consider two initial data* (u_{01},φ_{01}) *and* (u_{02},φ_{02})

such that $u_{0i} \in \mathbf{H}_\sigma$, $\varphi_{0i} \in V$, $\|\varphi_{0i}\|_{L^\infty(\Omega)} \leq 1$, *and* $\overline{\varphi_{01}} = \overline{\varphi_{02}} \in (-1,1)$. *The weak solutions* (u_1,φ_1), (u_2,φ_2) *on* $[0,T]$ *to* (1.1)--(1.2) *with initial data* (u_{01},φ_{01}) *and* (u_{02},φ_{02}), *respectively, satisfy the continuous dependence estimate*

$$\mathcal{H}(t) \leq C\Big(\frac{\mathcal{H}(0)}{C}\Big)^{\mathrm{e}^{-\int_0^t \mathcal{Y}_3(s)\,\mathrm{d}s}} \qquad \forall\, t \in [0,T] \tag{3.15}$$

where

$$\begin{aligned}
\mathcal{H}(t) &= \frac{1}{2}\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_\sharp^2 + \frac{1}{2}\|\varphi_1(t) - \varphi_2(t)\|_*^2,\\
\mathcal{Y}_3(t) &= C\Big(1 + \|\boldsymbol{u}_1(t)\|_{\mathbf{V}_\sigma}^2 + \|\boldsymbol{u}_2(t)\|_{\mathbf{V}_\sigma}^2 + \|\nabla\varphi_1(t)\|_{\mathbf{L}^\infty(\Omega)}^2\\
&\qquad + \|\nabla\varphi_2(t)\|_{\mathbf{L}^\infty(\Omega)}^2 + \|\varphi_1(t)\|_{H^2(\Omega)}^4\Big).
\end{aligned}$$

Here, C is a positive constant depending on the norms of the initial data.

Proof. The argument is based on an estimate of $\mathcal{I}_3$ different than that in the proof of Theorem 3.1. Thanks to the product estimate (C.5) in Appendix C, using the properties of A_0 and $\mathbf{A}$ (see Appendixes A and B) and (3.4), we have

$$\begin{aligned}
\mathcal{I}_3 &\leq C\|D\boldsymbol{u}_2\|\|\varphi\nabla\mathbf{A}^{-1}\boldsymbol{u}\|\\
&\leq C\|D\boldsymbol{u}_2\|\|\nabla\varphi\|\Big(\|\nabla\mathbf{A}^{-1}\boldsymbol{u}\| + \|\varphi\|_{-1}\Big)\Big[\log\Big(\mathrm{C}\frac{\|\boldsymbol{u}\| + \|\varphi\|_V}{\|\nabla\mathbf{A}^{-1}\boldsymbol{u}\| + \|\varphi\|_{-1}}\Big)\Big]^{\frac{1}{2}}\\
&\leq \frac{1}{4}\|\nabla\varphi\|^2 + C_9\|D\boldsymbol{u}_2\|^2\mathcal{H}\log\Big(\frac{C_{10}}{\mathcal{H}}\Big).
\end{aligned}$$

Noting that $\mathcal{H} \leq C_{11}$ by (3.4), we observe that C_{10} can be chosen sufficiently large such that $\log(\frac{C_{10}}{\mathcal{H}}) \geq 1$. Exploiting the above estimate in the proof of Theorem 3.1, we eventually deduce the refined differential inequality for the difference of two solutions (cf. (3.14))

$$\frac{d}{dt}\mathcal{H} \leq \mathcal{Y}_3 \mathcal{H} \log\Bigl(\frac{C_{10}}{\mathcal{H}}\Bigr). \tag{3.16}$$

After integration of (3.16), we obtain the following estimate

$$\mathcal{H}(t) \leq C_{10}\Bigl(\frac{\mathcal{H}(0)}{C_{10}}\Bigr)^{e^{-\int_0^t \mathcal{Y}_3(s)\,ds}} \quad \forall\, t \in [0,T], \tag{3.17}$$

where $\mathcal{Y}_3 \in L^1(0,T)$, for any $T > 0$, due to the regularity in Theorem 2.4. Noticing that $\mu(s) = s\log(\frac{C}{s})$ is an Osgood modulus of continuity, the above (3.17)–(3.16) also imply the uniqueness of weak solutions. □

Remark 3.6. We note that the estimate for the difference of two solutions (3.15) is not sufficient to guarantee the continuity of solutions with respect to the data in the norm of the energy space $\mathbf{H}_\sigma \times V$. A similar remark holds for the constant viscosity case (cf. Remark 3.2). Nevertheless, the continuous dependence in the energy space will be recovered by using the propagation of regularity and an interpolation technique in section 4.

4. Global strong solutions and regularity in two dimensions. In this section we prove the global well-posedness of strong solutions for the NSCH system with unmatched viscosities in dimension two. Later on, some consequences will be inferred regarding the regularity and continuous dependence from the initial data.

Theorem 4.1. *Let* $d = 2$, $u_0 \in \mathbf{V}_\sigma$ *and* $\varphi_0 \in H^2(\Omega)$ *be such that* $\|\varphi_0\|_{L^\infty(\Omega)} \leq 1$, $|\overline{\varphi_0}| < 1$, $\mu_0 = -\Delta\varphi_0 + \Psi'(\varphi_0) \in V$ *and* $\partial_n \varphi_0 = 0$ *on* $\partial\Omega$. *Then, for any* $T > 0$, *there exists a unique strong solution to* (1.1)–(1.2) *on* $[0,T]$ *such that*

$$u \in L^\infty(0,T;\mathbf{V}_\sigma) \cap L^2(0,T;\mathbf{W}_\sigma) \cap H^1(0,T;\mathbf{H}_\sigma), \ \pi \in L^2(0,T;V), \ \varphi \in L^\infty(0,T;W^{2,p}(\Omega)) \cap H^1(0,T;V),$$

$$\mu \in L^\infty(0,T;V) \cap L^2(0,T;H^3(\Omega)) \cap H^1(0,T;V'),$$

where $2 \leq p < \infty$. *The strong solution satisfies* (1.1) *a.e. in* $\Omega \times (0,T)$ *and* $\partial_n \mu = 0$ *a.e. on* $\partial\Omega \times (0,T)$. *In addition, given two strong solutions* (u_1,φ_1), (u_2,φ_2) *on* $[0,T]$ *with initial data* (u_{01},φ_{01}) *and* (u_{02},φ_{02}), *respectively, we have the continuous dependence estimate*

$$\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\| + \|\varphi_1(t) - \varphi_2(t)\| \leq C\|\boldsymbol{u}_{01} - \boldsymbol{u}_{02}\| + C\|\varphi_{01} - \varphi_{02}\| \quad \forall\, t \in [0,T], \tag{4.1}$$

where C is a positive constant depending on T and on the norms of the initial data.

Let us briefly explain some technical points of the proof of Theorem 4.1. The argument relies on a priori higher-order energy estimates in Sobolev spaces, combined with a suitable approximation of the logarithmic potential and the initial datum. More precisely, we approximate the logarithmic potential Ψ by means of a family of regular potentials Ψ_ε defined on the whole real line. Next, we need to perform a suitable cut- off procedure of the initial condition, since we cannot control immediately the norm of $\nabla \mu_\varepsilon(0) = \nabla(-\Delta\varphi_0 + \Psi'_\varepsilon(\varphi_0))$ with $\nabla\mu(0) = \nabla(-\Delta\varphi_0 + \Psi'(\varphi_0))$. To overcome this difficulty, we construct a preliminary approximation of the initial datum by exploiting the regularity theory of the Neumann problem with a logarithmic nonlinearity given in Appendix A. Our argument differs from the one used in [2], which is based on fractional time regularity and maximal regularity of a Stokes operator with variable viscosity.

Proof of Theorem 4.1. We divide the proof into several steps.

1. *Approximation of the logarithmic potential.* We introduce a family of regular potentials Ψ_ε that approximate the singular potential Ψ. For any $\varepsilon \in (0,1)$, we set

$$\Psi_\varepsilon(z) = F_\varepsilon(z) - \frac{\theta_0}{2} z^2 \quad \forall\, z \in \mathbb{R}, \tag{4.2}$$

where

$$F_\varepsilon(z) = \begin{cases} \sum_{j=0}^{2} \frac{1}{j!} F^{(j)}(1-\varepsilon)\,[z-(1-\varepsilon)]^j & \forall\, z \geq 1-\varepsilon, \\ F(z) & \forall\, z \in [-1+\varepsilon, 1-\varepsilon] \\ \sum_{j=0}^{2} \frac{1}{j!} F^{(j)}(-1+\varepsilon)\,[z-(-1+\varepsilon)]^j & \forall\, z \leq -1+\varepsilon. \end{cases} \tag{4.3}$$

By virtue of the assumptions on Ψ stated in section 2, we infer that there exists $\varepsilon^\ast \in (0,\gamma)$ (where γ is defined in section 2) such that, for any $\varepsilon \in (0,\varepsilon^\ast]$, the approximating function Ψ_ε satisfies $\Psi_\varepsilon \in \mathcal{C}^2(\mathbb{R})$ and

$$-\widetilde{\alpha} \leq \Psi_\varepsilon(z), \quad -\alpha \leq \Psi_\varepsilon''(z) \leq L \quad \forall\, z \in \mathbb{R} \tag{4.4}$$

where $\widetilde{\alpha}$ is a positive constant independent of ε, α is given by (2.10), $|\Psi_\varepsilon'(z)| \leq |\Psi'(z)|$ and L is a positive constant that may depend on ε. Moreover, we have that $\Psi_\varepsilon(z) \leq \Psi(z)$ for all $z \in [-1,1]$, and for all $z \in (-1,1)$ (see, e.g., [40]).

2. *Approximation of the initial datum.* We perform a cutoff procedure on the initial condition. To do so, we introduce the globally Lipschitz function $h_k : \mathbb{R} \rightarrow \mathbb{R}$, $k \in \mathbb{N}$, such that

$$h_k(z) = \left\{ \begin{array}{ll} -k, & z < -k, \\ z, & z \in [-k,k], \\ k, & z > k. \end{array} \right. \tag{4.5}$$

We define $\widetilde{\mu}_{0,k} = h_k \circ \widetilde{\mu}_0$, where $\widetilde{\mu}_0 = -\Delta \varphi_0 + F'(\varphi_0)$ (or equivalently $\widetilde{\mu}_0 = \mu_0 + \theta_0 \varphi$). Since $\widetilde{\mu}_0 \in V$, the classical result on compositions in Sobolev spaces [77] yields $\widetilde{\mu}_{0,k} \in V$, for any $k > 0$, and $\nabla \widetilde{\mu}_{0,k} = \nabla \widetilde{\mu}_0 \cdot \chi_{[-k,k]}(\widetilde{\mu}_0)$, which, in turn, gives

$$\| \widetilde{\mu}_{0,k} \|_V \leq \| \widetilde{\mu}_0 \|_V. \tag{4.6}$$

For $k \in \mathbb{N}$, we consider the Neumann problem

$$\begin{cases} -\Delta \varphi_{0,k} + F'(\varphi_{0,k}) = \tilde{\mu}_{0,k} \\ \partial_{\mathbf{n}} \varphi_{0,k} = 0, \end{cases}$$

, on $\partial\Omega$.

(4.7) in Ω,

$^2(\Omega)$, Thanks to Lemma A.1, there exists a unique solution to (4.7) such that $'(\varphi_{0,k}) \in H$, which satisfies (4.7) a.e. in Ω and $\partial_{\mathbf{n}}\varphi_{0,k} = 0$ a.e. on $\partial\Omega$. In addition, $\varphi_{0,k} \in H$

F

by (A.6) and (4.6), we have

$$\|\varphi_{0,k}\|_{H^2(\Omega)} \leq C(1 + \|\tilde{\mu}_0\|). \tag{4.8}$$

Since $\tilde{\mu}_{0,k} \rightarrow \tilde{\mu}_0$ in H, Lemma A.1 also entails that $\varphi_{0,k} \rightarrow \varphi_0$ in V. As a consequence, there exist $\tilde{m} \in (0,1)$, which is independent of k, and $\bar{k}$ sufficiently large such that

$$\|\varphi_{0,k}\|_V \leq 1 + \|\overline{\varphi_0}\|_V, |\varphi_{0,k}| \leq m < \tilde{1} \quad \forall k > k. \tag{4.9}$$

In addition, by Theorem A.2 with $f = \tilde{\mu}_{0k}$, we obtain

$$\|F'(\varphi_{0,k})\|_{L^\infty(\Omega)} \leq \|\tilde{\mu}_{0,k}\|_{L^\infty(\Omega)} \leq k.$$

As a byproduct, there exists $\delta = \delta(k) > 0$ such that

$$\|\varphi_{0,k}\|_{L^\infty(\Omega)} \leq 1 - \delta. \tag{4.10}$$

At this point, $\bar{\varepsilon} = \min\{\frac{1}{2}\delta(k), \varepsilon^*\}$ since $F'(\varphi_{0,k}) \in V$, it is easily seen that $\varphi_{0,k} \in H^3(\Omega)$. Finally, for any $\varepsilon \in (0,\varepsilon)$, where, since $F(z) = F_\varepsilon(z)$ for all $z \in [-1+\varepsilon, 1-\varepsilon]$, we infer $\|-\Delta\varphi_{0,k} + F'_\varepsilon(\varphi_{0,k})\|_V \leq \|\tilde{\mu}_0\|_V$ from (4.10) that $-\Delta\varphi_{0,k} + F_\varepsilon'(\varphi_{0,k}) = \tilde{\mu}_{0,k}$, which entails

(4.11).

3. *Approximating problems.* Let us introduce the Galerkin scheme. We consider the family of eigenfunctions $\{w_j\}_{j\geq 1}$ of the homogeneous Neumann operator $A_1 = -\Delta + I$ (see Appendix A) and the family of eigenfunctions $\{w_j\}_{j\geq 1}$ of the Stokes operator $\mathbf{A}$ (see Appendix B). In particular, we recall that $w_1 = 1$ while any w_i, $i > 1$, is

nonconstant with $w_i = 0$. For any integer $n \geq 1$, we define the finitedimensional subspaces of V and $\mathbf{V}_\sigma$, respectively, by $V_n = \mathrm{span}\{w_1,\dots,w_n\}$ and $\mathbf{V}_n = \mathrm{span}\{\boldsymbol{w}_1,\dots,\boldsymbol{w}_n\}$. We denote by Π_n and P_n the orthogonal projections on V_n and $\mathbf{V}_n$ with respect to the inner product in H and in $\mathbf{H}_\sigma$, respectively. We consider the approximating sequences

(4.12)

$$\boldsymbol{u}^n_{k,\varepsilon}(x,t) = \sum_{i=1}^n g_i(t)\boldsymbol{w}_i(x), \quad \varphi^n_{k,\varepsilon}(x,t) = \sum_{i=1}^n k_i(t)w_i(x), \quad \mu^n_{k,\varepsilon}(x,t) = \sum_{i=1}^n l_i(t)w_i(x),$$

solutions of the following approximating system

$$\langle \partial_t \boldsymbol{u}^n_{k,\varepsilon}, \boldsymbol{v}\rangle + b(\boldsymbol{u}^n_{k,\varepsilon}, \boldsymbol{u}^n_{k,\varepsilon}, \boldsymbol{v}) + (\nu(\varphi^n_{k,\varepsilon})D\boldsymbol{u}^n_{k,\varepsilon}, D\boldsymbol{v}) = (\mu^n_{k,\varepsilon}\nabla\varphi^n_{k,\varepsilon}, \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \mathbf{V}_n,$$

(4.14)

$$\langle \partial_t \varphi^n_{k,\varepsilon}, v\rangle + (\boldsymbol{u}^n_{k,\varepsilon}\cdot\nabla\varphi^n_{k,\varepsilon}, v) + (\nabla\mu^n_{k,\varepsilon}, \nabla v) = 0 \quad \forall\, v \in V_n,$$

(4.13)

where

(4.15)
$$\mu^n_{k,\varepsilon} = \Pi_n\big(-\Delta\varphi^n_{k,\varepsilon} + \Psi'_\varepsilon(\varphi^n_{k,\varepsilon})\big).$$

The initial conditions are defined as

(4.16)
$$\boldsymbol{u}^n_{k,\varepsilon}(0) = P_n\boldsymbol{u}_0 \quad \text{and} \quad \varphi^n_{k,\varepsilon}(0) = \Pi_n\varphi_{0,k}.$$

Let us notice that $\varphi_{0,k} \in H^3(\Omega)$ with $\partial_{\mathbf{n}}\varphi_{0,k} = 0$ on $\partial\Omega$. Since $D(A_1^{\frac{3}{2}}) = \{u \in H^3(\Omega) : \partial_{\mathbf{n}} u = 0 \text{ on } \partial\Omega\}$, we have that $\varphi^n_{k,\varepsilon}(0) \rightarrow \varphi_{0,k}$ in $H^3(\Omega)$ as $n \rightarrow \infty$. In turn, this gives $\varphi^n_{k,\varepsilon}(0) \rightarrow \varphi_{0,k}$ in $L^\infty(\overline{\Omega})$. Hence, there exist m, with $\tilde m < \overline{m} < 1$ (independent of n), and $\overline{n}$ such that

$$|\overline{\varphi}^n_{k,\varepsilon}(0)| \leq \overline{m}, \quad \|\varphi^n_{k,\varepsilon}(0)\|_{L^\infty(\Omega)} \leq 1 - \frac{1}{2}\delta(k) \quad \forall\, n > \overline{n}. \tag{4.17}$$

On account of Steps 1 and 2, for any $k > \overline{k}$, we fix $\varepsilon \in (0, \overline{\varepsilon})$ with $\overline{\varepsilon}$ depending on k, and $n > \overline{n}$ with $\overline{n}$ depending on k. The existence of a sequence of functions $u^n_{k,\varepsilon}$, $\varphi^n_{k,\varepsilon}$, and $\mu_{k,\varepsilon}$ of the form (4.12) which satisfy (4.13)--(4.16) for any $t \in [0,T]$ can be proved in a standard way (see, e.g., [81]). In particular, the system (4.13)--(4.16) is equivalent to a Cauchy problem for a nonlinear system of ordinary differential equations in the unknowns g_i, k_i and l_i, $i = 1,\dots,n$. Thanks to the Cauchy--Lipschitz theorem, for any $n > \overline{n}$, there exists a unique maximal solution to this system defined on some interval $[0,t_n]$. Moreover, by the energy estimates we shall prove in the next step (cf. (4.20)), it is clear that $t_n = T$.

4. *Energy estimates.* Let us recall the above choices of the parameters, namely, for any $k > \overline{k}$, we fix $\varepsilon \in (0, \overline{\varepsilon})$ and $n > \overline{n}$. We now show uniform energy estimates with respect to the approximating parameters k, ε, and n. In particular, c_i, $i \in \mathbb{N}$, denotes a positive constant, which depends on the parameters of the system, the constants arising from embedding and interpolation results and the energy $\mathcal{E}(u_0, \varphi_0)$, but is independent of the approximation parameters k, ε, and n.

First, by taking $v = 1$ in (4.14), we have $|\overline{\varphi}^n_{k,\varepsilon}(t)| = |\overline{\varphi}^n_{k,\varepsilon}(0)| \leq \overline{m}$ for all $t \in [0,T]$. We introduce the approximated energy

$$\mathcal{E}_\varepsilon(\boldsymbol{v}, \psi) = \frac{1}{2}\|\boldsymbol{v}\|^2 + \frac{1}{2}\|\nabla \psi\|^2 + \int_\Omega \Psi_\varepsilon(\psi)\,\mathrm{d}x.$$

In light of (4.9), (4.17), and $\Psi_\varepsilon(z) \leq \Psi(z)$ for all $z \in [-1,1]$, we deduce that

$$\begin{aligned}\mathcal{E}_\varepsilon(\boldsymbol{u}^n_{k,\varepsilon}(0), \varphi^n_{k,\varepsilon}(0)) &= \frac{1}{2}\|P_n \boldsymbol{u}_0\|^2 + \frac{1}{2}\|\nabla \Pi_n \varphi_{0,k}\|^2 + \int_\Omega \Psi_\varepsilon(\varphi^n_{k,\varepsilon}(0))\,\mathrm{d}x \\ &\leq \frac{1}{2}\|\boldsymbol{u}_0\|^2 + \frac{1}{2}\|\varphi_0\|^2_V + C.\end{aligned} \tag{4.18}$$

Here we have used that Ψ is bounded on $[-1,1]$. Taking $v = u^n_{k,\varepsilon}$ in (4.13), $v = \mu^n_{k,\varepsilon}$ in (4.14), multiplying (4.15) by $\partial_t \varphi^n_{k,\varepsilon}$, and summing up the resulting equations, we find

$$(4.19)\qquad \frac{d}{dt}\mathcal{E}_\varepsilon(u^n_{k,\varepsilon}, \varphi^n_{k,\varepsilon}) + \|\sqrt{\nu(\varphi^n_{k,\varepsilon})}\, Du^n_{k,\varepsilon}\|^2 + \|\nabla \mu^n_{k,\varepsilon}\|^2 = 0$$

for almost every $t \in (0,T)$. Owing to the Korn inequality and (4.18), after an integration in time, we have

$$(4.20)\qquad \mathcal{E}_\varepsilon(\boldsymbol{u}^n_{k,\varepsilon}(t), \varphi^n_{k,\varepsilon}(t)) + \int_0^t \Big(\nu_*\|\nabla \boldsymbol{u}^n_{k,\varepsilon}(s)\|^2 + \|\nabla \mu^n_{k,\varepsilon}(s)\|^2\Big)\,\mathrm{d}s \le c_0 \quad \forall\, t \in [0,T].$$

In particular, by using (4.4), we have

$$(4.21)\qquad \|\boldsymbol{u}^n_{k,\varepsilon}(t)\| + \|\varphi^n_{k,\varepsilon}(t)\|_V \le c_1 \quad \forall\, t \in [0,T].$$

In order to find an estimate on $\|\mu^n_{k,\varepsilon}\|_V$, we recall the inequality (see, e.g., [72, Proposition A.1])

$$\|\Psi_\varepsilon(\varphi^n_{k,\varepsilon})\|_{L^1(\Omega)} \le c_2\Big(1 + (\Psi'_\varepsilon(\varphi^n_{k,\varepsilon}), \varphi^n_{k,\varepsilon} - \overline{\varphi}^n_{k,\varepsilon})\Big),$$

where c_2 depends on $\overline{m}$. Testing (4.15) by $\varphi^n_{k,\varepsilon} - \overline{\varphi^n_{k,\varepsilon}}$, we obtain

$$\|\nabla \varphi^n_{k,\varepsilon}\|^2 + (\Psi'_\varepsilon(\varphi^n_{k,\varepsilon}), \varphi^n_{k,\varepsilon} - \overline{\varphi}^n_{k,\varepsilon}) = (\mu^n_{k,\varepsilon} - \overline{\mu}^n_{k,\varepsilon}, \varphi^n_{k,\varepsilon} - \overline{\varphi}^n_{k,\varepsilon}).$$

Thus, by the Poincaré inequality and (4.21), we have

$$(\Psi'_\varepsilon(\varphi^n_{k,\varepsilon}), \varphi^n_{k,\varepsilon} - \overline{\varphi}^n_{k,\varepsilon}) \le c_3\|\nabla \mu^n_{k,\varepsilon}\|.$$

Accordingly, since $|\overline{\mu_{k,\varepsilon}}| = |\overline{\Psi'(\varphi^n_{k,\varepsilon})}|$, we learn that

$$(4.22)\qquad \|\mu^n_{k,\varepsilon}\|_V \le c_4(1 + \|\nabla \mu^n_{k,\varepsilon}\|).$$

Next, testing (4.15) by $-\Delta\varphi^n_{k,\varepsilon}$ and integrating by parts, we get

$$\|\Delta\varphi^n_{k,\varepsilon}\|^2 + (\Psi''_\varepsilon(\varphi^n_{k,\varepsilon})\nabla\varphi^n_{k,\varepsilon},\nabla\varphi^n_{k,\varepsilon}) = (\nabla\mu^n_{k,\varepsilon},\nabla\varphi^n_{k,\varepsilon}).$$

By using (4.4) and (4.21), we deduce that

$$\|\varphi^n_{k,\varepsilon}\|^2_{H^2(\Omega)} \leq c_5(1+\|\nabla\mu^n_{k,\varepsilon}\|). \tag{4.23}$$

On the other hand, by comparison in (4.13) and in (4.14) and by exploiting (2.1), (4.21), and (4.22), we infer that

$$\|\partial_t u^n_{k,\varepsilon}\|_{\mathbf{V}_\sigma'} \leq c_6(1+\|\nabla u^n_{k,\varepsilon}\|+\|\nabla\mu^n_{k,\varepsilon}\|), \tag{4.24}$$

and

$$\|\partial_t\varphi^n_{k,\varepsilon}\|_\ast \leq c_7(\|\nabla u^n_{k,\varepsilon}\|+\|\nabla\mu^n_{k,\varepsilon}\|). \tag{4.25}$$

In light of the above estimates (4.20)--(4.25), we have

$u^n_{k,\varepsilon}$ is uniformly bounded in $L^\infty(0,T;\mathbf{H}_\sigma)\cap L^2(0,T;\mathbf{V}_\sigma)\cap H^1(0,T;\mathbf{V}_\sigma')$, $\varphi^n_{k,\varepsilon}$ is uniformly bounded in $L^\infty(0,T;V)\cap L^4(0,T;H^2(\Omega))\cap H^1(0,T;V')$, $\mu^n_{k,\varepsilon}$ is uniformly bounded in $L^2(0,T;V)$,

with respect to the parameters k, ε, and n.

5. *Higher-order energy estimates.* We are now in position to prove uniform higher-order Sobolev estimates. We will denote by c'_i, $i\in\mathbb{N}$, a positive constant, which depends on the parameters of the system, the constants arising from embedding and interpolation results, and $\mathscr{E}(u_0,\varphi_0)$, but are independent of the approximation parameters k, ε, and n and of the norms $\|u_0\|_{\mathbf{V}_\sigma}$ and $\|\mu_0\|_V$. Taking $v=\partial_t\mu^n_{k,\varepsilon}$ in (4.14), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mu^n_{k,\varepsilon}\|^2 + (\partial_t\mu^n_{k,\varepsilon},\partial_t\varphi^n_{k,\varepsilon}) + (\partial_t\mu^n_{k,\varepsilon}, \boldsymbol{u}^n_{k,\varepsilon}\cdot\nabla\varphi^n_{k,\varepsilon}) = 0.$$

Since $\overline{\partial_t\varphi^n_{k,\varepsilon}}(t)=0$ for all $t\in[0,T]$, we have

$$\alpha\|\partial_t\varphi^n_{k,\varepsilon}\|^2 \leq \frac{1}{2}\|\nabla\partial_t\varphi^n_{k,\varepsilon}\|^2 + \frac{\alpha^2}{2}\|\partial_t\varphi^n_{k,\varepsilon}\|^2_*.$$

Then, we infer from the assumptions on Ψ that

$$\begin{aligned}(\partial_t\mu^n_{k,\varepsilon}, \partial_t\varphi^n_{k,\varepsilon}) &= \|\nabla\partial_t\varphi^n_{k,\varepsilon}\|^2 + (\Psi''_\varepsilon(\varphi^n_{k,\varepsilon})\partial_t\varphi^n_{k,\varepsilon}, \partial_t\varphi^n_{k,\varepsilon})\\ &\geq \|\nabla\partial_t\varphi^n_{k,\varepsilon}\|^2 - \alpha\|\partial_t\varphi^n_{k,\varepsilon}\|^2\\ &\geq \frac{1}{2}\|\nabla\partial_t\varphi^n_{k,\varepsilon}\|^2 - \frac{\alpha^2}{2}\|\partial_t\varphi^n_{k,\varepsilon}\|^2_*.\end{aligned}$$

$$\begin{aligned}(\partial_t\mu^n_{k,\varepsilon}, \boldsymbol{u}^n_{k,\varepsilon}\cdot\nabla\varphi^n_{k,\varepsilon}) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big[(\boldsymbol{u}^n_{k,\varepsilon}\cdot\nabla\varphi^n_{k,\varepsilon}, \mu^n_{k,\varepsilon})\Big]\\ &\quad - (\partial_t\boldsymbol{u}^n_{k,\varepsilon}\cdot\nabla\varphi^n_{k,\varepsilon}, \mu^n_{k,\varepsilon}) - (\boldsymbol{u}^n_{k,\varepsilon}\cdot\nabla\partial_t\varphi^n_{k,\varepsilon}, \mu^n_{k,\varepsilon})\end{aligned}$$

$$\begin{aligned}(\mu^n_{k,\varepsilon}, \boldsymbol{u}^n_{k,\varepsilon}\cdot\nabla\partial_t\varphi^n_{k,\varepsilon}) &\leq \|\boldsymbol{u}^n_{k,\varepsilon}\|_{\mathbf{L}^3(\Omega)}\|\nabla\partial_t\varphi^n_{k,\varepsilon}\|\|\mu^n_{k,\varepsilon}\|_{L^6(\Omega)}\\ &\leq \frac{1}{4}\|\nabla\partial_t\varphi^n_{k,\varepsilon}\|^2 + c_1'\|\boldsymbol{u}^n_{k,\varepsilon}\|^2_{\mathbf{L}^3(\Omega)}(1+\|\nabla\mu^n_{k,\varepsilon}\|^2).\end{aligned}$$

Accordingly, by using (4.25), we arrive at

$$\Big[\frac{1}{\ }\|\nabla\ ^n\ \|^2 + (\boldsymbol{u}^n\ \cdot\nabla\ ^n\ \ ^n\ )\Big] + \frac{1}{\ }\|\nabla\partial\ ^n\ \|^2$$

Besides, we observe that

By (4.22), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\ 2\ \mu_{k,\varepsilon}\ \ {}_{k,\varepsilon}\ \ \varphi_{k,\varepsilon}, \mu_{k,\varepsilon}\ \ 4\ \ {}_t\varphi_{k,\varepsilon}$$

(4.26)\leq (\partial tunk,\varepsilon \cdot \nabla \varphi nk,\varepsilon ,\mu nk,\varepsilon) + c\prime 2(1 + \| unk,\varepsilon \| 2\bfL 3(\Omega))(1 + \| \nabla unk,\varepsilon \| 2 + \| \nabla \mu nk,\varepsilon \| 2).

Taking $v = \partial_t\boldsymbol{u}^n_{k,\varepsilon}$ in (4.13), we have

$$\| \partial_t u_{nk,\varepsilon} \|^2 + b(u_{nk,\varepsilon}, u_{nk,\varepsilon}, \partial_t u_{nk,\varepsilon}) - (\mathrm{div}(\nu(\varphi_{nk,\varepsilon}) D u_{k,\varepsilon n}), \partial_t u_{nk,\varepsilon}) = (\mu_{k,\varepsilon n} \nabla \varphi_{nk,\varepsilon}, \partial_t u_{nk,\varepsilon}).$$

By (2.1), (2.6), (2.7), and (4.21), we deduce that

$$
\begin{aligned}
b(\boldsymbol{u}^n_{k,\varepsilon}, \boldsymbol{u}^n_{k,\varepsilon}, \partial_t \boldsymbol{u}^n_{k,\varepsilon}) &\le \|\boldsymbol{u}^n_{k,\varepsilon}\|_{\mathbf{L}^4(\Omega)} \|\nabla \boldsymbol{u}^n_{k,\varepsilon}\|_{\mathbf{L}^4(\Omega)} \|\partial_t \boldsymbol{u}^n_{k,\varepsilon}\| \\
&\le \sqrt{c_1} C \|\nabla \boldsymbol{u}^n_{k,\varepsilon}\| \|\mathbf{A} \boldsymbol{u}^n_{k,\varepsilon}\|^{\frac{1}{2}} \|\partial_t \boldsymbol{u}^n_{k,\varepsilon}\| \\
&\le \frac{1}{6} \|\partial_t \boldsymbol{u}^n_{k,\varepsilon}\|^2 + c_3' \Big(\|\mathbf{A} \boldsymbol{u}^n_{k,\varepsilon}\|^2 + \|\nabla \boldsymbol{u}^n_{k,\varepsilon}\|^4 \Big),
\end{aligned}
$$

and

$$
\begin{aligned}
&(\operatorname{div}(\nu(\varphi^n_{k,\varepsilon}) D\boldsymbol{u}^n_{k,\varepsilon}), \partial_t \boldsymbol{u}^n_{k,\varepsilon}) \\
&\quad = \frac{1}{2} (\nu(\varphi^n_{k,\varepsilon}) \Delta \boldsymbol{u}^n_{k,\varepsilon}, \partial_t \boldsymbol{u}^n_{k,\varepsilon}) + (\nu'(\varphi^n_{k,\varepsilon}) D\boldsymbol{u}^n_{k,\varepsilon} \nabla \varphi^n_{k,\varepsilon}, \partial_t \boldsymbol{u}^n_{k,\varepsilon}) \\
&\quad \le C \|\mathbf{A} \boldsymbol{u}^n_{k,\varepsilon}\| \|\partial_t \boldsymbol{u}^n_{k,\varepsilon}\| + C \|\nabla \varphi^n_{k,\varepsilon}\|_{\mathbf{L}^4(\Omega)} \|D\boldsymbol{u}^n_{k,\varepsilon}\|_{\mathbf{L}^4(\Omega)} \|\partial_t \boldsymbol{u}^n_{k,\varepsilon}\| \\
&\quad \le \frac{1}{6} \|\partial_t \boldsymbol{u}^n_{k,\varepsilon}\|^2 + C \|\mathbf{A} \boldsymbol{u}^n_{k,\varepsilon}\|^2 + c_1 C \|\varphi^n_{k,\varepsilon}\|_{H^2(\Omega)} \|\nabla \boldsymbol{u}^n_{k,\varepsilon}\| \|\mathbf{A} \boldsymbol{u}^n_{k,\varepsilon}\| \\
&\quad \le \frac{1}{6} \|\partial_t \boldsymbol{u}^n_{k,\varepsilon}\|^2 + c_4' \Big(\|\mathbf{A} \boldsymbol{u}^n_{k,\varepsilon}\|^2 + \|\varphi^n_{k,\varepsilon}\|^2_{H^2(\Omega)} \|\nabla \boldsymbol{u}^n_{k,\varepsilon}\|^2 \Big).
\end{aligned}
$$

By (4.22), we have

$$
\begin{aligned}
(\mu^n_{k,\varepsilon} \nabla \varphi^n_{k,\varepsilon}, \partial_t \boldsymbol{u}^n_{k,\varepsilon}) &\le \|\mu^n_{k,\varepsilon}\|_{L^6(\Omega)} \|\nabla \varphi^n_{k,\varepsilon}\|_{\mathbf{L}^3(\Omega)} \|\partial_t \boldsymbol{u}^n_{k,\varepsilon}\| \\
&\le \frac{1}{6} \|\partial_t \boldsymbol{u}^n_{k,\varepsilon}\|^2 + c_5' \|\varphi^n_{k,\varepsilon}\|^2_{H^2(\Omega)} (1 + \|\nabla \mu^n_{k,\varepsilon}\|^2).
\end{aligned}
$$

Hence, we find

$$
\|\partial_t \boldsymbol{u}^n_{k,\varepsilon}\|^2 \le c_6' \Big(\|\mathbf{A} \boldsymbol{u}^n_{k,\varepsilon}\|^2 + \|\nabla \boldsymbol{u}^n_{k,\varepsilon}\|^4 + \|\varphi^n_{k,\varepsilon}\|^2_{H^2(\Omega)} (1 + \|\nabla \boldsymbol{u}^n_{k,\varepsilon}\|^2 + \|\nabla \mu^n_{k,\varepsilon}\|^2) \Big). \tag{4.27}
$$

Because of (4.12) and (4.20), we deduce that $g_i \in L^2(0,T)$ for all $i = 1,\dots,n$, and $u^n_{k,\varepsilon} \in L^2(0,T;D(\mathbf{A}))$, which implies that $\mathbf{A}u^n_{k,\varepsilon} \in L^2(0,T;\mathbf{H}_\sigma)$. By the theory of the Stokes operator (see Appendix B), there exists $p^n_{k,\varepsilon} \in L^2(0,T;V)$ such that $-\Delta u^n_{k,\varepsilon} + \nabla p^n_{k,\varepsilon} = \mathbf{A}u^n_{k,\varepsilon}$ a.e. in $\Omega \times (0,T)$. In particular, we have

$$
\|p^n_{k,\varepsilon}\| \le C \|\nabla \boldsymbol{u}^n_{k,\varepsilon}\|^{\frac{1}{2}} \|\mathbf{A} \boldsymbol{u}^n_{k,\varepsilon}\|^{\frac{1}{2}}, \quad \|p^n_{k,\varepsilon}\|_V \le C \|\mathbf{A} \boldsymbol{u}^n_{k,\varepsilon}\|, \tag{4.28}
$$

where C is independent of k, ε, and n. Now we take $v = \mathbf{A}u^n_{k,\varepsilon}$ in (4.13), and we obtain d

d

$$- (\mathrm{div}\,(\nu(\varphi^n_{k,\varepsilon}) D\boldsymbol{u}^n_{k,\varepsilon}), \mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}) = (\mu^n_{k,\varepsilon}\nabla\varphi^n_{k,\varepsilon}, \mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}).$$

$$\begin{aligned}
-(\mathrm{div}\,(\nu(\varphi^n_{k,\varepsilon}) D\boldsymbol{u}^n_{k,\varepsilon}), \mathbf{A}\boldsymbol{u}^n_{k,\varepsilon})
&= -\frac{1}{2}(\nu(\varphi^n_{k,\varepsilon})\Delta\boldsymbol{u}^n_{k,\varepsilon}, \mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}) - (\nu'(\varphi^n_{k,\varepsilon}) D\boldsymbol{u}^n_{k,\varepsilon}\nabla\varphi^n_{k,\varepsilon}, \mathbf{A}\boldsymbol{u}^n_{k,\varepsilon})\\
&= \frac{1}{2}(\nu(\varphi^n_{k,\varepsilon})\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}, \mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}) - (\nu(\varphi^n_{k,\varepsilon})\nabla p^n_{k,\varepsilon}, \mathbf{A}\boldsymbol{u}^n_{k,\varepsilon})\\
&\quad - (\nu'(\varphi^n_{k,\varepsilon}) D\boldsymbol{u}^n_{k,\varepsilon}\nabla\varphi^n_{k,\varepsilon}, \mathbf{A}\boldsymbol{u}^n_{k,\varepsilon})\\
&\geq \nu_*\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}\|^2 + (\nu'(\varphi^n_{k,\varepsilon})\nabla\varphi^n_{k,\varepsilon} p^n_{k,\varepsilon}, \mathbf{A}\boldsymbol{u}^n_{k,\varepsilon})\\
&\quad - (\nu'(\varphi^n_{k,\varepsilon}) D\boldsymbol{u}^n_{k,\varepsilon}\nabla\varphi^n_{k,\varepsilon}, \mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}).
\end{aligned}$$

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We observe that

By (2.1), (4.21), and (4.28), we have the following estimates:

$$\begin{aligned}
&-(\nu'(\varphi^n_{k,\varepsilon})\nabla\varphi^n_{k,\varepsilon}p^n_{k,\varepsilon}, \mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}) + (\nu'(\varphi^n_{k,\varepsilon})D\boldsymbol{u}^n_{k,\varepsilon}\nabla\varphi^n_{k,\varepsilon}, \mathbf{A}\boldsymbol{u}^n_{k,\varepsilon})\\
&\quad\leq C\|\nabla\varphi^n_{k,\varepsilon}\|_{\mathbf{L}^4(\Omega)}\Big(\|p^n_{k,\varepsilon}\|_{L^4(\Omega)} + \|D\boldsymbol{u}^n_{k,\varepsilon}\|_{\mathbf{L}^4(\Omega)}\Big)\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}\|\\
&\quad\leq \sqrt{c_1}C\|\varphi^n_{k,\varepsilon}\|^{\frac12}_{H^2(\Omega)}\Big(\|p^n_{k,\varepsilon}\|^{\frac12}\|p^n_{k,\varepsilon}\|^{\frac12}_{V} + \|\nabla\boldsymbol{u}^n_{k,\varepsilon}\|^{\frac12}\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}\|^{\frac12}\Big)\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}\|\\
&\quad\leq \sqrt{c_1}C\|\varphi^n_{k,\varepsilon}\|^{\frac12}_{H^2(\Omega)}\Big(\|\nabla\boldsymbol{u}^n_{k,\varepsilon}\|^{\frac14}\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}\|^{\frac34} + \|\nabla\boldsymbol{u}^n_{k,\varepsilon}\|^{\frac12}\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}\|^{\frac12}\Big)\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}\|\\
&\quad\leq \frac{\nu_*}{6}\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}\|^2 + c_7'(1+\|\varphi^n_{k,\varepsilon}\|^4_{H^2(\Omega)})\|\nabla\boldsymbol{u}^n_{k,\varepsilon}\|^2,
\end{aligned}$$

and

$$\begin{aligned}
b(\boldsymbol{u}^n_{k,\varepsilon}, \boldsymbol{u}^n_{k,\varepsilon}, \mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}) &\leq \|\boldsymbol{u}^n_{k,\varepsilon}\|_{\mathbf{L}^4(\Omega)}\|\nabla\boldsymbol{u}^n_{k,\varepsilon}\|_{\mathbf{L}^4(\Omega)}\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}\|\\
&\leq \frac{\nu_*}{6}\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}\|^2 + c_8'\|\nabla\boldsymbol{u}^n_{k,\varepsilon}\|^4.
\end{aligned}$$

Also, we have

$$\begin{aligned}
(\mu^n_{k,\varepsilon}\nabla\varphi^n_{k,\varepsilon}, \mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}) &\leq \|\mu^n_{k,\varepsilon}\|_{L^6(\Omega)}\|\nabla\varphi^n_{k,\varepsilon}\|_{\mathbf{L}^3(\Omega)}\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}\|\\
&\leq \frac{\nu_*}{6}\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}\|^2 + c_9'\|\varphi^n_{k,\varepsilon}\|^2_{H^2(\Omega)}(1+\|\nabla\mu^n_{k,\varepsilon}\|^2).
\end{aligned}$$

Hence, we are led to

$$\begin{aligned}
&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\boldsymbol{u}^n_{k,\varepsilon}\|^2 + \frac{\nu_*}{2}\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}\|^2\\
&\quad\leq c_{10}'\Big(\|\nabla\boldsymbol{u}^n_{k,\varepsilon}\|^4 + (1+\|\varphi^n_{k,\varepsilon}\|^4_{H^2(\Omega)})\|\nabla\boldsymbol{u}^n_{k,\varepsilon}\|^2 + \|\varphi^n_{k,\varepsilon}\|^2_{H^2(\Omega)}(1+\|\nabla\mu^n_{k,\varepsilon}\|^2)\Big) \qquad (4.29)
\end{aligned}$$

Multiplying (4.27) by $\varpi = \frac{\nu_*}{4c_6'} > 0$ and summing up with (4.29), we arrive at

$$\begin{aligned}
&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\boldsymbol{u}^n_{k,\varepsilon}\|^2 + \frac{\nu_*}{4}\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}\|^2 + \varpi\|\partial_t\boldsymbol{u}^n_{k,\varepsilon}\|^2\\
&\quad\leq c_{11}'\Big(\|\nabla\boldsymbol{u}^n_{k,\varepsilon}\|^4 + (1+\|\varphi^n_{k,\varepsilon}\|^4_{H^2(\Omega)})(1+\|\nabla\boldsymbol{u}^n_{k,\varepsilon}\|^2 + \|\nabla\mu^n_{k,\varepsilon}\|^2)\Big). \qquad (4.30)
\end{aligned}$$

Adding (4.26) and (4.30), we find the differential inequality

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Lambda(\boldsymbol{u}^n_{k,\varepsilon}, \varphi^n_{k,\varepsilon}) + \frac{\nu_*}{4}\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}\|^2 + \frac{\varpi}{2}\|\partial_t\boldsymbol{u}^n_{k,\varepsilon}\|^2 + \frac14\|\nabla\partial_t\varphi^n_{k,\varepsilon}\|^2\\
&\quad\leq (\partial_t\boldsymbol{u}^n_{k,\varepsilon}\cdot\nabla\varphi^n_{k,\varepsilon}, \mu^n_{k,\varepsilon}) + c_{12}'\|\nabla\boldsymbol{u}^n_{k,\varepsilon}\|^4 \ldots
\end{aligned}$$

$$+ C_{12}' \Bigl((1 + \| \boldsymbol{u}^n_{k,\varepsilon} \|^2_{\mathbf{L}^3(\Omega)} + \| \varphi^n_{k,\varepsilon} \|^4_{H^2(\Omega)})(1 + \| \nabla \boldsymbol{u}^n_{k,\varepsilon} \|^2 + \| \nabla \mu^n_{k,\varepsilon} \|^2) \Bigr), \tag{4.31}$$

where

$$\Lambda(\boldsymbol{u}^n_{k,\varepsilon}, \varphi^n_{k,\varepsilon}) = \frac{1}{2}\|\nabla \boldsymbol{u}^n_{k,\varepsilon}\|^2 + \frac{1}{2}\|\nabla \mu^n_{k,\varepsilon}\|^2 + (\boldsymbol{u}^n_{k,\varepsilon} \cdot \nabla \varphi^n_{k,\varepsilon}, \mu^n_{k,\varepsilon}). \tag{4.32}$$

We control the first term on the right-hand side of (4.31) as follows:

$$\begin{aligned}
(\partial_t \boldsymbol{u}^n_{k,\varepsilon} \cdot \nabla \varphi^n_{k,\varepsilon}, \mu^n_{k,\varepsilon}) &\le \|\partial_t \boldsymbol{u}^n_{k,\varepsilon}\| \|\nabla \varphi^n_{k,\varepsilon}\|_{\mathbf{L}^3(\Omega)} \|\mu^n_{k,\varepsilon}\|_{L^6(\Omega)} \\
&\le \frac{\varpi}{4}\|\partial_t \boldsymbol{u}^n_{k,\varepsilon}\|^2 + C\|\nabla \varphi^n_{k,\varepsilon}\|^2_{\mathbf{L}^3(\Omega)} \|\mu^n_{k,\varepsilon}\|^2_V \\
&\le \frac{\varpi}{4}\|\partial_t \boldsymbol{u}^n_{k,\varepsilon}\|^2 + c_{13}'\|\varphi^n_{k,\varepsilon}\|^2_{H^2(\Omega)}(1 + \|\nabla \mu^n_{k,\varepsilon}\|^2).
\end{aligned}$$

Then, we arrive at

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Lambda(\boldsymbol{u}^n_{k,\varepsilon}, \varphi^n_{k,\varepsilon}) + \frac{\nu_*}{4}\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}\|^2 + \frac{\varpi}{4}\|\partial_t \boldsymbol{u}^n_{k,\varepsilon}\|^2 + \frac{1}{4}\|\nabla \partial_t \varphi^n_{k,\varepsilon}\|^2 \\
&\quad \le c_{14}'\Bigl(\|\nabla \boldsymbol{u}^n_{k,\varepsilon}\|^4 + (1 + \|\boldsymbol{u}^n_{k,\varepsilon}\|^2_{\mathbf{L}^3(\Omega)} + \|\varphi^n_{k,\varepsilon}\|^4_{H^2(\Omega)})(1 + \|\nabla \boldsymbol{u}^n_{k,\varepsilon}\|^2 + \|\nabla \mu^n_{k,\varepsilon}\|^2)\Bigr).
\end{aligned} \tag{4.33}$$

Now we show that $\Lambda(\boldsymbol{u}^n_{k,\varepsilon}, \varphi^n_{k,\varepsilon})$ is bounded from below. By using (2.1) and exploiting (4.20)–(4.22), we have

$$\begin{aligned}
(\boldsymbol{u}^n_{k,\varepsilon} \cdot \nabla \varphi^n_{k,\varepsilon}, \mu^n_{k,\varepsilon}) &\le \|\boldsymbol{u}^n_{k,\varepsilon}\|_{\mathbf{L}^4(\Omega)} \|\nabla \varphi^n_{k,\varepsilon}\| \|\mu^n_{k,\varepsilon}\|_{L^4(\Omega)} \\
&\le c_1 C \|\boldsymbol{u}^n_{k,\varepsilon}\|^{\frac{1}{2}} \|\nabla \boldsymbol{u}^n_{k,\varepsilon}\|^{\frac{1}{2}} \; \|\mu^n_{k,\varepsilon}\|_V \\
&\le \frac{1}{4}\|\nabla \boldsymbol{u}^n_{k,\varepsilon}\|^2 + \frac{1}{4}\|\nabla \mu^n_{k,\varepsilon}\|^2 + c_{15}'.
\end{aligned}$$

Hence, we infer that

$$\Lambda(\boldsymbol{u}^n_{k,\varepsilon}, \varphi^n_{k,\varepsilon}) \ge \frac{1}{4}\|\nabla \boldsymbol{u}^n_{k,\varepsilon}\|^2 + \frac{1}{4}\|\nabla \mu^n_{k,\varepsilon}\|^2 - c_{15}' \tag{4.34}$$

Moreover, it is easily seen that

$$\Lambda(\boldsymbol{u}^n_{k,\varepsilon}, \varphi^n_{k,\varepsilon}) \le c_{16}'\Bigl(1 + \|\nabla \boldsymbol{u}^n_{k,\varepsilon}\|^2 + \|\nabla \mu^n_{k,\varepsilon}\|^2\Bigr) \tag{4.35}$$

In summary, exploiting (4.23) and the Sobolev embedding $V \hookrightarrow L^3(\Omega)$, we are led to rewrite (4.33) as

$$\frac{d}{dt}\Lambda(\boldsymbol{u}^n_{k,\varepsilon},\varphi^n_{k,\varepsilon}) + \nu \Bigl(\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}\|^2 + \|\partial_t \boldsymbol{u}^n_{k,\varepsilon}\|^2 + \|\nabla \partial_t \varphi^n_{k,\varepsilon}\|^2 \Bigr)$$

$$\leq c'_{17}\Bigl(1 + \Lambda^2(\boldsymbol{u}^n_{k,\varepsilon},\varphi^n_{k,\varepsilon})\Bigr), \tag{4.36}$$

where $\nu = \frac{1}{4}\min\{1,\nu_*,\varpi\}$. Owing to (4.4), (4.20), and (4.35), we infer that

$$\int_0^T \Lambda(\boldsymbol{u}^n_{k,\varepsilon}(s),\varphi^n_{k,\varepsilon}(s))\,ds \leq c'_{18}.$$

An application of the Gronwall lemma to (4.36) implies that

$$\Lambda(\boldsymbol{u}^n_{k,\varepsilon}(t),\varphi^n_{k,\varepsilon}(t)) \leq \Lambda(\boldsymbol{u}^n_{k,\varepsilon}(0),\varphi^n_{k,\varepsilon}(0))\mathrm{e}^{c'_{18}} + c'_{17}\mathrm{e}^{c'_{18}}T \quad \forall\, t\in[0,T]. \tag{4.37}$$

In order to find a uniform control of the right-hand side of (4.37), by using the Sobolev embedding $V \hookrightarrow L^6(\Omega)$, (4.8), and (4.9), we obtain

$$\begin{aligned}
\Lambda(\boldsymbol{u}^n_{k,\varepsilon}(0),\varphi^n_{k,\varepsilon}(0)) &= \Lambda(P_n\boldsymbol{u}_0,\Pi_n\varphi_{0,k})\\
&= \frac{1}{2}\|\nabla P_n\boldsymbol{u}_0\|^2 + \frac{1}{2}\|\nabla\mu^n_{k,\varepsilon}(0)\|^2 + (P_n\boldsymbol{u}_0\cdot\nabla\Pi_n\varphi_{0,k},\mu^n_{k,\varepsilon}(0))\\
&\leq \frac{1}{2}\|\nabla\boldsymbol{u}_0\|^2 + \frac{1}{2}\|\nabla\mu^n_{k,\varepsilon}(0)\|^2 + \|P_n\boldsymbol{u}_0\|_{\mathbf{L}^3(\Omega)}\|\nabla\Pi_n\varphi_{0,k}\|\|\mu^n_{k,\varepsilon}(0)\|_{L^6(\Omega)}\\
&\leq \|\nabla\boldsymbol{u}_0\|^2 + C\bigl(1+\|\varphi_{0,k}\|_V^2\bigr)\|\mu^n_{k,\varepsilon}(0)\|_V^2\\
&\leq \|\nabla\boldsymbol{u}_0\|^2 + C\bigl(1+\|\varphi_0\|_V^2\bigr)\|\mu^n_{k,\varepsilon}(0)\|_V^2.
\end{aligned}$$

In light of (4.8), (4.9), (4.11), and (4.16), we find

$$\begin{aligned}
\|\mu^n_{k,\varepsilon}(0)\|_V &= \|\Pi_n(-\Delta\varphi^n_{k,\varepsilon}(0) + \Psi'_\varepsilon(\varphi^n_{k,\varepsilon}(0)))\|_V\\
&\leq \|-\Delta\varphi^n_{k,\varepsilon}(0) + F'_\varepsilon(\varphi^n_{k,\varepsilon}(0))\|_V + \theta_0\|\varphi^n_{k,\varepsilon}(0)\|_V\\
&\leq \|-\Delta\varphi^n_{k,\varepsilon}(0) + F'_\varepsilon(\varphi^n_{k,\varepsilon}(0)) + \Delta\varphi_{0,k} - F'_\varepsilon(\varphi_{0,k})\|_V + C(\|\tilde{\mu}_{0,k}\|_V + \|\varphi_0\|_V)\\
&\leq \|\varphi^n_{k,\varepsilon}(0) - \varphi_{0,k}\|_{H^3(\Omega)} + \|F'_\varepsilon(\varphi^n_{k,\varepsilon}(0)) - F'_\varepsilon(\varphi_{0,k})\|_V + C(\|\tilde{\mu}_0\|_V + \|\varphi_0\|_V)\\
&= \|\Pi_n\varphi_{0,k} - \varphi_{0,k}\|_{H^3(\Omega)} + \|F'_\varepsilon(\Pi_n\varphi_{0,k}) - F'_\varepsilon(\varphi_{0,k})\|_V + C(\|\mu_0\|_V + \|\varphi_0\|_V).
\end{aligned} \tag{4.38}$$

Recalling the bounds (4.10) and (4.17) and the relation $F(z) = F_\varepsilon(z)$, for all $z \in [-1+\overline{\varepsilon}, 1-\overline{\varepsilon}]$ (cf. $0 < \varepsilon < \overline{\varepsilon}$), we deduce that

(4.39)
$$\begin{aligned}
&\|F'_\varepsilon(\Pi_n\varphi_{0,k}) - F'_\varepsilon(\varphi_{0,k})\|_V \\
&\qquad \le \|F'_\varepsilon(\Pi_n\varphi_{0,k}) - F'_\varepsilon(\varphi_{0,k})\| + \|F''_\varepsilon(\Pi_n\varphi_{0,k})\nabla(\Pi_n\varphi_{0,k} - \varphi_{0,k})\| \\
&\qquad\quad + \|(F''_\varepsilon(\Pi_n\varphi_{0,k}) - F''_\varepsilon(\varphi_{0,k}))\nabla\varphi_{0,k}\| \\
&\qquad \le C\Big(\max_{z\in[-1+\overline{\varepsilon},1-\overline{\varepsilon}]}|F''(z)| + \max_{z\in[-1+\overline{\varepsilon},1-\overline{\varepsilon}]}|F'''(z)|\Big)\|\Pi_n\varphi_{0,k} - \varphi_{0,k}\|_V .
\end{aligned}$$

We notice that the quantity between brackets in (4.39) is finite since $F \in \mathscr{C}^3(-1,1)$, and it only depends on k (cf. definition of $\overline{\varepsilon}$). Let us now recall that $\Pi_n\varphi_{0,k} \rightarrow \varphi_{0,k}$ in $H^3(\Omega)$ as $n \rightarrow \infty$. Thus, we infer from (4.38)–(4.39) that, for any fixed $k > \overline{k}$ (and $\varepsilon \in (0,\overline{\varepsilon})$), there exists $\overline{\overline{n}} > \overline{n}$ (cf. (4.16)) such that

$$\|\mu^n_{k,\varepsilon}(0)\|_V \le C(1 + \|\mu_0\|_V + \|\varphi_0\|_V) \quad \forall n > \overline{\overline{n}}, \tag{4.40}$$

where C is independent of k, n, and ε. Finally, for any fixed $k > \overline{k}$, $\varepsilon \in (0,\overline{\varepsilon})$, and $n > \overline{\overline{n}}$ (where $\overline{\varepsilon}$ and $\overline{\overline{n}}$ depends on k), we infer from (4.37) and (4.40) that

$$\Lambda(u^n_{k,\varepsilon}(t), \varphi^n_{k,\varepsilon}(t)) \le C(1 + \|\mu_0\|_V + \|\varphi_0\|_V)\mathrm{e}^{c'_{18}} + c'_{17}\mathrm{e}^{c'_{18}}T \quad \forall t \in [0,T].$$

In view of (4.34), we have

$$\sup_{t\in[0,T]}\|\nabla \boldsymbol{u}^n_{k,\varepsilon}(t)\| + \sup_{t\in[0,T]}\|\nabla\mu^n_{k,\varepsilon}(t)\| \le \overline{C}_1, \tag{4.41}$$

where $\overline{C}_1$ is a positive constant, which depends on T and $\mathscr{E}(u_0,\varphi_0)$, $\|u_0\|_{\mathbf{V}_\sigma}$, and $\|\mu_0\|_V$, but is independent of k, n, and ε. Moreover, an integration in time of (4.36) on the time interval $[0,T]$ yields

$$\int_0^T \Big(\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}(s)\|^2 + \|\partial_t\boldsymbol{u}^n_{k,\varepsilon}(s)\|^2 + \|\nabla\partial_t\varphi^n_{k,\varepsilon}(s)\|^2\Big)\,\mathrm{d}s \le \overline{C}_2, \tag{4.42}$$

where $\overline{C}_2$ is a positive constant depending on T and on the initial datum, but independent of k, ε, and n.

6. *Passage to the limit.* Thanks to the analysis performed in step 5, for any fixed $n > \overline{\overline{n}}$, $k > \overline{k}$, $\varepsilon \in (0,\overline{\varepsilon})$, we deduce from (4.41) and (4.42) that

$u^n_{k,\varepsilon}$ is uniformly bounded in $L^\infty(0,T;\mathbf{V}_\sigma) \cap L^2(0,T;\mathbf{W}_\sigma) \cap H^1(0,T;\mathbf{H}_\sigma)$, $\varphi^n_{k,\varepsilon}$ is uniformly bounded in $L^\infty(0,T;H^2(\Omega)) \cap H^1(0,T;V)$,

$\mu^n_{k,\varepsilon}$ is uniformly bounded in $L^\infty(0,T;V)$.

By a standard compactness method, we are in position to pass to the limit first as $n \rightarrow \infty$, then as $\varepsilon \rightarrow 0$, and finally, as $k \rightarrow \infty$. As a result, we obtain the existence of a pair (u,φ) such that

$$u \,{}^{\infty} \cap L^2(0,T;\mathbf{W}_\sigma) \cap H^1(0,T; \mathbf{H}_\sigma), \in L\,(0,T;\mathbf{V}_\sigma)$$

$$\varphi \in L^\infty(0,T;H^2(\Omega)) \cap H^1(0,T;V),$$

$\varphi \in L^\infty(\Omega \times (0,T))$, with $|\varphi(x,t)| < 1$ a.e. $(x,t) \in \Omega \times (0,T)$,

$^\prime(\varphi) \in L^\infty(0,T;V$). Morewhich satisfies (2.12) and (2.13), where $\mu = -\Delta\varphi + \Psi$ over, $\partial_{\mathbf{n}}\varphi = 0$ a.e. on $\partial\Omega \times (0,T)$, $u(\cdot,0) = u_0$, and $\varphi(\cdot,0) = \varphi_0$ in Ω. Since $\partial_t\varphi + u \cdot \nabla\varphi$ belongs to $L^2(0,T;V)$ owing to the above regularity properties, we in fer from the classical regularity theory of the homogeneous Neumann operator that

$\mu \in L^2(0,T;H^3(\Omega))$, $\partial_n\mu = 0$ a.e. on $\partial\Omega \times (0,T)$ and $\partial_t\varphi + u \cdot \nabla\varphi = \Delta\mu$ holds

a.e. in $\Omega \times (0,T)$. Finally, we can recover the pressure π arguing as in [81, Propositions 1.1 and 1.2, Chapter III]. In particular, it is possible to show that there exists

$\pi \in L^2(0,T;V)$ such that $\partial_t u + (u \cdot \nabla)u - \operatorname{div}(\nu(\varphi)D u) + \nabla\pi = \mu\nabla\varphi$ holds a.e. in $\Omega \times (0,T)$.

7. *Further regularity properties.* From the regularity $\mu \in L^\infty(0,T;V)$, Theorem A.2 entails that $\varphi \in L^\infty(0,T;W^{2,p}(\Omega))$ and $F^\prime(\varphi) \in L^\infty(0,T;L^p(\Omega))$ for any $2 \leq p < \infty$. Furthermore, thanks to the growth condition (2.11), we also deduce that $F^{\prime\prime}(\varphi) \in L^\infty(0,T;L^p(\Omega))$ for any $p \in (2,\infty)$. Next, as a consequence, we prove that

$\partial_t\mu$ exists and belongs to $L^2(0,T;V^\prime)$. To this aim, given $h > 0$, we denote the difference quotient of a function f by $\partial_t^h f = \frac{1}{h}\big(f(t+h) - f(t)\big)$. For any $v \in V$ $^h\mu, v) =$ with$^h\varphi,\|$ $v\|\nabla_v v) + (\leq 1$, by using the boundary condition on $\partial_t^h F^\prime(\varphi), v)$ $- \theta_0(\partial_t^h\varphi, v)$. Since $F^{\prime\prime}$ is convex, we find the control φ, we observe that $(\partial_t$ $(\nabla\partial_t$

$$(\partial_t^h F^\prime(\varphi), v) \leq \bigm\| \bigm\| \bigm\| \int_0^1 \Bigl(sF^{\prime\prime}(\varphi(\cdot + h)) + (1-s)F^{\prime\prime}(\varphi)\Bigr)\, ds \bigm\| \bigm\| \bigm\|_{L^3(\Omega)} \|\partial_t^h\varphi\| \, \|v\|_{L^6(\Omega)}$$

(4.43) $$\leq C\Bigl(\| F''(\varphi(\cdot + h))\|_{L^3(\Omega)} + \| F''(\varphi)\|_{L^3(\Omega)}\Bigr) \| \partial_t^h\varphi \|.$$

Recalling that $\partial_t^h\varphi \rightarrow \partial_t\varphi$ in $L^2(0,T;V)$ and $F''(\varphi) \in L^\infty(0,T;L^3(\Omega))$, there exists a positive constant C_3, independent of h, such that $\| \partial_t^h\mu \|_{L^2(0,T;V')} \leq C_3$. This implies that $\partial_t\mu \in L^2(0,T;V')$. In particular, we deduce that $\mu \in \mathscr{C}([0,T],V)$.

8. *Uniqueness and continuous dependence.* The uniqueness of strong solutions is an immediate consequence of Theorem 3.1. We conclude the proof by showing a continuous dependence estimate with respect to the initial conditions in higher-order norms than the dual norms employed in Theorem 3.1. We define $u = u_1 - u_2$ and $\varphi = \varphi_1 - \varphi_2$, where (u_1,φ_1) and (u_2,φ_2) are two strong solutions departing from (u_{01},φ_{01}) and (u_{02},φ_{02}) that satisfy $u_{0i} \in \mathbf{V}_\sigma$ and $\varphi_{0i} \in H^2(\Omega)$ such that $\|\varphi_{0i}\|_{L^\infty(\Omega)} \leq 1$, $|\overline{\varphi_{0i}}| < 1$, $\mu_{0i} = -\Delta\varphi_{0i} + \Psi'(\varphi_{0i}) \in V$ and $\partial_{\mathbf{n}}\varphi_{0i} = 0$ on $\partial\Omega$. We take $v = u$ and $v = \varphi$ in (3.1) and (3.2), respectively. Adding the resulting equalities, we find

$$\frac{d}{dt}\mathscr{H}_1 + (\nu(\varphi_1)Du,Du) + (\nabla\mu,\nabla\varphi) = \sum_{k=1}^{3}\mathscr{J}_k,$$

having set

$$\mathcal{H}_1 = \frac{1}{2}\|\boldsymbol{u}\|^2 + \frac{1}{2}\|\varphi\|^2,$$
$$\mathcal{J}_1 = -((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}_2, \boldsymbol{u}) - ((\nu(\varphi_1) - \nu(\varphi_2))D\boldsymbol{u}_2, \nabla\boldsymbol{u}),$$
$$\mathcal{J}_2 = (\nabla\varphi_1 \otimes \nabla\varphi, \nabla\boldsymbol{u}) + (\nabla\varphi \otimes \nabla\varphi_2, \nabla\boldsymbol{u}),$$
$$\mathcal{J}_3 = (\varphi\boldsymbol{u}_1, \nabla\varphi) + (\varphi_2\boldsymbol{u}, \nabla\varphi).$$

In light of the regularity of strong solutions, there exists a positive constant C_0 such that

(4.44) $$\|\boldsymbol{u}_i\|_{L^\infty(0,T;\mathbf{L}^3(\Omega))} + \|\varphi\|_{L^\infty(0,T;W^{2,3}(\Omega))} + \|\Psi(\varphi_i)\|_{L^\infty(0,T;L^3(\Omega))} \leq C_0.$$

In what follows the positive constant C_i, $i \in \mathbb{N}$ depends on ν_*, ν', C_0, and the constants appearing in embedding results. Due to the homogeneous Neumann boundary condition, we also recall the basic inequality

(4.45) $$\|\varphi\|_V^2 \leq \|\Delta\varphi\|\|\varphi\| + \|\varphi\|^2.$$

Integrating by parts and using the embedding $V \hookrightarrow L^6(\Omega)$, together with (4.44) and (4.45), we observe that

$$
\begin{aligned}
(\nabla\mu, \nabla\varphi) &\geq \|\Delta\varphi\|^2 - \big(\|\Psi''(\varphi_1)\|_{L^3(\Omega)} + \|\Psi''(\varphi_2)\|_{L^3(\Omega)}\big)\|\varphi\|_{L^6(\Omega)}\|\Delta\varphi\| \\
&\geq \frac{1}{2}\|\Delta\varphi\|^2 - C_1\|\varphi\|_V^2 \\
&\geq \frac{1}{4}\|\Delta\varphi\|^2 - C_2\|\varphi\|^2.
\end{aligned}
$$

Due to the Korn inequality and the above estimate, we obtain

$$
\frac{d}{dt}\mathcal{H}_1 + \nu_*\|\nabla \boldsymbol{u}\|^2 + \frac{1}{4}\|\Delta\varphi\|^2 \leq C_2\|\varphi\|^2 + \sum_{k=1}^{3}\mathcal{J}_k.
$$

We now address the terms $\mathcal{J}_k$. By using (4.45), we have

$$
\begin{aligned}
\mathcal{J}_1 &\leq \|\boldsymbol{u}\|\|\nabla \boldsymbol{u}_2\|_{\mathbf{L}^3(\Omega)}\|\boldsymbol{u}\|_{\mathbf{L}^6(\Omega)} + C\|\varphi\|_{L^6(\Omega)}\|D\boldsymbol{u}_2\|_{\mathbf{L}^3(\Omega)}\|\nabla \boldsymbol{u}\| \\
&\leq \frac{\nu_*}{4}\|\nabla \boldsymbol{u}\|^2 + C\|\nabla \boldsymbol{u}_2\|_{\mathbf{L}^3(\Omega)}^2\|\boldsymbol{u}\|^2 + C\|D\boldsymbol{u}_2\|_{\mathbf{L}^3(\Omega)}^2\|\varphi\|_V^2, \\
&\leq \frac{\nu_*}{4}\|\nabla \boldsymbol{u}\|^2 + \frac{1}{24}\|\Delta\varphi\|^2 + C_3\Big(\|\nabla \boldsymbol{u}_2\|_{\mathbf{L}^3(\Omega)}^2\|\boldsymbol{u}\|^2 + \|D\boldsymbol{u}_2\|_{\mathbf{L}^3(\Omega)}^4\|\varphi\|^2\Big).
\end{aligned}
$$

By (4.44) and (4.45) and the embedding $W^{2,3}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ valid in dimension two, we obtain

$$
\begin{aligned}
\mathcal{J}_2 &\leq \big(\|\nabla\varphi_1\|_{\mathbf{L}^\infty(\Omega)} + \|\nabla\varphi_2\|_{\mathbf{L}^\infty(\Omega)}\big)\|\nabla\varphi\|\|\nabla \boldsymbol{u}\| \\
&\leq \frac{\nu_*}{4}\|\nabla \boldsymbol{u}\|^2 + C_4\|\nabla\varphi\|^2 \\
&\leq \frac{\nu_*}{4}\|\nabla \boldsymbol{u}\|^2 + \frac{1}{24}\|\Delta\varphi\|^2 + C_5\|\varphi\|^2,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{J}_3 &\leq \|\varphi\|_{L^6(\Omega)}\|\boldsymbol{u}_1\|_{\mathbf{L}^3(\Omega)}\|\nabla\varphi\| + \|\boldsymbol{u}\|\|\nabla\varphi\| \\
&\leq C_6\|\varphi\|_V^2 + \|\boldsymbol{u}\|^2 \\
&\leq \frac{1}{24}\|\Delta\varphi\|^2 + C_7\Big(\|\varphi\|^2 + \|\boldsymbol{u}\|^2\Big).
\end{aligned}
$$

In view of the above estimates, we end up with the following differential inequality

$$
\frac{d}{dt}\mathcal{H}_1 + \frac{\nu_*}{2}\|\nabla \boldsymbol{u}\|^2 + \frac{1}{8}\|\Delta\varphi\|^2 \leq C_8\Big(1 + \|\boldsymbol{u}_2\|_{\mathbf{W}^{1,3}(\Omega)}^4\Big)\mathcal{H}_1.
$$

Therefore, since $\boldsymbol{u}_2 \in L^4(0,T;\mathbf{W}^{1,3}(\Omega))$, an application of the Gronwall lemma implies the desired stability inequality (4.1). ∎

By virtue of the energy identity (cf. (2.18)) and the global well-posedness of the strong solutions, we can prove that the (unique) weak solution regularizes instantaneously. That is, the weak solution is indeed a strong solution on $\Omega \times (\tau, \infty)$ for any $\tau > 0$.

Theorem 4.2. *Let $d = 2$, $R > 0$, $m \in (-1,1)$, and $\tau > 0$ be given. Assume that (u_0,φ_0) is an initial datum such that $\mathcal{E}(u_0,\varphi_0) \leq R$, $\|\varphi_0\|_{L^\infty(\Omega)} \leq 1$ and $\overline{\varphi_0} = m$, and (u,φ) is the weak solution departing from (u_0,φ_0). Then, there exist two positive constants $M_1 = M_1(R,m,\tau)$ and $M_2 = M_2(R,m,\tau)$, independent of the specific datum (u_0,φ_0), such that*

$$\sup_{t\geq\tau} \|\boldsymbol{u}(t)\|_{\mathbf{V}_\sigma} + \sup_{t\geq\tau} \|\mu(t)\|_V \leq M_1, \tag{4.46}$$

and

$$\|u\|_{L^2(t,t+1;\mathbf{W}_\sigma)} + \|\partial_t u\|_{L^2(t,t+1;\mathbf{H}_\sigma)} + \|\partial_t\varphi\|_{L^2(t,t+1;V)} \leq M_2 \quad \forall\, t \geq \tau. \tag{4.47}$$

In addition, for any $p \geq 2$, there exists a positive constant $M_3 = M_3(R,m,\tau,p)$ such that

$$\sup_{t\geq\tau} \|\varphi(t)\|_{W^{2,p}(\Omega)} + \|F''(\varphi)\|_{L^\infty(\tau,\infty;L^p(\Omega))} \leq M_3. \tag{4.48}$$

Proof. Let (u,φ) be the global weak solution with initial condition (u_0,φ_0) given by Theorem 2.4. Due to (2.18), for any $\tau > 0$, we infer from (2.18) that there exists $\tau_0 \in (0,\tau)$ such that $(u(\tau_0),\varphi(\tau_0))$ satisfies the assumptions of Theorem 4.1 and

$$\mathcal{E}(u(\tau_0),\varphi(\tau_0)) \leq R, \quad \overline{\varphi(\tau_0)} = m. \tag{4.49}$$

Taking $(u(\tau_0),\varphi(\tau_0))$ as initial datum, we have a global strong solution on the time interval $[\tau_0,\infty)$, which coincides with the weak solution due to Theorem 3.1. Now, in order to show the uniform estimates (4.46)--(4.48), we consider the approximating solutions $(u^n_{k,\varepsilon},\varphi^n_{k,\varepsilon})$ constructed in the proof of Theorem 4.1 on the time interval $[\tau_0,\infty)$ corresponding to the initial datum $(u(\tau_0),\varphi(\tau_0))$. Thanks to (4.18) and (4.19), we have

$$\mathcal{E}_\varepsilon(u^n_{k,\varepsilon}(t),\varphi^n_{k,\varepsilon}(t)) + \int_t^{t+1} \Bigl(\nu_\ast \|\nabla u^n_{k,\varepsilon}(s)\|^2 + \|\nabla \mu^n_{k,\varepsilon}(s)\|^2 \Bigr)\, ds \leq \tilde{c}_0 \quad \forall\, t \geq \tau_0, \tag{4.50}$$

where $\tilde{c}_0$ depends on R, but is independent of t. Then, following line by line steps 4 and 5 in the proof of Theorem 4.1, we deduce the differential inequality (cf. (4.36))

(4.51)

$$\frac{d}{dt}\Lambda(\boldsymbol{u}^n_{k,\varepsilon},\varphi^n_{k,\varepsilon}) + \nu\Big(\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}\|^2 + \|\partial_t \boldsymbol{u}^n_{k,\varepsilon}\|^2 + \|\nabla\partial_t\varphi^n_{k,\varepsilon}\|^2\Big) \leq \tilde{c}_1\Big(1+\Lambda^2(\boldsymbol{u}^n_{k,\varepsilon},\varphi^n_{k,\varepsilon})\Big),$$

where $\Lambda(\boldsymbol{u}^n_{k,\varepsilon},\varphi^n_{k,\varepsilon})$ is defined in (4.32). Here, the positive constants ν and $\tilde{c}_1$ depend on R, m, and the other parameters of the system but are independent of k, ε, and n.

By (2.7) and (4.50), we notice that

$$\int_t^{t+1} \Lambda(\boldsymbol{u}^n_{k,\varepsilon}(s),\varphi^n_{k,\varepsilon}(s))\, ds \leq \tilde{c}_2 \quad \forall\, t \geq \tau_0.$$

Hence, an application of the uniform Gronwall lemma (see [80, Chapter III, Lemma 1.1]) to (4.51) with $r = \tau - \tau_0$ entails

$$\|\boldsymbol{u}^n_{k,\varepsilon}(t)\|_{\mathbf{V}_\sigma} + \|\mu^n_{k,\varepsilon}(t)\|_V \leq M_1 \quad \forall\, t \geq \tau,$$

where M_1 depends on R, m, and τ, but is independent of $(\boldsymbol{u}(\tau_0),\varphi(\tau_0))$. In addition, integrating in time (4.51) on $(t,t+1)$, for any $t \geq \tau$, we are lead to

$$\|\boldsymbol{u}^n_{k,\varepsilon}\|_{L^2(t,t+1;\mathbf{W}_\sigma)} + \|\partial_t \boldsymbol{u}^n_{k,\varepsilon}\|_{L^2(t,t+1;\mathbf{H}_\sigma)} + \|\partial_t\varphi^n_{k,\varepsilon}\|_{L^2(t,t+1;V)} \leq M_2 \quad \forall\, t \geq \tau.$$

At this stage, passing to the limit in k, ε, and n as in the proof of Theorem 4.1 and using the regularity in time of the strong solutions, the estimates (4.46) and (4.47) easily follow. In turn, we also infer the estimate (4.48) from Theorem A.2. ∎

As a consequence of Proposition 3.5 and Theorem 4.2, we deduce the continuous dependence of weak solutions with respect to the initial data in the energy space.

Proposition 4.3. *Let $d = 2$. Assume that a sequence of initial data $(\boldsymbol{u}_{0n},\varphi_{0n})$ and $(\boldsymbol{u}_0,\varphi_0)$ are given such that $\boldsymbol{u}_{0n} \in \mathbf{H}_\sigma$, $\varphi_{0n} \in V$, $\|\varphi_{0n}\|_{L^\infty(\Omega)} \leq 1$ and $\overline{\varphi_{0n}} = m$ with $m \in (-1,1)$ for all n, and $(\boldsymbol{u}_{0n},\varphi_{0n})$ converges to $(\boldsymbol{u}_0,\varphi_0)$ in $\mathbf{H}_\sigma \times V$. Consider the solutions $(\boldsymbol{u}_n,\varphi_n)$, $(\boldsymbol{u},\varphi)$ to (1.1)–(1.2) with initial data $(\boldsymbol{u}_{0n},\varphi_{0n})$ and $(\boldsymbol{u}_0,\varphi_0)$, respectively. Then, for any $t > 0$, $(\boldsymbol{u}_n(t),\varphi_n(t))$ converges to $(\boldsymbol{u}(t),\varphi(t))$ in $\mathbf{H}_\sigma \times V$.*

Proof. Let us fix $t > 0$. By assumption there exists $N_0 > 0$ such that $\|\boldsymbol{u}_{0n}\| + \|\varphi_{0n}\|_V \leq N_0$ and $\|\boldsymbol{u}_0\| + \|\varphi_0\|_V \leq N_0$. By Theorem 4.2 (with $\tau = \frac{t}{2}$) there exists N_1 depending only on N_0,m,t such that $\|\boldsymbol{u}_n(t)\|_{\mathbf{V}_\sigma} + \|\varphi_n(t)\|_{H^2(\Omega)} \leq N_1$. Obviously, the same control in $\mathbf{V}_\sigma \times H^2(\Omega)$ holds for $(\boldsymbol{u},\varphi)$. By Proposition 3.5 we infer that there exists N_2 depending on N_0 and m such that

$$\|\boldsymbol{u}_n(t)-\boldsymbol{u}(t)\|_\sharp^2+\|\varphi_n(t)-\varphi(t)\|_*^2 \leq N_2\Big(\frac{\|\boldsymbol{u}_{0n}-\boldsymbol{u}_0\|_\sharp^2+\|\varphi_{0n}-\varphi_0\|_*^2}{N_2}\Big)^{\mathrm{e}^{-\int_0^t \mathcal{Y}(s)\,\mathrm{d}s}},$$

where

$$\mathcal{Y}(t)=N_2\Big(1+\|\boldsymbol{u}_n(t)\|_{\mathbf{V}_\sigma}^2+\|\boldsymbol{u}(t)\|_{\mathbf{V}_\sigma}^2+\|\nabla\varphi_n(t)\|_{\mathbf{L}^\infty(\Omega)}^2+\|\nabla\varphi(t)\|_{\mathbf{L}^\infty(\Omega)}^2+\|\varphi_n(t)\|_{H^2(\Omega)}^2\Big).$$

Noticing that $\mathcal{Y}(t) \geq N_2$, assuming that $\|\boldsymbol{u}_{0n}-\boldsymbol{u}_0\|_\sharp^2+\|\varphi_{0n}-\varphi_0\|_*^2 \leq 1$, by interpolation we have

$$\begin{aligned}
&\|\boldsymbol{u}_n(t)-\boldsymbol{u}(t)\|+\|\varphi_n(t)-\varphi(t)\|_V\\
&\leq C\Big(\|\boldsymbol{u}_n(t)-\boldsymbol{u}(t)\|_\sharp^{\frac12}+\|\varphi_n(t)-\varphi(t)\|_*^{\frac14}\Big)\\
&\quad\times\Big(\|\boldsymbol{u}_n(t)-\boldsymbol{u}(t)\|_{\mathbf{V}_\sigma}^{\frac12}+\|\varphi_n(t)-\varphi(t)\|_{H^2(\Omega)}^{\frac34}\Big)\\
&\leq C(N_1^{\frac12}+N_1^{\frac34})(N_2^{\frac14}+N_2^{\frac18})\Big(\frac{\|\boldsymbol{u}_{0n}-\boldsymbol{u}_0\|_\sharp^2+\|\varphi_{0n}-\varphi_0\|_*^2}{N_2}\Big)^{\frac14\mathrm{e}^{-N_2 t}}.
\end{aligned}$$

The above inequality implies the desired conclusion. ∎

Our next result concerns the propagation of regularity for any weak solution and the validity of the instantaneous separation property from the pure concentrations (i.e., ± 1) in dimension two. This is possible due to a suitable estimate of $\Psi''(\varphi)$ in L^p spaces, which allows us to show further a priori higher-order Sobolev estimates.

Theorem 4.4. *Let $d=2$, $R>0$, $m\in(-1,1)$, and $\tau>0$ be given. Assume that $(\boldsymbol{u}_0,\varphi_0)$ is an initial datum such that $\mathcal{E}(\boldsymbol{u}_0,\varphi_0)\leq R$, $\|\varphi_0\|_{L^\infty(\Omega)}\leq 1$ and $\overline{\varphi}_0=m$, and $(\boldsymbol{u},\varphi)$ is the weak solution departing from $(\boldsymbol{u}_0,\varphi_0)$. Then, there exists two positive constants $M_4=M_4(R,m,\tau)$ and $M_5=M_5(R,m,\tau)$, independent of the specific datum $(\boldsymbol{u}_0,\varphi_0)$, such that*

$$\|\partial_t\boldsymbol{u}\|_{L^\infty(\tau,\infty;\mathbf{H}_\sigma)}+\|\partial_t\varphi\|_{L^\infty(\tau,\infty;H)}\leq M_4, \tag{4.52}$$

and

$$\|\partial_t\boldsymbol{u}\|_{L^2(t,t+1;\mathbf{V}_\sigma)}+\|\partial_t\varphi\|_{L^2(t,t+1;H^2(\Omega))}\leq M_5\quad\forall\, t\geq\tau. \tag{4.53}$$

Furthermore, there exists $\delta=\delta(R,m,\tau)>0$ and $M_6=M_6(R,m,\tau)$ such that

$$\sup_{t\geq\tau}\|\varphi(t)\|_{C(\overline{\Omega})}\leq 1-\delta$$

and

$$\sup_{t\geq\tau}\|\boldsymbol{u}(t)\|_{\mathbf{W}_\sigma}+\sup_{t\geq\tau}\|\varphi(t)\|_{H^4(\Omega)}\leq M_6. \tag{4.54}$$

Proof. First, by replacing τ with $\frac{\tau}{2}$ in Theorem 4.2, we can assume that the solution $(\boldsymbol{u},\varphi)$ satisfies the uniform estimates (4.46)–(4.48) on the time interval $[\frac{\tau}{2},\infty)$. We proceed by showing additional higher-order a priori estimates on the solution. In the sequel, k_i, $i\in\mathbb{N}$, denotes a positive constant which depends on R, m, and τ but is independent of the specific initial datum. Given $h>0$, repeating line by line the proof of the stability result

(4.1) in Theorem 4.1 (cf. step 8), where the difference of two solutions $(\boldsymbol{u}_1 - \boldsymbol{u}_2, \varphi_1 - \varphi_2)$ is replaced by $(\partial_t^h \boldsymbol{u}, \partial_t^h \varphi)$, we deduce the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_2 + \frac{\nu_*}{2}\|\nabla \partial_t^h \boldsymbol{u}\|^2 + \frac{1}{8}\|\Delta \partial_t^h \varphi\|^2 \leq k_0\left(1 + \|\boldsymbol{u}\|^4_{\mathbf{W}^{1,3}(\Omega)}\right)\mathcal{H}_2, \tag{4.55}$$

where

$$\mathcal{H}_2 = \frac{1}{2}\|\partial_t^h \boldsymbol{u}\|^2 + \frac{1}{2}\|\partial_t^h \varphi\|^2,$$

and the positive constant k_0 is independent of h but depends on M_1 and M_3. Recalling that $\|\partial_t^h f\|_{L^2(t,t+1;H)} \leq \|\partial_t f\|_{L^2(t,t+2;H)}$, thanks to Theorem 4.2, we observe that

$$\int_t^{t+1} \left(\mathcal{H}_2(s) + \|\boldsymbol{u}(s)\|^4_{\mathbf{W}^{1,3}(\Omega)}\right) \mathrm{d}s \leq k_1 \quad \forall\, t \geq \frac{\tau}{2},$$

where k_1 is independent of h, but depends on M_2. Hence, the uniform Gronwall lemma (see [80, Chapter III, Lemma 1.1]) with $r = \frac{\tau}{2}$ yields

$$\|\partial_t^h \boldsymbol{u}\|_{L^\infty(\tau,\infty;\mathbf{H}_\sigma)} + \|\partial_t^h \varphi\|_{L^\infty(\tau,\infty;H)} \leq M_4,$$

and

$$\|\partial_t^h \boldsymbol{u}\|_{L^2(t,t+1;\mathbf{V}_\sigma)} + \|\partial_t^h \varphi\|_{L^2(t,t+1;H^2(\Omega))} \leq M_5 \quad \forall\, t \geq \tau,$$

where M_4 and M_5 depend on R, m, and τ but are independent of h, t, and the specific initial datum. A final passage to the limit as $h \rightarrow 0$ entails (4.52) and (4.53).

We are now in position to prove the separation property. In light of (4.52), it is immediate to deduce that $\partial_t \varphi + \boldsymbol{u}\cdot\nabla\varphi \in L^\infty(\tau,\infty;H)$. Then, the regularity theory of the Neumann problem implies that

$$\|\mu\|_{L^\infty(\tau,\infty;H^2(\Omega))} \leq k_2. \tag{4.56}$$

By Theorem A.2 with $f = \mu + \theta_0 \varphi \in L^\infty(\Omega \times (\tau,\infty))$, we find $\|F'(\varphi)\|_{L^\infty(\Omega\times(\tau,\infty))} \leq k_3$. This, in turn, entails that there exists $\delta > 0$ such that

$$\sup_{t\geq\tau} \|\varphi(t)\|_{C(\overline{\Omega})} \leq 1 - \delta. \tag{4.57}$$

Thanks to the regularity (4.48) and the separation property (4.57), and recalling

Thus, that $F \in \mathscr{C}^3([-1+\delta, 1-\delta])$, we deduce that $\|F'(\varphi)\|_{L^\infty(\tau,\infty;H^2(\Omega))} \leq k_4$. exploiting (4.56), the above control and the regularity theory of the Neumann problem, we get $\|\varphi\|_{L^\infty(\tau,\infty;H^4(\Omega))} \leq k_5$. Moreover, setting $f = \mu\nabla\varphi - \partial_t u - (u\cdot\nabla)u$, we infer from (4.46), (4.48), and (4.52) that, for any $1 < p < 2$, there exists k_6 such that $\|f\|_{L^\infty(\tau,\infty;\mathbf{L}^p(\Omega))} \leq k_6$, where k_6 depends on p. Then, in light of (4.48), an application of Theorem B.3 (with $r = \infty$) yields $\|u\|_{L^\infty(\tau,\infty;\mathbf{W}^{2,p}(\Omega))} \leq k_7$. Recalling the embedding $W^{1,p} \hookrightarrow L^{p^\ast}$, where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{2}$, and choosing $p = \frac{4}{3}$, we obtain $u \in L^\infty(\tau,\infty;\mathbf{W}^{1,4}(\Omega))$. Thanks to this regularity, we observe that $f \in L^\infty(\tau,\infty;\mathbf{H})$. Applying once again Theorem B.3, we find

$$\|u\|_{L^\infty(\tau,\infty;\mathbf{W}_\sigma)} \leq k_8.$$

Due to the continuity in time of the solution, we note that the above inequalities hold for any $t \geq \tau$, giving the desired estimate (4.54) with M_6 depending on k_5 and k_8. ∎

5. Local strong solutions in three dimensions.

In this section we study the well-posedness of strong solutions in dimension three.

Theorem 5.1. *Let* $d = 3$. *Assume that* $u_0 \in \mathbf{V}_\sigma$ *and* $\varphi_0 \in H^2(\Omega)$ *is such that*

$$\|\varphi_0\|_{L^\infty(\Omega)} \leq 1, \quad |\overline{\varphi_0}| < 1, \quad \mu_0 = -\Delta\varphi_0 + \Psi'(\varphi_0) \in V, \text{ and } \partial_n \varphi_0 = 0 \text{ on } \partial\Omega.$$

Then, there exist a time $T^\ast > 0$ *and a unique strong solution to* (1.1)–(1.2) *on* $[0,T^\ast]$ *satisfying*

$$u \in L^\infty(0,T^\ast;\mathbf{V}_\sigma) \cap L^2(0,T^\ast;\mathbf{W}_\sigma) \cap H^1(0,T^\ast;\mathbf{H}_\sigma), \quad \pi \in L^2(0,T^\ast;V),$$

$$\varphi \in L^\infty(0,T^\ast;W^{2,6}(\Omega)) \cap H^1(0,T^\ast;V),$$

$$\mu \in L^\infty(0,T^\ast;V) \cap L^2(0,T^\ast;H^3(\Omega)).$$

The strong solution satisfies (1.1) *a.e. on* $(x,t) \in \Omega \times (0,T^\ast)$ *and* $\partial_n \mu = 0$ *a.e. on* $\partial\Omega \times (0,T^\ast)$.

The proof of Theorem 5.1 relies on the argument employed in the proofs of Theorems 3.1 and 4.1. For the sake of brevity, we report only the main changes.

Proof. We follow the proof of Theorem 4.1. For the same values of k, ε, and n as defined in steps 1–3, we obtain the approximating sequences $(u^n_{k,\varepsilon}, \varphi^n_{k,\varepsilon})$ that solve (4.13)–(4.14) and (4.15). Before deriving uniform a priori estimates we specify that the constants c'_i, $i \in \mathbb{N}$, positive constant, depends on the parameters of the system, the constants in embedding and interpolation results, and \scrE (u_0,\varphi $_0$), but is independent of the approximation parameters k, \varepsilon , and n and of the norms \| u_0\| \bfV \sigma and \| \mu $_0$\| V . It is easily seen that the energy estimates (4.20)--(4.25) also hold. In particular, we have (5.1) $\|\boldsymbol{u}^n_{k,\varepsilon}(t)\| + \|\varphi^n_{k,\varepsilon}(t)\|_V \leq c_1' \quad \forall t \in [0,T]$.

Let us now proceed by showing higher-order Sobolev estimates. First, arguing as in step 5 we find

$$t\left[\frac{1}{2}\|\nabla\mu^n_{k,\varepsilon}\|^2 + (\boldsymbol{u}^n_{k,\varepsilon}\cdot\nabla\varphi^n_{k,\varepsilon},\mu^n_{k,\varepsilon})\right] + \frac{1}{4}\|\nabla\partial_t\varphi^n_{k,\varepsilon}\|— \ –$$

d2

d

$\boldsymbol{v} = \partial_t \boldsymbol{u}^n_{k,\varepsilon}$ in (4.13).

$$\leq (\partial_t\boldsymbol{u}^n_{k,\varepsilon}\cdot\nabla\varphi^n_{k,\varepsilon},\mu^n_{k,\varepsilon}) + c_2'(1+\|\boldsymbol{u}^n_{k,\varepsilon}\|^2_{\mathbf{L}^3(\Omega)})(1+\|\nabla\boldsymbol{u}^n_{k,\varepsilon}\|^2+\|\nabla\mu^n_{k,\varepsilon}\|^2). \tag{5.2}$$

In order to recover estimates on the velocity field, we take first This yields

$$\|\partial_t\boldsymbol{u}^n_{k,\varepsilon}\|^2 + b(\boldsymbol{u}^n_{k,\varepsilon},\boldsymbol{u}^n_{k,\varepsilon},\partial_t\boldsymbol{u}^n_{k,\varepsilon}) - (\operatorname{div}(\nu(\varphi^n_{k,\varepsilon})D\boldsymbol{u}^n_{k,\varepsilon}),\partial_t\boldsymbol{u}^n_{k,\varepsilon}) = (\mu^n_{k,\varepsilon}\nabla\varphi^n_{k,\varepsilon},\partial_t\boldsymbol{u}^n_{k,\varepsilon})$$

By using (2.2), (2.6), we have

$$\begin{aligned}
b(\boldsymbol{u}^n_{k,\varepsilon},\boldsymbol{u}^n_{k,\varepsilon},\partial_t\boldsymbol{u}^n_{k,\varepsilon}) &\leq \|\boldsymbol{u}^n_{k,\varepsilon}\|_{\mathbf{L}^6(\Omega)}\|\nabla\boldsymbol{u}^n_{k,\varepsilon}\|_{\mathbf{L}^3(\Omega)}\|\partial_t\boldsymbol{u}^n_{k,\varepsilon}\| \\
&\leq C\|\nabla\boldsymbol{u}^n_{k,\varepsilon}\|^{\frac{3}{2}}\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}\|^{\frac{1}{2}}\|\partial_t\boldsymbol{u}^n_{k,\varepsilon}\| \\
&\leq \frac{1}{6}\|\partial_t\boldsymbol{u}^n_{k,\varepsilon}\|^2 + c_3'\Big(\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}\|^2 + \|\nabla\boldsymbol{u}^n_{k,\varepsilon}\|^6\Big).
\end{aligned}$$

Exploiting once more (2.2) and (2.6), we obtain

$$\begin{aligned}
&(\operatorname{div}(\nu(\varphi^n_{k,\varepsilon})D\boldsymbol{u}^n_{k,\varepsilon}),\partial_t\boldsymbol{u}^n_{k,\varepsilon}) \\
&\quad= \frac{1}{2}(\nu(\varphi^n_{k,\varepsilon})\Delta\boldsymbol{u}^n_{k,\varepsilon},\partial_t\boldsymbol{u}^n_{k,\varepsilon}) + (\nu'(\varphi^n_{k,\varepsilon})D\boldsymbol{u}^n_{k,\varepsilon}\nabla\varphi^n_{k,\varepsilon},\partial_t\boldsymbol{u}^n_{k,\varepsilon}) \\
&\quad\leq C\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}\|\|\partial_t\boldsymbol{u}^n_{k,\varepsilon}\| + C\|\nabla\varphi^n_{k,\varepsilon}\|_{\mathbf{L}^6(\Omega)}\|D\boldsymbol{u}^n_{k,\varepsilon}\|_{\mathbf{L}^3(\Omega)}\|\partial_t\boldsymbol{u}^n_{k,\varepsilon}\| \\
&\quad\leq \frac{1}{6}\|\partial_t\boldsymbol{u}^n_{k,\varepsilon}\|^2 + C\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}\|^2 + C\|\varphi^n_{k,\varepsilon}\|^2_{H^2(\Omega)}\|\nabla\boldsymbol{u}^n_{k,\varepsilon}\|\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}\| \\
&\quad\leq \frac{1}{6}\|\partial_t\boldsymbol{u}^n_{k,\varepsilon}\|^2 + c_4'\Big(\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}\|^2 + \|\varphi^n_{k,\varepsilon}\|^4_{H^2(\Omega)}\|\nabla\boldsymbol{u}^n_{k,\varepsilon}\|^2\Big).
\end{aligned}$$

On the other hand, by (4.22) we have

$$\begin{aligned}
(\mu^n_{k,\varepsilon}\nabla\varphi^n_{k,\varepsilon},\partial_t\boldsymbol{u}^n_{k,\varepsilon}) &\leq \|\mu^n_{k,\varepsilon}\|_{L^6(\Omega)}\|\nabla\varphi^n_{k,\varepsilon}\|_{\mathbf{L}^3(\Omega)}\|\partial_t\boldsymbol{u}^n_{k,\varepsilon}\| \\
&\leq \frac{1}{6}\|\partial_t\boldsymbol{u}^n_{k,\varepsilon}\|^2 + c_5'\|\nabla\varphi^n_{k,\varepsilon}\|^2_{\mathbf{L}^3(\Omega)}(1+\|\nabla\mu^n_{k,\varepsilon}\|^2).
\end{aligned}$$

Collecting the above estimates, we arrive at

$$\begin{aligned}
\|\partial_t\boldsymbol{u}^n_{k,\varepsilon}\|^2 \leq c_6'\Big(&\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}\|^2 + \|\nabla\boldsymbol{u}^n_{k,\varepsilon}\|^6 \\
&+ \|\varphi^n_{k,\varepsilon}\|^4_{H^2(\Omega)}\|\nabla\boldsymbol{u}^n_{k,\varepsilon}\|^2 + \|\nabla\varphi^n_{k,\varepsilon}\|^2_{\mathbf{L}^3(\Omega)}(1+\|\nabla\mu^n_{k,\varepsilon}\|^2)\Big).
\end{aligned} \tag{5.3}$$

Next, we take v = **A**$u^n_{k,\varepsilon}$ in (4.13). We recall that there exists $p^n_{k,\varepsilon}$ \in $L^2(0,T;V)$ satisfying - \Delta $u^n_{k,\varepsilon}$ +\nabla $p^n_{k,\varepsilon}$ = **A**$u^n_{k,\varepsilon}$ a.e. in \Omega $\times (0,T$) and the estimates (4.28). Thus, we find

We address the right-hand side of the above differential inequality by using (2.2) and

$$\begin{aligned}
-(\nu'(\varphi^n_{k,\varepsilon})\nabla\varphi^n_{k,\varepsilon}p^n_{k,\varepsilon},\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}) &+ (\nu'(\varphi^n_{k,\varepsilon})D\boldsymbol{u}^n_{k,\varepsilon}\nabla\varphi^n_{k,\varepsilon},\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon})\\
&\le C\|\nabla\varphi^n_{k,\varepsilon}\|_{\mathbf{L}^6(\Omega)}\Big(\|p^n_{k,\varepsilon}\|_{L^3(\Omega)}+\|D\boldsymbol{u}^n_{k,\varepsilon}\|_{\mathbf{L}^3(\Omega)}\Big)\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}\|\\
&\le C\|\varphi^n_{k,\varepsilon}\|_{H^2(\Omega)}\Big(\|p^n_{k,\varepsilon}\|^{\frac12}\|p^n_{k,\varepsilon}\|_V^{\frac12}+\|\nabla\boldsymbol{u}^n_{k,\varepsilon}\|^{\frac12}\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}\|^{\frac12}\Big)\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}\|\\
&\le C\|\varphi^n_{k,\varepsilon}\|_{H^2(\Omega)}\Big(\|\nabla\boldsymbol{u}^n_{k,\varepsilon}\|^{\frac14}\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}\|^{\frac34}+\|\nabla\boldsymbol{u}^n_{k,\varepsilon}\|^{\frac12}\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}\|^{\frac12}\Big)\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}\|\\
&\le \frac{\nu_*}{6}\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}\|^2+c_7'(1+\|\varphi^n_{k,\varepsilon}\|^8_{H^2(\Omega)})\|\nabla\boldsymbol{u}^n_{k,\varepsilon}\|^2,
\end{aligned}$$

and

$$\begin{aligned}
b(\boldsymbol{u}^n_{k,\varepsilon},\boldsymbol{u}^n_{k,\varepsilon},\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}) &\le \|\boldsymbol{u}^n_{k,\varepsilon}\|_{\mathbf{L}^6(\Omega)}\|\nabla\boldsymbol{u}^n_{k,\varepsilon}\|_{\mathbf{L}^3(\Omega)}\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}\|\\
&\le \frac{\nu_*}{6}\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}\|^2+c_8'\|\nabla\boldsymbol{u}^n_{k,\varepsilon}\|^6.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
(\mu^n_{k,\varepsilon}\nabla\varphi^n_{k,\varepsilon},\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}) &\le \|\mu^n_{k,\varepsilon}\|_{L^6(\Omega)}\|\nabla\varphi^n_{k,\varepsilon}\|_{\mathbf{L}^3(\Omega)}\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}\|\\
&\le \frac{\nu_*}{6}\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}\|^2+c_9'\|\nabla\varphi^n_{k,\varepsilon}\|^2_{\mathbf{L}^3(\Omega)}(1+\|\nabla\mu^n_{k,\varepsilon}\|^2).
\end{aligned}$$

Combining these estimates, we obtain

$$\begin{aligned}
&1\,\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\boldsymbol{u}^n_{k,\varepsilon}\|^2+\nu_*\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}\|^2\\
&\quad\le c_{10}'\Big((1+\|\varphi^n_{k,\varepsilon}\|^8_{H^2(\Omega)})\|\nabla\boldsymbol{u}^n_{k,\varepsilon}\|^2+\|\nabla\boldsymbol{u}^n_{k,\varepsilon}\|^6+\|\nabla\varphi^n_{k,\varepsilon}\|^2_{\mathbf{L}^3(\Omega)}(1+\|\nabla\mu^n_{k,\varepsilon}\|^2)\Big).
\end{aligned}\tag{5.4}$$

Multiplying (5.3) by ϖ $\frac{\nu}{4c_6'}$ $>$

$$\begin{aligned}
&1\,\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\boldsymbol{u}^n_{k,\varepsilon}\|^2+\nu_*\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}\|^2+\varpi\|\partial_t\boldsymbol{u}^n_{k,\varepsilon}\|^2\\
&\quad\le c_{11}'\Big((1+\|\varphi^n_{k,\varepsilon}\|^8_{H^2(\Omega)})\|\nabla\boldsymbol{u}^n_{k,\varepsilon}\|^2+\|\nabla\boldsymbol{u}^n_{k,\varepsilon}\|^6+\|\nabla\varphi^n_{k,\varepsilon}\|^2_{\mathbf{L}^3(\Omega)}(1+\|\nabla\mu^n_{k,\varepsilon}\|^2)\Big).
\end{aligned}\tag{5.5}$$

Adding (5.2) to (5.5), we find the differential inequality

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Lambda(\boldsymbol{u}^n_{k,\varepsilon},\varphi^n_{k,\varepsilon})&+\frac{\nu_*}{8}\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}\|^2+\frac{\varpi}{2}\|\partial_t\boldsymbol{u}^n_{k,\varepsilon}\|^2+\frac14\|\nabla\partial_t\varphi^n_{k,\varepsilon}\|^2\\
&\le(\partial_t\boldsymbol{u}^n_{k,\varepsilon}\cdot\nabla\varphi^n_{k,\varepsilon},\mu^n_{k,\varepsilon})+c_{12}'\Big((1+\|\varphi^n_{k,\varepsilon}\|^8_{H^2(\Omega)})\|\nabla\boldsymbol{u}^n_{k,\varepsilon}\|^2+\|\nabla\boldsymbol{u}^n_{k,\varepsilon}\|^6\\
&\quad+(1+\|\nabla\varphi^n_{k,\varepsilon}\|^2_{\mathbf{L}^3(\Omega)}+\|\boldsymbol{u}^n_{k,\varepsilon}\|^2_{\mathbf{L}^3(\Omega)})(1+\|\nabla\boldsymbol{u}^n_{k,\varepsilon}\|^2+\|\nabla\mu^n_{k,\varepsilon}\|^2)\Big),
\end{aligned}\tag{5.6}$$

where $\Lambda(\boldsymbol{u}^n_{k,\varepsilon},\varphi^n_{k,\varepsilon}$
that

$$\begin{aligned}
(\boldsymbol{u}^n_{k,\varepsilon}\cdot\nabla\varphi^n_{k,\varepsilon},\mu^n_{k,\varepsilon}) &\le \|\boldsymbol{u}^n_{k,\varepsilon}\|_{\mathbf{L}^3(\Omega)}\|\nabla\varphi^n_{k,\varepsilon}\|\|\mu^n_{k,\varepsilon}\|_{L^6(\Omega)}\\
&\le \frac14\|\nabla\boldsymbol{u}^n_{k,\varepsilon}\|^2+\frac14\|\nabla\mu^n_{k,\varepsilon}\|^2+c_{13}'.
\end{aligned}$$

$$\begin{aligned}
\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\boldsymbol{u}^n_{k,\varepsilon}\|^2+\nu_*\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}\|^2 &\le -b(\boldsymbol{u}^n_{k,\varepsilon},\boldsymbol{u}^n_{k,\varepsilon},\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon})-(\nu'(\varphi^n_{k,\varepsilon})\nabla\varphi^n_{k,\varepsilon}p^n_{k,\varepsilon},\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon})\\
&\quad+(\nu'(\varphi^n_{k,\varepsilon})D\boldsymbol{u}^n_{k,\varepsilon}\nabla\varphi^n_{k,\varepsilon},\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon})+(\mu^n_{k,\varepsilon}\nabla\varphi^n_{k,\varepsilon},\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}).
\end{aligned}$$

(4.28). We have

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= \ast 0 and summing up to (5.4), we obtain

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) is the same as in (4.32). Owing to (2.2) and (5.1), we observe

Thus, we deduce that

(5.7) $$\Lambda (u^n_{k,\varepsilon}, \varphi^n_{k,\varepsilon}) \geq \frac{1}{4}\|\nabla u^n_{k,\varepsilon}\|^2 + \frac{1}{4}\|\nabla \mu^n_{k,\varepsilon}\|^2 - c'_{13}.$$

On the other hand, we have

$$\Lambda (u^n_{k,\varepsilon}, \varphi^n_{k,\varepsilon}) \leq C\|\nabla u^n_{k,\varepsilon}\|^2 + C\|\nabla \mu^n_{k,\varepsilon}\|^2 + c'_{14}.$$

Exploiting (4.23), we are led to

$$\frac{d}{dt}\Lambda (u^n_{k,\varepsilon}, \varphi^n_{k,\varepsilon}) + \overline{\nu} \Bigl(\|\mathbf{A}u^n_{k,\varepsilon}\|^2 + \|\partial_t u^n_{k,\varepsilon}\|^2 + \|\nabla \partial_t \varphi^n_{k,\varepsilon}\|^2\Bigr)$$

(5.8) $$\leq c'_{15}\Big(1 + \Lambda^3(u^n_{k,\varepsilon}, \varphi^n_{k,\varepsilon})\Big),$$

where $\overline{\nu} = \frac{1}{4}\min\{1, \nu_*, \varpi\}$. In addition, following line by line the estimates performed in the proof of Theorem 4.1 for a uniform bound of the initial condition, we easily get

(5.9) $$\Lambda (u^n_\lambda(0), \varphi^n_\lambda(0)) \leq C(1 + \|u_0\|_{\mathbf{V}_\sigma} + \|\mu_0\|_V),$$

where C is independent of k, ε, and n, provided that n is sufficiently large. Therefore, we infer from (5.8) and (5.9) that there exist a positive time $T^\ast$, depending on $\|u_0\|_{\mathbf{V}_\sigma}$ and $\|\mu_0\|_V$, and a positive constant $\overline{C}$ such that

$$\sup_{0\leq t\leq T^\ast}\Lambda(\boldsymbol{u}^n_{k,\varepsilon}(t),\varphi^n_{k,\varepsilon}(t))+\int_0^{T^\ast}\left(\|\mathbf{A}\boldsymbol{u}^n_{k,\varepsilon}(s)\|^2+\|\partial_t\boldsymbol{u}^n_{k,\varepsilon}(s)\|^2+\|\nabla\partial_t\varphi^n_{k,\varepsilon}(s)\|^2\right)\mathrm{d}s\leq \overline{C},$$

where $\overline{C}$ is independent of k, ε, and n. A final passage to the limit allows us to recover the existence of a strong solution to the original problem (1.1)--(1.2). Moreover, the additional claimed regularities for φ and μ can be easily deduced as in the proof of Theorem 4.1.

We are left to prove the uniqueness of strong solutions. Given two strong solutions $(\boldsymbol{u}_1,\varphi_1)$ and $(\boldsymbol{u}_2,\varphi_2)$, defined on the time interval $(0,T_0)$ with the same initial datum $(\boldsymbol{u}_0,\varphi_0)$, we define their difference $\boldsymbol{u}=\boldsymbol{u}_1-\boldsymbol{u}_2$ and $\varphi=\varphi_1-\varphi_2$. We observe that the regularity of strong solutions allows us to follow the argument in the proof of Theorem 3.1. Then, we have the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}+\frac{\nu_\ast}{2}\|\boldsymbol{u}\|^2+\|\nabla\varphi\|^2\leq \alpha_2\mathcal{H}+\sum_{k=1}^{7}\mathcal{I}_k, \tag{5.10}$$

where the terms $\mathcal{H}$ and $\mathcal{I}_k$ are defined as above. In light of the regularity $\boldsymbol{u}_i\in L^\infty(0,T_0;\mathbf{V}_\sigma)$ and $\varphi_i\in L^\infty(0,T_0;W^{2,6}(\Omega))$, $i=1,2$, we can easily infer that

$$\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_5+\mathcal{I}_6+\mathcal{I}_7\leq\frac{1}{6}\|\nabla\varphi\|^2+\frac{\nu_\ast}{8}\|\boldsymbol{u}\|^2+C_1\left(\|\varphi\|_\ast^2+\|\boldsymbol{u}\|_\sharp^2\right)$$

for some positive constant C_1. On the other hand, by using (2.2) and the boundedness of ν', we simply obtain

$$\begin{aligned}\mathcal{I}_3&\leq C\|\varphi\|_{L^6(\Omega)}\|D\boldsymbol{u}_2\|_{\mathbf{L}^3(\Omega)}\|\nabla\mathbf{A}^{-1}\boldsymbol{u}\|\\&\leq\frac{1}{12}\|\nabla\varphi\|^2+C_2\|D\boldsymbol{u}_2\|^2_{\mathbf{L}^3}\|\boldsymbol{u}\|_\sharp^2,\end{aligned}$$

and

$$\begin{aligned}\mathcal{I}_4 &\leq \Big(\|\boldsymbol{u}_1\|_{\mathbf{L}^6(\Omega)} + \|\boldsymbol{u}_2\|_{\mathbf{L}^6(\Omega)}\Big)\|\boldsymbol{u}\|\|\nabla \mathbf{A}^{-1}\boldsymbol{u}\|_{\mathbf{L}^3(\Omega)} \\ &\leq C\Big(\|\boldsymbol{u}_1\|_{\mathbf{L}^6(\Omega)} + \|\boldsymbol{u}_2\|_{\mathbf{L}^6(\Omega)}\Big)\|\boldsymbol{u}\|^{\frac{3}{2}}\|\nabla \mathbf{A}^{-1}\boldsymbol{u}\|^{\frac{1}{2}} \\ &\leq \frac{\nu_*}{8}\|\boldsymbol{u}\|^2 + C_3\|\boldsymbol{u}\|_\sharp^2\end{aligned}$$

for some positive constants C_2 and C_3. Collecting the above estimates together, we end up with

$$\frac{d}{dt}\mathcal{H} \leq C\left(1 + \|D\boldsymbol{u}\|^2_{\mathbf{L}^3(\Omega)}\right)\mathcal{H}.$$

Since $D\boldsymbol{u} \in L^2(0,T_0;\mathbf{L}^3(\Omega))$, the uniqueness of strong solutions immediately follows from the Gronwall lemma. ∎

Appendix A. On Neumann problems. For any $\lambda \geq 0$, let us consider the Neumann problem

$$\begin{cases} -\Delta u + \lambda u = f, & \text{in } \Omega, \\ \partial_{\mathbf{n}} u = 0, & \text{on } \partial\Omega. \end{cases} \tag{A.1}$$

We introduce the operator $B_\lambda \in \mathcal{L}(V, V')$ defined by

$$\langle B_\lambda u, v\rangle = \int_\Omega (\nabla u \cdot \nabla v + \lambda uv)\,\mathrm{d}x \quad \forall\, u, v \in V.$$

We consider the spaces

$$V_0 = \{v \in V : \overline{v} = 0\}, \quad V_0' = \{f \in V' : \overline{f} = 0\},$$

and we recall that $V = V_0 \oplus \mathbb{R}$ and $V' = V_0' \oplus \mathbb{R}$. The restriction A_0 of B_0 to V_0 being an isomorphism from V_0 onto V_0', we denote by $A_0^{-1} : V_0' \to V_0$ its inverse map. It is well known that for all $f \in V_0'$, $A_0^{-1} f$ is the unique $u \in V_0$ such that $\langle A_0 u, v\rangle = \langle f, v\rangle$ for all $v \in V$. On account of the above definitions, we observe that

$$\langle f, A_0^{-1} g\rangle = \int_\Omega \nabla(A_0^{-1} f)\cdot \nabla(A_0^{-1} g)\,\mathrm{d}x \quad \forall\, f, g \in V_0'. \tag{A.2}$$

Owing to (A.2), it is straightforward to prove that $\|f\|_* := \|\nabla A_0^{-1} f\| = \langle f, A_0^{-1} f\rangle^{\frac{1}{2}}$ is a norm on V_0' equivalent to the natural one. In addition, for any $u \in H^1(0,T;V_0')$, we have the chain rule

$$\langle u_t(t), A_0^{-1} u(t)\rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_*^2 \qquad \text{a.e. } t \in (0,T). \tag{A.3}$$

Furthermore, due to regularity theory of the Neumann problem, we know that

$$\|\nabla A_0^{-1} f\|_V \leq C\|f\| \quad \forall f \in H \cap V_0'. \tag{A.4}$$

For any $\lambda > 0$, we also consider the operator $A_\lambda = -\Delta + \lambda I$ as unbounded operator on H with domain $D(A_\lambda) = \{ u \in H^2(\Omega) : \partial_{\mathbf{n}} u = 0 \text{ on } \partial\Omega \}$. It is well-known that A_λ is positive, unbounded, self-adjoint operator in H with compact inverse (see, e.g., [80, Chapter II, section 2.2]).

Next, we introduce the homogeneous Neumann elliptic problem with a logarithmic convex nonlinear term, that is, with the same F as in (2.8)–(2.9),

$$\Biggl\{ \begin{array}{ll} -\Delta u + F'(u) = f, & \text{in } \Omega, \\ \partial_{\mathbf{n}} u = 0, & \text{on } \partial\Omega. \end{array} \tag{A.5}$$

Under the assumptions in section 2, we have the following well-posedness and approximation result.

Lemma A.1. *Let Ω be a bounded domain in $\mathbb{R}^d$, $d = 2,3$, with smooth boundary. Assume that $f \in H$. Then, there exists a unique solution u to problem* (A.5) *such that $u \in H^2(\Omega)$, $F'(u) \in H$ and satisfies $-\Delta u + F'(u) = f$ for almost every $x \in \Omega$ and $\partial_n u = 0$ for almost every $x \in \partial\Omega$. Moreover, we have*

$$\|u\|_{H^2(\Omega)} + \|F'(u)\| \leq C\bigl(1 + \|f\|\bigr). \tag{A.6}$$

Let us assume that the sequence $\{f_k\} \subset H$, and $f \in H$. We consider the solutions u_k and u to problem (A.5) *corresponding to f_k and f, respectively. Then, $f_k \rightarrow f$ in H, as $k \rightarrow \infty$, implies*

(A.7) $$\|u_k - u\|_V \rightarrow 0, \text{ as } k \rightarrow \infty.$$

Proof. The existence of a solution u to problem (A.5) can be proved relying on the theory of maximal monotone operator. We define the functional on H

$$\mathcal{F}(u) = \int_\Omega \frac{1}{2}\|\nabla u\|^2 + F(u)\,\mathrm{d}x,$$

with domain $D(\mathcal{F}) = \{u \in H^1(\Omega) : \|u\|_{L^\infty(\Omega)} \leq 1\}$. We observe that $\mathcal{F}$ is a proper, lower semicontinuous and convex functional. Now, we consider the subdifferential $\partial\mathcal{F}$ of $\mathcal{F}$, defined as $w \in \partial\mathcal{F}(u)$ if and only if, for all $v \in H$, $\mathcal{F}(v) \geq \mathcal{F}(u) + (w, v - u)$. Then, $\partial\mathcal{F}$ is a maximal monotone operator on H (see [20]). Moreover, it is well known that $D(\partial\mathcal{F}) = \{u \in H^2(\Omega) : F'(u) \in H, \partial_{\mathbf{n}} u = 0 \text{ on } \partial\Omega\}$ and $\partial\mathcal{F}(u) = -\Delta u + F'(u)$ (see [14, 9]). By (2.9), we deduce that $\partial\mathcal{F}$ is also coercive, namely, $(\partial\mathcal{F}(u) - \partial\mathcal{F}(v), u - v) \geq \theta\|u - v\|^2$ for all $u, v \in D(\partial\mathcal{F})$, where θ is the same as in (2.9). In turn, this implies that $\partial\mathcal{F}$ is surjective on H. In addition, the estimate (A.6) can be proved as in [9, 29]. Finally, exploiting (2.9) once more, we can easily infer the uniqueness of solutions and the approximation result (A.7) to problem (A.5). ∎

We now report some elliptic estimates, whose proofs can be found in [2, 29, 49].

Theorem A.2. *Let Ω be a bounded domain in $\mathbb{R}^d$ with smooth boundary. Assume that u is the solution to problem* (A.5)*. We have the following:*

- *Let $d = 2, 3$ and $f \in L^p(\Omega)$, where $2 \leq p \leq \infty$. Then, we have*

$$\|F'(u)\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)}.$$

- *Let $d = 2, 3$ and $f \in V$. Then, we have*

$$\|\Delta u\| \leq \|\nabla u\|^{\frac{1}{2}} \|\nabla f\|^{\frac{1}{2}}.$$

In addition, there exists a positive constant $C = C(p)$ such that

$$\|u\|_{W^{2,p}(\Omega)} + \|F'(u)\|_{L^p(\Omega)} \leq C\bigl(1 + \|f\|_V\bigr),$$

where $p = 6$ *if* $d = 3$ *and for any* $p \geq 2$ *if* $d = 2$*.*

- *Let* $d = 2$ *and* $f \in V$*. Assume that* F *satisfies*

$$F''(s) \leq e^{C|F'(s)|+C} \quad \forall s \in (-1,1)$$

for some positive constant C*. Then, for any* $p \geq 1$*, there exists a positive constant* $C = C(p)$ *such that*

$$\|F''(u)\|_{L^p(\Omega)} \leq C\bigl(1 + e^{C\|f\|^2_V}\bigr).$$

Appendix B. On Stokes operators. We consider the homogeneous Stokes problem

$$\text{(B.1)} \qquad \left\{ \begin{array}{ll} -\Delta u + \nabla p = f, & \text{in } \Omega, \\ \operatorname{div} u = 0, \quad u = 0, & \text{in } \Omega, \text{ on } \partial\Omega. \end{array} \right.$$

First, we introduce the Stokes operator as the map $\mathbf{A} : \mathbf{V}_\sigma \to \mathbf{V}'_\sigma$ such that

$$\langle \mathbf{A}\boldsymbol{u}, \boldsymbol{v}\rangle = (\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in \mathbf{V}_\sigma,$$

namely, $\mathbf{A}$ is the canonical isomorphism from $\mathbf{V}_\sigma$ onto $\mathbf{V}'_\sigma$. We denote by $\mathbf{A}^{-1} : \mathbf{V}'_\sigma \to \mathbf{V}_\sigma$ the inverse map of the Stokes operator. That is, given $f \in \mathbf{V}'_\sigma$, there exists a unique $u = \mathbf{A}^{-1} f \in \mathbf{V}_\sigma$ such that

$$(\nabla \mathbf{A}^{-1} f, \nabla v) = \langle f, v\rangle \quad \forall v \in \mathbf{V}_\sigma.$$

It follows that $\|\boldsymbol{f}\|_\sharp := \|\nabla \mathbf{A}^{-1}\boldsymbol{f}\| = \langle \boldsymbol{f}, \mathbf{A}^{-1}\boldsymbol{f}\rangle^{\frac{1}{2}}$ is an equivalent norm on $\mathbf{V}'_\sigma$ and the chain rule

$$\langle \boldsymbol{f}_t(t), \mathbf{A}^{-1}\boldsymbol{f}(t)\rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{f}(t)\|^2_\sharp, \quad \text{a.e. } t \in (0,T),$$

holds for any $f \in H^1(0,T;\mathbf{V}'_\sigma)$. In order to recover the pressure p, the well-known De Rham result implies that if $f \in \mathbf{H}^{-1}(\Omega)$, there exists $p \in H$ (such that $\overline{p} = 0$) such that $\nabla p = \Delta u + f$ in the distributional sense. In addition, by [81, Proposition 1.2] we know that

$$\|p\| \leq C\|f\|_{\mathbf{H}^{-1}(\Omega)}. \tag{B.2}$$

Let us now report the regularity theory of the Stokes problem (B.1) (see [24]). Assuming that $f \in \mathbf{H}$, then there exist a unique $u \in \mathbf{H}^2(\Omega) \cap \mathbf{V}_\sigma$ and $p \in V$ (unique up to a constant) such that $-\Delta u + \nabla p = f$ a.e. in Ω. Moreover, there exists a constant

C such that

$$\|u\|_{\mathbf{H}^2(\Omega)} + \|p\|_V \leq C\|f\|. \tag{B.3}$$

We denote by $P : \mathbf{H} \rightarrow \mathbf{H}_\sigma$ the Helmholtz--Leray orthogonal projection from $\mathbf{H}$ onto $\mathbf{H}_\sigma$. We recall that P is a bounded operator from $\mathbf{V}$ into $\mathbf{V} \cap \mathbf{H}_\sigma$, namely, there exists a positive constant C such that

$$\|Pv\|_{\mathbf{V}} \leq C\|v\|_{\mathbf{V}} \forall v \in \mathbf{V}.$$

We also report that $P\nabla v = \mathbf{0}$ for any $v \in V$. Next, we consider the Stokes operator as an unbounded operator on $\mathbf{H}_\sigma$ with domain $D(\mathbf{A}) = \{u \in \mathbf{V}_\sigma : \mathbf{A}u \in \mathbf{H}_\sigma\}$. It is well known that $\mathbf{A}$ is a positive, unbounded, self-adjoint operator in $\mathbf{H}_\sigma$ with compact

inverse (see, e.g., [81]). In particular, we have

$$\mathbf{A}u = P(-\Delta u) \forall u \in D(\mathbf{A}), \text{where} D(\mathbf{A}) = \mathbf{H}^2(\Omega) \cap \mathbf{V}_\sigma.$$

Thanks to the above regularity results, we deduce that the operator $\mathbf{A}^{-1} : \mathbf{H}_\sigma \rightarrow \mathbf{H}^2(\Omega) \cap \mathbf{V}_\sigma$ is such that, for any $f \in \mathbf{H}_\sigma$, there exist $\mathbf{A}^{-1}f \in D(\mathbf{A})$ and $p \in V$ that

solve

$$-\Delta \mathbf{A}^{-1}f + \nabla p = f. \tag{B.4}$$

In turn, this entails that $\mathbf{A}\mathbf{A}^{-1}f = f$. Owing to (B.3), we have

$$\|\mathbf{A}^{-1}f\|_{\mathbf{H}^2(\Omega)} + \|p\|_V \leq C\|f\|. \tag{B.5}$$

We are now in position to find an L^2-estimate of the pressure p in (B.4) in terms of $\|\nabla \mathbf{A}^{-1}f\|$. Let us first report a preliminary interpolation result (see [73]).

Lemma B.1. *Let Ω be a Lipschitz domain in $\mathbb{R}^d$, $d = 2,3$, with compact boundary.*

Then, there exists a positive constant C such that

$$\|f\|_{L^2(\partial\Omega)} \le C\|f\|^{\frac12}\|f\|_V^{\frac12} \quad \forall f \in V. \tag{B.6}$$

We have the following result.

Lemma B.2. *Let* $d = 2, 3$ *and* $f \in \mathbf{H}_\sigma$. *Then, there exists a positive constant C (independent of f) such that*

$$\|p\| \le C\|\nabla \mathbf{A}^{-1} \boldsymbol{f}\|^{\frac12}\|\boldsymbol{f}\|^{\frac12}. \tag{B.7}$$

Proof. Thanks to (B.2), we need to control $\|\boldsymbol{f}\|_{\mathbf{H}^{-1}(\Omega)}$ by means of $\|f\|_\sharp$. To this end, let us consider $v \in \mathbf{H}_0^1(\Omega)$ with $\|\boldsymbol{v}\|_{\mathbf{H}_0^1(\Omega)} \le 1$. By exploiting the integration by parts, we find

$$\begin{aligned}(f, v) &= (P(-\Delta)\mathbf{A}^{-1} f, v)\\ &= (-\Delta \mathbf{A}^{-1} f, Pv)\\ &= (\nabla \mathbf{A}^{-1} f, \nabla Pv) - \int_{\partial\Omega} \nabla \mathbf{A}^{-1} f \mathbf{n} \cdot Pv \, d\sigma.\end{aligned}$$

We recall that the classical trace theorem implies $\|P\boldsymbol{v}\|_{L^2(\partial\Omega)} \le C\|P\boldsymbol{v}\|_V$. In addition, by the properties of the Helmholtz--Leray operator and the Poincaré inequality, we have $\|P\boldsymbol{v}\|_{\mathbf{V}} \le C\|\boldsymbol{v}\|_{\mathbf{H}_0^1(\Omega)}$. Then, we deduce that

$$\|\boldsymbol{f}\|_{\mathbf{H}^{-1}(\Omega)} \le C\|\nabla \mathbf{A}^{-1}\boldsymbol{f}\| + C\|\nabla \mathbf{A}^{-1}\boldsymbol{f}\|_{\mathbf{L}^2(\partial\Omega)}.$$

An application of Lemma B.1, together with (B.5), implies that

$$\|f\|_{\mathbf{H}^{-1}(\Omega)} \le C\|\nabla \mathbf{A}^{-1} f\| + C\|\nabla \mathbf{A}^{-1} f\|^{\frac12}\|f\|^{\frac12}.$$

Thus, the desired inequality (B.7) immediately follows. ∎

Finally, we consider the homogeneous Stokes problem with nonconstant viscosity depending on a given measurable function φ. The system reads as follows

$$\left\{\begin{array}{ll} -\operatorname{div}(\nu(\varphi)Du) + \nabla \pi = f, & \text{in } \Omega,\\ \operatorname{div} u = 0, & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega,\end{array}\right. \tag{B.8}$$

where the coefficient ν fulfils the assumptions stated in section 2. We report a regularity result whose proof has been provided in [2, section 4, Lemma 4].

Theorem B.3. *Let* $d = 2$, $\varphi \in W^{1,r}(\Omega)$, *with* $2 < r \leq \infty$, *and* $f \in \mathbf{L}^p(\Omega)$, *with* $1 \leq p < \infty$. *Assume that* $u \in \mathbf{V}_\sigma$ *is a weak solution to* (B.8), *i.e.,*

$$(\nu(\varphi)Du, Dv) = (f, v) \quad \forall v \in \mathbf{V}_\sigma.$$

Then, there exists $C > 0$, *depending on r and p, such that*

$$\|u\|_{\mathbf{W}^{2,p'}(\Omega)} \leq C\bigl(1 + \|\nabla \varphi\|_{\mathbf{L}^r(\Omega)}\bigr)\bigl(\|f\|_{\mathbf{L}^p(\Omega)} + \|\nabla u\|\bigr), \tag{B.9}$$

where $\frac{1}{p'} = \frac{1}{p} + \frac{1}{r}$, *provided that* $p' > 1$.

Appendix C. A product estimate in two dimensions. We report here a logarithmic estimate of the product of two functions in two dimensions. The following proof is based on an idea developed in [30] and [82] to control the convective term of the Navier–Stokes equations.

Proposition C.1. *Let* Ω *be a bounded domain in* $\mathbb{R}^2$ *with smooth boundary. Assume that* $f \in V$ *and* $g \in V$. *Then, there exists a positive constant* C *such that*

$$\|fg\| \leq C\|f\|_V\|g\|\left[\log\left(e\frac{\|g\|_V}{\|g\|}\right)\right]^{\frac{1}{2}}. \tag{C.1}$$

Proof. Let us consider the operator $A_1 = -\Delta + I$ on H with domain $D(A_1) = \{u \in H^2(\Omega) : \partial_{\mathbf{n}} u = 0 \text{ on } \partial\Omega\}$ defined in Appendix A. By the spectral theory, there exists a sequence of positive eigenvalues λ_k ($k \in \mathbb{N}$) associated with A_1 such that $\lambda_1 = 1$, $\lambda_k \leq \lambda_{k+1}$ and $\lambda_k \rightarrow \infty$ as k goes to ∞. The sequence of eigenfunctions $w_k \in D(A_1)$ such that $A_1 w_k = \lambda_k w_k$ forms an orthonormal basis in $L^2(\Omega)$ and an orthogonal basis in $H^1(\Omega)$. In particular, we have the representation

$$f = \sum_{k=1}^{\infty} (f, w_k) w_k.$$

Let us fix $N \in \mathbb{N}$ whose value will be chosen later. We write f as follows

$$f = \sum_{n=0}^{N} f_n + f_N^{\bot}, \tag{C.2}$$

where

$$f_n = \sum_{k: \mathrm{e}^n \le \sqrt{\lambda_k} < \mathrm{e}^{n+1}} (f, w_k) w_k, \quad f_N^{\bot} = \sum_{k: \sqrt{\lambda_k} \ge \mathrm{e}^{N+1}} (f, w_k) w_k .$$

By using the above decomposition, the Hölder inequality, and subsequently (2.1) and (2.3), we find

$$\begin{aligned} \|fg\| &\le \sum_{n=0}^{N} \|f_n g\| + \|f_N^{\bot} g\| \\ &\le \sum_{n=0}^{N} \|f_n\|_{L^\infty(\Omega)} \|g\| + \|f_N^{\bot}\|_{L^4(\Omega)} \|g\|_{L^4(\Omega)} \\ &\le C \sum_{n=0}^{N} \|f_n\|^{\frac{1}{2}} \|f_n\|_{H^2(\Omega)}^{\frac{1}{2}} \|g\| + C \|f_N^{\bot}\|^{\frac{1}{2}} \|f_N^{\bot}\|_V^{\frac{1}{2}} \|g\|^{\frac{1}{2}} \|g\|_V^{\frac{1}{2}} . \end{aligned} \tag{C.3}$$

We now observe that

$$\|f_n\|_{L^2(\Omega)}^2 = \sum_{k: \mathrm{e}^n \le \sqrt{\lambda_k} < \mathrm{e}^{n+1}} |(f, w_k)|^2$$

$$\leq \frac{1}{e^{2n}} \sum_{k: e^n \leq \sqrt{\lambda_k} < e^{n+1}} \lambda_k |(f, w_k)|^2 = \frac{1}{e^{2n}} \|f_n\|^2_{H^1(\Omega)}.$$

Here we have used the fact that (^{2}) = $_n$ is a finite sum of w_k

$D\ A_1^1 \quad V$. Observing that f

's, by the regularity theory of the Neumann problem, we have

$$\|f_n\|^2_{H^2(\Omega)} \leq C\|A_1 f_n\|^2_{L^2(\Omega)} = C \sum_{k: e^n \leq \sqrt{\lambda_k} < e^{n+1}} \lambda_k^2 |(f, w_k)|^2$$

$$\leq C \sum_{k: e^n \leq \sqrt{\lambda_k} < e^{n+1}} e^{2(n+1)} \lambda_k |(f, w_k)|^2$$

$$\leq C e^{2(n+1)} \|f_n\|^2_V.$$

Then, we deduce that $\|f_n\|^{\frac{1}{2}} \|f_n\|^{\frac{1}{2}}_{H^2(\Omega)} \leq C e^{\frac{1}{2}} \|f_n\|_V$.

On the other hand, reasoning as above, we have

$$\|f_N^\bot\|_{L^2(\Omega)} \leq \frac{1}{e^{2(N+1)}} \|f_N^\bot\|_V.$$

Combining the above inequalities in (C.3) and applying the Cauchy--Schwarz inequality, we get

$$\|fg\| \leq C \sum_{n=0}^{N} e^{\frac{1}{2}} \|f_n\|_V \|g\| + C \frac{1}{e^{\frac{N+1}{2}}} \|f_N^\bot\|_V \|g\|^{\frac{1}{2}} \|g\|_V^{\frac{1}{2}}$$

$$\leq C\|g\| \left(\sum_{n=0}^{N} e^{\frac{1}{2}} \|f_n\|_V + \frac{1}{e^{\frac{N+1}{2}}} \frac{\|g\|_V^{\frac{1}{2}}}{\|g\|^{\frac{1}{2}}} \|f_N^\bot\|_V \right)$$

$$\leq C\|g\| \left(e(N+1) + \frac{1}{e^{N+1}} \frac{\|g\|_V}{\|g\|} \right)^{\frac{1}{2}} \left(\sum_{n=0}^{N} \|f_n\|_V^2 + \|f_N^\bot\|_V^2 \right)^{\frac{1}{2}}$$

$$\leq C\|g\| \left(e(N+1) + \frac{1}{e^{N+1}} \frac{\|g\|_V}{\|g\|} \right)^{\frac{1}{2}} \|f\|_V. \tag{C.4}$$

Now, we choose the integer N so that

$$\ln\left(e \frac{\|g\|_V}{\|g\|} \right) \leq N + 1 < 1 + \ln\left(e \frac{\|g\|_V}{\|g\|} \right).$$

By using the above choice of N in (C.4), we eventually infer that

$$\|fg\| \le C\|g\|\|f\|_V\left[\mathrm{e}\ln\left(\mathrm{e}^2\frac{\|g\|_V}{\|g\|}\right)+\frac{1}{\mathrm{e}}\right]^{\frac{1}{2}},$$

which implies the desired conclusion. □

For the purpose of this work we state an immediate generalization of (C.1), whose proof which can be inferred from that of Proposition C.2 is left to the interested reader.

Proposition C.2. *Let* Ω *be a bounded domain in* $\mathbb{R}^2$ *with smooth boundary. Assume that* $f \in V$, $\boldsymbol{g} \in \mathbf{V}$, *and* $h \in V$. *Then, there exists a positive constant C such that*

$$\|f\boldsymbol{g}\| \le C\|f\|_V\Big(\|\boldsymbol{g}\|+\|h\|\Big)\left[\log\left(\mathrm{e}\frac{\|\boldsymbol{g}\|_{\mathbf{V}}+\|h\|_V}{\|\boldsymbol{g}\|+\|h\|}\right)\right]^{\frac{1}{2}}. \quad \text{(C.5)}$$

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REFERENCES

[1] H. Abels, *Existence of weak solutions for a diffuse interface model for viscous, incompressible fluids with general densities*, Comm. Math. Phys., 289 (2009), pp. 45--73.

[2] H. Abels, *On a diffuse interface model for two-phase flows of viscous, incompressible fluids with matched densities*, Arch. Ration. Mech. Anal., 194 (2009), pp. 463--506.

[3] H. Abels, *Strong well-posedness of a diffuse interface model for a viscous, quasi-incompressible two-phase flow*, SIAM J. Math. Anal., 44 (2012), pp. 316--340.

[4] H. Abels, D. Depner, and H. Garcke, *Existence of weak solutions for a diffuse interface model for two-phase flows of incompressible fluids with different densities*, J. Math. Fluid Mech., 15 (2013), pp. 453--480.

[5] H. Abels, D. Depner, and H. Garcke, *On an incompressible Navier-Stokes/Cahn-Hilliard system with degenerate mobility*, Ann. Inst. H. Poincar\'e Anal. Non Lin\'eaire, 30 (2013), pp. 1175--1190.

[6] H. Abels, L. Diening, and Y. Terasawa, *Existence of weak solutions for a diffuse interface model of non-Newtonian two-phase flows*, Nonlinear Anal. Real World Appl., 15 (2014), pp. 149--157.

[7] H. Abels and E. Feireisl, *On a diffuse interface model for a two-phase flow of compressible viscous fluids*, Indiana Univ. Math. J., 57 (2008), pp. 659--698.

[8] H. Abels, H. Garcke, and G. Grun\" , *Thermodynamically consistent, frame indifferent diffuse interface models for incompressible two-phase flows with different densities*, Math. Models Methods Appl. Sci., 22 (2012), 1150013.

[9] H. Abels and M. Wilke, *Convergence to equilibrium for the Cahn-Hilliard equation with a logarithmic free energy*, Nonlinear Anal., 67 (2007), pp. 3176--3193.

[10] D. Anderson, G. McFadden, and A. Wheeler, *Diffuse-interface methods in fluid mechanics* , Annu. Rev. Fluid Mech., 30 (1998), pp. 139--165.
[11] L. Antanovskii, *A phase field model of capillarity*, Phys. Fluids, 7 (1995), pp. 747--753.
[12] V. Badalassi, H. Ceniceros, and S. Banerjee, *Computation of multiphase systems with phase field models*, J. Comput. Phys., 190 (2003), pp. 371--397.
[13] K. Bao, Y. Shi, S. Sun, and X. Wang, *A finite element method for the numerical solution of the coupled Cahn-Hilliard and Navier-Stokes system for moving contact line problems* , J. Comput. Phys., 231 (2012), pp. 8083--8099.

[14] V. Barbu, *Nonlinear Differential Equations of Monotone Types in Banach Spaces*, Springer,

New York, 2010.

[15] A. Berti, V. Berti, and D. Grandi, *Well-posedness of an isothermal diffusive model for binary mixtures of incompressible fluids*, Nonlinearity, 24 (2011), pp. 3143--3164.
[16] S. Bosia, M. Grasselli, and A. Miranville, *On the longtime behavior of a 2D hydrodynamic model for chemically reacting binary fluid mixtures*, Math. Methods Appl. Sci., 37 (2014), pp. 726--743.
[17] F. Boyer, *Mathematical study of multi-phase flow under shear through order parameter for- mulation*, Asympt. Anal., 20 (1999), pp. 175--212.
[18] F. Boyer, *Nonhomogeneous Cahn-Hilliard fluids*, Ann. Inst. H. Poincar\'e Anal. Non Lin\'eaire, 18 (2001), pp. 225--259.

[19] F. Boyer and S. Minjeaud, *Hierarchy of consistent n-component Cahn-Hilliard systems*,

Math. Models Methods Appl. Sci., 24 (2014), pp. 2885--2928.
[20] H. Brezis, *Op\'erateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Es- paces de Hilbert*, North-Holland, Amsterdam, 1973.
[21] H. Brezis and T. Gallouet, *Nonlinear Schr\"odinger evolution equations* , Nonlinear Anal., 4 (1980), pp. 677--681.
[22] J. Cahn and J. Hilliard, *Free energy of a nonuniform system.* I. *Interfacial free energy*, J. Chem. Phys., 28 (1958), pp. 258--267.
[23] C. Cao and C. Gal, *Global solutions for the 2D NS-CH model for a two-phase flow of viscous, incompressible fluids with mixed partial viscosity and mobility* , Nonlinearity, 25 (2012), pp. 3211--3234.
[24] L. Cattabriga, *Su un problema al contorno relativo al sistema di equazioni di Stokes* , Rendiconti del Seminario Matematico della Universit\`a di Padova, 31 (1961), pp. 308--340.
[25] R. Chella and J. V. nals, *Mixing of a two-phase fluid by cavity flow* , Phys. Rev. E, 53 (1996), pp. 3832--3840.

[26] W. Chen, C. Wang, X. Wang, and S. Wise, *Positivity-preserving, energy stable numerical schemes for the Cahn-Hilliard equation with logarithmic potential*, preprint,

arXiv:1712.03225v2, 2017, https://arxiv.org/abs/1712.03225v2.
[27] Y. Chen and J. Shen, *Efficient, adaptive energy stable schemes for the incompressible Cahn-*

Hilliard Navier-Stokes phase-field models, J. Comput. Phys., 308 (2016), pp. 40--56.
[28] C. Collins, J. Shen, and S. Wise, *An efficient, energy stable scheme for the Cahn-Hilliard- Brinkman system*, Commun. Comput. Phys., 13 (2013), pp. 929--957.
[29] M. Conti and A. Giorgini, *The three-dimensional Cahn-Hilliard-Brinkman system with un- matched viscosities*, (2018), https://hal.archives-ouvertes.fr/hal-01559179 .
[30] R. Dascaliuc, C. Foias, and M. Jolly, *Relations between energy and enstrophy on the global attractor of the 2-D Navier-Stokes equations*, J. Dynam. Differential Equations, 17 (2005), pp. 643--736.

[31] A. Diegel, X. Feng, and S. Wise, *Analysis of a mixed finite element method for a Cahn- Hilliard-Darcy-Stokes system*, SIAM J. Numer. Anal., 1 (2015), pp. 127--152.

[32] A. Diegel, C. Wang, X. Wang, and S. Wise, *Convergence analysis and error estimates for a second order accurate finite element method for the Cahn-Hilliard-Navier-Stokes system* , Numer. Math., 137 (2017), pp. 495--534.

[33] H. Ding, P. Spelt, and C. Shu, *Diffuse interface model for incompressible two-phase flows with large density ratios*, J. Comput. Phys., 226 (2007), pp. 2078--2095.

[34] M. Eleuteri, E. Rocca, and G. Schimperna, *On a non-isothermal diffuse interface model for two-phase flows of incompressible fluids*, Discrete Contin. Dyn. Syst., 35 (2015), pp. 2497--

2522.

[35] M. Eleuteri, E. Rocca, and G. Schimperna, *Existence of solutions to a two-dimensional model for nonisothermal two-phase flows of incompressible fluids* , Ann. Inst. H. Poincar\'e Anal. Non Lin\'eaire, 33 (2016), pp. 1431--1454.

[36] L. Espath, A. Sarmiento, P. Vignal, B. Varga, A. Cortes, L. Dalcin, and V. M. Calo , *Energy exchange analysis in droplet dynamics via the Navier-Stokes-Cahn-Hilliard model* , J. Fluid Mech., 797 (2016), pp. 389--430.

[37] X. Feng, *Fully discrete finite element approximation of the Navier-Stokes-Cahn-Hilliard dif- fuse interface model for two-phase flows*, SIAM J. Numer. Anal., 44 (2006), pp. 1049--1072.

[38] X. Feng, Y. He, and C. Liu, *Analysis of finite element approximations of a phase field model for two-phase fluids*, J. Math. Comput., 76 (2007), pp. 539--571.

[39] X. Feng and S. Wise, *Analysis of a Darcy-Cahn-Hilliard diffuse interface model for the Hele- Shaw flow and its fully discrete finite element approximation* , SIAM J. Numer. Anal., 50 (2012), pp. 1320--1343.

[40] S. Frigeri and M. Grasselli, *Nonlocal Cahn-Hilliard-Navier-Stokes systems with singular potential*, Dyn. Partial Differ. Equ., 24 (2012), pp. 827--856.

[41] C. Gal and M. Grasselli, *Asymptotic behavior of a Cahn-Hilliard-Navier-Stokes in* 2*d*, Ann. Inst. H. Poincar\'e Anal. Non Lin\'eaire, 27 (2010), pp. 401--436.

[42] C. Gal and M. Grasselli, *Longtime behavior for a model of homogeneous incompressible*

two-phase flows, Discrete Cont. Dyn. Systems, 28 (2010), pp. 1--39.

[43] C. Gal and M. Grasselli, *Trajectory and global attractors for binary mixtures fluid flows in* 3*d*, Chin. Ann. Math. Ser. B, 31 (2010), pp. 655--678.

[44] C. Gal and M. Grasselli, *Instability of two-phase flows: A lower bound on the dimension of the global attractor of the Cahn-Hilliard-Navier-Stokes system* , Phys. D, 240 (2011), pp. 629--635.

[45] C. Gal, M. Grasselli, and A. Miranville, *Cahn-Hilliard-Navier-Stokes systems with moving*

contact lines, Calculus of Variations Partial Differ. Eqns., 55 (2016), pp. 1--47.

[46] C. Gal and T. Tachim-Medjo, *On a regularized family of models for homogeneous incompressible two-phase flows*, J. Nonlinear Sci., 24 (2014), pp. 1033--1103.

[47] M. Gao and X.-P. Wang, *A gradient stable scheme for a phase field model for the moving contact line problem*, J. Comput. Phys., 231 (2012), pp. 1372--1386.

[48] J. Gibbon, A. Gupta, N. Pal, and R. Pandit, *The role of BKM-type theorems in* 3*D Euler, Navier-Stokes and Cahn-Hilliard-Navier-Stokes analysis* , Phys. D, 376/377 (2018), pp. 60--69.

[49] A. Giorgini, M. Grasselli, and A. Miranville, *The Cahn-Hilliard-Oono equation with sin- gular potential*, Math. Models Meth. Appl. Sci., 27 (2017), pp. 2485--2510.

[50] A. Giorgini, M. Grasselli, and H. Wu, *The Cahn-Hilliard-Hele-Shaw system with singular potential*, Ann. Inst. H. Poincar\'e Anal. Non Lin\'eaire, 35 (2018), pp. 1079--1118.

[51] Y. Gong, J. Zhao, and Q. Wang, *Linear second order in time energy stable schemes for hydrodynamic models of binary mixtures based on a spatially pseudospectral approximation* , Adv. Comput. Math., 44 (2018), pp. 1573--1600.

[52] Z. Guo, P. Lin, J. Lowengrub, and S. Wise, *Mass conservative and energy stable finite difference methods for the quasi-incompressible Navier-Stokes-Cahn-Hilliard system: Primitive variable and projection-type schemes*, Comput. Methods Appl. Mech. Engrg, 326 (2017), pp. 144--174.

[53] M. Gurtin, D. Polignone, and J. Vinals, *Two-phase binary fluids and immiscible fluids described by an order parameter*, Math. Models Methods Appl. Sci., 6 (1996), pp. 815--831.

[54] D. Han and X. Wang, *A second order in time, uniquely solvable, unconditionally stable nu- merical scheme for Cahn-Hilliard-Navier-Stokes equation* , J. Comput. Phys., 290 (2015), pp. 139--156.

[55] M. Hintermuller, M. Hinze, and C. Kahle\" , *An adaptive finite element Moreau-Yosida-based solver for a coupled Cahn-Hilliard/Navier-Stokes system* , J. Comput. Phys., 235 (2013), pp. 810--827.

[56] M. Hintermuller and D. Wegner\" , *Optimal control of a semidiscrete Cahn-Hilliard-Navier- Stokes system*, SIAM J. Control Optim., 52 (2014), pp. 747--772.

[57] P. Hohenberg and B. Halperin, *Theory of dynamic critical phenomena* , Rev. Mod. Phys., 49 (1977), pp. 435--479.

[58] K. K. J. Kim and J. Lowengrub, *Conservative multigrid methods for Cahn-Hilliard fluids* , 193 (2004), pp. 511--543.

[59] H. Y. J. Shen, X. Yang, *Efficient energy stable numerical schemes for a phase field moving contact line model*, J. Comput. Phys., 284 (2015), pp. 617--630.

[60] D. Jacqmin, *An energy approach to the continuum surface tension method* , in Proceedings of the 34th Aerosp. Sci. Meet. Exh., AIAA, 1996, pp. 96--0858.

[61] D. Jacqmin, *Calculation of two phase Navier-Stokes flows using phase-field modeling*, J. Comput. Phys., 155 (1999), pp. 96--127.

[62] D. Jasnow and J. Vinals, *Coarse-grained description of thermo-capillary flow* , Phys. Fluids, 8 (1996), pp. 660--669.

[63] D. Kay, V. Styles, and R. Welford, *Finite element approximation of a Cahn-Hilliard- Navier-Stokes system*, Interfaces Free Bound., 10 (2008), pp. 15--43.

[64] D. Kay and R. Welford, *Efficient numerical solution of Cahn-Hilliard-Navier-Stokes fluids in* 2*D*, SIAM J. Sci. Comput., 29 (2007), pp. 2241--2257.

[65] J. Kim, *Phase-field models for multi-component fluid flows* , Commun. Comput. Phys., 12 (2012), pp. 613--661.

[66] K. Lam and H. Wu, *Thermodynamically consistent Navier-Stokes-Cahn-Hilliard models with mass transfer and chemotaxis*, European J. Appl. Math., 29 (2018), pp. 595--644.

[67] J. Li and E. S. Titi, *A tropical atmosphere model with moisture: Global well-posedness and relaxation limit*, Nonlinearity, 29 (2016), pp. 2674--2714.

[68] C. Liu and J. Shen, *A phase field model for the mixture of two incompressible fluids and its approximation by a Fourier-spectral method*, Phys. D, 179 (2003), pp. 211--228.

[69] J. Lowengrub and L. Truskinovsky, *Quasi-incompressible Cahn-Hilliard fluids and topological transitions*, Proc. Roy. Soc. Lond. A, 454 (1998), pp. 2617--2654.

[70] A. Miranville, *The Cahn-Hilliard equation and some of its variants*, AIMS Math., 2 (2017), pp. 479--544.

[71] A. Miranville and R. Temam, *On the Cahn-Hilliard-Oono-Navier-Stokes equations wih sin- gular potentials*, Appl. Anal., 95 (2016), pp. 2609--2624.

[72] A. Miranville and S. Zelik, *Robust exponential attractors for Cahn-Hilliard type equations with singular potentials*, Math. Meth. Appl. Sci., 27 (2004), pp. 545--582.

[73] J. Necas \v , *Direct Methods in the Theory of Elliptic Equations*, Springer, Heidelberg, 2012.

[74] A. Salgado, *A diffuse interface fractional time-stepping technique for incompressible twophase flows with moving contact lines*, ESAIM Math. Model. Numer. Anal., 47 (2013), pp. 743--769.

[75] J. Shen and X. Yang, *Energy stable schemes for Cahn-Hilliard phase-field model of two-phase incompressible flows*, Chin. Ann. Math. Ser. B, 31 (2010), pp. 743--758.

[76] Y. Shi, K. Bao, and X.-P. Wang, 3*D adaptive finite element method for a phase field model for the moving contact line problems*, Inv. Probl. Imag., 7 (2013), pp. 947--959.

[77] G. Stampacchia, *Equations Elliptiques du Second Ordre a\` Coefficients Discontinus\`* , Presses de l'Universit\`e de Montr\'eal, 1966.

[78] V. N. Starovoitov, *Model of the motion of a two-component liquid with allowance of capillary forces*, J. Appl. Mech. Tech. Phys., 35 (1994), pp. 891--897.

[79] Z. Tan, K. Lim, and B. Khoo, *An adaptive mesh redistribution method for the incompressible mixture flows using phase-field model*, J. Comput. Phys., 225 (2007), pp. 1137--1158.

[80] R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics*, Springer-Verlag, New York, 1997.

[81] R. Temam, *Navier-Stokes equations*, AMS Chelsea Publishing, Providence, 2001.

[82] E. Titi, *On a criterion for locating stable stationary solutions to the Navier-Stokes equations*, Nonlinear Anal., 11 (1987), pp. 1085--1102.

[83] X. Yang, J. Feng, C. Liu, and J. Shen, *Numerical simulations of jet pinching-off and drop formation using an energetic variational phase-field method*, J. Comput. Phys., 218 (2006), pp. 417--428.

[84] P. Yue, J. Feng, C. Liu, and J. Shen, *A diffuse-interface method for simulating two-phase flows of complex fluids*, J. Fluid Mech., 515 (2004), pp. 293--317.

[85] L.-Y. Zhao, H. Wu, and H.-Y. Huang, *Convergence to equilibrium for a phase-field model for the mixture of two incompressible fluids*, Commun. Math. Sci., 7 (2009), pp. 939--962.