IMPERIAL COLLEGE LONDON

DOCTORAL THESIS

Applications of the amalgam method to the study of locally projective graphs

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A thesis submitted in fulfilment of the requirements
for the degree of Doctor of Philosophy

in the

Department of Mathematics
Imperial College London

April 23, 2022

Declaration of Originality

I, William GIULIANO, declare that this thesis and the research to which it refers are the product of my own work except where acknowledged in accordance with the standard referencing practices of the discipline.

Copyright Declaration

CARMELA - Ma comme, Marche'? Chillo ha faticato sempe...
EDUARDO - E ha fatto malissimo. Innanzi tutto, il lavoro fa male. Tanto vero, che quando un medico visita un ammalato, come prima cosa gli dice: "Riposo assoluto." Non gli ho mai sentito dire: "Lavoro assoluto."
VICENZINO - Mai, mai!
EDUARDO - E poi, il lavoro è un perditempo. Il tempo non bisogna perderlo in cose inutili, bisogna utilizzarlo. C'è gente che perde tutta la giornata a lavorare. Invece, guardate me: io non perdo un minuto. Da che m'alzo la mattina a che vado a letto la sera, utilizzo tutta la giornata a passeggiare, a pensare, a starmene seduto ai giardini pubblici guardando il mare, gli alberi. Se trovo qualcuno, faccio quattro chiacchiere, parlo coi discepoli. Insomma, io occupo tutto il mio tempo come va occupato. Non perdo neanche un'ora, nemmeno un minuto.
VICENZINO - (entusiasmato batte le mani) *Bene! Bene!*
CARMELA - Giesù, Giesù, Giesù!
EDUARDO - C'è gente che lavora tutta una vita per riposare a settant'anni. Lavorano magari quaranta o cinquant'anni, poi a settant'anni riposano. Io non discuto il loro metodo: sarà eccellente. Ma io ho un sistema diverso. Io riposo quaranta o cinquant'anni, poi a settant'anni, se sarà il caso, e se ci arriverò, forse comincerò a lavorare. (alzandosi e avvicinandosi alla porta, nel mentre guarda sulla strada) *Mi fanno ridere. Se voi vi affacciate sulla via, cosa vedete? Case, case, case. E in ogni casa c'è della gente che lavora. Il calzolaio fa le scarpe al barbiere, il barbiere fa la barba al sarto, il sarto cuce gli abiti al calzolaio e al barbiere. Tutti lavorano, eppure tutti non desiderano che il momento in cui potranno riposare. Quando, poi, dopo anni e anni di lavoro, si riposano, sono talmente abituati a lavorare che a riposare si annoiano. È naturale: non bisogna abituarsi al lavoro.*
VICENZINO - Non bisogna! Non bisogna!
CARMELA - (a Vicenzino) *Statte zitto, tu!*
EDUARDO - Quando uno invece è abituato al riposo, non si può mai annoiare. Alla domenica, per esempio, la gente sapete perché s'annoia? Perché non è abituata al riposo, manca del necessario allenamento all'ozio. Perciò che io dico: alleniamoci all'ozio e combatteremo la noia del giorno domenicale. È chiaro?
VICENZINO - (entusiasta) *Chiarissimo!*
EDUARDO - Del resto, voi credete che non far niente sia una cosa facile? Seh! C'è un'abilità, una tecnica del non far niente. Tutti sono capaci di non far niente. Bisogna vedere questo niente come lo fanno. Guardate Socrate, Platone, Diogene! Non facevano niente, ma lo facevano in un modo perfetto. Diogene se ne stava giornate intere seduto al sole, ogni tanto scambiava quattro chiacchiere con un discepolo, parlava del più e del meno. Sapete dove dormiva? In una botte.

E disprezzava il danaro: era orgoglioso della sua povertà. Diceva: "Omnia mea mecum porto." Per significare che tutta la sua ricchezza era il suo cervello. Che uomo! Un giorno lo videro girare per Atene con una lanterna in mano: cercava l'uomo. Sembra una sciocchezza, ma guardate che profondità. Uno gira per Atene con una lanterna in mano. Che cosa cerca? L'uomo. Un'altra volta l'Imperatore Alessandro si fermò davanti a lui e gli disse: "Diogene, cosa posso fare per te?" Sapete che gli rispose Diogene? "Voglio che tu ti tolga tra me e il sole." Avete capito che testa, che carattere? Un'altra volta vide un bimbo che beveva nel cavo della mano. Allora disse: "Quel ragazzo m'insegna che porto con me molte cose inutili." E spezzò il suo bicchiere. Ecco uno che non faceva niente tutto il giorno, ma quel niente lo sapeva fare. Se avesse fatto il tornitore, certo i suoi contemporanei avrebbero detto di lui: "Che tornitore, quel Diogene! È proprio un bravo artigiano, un lavoratore onesto e coscienzioso!" Ma i posteri non si sarebbero mai curati di lui. I posteri lo hanno esaltato, perché? Perché non faceva niente tutto il giorno, perché pensava, perché non perdeva il suo tempo a lavorare, ma lo utilizzava oziando. Di Eduardo Parascandolo, cara signorina Carmela, parleranno i posteri.

— Armando Curcio, *A che servono questi quattrini*

IMPERIAL COLLEGE LONDON

Abstract

Faculty of Natural Sciences

Department of Mathematics

Doctor of Philosophy

Applications of the amalgam method to the study of locally projective graphs

by William GIULIANO

Since its birth in 1980 with the seminal paper [Gol80] by Goldschmidt, the *amalgam method* has proved to be one of the most powerful tools in the modern study of groups, with interesting applications to graphs.

Consider a connected graph Γ with a family $\mathcal{L}$ of complete subgraphs (called lines) with $\alpha \in \{2, 3\}$ vertices each, and possessing a vertex- and edge-transitive group G of automorphisms preserving $\mathcal{L}$. It is assumed that for every vertex x of Γ, there is a bijection between the set of lines containing x and the point-set of a projective $\mathrm{GF}(2)$-space. There is a number of important examples of such *locally projective graphs*, studied and partly classified by Trofimov, Ivanov and Shpectorov, where both classical and sporadic simple groups appear among the automorphism groups.

To a locally projective graph one can associate the corresponding *locally projective amalgam* $\mathcal{A} = \{G(x), G\{l\}\}$ comprised of the stabilisers in G of a vertex x and of a line l containing it. The renowned Goldschmidt amalgams turn out to belong to this family ($\alpha = 3$), as well as their *densely embedded* Djoković-Miller subamalgams ($\alpha = 2$). We first determine all the embeddings of the Djoković-Miller amalgams in the Goldschmidt amalgams, by designing and implementing an algorithm in GAP [Gap] and MAGMA [BCP97]. This gives, as a by-product, a list of some finite completions for both the Goldschmidt and the Djoković-Miller amalgams.

Next, we consider two examples of locally projective graphs, special for being devoid of densely embedded subgraphs, and we extend their corresponding locally projective amalgams through the notion of a *geometric subgraph*. In both cases we find a *geometric presentation* of the amalgams, which we use to prove the simple connectedness of the corresponding geometry.

Finally, we use the Goldschmidt's lemma to classify, up to isomorphism, certain amalgams related to the Mathieu group M_{24} and the Held group He, as outlined in [Iva18], and we give an explicit construction of the cocycle whose existence and uniqueness is asserted in [Iva18, Lemma 8.5].

Acknowledgements

Thanking all the people that, in different ways and moments, have accompanied and supported me during these years would certainly result in a document longer than this thesis itself.

The tradition suggests that I should begin with my supervisor, *Professor Alexander A. Ivanov*, but in this case I really feel obliged to start with him. To say that he offered me continued support, encouragement and insights would be somehow belittling, as he literally spurred me on to continue my PhD when my motivation and interest were fading with each passing day. I will remember with great fondness our meetings during the pandemic, on a bench in our neighbourhood, as well as our chats about coins, our common hobby. Thank you very much again, Sasha!

Next, I would like to thank two people who are strangely related, and even more weirdly put together here. One is *Claudio*, who took care of my settling down in London by offering his essential help; I doubt that he will ever read these lines, but he perfectly knows what I mean. Grazie di tutto. The other person is *Elisabetta*, who has been an aunt and a friend. Her amazing culinary skills have often made me feel in Italy, and her extensive experience of London has helped me discover and appreciate this wonderful city. Thanks to her, I also met *Ludo*: a very nice new friend . . . and an incredibly young-looking uncle!

My PhD experience, as natural, has made me meet and know many people: some of them came close to me and then streaked past, like bright comets across the sky; others fell in my orbit and have now become new friends. The first of them is *Daniel*, with whom I spent my first year in Orient House, when everything was new for both of us. We supported each other and got to establish a great friendship: my presence at his wedding with Layla, followed by the birth of the beautiful Amira, is a proof of it! It is also certainly the case of *Faezeh*, whose company and kindness sweetened my days at the college; together with *Donya*, an excellent cook, I discovered a bit of the Iranian culture, especially its delicious cuisine. Thank you so much! It might seem that my years here in London have somehow had food as a common denominator, and it is actually true: another new and good friend, *Aluna*, revealed himself to be an experienced chef, as well as an extraordinary bargain hunter for theatre tickets. I'll always remember with pleasure our bike rides from Imperial College to Covent Garden to attend the shows. Grazie anche a te! I can't avoid mentioning *Mario*, met during a college tour and immediately become a caring and invaluable (although a bit crazy) friend. His and his family's generosity and hospitality in Albania will be forever remembered. Faleminderit shumë! Finally, *Carlos*: a trusted friend, a Spanish teacher and, when needed, an unexpectedly good amateur psychologist. Our long wanders through the parks, particularly during the recent lockdowns, have been an opportunity for interesting exchanges of ideas about life, in all its facets. Many other people met at the college made me feel these years really enjoyable; among them: *Melissa*, *Yibei*, *Kalle*, *Riikka*, *Francesco*, *Alessandro*, *Massimiliano*, *Ferdinando*, *Domenico*, *Imen* . . . and the list could certainly continue further.

The rest of the time not spent at university was at home, where I really found a pleasant happy place in which to live. A special thank you goes to *Myriam*, *Marco* and *Chiara*, a family (in all respects) that received me with open arms as a part of them. My heartfelt thanks also to *Laura*, an unforeseen surprise here in London: a really genuine person whose friendship, demonstrated in so many occasions, I'm

sure will last forever. ¡Muchas gracias, de verdad! …also for making me know other new good friends such as *James*, *Yoli* and *Víctor*. Many thanks also to my new flatmates *Carol*, *Estefa* and *Johnny* who, together with the rabbits *Venus* and *Estrella*, in the last three months gave me the necessary peacefulness to finish this thesis.

Before leaving London in this virtual tour, I would like to give a special thanks to *Ana*, who I used to call 'mi churri'. Even though she never helped me directly with this thesis, she did a lot for me in the last year, and she will always hold a special place in my heart. Finally, thank you to all my *friends in Milan*, far away but still so near, and to my *family*, to whom this effort is dedicated.

P.S. I feel a deep sense of gratitude to *OLIO*, a mobile app for food-sharing, aiming to reduce food waste. Since I (unfortunately late) discovered it, it changed my life in London through a new ideology that will hopefully spread around the world.

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List of Symbols

$\mathrm{GF}(q)$	the finite field (or Galois field) with q elements
$\mathrm{GL}_n(q)$	the general linear group of degree n over $\mathrm{GF}(q)$
$\mathrm{SL}_n(q)$	the special linear group of degree n over $\mathrm{GF}(q)$
$\mathrm{AGL}_n(q)$	the affine general linear group of degree n over $\mathrm{GF}(q)$
$\mathrm{PGL}_n(q)$	the projective general linear group of degree n over $\mathrm{GF}(q)$
$\mathrm{L}_n(q)$	the projective special linear group of degree n over $\mathrm{GF}(q)$
$\mathrm{P\Sigma L}_n(q)$	the extension of $\mathrm{L}_n(q)$ by the field automorphisms
$\mathrm{U}_n(q)$	the unitary group of degree n over $\mathrm{GF}(q)$
$\mathrm{Sp}_{2n}(q)$	the symplectic group of degree $2n$ over $\mathrm{GF}(q)$

To my family, sine qua non.

Chapter 1

Background

1.1 Notation for groups

Since there is no general consensus on notation for groups and their extensions, we begin by fixing our conventions.

If A and B are arbitrary groups, then $A \times B$ denotes a direct product, with normal subgroups A and B; a (non-trivial) semidirect product, or split extension, with a normal subgroup A and a complement B is denoted by $A : B$; a non-split extension with a normal subgroup A and a quotient (but no subgroup) B is denoted by $A \cdot B$. Finally, $A \circ B$ is the notation for a central product of A and B, and $A \wr B$ denotes the wreath product of A and B.

The expression $[n]$ denotes an unspecified group of order n, while n denotes the cyclic group of that order, and m^n a homocyclic group of order m^n, i.e. the direct product of n copies of the cyclic group of order m. In particular, if p is a prime, p^n denotes the elementary abelian group of order p^n, and p_ϵ^{2r+1} is the symbol for the extraspecial group of order p^{2r+1} of type $\epsilon \in \{+, -\}$. Furthermore, the symbols D_{2n}, $Q_8 \cong 2_-^{1+2}$ and QD_{2^n} indicate, respectively, the dihedral group of order $2n$, the quaternion group of order 8, and the quasidihedral (or semidihedral) group of order 2^n, with $n \geq 4$. Finally, S_n and A_n denote the symmetric and the alternating group on n letters, respectively.

We will often deal with groups that can be described as split or non-split extensions, and since this description is generally far form being exact, whenever it is possible, we will use the GAP [Gap] function `IdGroup` (or equivalently the MAGMA [BCP97] function `IdentifyGroup`) to identify the groups uniquely, up to isomorphism. For a group G, we will write $\mathrm{ID} = [a, b]$, where a is the order of G and b is the position in which G occurs in the list of groups with that order. Whenever the isomorphism type of G is not uniquely determined by its description as an extension, we prefer the symbol $\sim$ to $\cong$, and we rather use the more general term *shape*.

For other symbols which are not included here, the reader can consult the List of Symbols 1.

1.2 Amalgams: preliminaries

We start by introducing the notion of a *group amalgam* and its role in modern algebraic combinatorics and group theory. This section has been inspired by [BS04; IS04].

Definition 1. *Let* $n \in \mathbb{N}$ *and* $I = \{1, \dots, n\}$. *An (abstract)* amalgam $\mathcal{A}$ *of rank* n *is a set with a partial operation of multiplication and a collection of subsets* $\{A_i\}_{i \in I}$, *called the* members *of the amalgam, such that the following conditions hold:*

(a) $\mathcal{A} = \bigcup_{i \in I} A_i$;

(b) *the product* ab *is defined if and only if* $a, b \in A_i$ *for some* $i \in I$;

(c) *the restriction of the multiplication to each* A_i *turns* A_i *into a group;*

(d) *the intersection* $A_{ij} := A_i \cap A_j$ *is a subgroup in both* A_i *and* A_j *for all* $i, j \in I$.

It follows that the groups A_i share the same identity element, which is then the only identity element in $\mathcal{A}$, and that $a^{-1} \in \mathcal{A}$ is well-defined for every $a \in \mathcal{A}$. Abusing the notation, we often write $\mathcal{A} = \{(A_i, *_i) \mid 1 \leq i \leq n\}$ to indicate explicitly which groups constitute the union of $\mathcal{A}$. The main source of group amalgams is given by a *concrete* amalgam, which is a finite collection of subgroups of a group, where the group product in each member is the restriction of the product in the whole group.

One of the most important problems, if not the ultimate question, about an amalgam is to determine its completion(s), defined as follows. A *completion* of an amalgam $\mathcal{A}$ is a pair (A, φ), where $(A, \cdot)$ is a group (the *completion group*) and φ is a mapping (the *completion map*) from $\mathcal{A}$ to A, such that the restriction of φ to every member of $\mathcal{A}$ is a group homomorphism:

$$\varphi(a *_i b) = \varphi(a) \cdot \varphi(b),$$

for every $1 \leq i \leq n$ and all $a, b \in A_i$. The completion (A, φ) is said to be *faithful* if φ is injective and *generating* if A is generated by the image of φ. By abuse of notation, it is not uncommon to say that A is a completion of $\mathcal{A}$.

Among all completions of $\mathcal{A}$, besides the ever-present trivial one, there is a 'largest' one, defined as follows. A completion $(\widehat{A}, \widehat{\varphi})$ is called *universal* if for every completion (A, φ) there is a homomorphism ψ from $\widehat{A}$ into A, such that φ is the composition of $\widehat{\varphi}$ and ψ. The universal completion group, which is unique up to isomorphism, can be defined as the group $\mathcal{U}(\mathcal{A})$ having the following presentation [IS02, Lemma 1.3.2]:

$$\mathcal{U}(\mathcal{A}) = \langle t_h, h \in \mathcal{A} \mid t_x t_y = t_z, \text{ if } x, y, z \in A_i \text{ for some } i \text{ and } x *_i y = z \rangle .$$

Thus the generators of $\mathcal{U}(\mathcal{A})$ are indexed by the elements of $\mathcal{A}$ and the relations are all the equalities that can be seen in the groups constituting the amalgam. There is a natural bijection between the generating completions of $\mathcal{A}$ and the normal subgroups of the universal completion group $\mathcal{U}(\mathcal{A})$: if N is a normal subgroup of $\mathcal{U}(\mathcal{A})$, then the corresponding completion group is the quotient of $\mathcal{U}(\mathcal{A})$ over N.

We say that an amalgam $\mathcal{A}$ *collapses* if its universal completion group is trivial, i.e. $\mathcal{U}(\mathcal{A}) = 1$ [Gra06, Example 2.7]. The opposite extreme is represented by those amalgams having infinite completion groups that cannot be studied in any meaningful way. A wider definition of group amalgams can be given in category-theoretical terms, but since we never make use of it, we refer the interested reader to [Ser80; GGH10; GLS96; AS92].

Rank 1 amalgams are nothing but groups, while amalgams of rank 2 are usually treated in a slightly more refined, although equivalent, setting. They consist of three groups P_1, P_2 and B, together with a pair of injective group homomorphisms $\varphi_i\colon B \longrightarrow P_i$ for $i \in \{1,2\}$. In the concrete case, which is how amalgams are usually given, P_1 and P_2 are subgroups of an ambient group in such a way that $B = P_1 \cap P_2$ (the so-called *Borel subgroup* of the amalgam), and φ_1 and φ_2 are the inclusion mappings. The notation $\{P_1, P_2; B\}$ we use in this case, although suppressing (without forgetting) the monomorphisms, has the advantage of stressing the importance of the intersection, which indeed plays a crucial role in the whole theory. Sometimes, in place of the two members and their intersection, we simply write their isomorphism types, even though this may not specify the amalgam uniquely: for example, if we write $\{D_8, D_8; 2^2\}$, it is unclear which particular 2^2-subgroup of the two dihedral groups should become the intersection.

It is known (see [Kur60] or [Can05, Lemma 37]) that for an amalgam of rank 2, the universal completion is faithful and the universal completion group is isomorphic to $P_1 *_B P_2$, the free amalgamated product of P_1 and P_2 with respect to B, which is the *pushout* in the category of groups [Löh17]:

$$\begin{tikzcd}
B \arrow[r, "\varphi_2"] \arrow[d, "\varphi_1"'] & P_2 \arrow[d, "i_2"'] \arrow[rdd, bend left, "f_2"] & \\
P_1 \arrow[r, "i_1"] \arrow[rrd, bend right, "f_1"'] & P_1 *_B P_2 \arrow[rd, dashed, "\psi"] & \\
 & & G
\end{tikzcd}$$

FIGURE 1.1: The free product of P_1 and P_2 amalgamated over B.

The group $P_1 *_B P_2$, which is infinite whenever B is proper in both P_1 and P_2, contains subgroups isomorphic to P_1 and P_2 which intersect in a subgroup isomorphic to B (see [Rob96, Chapter 6, § 4]), and it is isomorphic to the free product of P_1 and P_2 factored by the normal subgroup generated by $\varphi_1(b)\varphi_2(b^{-1})$ where $b \in B$.

Following [Pot09; Gol80], we give some more useful definitions. Let $\mathcal{A} = \{P_1, P_2; B\}$ and $\mathcal{B} = \{P_1', P_2'; B'\}$ be two amalgams. Then $\mathcal{A}$ and $\mathcal{B}$ have the same *type* provided there exist isomorphisms $\tau_i\colon P_i \longrightarrow P_i'$ such that $\mathrm{Im}(\tau_i \circ \varphi_i) = \mathrm{Im}(\varphi_i')$, for $i \in \{1,2\}$. A *morphism* from $\mathcal{A}$ to $\mathcal{B}$ is a triple $f = (\alpha, \beta, \gamma)$ of group homomorphisms, as in Figure 1.2, such that $\alpha \circ \varphi_1 = \varphi_1' \circ \beta$ and $\gamma \circ \varphi_2 = \varphi_2' \circ \beta$.

$$\begin{tikzcd}
P_1 \arrow[d, "\alpha"] & B \arrow[l, "\varphi_1"'] \arrow[r, "\varphi_2"] \arrow[d, "\beta"] & P_2 \arrow[d, "\gamma"] \\
P_1' & B' \arrow[l, "\varphi_1'"'] \arrow[r, "\varphi_2'"] & P_2'
\end{tikzcd}$$

FIGURE 1.2: A morphism of amalgams.

Such a morphism f of amalgams is *injective*, *surjective* or an *isomorphism* whenever all the homomorphisms α, β and γ have the corresponding property. We say that $\mathcal{A}' = \{P_1', P_2'; B'\}$ is a *subamalgam* of $\mathcal{A} = \{P_1, P_2; B\}$ if $P_i' \leq P_i$ and $P_i' \cap B = B'$, for $i \in \{1,2\}$, in which case $\mathcal{U}(\mathcal{A}') \leq \mathcal{U}(\mathcal{A})$ (see [Can05, Lemma 54]). A *normal subgroup* of an amalgam $\mathcal{A} = \{P_1, P_2; B\}$ is, by definition, a subgroup N of B, such that $\varphi_i(N) \trianglelefteq P_i$ for $i \in \{1,2\}$. By Zorn's lemma, there exists a unique maximal normal subgroup of $\mathcal{A}$, which is called the *core* of $\mathcal{A}$. If the core is trivial, i.e. the identity subgroup, then the amalgam is said to be *simple*.[1] If N is a normal subgroup of $\mathcal{A} = \{P_1, P_2; B\}$, then $\{P_1/\varphi_1(N), P_2/\varphi_2(N); B/N\}$ is also an amalgam called the *quotient of $\mathcal{A}$ modulo N*. In particular, if N is the core of $\mathcal{A}$, then this quotient is a simple amalgam.

1.3 Some basic facts about graphs

We will recall here some basic facts about graphs, mainly to fix our terminology and notation. If the contrary is not stated explicitly, all graphs in question are assumed to be undirected, without loops and multiple edges.

For a graph Γ let $V(\Gamma)$ and $E(\Gamma)$ denote its vertex set and edge set respectively. An *automorphism* of Γ is any permutation of the vertices of Γ preserving adjacency. Under composition, the set of all such permutations of $V(\Gamma)$ forms a group known as the (full) automorphism group of Γ and denoted by $\mathrm{Aut}(\Gamma)$.

For any positive integer s, an *s-arc* (or a *path* of length s) in Γ is an ordered sequence $(x_0, x_1, \ldots, x_s)$ of $s+1$ vertices, such that $\{x_{i-1}, x_i\}$ is an edge of Γ for $1 \leq i \leq s$ and $x_{i-1} \neq x_{i+1}$ for $1 \leq i < s$, that is, such that any two consecutive vertices are adjacent and any three consecutive vertices are distinct. Arcs can be used to define the *distance* $d_\Gamma(x,y)$ between vertices $x, y \in V(\Gamma)$, which is the length of a shortest path from x to y.[2] If $x_0 = x_s$, then the arc is called a *cycle* of length s or simply an *s-cycle*, and the *girth* of Γ is the length of its shortest cycle. Γ is said to be *connected* if there is a path between every pair of its vertices, and a *tree* is a connected graph without cycles.

Let $\Gamma(x)$ denote the neighbourhood of $x \in V(\Gamma)$ in Γ, i.e. the set of vertices adjacent to x, and for a non-negative integer i put

$$\Gamma_i(x) = \{y \in V(\Gamma) \mid d_\Gamma(x,y) = i\},$$

so that $\Gamma_0(x) = \{x\}$ and $\Gamma_1(x) = \Gamma(x)$. A graph Γ is said to be *regular* if $|\Gamma(x)| = k$ does not depend on the particular choice of x. In this case k is called the *valency* (or *degree*) of Γ.

For a subset Ξ of the vertex set of Γ the *subgraph induced by* Γ on Ξ has Ξ as vertex set and $\{x,y\}$ is an edge in this subgraph if $x, y \in \Xi$ and $\{x,y\} \in E(\Gamma)$. A *clique* is a subset of vertices of a graph such that every two distinct vertices are adjacent, while an *independent set* (or *coclique*) is a set of vertices, no two of which are adjacent. A graph is called *k-partite* if its vertex set can be partitioned into k different non-empty independent sets, usually called the *parts* of the graph. When $k = 2$ (resp. $k = 3$), the name *bipartite* (resp. *tripartite*) is more common. A graph is *biregular of valency* $\{k_1, k_2\}$ if it is bipartite and vertices in the ith part of the bipartition have valency k_i, for $i = 1, 2$. A graph is called *complete* if every pair of distinct vertices is connected by a unique edge, and a complete graph on n vertices is denoted by K_n. A *complete k-partite graph* is a k-partite graph in which there is an edge between

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[1] Other authors use the terms *primitive* [Gol80] or *faithful* [Pot09] or *effective* [Gla03].

[2] If there is no path in Γ from x to y, then $d_\Gamma(x,y) = \infty$.

every pair of vertices from different independent sets. These graphs are described by notation with a capital letter K subscripted by a sequence of the sizes of each set in the partition. We will occasionally need the following definition, only for the case $k = 2$. For a connected graph Γ of diameter d, the *distance-k graph* of Γ, for $k = 1, \dots, d$, is a graph with the same vertex set and having edge set consisting of pairs of vertices that lie a distance k apart.

If G is a group of automorphisms of Γ, that is a subgroup of $\mathrm{Aut}(\Gamma)$, then G is said to be *vertex-transitive*, *edge-transitive* and *s-arc transitive* if it acts transitively on the vertex set, the edge set and the set of s-arcs of Γ, respectively. If $\Xi \subseteq V(\Gamma)$, then $G(\Xi)$ and $G\{\Xi\}$ denote the pointwise and the setwise stabilisers of Ξ in G, respectively. If $H \leq G\{\Xi\}$, we write H^{Ξ} for the permutation group induced by H on Ξ, so that abstractly $H^{\Xi} \cong H/H(\Xi)$. We write $G(x, y, \dots)$ instead of $G(\{x, y, \dots\})$ and $G\{x, y, \dots\}$ instead of $G\{\{x, y, \dots\}\}$. In particular, the permutation group $G(x)^{\Gamma(x)}$ induced by the vertex stabiliser $G(x)$ on $\Gamma(x)$ is known as the *subconstituent* of G on Γ. For a non-negative integer i let $G_i(x)$ denote the vertex-wise stabiliser in G of the ball of radius i centred at x, so that

$$G_i(x) = \bigcap_{d_\Gamma(x,y) \leq i} G(y).$$

Each $G_i(x)$ is clearly a normal subgroup of $G(x)$; in particular, $G_0(x) = G(x)$ and $G_1(x) = G(\{x\} \cup \Gamma(x))$ is the kernel of the action of $G(x)$ on $\Gamma(x)$. Then $G(x)^{\Gamma(x)}$ is abstractly isomorphic to $G(x)/G_1(x)$, and the quotients $G_i(x)/G_{i+1}(x)$ are known as the *distance factors*. It is well known (see [Iva99, Lemma 9.1.2]) that G is 2-arc transitive if and only if G is vertex-transitive and, for $x \in V(\Gamma)$, $G(x)^{\Gamma(x)}$ is a doubly transitive permutation group.

We conclude this section with the definition of a notion borrowed from algebraic topology [Lei82]. Let Γ and Γ' be two graphs, and let $f \colon V(\Gamma') \longrightarrow V(\Gamma)$ be a surjection. Then f is said to be a *covering map* from Γ' to Γ if for each $v \in V(\Gamma')$, the restriction of f to the neighbourhood of v is a bijection onto the neighbourhood of $f(v)$ in Γ; in other words, f maps edges incident to v one-to-one onto edges incident to $f(v)$. If there is a covering map from Γ' to Γ, we say that Γ' *covers* Γ, and this, intuitively, means that Γ looks everywhere locally like Γ'.

The *universal cover* $\mathcal{U}(\Gamma)$ of a connected graph Γ is the (possibly infinite) tree which covers Γ. Unless Γ itself is a tree, in which case $\mathcal{U}(\Gamma)$ can be identified with Γ, $\mathcal{U}(\Gamma)$ will be an infinite graph which covers any graph covering Γ.

1.4 Geometries: definitions and basic concepts

Geometries, as introduced by Tits in the 1950s, form a special class of incidence systems, where the incidence structure is expressed by a graph on a set of elements which generalises the set of subspaces of a classical geometry. Each element bears a type which is inspired by names such as ‘point’, ‘line’, ‘plane’ in elementary geometry or by the dimension of a subspace in classical geometries.

We begin with some definitions. Fairly comprehensive references for the material in this section are [BC13; Pas94; Shu11]. Let I be a set, called the *type set*. A triple $\Gamma = (X, *, \tau)$ is called an *incidence system* (or a *pregeometry*) over I if X is a set of *elements*, $*$ is a binary, symmetric and reflexive relation defined on X (called the *incidence relation*) and τ is a mapping from X onto I (called the *type function*), such that distinct elements $x, y \in X$ which are *incident*, i.e. with $x * y$, satisfy $\tau(x) \neq \tau(y)$.

If $A \subseteq X$, we say that $\tau(A)$ is the *type* of A and its cotype is $I \setminus \tau(A)$. The *rank* of A is $|\tau(A)|$ and its *corank* is $|I \setminus \tau(A)|$. The *rank* of Γ is the cardinality of I, that is the number of distinct types of elements. A *flag* of Γ is a (possibly empty) set of pairwise incident elements of Γ, and flags of type I are called *chambers*. By Zorn's lemma, every flag is contained in at least one maximal flag, that is a flag not properly contained in any other flag. In an incidence system Γ, chambers are maximal flags, and if also the converse holds, then Γ is called a *geometry* over I.

In an incidence system $\Gamma = (X, *, \tau)$ over I, for each $i \in I$, we write X_i to denote $\tau^{-1}(i)$, the set of elements of type i, and so X is the disjoint union $\bigcup_{i \in I} X_i$. Since different elements of the same type are never incident, the pair $(X, *)$, ignoring loops, is a multipartite graph, called the *incidence graph* of Γ, with partition $(X_i)_{i \in I}$. An incidence system is said to be *connected* if its incident graph is connected and non-empty.

A typical example of an incident system is given by the *classical projective geometries*: the elements are the non-trivial proper subspaces of a vector space, two subspaces are incident if one is contained in the other, and the type function records the algebraic dimension of any subspace.

While the elements of an incident system Γ over I remind us of the subspaces of a classical geometry, the latter are also naturally represented by the set of elements incident with them. This leads to the concept of a residue, central to any theory of geometries and defined as follows.

Let $\Gamma = (X, *, \tau)$ be an incident system over I, and let F be a flag of Γ. Then the *residue of F in* Γ is the incident system

$$\Gamma_F = (X_F, *|_{X_F}, \tau|_{X_F})$$

over the type set $I \setminus \tau(F)$, where $X_F := \{x \in X \setminus F \mid x * F\}$ is the set of all elements of $X \setminus F$ which are incident with every element of F. If Γ is a geometry, then so is Γ_F. An incident system Γ is said to be *residually connected* if for every flag F of Γ (including the empty one) such that Γ_F is of rank $r \geq 2$, the graph whose vertices are the elements of Γ_F and whose edges are the pairs of incident elements of Γ_F is connected.

Let us illustrate these definitions on a familiar example, taken from [Ueb11; F.86]. Here and elsewhere, for a flag $F = \{x\}$ consisting of a single element, its residue will be denoted by Γ_x rather than $\Gamma_{\{x\}}$. The cube determines a geometry Γ of rank 3 over $\{\texttt{vertex}, \texttt{edge}, \texttt{face}\}$ (see Figure 1.3) consisting of 8 vertices, 12 edges and 6 faces, with symmetrised inclusion as incidence and the obvious type map suggested by the names of the elements. There are 8 flags of type $\{\texttt{vertex}\}$, $8 \cdot 3 = 24$ flags of type $\{\texttt{vertex}, \texttt{edge}\}$ and $8 \cdot 3 \cdot 2 = 48$ chambers, which are the sets of pairwise incident elements consisting of exactly one vertex, one edge and one face.

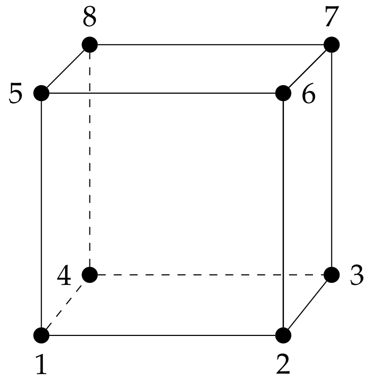


FIGURE 1.3: The cube as a geometry Γ of rank 3.

The residue Γ_1 of the first vertex consists of the edges 12, 14, 15 and of the faces 1234, 1485, 1562; hence, it is the triangle shown in Figure 1.4.

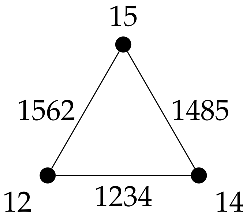


FIGURE 1.4: The residue of a vertex in Γ.

Let us now consider the residue of an edge. The residue Γ_{12} consists of the two vertices 1, 2 and of the two faces 1234, 1562; hence, it is the digon displayed in Figure 1.5.

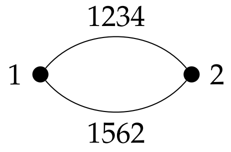


FIGURE 1.5: The residue of an edge in Γ.

Finally, the residue Γ_{1234} of the bottom face consists of the four vertices 1, 2, 3, 4 and of the four edges 12, 23, 34, 14; hence, it is the quadrangle shown in Figure 1.6.

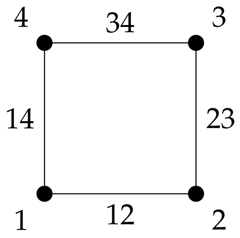


FIGURE 1.6: The residue of a face in Γ.

The residue of the empty flag is the full cube, the residue of a chamber is empty, and the other residues are less interesting. The cube geometry is residually connected.

In geometry, as in every structure theory, the concepts of homomorphism, isomorphism and automorphism are essential. We begin with the definition of homomorphisms of a more general kind, called weak homomorphisms. Let $\Gamma = (X, *, \tau)$ and $\Gamma' = (X', *', \tau')$ be two incident systems over some sets I and I', respectively. A *weak homomorphism* $\alpha \colon \Gamma \longrightarrow \Gamma'$ is a map $\alpha \colon X \longrightarrow X'$ such that, for all $x, y \in X$,

$$x * y \Longrightarrow \alpha(x) *' \alpha(y) \qquad \text{and} \qquad \tau(x) = \tau(y) \iff \tau'(\alpha(x)) = \tau'(\alpha(y)).$$

In other words, α preserves incidence and sends elements of the same type in I to elements of the same type in I'. If, in addition, $I = I'$ and $\tau(x) = \tau'(\alpha(x))$ for all $x \in X$, then α is called a *homomorphism*.

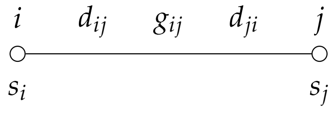
An injective homomorphism $\Gamma \longrightarrow \Gamma'$ of incident systems is also called an *embedding* of Γ into Γ'. A bijective weak homomorphism α whose inverse α^{-1} is also a weak homomorphism is called a *correlation*. If α is a homomorphism and a correlation, then we call α an *isomorphism* and write $\Gamma \cong \Gamma'$. The correlations of Γ onto itself, called *auto-correlations*, form a group under composition; similarly, the *automorphisms* of Γ, i.e. the isomorphisms of Γ onto itself, also form a group called the *automorphism group* of Γ and denoted by $\mathrm{Aut}(\Gamma)$. An automorphism group G of Γ, that is a subgroup of $\mathrm{Aut}(\Gamma)$, is said to be *flag-transitive* if any two flags in Γ of the same type are in the same G-orbit. A geometry Γ possessing a flag-transitive automorphism group is called *flag-transitive*.

A surjective homomorphism $\alpha \colon \Gamma \longrightarrow \Gamma'$ is said to be a *covering* of Γ' if for every nonempty flag F of Γ the restriction of α to the residue Γ_F is an isomorphism onto $\Gamma'_{\alpha(F)}$. In this case Γ is said to be a *cover* of Γ'. If every covering of Γ' is an isomorphism, then Γ' is said to be *simply connected*. If $\psi \colon \widetilde{\Gamma} \longrightarrow \Gamma$ is a covering and $\widetilde{\Gamma}$ is simply connected, then ψ is the *universal covering* and $\widetilde{\Gamma}$ is the *universal cover* of Γ.

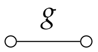
The rank 2 geometries are the building blocks for higher rank geometries, and their residues are commonly presented in the form of a diagram, a concise way of capturing some characteristics of the geometry. Let i and j be two different types of a geometry Γ, and let Γ_{ij} be a typical residue of Γ over $\{i, j\}$. Let us first introduce the following parameters as follows.

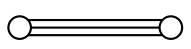
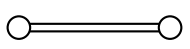
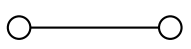
Assume that the shortest cycles (if there are any) of the incident graph of Γ_{ij} have length $2g_{ij}$. Then $g_{ij} = g_{ji}$ is the *gonality* of Γ_{ij}, and all of the elements of Γ_{ij} of type i (resp. j) have the same properties. Let p (resp. l) be a typical element of type i (resp. j) in Γ_{ij}. Then the *i-diameter d_{ij}* (resp. the *j-diameter d_{ji}*) of Γ_{ij} is the largest distance from p (resp. l) to any element in the incident graph of Γ_{ij}. The *i-order s_i* (resp. the *j-order s_j*) of Γ_{ij} is the number of elements of type j (resp. i) incident to p (resp. l) minus one, so that $s_i + 1$ is the number of chambers of Γ containing a given flag of type $I \smallsetminus \{i\}$. If $s_i = 1$ for all $i \in I$, then Γ is called *thin*.

We summarise the information provided by the parameters as follows

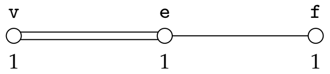


Important geometries are those where $d_{ij} = g_{ij} = d_{ji} = g$, which are knows as *generalised g-gons*. In this case we use the picture



which is reduced to  for $g = 6$, to  for $g = 4$, to  for $g = 3$ and to ○ ○ for $g = 2$.

For a geometry Γ of arbitrary rank, the corresponding diagram reveals at once the parameters of all Γ_{ij}, as one can see in the following diagram for the cube geometry, where in order to obtain the diagram of the residue of an element of type i (which is a flag of size 1) one has to remove from the diagram the node of type i along with all the edges incident to this node.



1.5 The interplay between amalgams, graphs and geometries

In this section we begin by describing how group amalgams and graphs are related from our point of view, through the definition of the coset graph. Following [KS04; PR02b; Gol80], we will introduce this notion for amalgams of rank 2, but everything can be generalised to amalgams of higher rank.

Let G be a group, and let P_1 and P_2 be two different finite subgroups of G with $P_1 \cap P_2 = B$. The (*right*) *coset graph* $\Gamma = \Gamma(G, P_1, P_2, B)$ of G with respect to P_1 and P_2 is the bipartite graph with vertex set

$$V(\Gamma) = \{P_i g \mid g \in G,\ i \in \{1, 2\}\}$$

and adjacency relation $\sim$ defined by $P_1 x \sim P_2 y$ if and only if $P_1 x \cap P_2 y \neq \emptyset$. It is easy to see that in this case $P_1 x \cap P_2 y$ is a coset of B, so that there is a one-to-one correspondence between edges of Γ and cosets of B in G. The group G acts naturally on Γ by right multiplication

$$g \colon V(\Gamma) \longrightarrow V(\Gamma) \qquad \text{with} \qquad P_i x \mapsto P_i x g \qquad (g \in G),$$

preserving the two parts together with the adjacency relation. We now collect some properties of this action and of the coset graph:

(a) G has two orbits on $V(\Gamma)$, and P_1 and P_2 are representatives of these orbits; for every $\alpha \in V(\Gamma)$, the stabiliser $G(\alpha)$ is G-conjugate to either P_1 or P_2.

(b) G acts transitively on $E(\Gamma)$ and every edge stabiliser in G is G-conjugate to B.

(c) Each vertex $P_i x$ lies on $[P_i : B]$ edges.

(d) $G(\alpha)$ acts transitively on $\Gamma(\alpha)$, $\alpha \in V(\Gamma)$, in particular

$$|\Gamma(\alpha)| = |G(\alpha) : G(\alpha) \cap G(\beta)| \quad \text{for } \beta \in \Gamma(\alpha).$$

(e) The kernel of the action of G on Γ is the normal core B_G of B in G.

(f) Γ is connected if and only if $G = \langle P_1, P_2 \rangle$.

(g) Γ is a tree if and only if G is the universal completion of the amalgam $\{P_1, P_2; B\}$, i.e. $G = P_1 *_B P_2$.

Amalgams can encode some of the information coming from a flag-transitive geometry, as explained below, where our main references is [Iva99]. The reader interested in further results on the interaction between group theory and incidence geometry is referred to [Asc83].

Let $\Gamma = (X, *, \tau)$ be a geometry over the set $I = \{1, \ldots, n\}$ and let G be a flag-transitive automorphism group of Γ. Corresponding to Γ and G, there is an amalgam $\mathcal{A} = \mathcal{A}(\Gamma, G)$ defined as follows. Let $F = \{x_1, x_2, \ldots, x_n\}$ be a maximal flag in Γ, and define $\mathcal{A}$ to be the union $\bigcup_{i=1}^{n} G_i$, where $G_i := G(x_i)$ denotes the stabiliser of x_i in G. Since G is flag-transitive, it follows that $\mathcal{A}$ is independent (up to conjugation) of the choice of F. In general, for $\emptyset \neq F_0 \subseteq F$, the stabiliser in G of F_0 is known as a *parabolic subgroup*, or just a parabolic. Parabolics are ordered by inclusion, which corresponds to the reverse inclusion of the associated flags. The *maximal parabolic subgroups*, or just maximal parabolics, are the stabilisers of one-element subflags, and thus we call $\mathcal{A}$ the *amalgam of maximal parabolics* in G associated with the flag F.

By the flag-transitivity assumption, G acts transitively on the set X_i of elements of type i in Γ, so that there is a canonical way to identify X_i with the set of right cosets of G_i in G by associating to $y \in X_i$ the coset $G_i h$ such that $x_i^h = y$. This coset consists of all the elements of G which map x_i onto y, and the mapping

$$y \mapsto G_i h$$

establishes an isomorphism of Γ onto $\Gamma(G, \mathcal{A})$, the incidence system whose elements of type i are the right cosets of G_i in G and in which two elements are incident if and only if the intersection of the corresponding cosets is non-empty.

Then on the one hand $\Gamma \cong \Gamma(G, \mathcal{A})$ and on the other hand G is a faithful completion group of $\mathcal{A}$. If G' is another faithful completion group of $\mathcal{A}$, the mapping of $\Gamma(G', \mathcal{A})$ onto $\Gamma(G, \mathcal{A})$ induced by a homomorphism $\varphi \colon G' \longrightarrow G$ is a covering of geometries. In particular, by taking G' to be the universal completion $\mathcal{U}(\mathcal{A})$ of $\mathcal{A}$, we obtain that $\Gamma(\mathcal{U}(\mathcal{A}), \mathcal{A})$ is the universal cover of $\Gamma \cong \Gamma(G, \mathcal{A})$ (see [Pas85; Tit86]). Therefore, a flag-transitive geometry is simply connected if and only if a flag-transitive automorphism group G of Γ is the universal completion of the amalgam of maximal parabolic subgroups associated with the action of G on Γ.

Chapter 2

The Djoković-Miller subamalgams of the Goldschmidt amalgams

In this chapter we introduce the objects at the heart of our discussion, namely the *locally projective graphs* and their corresponding amalgams. Our main references for this chapter are [Iva21] and [Iva18, Chapter 10].

2.1 Locally projective graphs and amalgams

The amalgams in which we are interested come from a particular class of graphs, introduced in the following definition.

Definition 2. *Let Γ be a connected graph, let G be a group of automorphisms of Γ and let $n, \alpha \in \mathbb{N}$, with $n, \alpha \geq 2$. Then Γ is said to be* locally projective *of type (n, α) with respect to the action of G if the following conditions hold:*

(i) *G acts vertex- and edge-transitively on Γ;*

(ii) *there is a family $\mathcal{L}$ of complete subgraphs in Γ (called* lines*) having α vertices each, such that $\mathcal{L}$ is preserved by G and every edge of Γ is contained in a unique line from $\mathcal{L}$;*

(iii) *every vertex x of Γ is contained in exactly $2^n - 1$ lines, and the stabiliser $G(x)$ of x in G induces on this $(2^n - 1)$-set of lines the natural doubly transitive action of the group $\mathrm{L}_n(2)$ as on the set of points of the corresponding projective $\mathrm{GF}(2)$-geometry π_x;*

(iv) *the stabiliser in G of a line acts doubly transitively on the vertex set of the line;*

(v) *if $\alpha = 2$, then G is not transitive on 3-paths in Γ and, for $\{x, y\} \in E(\Gamma)$, an element swapping x and y induces a collineation*[1] *between the residue of y in π_x and the residue of x in π_y.*

We assume that α is either 2 or 3. If $\alpha = 2$, then $\mathcal{L} = E(\Gamma)$; if $\alpha = 3$, then $\mathcal{L}$ is a family of triangles in Γ and the setwise stabiliser $G\{l\}$ in G of a line-triangle l induces on its vertices the symmetric group $S_3 \cong \mathrm{L}_2(2)$. Since any two lines intersect in at most one vertex, the valency of Γ is $(\alpha - 1) \cdot (2^n - 1)$. The $\mathrm{GF}(2)$-vector space whose non-zero vectors are indexed by the lines passing through x will be called the *natural module* of the group $\mathrm{L}_n(2)$ induced by $G(x)$ on the set of these lines.

---

[1] A *collineation* is an isomorphism between two projective geometries.

Definition 3. *Let* Γ *be a locally projective graph of type* (n, α) *with respect to a group* G*, let* $x \in V(\Gamma)$ *and let* l *be a line containing* x*. Then the amalgam*

$$\mathcal{A} = \{G(x), G\{l\}\}$$

is said to be a locally projective amalgam *of type* (n, α)*.*

The locally projective amalgams of type $(2, 2)$ were classified at the end of the 1970s by Djoković and Miller [DM80] and for this reason we will call them the *Djoković-Miller amalgams*. We will describe them in the following section, together with the so-called *Goldschmidt amalgams* [Gol80], which are, with only a few small exceptions, the locally projective amalgams of type $(2, 3)$. More recently Ivanov and Shpectorov [IS04] gave a complete classification of those of type $(n, 2)$ for all $n \geq 3$. The classification, which makes use of a result of Trofimov [Tro03], is given in the following theorem [IS04, Theorem 1].

Theorem 1. *Let* G *be a group acting locally projectively of type* $(n, 2)$ *on a graph* Γ *for some* $n \geq 3$*, and let* $\mathcal{A} = \{G(x), G\{l\}\}$ *be the corresponding locally projective amalgam. Then one of the following three possibilities holds:*

(i) $\mathcal{A}$ *is isomorphic to the locally projective amalgam associated with the natural action of the affine group* $\mathrm{AGL}_n(2)$ *on the corresponding* n*-dimensional* $\mathrm{GF}(2)$*-vector space;*

(ii) $\mathcal{A}$ *is isomorphic to the locally projective amalgam associated with the natural action of the orthogonal group* $\mathrm{O}_{2n}^{+}(2)$ *on the corresponding dual polar space graph;*

(iii) $\mathcal{A}$ *is isomorphic to one of the twelve exceptional amalgams in* [IS04, Table 1, p. 31].

2.2 The Djoković-Miller and the Goldschmidt amalgams

In [DM80] the authors consider a connected graph Γ of valency three and a subgroup G of $\mathrm{Aut}(\Gamma)$ acting regularly, i.e. sharply transitively, on the set of s-arcs of Γ. The study of the subject was started by Tutte [Tut47; Tut66], who proved that $s \leq 5$ and that the girth of Γ is at least $2s - 2$. The s-regularity of G also implies that $|G(x)| = 3 \cdot 2^{s-1}$ for every $x \in V(\Gamma)$ and $|G(x, y)| = 2^{s-1}$ for every $\{x, y\} \in E(\Gamma)$.

Djoković and Miller associate to G the rank 2 amalgam

$$\{G(x), G\{x, y\}; G(x, y)\},$$

which is independent (up to isomorphism) of the chosen edge $\{x, y\}$, and succeed in describing the structure of its members in terms of certain canonical generators. The conclusion is summarised in Table 2.1 , which contains the list of the *Djoković-Miller amalgams*, denoted by $\mathcal{DM}_i$ for $0 \leq i \leq 6$, corresponding to the possible values of s, and some finite simple groups as completions. We notice that the first amalgam $\mathcal{DM}_0$ does not fall within the locally projective class, and that the second member of $\mathcal{DM}_6$, with ID $= [32, 43]$, can be equivalently described as the holomorph of the cyclic group of order 8 or the automorphism group of D_{16}.

We now move to a brief description of the *Goldschmidt amalgams* and to their relationship with the Djoković-Miller amalgams, through the introduction of certain subgraphs of locally projective graphs which deserve a special name, also used for the corresponding amalgams.

s	Djoković-Miller amalgams	Some simple completion groups
1	$\mathcal{DM}_0 = \{3, 2; 1\}$	A_9, A_{10}, A_{11}, A_{12}, M_{12}, M_{24}, J_1, J_2, J_3, $\mathrm{G}_2(3)$
2	$\mathcal{DM}_1 = \{S_3, 2^2; 2\}$	A_{10}, J_1, J_2, J_3, $\mathrm{G}_2(3)$, $\mathrm{G}_2(4)$, ${}^2\mathrm{F}_4(2)'$, HS
2	$\mathcal{DM}_2 = \{S_3, 4; 2\}$	A_9, $\mathrm{U}_3(4)$, M_{24}, J_3, ${}^2\mathrm{F}_4(2)'$, $\Omega_8^-(2)$, ${}^3\mathrm{D}_4(2)$
3	$\mathcal{DM}_3 = \{D_{12}, D_8; 2^2\}$	$\mathrm{L}_2(71)$, $\mathrm{L}_2(73)$, $\mathrm{L}_2(97)$, $\mathrm{L}_2(167)$, ${}^3\mathrm{D}_4(2)$
4	$\mathcal{DM}_4 = \{S_4, D_{16}; D_8\}$	A_{10}, A_{13}, $\mathrm{L}_2(17)$, $\mathrm{L}_2(31)$, $\mathrm{L}_2(47)$, $\mathrm{L}_2(79)$
4	$\mathcal{DM}_5 = \{S_4, QD_{16}; D_8\}$	A_{26}, A_{29}, $\mathrm{L}_3(3)$, $\mathrm{L}_3(11)$, $\mathrm{U}_3(7)$, $\mathrm{U}_3(13)$, J_3
5	$\mathcal{DM}_6 = \{2 \times S_4, 8 : 2^2; 2 \times D_8\}$	A_{26}, A_{29}, A_{36}, A_{42}, A_{44}, A_{45}, A_{48}

TABLE 2.1: The Djoković-Miller amalgams.

In [Gol80], a remarkable paper that marked the birth of the amalgam method, Goldschmidt considered the situation of a group generated by two finite subgroups P_1 and P_2 which satisfy

(i) $P_1 \cap P_2 = B$;

(ii) $[P_1 : B] = 3 = [P_2 : B]$;

(iii) no non-trivial subgroup of B is normal in both P_1 and P_2.

The approach adopted by Goldschmidt used the embeddings of B into P_1 and P_2 to construct the universal completion group $\widehat{G} = P_1 *_B P_2$ of the corresponding amalgam and then examine its action on the coset graph $\Gamma(\widehat{G}, P_1, P_2, B)$, which is a tree by the universality property. Using this geometric framework Goldschmidt was able to successfully determine all the fifteen possibilities for the triple of groups $\{P_1, P_2; B\}$; one consequence of his result is that B is a 2-group of order at most 2^7.

We now introduce the notion of a *densely embedded subgraph*, for which we require a further piece of notation. For a group G acting on a graph Γ locally projectively of type $(n, 3)$ for $n \geq 2$, we denote by $G_{1/2}(x)$ the largest subgroup of $G(x)$ which stabilises every line passing through $x \in V(\Gamma)$, so that we have the following sequence of subgroups

$$G(x) \trianglerighteq G_{1/2}(x) \trianglerighteq G_1(x) \trianglerighteq G_2(x) \trianglerighteq \cdots$$

which always terminates at $G_6(x) = 1$ in the case $\alpha = 2$[2], due to a remarkable result of Trofimov, announced and proved in a sequence of papers from the beginning of the 1990s.

Definition 4. *Suppose that G acts on Γ locally projectively of type $(n, 3)$ for $n \geq 2$, and let Δ be a connected subgraph in Γ. Then Δ is said to be* densely embedded *in Γ if the following conditions hold:*

---

[2] A similar bound for locally projective graphs of type $(n, 3)$ is still not known and considered an important open question.

(i) *the subgroup H of G which stabilises Δ as a whole induces on it a locally projective action of type $(n, 2)$, possibly with a non-trivial kernel;*

(ii) *if $x \in \Delta$, then $H(x)$ contains $G_1(x)$ and $H(x)/G_1(x)$ is an $\mathrm{L}_n(2)$-complement to $G_{1/2}(x)/G_1(x)$ in $G(x)/G_1(x)$.*

It is implicit in the definition above that a densely embedded subgraph exists only if $G(x)/G_1(x)$ splits over $G_{1/2}(x)/G_1(x)$. In fact densely embedded subgraphs exist quite often [Iva21, Table 2] and their existence for $n \geq 3$ has been recently established under certain hypotheses [Iva21]. We analyse the case $n = 2$ and give a complete list of the Goldschmidt amalgams that admit Djoković-Miller densely embedded subamalgams. The procedure for constructing densely embedded subgraphs consists of the following steps [Iva18, p. 145], motivated in [Iva21, Theorem 16]:

(a) assuming that $G(x)$ splits over $G_{1/2}(x)$, take the preimage under the mapping $q \colon G(x) \to G(x)/G_1(x)$ of an $\mathrm{L}_n(2)$-complement of the subgroup $G_{1/2}(x)/G_1(x)$ and denote it by H_1;

(b) intersect H_1 with the pointwise stabiliser $G(l) = G(x, y, z)$ of $l = \{x, y, z\}$ to obtain the subgroup H_{12};

(c) search for elements $\sigma \in G\{l\}$ which normalise H_{12} and swap x either with y or with z;

(d) if and when the required σ has been found, put $H_2 = \langle H_{12}, \sigma \rangle_G$, $H = \langle H_1, H_2 \rangle_G$ and define Δ to be the subgraph on the set of images of x under H.

The conditions listed above are only necessary, so that if no σ is found, then the graph Γ does not admit any densely embedded subgraphs. If, instead, such an element σ exists, then the corresponding subamalgam of $\{G(x), G\{l\}\}$ is $\{H_1, H_2; H_{12}\}$, where $H_1 = H(x)$, $H_{12} = H(x, y, z)$ and H_2 is either $H\{x, y\}$ or $H\{x, z\}$. We also notice that H represents an instance of a completion of $\{H_1, H_2; H_{12}\}$ which, in the case where G is the universal completion of $\{G(x), G\{l\}\}$, is indeed the universal completion $H_1 *_{H_{12}} H_2$ of the corresponding densely embedded subamalgam (cf. 10).

We systematically applied the recipe above to the Goldschmidt amalgams, by first constructing the corresponding graphs as follows. Any faithful generating completion group G of a Goldschmidt amalgam $\{P_1, P_2; B\}$ gives rise to the coset graph $\Gamma = \Gamma(G, P_1, P_2, B)$, which is a connected 3-regular graph on which G operates as an edge-transitive group of automorphisms with vertex stabilisers isomorphic to P_1 or P_2 and edge-stabiliser isomorphic to B. Since Γ is connected and bipartite, its distance-2 graph Ξ has two connected components $\Xi^{(1)}$ and $\Xi^{(2)}$, each of which is a 6-regular graph acted on by G locally projectively of type $(2, 3)$ in most cases. The lines of each component are the (maximal) cliques of size 3 and correspond to the vertices of the other component, in a duality reflected also by the corresponding stabilisers, isomorphic to P_1 and P_2.

Figure 2.1 shows the neighbourhood of a vertex in a locally projective graph of type $(2, 3)$, with the three lines containing it and the green part representing its intersection with a densely embedded subgraph.

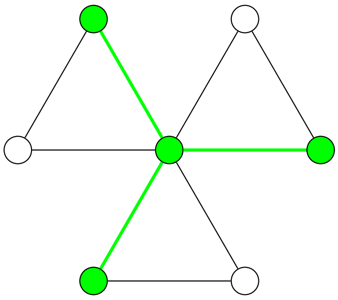


FIGURE 2.1: The neighbourhood of a vertex with the three lines.

We now list explicitly the fifteen Goldschmidt amalgams, organised into five tables, one for each class[3]. For each member P_i we give the structure of its distance factors

$$M_\alpha^\beta := G_\alpha(x)/G_\beta(x),$$

where $x \in \Xi^{(i)}$ and G is a completion group. Unlike other properties of Ξ, such as the number of vertices and edges or the diameter, these quotients do not depend on G, but only on the local structure of Ξ. The entry in the last column and in the corresponding row is the Djoković-Miller densely embedded subamalgam if it exists, in which case we give a proof. As the embeddings are independent of the completion of the Goldschmidt amalgam, in each proof we choose one of it, G, not necessarily the one indicated in the third column of [Gol80, Table 1], and with the aid of MAGMA [BCP97] we explicitly construct the densely embedded Djoković-Miller subamalgam, together with the corresponding completion group H. A line — indicates that no Djoković-Miller subamalgam is densely embedded in the corresponding Goldschmidt amalgam, mainly because no σ as in the condition (c) above can be found. The list of embeddings shown in each table is complete, in the sense that no other embeddings are possible: the 'negative' cases are not accompanied by a corresponding proof, although they have all been checked computationally. Therefore in each of the following theorems, a statement like 'the Djoković-Miller amalgam $\mathcal{Y}$ is densely embedded in the Goldschmidt amalgam $\mathcal{X}$' must be interpreted as '$\mathcal{Y}$ is the only Djoković-Miller amalgam which is embedded in the Goldschmidt amalgam $\mathcal{X}$'.

What follows is our main result of this chapter: the information on the distance factors verifies (with a different method) and completes the content of [Iva18, Table 10.3], while the embedding of the Djoković-Miller amalgams into the Goldschmidt amalgams represents a modest contribution towards the classification project of locally projective graphs.

Table 2.2 shows the four Goldschmidt amalgams of class 1, namely G_1, G_1^1, G_1^2 and G_1^3, whose Borel subgroup B is isomorphic to 1, 2, 2 and 2^2 respectively. We notice that G_1 and G_1^2 do not satisfy the defining conditions for a locally projective amalgam, but they are included anyway for completeness. For additional material on completions of the Goldschmidt amalgams of class 1 we direct the reader to [BR06].

Theorem 2. *The Djoković-Miller amalgam* $\mathcal{DM}_1 = \{S_3, 2^2; 2\}$ *is densely embedded in the Goldschmidt* G_1^3*-amalgam* $\{D_{12}, D_{12}; 2^2\}$.

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[3] Following [Che86] we say that a Goldschmidt amalgam is of *class n* if it isomorphic to G_n or G_n^i for some i.

Name	P_i	$M_0^{1/2}$	$M^1_{1/2}$	M_0^1	M_1^2	M_2^3	$\mathcal{DM}_i$
G_1	3	3	1	3	1	1	—
	3	3	1	3	1	1	—
G_1^1	S_3	S_3	1	S_3	1	1	—
	S_3	S_3	1	S_3	1	1	—
G_1^2	S_3	S_3	1	S_3	1	1	—
	6	3	2	6	1	1	—
G_1^3	D_{12}	S_3	2	D_{12}	1	1	$\mathcal{DM}_1$
	D_{12}	S_3	2	D_{12}	1	1	$\mathcal{DM}_1$

TABLE 2.2: The Goldschmidt amalgams of class 1.

Proof. As the Goldschmidt G_1^3-amalgam is 'symmetric'[4], we can consider only one connected component, say $\Xi^{(1)}$, of the distance-2 graph Ξ of $\Gamma(G, P_1, P_2, B)$, where $G = \langle (3,11,9,7,5)(4,12,10,8,6), (1,2,8)(3,7,9)(4,10,5)(6,12,11) \rangle \cong \mathrm{L}_2(11)$ is a completion group. For a vertex $x \in \Xi^{(1)}$ and for a line $l = \{x, y, z\}$ we have:

$$G(x) = \langle (1,5)(2,6)(3,4)(7,8)(9,11)(10,12), (1,9)(2,11)(3,7)(4,12)(5,10)(6,8) \rangle,$$
$$G\{l\} = \langle (1,8)(2,9)(3,5)(4,12)(6,7)(10,11), (1,6,11,12,9,5)(2,4,10,7,8,3) \rangle,$$

both isomorphic to $D_{12} \cong 2 \times S_3$, with intersection $G(x) \cap G\{l\} \cong 2^2$. The subgroup $G_{1/2}(x) = Z(G(x)) = \langle (1,7)(2,12)(3,9)(4,11)(5,8)(6,10) \rangle \cong 2$ has two non-conjugate complements in $G(x)$, but only one of them,

$$H_1 = \langle (1,8)(2,10)(3,11)(4,9)(5,7)(6,12), (1,4,6)(2,3,5)(7,11,10)(8,12,9) \rangle \cong S_3,$$

intersects $G(l) = G(x,y,z) = \langle (1,12)(2,7)(3,10)(4,8)(5,11)(6,9) \rangle \cong 2$ non trivially in $H_{12} = G(l)$. The permutation $\sigma = (1,8)(2,9)(3,5)(4,12)(6,7)(10,11) \in G\{l\}$ normalises H_{12} and swaps x with y, so that $H_2 = \langle H_{12}, \sigma \rangle \cong 2^2$ and $H \cong A_5$. Further details of the embedding of the Djoković-Miller amalgam $\mathcal{DM}_1$ in the Goldschmidt G_1^3-amalgam can be found in Appendix A. ☐

Table 2.3 shows the five Goldschmidt amalgams of class 2, namely G_2, G_2^1, G_2^2, G_2^3 and G_2^4, whose Borel subgroup B is isomorphic to 2^2, D_8, D_8 , 2^3 and $2 \times D_8$ respectively. Before proving the next theorem, we notice that although it is about a case which is not a locally projective one, the usual technique works and produces an unexpected embedding.

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[4] By this we mean that there is an automorphism of the amalgam which permutes P_1 and P_2. See Appendix A for some remarks about this fact.

Name	P_i	$M_0^{1/2}$	$M^1_{1/2}$	M^1_0	M^2_1	M^3_2	$\mathcal{DM}_i$
G_2	A_4	3	2^2	A_4	1	1	$\mathcal{DM}_0$
	D_{12}	S_3	1	S_3	2	1	—
G_2^1	S_4	S_3	2^2	S_4	1	1	—
	D_{24}	S_3	2	D_{12}	2	1	$\mathcal{DM}_3$
G_2^2	S_4	S_3	2^2	S_4	1	1	—
	$(2 \times 6) : 2$	S_3	2	D_{12}	2	1	$\mathcal{DM}_3$
G_2^3	$2 \times A_4$	3	2^3	$2 \times A_4$	1	1	—
	$2^2 \times S_3$	S_3	1	S_3	2^2	1	—
G_2^4	$2 \times S_4$	S_3	2^3	$2 \times S_4$	1	1	$\mathcal{DM}_1$
	$S_3 \times D_8$	S_3	2	D_{12}	2^2	1	—

TABLE 2.3: The Goldschmidt amalgams of class 2.

Theorem 3. *The Djoković-Miller amalgam* $\mathcal{DM}_0 = \{3, 2; 1\}$ *is densely embedded in the Goldschmidt* G_2*-amalgam* $\{A_4, D_{12}; 2^2\}$.

Proof. We consider the connected component of the distance-2 graph Ξ of the coset graph $\Gamma(G, P_1, P_2, B)$, with $G = \langle (1,2,3,4,5,6,7), (1,2,3) \rangle \cong A_7$, where the stabilisers in G of a vertex x and of a line $l = \{x, y, z\}$ can be described respectively as the following subgroups of G:

$$G(x) = \langle (1,5,6)(2,4,3), (1,2)(4,5) \rangle \cong A_4,$$
$$G\{l\} = \langle (3,6)(5,7), (1,2)(3,6)(4,5,7) \rangle \cong D_{12}.$$

As $G_1(x) = Z(G(x)) \cong 1$ and $G_{1/2}(x) = \langle (3,6)(4,5), (1,2)(4,5) \rangle \cong 2^2$, we can still consider representatives for the conjugacy classes of complements of $G_{1/2}(x)/G_1(x)$ in $G(x)/G_1(x)$, as subgroups of $G(x)$, and we obtain $H_1 = \langle (1,5,6)(2,4,3) \rangle \cong 3$. The intersection of H_1 with $G(l) = \langle (1,2)(3,6) \rangle \cong 2$ gives the trivial subgroup H_{12}, and the element $\sigma = (3,6)(5,7) \in G\{l\}$ swaps x with y generating $H_2 \cong 2$. Finally, the subgroup H of G generated by H_1 and H_2 is isomorphic to $L_3(2)$ and stabilises a 3-regular subgraph with 56 vertices and 84 edges. Further examples of embeddings are given in Table A.2. ☐

Theorem 4. *The Djoković-Miller amalgam* $\mathcal{DM}_3 = \{D_{12}, D_8; 2^2\}$ *is densely embedded in the Goldschmidt* G_2^1*-amalgam* $\{D_{24}, S_4; D_8\}$.

Proof. We do our analysis in the group $G = \langle (1,6), (1,2,7,4)(3,5,6) \rangle \cong S_7$, which is a faithful completion of Goldschmidt G_2^1-amalgam. We consider the connected component of the distance-2 graph Ξ of $\Gamma(G, P_1, P_2, B)$ where the stabilisers in G of a vertex x and of a line $l = \{x, y, z\}$ can be described respectively as the following subgroups of G:

$$\begin{aligned} G(x) &= \langle (4,5)(6,7), (1,7,6)(2,5,3,4) \rangle \cong D_{24}, \\ G\{l\} &= \langle (1,4,3)(2,6,5), (1,4)(2,3)(5,6) \rangle \cong S_4. \end{aligned}$$

The vertex-wise stabilisers of the neighbourhood of x and of l are respectively

$$\begin{aligned} G_1(x) &= Z(G(x)) = \langle (2,3)(4,5) \rangle \cong 2, \\ G(l) &= G(x,y,z) = \langle (2,3)(4,5), (1,6)(4,5) \rangle \cong 2^2. \end{aligned}$$

The group $M_0^1 := G(x)/G_1(x) \cong D_{12} \cong 2 \times S_3$ possesses two (normal) S_3-subgroups, whose preimages under the quotient map $q\colon G(x) \longrightarrow M_0^1$ are dihedral groups of order 12: one intersects $G(l)$ in $G_1(x)$, while the other one, which is the required H_1, in $H_{12} = G(l)$. The element $\sigma = (1,2,6,3) \in N_{G\{l\}}(H_{12})$ fixes y, swaps x with z and gives $H_2 = \langle H_{12}, \sigma \rangle \cong D_8$. Finally, $H \cong 2 \times S_5$ and the subgraph stabilised by H is 3-regular with 20 vertices and 30 edges. For more embeddings of the Djoković-Miller amalgam $\mathcal{DM}_3$ in the Goldschmidt G_2^1-amalgam, see Table A.3.

☐

Theorem 5. *The Djoković-Miller amalgam* $\mathcal{DM}_3 = \{D_{12}, D_8; 2^2\}$ *is densely embedded in the Goldschmidt* G_2^2*-amalgam* $\{S_4, (2 \times 6) : 2; D_8\}$.

Proof. For the Goldschmidt G_2^2-amalgam we choose the same completion group indicated by Goldschmidt [Gol80, Table 1], that is $G = \langle (1,2,3,4,5,6,7), (5,6,7) \rangle \cong A_7$. The corresponding coset graph $\Gamma(G, P_1, P_2, B)$ has $105 + 105$ vertices and 315 edges, and we consider the connected component of its distance-2 graph where the stabilisers in G of a vertex and of a line-triangle have, respectively, ID $= [24, 8]$ and ID $= [24, 12]$. More explicitly, if $l = \{x, y, z\}$ is such a triangle, we have:

$$\begin{aligned} G(x) &= \langle (3,4)(6,7), (1,3)(2,4)(5,6,7) \rangle, \\ G(y) &= \langle (4,7)(5,6), (1,5)(2,6)(3,4,7) \rangle, \\ G(z) &= \langle (2,7)(3,5,4,6), (1,7)(5,6) \rangle, \end{aligned}$$

with shape $(2 \times 6) : 2 \sim 3 : D_8 \sim 2^2 : S_3$, and

$$\begin{aligned} G\{l\} &= \langle (1,3,5)(2,4,6), (3,6)(4,5) \rangle \cong S_4, \\ G(x) \cap G\{l\} &= \langle (1,3,2,4)(5,6), (1,2)(5,6) \rangle \cong D_8. \end{aligned}$$

The vertex-wise stabiliser of the neighbourhood of x, which coincides with the centre and the Frattini subgroup of $G(x)$, is $G_1(x) = \langle (1,2)(3,4) \rangle \cong 2$, which gives the natural projection homomorphism $q\colon G(x) \longrightarrow M_0^1 \cong D_{12}$. The complete preimages in $G(x)$ of the two S_3-subgroups of M_0^1 are $\langle (5,6,7), (1,3,2,4)(5,7) \rangle \cong 3 : 4$ and

$$H_1 = \langle (3,4)(6,7), (5,6,7), (1,2)(3,4) \rangle \cong D_{12}.$$

As the latter contains the subgroup $G(l) = \langle (1,2)(5,6), (3,4)(5,6) \rangle \cong 2^2$, this one is also H_{12}. The element $\sigma = (1,2)(3,6,4,5) \in N_{G\{l\}}(H_{12})$, which swaps x with y, generates with H_{12} the subgroup $H_2 \cong D_8$. Finally, $H = \langle H_1, H_2 \rangle_G \cong S_5$ is a completion group of the $\mathcal{DM}_3$-subamalgam corresponding to the densely embedded subgraph on x^H, which is isomorphic to the Petersen graph. ☐

We conclude the analysis of the Goldschmidt amalgams of class 2 with the following theorem.

Theorem 6. *The Djoković-Miller amalgam* $\mathcal{DM}_1 = \{S_3, 2^2; 2\}$ *is densely embedded in the Goldschmidt* G_2^4*-amalgam* $\{2 \times S_4, S_3 \times D_8; 2 \times D_8\}$.

Proof. Also for the Goldschmidt G_2^4-amalgam we abide by [Gol80, Table 1] and consider $G = \langle (1,2,3,4,5,6,7), (1,2) \rangle \cong S_7$ as a faithful completion group. For one of the two connected components of the distance-2 graph of $\Gamma(G, P_1, P_2, B)$ we have:

$$\begin{aligned} G(x) &= \langle (1,6,2,5,3,4), (1,5)(2,3)(4,6) \rangle \cong 2 \times S_4, \\ G\{l\} &= \langle (1,4)(2,5), (6,7), (2,4)(3,7,6) \rangle \cong S_3 \times D_8, \\ G(l) &= \langle (1,2)(4,5), (1,5) \rangle \cong D_8. \end{aligned}$$

As $G_1(x)$ is the trivial subgroup, $M_0^1 \cong G(x)$. This group possesses eight S_3-subgroups evenly divided into two conjugacy classes, whose representatives intersect $G(l)$ either trivially or in a subgroup of order 2. We choose

$$H_1 = \langle (2,3)(4,6), (1,3,2)(4,5,6) \rangle \cong S_3,$$

so that $H_{12} = \langle (1,2)(4,5) \rangle \cong 2$. The transposition $\sigma = (6,7)$ normalises H_{12}, swaps x with y yielding $H_2 = \langle H_{12}, \sigma \rangle \cong 2^2$ and $H \cong S_3 \times S_4$. For further embeddings of $\mathcal{DM}_1$ in the Goldschmidt G_2^4-amalgam, see Table A.5. ☐

We now move to the Goldschmidt amalgams of class 3, namely G_3 and G_3^1, whose Borel subgroup is isomorphic to D_8 and $2 \times D_8$ respectively. The completions of these two amalgams, which both contain densely embedded subamalgams, have been extensively studied in [Thi93; PR02a; PR01b; PR00; PR02b; Vas14], and G_3^1 will reappear in a different form in the last chapter. We begin with G_3, for which the following theorem holds.

Name	P_i	$M_0^{1/2}$	$M_{1/2}^1$	M_0^1	M_1^2	M_2^3	$\mathcal{DM}_i$
G_3	S_4	S_3	2^2	S_4	1	1	$\mathcal{DM}_1, \mathcal{DM}_2$
	S_4	S_3	2^2	S_4	1	1	$\mathcal{DM}_1, \mathcal{DM}_2$
G_3^1	$2 \times S_4$	S_3	2^2	S_4	2	1	$\mathcal{DM}_3$
	$2 \times S_4$	S_3	2^2	S_4	2	1	$\mathcal{DM}_3$

TABLE 2.4: The Goldschmidt amalgams of class 3.

Theorem 7. *The Djoković-Miller amalgams* $\mathcal{DM}_1 = \{S_3, 2^2; 2\}$ *and* $\mathcal{DM}_2 = \{S_3, 4; 2\}$ *are densely embedded in the Goldschmidt* G_3*-amalgam* $\{S_4, S_4; D_8\}$.

Proof. It is well known [Vas14] that, for certain values of n, the alternating group A_n is a completion group of the Goldschmidt G_3-amalgam. We choose the minimal n and take $G = \langle (1,2,3,4,5),(4,5,6) \rangle \cong A_6$, so that we have:

$$\begin{aligned} G(x) &= \langle (1,3,5)(2,4,6),(1,6)(2,5) \rangle \cong S_4, \\ G\{l\} &= \langle (2,6,5),(1,6,2,5)(3,4) \rangle \cong S_4, \\ G(x) \cap G\{l\} &= \langle (3,4)(5,6),(1,5)(2,6) \rangle \cong D_8. \end{aligned}$$

By taking the complete preimage in $G(x)$ under $q\colon G(x) \longrightarrow M_0^1 \cong S_4$ of an S_3-subgroup, we obtain $H_1 = \langle (1,5)(2,6),(1,5,4)(2,6,3) \rangle \cong S_3$. This subgroup intersects $G(l) = \langle (1,5)(2,6),(1,6)(2,5) \rangle \cong 2^2$ in $H_{12} = \langle (1,5)(2,6) \rangle \cong 2$. The elements $\sigma_1 = (2,6)(3,4)$ and $\sigma_2 = (1,6,5,2)(3,4)$ both normalise H_{12} and swap x with y, yielding

$$H_2^{(1)} = \langle H_{12}, \sigma_1 \rangle \cong 2^2, \qquad H_2^{(2)} = \langle H_{12}, \sigma_2 \rangle \cong 4$$

and

$$H^{(1)} = \langle H_1, \sigma_1 \rangle \cong A_5, \qquad H^{(2)} = \langle H_1, \sigma_2 \rangle \sim 3^2 : 4,$$

where $H^{(2)}$ has ID $= [36,9]$. For further embeddings of $\mathcal{DM}_1$ and $\mathcal{DM}_2$ in the Goldschmidt G_3-amalgam, see Table A.6. ☐

Theorem 8. *The Djoković-Miller amalgam* $\mathcal{DM}_3 = \left\{D_{12}, D_8; 2^2\right\}$ *is densely embedded in the Goldschmidt* G_3^1*-amalgam* $\{2 \times S_4, 2 \times S_4; 2 \times D_8\}$.

Proof. As the Goldschmidt G_3-amalgam, also G_3^1 is 'symmetric', so that we can consider only one connected component of the distance-2 graph of $\Gamma(G, P_1, P_2, B)$, where $G = \langle (1,2,3,4,5,6),(1,2) \rangle \cong S_6$ is the chosen completion group. For a vertex x and for a line-triangle $l = \{x, y, z\}$ we have:

$$\begin{aligned} G(x) &= \langle (4,6),(1,6,2)(3,5) \rangle \cong 2 \times S_4, \\ G\{l\} &= \langle (3,4)(5,6),(1,5,6,2,3,4) \rangle \cong 2 \times S_4, \\ G(x) \cap G\{l\} &= \langle (1,4)(2,6),(3,5),(1,2) \rangle \cong 2 \times D_8. \end{aligned}$$

We notice that, although isomorphic, $G(x)$ and $G\{l\}$ play different roles and are not conjugate in G. The kernel of the action of $G(x)$ on the neighbourhood of x is $G_1(x) = Z(G(x)) = \langle (3,5) \rangle \cong 2$, so that $M_0^1 \cong S_4 \cong 2^2 : S_3$. The complete preimage of one of the four S_3-subgroups of M_0^1 is $H_1 = \langle (2,4),(1,2,4)(3,5) \rangle \cong D_{12}$, which intersects $G(l) = \langle (4,6),(3,5),(1,2)(3,5)(4,6) \rangle \cong 2^3$ in $H_{12} = \langle (1,2),(3,5) \rangle \cong 2^2$. The elements $\sigma_1 = (1,4)(2,6)(3,5)$ and $\sigma_2 = (1,4,2,6)$ both normalise H_{12} and swap x with z, yielding respectively

$$H^{(1)} = \langle H_1, \sigma_1 \rangle \cong S_3 \wr 2 \quad \text{and} \quad H^{(2)} = \langle H_1, \sigma_2 \rangle \cong S_5.$$

The group $H^{(1)}$, which has ID $= [72,40]$, stabilises a 3-regular bipartite graph, while the subgraph on the set of images of x under $H^{(2)}$ is the Petersen graph. ☐

The Goldschmidt amalgams of class 4 and 5 exhibit a much more complicated structure than the others and are constructed using certain automorphisms of the direct product of two cyclic groups of order 4 [Gol80, (3.6)]. With the aid of [Dok], we begin with a few comments on the structure of the groups shown in Tables 2.5 and 2.6.

In the Goldschmidt G_4-amalgam[5] the first member P_1 has ID $= [96, 64]$ and shape $4^2 : S_3 \sim 2^2 \cdot S_4$, where the action of S_3 on 4^2 is faithful; the second member P_2, with ID $= [96, 67]$, is the unitary group on $\mathrm{GF}(3)^2$ and its shape can be described as $(4 \circ Q_8) \cdot S_3 \sim (4 \circ D_8) \cdot S_3 \sim 4 \cdot S_4$; the Borel subgroup $B \cong 4 \wr 2$ has ID $= [32, 11]$. As for the G_4^1-amalgam, P_1 has ID $= [192, 956]$ and shape $4^2 : D_{12} \sim 2^3 \cdot S_4$, where the action of D_{12} on 4^2 is faithful; P_2 has ID $= [192, 988]$ and shape $2_+^{1+4} : S_3 \sim (D_8 \circ D_8) : S_3 \sim (Q_8 \circ Q_8) : S_3 \sim Q_8 \cdot S_4$; the Borel subgroup $B \sim 2^3 \cdot D_8 \sim 4^2 : 2^2$ has ID $= [64, 134]$ and can be identified with the holomorph of D_8.

Finally, there are two Goldschmidt amalgams of type 5, namely G_5 and G_5^1. In the former, P_1 and B are the same as in the G_4^1-amalgam, while P_2 has ID $= [192, 1494]$, shape $2_+^{1+4} : S_3 \sim Q_8 : S_4 \sim 2^3 \cdot S_4$ and can be identified with the holomorph of Q_8. As for the G_5^1-amalgam, P_1 has ID $= [384, 5677]$, P_2 has ID $= [384, 5608]$ and the Borel subgroup $B \sim 4^2 : D_8$ has ID $= [128, 932]$.

Name	P_i	$M_0^{1/2}$	$M_{1/2}^1$	M_0^1	M_1^2	M_2^3	$\mathcal{DM}_i$
G_4	$4^2 : S_3$	S_3	2^2	S_4	2^2	1	—
	$(4 \circ Q_8) \cdot S_3$	S_3	2^2	S_4	4	1	—
G_4^1	$4^2 : D_{12}$	S_3	2^2	S_4	2^3	1	—
	$2_+^{1+4} : S_3$	S_3	2^3	$2 \times S_4$	4	1	—

TABLE 2.5: The Goldschmidt amalgams of class 4.

Name	P_i	$M_0^{1/2}$	$M_{1/2}^1$	M_0^1	M_1^2	M_2^3	$\mathcal{DM}_i$
G_5	$4^2 : D_{12}$	S_3	2^2	S_4	2^3	1	—
	$(Q_8 \circ Q_8) : S_3$	S_3	2^2	S_4	2^2	2	$\mathcal{DM}_6$
G_5^1	$4^2 : ((2 \times 6) : 2)$	S_3	2^3	$2 \times S_4$	2^3	1	—
	$(Q_8 \circ Q_8) \cdot D_{12}$	S_3	2^2	S_4	2^3	2	—

TABLE 2.6: The Goldschmidt amalgams of class 5.

Theorem 9. *The Djoković-Miller amalgam $\mathcal{DM}_6 = \{2 \times S_4, 8 : 2^2; 2 \times D_8\}$ is densely embedded in the Goldschmidt G_5-amalgam.*

---

[5] For completions of this amalgam in dimension 3 see [PR01a].

Proof. We work inside the completion group $G = \langle a, b, c\rangle \cong M_{12}$, where

$$\begin{aligned} a &:= (1,2,3,4,5,6,7,8,9,10,11), \\ b &:= (3,7,11,8)(4,10,5,6), \\ c &:= (1,12)(2,11)(3,6)(4,8)(5,9)(7,10). \end{aligned}$$

The coset graph $\Gamma(G, P_1, P_2, B)$ has 990 vertices, 1485 edges, diameter 12 and girth 16. For one of the connected components of its distance-2 graph we have:

$$\begin{aligned} G(x) &= \left\langle a^{-1}b^{-1}cb^2acba,\ bc(b^{-1}a^{-1})^2a^{-1}c \right\rangle \sim 2_+^{1+4} : S_3, \\ G\{l\} &= \left\langle cab^{-1}cb^2ac,\ cbca^{-1}b^{-1}abca \right\rangle \sim 4^2 : D_{12}, \\ G(x) \cap G\{l\} &= \left\langle (a^{-1}b)^2a^{-4},\ cb^2a^{-1}b^{-1}ab^2c,\ ab^{-1}ca^{-1}b^{-1}a^{-2}c \right\rangle \sim 4^2 : 2^2. \end{aligned}$$

The subgroup $G_1(x) = \langle abca^{-1}b^2acb^{-1}a^{-1},\ cab^2a^2b^2c,\ caba^2b^{-1}a^2cb\rangle \cong 2^3$ is characteristic in $G(x)$ and gives $M_0^1 \cong S_4$. By taking the complete preimage of an S_3-subgroup of M_0^1, we obtain:

$$H_1 = \left\langle b^{-1}a^{-1}(bc)^2ab,\ a^{-2}ca^2caba^{-1} \right\rangle \cong 2 \times S_4,$$

which intersects $G(l) = \langle cb^2a^{-1}b^{-1}ab^2c,\ bacb^2aca^{-1}b,\ (ba^{-1})^2a^{-2}b^{-1}a^2\rangle \sim 4 : D_8$ in

$$H_{12} = \left\langle ca^2(cb^{-1}a^{-1})^2b,\ abaca^{-1}bcacb,\ ba^{-1}b^{-1}cb^{-1}a^{-1}b^{-1}ca^{-1} \right\rangle \cong 2 \times D_8.$$

The element $\sigma = ab^{-1}aca^2b \in N_{G\{l\}}(H_{12})$ swaps x with z and together with H_{12} generates $H_2 \cong 8 : 2^2$, which has ID $= [32, 43]$. The subgroup $H = \langle H_1, H_2\rangle \leq G$, with ID $= [1440, 5841]$, is isomorphic to $\mathrm{Aut}(S_6)$, and stabilises a 3-regular subgraph with 30 vertices, 45 edges, diameter 4 and girth 8. □

Chapter 3

An exceptional example related to the group $G_2(3)$

The purpose of this chapter is to describe a locally projective amalgam of type $(3,3)$ considered in [AS83; Iva18] and coming from a geometry discovered by Cooperstein [Coo89]. Before describing this amalgam, whose exceptionality mainly lies in the number of classes of *geometric subgraphs* contained in its universal cover graph, we present a classical example related to symplectic and orthogonal spaces.

3.1 Geometric subgraphs and a classical example

In this section we introduce the notion of a *geometric subgraph*, which plays an important role in the theory of locally projective graphs. These subgraphs were introduced in [Iva99, Chapter 9, § 5] for locally projective graphs of type $(n,2)$ defined more generally over $\mathrm{GF}(q)$, and for some classes of geometries, including the classical ones, they enable to reconstruct the elements of higher type. After recalling that for a locally projective graph Γ of type $(n,3)$ with respect to the action of a group $G \leq \mathrm{Aut}(\Gamma)$, every vertex x is equipped with a projective geometry π_x on $\Gamma(x)$ invariant under the action of $G(x)$, we give the following definition, following [Iva21].

Definition 5. *A connected subgraph $\Xi^{(k)}$ in Γ is said to be* geometric *at level k, where $1 \leq k \leq n-1$, whenever together with an edge it always contains the line on this edge, and the following conditions hold:*

- (i) *if $x \in \Xi^{(k)}$, then the set of neighbours $\Xi^{(k)}(x)$ of x in $\Xi^{(k)}$ is a k-dimensional subspace in π_x and the setwise stabiliser of $\Xi^{(k)}(x)$ in $G(x)$ stabilises $\Xi^{(k)}$;*
- (ii) *the stabiliser $X^{(k)}$ of $\Xi^{(k)}$ in G acts on $\Xi^{(k)}$ locally projectively of type $(k,3)$.*

The kernel of the action of $X^{(k)}$ on $\Xi^{(k)}$ will be denoted by $K^{(k)}$. For $k=1$, the geometric subgraphs are just the lines, while a geometric subgraph $\Xi^{(2)}$ at level 2 (called a *plane*) is regular of valency 6, and its stabiliser $X^{(2)}$ modulo its vertex-wise stabiliser $K^{(2)}$ is a completion of a Goldschmidt amalgam

$$\left\{X^{(2)}(x)/K^{(2)}, X^{(2)}\{l\}/K^{(2)}\right\}.$$

In general, geometric subgraphs might not exist, although in most cases a locally projective graph Γ contains at least one family of them, and the universal cover of Γ contains a complete set of geometric subgraphs for all levels [Iva18, Theorem 10.11].

In the second part of this section, following [Iva21] and [Iva18, Chapter 10, § 7], we present a classical example of a locally projective graph of type $(3,3)$ and describe its densely embedded and geometric subgraphs. For a similar example in higher dimension and with more details the reader is advised to consult [Iva04, Chapter 2].

Let V be a 6-dimensional GF(2)-vector space equipped with a non-degenerate symplectic form f, that is a map

$$f\colon V \times V \longrightarrow \mathrm{GF}(2)$$

satisfying the following conditions:

(a) $f(u+v,w) = f(u,w)+f(v,w)$ and $f(u,v+w) = f(u,v)+f(u,w)$ for all $u,v,w \in V$ (bilinear),

(b) $f(u,v) = f(v,u)$ for all $u,v \in V$ (symmetric),

(c) $f(u,u) = 0$ for all $u \in V$ (alternating),

(d) $f(u,v) = 0$ for all $v \in V$ implies $u = 0$ (non-degenerate).

The *dual polar graph* associated with the pair (V,f) is the graph Γ defined as follows: its vertices are the maximal totally isotropic subspaces of V with respect to f, that is the 3-dimensional subspaces on which the form f vanishes completely; two such subspaces are adjacent in Γ if and only if their intersection has codimension 1 in each of them. It is well known that the number of 3-dimensional subspaces of V is given by the Gaussian binomial coefficient

$$\binom{6}{3}_2 = \frac{(1-2^6)(1-2^5)(1-2^4)}{(1-2)(1-2^2)(1-2^3)} = 1395,$$

and that only $\prod_{i=0}^{2}(2^{3-i}+1) = 135$ of them are totally isotropic with respect to f (see, for example, [BCN89, Lemma 9.4.1]). The graph Γ, constructed with the aid of MAGMA [BCP97], is 14-regular with 135 vertices, 945 edges, diameter 3, girth 3, and it is locally projective of type $(3,3)$ with respect to $G = \mathrm{Aut}(\Gamma) \cong \mathrm{Sp}_6(2)$. The corresponding locally projective amalgam is $\{G(x), G\{l\}\}$, with $G(x) \sim 2^6 : \mathrm{L}_3(2)$ and $G\{l\} \sim 2_+^{1+4} : (S_3 \times S_4) \sim (2^2 \times 2_+^{1+4}) : (S_3 \times S_3)$ intersecting in a subgroup of shape $2_+^{1+4} : (2 \times S_4)$.

Now we choose a quadratic form q of plus type (see [Iva04, Chapter 1] or [Rot95, Chapter 8] for details) whose associated bilinear form is f and consider the subgraph Δ of Γ formed by the 3-dimensional subspaces totally singular with respect to q, i.e. those on which q vanishes totally. This subgraph is locally projective of type $(3,2)$ with respect to the group $H = \mathrm{Aut}(\Delta) \cong \mathrm{O}_6^+(2) \cong S_8$, and it is densely embedded in Γ, as verified with the code in Appendix A. The group H turns out to be a completion of the corresponding subamalgam $\{H_1, H_2; H_{12}\}$, where $H_1 \cong \mathrm{AGL}_3(2) \cong 2^3 : \mathrm{L}_3(2)$ has ID $= [1344, 11686]$, $H_2 \sim 2^4 : S_4 \sim 2_+^{1+4} : D_{12}$ has ID $= [384, 5602]$ and their intersection $H_{12} \sim 2^3 : S_4 \sim 2_+^{1+4} : S_3$ has ID $= [192, 1493]$.

The only interesting geometric subgraphs of Γ are those at level 2, which are generalised quadrangles of order $(2,2)$ associated with the group $\mathrm{Sp}_4(2) \cong S_6$, completion of the Goldschmidt G_3^1-amalgam. We conclude with Figure 3.1, which shows the neighbourhood of a vertex in Γ, with the seven lines containing it and the green part representing its intersection with the subgraph Δ.

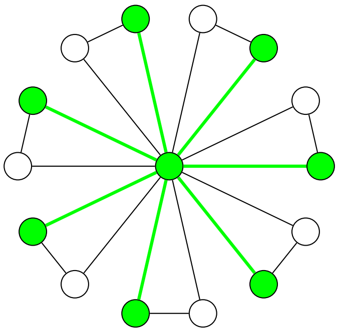

FIGURE 3.1: The neighbourhood of a vertex with the seven lines.

3.2 The octonion algebra over GF(3)

Since the amalgam we want to consider comes from a geometry for $G_2(3)$, we begin with the description of this group in its relation to the octonion algebra over GF(3) following [Wil09; Coh80; Bae02].

The octonions over GF(3)[1] form a non-commutative, non-associative, 8-dimensional unital algebra $\mathbb{O}$ of size 3^8 with basis $\{e_0 = 1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$. The most elementary way to construct this algebra is through its multiplication table, given for reference in Table 3.1, from which one can learn the following pieces of information:

- $e_1, \ldots, e_7$ are square roots of -1:

$$e_i^2 = -1 \quad \text{for} \quad 1 \leq i \leq 7,$$

- e_i and e_j anticommute when $i \neq j$:

$$e_i e_j = -e_j e_i = e_k$$

  whenever (i, j, k) is one of the 3-cycles $(1 + r, 2 + r, 4 + r)$, with i, j, k and r running through the integers modulo 7 and taking their values in $\{1, 2, \ldots, 7\}$,

- the multiplication is non-associative:

$$(e_1 e_2) e_3 = e_4 e_3 = -e_6 \quad \text{but} \quad e_1(e_2 e_3) = e_1 e_5 = e_6,$$

- the 'index cycling' identity holds:

$$e_i e_j = e_k \Longrightarrow e_{i+1} e_{j+1} = e_{k+1},$$

  i.e. the table is invariant under the map $\alpha \colon e_t \mapsto e_{t+1}$,

- the 'index doubling' identity holds:

$$e_i e_j = e_k \Longrightarrow e_{2i} e_{2j} = e_{2k}$$

---

[1] The same construction works over any field of characteristic not 2.

i.e. the table is invariant under the map $\beta\colon e_t \mapsto e_{2t}$,

- the table is also invariant under the map

$$\gamma\colon (e_1, e_2, e_3, e_4, e_5, e_6, e_7) \mapsto (e_1, e_4, -e_3, -e_2, e_6, -e_5, -e_7)$$

in the sense that $e_ie_j = e_k \Longrightarrow \gamma(e_i)\gamma(e_j) = \gamma(e_k)$.

	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
e_2	e_2	$-e_4$	-1	e_5	e_1	$-e_3$	e_7	$-e_6$
e_3	e_3	$-e_7$	$-e_5$	-1	e_6	e_2	$-e_4$	e_1
e_4	e_4	e_2	$-e_1$	$-e_6$	-1	e_7	e_3	$-e_5$
e_5	e_5	$-e_6$	e_3	$-e_2$	$-e_7$	-1	e_1	e_4
e_6	e_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	-1	e_2
e_7	e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	-1

TABLE 3.1: The octonion multiplication table.

The full multiplication table is conveniently encoded in the Fano plane, shown in Figure 3.2. The product of any two 'imaginary units' is given by the third one on the unique line connecting them, with the sign determined by the relative orientation.

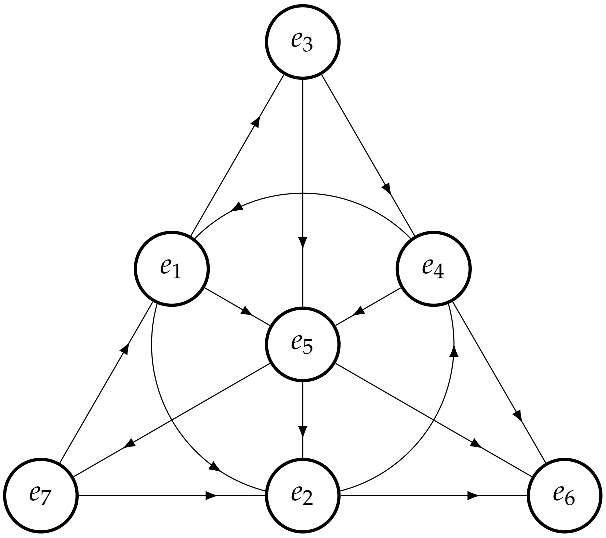

FIGURE 3.2: The octonion multiplication through the Fano plane.

The *conjugate* of an octonion $x = \sum_{i=0}^{7} \alpha_i e_i,\ \alpha_i \in \mathrm{GF}(3)$, is given by

$$\bar{x} = \alpha_0 e_0 - \sum_{i=1}^{7} \alpha_i e_i$$

so that conjugation $x \mapsto \bar{x}$, which is the GF(3)-linear map fixing $e_0 = 1$ and negating $e_1, \ldots, e_7$, is an antiautomorphism (i.e. $\overline{xy} = \bar{y}\bar{x}\ \forall x, y \in \mathbb{O}$) of order 2 (i.e. $\bar{\bar{x}} = x$ $\forall x \in \mathbb{O}$). We also define the *real part* of x by

$$\mathrm{Re}(x) = \tfrac{1}{2}(x + \bar{x}) = \alpha_0 e_0$$

and the *imaginary part* of x by

$$\mathrm{Im}(x) = \tfrac{1}{2}(x - \bar{x}) = \sum_{i=1}^{7} \alpha_i e_i$$

so that $\bar{x} = 2\,\mathrm{Re}(x) - x = x - 2\,\mathrm{Im}(x)$ is expressible as a linear combination of 1 and x. There is a natural non-singular quadratic form (the *norm*)

$$q \colon \mathbb{O} \longrightarrow \mathrm{GF}(3), \qquad x \mapsto x\bar{x} = \bar{x}x = \sum_{i=0}^{7} \alpha_i^2 \quad \forall x \in \mathbb{O}$$

which is multiplicative, i.e. $q(xy) = q(x)q(y)$ for any $x, y \in \mathbb{O}$, thus endowing $\mathbb{O}$ with the structure of a *composition algebra* over GF(3). The bilinear form associated to q is given by

$$f(x, y) := q(x + y) - q(x) - q(y) = 2\,\mathrm{Re}(x\bar{y}) = x\bar{y} + y\bar{x}$$

and therefore it is twice the usual inner product, under which $\{e_0 = 1, e_1, \ldots, e_7\}$ is an orthonormal basis of $\mathbb{O}$.

It is well known (see for example [Wil09; Con+85]) that $G = \mathrm{Aut}(\mathbb{O})$ is the exceptional group of Lie type $G_2(3)$, which is simple of order 4 245 696 $= 2^6 \cdot 3^6 \cdot 7 \cdot 13$. Of interest to us will be the set $B = \{\pm e_0, \pm e_1, \ldots, \pm e_7\}$ of the basis octonions and their additive inverses. This set, which we will call a *base*, is closed under multiplication and in fact it has the structure of a *Moufang loop*, which means that it is a quasigroup with an identity element which satisfies certain identities knows as the *Moufang laws*. Ignoring the signs for the moment, we see that the maps α, β and γ correspond respectively to the permutations $(1, 2, 3, 4, 5, 6, 7)$, $(1, 2, 4)(3, 6, 5)$ and $(2, 4)(5, 6)$, which generate $L_3(2)$. Thus there is a homomorphism from the stabiliser of B in G onto $L_3(2)$ whose kernel is a group of order 2^3, as we may change sign independently on e_1, e_2 and e_4, and then the other signs are determined. In fact, the resulting group is the unique[2] non-split extension $2^3 \cdot L_3(2)$ (see [Coh80; Cox46; Abb+99]).

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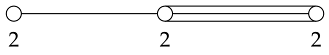
[2] To be precise, there is another group with this shape, but it is isomorphic to $2^2 \times \mathrm{SL}_2(7)$.

3.3 The Cooperstein geometry

The purpose of this section is to describe the *Cooperstein geometry*, a flag-transitive GAB[3] having $G_2(3)$ as automorphism group. This rank 3 geometry, explicitly constructed in [Coo89] making use of the octonion algebra $\mathbb{O}$ over GF(3), consists of three types of objects, called *points*, *lines* and *planes*, so that the residue of a point is a generalised hexagon, the residue of a line is a complete bipartite graph, and the residue of a plane is a projective plane, as shown in the following diagram.



Before describing the geometry, for an alternative and equivalent construction of which the reader is referred to [DHV05], we remind a few definitions. Generalised polygons, introduced by Tits [Tit59] in an attempt to find geometric models for simple groups of Lie type, have already been mentioned in Section 1.4, and reasonably comprehensive references for them can be found in [BM94; Mal11]. A *generalised d-gon* can be defined as a point-line geometry whose bipartite incidence graph has diameter d and girth $2d$. If we assume, to exclude trivial cases, that the geometry is *thick*, namely that each line contains at least three points and each point lies on at least three lines, then there are constants $s \geq 2$ and $t \geq 2$ such that each line contains exactly $s+1$ points and each point lies on exactly $t+1$ lines, and (s,t) is called the *order* of the generalised d-gon. The celebrated theorem of Feit and Higman [FH64] shows that a finite thick generalised d-gon can only exist when $d \in \{2,3,4,6,8\}$. Generalised digons (2-gons) are geometries whose incidence graphs are complete bipartite, while a generalised triangle (3-gon) is precisely a projective plane. As far as generalised hexagons (6-gons) are concerned, there are only two known infinite families of them with an order (s,t), each parametrised by a finite field. The family in which we are interested is comprised of the so-called *split Cayley hexagons* of order (q,q), which are related to the group $G_2(q)$ in the sense that they are rank 2 geometries defined by the two classes of maximal parabolic subgroups of $G_2(q)$. In particular, for $q=2$, as proved in [CT85], up to isomorphism there are exactly two generalised hexagons of order $(2,2)$. Each is the dual[4] of the other and is related to the group $G_2(2)$, hence the name $G_2(2)$*-hexagon* we use for one of them. There are 63 points and 63 lines, and each point (respectively line) is incident with exactly 3 lines (respectively points). For more details and a nice visual presentation of the $G_2(2)$-hexagons, see [Sch99] or [Bam+17, Fig. 1].

Following [Coo89; BC13], we now give a construction of the $G_2(2)$-hexagon as the geometry of non-isotropic points in a 3-dimensional unitary space. Let $K = \mathrm{GF}(9)$ and equip $V = K^3$ with the standard hermitian form[5] $h\colon V \times V \longrightarrow K$ defined as

$$h(x,y) = \sum_{i=1}^{3} x_i^3 y_i \qquad \text{for all } x,y \in V.$$

---

[3] Geometries that are almost buildings (GABs), introduced by Tits, are special geometries in which all rank 2 residues are generalised polygons.

[4] The *dual* of a point-line geometry is obtained by interchanging the roles of points and lines.

[5] The same construction works with any non-degenerate hermitian form.

Let P be the the set of non-isotropic points of the underlying projective space, that is, $P = \{\langle x\rangle \mid x \in V, h(x,x) \neq 0\}$, and define a graph on P as follows: $\langle x\rangle \sim \langle y\rangle$ if and only if $h(x,y) = 0$. Let L be the collection of maximal cliques in $\Gamma = (P, \sim)$, which have size three and correspond to orthonormal bases of V (up to scalar multiples for the basis vectors). Then, as shown in [BC13, Example 2.2.15], $(P, L, *)$, where * is the symmetrised containment, is a generalised hexagon.

We first collect some facts about the graph Γ, starting with its vertex set $V(\Gamma) = P$. The following Gaussian binomial coefficient counts the number of 1-dimensional subspaces of V

$$\binom{3}{1}_9 = \frac{1-9^3}{1-9} = 91,$$

among which $9^{3/2} + 1 = 28$ (see, for example, [BCN89, Lemma 9.4.1]) are totally isotropic with respect to h. With the aid of GAP [Gap] and MAGMA [BCP97], we check some other properties of Γ, shown in Table 3.2.

$\lvert V(\Gamma)\rvert$	63
$\lvert E(\Gamma)\rvert$	189
$G = \mathrm{Aut}(\Gamma)$	$G_2(2)$
diameter	3
girth	3
valency	6

TABLE 3.2: Some properties of the graph $\Gamma = (P, \sim)$.

If $x \in P$ is a point and $l \in L$ is a line, i.e. a maximal clique of Γ, we have:

$$G(x) \sim 2_+^{1+4} : S_3 \qquad \text{and} \qquad G\{l\} \sim 4^2 : D_{12},$$

with ID $= [192, 988]$ and ID $= [192, 956]$ respectively. These two subgroups are precisely the maximal parabolics of G (see, for example, [Wil09]), so that the $G_2(2)$-hexagon $(P, L, *)$ can be equivalently constructed as the coset graph of G with respect to $G(x)$ and $G\{l\}$, with Γ being (isomorphic to) one of the two connect components of its distance-2 graph.

We can now describe the Cooperstein geometry, following [Coo89] and retaining the notation of the previous section. Let $K = \langle e_0, e_1\rangle$, considered as the field with nine elements, and let $W = e_0^\perp = \langle e_1, \dots, e_7\rangle$, so that $q|_W$ is a non-degenerate quadratic form with maximal Witt index. Set $V = \langle e_2, \dots, e_7\rangle = W \cap e_1^\perp$, so that V becomes a 3-dimensional vector space over K by restriction of the multiplication μ of $\mathbb{O}$ to $K \times V$. Next define $h \colon V \times V \longrightarrow K$ to be $p \circ \mu$, where p is the projection of $\mathbb{O}$ onto K, so that h is a non-degenerate hermitian form on V with associated automorphism given by the restriction of octonion conjugation to K.

Let us now consider the following three sets of *points*, *lines* and *planes*, respectively:

$$\begin{aligned}P &= \langle e_1 \rangle^G = \{\langle w \rangle \mid w \in W,\ q(w) = 1\},\\ L &= \{\langle e_1 \rangle, \langle e_2 \rangle, \langle e_4 \rangle\}^G,\\ \Pi &= \{\langle e_1 \rangle, \langle e_2 \rangle, \langle e_3 \rangle, \langle e_4 \rangle, \langle e_5 \rangle, \langle e_6 \rangle, \langle e_7 \rangle\}^G.\end{aligned}$$

Then [Coo89, Theorem 5.1] shows that $\Gamma = (P \cup L \cup \Pi, *, \tau)$, where * is the symmetrised inclusion and τ the obvious type function, is a rank 3 geometry; moreover, if $p \in P$, $l \in L$, $\pi \in \Pi$, then Γ_π is a projective plane of order 2, Γ_p is a generalised hexagon and Γ_l is the complete bipartite graph $K_{3,3}$.

3.4 The amalgam $\mathcal{A}$ and the Cooperstein graph

The amalgam in which we are interested is comprised of the stabilisers of the objects in a maximal flag, as described in the previous section. We first begin with the rank 2 amalgam $\mathcal{A} := \{G_1, G_2; G_{12}\}$, whose members G_1 and G_2 are the following maximal subgroups of $G \cong \mathrm{G}_2(3)$:

- $G_1 \sim 2^3 \cdot \mathrm{L}_3(2)$,
- $G_2 \sim (2_+^{1+4} : 3^2) : 2 \sim 2_+^{1+4} : (3^2 : 2) \sim \mathrm{SL}_2(3) : S_4 \sim Q_8 \cdot (3 : S_4)$,
- $G_{12} := G_1 \cap G_2 \sim 2_+^{1+4} : S_3 \sim Q_8 : S_4 \sim 2^3 \cdot S_4$.

The group G_1 has ID $= [1344, 814]$ and, as already mentioned, is the stabiliser in G of the base $\{\pm e_0, \pm e_1, \dots, \pm e_7\}$. The second member G_2, with ID $= [576, 8282]$, is the stabiliser of a quaternion subalgebra of $\mathbb{O}$ and it is isomorphic to $\mathrm{SO}_4^+(3)$ (see [Wil09, Chapter 4, § 3.6]). Their intersection G_{12}, with ID $= [192, 1494]$, already appeared in Chapter 2 as a member of the Goldschmidt G_5-amalgam and is chosen such that $[G_1 : G_{12}] = 7$ and $[G_2 : G_{12}] = 3$.

Following [AS83], we provide a few details and remarks about the groups involved in our construction. If we denote by B a Sylow 2-subgroup of G, then $B \cong \mathrm{Hol}(D_8)$ has ID = [64, 134] and coincides with the Borel subgroup of the Goldschmidt G_4^1- and G_5-amalgams. Its centre $Z(B) \cong 2$ contains an involution w such that $G_2 = C_G(w)$ has two subnormal subgroups[6] $K_i \cong \mathrm{SL}_2(3)$, $1 \leq i \leq 2$. Let $Q_i := K_i \cap B \cong Q_8$ and let $W \cong 4^2$ be the largest abelian normal subgroup of B. Then $N := N_G(W)$, having shape $4^2 : D_{12} \sim 2^3 \cdot S_4$ and ID = [192, 956], is the first member of the Goldschmidt G_4^1- and G_5-amalgams.

Before we proceed, we remind some standard notation. If p is a prime number and G a finite group, $O_p(G)$ denotes the p-core of G, which is the largest normal p-subgroup of G, and $O^p(G)$ the smallest normal subgroup H of G such that G/H is a p-group. The group G_2 has four subgroups X_1, X_2, Y_1 and Y_2 of index 3 containing B, all of shape $2_+^{1+4} : S_3$. The subgroups X_1 and X_2 have ID = [192, 988] and $O^2(X_i) = K_i$, while Y_1 and Y_2 have ID = [192, 1494], with $O^2(Y_i) \sim 2_+^{1+4} : 3 \sim Q_8 : A_4$ and $O_2(O^2(Y_i)) = Q_1Q_2 \cong Q_8 \circ Q_8 \cong D_8 \circ D_8 \cong 2_+^{1+4}$. There is an outer automorphism α of G acting on B, W, Y_1 and Y_2, and interchanging X_1 and X_2. We have

---

[6] A subgroup H of a group G is termed *subnormal* if there exists a finite chain of subgroups of G, each one normal in the next, beginning at H and ending at G.

that

$$M_i = \langle X_i, N \rangle_G \cong \mathrm{G}_2(2) \cong \mathrm{U}_3(3):2,$$

and evidently α interchanges M_1 and M_2. Finally,

$$G_1 = \langle Y_1, N \rangle_G \cong 2^3 \cdot \mathrm{L}_3(2) \qquad \text{and} \qquad \langle Y_2, N \rangle_G = G \cong \mathrm{G}_2(3).$$

The triple $\{G_1, G_2, M_i\}$ for $i = 1$ or 2 yields the Cooperstein geometry, while the choice $\{M_1, G_2, M_2\}$ leads to another geometry for $\mathrm{G}_2(3)$, the third listed in [AS83, Table 1].

Using the functions `Amalgams` and `Simple` [Can05] and `Goldschmidt` in Appendix B, we verify that there is a unique isomorphism class of (simple) amalgams having the type of $\mathcal{A}$, and we find a presentation for its universal completion group $G_1 *_{G_{12}} G_2$. The coset graph $\Gamma = \Gamma(G, G_1, G_2, G_{12})$ has 10530 vertices, 22113 edges, diameter 10 and girth 12. The 14-regular connected component of the distance-2 graph of Γ is locally projective of type $(3,3)$ with respect to the action of $G \cong \mathrm{G}_2(3)$, and we will call it the *Cooperstein graph*. Following [Iva18, Chapter 10, § 6], we notice that in the Cooperstein graph there are three G-orbits of planes: the representatives of two of them are isomorphic to the point graph of the $\mathrm{G}_2(2)$-generalised hexagon (realising the Goldschmidt G_4^1-amalgam), while the representatives of the third orbit realise the Goldschmidt G_5-amalgam.

The full automorphism group of Γ, which is $\mathrm{Aut}(G) \cong \mathrm{G}_2(3):2$, leads to another locally projective action [Iva18, Chapter 10, § 6.3], permutes the two orbits of planes and, when lifted to an automorphism of the universal cover of Γ, stabilises the third one, thus realising the Goldschmidt G_5^1-amalgam.

3.5 Some presentations following Goldschmidt

Throughout this subsection we continue the earlier notation for the groups introduced above and, following Goldschmidt [Gol80], we give a presentation of the amalgam $\mathcal{A}$ together with its four subamalgams listed below

$$\mathcal{B}_1 = \{N, X_1; B\}, \quad \mathcal{B}_2 = \{N, X_2; B\}, \quad \mathcal{B}_3 = \{N, Y_1; B\}, \quad \mathcal{B}_4 = \{N, Y_2; B\}.$$

We start with the following two elements of G, realised as a group of 8×8 matrices over $\mathrm{GF}(3)$:

$$a = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 & 2 & 0 & 0 & 2 \\ 1 & 2 & 2 & 2 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 2 & 1 & 2 & 2 & 1 \\ 1 & 2 & 0 & 2 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 2 & 2 & 2 & 1 & 1 & 2 & 2 & 0 \\ 2 & 2 & 1 & 2 & 2 & 1 & 1 & 0 \\ 1 & 2 & 2 & 0 & 2 & 2 & 2 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 2 & 2 & 1 \\ 2 & 1 & 0 & 0 & 1 & 2 & 0 & 2 \\ 1 & 2 & 2 & 2 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

One can check that a and b have both order 4, commute and therefore generate the subgroup $W \cong 4^2$, whose automorphism group is used by Goldschmidt [Gol80] to define the amalgams of class 4 and 5.

The automorphisms of W which will play an important role in our construction are the following maps:

$$s : \begin{cases} a \mapsto b \\ b \mapsto a \end{cases}$$
$$t : \begin{cases} a \mapsto a^{-1} = a^3 \\ b \mapsto b^{-1} = b^3 \end{cases}$$
$$x : \begin{cases} a \mapsto b \\ b \mapsto a^{-1}b^{-1} = a^3b^3 \end{cases}$$

which are realised as the following matrices

$$s = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 2 & 2 & 0 & 0 & 1 \\ 1 & 2 & 0 & 1 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 & 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 1 & 2 & 2 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 & 2 & 1 \end{pmatrix}, \qquad t = \begin{pmatrix} 2 & 2 & 2 & 2 & 0 & 0 & 0 & 2 \\ 2 & 2 & 1 & 1 & 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 & 1 & 1 & 1 & 2 \\ 2 & 1 & 2 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 & 2 \\ 1 & 2 & 1 & 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 & 2 & 2 & 2 & 2 \end{pmatrix},$$

$$x = \begin{pmatrix} 2 & 2 & 0 & 2 & 1 & 2 & 2 & 2 \\ 2 & 2 & 0 & 1 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 & 1 & 2 & 0 \\ 2 & 1 & 0 & 2 & 2 & 2 & 2 & 1 \\ 2 & 1 & 0 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 2 & 2 & 0 & 0 & 2 \\ 0 & 0 & 2 & 1 & 1 & 1 & 2 & 0 \end{pmatrix}.$$

The elements s and t are involutions, while x has order 3, and together they generate a subgroup isomorphic to D_{12}. The following set of relators

$$R_B = \left\{a^4, b^4, [a,b], s^2, t^2, [s,t], a^s b^{-1}, b^s a^{-1}, a^t a, b^t b\right\}$$

gives a presentation of the group

$$B = \langle a, b, s, t \mid R \rangle \sim 4^2 : 2^2$$

which, as explained in [Gol80, p. 390], admits also the following description

$$B = \langle Q_1 Q_2, bt \rangle \sim 2_+^{1+4} : 2,$$

where $Q_1 = \left\langle abs, sa^2 \right\rangle \cong Q_8$, $Q_2 = \left\langle ab, tsa^2 \right\rangle \cong Q_8$, $Q_1Q_2 \cong Q_8 \circ Q_8 \cong 2_+^{1+4}$, $(bt)^2 = 1$, and $Z(B) = \left\langle a^2b^2 \right\rangle \cong 2$.

By adjoining x and the following set of relators

$$R_N = \left\{x^3, xx^s, [x,s]^3, [x,t], a^x b^{-1}, b^x ba\right\},$$

we obtain a presentation of the group

$$N = \langle a,b,s,t,x \mid R_B \cup R_N \rangle \sim 4^2 : D_{12}.$$

Next we find presentations for the groups X_1, X_2, Y_1 and Y_2, by describing the action of an element of order 3 on the generators abs, sa^2, ab, tsa^2 and bt of the common subgroup B:

$$\begin{aligned} Y_1 &= \langle a,b,s,t,y \mid R_{Y_1} \rangle\,, \\ X_1 &= \langle a,b,s,t,z \mid R_{X_1} \rangle\,, \\ Y_2 &= \langle a,b,s,t,u \mid R_{Y_2} \rangle\,, \\ X_2 &= \langle a,b,s,t,v \mid R_{X_2} \rangle\,, \end{aligned}$$

where

$$\begin{aligned} R_{Y_1} &= R_B \ \cup \ \left\{y^3, (abs)^y ba^3, (sa^2)^y sba, (tsa^2)^y tsba^3, (ab)^y tsa^2, y^{bt} ytb^2\right\}, \\ R_{X_1} &= R_B \ \cup \ \left\{z^3, (abs)^z ba^3, (sa^2)^z sb^3a^3, (tsa^2)^z tsb^2, (ab)^z b^3a^3, z^{bt} z\right\}, \\ R_{Y_2} &= R_B \ \cup \ \left\{u^3, (abs)^u ba^3, (sa^2)^u sb^3a^3, (tsa^2)^u b^3a^3, (ab)^u tsba^3, u^{bt} utsb^2\right\}, \\ R_{X_2} &= R_B \ \cup \ \left\{v^3, (abs)^v sb^3a^3, (sa^2)^v sb^2, (tsa^2)^v tsba^3, (ab)^v tsa^2, v^{bt} vtsba^3\right\}. \end{aligned}$$

The group Y_1 forms with N the subamalgam $\mathcal{B}_3$, which has the same type of the Goldschmidt G_5-amalgam. However, as described in [Gol80, p. 391], $\mathcal{B}_3$ is not simple as the subgroup $\langle a^2, b^2, t \rangle \cong 2^3$ is normal in both of its members, and has G_1 as a completion group:

$$G_1 = \langle N, Y_1 \rangle = \left\langle a,b,s,t,x,y \mid R_N \cup R_{Y_1} \cup \{byx^2yxy^2xa^3\} \right\rangle \cong 2^3 \cdot \mathrm{L}_3(2).$$

The groups X_1 and X_2 form with N the subamalgams $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively. These can be identified with the amalgam of maximal parabolics in $\mathrm{G}_2(2)$, and with the Goldschmidt G_4^1-amalgam, with completions given, respectively, by

$$\begin{aligned} M_1 &= \langle N, X_1 \rangle = \left\langle a,b,s,t,x,z \mid R_N \cup R_{X_1} \cup \{(x^2z^2)^3 b^2 ax (z^2x^2)^2 z\} \right\rangle \cong \mathrm{G}_2(2), \\ M_2 &= \langle N, X_2 \rangle = \left\langle a,b,s,t,x,v \mid R_N \cup R_{X_2} \cup \{(x^2v^2)^3 b^3 x^2 v^2 a^3 x v^2 x v\} \right\rangle \cong \mathrm{G}_2(2). \end{aligned}$$

The groups Y_2 and N, which generate in G the whole group

$$\left\langle a,b,s,t,x,u \mid R_N \cup R_{Y_2} \cup \{(x^2u^2)^{12}, (xu^2)^2(x^2u^2)^4 xa^2u^2(x^2u)^3 xux^2u\} \right\rangle \cong \mathrm{G}_2(3),$$

are the members of the subamalgam $\mathcal{B}_4$, which is the Goldschmidt G_5-amalgam. Altogether the four groups X_1, X_2, Y_1 and Y_2 generate the group G_2, which intersects G_1 in $G_{12} = N$:

$$G_2 = \langle X_1, X_2, Y_1, Y_2 \rangle = \langle a,b,s,t,y,z,u,v \mid R_{X_1} \cup R_{X_2} \cup R_{Y_1} \cup R_{Y_2} \ \cup R_2 \rangle\,,$$

where

$$R_2 = \{v^z v^2, za^2b^2u^2(sty^2u)^{-1}, za^3b^2st(by^2zu^2)^{-1}, zsu^2(styz^2)^{-1}, z^{-1}avu^{-1}va\}.$$

Finally

$$\langle G_1, G_2 \rangle = \langle a, b, s, t, x, y, z, u, v \mid R_{G_1} \cup R_{G_2} \cup R_G \rangle \cong \mathrm{G}_2(3),$$

where R_{G_1} and R_{G_2} denote respectively all the relators of G_1 and G_2, and

$$\begin{aligned} R_G = \{ & xby^2x^2y^2b^3y^2x^2y^2byxyx^2(avu^2)^{-2}, \\ & y(uv^2a^2)^{-1}, \\ & xbx^2a^3, \\ & (xy)^2x^2(avu^2a^2)^{-1}, \\ & xu^2x^2v^2(x^2u^2)^2x^2v^2b^3xu^2x^2vx^2u, \\ & x^2vxv^2x^2v^2xvx^2v^2xv^2\}. \end{aligned}$$

The first four relators describe the intersection G_{12}, while the last two are needed to obtain $\mathrm{G}_2(3)$.

3.6 Incorporating G_3

In this section we address a question that has its origin in the observation that $\mathcal{B}_4$ is isomorphic to an amalgam realised in the Mathieu group M_{12}. It is well known [Con+85; Wil09] that M_{12} possesses two conjugacy classes of maximal subgroups of order 192. One class consists of the 495 stabilisers of tetrads, i.e. 4-subsets of the 12-set upon which M_{12} acts, and they have ID = [192, 1494]; the subgroups in the other class stabilise a partition of the 12-set into three 4-subsets and have ID = [192, 956].

We aim to extend $\mathcal{A}$ to a rank 3 amalgam by adjoining as a third member the stabiliser G_3 of a geometric subgraph at level 2. As we have already mentioned, if G_3 is chosen to be M_1 or M_2, then the triple $\{G_1, G_2, G_3\}$ is the amalgam of maximal parabolics in $\mathrm{G}_2(3)$ associated to the Cooperstein geometry. We consider the case $G_3 \cong \mathrm{M}_{12}$ and construct the universal completion of the corresponding amalgam. In order to do that we first need to recall the following quite old result [Neu54, Theorem 4.1], which describes some special subgroups of a free amalgamated product.

Theorem 10 (Hanna Neumann, 1948). *Let $P = G_1 *_{G_{12}} G_2$ be the free product of groups G_1 and G_2 with an amalgamated subgroup G_{12}. In G_1 (resp. G_2) let there be given a subgroup H_1 (resp. H_2) which intersects G_{12} in a fixed subgroup H_{12},*

$$H_i \leq G_i, \qquad H_i \cap G_{12} = H_{12}, \qquad i = 1, 2.$$

Then the subgroup of P generated by H_1 and H_2 is their free product with amalgamated H_{12},

$$\langle H_1, H_2 \rangle_P \cong H_1 *_{H_{12}} H_2.$$

If, in particular, the subgroups H_1 and H_2 have trivial intersection with G_{12}, then they generate their ordinary free product.

We apply the result above to the case where P is the universal completion group of $\mathcal{A}$ and H_1, H_2 the two members of the subamalgam $\mathcal{B}_4$. The group $G \cong \mathrm{G}_2(3)$ is a completion of $\mathcal{A}$, so that there is a surjective homomorphism

$$\varphi\colon \widehat{G} := G_1 *_{G_{12}} G_2 \longrightarrow G,$$

which can be restricted to the stabiliser $\langle N, Y_2\rangle_{\widehat{G}} \cong N *_B Y_2$ of the geometric subgraph in the universal completion group, yielding

$$\psi := \varphi|_{N *_B Y_2}\colon N *_B Y_2 \longrightarrow \mathrm{M}_{12}.$$

By taking its kernel and its normal closure, we notice that

$$\left\langle \mathrlap{\overbrace{}^{N}}x,\underbrace{a,b,s,t,u}_{Y_2} \mid R_N \cup R_{Y_2} \cup \{r_1\} \right\rangle \cong \mathrm{M}_{12},$$

where $r_1 = x^{-1}u^{-1}xu^{-1}x^{-1}u^{-1}x^{-1}u^{-1}b^{-1}xuxu^{-1}x^{-1}u^{-1}x^{-1}u$, and

$$\left\langle \mathrlap{\overbrace{}^{N}}x,\underbrace{a,b,s,t,u}_{Y_2} \mid R_N \cup R_{Y_2} \cup \{r_2, r_3\} \right\rangle \cong \mathrm{G}_2(3),$$

where $r_2 = (x^2u^2)^{12}$ and $r_3 = (xu^2)^2(x^2u^2)^4xa^2u^2(x^2u)^3xux^2u$.

We addressed the problem of determining structural information about the following finitely presented groups:

$$X = \langle a, b, s, t, x, y, z, u, v \mid R_{G_1} \cup R_{G_2} \cup \{r_1\}\rangle$$

and

$$Y = \langle a, b, s, t, x, y, z, u, v \mid R_{G_1} \cup R_{G_2} \cup \{r_2, r_3\}\rangle,$$

the former being the universal completion group of the amalgam $\{G_1, G_2, G_3\}$ with $G_3 \cong \mathrm{M}_{12}$. The main result of this chapter is given by the following theorem.

Theorem 11. *The group X is perfect, i.e. $X = X'$, and has no simple quotients up to order 10^8, and the group Y is not isomorphic to* $\mathrm{G}_2(3)$.

Proof. With the aid of MAGMA [BCP97], we checked that X is perfect and has no simple quotients up to order 10^8:

```
> load "X_and_Y";
Loading "X_and_Y"
> IsPerfect(X);
true
> x:=Simplify(X);
> S := SimpleQuotients(x,1,10^8);
> #S;
0
```

but unfortunately we were not able to say more about X, in particular if it trivial or not.

More successful was the analysis of the group Y, which we managed to prove not isomorphic to $G_2(3)$. The strategy adopted, which comes from an idea suggested by D. Holt [7], is described as follows. For a homomorphism f from a finitely presented group G onto a transitive permutation group H, the MAGMA [BCP97] command `sub< G | f >`, which is very useful and does not appear to be widely known, returns the inverse image under f of the point stabiliser in H. In our case MAGMA [BCP97] finds two such surjective homomorphisms $Y \longrightarrow H \cong G_2(3)$, where H is chosen to be `PrimitiveGroup(2808,1)`, and for each of them we compute the elementary divisors of the quotient groups V/V' and V'/V'', where V is the corresponding subgroup of Y of index 2808. In both cases we find $V/V' \cong 3$ and $V'/V'' \cong 7$, thus contradicting the information contained in Table 3.3. This table lists the 10 conjugacy classes of maximal subgroups of $G_2(3)$; for each class we choose a representative A, whose index and derived length are shown in the second and third column respectively, while in the last two columns we give the isomorphism type of A/A' and A'/A''.

#	Index	Derived length	A/A'	A'/A''
1	351	1	2	1
2	351	1	2	1
3	364	6	2	3
4	364	6	2	3
5	378	1	2	1
6	378	1	2	1
7	2808	1	3	1
8	3159	0	1	1
9	3888	0	1	1
10	7371	4	2	3^2

TABLE 3.3: The conjugacy classes of maximal subgroups of $G_2(3)$.

We conclude with the MAGMA [BCP97] output which shows the result presented above.

```
YY := Simplify(Y);
h := Homomorphisms(YY,PrimitiveGroup(2808,1)); /* #h; 2 */
/* CompositionFactors(Image(h[1]));
    G
```

---

[7] D. Holt, private communication, 2019.

```
     |  G(2, 3)
    1
*/
V := sub< YY | h[1] >;
/* AQInvariants(V); [ 3 ] */
V := Rewrite(YY,V); D:=DerivedGroup(V);
/* AQInvariants(D); [ 7 ] */

/* CompositionFactors(Image(h[2]));
    G
     |  G(2, 3)
    1
*/
V := sub< YY | h[2] >;
/* AQInvariants(V); [ 3 ] */
V := Rewrite(YY,V); D:=DerivedGroup(V);
/* AQInvariants(D); [ 7 ] */
```

□

3.7 One is not enough

This last section, in which we continue our earlier notation, is devoted to answer a question about another geometry related to the group $\mathrm{G}_2(3)$ described in [HS05]. Here the authors study the amalgam

$$\widehat{\mathcal{B}} := \left\{\widehat{G}_1, \widehat{G}_2, M_1\right\},$$

where $\widehat{G}_1 \sim (2^3 \cdot \mathrm{L}_3(2)) : 2$ and $\widehat{G}_2 \sim 2_+^{1+4} \cdot (S_3 \times S_3)$ are maximal subgroups of $\mathrm{Aut}(\mathrm{G}_2(3))$. This amalgam corresponds to the amalgam

$$\mathcal{B} := \{G_1, G_2, M_1, M_2\}$$

of maximal subgroups of $G \cong \mathrm{G}_2(3)$.

The cosets geometries corresponding to the amalgams $\{G_1, G_2, M_1\}$ and $\{G_1, G_2, M_2\}$ are isomorphic to the Cooperstein geometry, which is not simply connected. Using the presentations of the groups in Section 5, we first check that $\mathcal{U}(\mathcal{B}) \cong \mathrm{G}_2(3)$, and then we prove that the universal completion of the amalgam obtained from $\mathcal{B}$ by removing one $M_i \cong \mathrm{G}_2(2)$, is not isomorphic to $\mathrm{G}_2(3)$[8].

Theorem 12. *The universal completion of the amalgam $\mathcal{B}$ is the following finitely presented group*

$$U = \langle a, b, s, t, x, y, z, u, v \mid R_U \rangle,$$

where

$$R_U = R_{G_1} \cup R_{G_2} \cup \{(x^2z^2)^3b^2ax(z^2x^2)^2z, (x^2v^2)^3b^3x^2v^2a^3xv^2xv\}.$$

The group U is isomorphic to $\mathrm{G}_2(3)$.

---

[8] In [Gra+08] it is wrongly stated that the universal completion group of the amalgam $\{G_1, G_2, M_i\}$ is the group $\mathrm{G}_2(\mathbb{Q}_2)$.

Proof. We construct the group U in MAGMA [BCP97] and apply to it the command `Simplify`, which returns a new group isomorphic to U defined by a simpler presentation. We check that the order of this group is 4 245 696 and we explicitly find an isomorphism with $\mathrm{G}_2(3)$.

```
UU := Simplify(U);
/* Order(UU); 4245696 */
S := SimpleQuotients(Simplify(UU),1,10^8);
/* #S; 1 */
c := CompositionFactors(Image(S[1][1]));
/* c;
    G
     |  G(2, 3)
    1
*/
K := Kernel(S[1][1]);
/* Order(K); 1 */
```

□

By removing one of the last two relators in R_U, we get a group which is not isomorphic to $\mathrm{G}_2(3)$.

Theorem 13. *The universal completions of the amalgams* $\{G_1, G_2, M_1\}$ *and* $\{G_1, G_2, M_2\}$ *are, respectively, the following finitely presented groups*

$$T_1 = \langle a, b, s, t, x, y, z, u, v \mid R_{G_1} \cup R_{G_2} \cup \{(x^2z^2)^3b^2ax(z^2x^2)^2z\}\rangle$$

and

$$T_2 = \langle a, b, s, t, x, y, z, u, v \mid R_{G_1} \cup R_{G_2} \cup \{(x^2v^2)^3b^3x^2v^2a^3xv^2xv\}\rangle .$$

Each of them is not isomorphic to $\mathrm{G}_2(3)$.

Proof. We adopt the same technique used in Section §6 for the group Y and for each $i \in \{1, 2\}$ we find two homomorphisms $T_i \longrightarrow H \cong \mathrm{G}_2(3)$. In both cases we find a contradiction with Table 3.3. □

Chapter 4

Another 'special' example of a locally projective graph of type (3,3)

The list of all known locally projective graphs of type $(n, 3)$ is shown in [Iva21, Table 2], from which one can see that most of them admit densely embedded subgraphs. Among the few exceptions, the graph considered in Chapter 3, and the one described here and constructed in [GLP05]. The first sections of this chapter are devoted to the description of such a graph, whose automorphism group, $\Omega_8^+(2) : S_3$, is a maximal subgroup of the Fisher group Fi_{22} (see [Con+85]). In the final section we address the problem of finding a *geometric presentation* of a certain rank 3 amalgam related to a geometry coming from the locally projective graph, and we prove computationally the simple connectedness of the geometry.

4.1 Construction and some properties

In this section we describe a 3-arc transitive locally projective graph of type (3,3) arising from a biregular graph of valency $\{3, 7\}$ constructed in [GLP05]. We first provide a brief outline of the geometry associated with the 8-dimensional orthogonal groups. For more details, the reader may refer to [Kle87].

Let V be an 8-dimensional vector space over $\mathrm{GF}(2)$ equipped with a non-degenerate quadratic form Q, that is, $Q\colon V \longrightarrow \mathrm{GF}(2)$ is a function such that

$$Q(\lambda v) = \lambda^2 Q(\lambda) \quad \text{for all } \lambda \in \mathrm{GF}(2) \text{ and } v \in V$$

and the associated bilinear form

$$\begin{aligned} &V \times V \longrightarrow \mathrm{GF}(2) \\ &(v, w) \mapsto Q(v + w) - Q(v) - Q(w) \end{aligned}$$

is non-degenerate. A subspace W of V is called *totally singular* if $Q(v) = 0$ for all vectors $v \in W$. We let Q have *maximal Witt index*, that is the maximal totally singular subspaces of V have dimension 4.

We denote the set of all totally singular 1-spaces of V by $\mathcal{P}$ and the set of totally singular 4-spaces by $\mathcal{S}$. The simple group $T \cong \Omega_8^+(2) \cong D_4(2)$ [1] acts transitively on

---

[1] In [Con+85] this group is denoted by $\mathrm{O}_8^+(2)$.

$\mathcal{P}$ and has two orbits $\mathcal{S}_1$ and $\mathcal{S}_2$ on $\mathcal{S}$. Two totally singular 4-spaces lie in the same T-orbit if and only if their intersection has even dimension. Let

$$\begin{aligned}
\Delta_1 &= \{\{U,S,R\} \mid U \in \mathcal{P},\ S \in \mathcal{S}_1,\ R \in \mathcal{S}_2,\ \dim(S \cap R) = 3 \text{ and } U < S \cap R\},\\
B_1 &= \{\{U,S\} \mid U \in \mathcal{P},\ S \in \mathcal{S}_1 \text{ and } U < S\},\\
B_2 &= \{\{S,R\} \mid S \in \mathcal{S}_1,\ R \in \mathcal{S}_2, \text{ and } \dim(S \cap R) = 3\},\\
B_3 &= \{\{U,R\} \mid U \in \mathcal{P},\ R \in \mathcal{S}_2 \text{ and } U < R\},
\end{aligned}$$

and let $\Delta_2 = B_1 \cup B_2 \cup B_3$. We now define Γ to be the bipartite graph with vertex set $\Delta_1 \cup \Delta_2$, such that two vertices $\{U,S,R\}$ and $\{X,Y\}$ are adjacent if and only if $\{X,Y\} \subseteq \{U,S,R\}$.

It is easy to check that there are 270 totally singular 4-spaces in V and that these are divided equally between $\mathcal{S}_1$ and $\mathcal{S}_2$. Moreover, we have the following numerology:

$$|\Delta_1| = 14175, \quad |B_1| = |B_2| = |B_3| = 2025, \quad \text{so that} \quad |\Delta_2| = 6075.$$

Each vertex in Δ_1 is adjacent to precisely three vertices in Δ_2 while every vertex in Δ_2 is adjacent to 7 vertices in Δ_1. Thus Γ is biregular of valency $\{3,7\}$.

Table 4.1 summarises some properties of the graph Γ, which we checked using MAGMA [BCP97].

$\lvert V(\Gamma)\rvert$	20250
$\lvert E(\Gamma)\rvert$	42525
$G = \mathrm{Aut}(\Gamma)$	$\Omega_8^+(2) : S_3$
diameter	12
girth	8

TABLE 4.1: Some properties of the graph Γ.

Given a vertex v of valency 3 and an adjacent vertex w of valency 7, we have

$$G(v) \sim [2^{11}] : (S_3 \times S_3), \qquad G(w) \sim [2^9] : (2 \times \mathrm{L}_3(2)), \qquad G(v,w) \sim [2^{11}] : D_{12},$$

where the normal subgroup in each semidirect product is not elementary abelian.

The distance factors are as follows:

- $G(v)/G_1(v) \cong S_3$;
- $G_1(v)/G_2(v) \sim 2^6 : S_3$;
- $G_2(v)/G_3(v) \cong 1$;
- $G_3(v)/G_4(v) \cong 2^3$;
- $G_4(v)/G_5(v) \cong 1$;
- $G_6(v) \cong 1$.

- $G(w)/G_1(w) \cong \mathrm{L}_3(2)$;
- $G_1(w)/G_2(w) \cong 2$;
- $G_2(w)/G_3(w) \cong 2^6$;
- $G_3(w)/G_4(w) \cong 1$;
- $G_5(w) \cong 1$.

where $G_1(v)/G_2(v) \sim 2^6 : S_3 \sim 2^4 : S_4$ has ID $= [384, 20164]$.

4.2 The distance-2 graph of Γ and the graph Ξ

As we did already for the Goldschmidt amalgams in Chapter 2, we carry on our analysis in the distance-2 graph of Γ. Since Γ is connected and bipartite, the distance-2 graph of Γ has two connected components, and let Ξ be the one containing all the vertices of valency 7. Then Ξ is locally projective of type $(3,3)$ and some of its properties are summarised in Table 4.2.

$\lvert V(\Xi)\rvert$	6075
$\lvert E(\Xi)\rvert$	42525
$G = \mathrm{Aut}(\Xi)$	$\Omega_8^+(2) : S_3$
valency	14
diameter	6
girth	3

TABLE 4.2: Some properties of the graph Ξ.

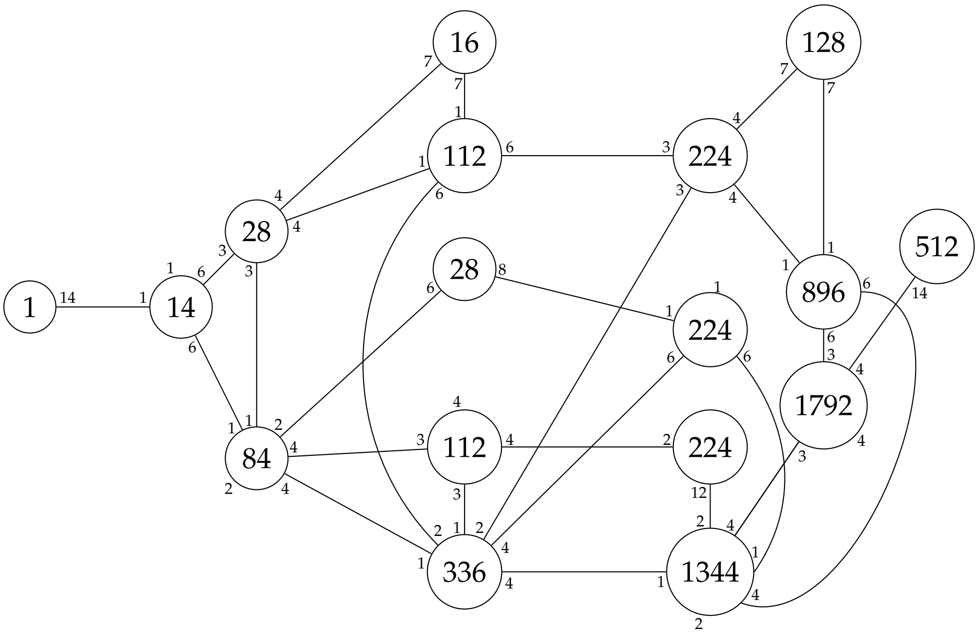


FIGURE 4.1: The distance diagram of Ξ.

Figure 4.1 gives the *distance diagram* of the graph Ξ according to the orbits of a vertex stabiliser as determined using MAGMA [BCP97]. Each orbit of $G(x)$ on $V(\Xi)$ is denoted by a circle containing the number of vertices in the orbit. An edge from an orbit S to an orbit T with number a attached at the end connected to S means that each vertex in S is adjacent to a vertices in T. The remaining number next to the orbit S is the number of vertices of S adjacent to a fixed vertex of S. If this number is zero, then we do not write anything.

4.3 The subgraph Λ of Ξ

We now construct a subgraph of Γ, following the notation in [GLP05]. We fix a totally singular 2-space $L \in \mathcal{L}$ and consider the vertices of Γ incident to L. There are three totally singular 1-spaces $U_1, U_2, U_3 \in \mathcal{P}$ contained in L and six totally singular 4-spaces containing L: $S_1, S_2, S_3 \in \mathcal{S}_1$ and $R_1, R_2, R_3 \in \mathcal{S}_2$.
These nine subspaces together with the twenty-seven pairs $\{U_i, S_j\}$, $\{U_i, R_k\}$, $\{S_j, R_k\}$ with $1 \leq i, j, k \leq 3$ form respectively the vertices and the edges of a complete tripartite graph $K_{3,3,3}$, which is 6-regular with automorphism group of shape $(S_3 \times S_3 \times S_3) : S_3$. We now define K to be the subgraph of Γ induced on the subset of $V(\Gamma)$ whose elements are the triples of vertices of $K_{3,3,3}$ taken from pairwise different parts (vertices of valency 3) and the edges of $K_{3,3,3}$ (vertices of valency 7). Since K is bipartite, we isolate again the vertices of valency 7 by considering the distance two graph of K, and define Λ to be corresponding subgraph of Ξ.

4.4 The amalgam $\mathcal{A}$ and its presentation

In this section we construct a rank 3 amalgam $\mathcal{A} = \{G_1, G_2, G_3\}$ inside the group $G = \mathrm{Aut}(\Xi) \sim \Omega_8^+(2) : S_3$, where the three members are defined as follows:

- $G_1 \sim [2^9] : (2 \times \mathrm{L}_3(2))$ is the stabiliser in G of a vertex of Ξ;
- $G_2 \sim [2^{11}] : (S_3 \times S_3)$ is the stabiliser in G of a line-triangle of Ξ;
- $G_3 \sim (2_+^{1+8} : (S_3 \times S_3 \times S_3)) : S_3$ is the stabiliser in G of the subgraph Λ.

In order to consider the correct intersections $G_{ij} := G_i \cap G_j$ $(1 \leq i < j \leq 3)$ shown in Table 4.3, where the entry in the ith row and jth column is the index $[G_i : G_{ij}]$, we proceed as follows.

	G_{1i}	G_{2i}	G_{3i}
G_1	1	7	7
G_2	3	1	3
G_3	27	27	1

TABLE 4.3

We fix $\{U, S, R\} \in \Delta_1$ and look for the totally singular 2-spaces L containing U and contained in S and R. It turns out that there are three such L and, if $T \cong \Omega_8^+(2)$, we have

$$|(T_S \cap T_R) \cap T_L| = 12288, \qquad |(T_U \cap T_S \cap T_R) \cap T_L| = 4096,$$

where T_U, T_S and T_R have all shape $2^6 : A_8$, while $T_L \sim 2^{1+8}_+ : (S_3 \times S_3 \times S_3)$.

We now give a *geometric presentation* of the amalgam $\mathcal{A}$, by which we mean the following: a set of generators and a set of relations such that, for every member G_i of the amalgam, the generators contained in G_i together with the relations only involving the elements of G_i constitute a presentation of G_i. Tits's lemma [Tit86] (see also [Pas94, Theorem 12.28]) provides a geometric way to prove that certain groups can be identified as universal completion groups of certain amalgams. More precisely, the universal completion group of the amalgam of the parabolic subgroups of a group G acting flag-transitively on a geometry Γ equals G if and only if Γ is simply connected. We apply Tits's lemma to the amalgam $\mathcal{A}$ and in Theorem 14 show that the corresponding geometry is simply connected.

We start with $G_{123} := G_1 \cap G_2 \cap G_3$ and extend it to G_{12}, G_{13} and G_{23} by adjoining in each case an element of order 3 together with a few extra relations. The group G_{123} admits various descriptions as a split and non-split extension, among which

$$2^7 : (2 \wr 2^2) \sim (2^2 \times 2^{1+4}_+) \cdot (D_8 \times D_8) \sim 2^6 : (D_8 \wr 2) \sim 2^5 \cdot ((2^2 \times D_8) : D_8),$$

and can be presented as follows

$$G_{123} = \langle a, b, c, d \mid R \rangle ,$$

where

$$\begin{aligned} R =\{&a^2,\ b^4,\ c^2,\ d^2,\ (ac)^2,\ (db^2)^2,\ (b^{-1}d)^4,\ (abdb)^2,\ (ad)^4,\ (cb^{-1}cb)^2,\ (cb^{-1})^4,\ (cd)^4,\\ &(dabda)^2,\ cb^{-1}cdcbcb^{-1}db,\ abab^2ab^{-1}ab^2,\ cdbadcdab^{-1}d,\ acbab^{-1}ab^{-1}ab^{-1}cb^2,\\ &acbcdcb^{-1}acb^{-1}db,\ (cbdadb^{-1})^2\}.\end{aligned}$$

The group G_{12}, obtained by adding an element x together with the set R_x of relators, has the following structure

$$2^7 : (2^3 : S_4) \sim 2^6 \cdot (2^4 : S_4) \sim 2^5 \cdot (2^6 : D_{12}) \sim 2^4 \cdot (2^6 : S_4)$$

and presentation

$$G_{12} = \langle a, b, c, d, x \mid R \cup R_x \rangle ,$$

where

$$R_x = \{x^3,\ x^{-1}caxa,\ xb^2x^{-1}ab^2a,\ xdbcb^{-1}dx^{-1}c,\ dx^{-1}bcb^{-1}xdc,\ axbdab^2xd\}.$$

The group G_{13}, obtained by adding an element y together with the set R_y of relators, has the following structure

$$(2^3 : 2^{1+4}_+) \cdot (2^2 \times S_4) \sim 2^7 : (2^4 : D_{12}) \sim 2^6 \cdot (2^5 : D_{12}) \sim 2^5 \cdot (2^4 : (D_8 \times S_3))$$

and presentation

$$G_{13} = \langle a, b, c, d, y \mid R \cup R_y \rangle ,$$

where

$$\begin{aligned}R_y =\{&y^3,\ (cy^{-1})^2,\ ycdcy^{-1}d,\ (y^{-1}d)^3,\ cb^2ycb^2y,\ b^{-1}dycb^{-1}cy^{-1}d,\ b^{-1}yb^2y^{-1}byb^2y^{-1},\\ &ayb^2y^{-1}ayb^2y^{-1},\ (ayay^{-1})^2,\ y^{-1}b^2y^{-1}b^2y^{-1}ab^2a,\ ay^{-1}cab^{-1}cdyb, dycab^{-1}cy^{-1}ba,\\ &y^{-1}by^{-1}b^{-1}aby^{-1}b^{-1}da,\ ab^{-1}ab^{-1}ab^{-1}ab^{-1}yb^2y^{-1}\}.\end{aligned}$$

Finally, the group G_{23}, obtained by adding an element z together with the set R_z of relators, has the following structure

$$(2_+^{1+4} : 2^3) \cdot (2^2 \times S_4) \sim (2^2 \times 2_+^{1+4}) \cdot (D_8 \times S_4) \sim (2^2 \times 4^2) : (2^4 : S_4)$$

and presentation

$$G_{23} = \langle a, b, c, d, z \mid R \cup R_z \rangle ,$$

where

$$\begin{aligned}R_z =\{&z^3,\ z^{-1}adazd, bcdcb^{-1}z^{-1}dz,\ dab^2z^{-1}b^2daz^{-1},\ azb^2z^{-1}az^{-1}b^2z,\ czb^2z^{-1}czb^2z^{-1},\\ &(az^{-1})^4,\ (czcz^{-1})^2,\ zb^2adb^{-1}adz^{-1}c\}.\end{aligned}$$

We now observe that

$$G_1 = \langle G_{12}, G_{13} \rangle , \qquad G_2 = \langle G_{12}, G_{23} \rangle , \qquad G_3 = \langle G_{13}, G_{23} \rangle ,$$

so that it is not difficult to find presentations for the three members.

The group $G_1 \sim 2^7 : \mathrm{AGL}_3(2)$ can be presented as

$$G_1 = \left\langle a, b, c, d, x, y \mid R \cup R_x \cup R_y \cup R_{xy} \right\rangle ,$$

where

$$R_{xy} = \{y^{-1}xbcb^{-1}x^{-1}yd,\ xy^{-1}x^{-1}cy^{-1}x^{-1}y^{-1},\ xay^{-1}x^{-1}ayxy\}.$$

The group $G_2 \sim \left[2^9\right] \cdot (S_3 \times S_4) \sim (2^2 \times 4^2) \cdot (2_+^{1+4} : (S_3 \times S_3))$ can be presented as

$$G_2 = \langle a, b, c, d, x, z \mid R \cup R_x \cup R_z \cup R_{xz} \rangle ,$$

where

$$R_{xz} = \{[x, z] ,\ czxdcxdz^{-1},\ cz^{-1}cx^{-1}az^{-1}ax\}.$$

The third member $G_3 \sim \left[2^{11} \cdot 3^3\right] : D_{12} \sim \left[2^{10} \cdot 3^3\right] : S_4$ can be presented as

$$G_3 = \left\langle a, b, c, d, y, z \mid R \cup R_y \cup R_z \cup R_{yz} \right\rangle ,$$

where

$$R_{yz} = \{zdyday^{-1}za,\ y^{-1}zcz^{-1}yzcz^{-1},\ y^{-1}zy^{-1}z^{-1}yzyz^{-1}\}.$$

The universal completion of $\mathcal{A}$ is the group A with the following presentation

$$A = \left\langle a, b, c, d, x, y, z \mid R \cup R_x \cup R_y \cup R_z \cup R_{xy} \cup R_{xz} \cup R_{yz} \right\rangle .$$

The following theorem is the final and main result of the chapter.

Theorem 14. *The group A is isomorphic to $G \cong \Omega_8^+(2) : S_3$.*

Proof. By enumerating the cosets of A over the identity subgroup[2], it turns out that $A \cong G$, so that the G is the universal completion of $\mathcal{A}$, thus confirming a result of Pasini [Pas]. $\square$

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[2] The computation was performed by Prof. Eamonn O'Brien on machines with large resources (1 TB of RAM).

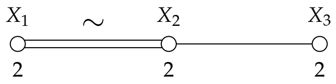
Chapter 5

The M_{24}-He dichotomy

The purpose of this chapter is to answer a question exposed in [Iva18, Chapter 8] about two locally isomorphic geometries related to the Mathieu group of degree 24 and to the sporadic Held group. In the first two sections we give a brief overview of the origin of the problem, describe a rank 2 amalgam and apply to it the powerful tool of Goldschmidt's lemma; in the last section we explicitly find the generating cocycle whose existence is asserted in [Iva18, Lemma 8.4].

5.1 Introduction

In [RS84] the authors consider various minimal parabolic geometries for sporadic simple groups, among which two locally isomorphic ones having the following diagram



where the subdiagram $\overset{\sim}{\circ\!=\!=\!\circ}$ indicates the rank 2 *tilde* geometry, which is a triple cover of the classical generalised quadrangle of order $(2, 2)$ associated with the non-split extension $3 \cdot S_6 \cong 3 \cdot \mathrm{Sp}_4(2)$.

In both cases the stabilisers of a point, of a line and of a plane have the following structure, respectively:

$$X_1 \sim 2_+^{1+6} : \mathrm{L}_3(2), \qquad X_2 \sim 2^6 : (S_3 \times S_4), \qquad X_3 \sim 2^6 : (3 \cdot S_6).$$

More information about these geometries, one related to the Mathieu group M_{24} and the other one to the Held group He, can be found in [Iva99; Giu+16; Iva18]. In particular, in [Iva18, Chapter 8] the author intends to prove what he calls the *dichotomy* principle, i.e. Theorem 8.2. Before reformulating it in our notation, we introduce the amalgam $\mathcal{A} = \{G_1, G_2; G_{12}\}$, which is strictly related to the Goldschmidt G_3^1-amalgam. The members of $\mathcal{A}$ are described as follows, where $G_3 \cong X_3 \sim 2^6 : (3 \cdot S_6)$ is isomorphic to the sextet subgroup of M_{24} and $N = O_2(G_3) \cong 2^6$:

- $G_1 = N : H \sim 2^6 : (2 \times S_4)$ is the normaliser in G_3 of a 2^2-subgroup of N;
- $G_2 = N : K \sim 2^6 : (2 \times S_4)$ is the centraliser in G_3 of an involution of N;

- $G_{12} = N : (H \cap K) \sim 2^6 : (2 \times D_8)$ is a Sylow 2-subgroup of G_1 and G_2.

The amalgam $\mathcal{A}$ does not fall within the list given by Goldschmidt [Gol80], even if $[G_1 : G_{12}] = [G_2 : G_{12}] = 3$, as $N \lhd G_{12}$ is normal in both G_1 and G_2. Moreover, we notice that the two members G_1 and G_2, despite having the same shape, are not isomorphic.

We are now ready to formulate the following theorem [Iva18, Theorem 8.2], whose motivation lies in [Iva18, Theorem 2.2].

Theorem 15. *The universal completion* $\widehat{G}$ *of the amalgam* $\mathcal{A}$ *contains precisely two complements* $E^{(1)}$ *and* $E^{(2)}$ *to* N *in* $C_{\widehat{G}}(N)$*. If*

$$\mathcal{D}^{(i)} := \{G_1, G_2, \widehat{G}/E^{(i)}\},$$

then $\mathcal{D}^{(1)}$ *and* $\mathcal{D}^{(2)}$ *are the rank 3 amalgams related to the above-mentioned geometries with* M_{24} *and* He *being the corresponding universal completion groups.*

The amalgams $\mathcal{D}^{(1)}$ and $\mathcal{D}^{(2)}$ are not isomorphic; however,

$$\widehat{G}/E^{(1)} \cong \widehat{G}/E^{(1)} \cong G_3.$$

Furthermore, $G_3/N \cong 3 \cdot S_6$ is a faithful generating completion of the quotient of $\mathcal{A}$ modulo N, which is the Goldschmidt G_3^1-amalgam $\{2 \times S_4, 2 \times S_4; 2 \times D_8\}$.

5.2 Presenting $\mathcal{A}$ and Goldschmidt's lemma

In this section, inspired by and using the same notation of [Iva18], we construct the amalgam $\mathcal{A}$ inside the group $G \cong \mathrm{L}_5(2)$, which is the only simple group, besides M_{24} and He, having the centraliser of an involution with shape $2_+^{1+6} : \mathrm{L}_3(2)$ [Hel69]. It is well known that, for $n \geq 2$, the group $\mathrm{L}_n(q)$ is generated by elementary transvections, which we denote by τ_{ij} $(i \neq j)$. Thus, τ_{ij} is the $n \times n$ matrix that differs from the identity matrix in that it has 1 as its ij entry.

By keeping the same notation of the previous section, we have the following:

- $G = \langle \tau_{ij} \mid 1 \leq i, j \leq 5,\ i \neq j \rangle \cong \mathrm{L}_5(2)$;
- $N = \langle \tau_{31}, \tau_{32}, \tau_{41}, \tau_{42}, \tau_{51}, \tau_{52} \rangle \cong 2^6$;
- $H = \langle \tau_{21}, \tau_{45}, \tau_{53}, \tau_{54} \rangle \cong 2 \times S_4$;
- $K = \langle \tau_{21}, \tau_{34}, \tau_{43}, \tau_{54} \rangle \cong 2 \times S_4$;
- $Z(H) = Z(K) = \langle \tau_{21} \rangle \cong 2$;
- $H \cap K = \langle \tau_{21}, \tau_{43}, \tau_{54} \rangle \cong 2 \times D_8$;
- $G_1 = \langle N, H \rangle \sim 2^6 : (2 \times S_4)$;
- $G_2 = \langle N, K \rangle \sim 2^6 : (2 \times S_4)$;
- $G_{12} = \langle N, H \cap K \rangle \sim 2^6 : (2 \times D_8)$.

We now wish to determine the number of isomorphism classes of amalgams of the same type as $\mathcal{A}$; this task, which is one of the most important goals in the whole theory of amalgams, has its motivation in the following situation. Let P_1 and P_2 be two groups, and let B_1 and B_2 be two isomorphic subgroups of P_1 and P_2 respectively. If ψ is an isomorphism from B_1 to B_2, then we can construct an amalgam $P_1 \cup P_2$ by identifying $x \in B_1$ with $\psi(x) \in B_2$. A natural question is the following: given P_1, P_2, B_1 and B_2, how many non-isomorphic amalgams can be constructed in this way, when we take all possible ψ? The answer is given by Theorem 16 [Gol80, (2.7)].

Theorem 16 (Goldschmidt's lemma)**.** *Let $\mathcal{A} = \{P_1, P_2; B\}$ be an amalgam of rank 2. For $i \in \{1,2\}$, let $N_i := N_{\mathrm{Aut}(P_i)}(B) = \{\alpha \in \mathrm{Aut}(P_i) \mid \alpha(B) = B\}$, let $\varphi_i \colon N_i \to \mathrm{Aut}(B)$ be the homomorphisms mapping $a \in N_i$ onto its restriction to B, and let $A_i := \varphi_i(N_i)$. Then two elements α and β of $\mathrm{Aut}(B)$ produce isomorphic amalgams if and only if $A_1\alpha A_2 = A_1\beta A_2$. In other words, the number of non-isomorphic amalgams with the type of $\mathcal{A}$ coincides with the number $|A_1\backslash\mathrm{Aut}(B)/A_2|$ of double cosets of A_1 and A_2 in $\mathrm{Aut}(B)$.*

We notice that, as A_1 and A_2 both contain $\mathrm{Inn}(B)$, the computation can be performed in $\mathrm{Out}(B)$ instead of $\mathrm{Aut}(B)$, by considering the number of double cosets of the images O_1 and O_2 of A_1 and A_2 respectively in $\mathrm{Out}(B)$.

We apply Goldschmidt's lemma to the amalgam $\mathcal{A}$, using the function `Goldschmidt` designed and implemented in [Gap] (see Appendix B), and checking the result with the MAGMA [BCP97] function `Amalgams` [Can05]. The final result

$$|A_1\backslash\mathrm{Aut}(G_{12})/A_2| = |O_1\backslash\mathrm{Out}(G_{12})/O_2| = 6$$

is somehow expected, being the same for an amalgam of type $\{2 \times S_4, 2 \times S_4; 2 \times D_8\}$ (see [Can05, Example 2]). More details about the output are given in Table 5.1.

Group	Shape(s)
$\mathrm{Aut}(G_1)$	$(2^6 : 3) : 2_+^{1+4} \sim 2^7 : (2 \times S_4) \sim 2^5 : (2^4 : D_{12})$
$\mathrm{Aut}(G_2)$	$(2^2 \times 2_+^{1+4}) : (2^3 \times S_3) \sim 2^6 : (2^2 \times S_4) \sim 2^5 : (2^3 : S_4)$
$\mathrm{Aut}(G_{12})$	$\left[2^{11}\right] : 3 \sim 2^8 \cdot \mathrm{Hol}(D_8) \sim 2^7 \cdot (2^3 \wr 2)$
A_1	$(2^5 : 4) : 2^3 \sim 2^6 : (2 \times D_8) \sim 2^5 : 2_+^{1+4} \sim 2^4 : (2 \wr 2^2)$
A_2	$(2^2 \times 2_+^{1+4}) : 2^4 \sim 2^6 : (2^2 \times D_8) \sim 2^5 : (2 \wr 2^2)$
$\mathrm{Out}(G_{12})$	$2^2 \wr 2$
O_1	2
O_2	2^2

TABLE 5.1: The machinery of Goldschmidt's lemma applied to $\mathcal{A}$.

The other important piece of information coming from the application of Goldschmidt's lemma is given by the representatives φ_i $(1 \leq i \leq 6)$ of the double cosets, which are automorphisms of G_{12}, with $\varphi_1 = \mathrm{id}_{\mathrm{Aut}(G_{12})}$. By using each of these automorphisms φ_i's we construct the universal completion $\widehat{A}_i = G_1 *_{\varphi_i} G_2$ of the corresponding amalgam $\mathcal{A}_i$, shown in Appendix B. In the following two tables we give the images under the φ_i's of the generators τ_{ij} (denoted by ij for brevity) of N and $H \cap K$ respectively.

	31	32	41	42	51	52
φ_1	31	32	41	42	51	52
φ_2	$31 \cdot 51 \cdot 52$	32	41	42	51	52
φ_3	$51 \cdot 52 \cdot 53$	$41 \cdot 42 \cdot 43 \cdot 51 \cdot 52 \cdot 53$	$51 \cdot 52$	$41 \cdot 42 \cdot 51 \cdot 52$	51	$41 \cdot 51$
φ_4	$51 \cdot 52 \cdot 53$	$41 \cdot 42 \cdot 43 \cdot 51 \cdot 52 \cdot 53$	$51 \cdot 52$	$41 \cdot 42 \cdot 51 \cdot 52$	51	$41 \cdot 51$
φ_5	31	$32 \cdot 51$	41	42	51	52
φ_6	$31 \cdot 51 \cdot 52$	$32 \cdot 51$	41	42	51	52

TABLE 5.2: The images of the generators of N under the φ_i's.

	21	43	54
φ_1	21	43	54
φ_2	$21 \cdot 53$	43	54
φ_3	$53 \cdot 54$	$31 \cdot 32 \cdot 41 \cdot 42$	$21 \cdot 31 \cdot 41$
φ_4	$53 \cdot 54$	$31 \cdot 32 \cdot 41 \cdot 42 \cdot 51 \cdot 52$	$21 \cdot 31 \cdot 41 \cdot 52 \cdot 53$
φ_5	21	43	54
φ_6	$21 \cdot 53$	43	54

TABLE 5.3: The images of the generators of $H \cap K$ under the φ_i's.

In [Giu+16] the authors consider the amalgam $\mathcal{B} = \{X_1, X_2; X_{12}\}$, with the intersection $X_{12} \sim 2_+^{1+6} : S_4$, and use Goldschmidt's lemma to obtain the amalgams of the same type as $\mathcal{B}$ in the different isomorphism classes. The result is a list of four amalgams $\mathcal{B}^\sigma$ for $\sigma \in \{\mathrm{id}, \alpha, \beta, \alpha\beta\}$, where α and β are suitable automorphisms of X_{12}. Among the four amalgams, the simple ones are $\mathcal{B}^\beta$ and $\mathcal{B}^{\alpha\beta}$; the former admits A_{16} as a completion group, while the sporadic groups M_{24} and He are completions of the latter. We notice that in our notation φ_2 corresponds to β, while φ_6 corresponds to $\alpha\beta$.

5.3 The strategy

In this section we classify the completions of the Goldschmidt G_3^1-amalgam isomorphic to $G := G_3$, by constructing explicitly the generating cocycle whose existence is asserted in [Iva18, Lemma 8.4]. The starting point is the observation that the Goldschmidt G_3^1-amalgam possesses two inequivalent completion groups isomorphic to G, in the sense explained in Appendix A. These two completions give rise to isomorphic coset graphs with 5760 vertices, 8640 edges, diameter 16 and girth 16, as shown in Table A.7. However, when considering the two connected components of their distance-2 graph, the resulting locally projective graphs are not isomorphic, as also reflected in the different completion groups of the corresponding densely embedded subgraphs. The same phenomenon occurs for the completion group $2^6 : (3 \cdot A_6)$ of the Goldschmidt G_3-amalgam (see Table A.6).

The second observation is the fact that among the amalgams $\mathcal{A}_i$, only $\mathcal{A}_2$ and $\mathcal{A}_6$ admit $3 \cdot S_6$ and G as completion groups. The latter completion makes $\mathcal{A}_2$ and $\mathcal{A}_6$ distinguishable from each other, as in this case we find four inequivalent epimorphisms

$$\psi_i \colon \widehat{A}_6 \longrightarrow H \cong G \qquad (1 \leq i \leq 4),$$

while only two from $\widehat{A}_2$; here H is chosen to be `PrimitiveGroup(64,47)`. By adopting the same technique explained in Chapter 3, we use the MAGMA [BCP97] command `sub< G | f >` to determine the preimage in $\widehat{A}_6$ of the point stabiliser of H, which is isomorphic to $3 \cdot S_6$. If V_i denotes such a preimage under ψ_i, then V_i is a subgroup of $\widehat{A}_6$ of index 64 and

$$V_i/V_i' \cong \begin{cases} 2, & \text{if } i \in \{1,4\} \\ 6^2, & \text{if } i \in \{2,3\} \end{cases}$$

We now describe the strategy followed to find the cocycle. We start with a generating Goldschmidt G_3^1-amalgam $\mathcal{T}^{(0)} = \{H^{(0)}, K^{(0)}; H^{(0)} \cap K^{(0)}\}$ inside the group G, where $H^{(0)} := j_1(P_1) \cong 2 \times S_4$ and $K^{(0)} := j_2(P_2) \cong 2 \times S_4$, so that $H^{(0)} \cap K^{(0)} \cong 2 \times D_8$ and $\left\langle H^{(0)}, K^{(0)} \right\rangle_G = G$, according to the commutative diagram shown in Figure 5.1.

$$\begin{tikzcd}
 & P_1 \arrow[dr, "e_1"] \arrow[drr, bend left, "j_1"] & & \\
B \arrow[ur, "\varphi_1"] \arrow[dr, "\varphi_2"'] & & P_1 *_B P_2 \arrow[r] & G \\
 & P_2 \arrow[ur, "e_2"] \arrow[urr, bend right, "j_2"'] & &
\end{tikzcd}$$

FIGURE 5.1: The group G as a completion of the amalgam $\{P_1, P_2; B\}$.

We can present G as follows:

$$G = \langle a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10} \mid R_1 \cup R_2 \cup R_3 \rangle \,,$$

where

$$\begin{aligned}R_1 = \{&a_5^2,\ a_6^2,\ [a_6, a_5],\ a_7^2,\ [a_7, a_5],\ [a_7, a_6],\ a_8^2,\ [a_8, a_5],\ [a_8, a_6],\ [a_8, a_7],\ a_9^2,\ [a_9, a_5],\\ &[a_9, a_6],\ [a_9, a_7],\ [a_9, a_8],\ a_{10}^2,\ [a_{10}, a_5],\ [a_{10},\ a_6],\ [a_{10},\ a_7],\ [a_{10},\ a_8],\ [a_{10},\ a_9]\}\end{aligned}$$

is the set of relators defining $N = O_2(G) = \langle a_5, a_6, a_7, a_8, a_9, a_{10}\rangle \cong 2^6$,

$$\begin{aligned}R_2 = \Big\{&a_1^2a_2a_3^{-1}a_2a_3^2a_2^{-1}a_3^{-1}a_4^{-1},\ a_1^{-1}a_2a_1a_2^{-1}a_3^{-1}a_2a_4^{-2},\ a_1^{-1}a_3a_1a_3a_2a_3^{-2}a_2^2a_3a_2^{-1}a_4^{-1},\\ &a_2^5a_3^{-5},\ a_2^5a_3^{-1}a_2^{-1}a_3^{-1}a_2^{-1}a_3^{-1}a_2^{-1}a_4^{-1},\ a_2^{-1}a_3^{-1}a_2a_3a_2^{-1}a_3^{-1}a_2a_3a_4^{-2},\\ &a_2^{-2}a_3^{-1}a_2a_3^2a_2^{-2}a_3^{-1}a_2a_3^2a_4^{-1},\ a_1^{-1}a_4a_1a_4^{-2},\ \left[a_2, a_4^{-1}\right],\ \left[a_3, a_4^{-1}\right],\ a_4^3\Big\}\end{aligned}$$

are the relators defining a complement $C = \langle a_1, a_2, a_3, a_4\rangle \cong 3 \cdot S_6$ to N in G, and

$$\begin{aligned}R_3 = \{\ &[a_1, a_5],\ a_1^{-1}a_6a_1a_9^{-1}a_8^{-1}a_7^{-1}a_6^{-1}a_5^{-1},\ a_1^{-1}a_7a_1a_{10}^{-1}a_8^{-1},\ a_1^{-1}a_8a_1a_{10}^{-1}a_9^{-1}a_8^{-1},\\ &a_1^{-1}a_9a_1a_7^{-1}a_5^{-1},\ a_1^{-1}a_{10}a_1a_{10}^{-1}a_9^{-1}a_8^{-1}a_7^{-1}a_6^{-1}a_5^{-1},\ a_2^{-1}a_5a_2a_{10}^{-1}a_8^{-1}a_7^{-1}a_5^{-1},\\ &a_2^{-1}a_6a_2a_{10}^{-1}a_5^{-1},\ a_2^{-1}a_7a_2a_{10}^{-1}a_9^{-1}a_8^{-1}a_7^{-1}a_6^{-1},\ a_2^{-1}a_8a_2a_{10}^{-1}a_7^{-1}a_5^{-1},\\ &a_2^{-1}a_9a_2a_{10}^{-1}a_9^{-1}a_8^{-1}a_5^{-1},\ a_2^{-1}a_{10}a_2a_{10}^{-1}a_9^{-1}a_8^{-1}a_6^{-1}a_5^{-1},\ a_3^{-1}a_5a_3a_8^{-1}a_6^{-1}a_5^{-1},\\ &a_3^{-1}a_6a_3a_8^{-1}a_6^{-1},\ a_3^{-1}a_7a_3a_6^{-1}, a_3^{-1}a_8a_3a_7^{-1}a_5^{-1},\ a_3^{-1}a_9a_3a_{10}^{-1}a_7^{-1}a_5^{-1},\\ &a_3^{-1}a_{10}a_3a_{10}^{-1}a_9^{-1}a_8^{-1}a_6^{-1}a_5^{-1},\ a_4^{-1}a_5a_4a_8^{-1}a_7^{-1}a_6^{-1}a_5^{-1},\ a_4^{-1}a_6a_4a_7^{-1}a_5^{-1},\\ &a_4^{-1}a_7a_4a_8^{-1},\ a_4^{-1}a_8a_4a_8^{-1}a_7^{-1},\ a_4^{-1}a_9a_4a_{10}^{-1}a_9^{-1}a_8^{-1},\ a_4^{-1}a_{10}a_4a_9^{-1}a_8^{-1}a_7^{-1}\}\end{aligned}$$

are those defining the action of C on N.

As for the members of $\mathcal{T}^{(0)}$, they have the following presentation:

$$H^{(0)} = \left\langle a_2^{-1}a_1a_2a_3^{-1}a_1^{-1}a_{10}^{-1}a_2^{-1}a_1,\ a_1^{-3}a_4^{-1}a_3a_2a_1a_3^{-1}a_4a_9^{-1}a_7^{-1}\right\rangle$$

and

$$K^{(0)} = \left\langle a_2^{-1}a_6a_4^{-1}a_2^{-1}a_1^{-1}a_3^{-1}a_2^{-1}a_3a_1^2,\ a_2^{-1}a_3a_1^{-1}a_3^2a_1^{-1}a_4^{-1}a_5^{-1}a_4\right\rangle$$

intersecting in

$$H^{(0)} \cap K^{(0)} = \left\langle a_1^{-3}a_4^{-1},\ a_1^{-3}a_2^{-1}a_3a_1^{-1}a_3^2a_1^{-1}a_4a_5^{-1}a_4,\ a_1^{-1}a_2^{-1}a_7a_3a_2^{-1}a_3^{-1}a_1^4\right\rangle.$$

We are looking for amalgams $\mathcal{T}^{(s)} = \{H^{(s)}, K^{(s)}; H^{(s)} \cap K^{(s)}\}$ in G isomorphic to $\mathcal{T}^{(0)}$ and having the same image in $G/N \cong 3 \cdot S_6$. Thus we need an isomorphism

$$\sigma\colon \mathcal{T}^{(0)} \longrightarrow \mathcal{T}^{(s)}$$

such that $t \in \mathcal{T}^{(0)}$ and $\sigma(t) \in \mathcal{T}^{(s)}$ induce the same automorphism of N, i.e. such that they have the same image under the map

$$\psi\colon G \longrightarrow \mathrm{Aut}(N),$$

defined for each $g \in G$ by $n \mapsto gng^{-1}$ for all $n \in N$. Therefore,

$$\psi(t) = \psi(\sigma(t)) \Longrightarrow \psi(t^{-1}\sigma(t)) = 1_{\mathrm{Aut}(N)} \Longrightarrow t^{-1}\sigma(t) \in \ker(\psi) = C_G(N) = N$$

which implies $\sigma(t) = ts(t)$ for some $s(t) \in N$. The function

$$s \colon \mathcal{T}^{(0)} \longrightarrow N,$$

defined by $t \mapsto s(t)$, is a 1-cocycle (or crossed homomorphism) when restricted to each of $H^{(0)}$ and $K^{(0)}$, as

$$s(xy) = (xy)^{-1}\sigma(xy) = y^{-1}x^{-1}\sigma(x)\sigma(y) = y^{-1}s(x)\sigma(y) = y^{-1}s(x)yy^{-1}\sigma(y) = s(x)^y s(y)$$

for all x and y in $H^{(0)}$ and $K^{(0)}$ separately.

We first observe that $H^{(0)}$ normalises the subgroup $T = \langle a_5a_6a_7a_8,\ a_5a_{10}\rangle \cong 2^2$ of N, while $K^{(0)} \leq C_G(a_5a_6a_7a_8)$, and then we find in $C = \langle a_1, a_2, a_3, a_4\rangle \cong 3 \cdot S_6$ the unique amalgam $\mathcal{T}^{(s)} = \{H^{(s)}, K^{(s)}; H^{(s)} \cap K^{(s)}\}$ which corresponds to $\mathcal{T}^{(0)}$, in the sense that $H^{(s)} \leq N_G(T)$ and $K^{(s)} \leq C_G(a_5a_6a_7a_8)$, and such that $\left\langle H^{(s)}, K^{(s)} \right\rangle_G = C$. This gives:

$$H^{(s)} = \left\langle a_3a_2^{-1}a_3^{-1}a_1,\ a_4^{-1}a_2^{-2}a_3^{-1}a_4 \right\rangle$$

and

$$K^{(s)} = \left\langle a_1^{-3}a_2^{-2}a_1a_3^{-1}a_1a_4^{-1},\ a_3^{-1}a_1^2a_3a_1^3 \right\rangle$$

intersecting in

$$H^{(s)} \cap K^{(s)} = \left\langle a_2^{-2}a_1a_3^{-1}a_1,\ a_3^{-2}a_2^{-1}a_1a_2^{-1}a_4^{-1},\ a_4^{-1}a_3a_2a_1a_3^{-1} \right\rangle.$$

In order to establish the isomorphism σ described above, we look at all surjective homomorphisms $f \colon G \longrightarrow C$ that map $H^{(0)}$ to $H^{(s)}$ and $K^{(0)}$ to $K^{(s)}$. There are 16 of them, which correspond to the 16 inner automorphisms induced by the elements of $H^{(s)} \cap K^{(s)}$, and each f, by restriction and corestriction, establishes an isomorphism $\sigma = (\sigma_H, \sigma_K)$ from $\mathcal{T}^{(0)}$ to $\mathcal{T}^{(s)}$, with $\sigma_H \colon H^{(0)} \xrightarrow{\cong} H^{(s)}$ and $\sigma_K \colon K^{(0)} \xrightarrow{\cong} K^{(s)}$. Only one of them is such that

$$\psi(h_0) = \psi(\sigma_H(h_0)) \quad \text{and} \quad \psi(k_0) = \psi(\sigma_H(k_0)) \quad \text{for all } h_0 \in H^{(0)},\ k_0 \in K^{(0)}.$$

h_0	$s(h_0)$
$a_2^{-1}a_1a_2a_3^{-1}a_1^{-1}a_{10}^{-1}a_2^{-1}a_1$	$a_1^{-2}a_2^{-1}a_1^{-1}a_{10}^{-1}a_2^{-1}a_1$
$a_1^{-3}a_4^{-1}a_3a_2a_1a_3^{-1}a_4a_9^{-1}a_7^{-1}$	$a_9^{-1}a_7^{-1}$

TABLE 5.4: The cocyle s on $H^{(0)}$.

In Table 5.4 we list the generators of $H^{(0)}$ with their images under the cocycle $s \colon H^{(0)} \longrightarrow N$, and in Table 5.5 we display the same for $K^{(0)}$.

By the existence and uniqueness (up to conjugation) of the generating cocycle s, we know that N possesses exactly two complements $E^{(1)}$ and $E^{(2)}$ to N in the centraliser of N of the universal completion group $\widehat{A}_6$ of the amalgam $\mathcal{A}_6$.

Finally, following [Iva18, § 8.5], we produce an explicit construction of the quotient

$$F = \widehat{A}_6/(E^{(1)} \cap E^{(2)}) \sim 2^{12} : (3 \cdot S_6) \sim 2^6 : (2^6 : (3 \cdot S_6)).$$

k_0	$s(k_0)$
$a_2^{-1}a_6a_4^{-1}a_2^{-1}a_1^{-1}a_3^{-1}a_2^{-1}a_3a_1^2$	$a_9^{-1}a_7^{-1}$
$a_2^{-1}a_3a_1^{-1}a_3^2a_1^{-1}a_4^{-1}a_5^{-1}a_4$	$a_4^{-1}a_5^{-1}a_4$

TABLE 5.5: The cocyle s on $K^{(0)}$.

Using the GAP [Gap] function `SubdirectProduct`, we find F as the subdirect product of G and G with respect to the canonical epimorphisms, i.e. the subgroup of the direct product $G \times G = \{(g_1, g_2) \mid g_1, g_2 \in G\}$ consisting of the elements (g_1, g_2) for which $q_1(g_1) = q_2(g_2)$. In category theory this is known as the *pullback* of the diagram consisting of the two morphisms q_1 and q_2 having C as a common codomain, as illustrated in Figure 5.2.

$$\begin{tikzcd}
F = G \times_C G \arrow[r, "p_2"] \arrow[d, "p_1"'] & G \arrow[d, "q_2"] \\
G \arrow[r, "q_1"] & C
\end{tikzcd}$$

FIGURE 5.2: The group F as a subdirect product of G and G.

We check that $O_2(F) \cong 2^{12}$ contains three 2^6-subgroups that are normal in F; they are the images of N, $E^{(1)}$ and $E^{(2)}$. To conclude, the group F is not a completion of the Goldschmidt G_3^1-amalgam, and MAGMA [BCP97] finds six inequivalent epimorphisms $\widehat{A}_6 \longrightarrow F$ whose kernels have all abelianisation isomorphic to $\underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{2881}$.

Appendix A

Some further embeddings

In this Appendix we give some further examples of embeddings of the Djoković-Miller subamalgams in the Goldschmidt amalgams. In the following tables the first and the last columns give respectively a completion group G of the given Goldschmidt amalgam and the subgroup H (obtained as explained in Chapter 2), which is a completion group of the corresponding Djoković-Miller densely embedded subamalgam. When the isomorphism type of H is not uniquely determined by its structure, whenever possible and only the first time it appears, we add its ID description; sometimes, for typographical reasons, we abbreviate as A^2 the direct square $A \times A$ of a group A. The remaining columns of the tables record the number of vertices, edges, the diameter $d(\Gamma)$ and the girth $g(\Gamma)$ of the associated coset graph Γ, obtained with the aid of the algebra package MAGMA [BCP97]. Note that as Γ is bipartite, its girth is always even, and that in some cases the subgroup H is the whole of G.

Sometimes, for a given completion group G, we found two different subgroups H. These cases correspond to inequivalent epimorphisms of the universal completion $P_1 *_B P_2$ of the Goldschmidt amalgam onto G, where two such homomorphisms $f_1, f_2 \colon P_1 *_B P_2 \longrightarrow G$ are considered equivalent if they differ by an automorphism of G, i.e. if there exists an element $\alpha \in \mathrm{Aut}(G)$ such that $f_1(x) = f_2(x)^\alpha$ for all $x \in P_1 *_B P_2$. For a 'symmetric' Goldschmidt amalgam, the presence of these inequivalent completions breaks its symmetry when mapped to G, and results in two non-isomorphic locally projective graphs $\Xi^{(1)}$ and $\Xi^{(2)}$. In a few cases, typically when there are more than two inequivalent (isomorphic) completion groups, the corresponding coset graphs are not isomorphic, usually with different diameter or girth; in these cases, in the table we devote a row for each isomorphism class of Γ.

For each of the following embeddings of a Djoković-Miller subamalgam in a Goldschmidt amalgam $(P_1, P_2; B)$, the following approach, inspired by [PR01b], was adopted. Starting with the free amalgamated product $P_1 *_B P_2$, given as a finitely presented group, the MAGMA [BCP97] `LowIndexNormalSubgroups` routine was used to produce all normal subgroups of $P_1 *_B P_2$ of index less than or equal to n, for a suitable $n \in \mathbb{N}$. The completion groups G were obtained as the corresponding factor groups, as explained in Section 1.2.

A.1 Embeddings of $\mathcal{DM}_1$ in G_1^3

In Table A.1 we record the complete list of the embeddings of the Djoković-Miller subamalgam $\mathcal{DM}_1$ in the Goldschmidt G_1^3-amalgam with completion group G, such that $120 \leq |G| \leq 36\,050$, obtained from the free amalgamated product $D_{12} *_{2^2} D_{12}$

with the following presentation

$$\left\langle a, b, c, d \mid a^2,\ b^3,\ c^2,\ d^3,\ (ac)^2,\ (ba)^2,\ (dc)^2,\ adad^{-1},\ cbcb^{-1} \right\rangle.$$

An asterisk after the structure of a group indicates the presence of three inequivalent completions which give rise to two non-isomorphic coset graphs, indistinguishable from each other in terms of the properties that we have considered. In the last part of the table, after the dashed line, we add a few more completions found with the GAP [Gap] function `GQuotients`.

G	$\lvert V(\Gamma)\rvert$	$\lvert E(\Gamma)\rvert$	$d(\Gamma)$	$g(\Gamma)$	H
S_5	20	30	5	6	S_4
$2 \times S_5$	40	60	6	8	$2 \times S_4$
$3_+^{1+2} : D_{12}$	54	81	6	8	$S_3 \times S_3$
					$3^2 : D_{12}$, [108, 17]
$\mathrm{PGL}_2(7)$	56	84	7	8	$\mathrm{PGL}_2(7)$
$4 \cdot S_5$	80	120	8	10	$4 \cdot S_4$, [96, 193]
$(A_4 \times A_4) : 2^2$	96	144	7	8	$2 \times S_4$
$\mathrm{L}_2(11)$	110	165	7	10	A_5
$2 \times \mathrm{PGL}_2(7)$	112	168	10	8	$2 \times \mathrm{PGL}_2(7)$
$S_3 \times S_5$	120	180	8	8	$2 \times S_4$
					$S_3 \times S_4$
$(3 \times 3_+^{1+2}) : D_{12}$	162	243	8	12	$3^2 : D_{12}$
$\mathrm{L}_2(13)$	182	273	9	12	$\mathrm{L}_2(13)$
$2_+^{1+4} : (S_3 \times S_3)$	192	288	12	8	$2 \times S_4$
$2 \times \mathrm{L}_2(11)^*$	220	330	10	10	$2 \times A_5$
$\mathrm{SL}_2(7) : 2^2$	224	336	10	12	$\mathrm{SL}_2(7) : 2^2$
$\mathrm{SL}_2(5) : D_{12}$	240	360	10	12	$4 \cdot S_4$
					$\mathrm{GL}_2(3) : S_3$, [288, 847]
$(2^3 \times 6) \cdot (S_3 \times S_3)$	288	432	9	12	$S_3 \times S_4$
$S_3 \times \mathrm{PGL}_2(7)$	336	504	10	12	$2 \times \mathrm{PGL}_2(7)$

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					$S_3 \times \mathrm{PGL}_2(7)$
$(3^2 \times A_5) : 2^2$	360	540	10	12	$S_3 \times S_4$
$2 \times \mathrm{L}_2(13)^*$	364	546	12	12	$2 \times \mathrm{L}_2(13)$
$(\mathrm{SL}_2(3) \times \mathrm{SL}_2(3)) : 2^2$	384	576	12	12	$4 \cdot S_4$
$(2^4 : 2^2) : (S_3 \times S_3)$	384	576	10	12	$4 \cdot S_4$
					$4^2 : D_{12}$, [192, 956]
$2^2 \times \mathrm{L}_2(11)$	440	660	10	12	$2^2 \times A_5$
$(3^2 \times 9) \cdot (S_3 \times S_3)$	486	729	12	12	$3_+^{1+2} \cdot D_{12}$, [324, 41]
$3^4 \cdot (S_3 \times S_3)$	486	729	12	12	$3^2 : D_{12}$
$2_+^{1+4} : (3^2 : D_{12})$	576	864	14	12	$S_3 \times S_4$
$2^5 : S_5$	640	960	12	10	$4^2 : D_{12}$
$S_3 \times \mathrm{L}_2(11)$	660	990	12	12	$2 \times A_5$
					$S_3 \times A_5$
$\mathrm{SL}_2(7) : D_{12}$	672	1008	14	12	$\mathrm{SL}_2(7) : 2^2$
					$\mathrm{SL}_2(7) : D_{12}$
$\mathrm{SL}_2(5) : (S_3 \times S_3)$	720	1080	12	12	$\mathrm{GL}_2(3) : S_3$
$2^2 \times \mathrm{L}_2(13)$	728	1092	14	12	$2^2 \times \mathrm{L}_2(13)$
$2^3 \cdot ((A_4 \times A_4) : 2^2)$	768	1152	14	12	$4 \cdot S_4$
					$4^2 : D_{12}$
$\mathrm{PGL}_2(17)$	816	1224	11	14	$\mathrm{PGL}_2(17)$
S_7	840	1260	14	10	$2 \times A_5$
					S_7
$(2^2 \times 6^2) \cdot (S_3 \times S_3)$	864	1296	12	12	$S_3 \times S_4$
					$6^2 : D_{12}$, [432, 523]
$\mathrm{SL}_2(11) : 2^2$	880	1320	12	14	$Q_8 \cdot A_5$, [480, 959]
$\mathrm{L}_3(2) : (S_3 \times S_3)$	1008	1512	12	16	$\mathrm{L}_3(2) : D_{12}$
$\mathrm{L}_2(23)$	1012	1518	12	16	$\mathrm{L}_2(23)$

continues on next page

$(3^2 \times A_5) : D_{12}$	1080	1620	12	12	$S_3 \times S_4$
$S_3 \times \mathrm{L}_2(13)$	1092	1638	12	12	$2 \times \mathrm{L}_2(13)$ $S_3 \times \mathrm{L}_2(13)$
$\mathrm{PGL}_2(19)$	1140	1710	12	14	$\mathrm{PGL}_2(19)$
$(3 \times Q_8^2) \cdot S_3^2$	1152	1728	14	14	$\mathrm{GL}_2(3) : S_3$
$(2 \times 6) \cdot (A_4^2 : 2^2)$	1152	1728	14	12	$\mathrm{GL}_2(3) : S_3$ $4^2 : S_3^2$, [576, 5053]
$(2^6 : 3^2) : D_{12}$	1152	1728	13	12	$S_3 \times S_4$
$2^5 \cdot (2 \times S_5)$	1280	1920	13	16	$2^3 \cdot (2 \times S_4)$
$2^6 : S_5$	1280	1920	13	12	$4^2 : D_{12}$
$(2^4 \times 4) \cdot S_5$	1280	1920	13	10	$2^3 \cdot (2 \times S_4)$
$D_{12} \times \mathrm{L}_2(11)$	1320	1980	14	12	$2^2 \times A_5$ $D_{12} \times A_5$
$\mathrm{L}_2(16) : 2$	1360	2040	17	10	$2 \times A_5$
$\mathrm{SL}_2(13) : 2^2$	1456	2184	14	14	$\mathrm{SL}_2(13) : 2^2$
$3^4 \cdot (3^2 : D_{12})$	1458	2187	12	12	$3^2 : D_{12}$ $3_+^{1+2} \cdot D_{12}$
$(3^3 \times 9) \cdot (S_3 \times S_3)$	1458	2187	16	12	$3_+^{1+2} \cdot D_{12}$
$(3^2 \times 9) \cdot (3^2 : D_{12})$	1458	2187	12	12	$3^3 \cdot S_3^2$, [972, 100] $9^2 : D_{12}$, [972, 115]
$2^4 \cdot ((A_4 \times A_4) : 2^2)$	1536	2304	14	12	$4 \cdot S_4$ $2^3 \cdot (2 \times S_4)$
$((2^2 \times D_8) : D_8) : S_3^2$	1536	2304	13	12	$2^3 \cdot (2 \times S_4)$
$4^4 : (S_3 \times S_3)$	1536	2304	15	12	$4^2 : D_{12}$
$3^4 : S_5$	1620	2430	12	10	$3^3 : S_4$, [648, 703]
$2 \times \mathrm{PGL}_2(17)$	1632	2448	14	16	$2 \times \mathrm{PGL}_2(17)$
$2 \times S_7$	1680	2520	14	14	$2^2 \times A_5$

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					$2 \times S_7$
$(3 \times 6) \cdot ((A_4 \times A_4) : 2^2)$	1728	2592	14	12	$S_3 \times S_4$
					$6^2 : D_{12}$
$\mathrm{L}_3(3) : 2$	1872	2808	13	12	$\mathrm{L}_3(3) : 2$
$\mathrm{L}_3(3) : 2$	1872	2808	14	12	$4 \cdot S_4$
					$\mathrm{L}_3(3) : 2$
$(2^4 \times S_3) : S_5$	1920	2880	13	12	$4^2 : D_{12}$
					$4^2 : (S_3 \times S_3)$
$\mathrm{U}_3(3) : 2 \cong \mathrm{G}_2(2)$	2016	3024	15	12	$4 \cdot S_4$
					$\mathrm{PGL}_2(7)$
$\mathrm{SL}_2(7) : (S_3 \times S_3)$	2016	3024	16	16	$\mathrm{SL}_2(7) : D_{12}$
$2 \times \mathrm{L}_2(23)^*$	2024	3036	16	16	$2 \times \mathrm{L}_2(23)$
$(3^2 \times \mathrm{SL}_2(5)) : D_{12}$	2160	3240	16	16	$\mathrm{GL}_2(3) : S_3$
$D_{12} \times \mathrm{L}_2(13)$	2184	3276	14	16	$2^2 \times \mathrm{L}_2(13)$
					$D_{12} \times \mathrm{L}_2(13)$
$2 \times \mathrm{PGL}_2(19)$	2280	3420	14	16	$2 \times \mathrm{PGL}_2(19)$
$(2^2 \times 6) \cdot ((A_4 \times A_4) : 2^2)$	2304	3456	16	12	$\mathrm{GL}_2(3) : S_3$
					$4^2 : (S_3 \times S_3)$
$2_-^{1+6} \cdot (3^2 : D_{12})$	2304	3456	14	16	$\mathrm{GL}_2(3) : S_3$
$5^3 \cdot S_5$	2500	3750	17	16	$5^3 : S_4$
$3 \cdot S_7$	2520	3780	14	14	$S_3 \times A_5$
					$3 \cdot S_7$
$3 \cdot S_7$	2520	3780	14	16	$S_3 \times A_5$
					$3 \cdot S_7$
$3 \cdot S_7$	2520	3780	14	10	$2 \times A_5$
					$3 \cdot S_7$
$3 : S_7 \sim A_7 : S_3$	2520	3780	14	16	$S_3 \times A_5$

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					S_7
$(2^5 : \mathrm{SL}_2(5)) : 2^2$	2560	3840	16	16	$2^3 \cdot (2 \times S_4)$
$(4 \circ 2_+^{1+4}) \cdot (2 \times S_5)$	2560	3840	16	12	$4^2 : D_{12}$
$2_-^{1+6} : S_5$	2560	3840	13	16	$2^3 \cdot (2 \times S_4)$
$2^7 \cdot S_5$	2560	3840	16	12	$4^2 : D_{12}$
$(2^5 \times 4) : S_5$	2560	3840	13	16	$2^3 \cdot (2 \times S_4)$
$(2 \times 6^3) \cdot (S_3 \times S_3)$	2592	3888	16	12	$6^2 : D_{12}$
$\mathrm{P\Sigma L}_2(25)$	2600	3900	15	10	$2 \times A_5$
$\mathrm{P\Sigma L}_2(25)$	2600	3900	14	12	$5^2 : D_{12}$, $[300, 25]$
$\mathrm{SL}_2(11) : D_{12}$	2640	3960	16	16	$Q_8 \cdot A_5$
					$\mathrm{SL}_2(5) : D_{12}$
$\mathrm{L}_2(16) : 2^2$	2720	4080	17	16	$2^2 \times A_5$
$(3^2 \times \mathrm{L}_3(2)) : D_{12}$	3024	4536	16	18	$(3^2 \times \mathrm{L}_3(2)) : 2^2$
$2^5 \cdot ((A_4 \times A_4) : 2^2)$	3072	4608	16	16	$2^3 \cdot (2 \times S_4)$
$2^4 \cdot (2_+^{1+4} : S_3^2)$	3072	4608	15	12	$4^2 : D_{12}$
$(3^3 \times A_5) : D_{12}$	3240	4860	16	12	$S_3 \times S_4$
					$6^2 : D_{12}$
$(3^3 \times 6) : S_5$	3240	4860	18	10	$(3^2 \times 6) : S_4$
$\mathrm{SL}_2(17) : 2^2$	3264	4896	16	16	$\mathrm{SL}_2(17) : 2^2$
$\mathrm{L}_3(2) : S_5$	3360	5040	13	16	$\mathrm{L}_3(2) : S_4$
$(2 \cdot A_7) : 2^2$	3360	5040	14	16	$Q_8 \cdot A_5$
					$(2 \cdot A_7) : 2^2$
$(3^2 \times Q_8 \times Q_8) \cdot S_3^2$	3456	5184	16	16	$\mathrm{GL}_2(3) : S_3$
					$(3 \times \mathrm{SL}_2(3)) : D_{12}$
$6^2 \cdot ((A_4 \times A_4) : 2^2)$	3456	5184	14	12	$\mathrm{GL}_2(3) : S_3$
					$12^2 : D_{12}$
$(2^4 : 2^2) : (3_+^{1+2} : D_{12})$	3456	5184	14	12	$4^2 : (S_3 \times S_3)$

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					$(3 \times \mathrm{SL}_2(3)) : D_{12}$
$(2^5 \times 6) \cdot (3^2 : D_{12})$	3456	5184	14	12	$S_3 \times S_4$
					$6^2 : D_{12}$
$(A_4 \times A_4 \times A_4) : D_{12}$	3456	5184	16	12	$S_3 \times S_4$
					$(2^6 : 3^2) : D_{12}$
$(A_4 \times A_4 \times A_4) : D_{12}$	3456	5184	18	12	$3^2 : D_{12}$
					$S_3 \times S_4$
$2^6 \cdot \mathrm{PGL}_2(7)$	3584	5376	15	14	$2^6 \cdot \mathrm{PGL}_2(7)$
$\mathrm{GL}_3(3) : 2$	3744	5616	16	12	$\mathrm{GL}_3(3) : 2$
$\mathrm{GL}_3(3) : 2$	3744	5616	18	12	$4 \cdot S_4$
					$\mathrm{GL}_3(3) : 2$
$5^4 : (S_3 \times S_3)$	3750	5625	19	12	$5^2 : D_{12}$
$(2^4 : \mathrm{SL}_2(5)) : D_{12}$	3840	5760	16	16	$2^3 \cdot (2 \times S_4)$
					$(2^2 \times 6) \cdot (2 \times S_4)$
$2^5 \cdot (S_3 \times S_5)$	3840	5760	16	16	$2^3 \cdot (2 \times S_4)$
					$(2^2 \times 6) \cdot (2 \times S_4)$
$(2^5 \times S_3) : S_5$	3840	5760	18	12	$4^2 : D_{12}$
					$4^2 : (S_3 \times S_3)$
$S_3 \times S_3 \times \mathrm{L}_2(11)$	3960	5940	16	12	$D_{12} \times A_5$
$\mathrm{U}_3(3) : 2^2$	4032	6048	18	12	$4 \cdot S_4$
					$2 \times \mathrm{PGL}_2(7)$
$2^2 \times \mathrm{L}_2(23)$	4048	6072	18	16	$2^2 \times \mathrm{L}_2(23)$
$\mathrm{PGL}_2(29)$	4060	6090	16	16	$\mathrm{PGL}_2(29)$
$\mathrm{L}_2(37)$	4218	6327	14	18	$\mathrm{L}_2(37)$
$\mathrm{SL}_2(13) : D_{12}$	4368	6552	18	16	$\mathrm{SL}_2(13) : 2^2$
					$\mathrm{SL}_2(13) : D_{12}$
$(3^3 \times 9) \cdot (3^2 : D_{12})$	4374	6561	16	12	$3^3 \cdot (S_3 \times S_3)$

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					$9^2 : D_{12}$
$(3 \times 9^2) \cdot (3^2 : D_{12})$	4374	6561	14	16	$(3^2 \times 9) \cdot S_3^2$
$3^5 \cdot (3^2 : D_{12})$	4374	6561	16	12	$3_+^{1+2} \cdot D_{12}$
					$3^3 \cdot (S_3 \times S_3)$
$3^4 \cdot (3_+^{1+2} : D_{12})$	4374	6561	16	12	$3^3 \cdot (S_3 \times S_3)$
					$9^2 : D_{12}$
$(3^2 \times 9^2) \cdot (S_3 \times S_3)$	4374	6561	18	12	$3_+^{1+2} \cdot D_{12}$
					$9^2 : D_{12}$
$3^3 \cdot (3^3 \cdot (S_3 \times S_3))$	4374	6561	14	12	$3_+^{1+2} \cdot D_{12}$
					$3^3 \cdot (S_3 \times S_3)$
$(3^2 \times 3_+^{1+2}) \cdot (3^2 : D_{12})$	4374	6561	14	12	$3_+^{1+2} \cdot D_{12}$
					$3^3 \cdot (S_3 \times S_3)$
$(3^3 : 3_+^{1+2}) \cdot (S_3 \times S_3)$	4374	6561	20	12	$3^2 : D_{12}$
					$3_+^{1+2} \cdot D_{12}$
$(3^3 : 3^3) \cdot (S_3 \times S_3)$	4374	6561	14	12	$3_+^{1+2} \cdot D_{12}$
$\mathrm{SL}_2(19) : 2^2$	4560	6840	16	18	$\mathrm{SL}_2(19) : 2^2$
$(2^3 \times 6) \cdot ((A_4 \times A_4) : 2^2)$	4608	6912	16	16	$\mathrm{GL}_2(3) : S_3$
					$(2^2 \times 6) \cdot (2 \times S_4)$
$((2 \times 6 \times D_8) : D_8) \cdot S_3^2$	4608	6912	16	12	$(2^2 \times 6) \cdot (2 \times S_4)$
$(4^3 \times 12) \cdot (S_3 \times S_3)$	4608	6912	20	12	$4^2 : (S_3 \times S_3)$
$((Q_8 \times Q_8) : A_4) \cdot S_3^2$	4608	6912	16	12	$\mathrm{GL}_2(3) : S_3$
					$S_4 \times S_4$
$3^5 : S_5$	4860	7290	20	10	$3^3 : S_4$, [648, 703]
$\mathrm{L}_2(17) : D_{12}$	4896	7344	16	16	$2 \times \mathrm{PGL}_2(17)$
					$\mathrm{L}_2(17) : D_{12}$
$\mathrm{PGL}_2(31)$	4960	7440	16	16	$\mathrm{PGL}_2(31)$
$(5^2 \times 10) \cdot S_5$	5000	7500	18	16	$(5^2 \times 10) : S_4$

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$S_3 \times S_7$	5040	7560	18	16	$2^2 \times A_5$ $S_3 \times S_7$
$2 \times (3 : S_7)$	5040	7560	18	16	$D_{12} \times A_5$ $2 \times S_7$
$(3 \cdot A_7) : 2^2$	5040	7560	16	14	$D_{12} \times A_5$ $(3 \cdot A_7) : 2^2$
$(3 \cdot A_7) : 2^2$	5040	7560	16	16	$D_{12} \times A_5$ $(3 \cdot A_7) : 2^2$
$(3 \cdot A_7) : 2^2$	5040	7560	16	16	$2^2 \times A_5$ $(3 \cdot A_7) : 2^2$
$(2 \times 4) \cdot (2^5 : S_5)$	5120	7680	16	16	$2^3 \cdot (2 \times S_4)$ $2^3 \cdot (2 \times S_4)$
$(2^6 : \mathrm{SL}_2(5)) : 2^2$	5120	7680	16	16	$2^3 \cdot (2 \times S_4)$
$((2 \times 4 \times D_8) : 2^2) : S_5$	5120	7680	16	16	$2^3 \cdot (2 \times S_4)$
$((2^3 \times D_8) : 2^2) \cdot S_5$	5120	7680	18	12	$4^2 : D_{12}$
$(2_-^{1+4} : 2^3) \cdot S_5$	5120	7680	16	16	$2^3 \cdot (2 \times S_4)$
$2_+^{1+4} : ((3 \times 3_+^{1+2}) : D_{12})$	5184	7776	20	12	$6^2 : D_{12}$
$2 \times \mathrm{P\Sigma L}_2(25)$	5200	7800	15	16	$2^2 \times A_5$
$2 \times \mathrm{P\Sigma L}_2(25)$	5200	7800	20	12	$5^2 : D_{12}$
$3 : (\mathrm{L}_3(3) : 2)$	5616	8424	16	16	$\mathrm{GL}_2(3) : S_3$ $\mathrm{L}_3(3) : 2$
$(A_4 \times A_4) : (2 \times S_5)$	5760	8640	16	16	$S_4 \times S_4$
$(2^4 : A_5) : (S_3 \times S_3)$	5760	8640	16	12	$4^2 : (S_3 \times S_3)$
$2 \times \mathrm{L}_2(37)^*$	8436	12654	18	18	$2 \times \mathrm{L}_2(37)$
$\mathrm{L}_2(47)$	8648	12972	19	20	$\mathrm{L}_2(47)$
$\mathrm{L}_2(59)$	17110	25665	17	20	$\mathrm{L}_2(59)$
$\mathrm{L}_2(61)$	18910	28365	17	18	$\mathrm{L}_2(61)$

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$L_2(71)$	29820	44730	23	14	$L_2(71)$
$L_2(73)$	32412	48618	21	22	$L_2(73)$
$L_2(83)$	47642	71463	20	18	$L_2(83)$
$PGL_2(41)$	11480	17220	17	18	$PGL_2(41)$
$PGL_2(43)$	13244	19866	18	18	$PGL_2(43)$
$PGL_2(53)$	24804	37206	19	22	$PGL_2(53)$
$PGL_2(67)$	50116	75174	21	22	$PGL_2(67)$
S_8	6720	10080	16	14	$PGL_2(7)$
$2 \times S_8$	13440	20160	18	16	$2 \times PGL_2(7)$
S_9	60480	90720	20	16	$S_4 \times A_5$ S_9
S_9	60480	90720	22	16	$S_4 \times A_5$ S_9
S_9	60480	90720	20	18	S_9

TABLE A.1: Some more embeddings of $\mathcal{DM}_1$ in G_1^3.

A.2 Embeddings of $\mathcal{DM}_0$ in G_2

In Table A.2 we record the complete list of the embeddings of the Djoković-Miller subamalgam $\mathcal{DM}_0$ in the Goldschmidt G_2-amalgam with completion group G, such that $324 \leq |G| \leq 50\,500$ obtained from the free amalgamated product $A_4 *_{2^2} D_{12}$ with the following presentation

$$\left\langle a, b, c \mid a^3,\ b^3,\ c^2,\ cbcb^{-1},\ (a^{-1}c)^3,\ (aca^2)^2,\ (ba^{-1}ca)^2 \right\rangle .$$

In this case an asterisk after the structure description of G indicates the presence of two inequivalent completions isomorphic to G, which results in two choices for H: apart form the one written in the corresponding row, there is always G itself, which is omitted. We notice that in these cases the corresponding coset graphs are always isomorphic.

G	$\|V(\Gamma)\|$	$\|E(\Gamma)\|$	$d(\Gamma)$	$g(\Gamma)$	H
$3^3 : A_4$	54	81	6	8	$3^3 : A_4$
$\mathrm{L}_2(11)^*$	110	165	7	10	A_5
$\mathrm{L}_2(13)^*$	182	273	8	12	$13 : 6$
$3 \times \mathrm{L}_2(11)^*$	330	495	10	12	$3 \times A_5$
A_7	420	630	10	12	$\mathrm{L}_3(2)$
$3 \times \mathrm{L}_2(13)^*$	546	819	12	12	$13 : 6$
$\mathrm{L}_3(3)^*$	936	1404	12	12	$\mathrm{AGL}_2(3)$
$\mathrm{L}_2(23)^*$	1012	1518	12	16	$\mathrm{L}_2(23)$
$3 \times A_7$	1260	1890	14	12	$3 \times \mathrm{L}_3(2)$
$3 \cdot A_7$	1260	1890	16	12	$3 \times \mathrm{L}_3(2)$
$3 \cdot A_7$	1260	1890	14	12	$\mathrm{L}_3(2), 3 \times \mathrm{L}_3(2)$
$\mathrm{L}_2(25)^*$	1300	1950	14	12	$\mathrm{L}_2(25)$
$A_4 \times \mathrm{L}_2(11)$	1320	1980	12	16	$A_4 \times A_5$
$(3^3 : 3_+^{1+2}) : A_4$	1458	2187	12	12	$(3^3 : 3_+^{1+2}) : A_4$
$A_4 \times \mathrm{L}_2(13)^*$	2184	3276	16	12	$D_{26} : A_4$
$\mathrm{SL}_2(11) : A_4^*$	2640	3960	16	16	$\mathrm{SL}_2(5) : A_4$
$3 \times \mathrm{L}_3(3)^*$	2808	4212	16	12	$3^3 : \mathrm{GL}_2(3)$
$3 \times \mathrm{L}_2(23)^*$	3036	4554	16	16	$3 \times \mathrm{L}_2(23)$
$3 \times (3 \cdot A_7)$	3780	5670	16	12	$3 \times \mathrm{L}_3(2)$
$3 \times \mathrm{L}_2(25)$	3900	5850	16	12	$3 \times \mathrm{L}_2(25)$
$\mathrm{L}_2(37)^*$	4218	6327	15	16	$\mathrm{L}_2(37)$
$\mathrm{SL}_2(13) : A_4^*$	4368	6552	16	16	$(13 \times Q_8) \cdot 6$
$3^4 \cdot (3^3 : A_4)$	4374	6561	20	12	$3^4 \cdot (3^3 : A_4)$
$A_4 \times A_7$	5040	7560	18	16	$A_4 \times \mathrm{L}_3(2)$

TABLE A.2: Embeddings of $\mathcal{DM}_0$ in G_2, with $324 \leq |G| \leq 50\,500$.

A.3 Embeddings of $\mathcal{DM}_3$ in G_2^1

In Table A.3 we record the complete list of the embeddings of the Djoković-Miller subamalgam $\mathcal{DM}_3$ in the Goldschmidt G_2^1-amalgam with completion group G, such that $648 \leq |G| \leq 100\,000$ obtained from the free amalgamated product $S_4 *_{D_8} D_{24}$ with the following presentation

$$\left\langle a,b \mid a^2,\ (b^3a)^3,\ b^{12},\ ab^4(b^2a)^2b^3ab^{-3}ab^{-1},\ (ab^3ab^{-3})^3 \right\rangle.$$

G	$\lvert V(\Gamma)\rvert$	$\lvert E(\Gamma)\rvert$	$d(\Gamma)$	$g(\Gamma)$	H
$3^3 : S_4$	54	81	6	8	$S_3 \wr 2$
$\mathrm{PGL}_2(11)^*$	110	165	7	10	$\mathrm{PGL}_2(11)$
$\mathrm{PGL}_2(13)^*$	182	273	8	12	$\mathrm{PGL}_2(13)$
$\mathrm{L}_2(11) : S_3^*$	330	495	10	12	$\mathrm{PGL}_2(11)$
S_7	420	630	10	12	$2 \times S_5$
$\mathrm{L}_2(23)^*$	506	759	11	12	$\mathrm{L}_2(23)$
$\mathrm{L}_2(13) : S_3^*$	546	819	12	12	$\mathrm{PGL}_2(13)$
$\mathrm{L}_2(25)$	650	975	13	10	S_5
$\mathrm{L}_3(3) : 2^*$	936	1404	12	12	$3_+^{1+2} : D_8$, $[216, 87]$
$2 \times \mathrm{L}_2(23)^*$	1012	1518	12	16	$2 \times \mathrm{L}_2(23)$
$A_7 : S_3 \sim 3 : S_7$	1260	1890	14	12	$2 \times S_5$
$3 \cdot S_7^*$	1260	1890	14	12	$2 \times S_5$
$3 \cdot S_7$	1260	1890	16	12	$2 \times S_5$
$2 \times \mathrm{L}_2(25)$	1300	1950	14	12	$2 \times S_5$
$\mathrm{L}_2(11) : S_4^*$	1320	1980	12	16	$\mathrm{L}_2(11) : D_8$
$(3^3 : 3_+^{1+2}) : S_4$	1458	2187	12	12	$3_+^{1+2} : D_8$
$\mathrm{L}_2(13) : S_4^*$	2184	3276	16	12	$\mathrm{L}_2(13) : D_8$
$\mathrm{SL}_2(11) : S_4^*$	2640	3960	16	16	$\mathrm{SL}_2(11) : D_8$
$\mathrm{L}_3(3) : S_3^*$	2808	4212	16	12	$3_+^{1+2} : D_8$
$S_3 \times \mathrm{L}_2(23)^*$	3036	4554	16	16	$2 \times \mathrm{L}_2(23)$

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$(3 \cdot A_7) : S_3$	3780	5670	16	12	$2 \times S_5$
$S_3 \times \mathrm{L}_2(25)$	3900	5850	16	12	$2 \times S_5$
$\mathrm{PGL}_2(37)^*$	4218	6327	15	16	$\mathrm{PGL}_2(37)$
$\mathrm{L}_2(47)^*$	4324	6486	16	16	$\mathrm{L}_2(47)$
$\mathrm{SL}_2(13) : S_4^*$	4368	6552	16	16	$\mathrm{SL}_2(13) : D_8$
$3^4 \cdot (3^3 : S_4)$	4374	6561	20	12	$3_+^{1+2} : D_8$
$\mathrm{L}_2(49)$	4900	7350	15	14	$\mathrm{PGL}_2(7)$
$A_7 : S_4 \sim A_4 : S_7$	5040	7560	18	16	$2^2 : S_5$, [480, 951]

TABLE A.3: Embeddings of $\mathcal{DM}_3$ in G_2^1, with $648 \leq |G| \leq 100\,000$.

A.4 Embeddings of $\mathcal{DM}_3$ in G_2^2

In Table A.4 we record the complete list of the embeddings of the Djoković-Miller subamalgam $\mathcal{DM}_3$ in the Goldschmidt G_2^2-amalgam with completion group G, such that $648 \leq |G| \leq 100\,000$ obtained from the universal completion group with the following presentation

$$\left\langle a, b, c, d \mid a^2,\ b^2,\ c^2,\ d^3,\ dbd^{-1}b,\ (cd^{-1})^2,\ (ba)^3, cbcacab, acbabcab \right\rangle .$$

G	$\lvert V(\Gamma)\rvert$	$\lvert E(\Gamma)\rvert$	$d(\Gamma)$	$g(\Gamma)$	H
$3^3 : S_4$	54	81	6	8	$S_3 \wr 2$
A_7	210	315	8	10	S_5
$2 \times A_7$	420	630	10	12	$2 \times S_5$
$3 \cdot A_7$	630	945	14	10	S_5
$S_3 \times A_7$	1260	1890	14	12	$2 \times S_5$
$2 \times (3 \cdot A_7)$	1260	1890	16	12	$2 \times S_5$
$3^3 \cdot (3^3 : S_4)$	1458	2187	12	12	$3_+^{1+2} : D_8$, [216, 87]
$S_3 \times (3 \cdot A_7)$	3780	5670	16	12	$2 \times S_5$
$\mathrm{L}_2(25) : S_3$	3900	5850	16	12	$2 \times S_5$

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$3^3 \cdot (3^3 : S_4)$	4374	6561	20	12	$3^{1+2}_{+} : D_8$
$S_4 \times A_7$	5040	7560	18	16	$2^2 : S_5$, $[480, 951]$

TABLE A.4: Embeddings of $\mathcal{DM}_3$ in G_2^2, with $648 \leq |G| \leq 100\,000$.

A.5 Embeddings of $\mathcal{DM}_1$ in G_2^4

In Table A.5 we record the complete list of the embeddings of the Djoković-Miller subamalgam $\mathcal{DM}_1$ in the Goldschmidt G_2^4-amalgam with completion group G, such that $648 \leq |G| \leq 100\,000$ obtained from the universal completion group with the following presentation

$$\langle a, b, c, d, e \mid R \rangle\,,$$

where

$$R = \{a^2, b^3, c^2, d^3, e^2, (ce)^2, (cd^2)^2, [d,a]\,, [c,b]\,, (ca)^2, (ab^2)^2, [e,d]\,, (caeb)^2, (ea)^4, (cb^2e)^3\}.$$

G	$\lvert V(\Gamma)\rvert$	$\lvert E(\Gamma)\rvert$	$d(\Gamma)$	$g(\Gamma)$	H
$3^3 : (2 \times S_4)$	54	81	6	8	$2 \times S_4$, $3^2 : D_{12}$
S_7	210	315	8	10	$S_3 \times S_4$, S_7
$2 \times S_7$	420	630	10	12	$S_3 \times S_4$, $2 \times S_7$
$(3 \cdot A_7) : 2$	630	945	14	10	$6^2 : D_{12}$, $(3 \cdot A_7) : 2$
$\mathrm{P\Sigma L}_2(25)$	650	975	13	10	$2 \times A_5$, $5^2 : D_{12}$
$S_3 \times S_7$	1260	1890	14	12	$S_3 \times S_4$, $S_3 \times S_7$
$\mathrm{Aut}(\mathrm{M}_{12})$	7920	11880	18	12	$\mathrm{PGL}_2(11)$, $\mathrm{Aut}(\mathrm{M}_{12})$

TABLE A.5: A few more embeddings of $\mathcal{DM}_1$ in G_2^4.

A.6 Embeddings of $\mathcal{DM}_1$ and $\mathcal{DM}_2$ in G_3

Table A.6 shows some more embeddings of the Djoković-Miller subamalgams $\mathcal{DM}_1$ and $\mathcal{DM}_2$ in the Goldschmidt G_3-amalgam. In the first part of the table we give the complete list of the completion groups of the Goldschmidt G_3-amalgam, with $360 \leq |G| \leq 67\,500$, while in the second part, after the dashed line, we add some more linear groups as completions. The completion groups are obtained as quotients of the universal completion group with the following presentation

$$\left\langle a,b,c,d \mid a^2,b^2,c^2,d^2,(db)^2,(dc)^3,(cb)^3,(dbc)^4,(acbcdc)^3,(abacbdcbd)^2,(cbdcbda)^3 \right\rangle .$$

An asterisk following the group structure indicates the presence of two inequivalent completions which, however, give rise to isomorphic coset graphs.

G	$\lvert V(\Gamma)\rvert$	$\lvert E(\Gamma)\rvert$	$d(\Gamma)$	$g(\Gamma)$	$H^{(1)}$	$H^{(2)}$
A_6	30	45	4	8	A_5	$3^2:4$
$3\cdot A_6$	90	135	8	10	A_5	$3_+^{1+2}:4$
$\mathrm{L}_2(17)^*$	204	306	9	12	$\mathrm{L}_2(17)$	$\mathrm{L}_2(17)$
$\mathrm{L}_3(3)$	468	702	13	12	$3^2:D_{12}$	$\mathrm{L}_3(3)$
$\mathrm{L}_2(23)^*$	506	759	10	14	$\mathrm{L}_2(23)$	$\mathrm{L}_2(23)$
$\mathrm{L}_2(25)$	650	975	11	12	$\mathrm{L}_2(25)$	S_5
$\mathrm{L}_2(31)^*$	1240	1860	15	16	$\mathrm{L}_2(31)$	$\mathrm{L}_2(31)$
$\mathrm{L}_3(2)\times\mathrm{L}_3(2)$	2352	3528	15	14	$S_4\times S_4$	$\mathrm{L}_3(2)$
$\mathrm{L}_2(41)^*$	2870	4305	14	14	$\mathrm{L}_2(41)$	$\mathrm{L}_2(41)$
$2^8:\mathrm{L}_3(2)^*$	3584	5376	17	12	$4^2:D_{12}$	$2^8:\mathrm{L}_3(2)$
$\mathrm{L}_2(47)^*$	4324	6486	15	18	$\mathrm{L}_2(47)$	$\mathrm{L}_2(47)$
$\mathrm{SL}_2(7):\mathrm{L}_3(2)$	4704	7056	15	16	$\mathrm{SL}_2(3)\cdot(2\times S_4)$	$2\times\mathrm{L}_3(2)$
$7^3\cdot\mathrm{L}_3(2)^*$	4802	7203	14	18	$7^3:S_4$	$7^3\cdot\mathrm{L}_3(2)$
$\mathrm{L}_3(2)\times A_6^*$	5040	7560	14	18	$S_4\times A_5$	$(3^2:4)\times\mathrm{L}_3(2)$
$2^6:(3\cdot A_6)^*$	5760	8640	16	16	$2^2\times A_5$	$2^6:(3_+^{1+2}:4)$
					$2^6:A_5$	$2^6:(3_+^{1+2}:4)$
$\mathrm{L}_2(71)^*$	14910	22365	18	20	$\mathrm{L}_2(71)$	$\mathrm{L}_2(71)$
$\mathrm{L}_2(73)^*$	16206	24309	16	22	$\mathrm{L}_2(73)$	$\mathrm{L}_2(73)$
$\mathrm{L}_2(79)^*$	20540	30810	18	20	$\mathrm{L}_2(79)$	$\mathrm{L}_2(79)$
$\mathrm{L}_2(89)^*$	29370	44055	19	22	$\mathrm{L}_2(89)$	$\mathrm{L}_2(89)$
$\mathrm{L}_3(5)^*$	31000	46500	22	12	$5^2:D_{12}$	$5^2:\mathrm{GL}_2(5)$
$\mathrm{L}_2(97)^*$	38024	57036	21	20	$\mathrm{L}_2(97)$	$\mathrm{L}_2(97)$

continues on next page

$\mathrm{L}_2(103)^*$	45526	68289	18	22	$\mathrm{L}_2(103)$	$\mathrm{L}_2(103)$
$\mathrm{L}_2(113)^*$	60116	90174	21	22	$\mathrm{L}_2(113)$	$\mathrm{L}_2(113)$

TABLE A.6: A few more embeddings of $\mathcal{DM}_1$ and $\mathcal{DM}_2$ in G_3.

A.7 Embeddings of $\mathcal{DM}_3$ in G_3^1

Table A.7 shows some more embeddings of the Djoković-Miller subamalgam $\mathcal{DM}_3$ in the Goldschmidt G_3^1-amalgam with completion group G, such that $720 \leq |G| \leq 100\,000$ obtained from the universal completion group with the following presentation

$$\langle a, b, c \mid R \rangle ,$$

where

$$\begin{aligned} R = \{ & a^6, b^2, c^6, a^3c^{-1}bcbc^{-1}, (c^{-1}bc^{-1}a)^2, bc^3bc^{-3}, (bc^{-1})^4, a^{-1}cbc^{-2}ba^{-1}c^{-1}bc^{-1}, \\ & cbacbcac^{-2}b, (bc^{-1}bc)^3, c^{-1}a^{-3}ca^{-1}c^{-2}bcba^{-1}bc^2bc^{-1}a^{-1}c^{-2}bcba^{-1}\}. \end{aligned}$$

G	$\lvert V(\Gamma)\rvert$	$\lvert E(\Gamma)\rvert$	$d(\Gamma)$	$g(\Gamma)$	H
S_6	30	45	4	8	$S_3 \wr 2$, S_5
$3 \cdot S_6$	90	135	8	10	S_5, $3_+^{1+2} : D_8$
$\mathrm{L}_3(3) : 2$	468	702	13	12	$3_+^{1+2} : D_8$, $\mathrm{L}_3(3) : 2$
$\mathrm{P\Sigma L}_2(25)$	650	975	11	12	$2 \times S_5$, $\mathrm{P\Sigma L}_2(25)$
$\mathrm{L}_3(2) \wr 2$	2352	3528	15	14	$\mathrm{PGL}_2(7)$, $S_4 \wr 2$
$2^6 : (3 \cdot S_6)$	5760	8640	16	16	$2^2 : S_5$, $2^6 : (3_+^{1+2} : D_8)$ $2^6 : S_5$, $2^6 : (3_+^{1+2} : D_8)$

TABLE A.7: A few more embeddings of $\mathcal{DM}_3$ in G_3^1.

A.8 The MAGMA code

In this last section we show the MAGMA[BCP97] code used to construct the locally projective graphs in Chapter 2.

```
e := [<x,Sphere(x,2)> :  x in V];

Gamma_2, V2, E2 := Graph< Set(V) | e>;

Xi, VV, EE := StandardGraph(sub< Gamma_2 | Components(Gamma_2)[i]>);
```

```
A, GV, GE := AutomorphismGroup(Xi);
G:=[x : x in LowIndexSubgroups(A,1000) | IsIsomorphic(x,g) ne false][1];

T:=AllCliques(Xi,3);

l:=T[1];

x:=VV!SetToIndexedSet(l)[1];
y:=VV!SetToIndexedSet(l)[2];
z:=VV!SetToIndexedSet(l)[3];

T_x:=[ t : t in T | x in t and t ne l];
m:=T_x[1];
n:=T_x[2];

G_x:=Stabiliser(G,GV,x);
G_y:=Stabiliser(G,GV,y);
G_z:=Stabiliser(G,GV,z);

G1_x:=&meet{Stabiliser(G,GV,SetToIndexedSet(Ball(x,1))[i]) : i in [1..#Ball(x,1)]};
G2_x:=&meet{Stabiliser(G,GV,SetToIndexedSet(Ball(x,2))[i]) : i in [1..#Ball(x,2)]};
G3_x:=&meet{Stabiliser(G,GV,SetToIndexedSet(Ball(x,3))[i]) : i in [1..#Ball(x,3)]};

G_l:=Stabiliser(G,GV,l);
G_m:=Stabiliser(G,GV,m);
G_n:=Stabiliser(G,GV,n);


G12_x:=G_l meet G_m meet G_n;
G_xyz:=G_x meet G_y meet G_z;

M_0_12:=G_x/G12_x;
M_12_1:=G12_x/G1_x;
M_0_1:=G_x/G1_x;
M_1_2:=G1_x/G2_x;
M_2_3:=G2_x/G3_x;

print IdentifyGroup(G_x), IdentifyGroup(M_0_12), IdentifyGroup(M_12_1),
IdentifyGroup(M_0_1), IdentifyGroup(M_1_2), IdentifyGroup(M_2_3);
if CanIdentifyGroup(Order(G)) then print
IdentifyGroup(G), #V, #E, Diameter(Gamma), Girth(Gamma);
else print Order(G), #V, #E, Diameter(Gamma), Girth(Gamma);
end if;
```

And finally the MAGMA[BCP97] code used to find the densely embedded subamalgams.

```
H1 := Complements(G_x,G12_x,G1_x);

H12 := [h1 meet G_xyz : h1 in H1];

j:=[j: j in [1..#H1] | IdentifyGroup(H12[j]) eq ID][1];
O := Orbits(H1[j],GV);
S:=[[a,b] : a, b in [1..#O] | a le b and O[a] join O[b] eq Sphere(x,1)];
if H1[j] eq Stabiliser(G_x,GV,O[S[1][1]]) and
H1[j] eq Stabiliser(G_x,GV,O[S[1][2]]) then
```

```
SIGMA := [sigma : sigma in G_l | H12[j]^sigma eq H12[j] and
(Image(sigma,GV,x) eq y or Image(sigma,GV,x) eq z)];

H2 := [sub<G_l | H12[j], sigma> : sigma in SIGMA];

H := [sub<G|H1[j],Generators(h2)> :  h2 in H2];
HH := [sub<G|H1[j],sigma > :  sigma in SIGMA];
if H eq HH then

print "H1: ", [IdentifyGroup(x) : x in H1];
print "H12: ", [IdentifyGroup(x) : x in H12];
print "H2: ", [IdentifyGroup(x) : x in H2];
if forall(k){ H[x] : x in [1..#H] | CanIdentifyGroup(Order(H[x]))}
then print "H: ", [IdentifyGroup(x) : x in H];
else print "Order of H: ", [Order(x) : x in H];
end if; end if; end if;
```

Appendix B

Some presentations in MAGMA

B.1 The groups from Chapter 3

```
G2_2 := Group< c, d |
c^2, d^4, (c*d)^7, (c,d)^6, (c*d*(c*d^2)^3)^2, (d^2,c*d*c)^3 >;

B<a,b,s,t> := Group< a, b, s, t |
a^4, b^4, (a,b), s^2, t^2, (s,t), a^s*b^-1, b^s*a^-1, a^t*a,b^t*b >;
/* IdentifyGroup(B); <64, 134> */

N<a,b,s,t,x> := Group< a, b, s, t, x |
a^4, b^4, (a,b), s^2, t^2, (s,t), a^s*b^-1, b^s*a^-1, a^t*a, b^t*b,
x^3, x*x^s, (x,s)^3, (x,t), a^x*b^-1, b^x*b*a >;
/* IdentifyGroup(N); <192, 956> */

Y1<a,b,s,t,y> := Group< a, b, s, t, y |
a^4, b^4, (a,b), s^2, t^2, (s,t), a^s*b^-1, b^s*a^-1, a^t*a,b^t*b,
y^3, (a*b*s)^y*b*a^3, (s*a^2)^y*s*b*a, (t*s*a^2)^y*t*s*b*a^3,
(a*b)^y*t*s*a^2, y^(b*t)*y*t*b^2 >;
/* IdentifyGroup(Y1); <192, 1494> */

X1<a,b,s,t,z> := Group< a, b, s, t, z |
a^4, b^4, (a,b), s^2, t^2, (s,t), a^s*b^-1, b^s*a^-1, a^t*a,b^t*b,
z^3, (a*b*s)^z*b*a^3, (s*a^2)^z*s*b^3*a^3, z^(b*t)*z, (a*b)^z*b^3*a^3,
(t*s*a^2)^z*t*s*b^2 >;
/* IdentifyGroup(X1); <192, 988> */

Y2<a,b,s,t,u> := Group< a, b, s, t, u |
a^4, b^4, (a,b), s^2, t^2, (s,t), a^s*b^-1, b^s*a^-1, a^t*a,b^t*b,
u^3, (a*b*s)^u*b*a^3, (s*a^2)^u*s*b^3*a^3, (a*b)^u*t*s*b*a^3,
u^(b*t)*u*t*s*b^2, (t*s*a^2)^u*b^3*a^3 >;
/* IdentifyGroup(Y2); <192, 1494> */

X2<a,b,s,t,v> := Group< a, b, s, t, v |
a^4, b^4, (a,b), s^2, t^2, (s,t), a^s*b^-1, b^s*a^-1, a^t*a,b^t*b,
v^3, (a*b*s)^v*s*b^3*a^3, (s*a^2)^v*s*b^2, (a*b)^v*t*s*a^2,
(t*s*a^2)^v*t*s*b*a^3, v^(b*t)*v*t*s*b*a^3 >;
/* IdentifyGroup(X2); <192, 988> */

G1<a,b,s,t,x,y> := Group< a, b, s, t, x, y |
a^4, b^4, (a,b), s^2, t^2, (s,t), a^s*b^-1, b^s*a^-1, a^t*a, b^t*b,
x^3, x*x^s, (x,s)^3, (x,t), a^x*b^-1, b^x*b*a,
y^3, (a*b*s)^y*b*a^3, (s*a^2)^y*s*b*a, (t*s*a^2)^y*t*s*b*a^3,
(a*b)^y*t*s*a^2, y^(b*t)*y*t*b^2, b*y*x^2*y*x*y^2*x*a^3 >;
```

```
/* IdentifyGroup(G1); <1344, 814> */

M1<a,b,s,t,x,z> := Group< a, b, s, t, x, z |
a^4, b^4, (a,b), s^2, t^2, (s,t), a^s*b^-1, b^s*a^-1, a^t*a, b^t*b,
x^3, x*x^s, (x,s)^3, (x,t), a^x*b^-1, b^x*b*a,
z^3, (a*b*s)^z*b*a^3, (s*a^2)^z*s*b^3*a^3, z^(b*t)*z, (a*b)^z*b^3*a^3,
(t*s*a^2)^z*t*s*b^2, (x^2*z^2)^3*b^2*a*x*(z^2*x^2)^2*z >;
/* CompositionFactors(PermutationGroup(M1));
    G
     |  Cyclic(2)
     *
     |  2A(2, 3)               = U(3, 3)
    1

#Kernel(Homomorphisms(M1,PermutationGroup(G2_2))[1]);
1 */

M2<a,b,s,t,x,v> := Group< a, b, s, t, x, v |
a^4, b^4, (a,b), s^2, t^2, (s,t), a^s*b^-1, b^s*a^-1, a^t*a, b^t*b,
x^3, x*x^s, (x,s)^3, (x,t), a^x*b^-1, b^x*b*a,
v^3, (a*b*s)^v*s*b^3*a^3, (s*a^2)^v*s*b^2, (a*b)^v*t*s*a^2,
(t*s*a^2)^v*t*s*b*a^3, v^(b*t)*v*t*s*b*a^3,
(x^2*v^2)^3*b^3*x^2*v^2*a^3*x*v^2*x*v >;
/* CompositionFactors(PermutationGroup(M2));
    G
     |  Cyclic(2)
     *
     |  2A(2, 3)               = U(3, 3)
    1

#Kernel(Homomorphisms(M2,PermutationGroup(G2_2))[1]);
1 */

G2<a,b,s,t,y,z,u,v> := Group< a, b, s, t, y, z, u, v |
a^4, b^4, (a,b), s^2, t^2, (s,t), a^s*b^-1, b^s*a^-1, a^t*a,b^t*b,
y^3, (a*b*s)^y*b*a^3, (s*a^2)^y*s*b*a, (t*s*a^2)^y*t*s*b*a^3,
(a*b)^y*t*s*a^2,y^(b*t)*y*t*b^2, z^3, (a*b*s)^z*b*a^3, (s*a^2)^z*s*b^3*a^3,
z^(b*t)*z,(a*b)^z*b^3*a^3, (t*s*a^2)^z*t*s*b^2, u^3, (a*b*s)^u*b*a^3,
(s*a^2)^u*s*b^3*a^3, (a*b)^u*t*s*b*a^3, u^(b*t)*u*t*s*b^2,
(t*s*a^2)^u*b^3*a^3, v^3, (a*b*s)^v*s*b^3*a^3, (s*a^2)^v*s*b^2,
(a*b)^v*t*s*a^2, (t*s*a^2)^v*t*s*b*a^3, v^(b*t)*v*t*s*b*a^3, v^z*v^2,
z*a^2*b^2*u^2*(s*t*y^2*u)^-1, z*a^3*b^2*s*t*(b*y^2*z*u^2)^-1,
z*s*u^2*(s*t*y*z^2)^-1, z^-1*a*v*u^-1*v*a >;
 */ IdentifyGroup(G2); <576, 8282> */

G<a,b,s,t,x,y,z,u,v> := Group< a, b, s, t, x, y, z, u, v |
a^4, b^4, (a,b), s^2, t^2, (s,t), a^s*b^-1, b^s*a^-1, a^t*a,b^t*b,
x^3, x*x^s, (x,s)^3, (x,t), a^x*b^-1, b^x*b*a,
y^3, (a*b*s)^y*b*a^3, (s*a^2)^y*s*b*a, (t*s*a^2)^y*t*s*b*a^3,
(a*b)^y*t*s*a^2, y^(b*t)*y*t*b^2, b*y*x^2*y*x*y^2*x*a^3,
z^3, (a*b*s)^z*b*a^3, (s*a^2)^z*s*b^3*a^3,
z^(b*t)*z,(a*b)^z*b^3*a^3, (t*s*a^2)^z*t*s*b^2, u^3, (a*b*s)^u*b*a^3,
(s*a^2)^u*s*b^3*a^3, (a*b)^u*t*s*b*a^3, u^(b*t)*u*t*s*b^2,
(t*s*a^2)^u*b^3*a^3, v^3, (a*b*s)^v*s*b^3*a^3, (s*a^2)^v*s*b^2,
(a*b)^v*t*s*a^2, (t*s*a^2)^v*t*s*b*a^3, v^(b*t)*v*t*s*b*a^3, v^z*v^2,
z*a^2*b^2*u^2*(s*t*y^2*u)^-1, z*a^3*b^2*s*t*(b*y^2*z*u^2)^-1,
z*s*u^2*(s*t*y*z^2)^-1, z^-1*a*v*u^-1*v*a,
```

```
x*b*y^-1*x^-1*y^-1*b^-1*y^-1*x^-1*y^-1*b*y*x*y*x^-1*(a*v*u^-1)^-2,
y*(u*v^-1*a^2)^-1, x*b*x^-1*a^-1, (x*y)^2*x^-1*(a*v*u^-1*a^2)^-1,
x*u^-1*x^-1*v^-1*x^-1*u^-1*x^-1*u^-1*x^-1*v^-1*b^-1*x*u^-1*x^-1*v*x^-1*u,
x^-1*v*x*v^-1*x^-1*v^-1*x*v*x^-1*v^-1*x*v^-1 >;
```

B.2 The Goldschmidt's lemma and the six amalgams $\mathcal{A}_i$ from Chapter 5

In this section we list the presentations of the universal completion groups of the six amalgams $\mathcal{A}_i$ from Chapter 5, preceded by a GAP [Gap] implementation of the Goldschmidt's lemma.

```
Goldschmidt := function (G1,G2,G12)
local Aut_1, Aut_2, Aut_12, iso1, iso2, N_1, N_2, psi_1, psi_2, A_1, A_2, Inn_12, q, Out_12, O_1, O_2, n, m, i, fpa;
fpa:=[];;
Aut_1 := AutomorphismGroup(G1);;
Aut_2 := AutomorphismGroup(G2);;
Aut_12 := AutomorphismGroup(G12);;
iso1 := NiceMonomorphism(Aut_1);;
iso2 := NiceMonomorphism(Aut_2);;
N_1 := PreImage(iso1,AsGroup(Filtered(AsList(Image(iso1)), x->Image(PreImage(iso1,x),G12)=G12)));;
N_2 := PreImage(iso2,AsGroup(Filtered(AsList(Image(iso2)), x->Image(PreImage(iso2,x),G12)=G12)));;
psi_1 := GroupHomomorphismByFunction(N_1,Aut_12,a->RestrictedMapping(a,G12));;
psi_2 := GroupHomomorphismByFunction(N_2,Aut_12,a->RestrictedMapping(a,G12));;
A_1 := Image(psi_1);;
A_2 := Image(psi_2);;
Inn_12 := InnerAutomorphismsAutomorphismGroup(Aut_12);;
q := NaturalHomomorphismByNormalSubgroup(Aut_12,Inn_12);;
Out_12:=Image(q);;
O_1 := Image(q,A_1);;
O_2 := Image(q,A_2);;
n := DoubleCosetRepsAndSizes(Aut_12,A_1,A_2);;
m := DoubleCosetRepsAndSizes(Out_12,O_1,O_2);;
if Length(n)=Length(m) then
for i in [1..Length(n)] do
fpa[i]:=FreeProductWithAmalgamation(G1,G2,n[i][1]);;
Print([i,GeneratorsOfGroup(fpa[i]),RelatorsOfFpGroup(fpa[i])],"\n");od;fi;
return;
end;;
```

```
> A1;
Finitely presented group A1 on 5 generators
Relations
    f1^2 = Id(A1)
    (f3 * f2)^2 = Id(A1)
    f3^4 = Id(A1)
    (f3 * f2^-2)^2 = Id(A1)
    f2^6 = Id(A1)
    f2^-1 * f1 * f2^2 * f1 * f2^-1 = Id(A1)
    f1 * f2^-1 * f1 * f2^-1 * f1 * f2 * f1 * f2 = Id(A1)
    (f3 * f1)^4 = Id(A1)
    (f3 * f1 * f3^-1 * f1)^2 = Id(A1)
    f3^-1 * f2^-1 * f3^2 * f2^-3 * f3 * f2^2 = Id(A1)
    f1 * f3 * f1 * f2^-1 * f3 * f2^-1 * f1 * f3 * f1 * f2 * f3^-1 * f2 = Id(A1)
    (f3^-1 * f1 * f2^-1)^4 = Id(A1)
    f3^2 * f2^-1 * f3 * f2^-1 * f1 * f3^-1 * f2 * f3^-2 * f2^-1 * f1 = Id(A1)
    (f1 * f3 * f1 * f2)^4 = Id(A1)
    f4^4 = Id(A1)
    f5^6 = Id(A1)
    (f4 * f5^-1)^4 = Id(A1)
    (f4^-1 * f5 * f4 * f5^-1 * f4^-1)^2 = Id(A1)
    (f4^-1 * f5^-1 * f4^-1 * f5 * f4^-1)^2 = Id(A1)
    (f4^-1 * f5^2)^4 = Id(A1)
    (f4^-1 * f5^-2)^4 = Id(A1)
    f4^-1 * f5^-1 * f4^2 * f5^-2 * f4^-2 * f5^-1 * f4^-2 * f5^-2 * f4^-1 = Id(A1)
    f4^-1 * f5^-1 * f4^-1 * f5^-1 * f4^-1 * f5^-1 * f4 * f5^3 * f4^-2 * f5^2 = Id(A1)
    f5^-2 * f4 * f5^-1 * f4^-1 * f5^-1 * f4^-1 * f5^-1 * f4^-1 * f5^2 * f4^2 * f5^-1 = Id(A1)
    f5 * f4 * f5^-1 * f4^-1 * f5^2 * f4 * f5 * f4^-1 * f5^2 * f4 * f5^-1 * f4^-1 = Id(A1)
    f5^-2 * f4^-1 * f5^-1 * f4^-1 * f5 * f4^-1 * f5^-2 * f4^-1 * f5 * f4^-1 * f5^2 * f4 * f5^-1 = Id(A1)
    f3^-1 * f2 * f3^-2 * f2 * f4^2 * f5 * f4^2 * f5^-1 = Id(A1)
    f3 * f2 * f3^-1 * f2 * f4^2 = Id(A1)
    f1^-1 * f3^-1 * f1^-1 * f3 * f5^-2 * f4 * f5^-1 * f4^-1 * f5^-1 * f4 * f5^-1 * f4^-1 * f5 = Id(A1)
    f2 * f1^-1 * f2^-1 * f1^-1 * f3^-1 * f1^-1 * f2^-1 * f1^-1 * f3^-1 * f2^-1 * f4^2 * f5^-3 * f4^-2 * f5^3 = Id(A1)
    f1^-1 * f4 * f5 * f4^2 * f5 * f4^2 * f5^-1 * f4 * f5^-1 * f4 * f5^-1 * f4^-1 * f5 = Id(A1)
    f2 * f1^-1 * f2^-1 * f1^-1 * f4 * f5 * f4 * f5^2 * f4^2 * f5^-1 * f4^-1 * f5^-2 * f4^-1 * f5^4 = Id(A1)
    f3^-1 * f2 * f3^-3 * f2 * f4 * f5 * f4^2 * f5^-1 = Id(A1)
    f2^-3 * f4 * f5 * f4 * f5^2 * f4^2 * f5^-1 * f4^-1 * f5^-2 * f4^-1 * f5 = Id(A1)

> A2;
Finitely presented group A2 on 5 generators
Relations
    f1^2 = Id(A2)
    f3^4 = Id(A2)
    (f3 * f2)^2 = Id(A2)
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    f1 * f2^2 * f1 * f2^-2 = Id(A2)
    (f2^2 * f3^-1)^2 = Id(A2)
    f2^6 = Id(A2)
    f1 * f2 * f1 * f2 * f1 * f2^-1 * f1 * f2^-1 = Id(A2)
    (f3 * f1)^4 = Id(A2)
    f3^2 * f2^-1 * f3 * f1 * f2^-1 * f3 * f2^-1 * f3^-2 * f1 * f2 = Id(A2)
    f3^-1 * f2 * f3^-1 * f1 * f3^2 * f2^-1 * f3 * f2^-1 * f1 * f2^-1 * f3^-1 = Id(A2)
    (f3 * f2 * f1 * f2^-1 * f3^-1 * f1)^2 = Id(A2)
    f3^-1 * f2^2 * f1 * f3 * f2^-1 * f3 * f2^-1 * f1 * f3 * f2^-1 * f3 * f2 * f1 * f2 * f3^-1 * f2^-1 * f1 = Id(A2)
    f4^4 = Id(A2)
    f5^6 = Id(A2)
    (f4^-2 * f5^-1 * f4 * f5)^2 = Id(A2)
    (f4^-2 * f5 * f4^-1 * f5^-1)^2 = Id(A2)
    (f4^-1 * f5^-2)^4 = Id(A2)
    (f5^2 * f4 * f5^-2 * f4^-2)^2 = Id(A2)
    (f4^-2 * f5^-2 * f4 * f5^2)^2 = Id(A2)
    f4 * f5^-1 * f4^-1 * f5 * f4^-1 * f5^-2 * f4 * f5^-1 * f4^-1 * f5^-2 * f4^-1 * f5 = Id(A2)
    f4 * f5^3 * f4^-1 * f5^-3 * f4 * f5^-3 * f4^-1 * f5^-3 = Id(A2)
    f5 * f4 * f5 * f4 * f5^-2 * f4 * f5^2 * f4^-1 * f5 * f4^-1 * f5^-2 * f4^-1 * f5 = Id(A2)
    f5^-1 * f4^-1 * f5^2 * f4 * f5^-2 * f4^-1 * f5^-1 * f4 * f5^2 * f4^-1 * f5^-2 * f4 = Id(A2)
    f5^2 * f4^-2 * f5^-1 * f4^-2 * f5^-2 * f4^-1 * f5 * f4^-1 * f5 * f4^-1 * f5 * f4 * f5 * f4^-2 * f5 = Id(A2)
    f3^-1 * f2 * f3^-2 * f2 * f4^2 * f5 * f4^2 * f5^-1 = Id(A2)
    f3 * f2 * f3^-1 * f2 * f4^2 = Id(A2)
    f1^-1 * f3^-1 * f1^-1 * f3 * f5^-2 * f4 * f5^-1 * f4^-1 * f5^-1 * f4 * f5^-1 * f4^-1 * f5 = Id(A2)
    f2 * f1^-1 * f2^-1 * f1^-1 * f3^-1 * f1^-1 * f2^-1 * f1^-1 * f3^-1 * f2^-1 * f4^2 * f5^-3 * f4^-2 * f5^3 = Id(A2)
    f1^-1 * f4 * f5 * f4^2 * f5 * f4^2 * f5^-1 * f4 * f5^-1 * f4 * f5^-1 * f4^-1 * f5 = Id(A2)
    f2 * f1^-1 * f2^-1 * f1^-1 * f4 * f5 * f4 * f5 * f4^2 * f5 * f4 * f5^-1 * f4 * f5^-1 * f4 * f5 * f4 = Id(A2)
    f3^-1 * f2 * f3^-3 * f2 * f4 * f5 * f4^2 * f5^-1 = Id(A2)
    f2^-3 * f4 * f5 * f4 * f5^2 * f4 * f5^-1 * f4 * f5^-1 * f4 * f5 * f4 * f5^3 = Id(A2)

> A3;
Finitely presented group A3 on 5 generators
Relations
    f1^2 = Id(A3)
    f3^4 = Id(A3)
    (f3 * f2)^2 = Id(A3)
    (f3 * f2^-2)^2 = Id(A3)
    f1 * f2^2 * f1 * f2^-2 = Id(A3)
    f2^6 = Id(A3)
    f1 * f2 * f1 * f2 * f1 * f2^-1 * f1 * f2^-1 = Id(A3)
    (f3 * f1)^4 = Id(A3)
    (f1 * f3 * f1 * f3^-1)^2 = Id(A3)
    (f2^-1 * f3^-1 * f1)^4 = Id(A3)
    f3^2 * f2^-1 * f3 * f1 * f3 * f2 * f3^-1 * f2^-1 * f3 * f1 * f2^-1 = Id(A3)
    f4^4 = Id(A3)
    f5^6 = Id(A3)
    (f5^-1 * f4)^4 = Id(A3)
    (f4^-2 * f5 * f4^-1 * f5^-1)^2 = Id(A3)
    (f5 * f4^-2 * f5^-1 * f4)^2 = Id(A3)
    (f4 * f5^2)^4 = Id(A3)
    f4 * f5^2 * f4^2 * f5^-2 * f4 * f5^2 * f4^-2 * f5^-2 = Id(A3)
    (f4^-2 * f5^-1 * f4^-2 * f5^-2)^2 = Id(A3)
    f5^-2 * f4^2 * f5^2 * f4^-1 * f5^-2 * f4^-2 * f5^2 * f4^-1 = Id(A3)
    f4^-1 * f5^-1 * f4 * f5 * f4 * f5^-2 * f4 * f5^-1 * f4^-1 * f5^-2 * f4^-1 * f5 = Id(A3)
    f4^2 * f5^-1 * f4^-1 * f5^-1 * f4^-2 * f5^-1 * f4 * f5^-2 * f4^-1 * f5^-2 * f4 * f5^-1 = Id(A3)
    f5^3 * f4^-1 * f5^-3 * f4 * f5^-3 * f4^-1 * f5^-3 * f4 = Id(A3)
    f3^-1 * f2 * f3^-2 * f2 * f4 * f5 * f4 * f5 * f4 * f5^-2 * f4 * f5^2 * f4 * f5^-3 * f4^-1 * f5^2 = Id(A3)
    f3 * f2 * f3^-1 * f2 * f4^2 * f5^-3 * f4^-2 * f5 * f4 * f5^-3 * f4^-1 * f5^-1 = Id(A3)
    f1^-1 * f3^-1 * f1^-1 * f3 * f4 * f5 * f4 * f5^2 * f4^2 * f5^-2 * f4 * f5^2 * f4 * f5^3 = Id(A3)
    f2 * f1^-1 * f2^-1 * f1^-1 * f3^-1 * f1^-1 * f2^-1 * f1^-1 * f3^-1 * f2^-1 * f4^2 * f5^-3 * f4^-2 * f5^3 = Id(A3)
    f1^-1 * f4^-1 * f5^-1 * f4 * f5 * f4^2 * f5^-2 * f4 * f5 * f4^-1 * f5 * f4^-1 = Id(A3)
    f2 * f1^-1 * f2^-1 * f1^-1 * f5 * f4^-1 * f5^-1 * f4 * f5^-1 * f4 * f5^-1 * f4^-1 * f5^4 = Id(A3)
    f3^-1 * f2 * f3^-3 * f2 * f4^2 * f5 * f4^-2 * f5^-2 * f4^-1 * f5^-1 * f4^2 * f5^3 * f4^-1 * f5^3 = Id(A3)
    f2^-3 * f5 * f4^2 * f5^-1 = Id(A3)

> A4;
Finitely presented group A4 on 5 generators
Relations
    f1^2 = Id(A4)
    f3^4 = Id(A4)
    (f3 * f2)^2 = Id(A4)
    (f3 * f2^-2)^2 = Id(A4)
    f1 * f2^2 * f1 * f2^-2 = Id(A4)
    f2^6 = Id(A4)
    f1 * f2 * f1 * f2 * f1 * f2^-1 * f1 * f2^-1 = Id(A4)
    (f3 * f1)^4 = Id(A4)
    (f1 * f3 * f1 * f3^-1)^2 = Id(A4)
    f3^2 * f2^-1 * f3 * f1 * f3 * f2 * f3^-1 * f2^-1 * f3 * f1 * f2^-1 = Id(A4)
    (f2^-1 * f3^-1 * f1)^4 = Id(A4)
    f4^4 = Id(A4)
    f5^6 = Id(A4)
    (f4^-2 * f5^-1 * f4 * f5)^2 = Id(A4)
    f4^2 * f5 * f4^-1 * f5^-1 * f4^-2 * f5 * f4^-1 * f5^-1 = Id(A4)
    f4^2 * f5^-1 * f4^-2 * f5^-2 * f4^-2 * f5^-1 * f4^-2 * f5^-2 = Id(A4)
    (f4 * f5^2 * f4^-1 * f5^-2 * f4)^2 = Id(A4)
    f5^-1 * f4^-1 * f5^2 * f4^2 * f5^3 * f4 * f5^-1 * f4^-1 * f5^-1 * f4^-1 = Id(A4)
    f4^-1 * f5^2 * f4^-1 * f5 * f4 * f5^-1 * f4 * f5^-1 * f4 * f5 * f4^-1 * f5^2 = Id(A4)
    f4^-1 * f5^-1 * f4 * f5 * f4^-1 * f5^-2 * f4 * f5^-1 * f4^-1 * f5^-2 * f4 * f5 = Id(A4)
    f4^-1 * f5^2 * f4 * f5^-3 * f4^-1 * f5^2 * f4^-1 * f5^-2 * f4^-1 * f5 * f4^-1 * f5^-2 = Id(A4)
    f5^-2 * f4^2 * f5^3 * f4^-1 * f5^-1 * f4^-1 * f5^-1 * f4^-1 * f5^-1 * f4^-1 * f5^-1 * f4^-2 * f5^-1 = Id(A4)
    f5 * f4^-1 * f5 * f4^2 * f5^2 * f4^-1 * f5 * f4^-2 * f5^2 * f4^-1 * f5 * f4^-2 * f5^2 * f4^-1 * f5 * f4^-2 * f5 = Id(A4)
    f3^-1 * f2 * f3^-2 * f2 * f4 * f5 * f4 * f5 * f4^2 * f5 * f4^2 * f5^-2 * f4 * f5^2 * f4 = Id(A4)
    f3 * f2 * f3^-1 * f2 * f5^-2 * f4 * f5^-1 * f4^-2 * f5^-1 * f4 * f5 * f4^-1 * f5 * f4^-1 = Id(A4)
    f1^-1 * f3^-1 * f1^-1 * f3 * f4 * f5 * f4 * f5^2 * f4^2 * f5^-2 * f4 * f5^2 * f4 * f5^3 = Id(A4)
    f2 * f1^-1 * f2^-1 * f1^-1 * f3^-1 * f1^-1 * f2^-1 * f1^-1 * f3^-1 * f2^-1 * f4^2 * f5^-3 * f4^-2 * f5^3 = Id(A4)
    f1^-1 * f4^-1 * f5^-1 * f4 * f5 * f4^2 * f5^-2 * f4 * f5 * f4^-1 * f5 * f4^-1 = Id(A4)
    f2 * f1^-1 * f2^-1 * f1^-1 * f5 * f4^-1 * f5^-1 * f4 * f5^-1 * f4 * f5^-1 * f4^-1 * f5^4 = Id(A4)
    f3^-1 * f2 * f3^-3 * f2 * f5^-1 * f4 * f5^2 * f4 * f5^3 = Id(A4)
```

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    f2^-3 * f5 * f4^2 * f5^-1 = Id(A4)

> A5;
Finitely presented group A5 on 5 generators
Relations
    f1^2 = Id(A5)
    (f3 * f2)^2 = Id(A5)
    f3^4 = Id(A5)
    (f3 * f2^-2)^2 = Id(A5)
    f1 * f2^2 * f1 * f2^-2 = Id(A5)
    f2^6 = Id(A5)
    f1 * f2^-1 * f1 * f2^-1 * f1 * f2 * f1 * f2 = Id(A5)
    (f3 * f1)^4 = Id(A5)
    (f1 * f3 * f1 * f3^-1)^2 = Id(A5)
    f3^-1 * f2^-1 * f3^2 * f2^-3 * f3 * f2^2 = Id(A5)
    f3^-2 * f2^-1 * f3 * f2^-1 * f1 * f2 * f3^-1 * f2 * f3^-2 * f1 = Id(A5)
    (f2^-1 * f3^-1 * f1)^4 = Id(A5)
    f4^4 = Id(A5)
    f5^6 = Id(A5)
    (f4 * f5^-1 * f4^-2 * f5)^2 = Id(A5)
    (f5 * f4^-2 * f5^-1 * f4)^2 = Id(A5)
    (f4^-1 * f5^-2)^4 = Id(A5)
    f5^2 * f4^2 * f5^-2 * f4 * f5^2 * f4^-2 * f5^-2 * f4 = Id(A5)
    f4 * f5^2 * f4^2 * f5 * f4^-2 * f5^2 * f4^-2 * f5 * f4 = Id(A5)
    f4^-1 * f5^2 * f4^-1 * f5^-1 * f4 * f5 * f4^-1 * f5^-1 * f4^-1 * f5^2 * f4 * f5 = Id(A5)
    f5 * f4 * f5^3 * f4^-1 * f5^-3 * f4 * f5^-3 * f4^-1 * f5^2 = Id(A5)
    f4 * f5^2 * f4^-1 * f5^-3 * f4 * f5 * f4^-1 * f5^-2 * f4 * f5^-3 * f4^-1 * f5^-1 = Id(A5)
    (f5^2 * f4 * f5^-2 * f4^-1 * f5^-3 * f4^-1)^2 = Id(A5)
    f3^-1 * f2 * f3^-2 * f2 * f4^2 * f5 * f4^2 * f5^-1 = Id(A5)
    f3 * f2 * f3^-1 * f2 * f4^2 = Id(A5)
    f1^-1 * f3^-1 * f1^-1 * f3 * f5^-2 * f4 * f5^-1 * f4^-1 * f5^-1 * f4 * f5^-1 * f4^-1 * f5 = Id(A5)
    f2 * f1^-1 * f2^-1 * f1^-1 * f3^-1 * f1^-1 * f2^-1 * f1^-1 * f3^-1 * f2^-1 * f4^2 * f5^-3 * f4^-2 * f5^3 = Id(A5)
    f1^-1 * f4 * f5 * f4^-1 * f5^-2 * f4 * f5 * f4 * f5^-1 * f4 * f5 * f4 = Id(A5)
    f2 * f1^-1 * f2^-1 * f1^-1 * f4 * f5 * f4 * f5^2 * f4^2 * f5^-1 * f4^-1 * f5^-2 * f4^-1 * f5^4 = Id(A5)
    f3^-1 * f2 * f3^-3 * f2 * f4 * f5 * f4^2 * f5^-1 = Id(A5)
    f2^-3 * f4 * f5 * f4 * f5^2 * f4^2 * f5^-1 * f4^-1 * f5^-2 * f4^-1 * f5 = Id(A5)

> A6;
Finitely presented group A6 on 5 generators
Relations
    f1^2 = Id(A6)
    f3^4 = Id(A6)
    (f3 * f2)^2 = Id(A6)
    f1 * f2^-2 * f1 * f2^2 = Id(A6)
    f2^6 = Id(A6)
    (f3 * f2^-2)^2 = Id(A6)
    f1 * f2 * f1 * f2 * f1 * f2^-1 * f1 * f2^-1 = Id(A6)
    (f1 * f3^-1 * f1 * f3)^2 = Id(A6)
    (f3 * f1)^4 = Id(A6)
    f3^2 * f2^-1 * f3 * f1 * f3^-2 * f2^-2 * f3^-1 * f1 * f2^-1 = Id(A6)
    f3 * f2^-1 * f1 * f2 * f3^-1 * f1 * f2^-1 * f3^-1 * f1 * f2^-1 * f3^-1 * f1 = Id(A6)
    f4^4 = Id(A6)
    f5^6 = Id(A6)
    (f4^-1 * f5)^4 = Id(A6)
    f5^-1 * f4^2 * f5 * f4^-1 * f5^-1 * f4^-2 * f5 * f4^-1 = Id(A6)
    (f5 * f4^-2 * f5^-1 * f4)^2 = Id(A6)
    (f5^2 * f4)^4 = Id(A6)
    (f4^-2 * f5^-1 * f4^-2 * f5^-2)^2 = Id(A6)
    f4^-1 * f5^-3 * f4^-1 * f5 * f4 * f5 * f4 * f5 * f4 * f5^-2 * f4^-1 = Id(A6)
    f4 * f5^-1 * f4^-1 * f5 * f4 * f5^-2 * f4 * f5^-1 * f4^-1 * f5^-2 * f4 * f5 = Id(A6)
    f5^3 * f4 * f5^-3 * f4^-1 * f5^-3 * f4 * f5^-3 * f4^-1 = Id(A6)
    f5 * f4^-1 * f5^3 * f4 * f5^-1 * f4 * f5 * f4 * f5^-3 * f4^-1 * f5^-1 * f4^-1 = Id(A6)
    f4 * f5 * f4^-2 * f5^3 * f4^-2 * f5^-3 * f4 * f5 * f4 * f5 * f4 * f5 = Id(A6)
    f3^-1 * f2 * f3^-2 * f2 * f4^2 * f5 * f4^2 * f5^-1 = Id(A6)
    f3 * f2 * f3^-1 * f2 * f4^2 = Id(A6)
    f1^-1 * f3^-1 * f1^-1 * f3 * f5^-2 * f4 * f5^-1 * f4^-1 * f5^-1 * f4 * f5^-1 * f4^-1 * f5 = Id(A6)
    f2 * f1^-1 * f2^-1 * f1^-1 * f3^-1 * f1^-1 * f2^-1 * f1^-1 * f3^-1 * f2^-1 * f4^2 * f5^-3 * f4^-2 * f5^3 = Id(A6)
    f1^-1 * f4 * f5 * f4^-1 * f5^-2 * f4 * f5 * f4 * f5^-1 * f4 * f5 * f4 = Id(A6)
    f2 * f1^-1 * f2^-1 * f1^-1 * f4 * f5 * f4 * f5 * f4^2 * f5 * f4 * f5^-1 * f4 * f5^-1 * f4 * f5 * f4 = Id(A6)
    f3^-1 * f2 * f3^-3 * f2 * f4 * f5 * f4^2 * f5^-1 = Id(A6)
    f2^-3 * f4 * f5 * f4 * f5^2 * f4 * f5^-1 * f4 * f5^-1 * f4 * f5 * f4 * f5^3 = Id(A6)
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