

# Scalar correlation functions in de Sitter space from the stochastic spectral expansion

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**Abstract.** We consider light scalar fields during inflation and show how the stochastic spectral expansion method can be used to calculate two-point correlation functions of an arbitrary local function of the field in de Sitter space. In particular, we use this approach for a massive scalar field with quartic self-interactions to calculate the fluctuation spectrum of the density contrast and compare it to other approximations. We find that neither Gaussian nor linear approximations accurately reproduce the power spectrum, and in fact always overestimate it. For example, for a scalar field with only a quartic term in the potential,  $V = \lambda\phi^4/4$ , we find a blue spectrum with spectral index  $n - 1 = 0.579\sqrt{\lambda}$ .

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## 1 Introduction

Light scalar spectator fields acquire super-horizon fluctuations during cosmological inflation with potentially observable consequences. For example, they can give rise to curvature perturbations through the curvaton mechanism [1–4] or through their effect on the non-equilibrium reheating dynamics at the end of inflation [5, 6]. They may also influence the dark matter abundance [7–9], the anisotropy of the gravitational wave background [10], generate primordial black holes [11–14] or source Dark Energy [15]. In particular, the inflationary dynamics of the Standard Model Higgs [16–18] have a wide variety of possible ramifications, from gravitational waves [19, 20] and leptogenesis [21, 22] to triggering electroweak vacuum decay [23–25].

In the case of free fields the scalar field fluctuations can be computed by solving the mode equations on a curved background [26]. However, when self-interactions are important it is well-known that for light fields the perturbative expansion fails [27, 28], requiring one to employ other techniques. The *stochastic approach* [29, 30] provides a powerful tool for such computations and is in agreement with, and in fact often superior to, other approaches to quantum field theory in de Sitter space [31–39]. In particular we note the direct QFT approaches making use of the Schwinger-Dyson equations: in Ref. [40] to two-loop order in perturbation theory and in Ref. [41] to Next-To-Leading order in a  $1/N$  expansion, where the latter is also in agreement with the recent Euclidean analysis of Ref. [42].

In many inflationary models, the spacetime is well approximated by a de Sitter space and inflation lasts for sufficiently long for the field fluctuations to equilibrate [43, 44]. In that case, the problem of finding scalar correlators reduces to finding the equilibrium state. The stochastic approach gives an exact expression for the one-point equilibrium probability distribution of the field, from which a spectral expansion for two-point correlation functions can be obtained [30].

The one-point probability distribution is well-known and widely used, see for example Ref. [45]. Investigating correlators via the stochastic formalism has been less common, however see [7, 9, 33, 46–57], even though it is the correlators that are more often directly related to observations. In particular the isocurvature perturbations, which are heavily constrained by observations [58], were studied in Refs. [9, 59–63] with the stochastic approach.

The aim of this paper is to use the stochastic approach to compute scalar field correlators in de Sitter space for a scalar field  $\phi$  with a mass term  $\frac{1}{2}m^2\phi^2$  and a quartic self-interaction term  $\frac{\lambda}{4}\phi^4$ . Up to trivial scaling, these are parameterised by a single dimensionless parameter  $\alpha \equiv m^2/(\sqrt{\lambda}H^2)$ . The method involves finding eigenfunctions and eigenvalues of a Schrödinger-like equation, which is easy to do numerically to high precision, and which give the coefficients and exponents for an asymptotic long-distance expansion of correlators. We compute these numerically for  $0 \leq \alpha \leq \infty$ . We also consider perturbative expansions around the two limits  $\alpha \rightarrow 0$  and  $\alpha \rightarrow \infty$ , which correspond to the massless and free cases, which have been studied in the literature earlier. Comparison of the results shows that the commonly used mean field and Gaussian approximations do not describe the field correlators correctly away from the free limit.

The paper is organised as follows: in Section 2, we summarise the stochastic approach and discuss one- and two-point correlators of local functions of a generic light scalar field in de Sitter space. In Section 3, we discuss as an example the case of a massive self-interacting field, computing the relevant eigenvalues and eigenfunctions related to the spectral expansion of correlators. In Section 4, we compare the results with different approximations. Finally, in Section 5, we conclude with an outlook.

## 2 The stochastic approach

### 2.1 One-point probability distribution

It was shown by Starobinsky and Yokoyama [29, 30] that on super-horizon scales, the behaviour of a light and energetically subdominant scalar field  $\phi$  in de Sitter spacetime is described by a stochastic Langevin equation

$$\dot{\phi}(t) = -\frac{1}{3H}V'(\phi) + \xi(t), \quad (2.1)$$

where  $H = \dot{a}/a$  is the Hubble rate,  $V(\phi)$  is the potential of the scalar field and prime denotes derivative with respect to the field, and  $\xi(t)$  is a Gaussian white noise term with two-point correlator

$$\langle \xi(t_1)\xi(t_2) \rangle = \frac{H^3}{4\pi^2}\delta(t_1 - t_2). \quad (2.2)$$

The condition for the field being light is  $V''(\phi) < H$ , whereas the field is energetically subdominant when  $V < 3H^2M_{\text{P}}^2$ , where  $M_{\text{P}}$  is the reduced Planck mass.

Let us denote the one-point probability distribution of the scalar field  $\phi$  at time  $t$  by  $P(t; \phi)$ . Starobinsky and Yokoyama showed [30] that it satisfies the Fokker-Planck equation

$$\frac{\partial P(t; \phi)}{\partial t} = D_\phi P(t; \phi), \quad (2.3)$$

where  $D_\phi$  is the differential operator

$$D_\phi \equiv \frac{V''(\phi)}{3H} + \frac{V'(\phi)}{3H} \frac{\partial}{\partial \phi} + \frac{H^3}{8\pi^2} \frac{\partial^2}{\partial \phi^2}. \quad (2.4)$$

From this it is straightforward to find the equilibrium distribution

$$P_{\text{eq}}(\phi) \propto \exp\left(-\frac{8\pi^2 V(\phi)}{3H^4}\right), \quad (2.5)$$

up to a normalization factor which guarantees total probability equal to unity.

If the potential is symmetric under  $\phi \rightarrow -\phi$ , the average value of the field vanishes,

$$\langle \phi \rangle = 0, \quad (2.6)$$

but in any given Hubble volume the field is non-zero. Note that because of the non-Gaussian form of the equilibrium distribution and its zero mean, it would be incorrect to use results that assume either Gaussianity or linear fluctuations around a non-zero mean field value.

Considering, as an example, a simple quartic potential,

$$V(\phi) = \frac{1}{4}\lambda\phi^4, \quad (2.7)$$

one finds the field variance [30]

$$\langle \phi^2 \rangle = \left(\frac{3}{2\pi^2}\right)^{1/2} \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \frac{H^2}{\lambda^{1/2}}, \quad (2.8)$$

and the average potential energy of the field [30]

$$\langle V(\phi) \rangle = \frac{3H^4}{32\pi^2}. \quad (2.9)$$

## 2.2 Temporal correlation functions

Let us now move to consider two-point correlation functions. We are interested in the general two-point function of some local function  $f(\phi)$  of the field,

$$G_f(t_1, t_2; \vec{r}_1, \vec{r}_2) \equiv \langle f(\phi(t_1, \vec{r}_1)) f(\phi(t_2, \vec{r}_2)) \rangle. \quad (2.10)$$

Starobinsky and Yokoyama developed a method for calculating them [30], but it has been used much less than the one-point probability distribution (2.5). In the following we present their calculation.

First, note that because of de Sitter invariance [30], any correlator of a scalar observable  $f(\phi)$  can only depend on the de Sitter invariant quantity

$$y = \cosh H(t_1 - t_2) - \frac{H^2}{2} e^{H(t_1+t_2)} |\vec{r}_1 - \vec{r}_2|^2, \quad (2.11)$$

where  $\vec{r}_1$  and  $\vec{r}_2$  are comoving position vectors. As long as  $|y| \gg 1$ , both time-like and space-like separations can be expressed as

$$G_f(t_1, t_2; \vec{r}_1, \vec{r}_2) = G_f(H^{-1} \ln |2y - 1|), \quad (2.12)$$

where the right-hand-side is the temporal correlation function

$$G_f(t) \equiv G_f(t; 0) = \langle f(\phi(0))f(\phi(t)) \rangle, \quad (2.13)$$

corresponding to  $\vec{r}_2 = \vec{r}_1$ , which can be obtained easily from the stochastic approach.

Following Starobinsky and Yokoyama [30], let us define

$$\tilde{P}(t; \phi) = e^{\frac{4\pi^2 V(\phi)}{3H^4}} P(t; \phi), \quad (2.14)$$

which satisfies the equation

$$\frac{\partial \tilde{P}(t; \phi)}{\partial t} = \frac{3H^3}{4\pi^2} \tilde{D}_\phi \tilde{P}(t; \phi), \quad (2.15)$$

with

$$\tilde{D}_\phi = \frac{1}{2} \frac{\partial^2}{\partial \phi^2} - \frac{1}{2} (v'(\phi)^2 - v''(\phi)), \quad v(\phi) = \frac{4\pi^2}{3H^4} V(\phi). \quad (2.16)$$

Because this is a linear equation, we can use separation of variables to find independent solutions of the form  $\tilde{P}_n(t; \phi) = e^{-\Lambda_n t} \psi_n(\phi)$ , where  $\psi_n(\phi)$  satisfies the time-independent Schrödinger-like eigenvalue equation

$$\tilde{D}_\phi \psi_n(\phi) = -\frac{4\pi^2 \Lambda_n}{H^3} \psi_n(\phi). \quad (2.17)$$

We assume that the eigenfunctions  $\psi_n(\phi)$  are orthonormal,

$$\int d\phi \psi_m(\phi) \psi_n(\phi) = \delta_{m,n}, \quad (2.18)$$

and complete

$$\sum_n \psi_n(\phi) \psi_n(\phi_0) = \delta(\phi - \phi_0). \quad (2.19)$$

The lowest eigenvalue, with  $\Lambda_0 = 0$ , is

$$\psi_0(\phi) \propto e^{-\frac{4\pi^2 V(\phi)}{3H^4}}, \quad (2.20)$$

and comparing with Eq. (2.5), one can see that

$$P_{\text{eq}}(\phi) = \psi_0(\phi)^2. \quad (2.21)$$

It is convenient to introduce Dirac-like notation

$$\langle n|f|m \rangle \equiv \int d\phi \psi_n(\phi) f(\phi) \psi_m(\phi). \quad (2.22)$$

In this way we can, for example, write local equilibrium expectation values as

$$\langle f(\phi) \rangle = \int d\phi \psi_0(\phi) f(\phi) \psi_0(\phi) = \langle 0|f|0 \rangle. \quad (2.23)$$

Because of the linearity, the probability distributions between any two times are related linearly,

$$\tilde{P}(t + \Delta t; \phi) = \int d\phi_0 \tilde{U}(\Delta t; \phi, \phi_0) \tilde{P}(t; \phi_0), \quad (2.24)$$

by a transfer matrix  $\tilde{U}(\Delta t; \phi, \phi_0)$ , which satisfies the same differential equation

$$\frac{\partial \tilde{U}(t; \phi, \phi_0)}{\partial t} = \frac{3H^3}{4\pi^2} \tilde{D}_\phi \tilde{U}(t; \phi, \phi_0). \quad (2.25)$$

Using this we can also write for the unscaled probability distribution

$$\begin{aligned} P(t + \Delta t; \phi) &= \int d\phi_0 e^{\frac{-4\pi^2 V(\phi)}{3H^4}} \tilde{U}(\Delta t; \phi, \phi_0) e^{\frac{4\pi^2 V(\phi_0)}{3H^4}} P(t; \phi_0) \\ &= \int d\phi_0 U(\Delta t; \phi, \phi_0) P(t; \phi_0), \end{aligned} \quad (2.26)$$

where we have defined

$$U(\Delta t; \phi, \phi_0) \equiv e^{\frac{-4\pi^2 V(\phi)}{3H^4}} \tilde{U}(\Delta t; \phi, \phi_0) e^{\frac{4\pi^2 V(\phi_0)}{3H^4}}. \quad (2.27)$$

The transfer matrix satisfies the initial condition

$$U(0; \phi, \phi_0) = \tilde{U}(0; \phi, \phi_0) = \delta(\phi - \phi_0), \quad (2.28)$$

and can be written in terms of the eigenfunctions as

$$\tilde{U}(t; \phi, \phi_0) = \sum_n e^{-\Lambda_n t} \psi_n(\phi) \psi_n(\phi_0). \quad (2.29)$$

To obtain the temporal correlation function (2.13), we integrate over  $\phi_0$  weighted by the equilibrium distribution  $P_{\text{eq}}(\phi_0)$ ,

$$\begin{aligned} G_f(t) &= \int d\phi \int d\phi_0 P_{\text{eq}}(\phi_0) f(\phi_0) U(t; \phi, \phi_0) f(\phi) \\ &= \int d\phi \int d\phi_0 P_{\text{eq}}(\phi_0) f(\phi_0) e^{\frac{4\pi^2}{3H^4}(V(\phi_0) - V(\phi))} \tilde{U}(t; \phi, \phi_0) f(\phi). \end{aligned} \quad (2.30)$$

Substituting Eq. (2.21) into Eq. (2.30), we find

$$G_f(t) = \int d\phi \int d\phi_0 \psi_0(\phi_0) f(\phi_0) \tilde{U}(t; \phi, \phi_0) \psi_0(\phi) f(\phi). \quad (2.31)$$

Now, we substitute the spectral expansion (2.29)

$$G_f(t) = \sum_n \langle 0|f|n \rangle e^{-\Lambda_n t} \langle n|f|0 \rangle = \sum_n f_n^2 e^{-\Lambda_n t}, \quad (2.32)$$

where

$$f_n = \langle 0|f|n \rangle = \int d\phi \psi_0(\phi) f(\phi) \psi_n(\phi). \quad (2.33)$$

Therefore the asymptotic behaviour is given by the lowest eigenvalues  $\Lambda_n$ .

If the potential  $V(\phi)$  is symmetric under parity  $\phi \rightarrow -\phi$ , the eigenfunctions are either odd or even. The lowest eigenfunction  $\psi_0$  is even, so if we are considering an even/odd function  $f(\phi)$ , only the even/odd eigenvalues contribute, respectively. For example, for the field correlator  $f(\phi) = \phi$ , the lowest contributing eigenvalue is  $\Lambda_1$ . This gives

$$\langle \phi(0)\phi(t) \rangle = G_\phi(t) = \phi_1^2 e^{-\Lambda_1 t} + \mathcal{O}(e^{-\Lambda_3 t}), \quad (2.34)$$

where

$$\phi_1 = \int d\phi \psi_0(\phi) \phi \psi_1(\phi) = \langle 0|\phi|1 \rangle. \quad (2.35)$$

In contrast, for the potential energy we get contributions from  $\Lambda_0 = 0$ , which gives the disconnected part of the correlator, and  $\Lambda_2$  which gives the asymptotic behaviour

$$G_V(t) = \langle V(\phi) \rangle^2 + V_2^2 e^{-\Lambda_2 t} + \mathcal{O}(e^{-\Lambda_4 t}), \quad (2.36)$$

where

$$V_2 = \langle 0|V|2 \rangle = \int d\phi \psi_0(\phi) V(\phi) \psi_2(\phi). \quad (2.37)$$

### 2.3 Equal-time correlation function, power spectrum and spectral index

Because of de Sitter invariance, the equal-time correlation function between two different points in space can be obtained from the temporal correlation function  $G_f(t)$ , through

$$G_f(0; \vec{x}_1, \vec{x}_2) = G_f \left( \frac{2}{H} \ln(|\vec{x}_1 - \vec{x}_2| H) \right), \quad (2.38)$$

which is valid at distances  $|\vec{x}_1 - \vec{x}_2| \gg 1/H$ , and  $\vec{x} = a\vec{r}$  is the physical, non-comoving coordinate. In the rest of the paper, we use only physical distances, because that makes the expressions for equal-time correlation functions time-independent.

Using Eq. (2.38), the spectral expansion (2.32) gives

$$G_f(0; \vec{x}_1, \vec{x}_2) = \sum_n \frac{f_n^2}{(|\vec{x}_1 - \vec{x}_2| H)^{2\Lambda_n/H}}. \quad (2.39)$$

At asymptotically long distances, the correlator therefore has a power-law form

$$G_f(0; \vec{x}) \sim \frac{A_f}{(|\vec{x}| H)^{n_f - 1}}, \quad (2.40)$$

with constant parameters  $A_f = f_n^2$  and

$$n_f = 1 + \frac{2\Lambda_n}{H}, \quad \text{for } k \ll H, \quad (2.41)$$

for the lowest  $n$  with  $f_n \neq 0$ .

In cosmology, equal-time correlation functions are often described in terms of their power spectrum  $\mathcal{P}_f(k)$ , defined by

$$\mathcal{P}_f(k) = \frac{k^3}{2\pi^2} \int d^3x e^{-i\vec{k}\cdot\vec{x}} G_f(0; \vec{x}). \quad (2.42)$$

Substituting Eq. (2.39) gives

$$\mathcal{P}_f(k) = \sum_n \frac{2}{\pi} f_n^2 \Gamma\left(2 - 2\frac{\Lambda_n}{H}\right) \sin\left(\frac{\Lambda_n \pi}{H}\right) \left(\frac{k}{H}\right)^{2\Lambda_n/H}. \quad (2.43)$$

At long distances,  $k \ll H$ , the power spectrum (2.43) is also dominated by the leading term and has the power-law form,

$$\mathcal{P}_f(k) \sim \frac{2}{\pi} A_f \Gamma[2 - (n_f - 1)] \sin\left(\frac{\pi(n_f - 1)}{2}\right) \left(\frac{k}{H}\right)^{n_f - 1} \approx A_f (n_f - 1) \left(\frac{k}{H}\right)^{n_f - 1}, \quad (2.44)$$

where the constants  $A_f$  and  $n_f$  are the same as in Eq. (2.40), and the last form is valid when  $|n_f - 1| \ll 1$ . In particular, this shows that  $n_f$  is the spectral index, commonly defined as

$$\frac{\ln \mathcal{P}_f(k)}{\ln k} = n_f - 1. \quad (2.45)$$

### 3 Example: a massive self-interacting field

#### 3.1 Eigenvalue equation

As an example of the formalism presented in the previous section we will discuss a potential with quadratic and quartic contributions

$$V(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4} \phi^4, \quad (3.1)$$

with the assumption  $m^2 > 0$ . The analysis required for the double well potential,  $m^2 < 0$ , is significantly more complicated, which we will investigate in a separate publication [64].

For the potential in Eq. (3.1) the eigenvalue equation (2.17) becomes

$$\frac{1}{2} \left\{ \frac{\partial^2}{\partial \phi^2} - \left( \frac{4\pi^2}{3H^4} \right)^2 (m^4 \phi^2 + 2\lambda m^2 \phi^4 + \lambda^2 \phi^6) + \frac{4\pi^2}{3H^4} (m^2 + 3\lambda \phi^2) \right\} \psi_n(\phi) = -\frac{4\pi^2}{H^3} \Lambda_n \psi_n(\phi). \quad (3.2)$$

It is convenient to introduce a scaled version of the above equation expressed with only dimensionless parameters

$$\left\{ \frac{\partial^2}{\partial x^2} - U(\alpha; x) \right\} \psi_n(\alpha; x) = -8\pi^2 \frac{\Lambda_n(\alpha)}{\lambda^{1/2} H} \psi_n(\alpha; x), \quad (3.3)$$

where

$$x \equiv \frac{\lambda^{1/4}}{H} \phi, \quad \alpha \equiv \frac{m^2}{\sqrt{\lambda} H^2}, \quad (3.4)$$

and

$$U(\alpha; x) = \left( \frac{4\pi^2}{3} \right)^2 x^2 (\alpha + x^2)^2 - \frac{4\pi^2}{3} (\alpha + 3x^2). \quad (3.5)$$

In this form it is apparent that up to an overall scale, the eigenvalues  $\Lambda_n$  and the eigenfunctions  $\psi_n$  depend only on one dimensionless parameter  $\alpha$ . In the next subsection, we will consider the limits of small and large  $\alpha$  using perturbation theory, and the case of an arbitrary  $\alpha$  numerically. From now on throughout this section we will drop the explicit  $x$  dependence from the eigenfunctions.



### 3.2 Massless limit ( $\alpha \ll 1$ )

Near the massless limit  $m^2 \ll \lambda^{1/2}H^2$ , or equivalently  $\alpha \ll 1$ , we can find the eigenvalues and eigenfunctions as perturbative expansions in powers of  $\alpha$ ,

$$\Lambda_n(\alpha) = \Lambda_n^{(0)} + \alpha\Lambda_n^{(1)} + O(\alpha^2), \quad \psi_n(\alpha) = \psi_n^{(0)} + \alpha\psi_n^{(1)} + O(\alpha^2). \quad (3.6)$$

To do this, we Taylor expand Eq. (3.3) in powers of  $\alpha$ . This gives

$$U(\alpha; x) = U^{(0)}(x) + \alpha U^{(1)}(x) + O(\alpha^2), \quad (3.7)$$

where

$$U^{(0)}(x) = U(0; x) = \left(\frac{4\pi^2}{3}\right)^2 x^6 - 4\pi^2 x^2, \quad U^{(1)}(x) = 2\left(\frac{4\pi^2}{3}\right)^2 x^4 - \frac{4\pi^2}{3}. \quad (3.8)$$

The zeroth order case,  $\alpha = 0$ , was first discussed in Ref. [30]. The lowest eigenvalues and eigenfunctions can be solved numerically with the overshoot/undershoot method from the eigenvalue equation

$$\left\{ \frac{\partial^2}{\partial x^2} - U^{(0)}(x) \right\} \psi_n^{(0)} = \left\{ \frac{\partial^2}{\partial x^2} + 4\pi^2 x^2 - \frac{16\pi^4 x^6}{9} \right\} \psi_n^{(0)} = -8\pi^2 \frac{\Lambda_n^{(0)}}{\lambda^{1/2}H} \psi_n^{(0)}. \quad (3.9)$$

The lowest five eigenvalues are

$$\begin{aligned} \Lambda_0^{(0)} &= 0, \\ \Lambda_1^{(0)} &\approx 0.08892\sqrt{\lambda}H, \\ \Lambda_2^{(0)} &\approx 0.28938\sqrt{\lambda}H, \\ \Lambda_3^{(0)} &\approx 0.53667\sqrt{\lambda}H, \\ \Lambda_4^{(0)} &\approx 0.82895\sqrt{\lambda}H. \end{aligned} \quad (3.10)$$

By making use of a standard result from time independent perturbation theory for quantum mechanics the first order correction to the eigenvalue may be expressed as (see e.g. Ref. [65])

$$\Lambda_n^{(1)} \frac{8\pi^2}{\lambda^{1/2}H} = \langle \psi_n^{(0)} | U^{(1)} | \psi_n^{(0)} \rangle. \quad (3.11)$$

With the above the  $O(\alpha)$  corrections to the eigenvalues are given by

$$\Lambda_n^{(1)} = \frac{\lambda^{1/2}H \int_{-\infty}^{\infty} dx \psi_n^{(0)} \left[ \frac{8\pi^2}{3}x^4 - 1 \right] \psi_n^{(0)}}{6 \int_{-\infty}^{\infty} dx \left( \psi_n^{(0)} \right)^2}. \quad (3.12)$$

Eq. (3.12) can only be evaluated analytically for the  $n = 0$  eigenvalue, for which the result is zero (which is to be expected, since the perturbation does not change the normalisability of the zero eigenfunction, so the lowest eigenvalue must be zero). The results for the lowest five eigenvalues are given in Table 1.

One can also calculate the coefficients (2.33) for  $f(\phi) = \phi^j$  to zeroth order in  $\alpha$  from the integral

$$(\phi^j)_n = \int d\phi \psi_0^{(0)}(\phi) \phi^j \psi_n^{(0)}(\phi) + O(\alpha), \quad (3.13)$$

using the numerically calculated zeroth order eigenfunctions  $\psi_n^{(0)}$ . For the lowest  $j$  and  $n$ , values are given in Table 2.

$\Lambda_n$	$\alpha \ll 1$	$\alpha \gg 1$
$n = 0$	0	0
$n = 1$	$0.08892\sqrt{\lambda}H + 0.21478\sqrt{\lambda}H\alpha + O(\alpha^2)$	$\frac{m^2}{H} \left( \frac{1}{3} + \frac{3}{8\pi^2\alpha^2} \right) + O\left(\frac{1}{\alpha^4}\right)$
$n = 2$	$0.28938\sqrt{\lambda}H + 0.35152\sqrt{\lambda}H\alpha + O(\alpha^2)$	$\frac{m^2}{H} \left( \frac{2}{3} + \frac{3}{2\pi^2\alpha^2} \right) + O\left(\frac{1}{\alpha^4}\right)$
$n = 3$	$0.53667\sqrt{\lambda}H + 0.52930\sqrt{\lambda}H\alpha + O(\alpha^2)$	$\frac{m^2}{H} \left( 1 + \frac{27}{8\pi^2\alpha^2} \right) + O\left(\frac{1}{\alpha^4}\right)$
$n = 4$	$0.82895\sqrt{\lambda}H + 0.70696\sqrt{\lambda}H\alpha + O(\alpha^2)$	$\frac{m^2}{H} \left( \frac{4}{3} + \frac{6}{\pi^2\alpha^2} \right) + O\left(\frac{1}{\alpha^4}\right)$

**Table 1.** Lowest eigenvalues  $\Lambda_n$  in the limits of large and small  $\alpha = m^2/\lambda^{1/2}H^2$ .

$ (\phi^j)_n $	$j = 1$	$j = 2$	$j = 3$	$j = 4$
$n = 1$	$0.36016 \lambda^{-1/4} H$	0	$0.09608(\lambda^{-1/4} H)^3$	0
$n = 2$	0	$0.13929(\lambda^{-1/4} H)^2$	0	$0.06046(\lambda^{-1/4} H)^4$
$n = 3$	$0.04475 \lambda^{-1/4} H$	0	$0.07203(\lambda^{-1/4} H)^3$	0
$n = 4$	0	$0.03435(\lambda^{-1/4} H)^2$	0	$0.04271(\lambda^{-1/4} H)^4$

**Table 2.** Numerically computed coefficients  $|(\phi^j)_n|$  in the massless self-interacting limit ( $\alpha = 0$ ).

### 3.3 Free limit ( $\alpha \gg 1$ )

In the opposite limit  $\lambda \ll (m/H)^4$ , or equivalently  $\alpha \gg 1$ , the coefficients of Eq. (3.3) diverge, so it is convenient to rescale again, and define

$$\tilde{x} = \alpha^{1/2}x = \frac{m}{H^2}\phi. \quad (3.14)$$

The eigenvalue equation then becomes

$$\left\{ \frac{\partial^2}{\partial \tilde{x}^2} - \tilde{U}(\alpha; \tilde{x}) \right\} \psi_n(\alpha) = -8\pi^2 \frac{\Lambda_n(\alpha)}{\alpha\lambda^{1/2}H} \psi_n(\alpha), \quad (3.15)$$

where

$$\tilde{U}(\alpha; \tilde{x}) = \left( \frac{4\pi^2}{3} \right)^2 \tilde{x}^2 \left( 1 + \frac{\tilde{x}^2}{\alpha^2} \right)^2 - \frac{4\pi^2}{3} \left( 1 + \frac{3\tilde{x}^2}{\alpha^2} \right). \quad (3.16)$$

We want to solve this perturbatively in powers of  $\alpha^{-2}$ , so we write

$$\Lambda_n(\alpha) = \Lambda_n^{(0)} + \frac{1}{\alpha^2}\Lambda_n^{(1)} + O\left(\frac{1}{\alpha^4}\right), \quad \psi_n(\alpha) = \psi_n^{(0)} + \frac{1}{\alpha^2}\psi_n^{(1)} + O\left(\frac{1}{\alpha^4}\right), \quad (3.17)$$

and expand the potential as

$$\tilde{U}(\alpha; \tilde{x}) = \tilde{U}^{(0)}(\tilde{x}) + \frac{1}{\alpha^2}\tilde{U}^{(1)}(\tilde{x}) + O\left(\frac{1}{\alpha^4}\right), \quad (3.18)$$

where

$$\tilde{U}^{(0)}(\tilde{x}) = \left( \frac{4\pi^2}{3} \right)^2 \tilde{x}^2 - \frac{4\pi^2}{3}, \quad \tilde{U}^{(1)}(\tilde{x}) = 2 \left( \frac{4\pi^2}{3} \right)^2 \tilde{x}^4 - 4\pi^2 \tilde{x}^2. \quad (3.19)$$

The zeroth order equation is nothing more than the standard equation for a harmonic oscillator, and eigenfunctions can be written in terms of the Hermite polynomials as<sup>1</sup>

$$\psi_n^{(0)} = \frac{\sqrt{m}}{H} \frac{1}{\sqrt{2^n n!}} \left(\frac{4\pi}{3}\right)^{1/4} e^{-\frac{2\pi^2 \tilde{x}^2}{3}} H_n \left(\frac{2\pi \tilde{x}}{\sqrt{3}}\right), \quad (3.20)$$

corresponding to the zeroth-order eigenvalues

$$\Lambda_n^{(0)} = \frac{n}{3} \alpha \lambda^{1/2} H = \frac{n}{3} \frac{m^2}{H}. \quad (3.21)$$

The perturbative correction to the eigenvalues again comes via Eq. (3.11)

$$\Lambda_n^{(1)} = \frac{m^2}{8\pi^2 H} \frac{1}{\sqrt{\pi} 2^n n!} \int_{-\infty}^{\infty} du H_n(u)^2 (2u^4 - 3u^2) e^{-u^2} = \frac{3n^2 m^2}{8\pi^2 H}. \quad (3.22)$$

The first five eigenvalues and their corrections are given in Table 1.

Again, one can calculate the coefficients (2.33) to zeroth order in  $\alpha^{-2}$  using the integral (3.13). This gives

$$(\phi^j)_1 = \begin{cases} \sqrt{\frac{2}{\pi}} \left(\frac{3H^4}{4\pi^2 m^2}\right)^{j/2} \Gamma\left(\frac{j}{2} + 1\right) + O(\alpha^{-2}), & \text{for odd } j, \\ 0 + O(\alpha^{-2}), & \text{for even } j, \end{cases} \quad (3.23)$$

and

$$(\phi^j)_2 = \begin{cases} 0 + O(\alpha^{-2}), & \text{for odd } j, \\ \frac{j}{2} \sqrt{\frac{2}{\pi}} \left(\frac{3H^4}{4\pi^2 m^2}\right)^{j/2} \Gamma\left(\frac{j+1}{2}\right) + O(\alpha^{-2}), & \text{for even } j. \end{cases} \quad (3.24)$$

In principle it is possible to calculate perturbative corrections to the eigenfunctions. However, we will omit this from our discussion as they are of limited use due to their analytically involved nature and limited applicability [65], especially since as we demonstrate in the next section full numerical solutions are readily available.

### 3.4 Arbitrary $\alpha$

For a generic value of  $\alpha$ , away from the two limits, the eigenvalue equation (3.3) can be solved numerically. It is convenient to do a further rescaling,

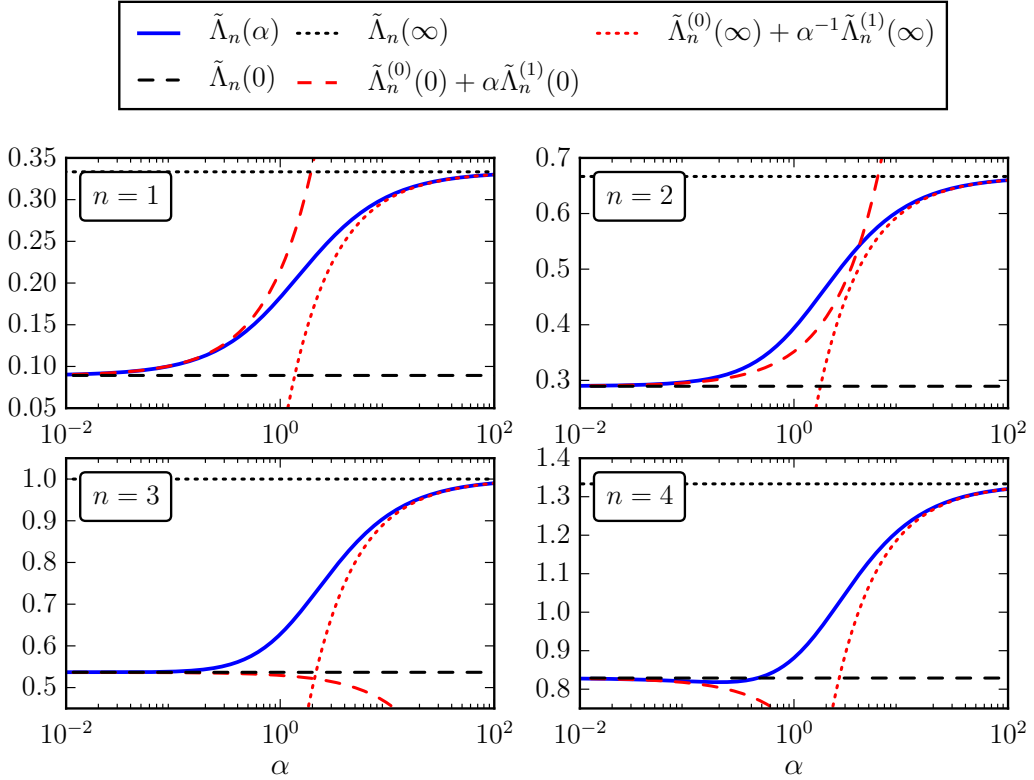
$$z \equiv \Omega x \equiv \frac{\lambda^{1/4} \Omega}{H} \phi, \quad \Omega \equiv \left(1 + \frac{m}{H\lambda^{1/4}}\right) \equiv (1 + \sqrt{\alpha}), \quad (3.25)$$

so that the equation becomes

$$\left\{ \frac{\partial^2}{\partial z^2} + \frac{4\pi^2 \alpha}{3\Omega^2} + 4\pi^2 \frac{1 - \frac{4\pi^2}{9} \alpha^2}{\Omega^4} z^2 - \frac{32\pi^4 \alpha z^4}{9\Omega^6} - \frac{16\pi^4 z^6}{9\Omega^8} + \frac{8\pi^2(1 + \alpha) \tilde{\Lambda}_n(\alpha)}{\Omega^2} \right\} \psi_n(z) = 0, \quad (3.26)$$

where

$$\tilde{\Lambda}_n(\alpha) = \frac{\Lambda_n}{\lambda^{1/2} H (1 + \alpha)} = \frac{\Lambda_n}{\lambda^{1/2} H + m^2/H}. \quad (3.27)$$



**Figure 1.** The dimensionless eigenvalues from Eq. (3.27) for the potential (3.1) solved from Eq. (3.3) as a function of  $\alpha \equiv m^2/(H^2\sqrt{\lambda})$ . The interpolation between quadratic (3.21) and quartic (3.10) results is clearly evident. The red curves correspond to the leading perturbative corrections, which can be obtained from Table 1 with Eq. (3.27).

The eigenvalues and -functions from Eq. (3.26) can be solved for example with the overshoot/undershoot method.<sup>2</sup> The results for the first four non-zero eigenvalues are presented in Fig. 1, which clearly coincide with Eqs. (3.21) and (3.10) as the limiting cases. In a similar fashion one may introduce scaled eigenfunctions satisfying

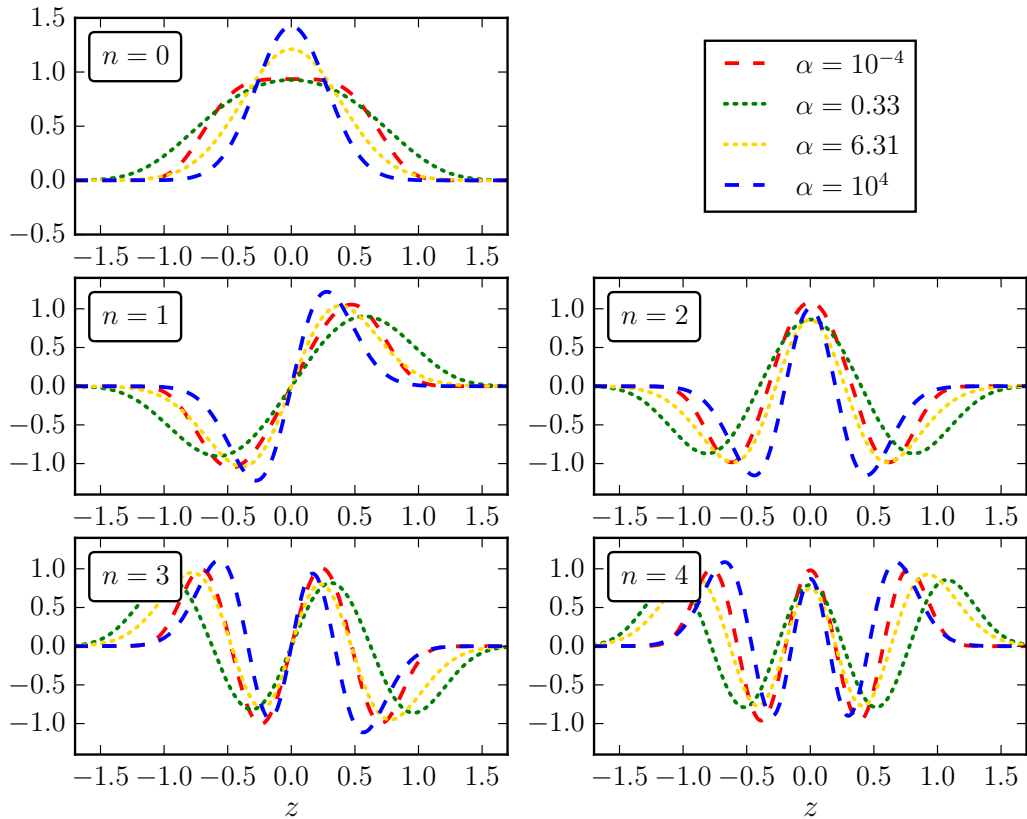
$$\psi_n \equiv \sqrt{\frac{\lambda^{1/4}\Omega}{H}} \tilde{\psi}_n(\alpha) \quad \Rightarrow \quad \int_{-\infty}^{\infty} d\phi |\psi_n|^2 = \int_{-\infty}^{\infty} dz |\tilde{\psi}_n(\alpha)|^2 = 1, \quad (3.28)$$

where we have again dropped the the explicit  $z$  and  $\phi$  dependences from the eigenfunctions. The results for the first five eigenfunctions are presented in Fig. 2.

In order to calculate correlators with the spectral expansion (2.32) one needs expressions for the coefficients  $f_n$  defined in Eq. (2.33). They depend on the specific form of  $f(\phi)$  and thus cannot be given independently of the correlator unlike the eigenvalues. As an important

<sup>1</sup> $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$

<sup>2</sup>When calculating the eigenfunctions with the overshoot/undershoot method we have truncated the solutions to a finite range  $z \in [-z_0, z_0]$  surrounded by  $\psi_n(z) = 0$  where the cut-off  $z_0$  is set by finding when the numerical solution  $\psi_n(z)$  is closest to zero before diverging.



**Figure 2.** The scaled eigenfunctions  $\tilde{\psi}_n(\alpha; z)$  defined in Eq. (3.28) for the potential (3.1) solved from (3.26) by making use of the definitions (3.25) and (3.27). The quartic and quadratic limits are reached at  $\alpha = 0$  and  $\alpha = \infty$ , respectively.

special case sufficient for most applications here we show the leading  $f_n$  contributions when  $f(\phi)$  is a monomial  $\phi^j$ , with  $j \leq 4$ . A convenient dimensionless version of the polynomial coefficients comes with the help of Eq. (3.25)

$$(\tilde{\phi}^j)_n \equiv \left(\frac{\lambda^{1/4}\Omega}{H}\right)^j (\phi^j)_n = \left(\frac{\lambda^{1/4}\Omega}{H}\right)^j \int_{-\infty}^{\infty} d\phi \psi_0 \phi^j \psi_n = \int_{-\infty}^{\infty} dz \tilde{\psi}_0(\alpha) z^j \tilde{\psi}_n(\alpha), \quad (3.29)$$

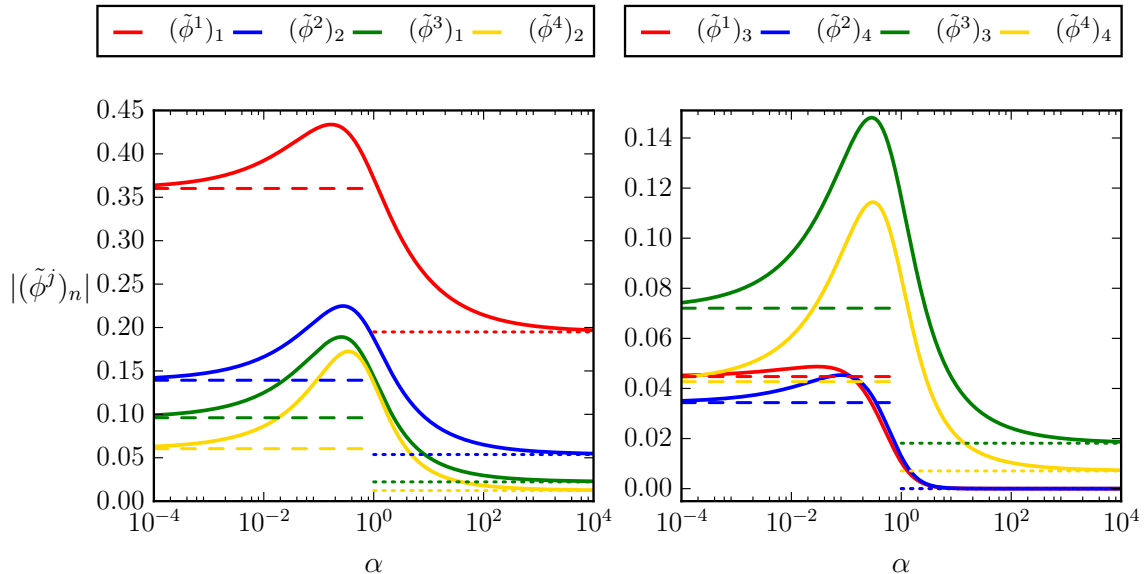
which are shown in Fig. 3. The quartic and quadratic limits for the first non-zero coefficients are given in Table 2 and Eqs. (3.23) and (3.24).

## 4 Validity of common approximations

### 4.1 Full results

As an illustration of the presented formalism and how it relates to other often implemented approximations, we consider equal-time correlators of the field  $\phi$  and its density contrast

$$\delta \equiv \frac{\delta\rho_\phi}{\langle\rho_\phi\rangle} = \frac{V(\phi) - \langle V(\phi)\rangle}{\langle V(\phi)\rangle}, \quad (4.1)$$



**Figure 3.** The leading terms in the spectral expansion when  $f(\phi) = \phi^j$  with  $j \leq 4$ , scaled according to Eq. (3.29). The dashed and dotted lines correspond to the quartic ( $\alpha \rightarrow 0$ ) and quadratic ( $\alpha \rightarrow \infty$ ) limits, respectively, given in Table 2 and Eqs. (3.23) and (3.24) for the leading results (left).

where as usual in cosmology the fluctuation is defined as the difference to the mean  $\delta f(\phi) \equiv f(\phi) - \langle f(\phi) \rangle$  and we have neglected the kinetic term in the energy density.

Assuming that the energy density  $\rho_\phi$  is an even function of the field  $\phi$ , Eq. (2.39) shows that at long distances, the correlators are given by

$$\langle \phi(0)\phi(\vec{x}) \rangle \stackrel{|\vec{x}|H \gg 1}{\approx} \frac{(\phi_1)^2}{(|\vec{x}|H)^{2\Lambda_1/H}}, \quad (4.2)$$

$$\langle \delta(0)\delta(\vec{x}) \rangle \stackrel{|\vec{x}|H \gg 1}{\approx} \frac{(\delta_2)^2}{(|\vec{x}|H)^{2\Lambda_2/H}}, \quad (4.3)$$

where  $\phi_1$  and  $\delta_2$  are given by Eq. (2.33), and  $\Lambda_1$  and  $\Lambda_2$  are eigenvalues obtained from Eq. (2.17). In the full stochastic approach, the spectral indices  $n_f$  and amplitudes  $A_f$  defined by Eq. (2.40) are therefore

$$n_\phi = 1 + 2\frac{\Lambda_1}{H}, \quad A_\phi = (\phi_1)^2, \quad (4.4)$$

for the field and

$$n_\delta = 1 + 2\frac{\Lambda_2}{H}, \quad A_\delta = (\delta_2)^2, \quad (4.5)$$

for the density contrast. The corresponding power spectra are given by Eq. (2.44) as

$$\begin{aligned} \mathcal{P}_\phi(k) \stackrel{k/H \ll 1}{\approx} & \approx \frac{2\Lambda_1}{H} (\phi_1)^2 \left(\frac{k}{H}\right)^{2\Lambda_1/H}, \\ \mathcal{P}_\delta(k) \stackrel{k/H \ll 1}{\approx} & \approx \frac{2\Lambda_2}{H} (\delta_2)^2 \left(\frac{k}{H}\right)^{2\Lambda_2/H}, \end{aligned} \quad (4.6)$$

where we have assumed  $\Lambda_{1,2} \ll H$ .

For a quadratic potential,  $V(\phi) = (m^2/2)\phi^2$ , the spectral coefficients are given by Eqs. (3.23) and (3.24) and the eigenvalues by Eq. (3.21). They give

$$n_\phi = 1 + \frac{2}{3} \frac{m^2}{H^2}, \quad A_\phi = (\phi_1)^2 = \frac{3}{8\pi^2} \frac{H^4}{m^2}, \quad (4.7)$$

$$n_\delta = 1 + \frac{4}{3} \frac{m^2}{H^2}, \quad A_\delta = (\delta_2)^2 = \left( \frac{(\phi^2)_2}{\langle \phi^2 \rangle} \right)^2 = 2. \quad (4.8)$$

The full expressions for the power spectra are

$$\mathcal{P}_\phi = \frac{H^2}{4\pi^2} \left( \frac{k}{H} \right)^{2m^2/3H^2}, \quad \mathcal{P}_\delta = \frac{8m^2}{3H^2} \left( \frac{k}{H} \right)^{4m^2/3H^2}. \quad (4.9)$$

For a quartic potential,  $V(\phi) = (\lambda/4)\phi^4$ , the spectral coefficients are given in Table 2 and the eigenvalues by Eq. (3.10) which lead to

$$n_\phi \approx 1 + 0.17784\lambda^{1/2}, \quad A_\phi = (\phi_1)^2 = 0.12972 \frac{H^2}{\lambda^{1/2}}, \quad (4.10)$$

$$n_\delta \approx 1 + 0.57876\lambda^{1/2}, \quad A_\delta = (\delta_2)^2 = \left( \frac{(\phi^4)_2}{\langle \phi^4 \rangle} \right)^2 = \left( \frac{8\pi^2 (\tilde{\phi}^4)_2}{3} \right)^2 \approx 2.53198. \quad (4.11)$$

The full expressions for the power spectra are then

$$\begin{aligned} \mathcal{P}_\phi &\approx 0.0231 H^2 \left( \frac{k}{H} \right)^{0.17784\lambda^{1/2}} + 0.0021 H^2 \left( \frac{k}{H} \right)^{1.07334\lambda^{1/2}} + \dots, \\ \mathcal{P}_\delta &\approx 1.4655 \lambda^{1/2} \left( \frac{k}{H} \right)^{0.57876\lambda^{1/2}} + 2.0949 \lambda^{1/2} \left( \frac{k}{H} \right)^{1.65790\lambda^{1/2}} + \dots, \end{aligned} \quad (4.12)$$

where for completeness we have also included the next-to-leading order terms. From them we can see that the leading term dominates the field spectrum  $\mathcal{P}_\phi$  on all superhorizon scales, but in the density spectrum  $\mathcal{P}_\delta$  only when  $k \lesssim 10^{-0.144\lambda^{-1/2}} H$ . If  $\lambda$  is sufficiently small, this can become important.

The results for the asymptotic amplitude  $A_\delta$  and the spectral index  $n_\delta$  for the density contrast when the potential has a quartic and a quadratic term (3.1) are shown in Fig. 4.

## 4.2 Approximation schemes

It is instructive to compare the stochastic results with different approximations used in the literature. In the following we will consider four such approximations, which can be thought of as the four different outcomes of two separate binary choices: The first choice is how to express the density contrast correlator in terms of the field correlators. For this we consider two approaches, which we refer to as the *mean field (MF)* and *Gaussian* approximations. The second choice is how to compute the field correlator, and for that we consider the conventional *linear* approximation and the *stochastic* approach. We will refer to the resulting four approximations as the linear-MF, stochastic-MF, linear-Gaussian, and stochastic-Gaussian approximations. In Fig. 4 we compare the amplitude and the spectral index of the density contrast computed using the full stochastic approach (4.5) to these four different approximations. We find that all of the approximations consistently overestimate the amplitude and underestimate the spectral tilt, leading to an overall overestimation of power on cosmological scales.

### 4.2.1 Linear-MF approximation

One way some authors have tried to compute the density contrast correlator is by treating the root-mean-square value of the field,  $\varphi \equiv (\langle \phi^2 \rangle)^{1/2}$ , as if it was constant homogeneous background field, and expanding the field fluctuations around it

$$\phi = \varphi + \delta\phi, \quad \delta\phi \ll \varphi. \quad (4.13)$$

We refer to this as the mean field (MF) approximation. The density contrast is then expressed to linear order in  $\delta\phi$  as

$$\delta^{\text{MF}} = \frac{V'(\varphi)}{V(\varphi)} \delta\phi, \quad (4.14)$$

and therefore its correlator would be

$$\langle \delta^{\text{MF}}(0) \delta^{\text{MF}}(\vec{x}) \rangle = \left( \frac{V'(\varphi)}{V(\varphi)} \right)^2 \langle \delta\phi(0) \delta\phi(\vec{x}) \rangle. \quad (4.15)$$

This would imply

$$n_{\delta}^{\text{MF}} = n_{\phi}, \quad A_{\delta}^{\text{MF}} = \left( \frac{V'(\varphi)}{V(\varphi)} \right)^2 A_{\phi} = 4 \frac{\left(1 + \frac{\lambda}{m^2} \varphi^2\right)^2 A_{\phi}}{\left(1 + \frac{\lambda}{2m^2} \varphi^2\right)^2 \varphi^2}. \quad (4.16)$$

The next step is to compute the field correlator and, hence, the parameters  $n_{\phi}$ ,  $A_{\phi}$  and  $\varphi$ . This requires a second choice of an approximation. In the linear-MF approximation the field correlator is taken to be the linear solution to the equation of motion, which then means that the different comoving modes decouple. By solving the linear mode functions exactly one obtains the 2-point correlator for the field [26, 66, 67]

$$\begin{aligned} \langle \phi(0) \phi(\vec{x}) \rangle^{\text{lin}} &= \frac{H^2(1-c)(2-c)}{16\pi \sin(\pi(1-c))} {}_2F_1 \left[ 3-c, c; 2; 1 - \left( \frac{\vec{x}H}{2} \right)^2 \right]; \quad c = \frac{3}{2} - \sqrt{\frac{9}{4} - \frac{M^2(\varphi)}{H^2}} \\ &\underset{|\vec{x}|H \rightarrow \infty}{=} \frac{H^2(1-c)(2-c)4^c}{16\pi \sin(\pi(1-c))} \frac{\Gamma(3-2c)}{\Gamma(2-c)\Gamma(3-c)} |\vec{x}H|^{-2c}, \end{aligned} \quad (4.17)$$

with the effective mass

$$M^2(\varphi) = V''(\varphi) = m^2 + 3\lambda\varphi^2. \quad (4.18)$$

The field variance  $\langle \phi^2 \rangle$  can be obtained from Eq. (4.17). The full expression is ultraviolet divergent, but the leading term in powers of  $M^2/H^2$  is finite,

$$\varphi^2 = \langle \phi^2 \rangle^{\text{lin}} = \frac{(2-c)(c-1)H^2 (H_{2-c} + \psi^{(0)}(c) + \gamma_e - 1)}{16\pi^2} = \frac{3}{8\pi^2} \frac{H^4}{M^2(\varphi)} \left( 1 + \mathcal{O}\left(\frac{M^2}{H^2}\right) \right), \quad (4.19)$$

where  $H_n$  is a harmonic number and  $\psi^{(i)}(z)$  the polygamma function. The background field value  $\varphi$  is obtained by solving this equation.

The spectral index and amplitude for the field correlator can be read off from Eq. (4.17)

$$\begin{aligned} n_{\phi}^{\text{lin}} - 1 &= 2c = \frac{2}{3} \frac{M^2(\varphi)}{H^2} \left( 1 + \frac{1}{9} \frac{M^2(\varphi)}{H^2} + \dots \right), \\ A_{\phi}^{\text{lin}} &= \frac{3}{8\pi^2} \frac{H^4}{M^2(\varphi)} \left( 1 + \frac{(3 \ln 4 - 7)}{9} \frac{M^2(\varphi)}{H^2} + \dots \right). \end{aligned} \quad (4.20)$$



Comparison with Eqs. (4.8) shows that in the quadratic limit ( $\alpha \rightarrow \infty$  or equivalently  $\lambda \rightarrow 0$ ) the stochastic results agree with the above only to linear order in  $m^2/H^2$ . This is because its starting point, Eq. (2.1), relies on the slow roll approximation [30].

The spectral index and amplitude for the density contrast are then given by Eqs. (4.16), (4.19) and (4.20), which to leading order in  $M^2/H^2$  give

$$n_\delta^{\text{lin-MF}} - 1 = \frac{2}{3} \frac{m^2 + 3\lambda\varphi^2}{H^2}, \quad A_\delta^{\text{lin-MF}} = 4 \frac{\left(1 + \frac{\lambda}{m^2}\varphi^2\right)^2}{\left(1 + \frac{\lambda}{2m^2}\varphi^2\right)^2}. \quad (4.21)$$

In the quadratic limit,  $\alpha \rightarrow \infty$  or  $\lambda \rightarrow 0$ , this gives

$$n_\delta^{\text{lin-MF}} = 1 + \frac{2}{3} \frac{m^2}{H^2}, \quad A_\delta^{\text{lin-MF}} = 4. \quad (4.22)$$

In the quartic limit,  $\alpha \rightarrow 0$  or  $m \rightarrow 0$ , one finds

$$n_\delta^{\text{lin-MF}} = 1 + \frac{2\lambda\varphi^2}{H^2} \approx 1 + 0.22508\lambda^{1/2}, \quad A_\delta^{\text{lin-MF}} = 16, \quad (4.23)$$

where  $\varphi$  was obtained from Eq. (4.19) to leading order in  $M^2/H^2$ . As comparison with the correct values (4.8) and (4.11) shows, this is not a good approximation in either case.

#### 4.2.2 Stochastic-MF approximation

An alternative way of proceeding from Eq. (4.15) is to use the full stochastic field correlator (4.2) for the right-hand side. The relations (4.16) are still valid, so using Eq. (4.10) one obtains

$$n_\delta^{\text{sto-MF}} = n_\phi = 1 + 2\frac{\Lambda_1}{H}, \quad A_\delta^{\text{sto-MF}} = \left(\frac{V'(\varphi)}{V(\varphi)}\right)^2 (\phi_1)^2. \quad (4.24)$$

In the limit  $\alpha \rightarrow \infty$ , Eq. (4.24) gives

$$n_\delta^{\text{sto-MF}} = 1 + \frac{2}{3} \frac{m^2}{H^2}, \quad A_\delta^{\text{sto-MF}} = 4 \frac{(\phi_1)^2}{\varphi^2} = 4, \quad (4.25)$$

and in the limit  $\alpha \rightarrow 0$ ,

$$n_\delta^{\text{sto-MF}} = 1 + 0.17785\lambda^{1/2}, \quad A_\delta^{\text{sto-MF}} = 16 \frac{(\phi_1)^2}{\varphi^2} \approx 15.751, \quad (4.26)$$

from (2.8) and (3.10). Again, comparing with (4.11), we see that this is a poor approximation.

#### 4.2.3 Linear-Gaussian approximation

Another way to express the density contrast correlator in terms of the field correlators is to assume that the field distribution is Gaussian. Then there is no need to introduce a non-zero background field. Instead, the density contrast correlator can be computed using Wick's theorem,

$$\begin{aligned} \langle \delta(0)\delta(\vec{x}) \rangle^{\text{Gauss}} &= \frac{\langle V(\phi(0))V(\phi(\vec{x})) \rangle^{\text{Gauss}}}{(\langle V(\phi) \rangle^{\text{Gauss}})^2} - 1 \\ &= 2 \frac{\langle \phi(0)\phi(\vec{x}) \rangle^2 + 6 \frac{\lambda}{m^2} \langle \phi(0)\phi(\vec{x}) \rangle^2 \langle \phi^2 \rangle + 9 \frac{\lambda^2}{m^4} \langle \phi(0)\phi(\vec{x}) \rangle^2 \langle \phi^2 \rangle^2 + 3 \frac{\lambda^2}{m^4} \langle \phi(0)\phi(\vec{x}) \rangle^4}{(\langle \phi^2 \rangle + \frac{3\lambda}{2m^2} \langle \phi^2 \rangle^2)^2} \\ &\sim \frac{2}{\langle \phi^2 \rangle^2} \frac{\left(1 + 3 \frac{\lambda}{m^2} \langle \phi^2 \rangle\right)^2}{\left(1 + \frac{3}{2} \frac{\lambda}{m^2} \langle \phi^2 \rangle\right)^2} \langle \phi(0)\phi(\vec{x}) \rangle^2, \end{aligned} \quad (4.27)$$

where in the last expression we have dropped the last term because it is subdominant at long distances. From this we see that the Gaussian approximation implies an exact relationship between the spectral indices of the field and the density contrast,

$$n_{\delta}^{\text{Gauss}} - 1 = 2(n_{\phi} - 1), \quad (4.28)$$

which is correctly reproduced by the stochastic results (4.8) in the free limit  $\alpha \rightarrow \infty$ , but is not true for general  $\alpha$ , as Eqs. (4.4)–(4.5) show.

In the linear-Gaussian approximation we make use of the linear result (4.17) (with no mean field) and express the amplitude and the spectral index of the density contrast correlator to leading order in  $m^2/H^2$  as

$$n_{\delta}^{\text{lin-Gauss}} - 1 \approx \frac{4}{3} \frac{m^2}{H^2},$$

$$A_{\delta}^{\text{lin-Gauss}} = 2 \frac{\left(1 + 3 \frac{\lambda}{m^2} \langle \phi^2 \rangle\right)^2}{\left(1 + \frac{3}{2} \frac{\lambda}{m^2} \langle \phi^2 \rangle\right)^2} \left(\frac{A_{\phi}^{\text{lin}}}{\langle \phi^2 \rangle}\right)^2 \approx 2 \frac{\left(1 + \frac{9}{8\pi^2 \alpha^2}\right)^2}{\left(1 + \frac{9}{16\pi^2 \alpha^2}\right)^2} \rightarrow \begin{cases} 2 & \text{as } \alpha \rightarrow \infty, \\ 8 & \text{as } \alpha \rightarrow 0, \end{cases} \quad (4.29)$$

where the spectral index is independent of  $\alpha$ . Comparing with the full results (4.8) and (4.11), we conclude that the linear-Gaussian approximation (unsurprisingly) works perfectly in the free case but fails when interactions are important.

#### 4.2.4 Stochastic-Gaussian approximation

As in Section 4.2.2, one can also use the full stochastic field correlator (4.2) instead of the linearised solution (4.17). If one still assumes a Gaussian field distribution, one can use Wick's theorem (4.27) to express the density contrast correlators in terms of it.

In this stochastic-Gaussian approximation the field amplitude  $A_{\phi}$  and spectral index  $n_{\phi}$  are, of course, the same as in the full stochastic case (4.5). The spectral index and amplitude for the density contrast are given by Eq. (4.28) and (4.27)

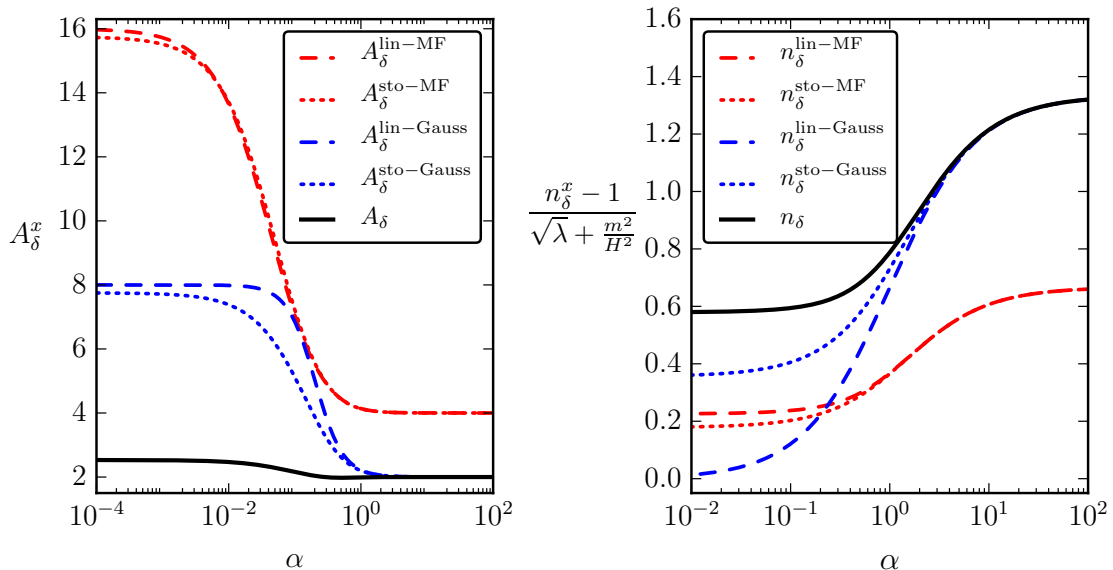
$$n_{\delta}^{\text{sto-Gauss}} - 1 = 4 \frac{\Lambda_1}{H}, \quad A_{\delta}^{\text{sto-Gauss}} = 2 \frac{\left(1 + 3 \frac{\lambda}{m^2} \langle \phi^2 \rangle\right)^2}{\left(1 + \frac{3}{2} \frac{\lambda}{m^2} \langle \phi^2 \rangle\right)^2} \frac{(\phi_1)^4}{\langle \phi^2 \rangle^2}. \quad (4.30)$$

In the free limit,  $\alpha \rightarrow \infty$  or equivalently  $\lambda \rightarrow 0$ , this reproduces the full stochastic result (4.8). In the quartic limit,  $\alpha \rightarrow 0$  or equivalently  $m \rightarrow 0$ , it gives

$$n_{\delta}^{\text{sto-Gauss}} \equiv 1 + 4 \frac{\Lambda_1}{H} \approx 1 + 0.35570 \lambda^{1/2}, \quad A_{\delta}^{\text{sto-Gauss}} \equiv 8 \frac{(\phi_1)^4}{\langle \phi^2 \rangle^2} \approx 7.753, \quad (4.31)$$

which again fails to reproduce the full result (4.11) but fares better than the other approximations.

In Fig. 4, we can see that both mean field approximations, linear-MF and stochastic-MF, perform poorly at all values of  $\alpha$ . This is largely because the assumption of a non-zero mean field is not justified. The Gaussian approximations, linear-Gaussian and stochastic-Gaussian, work well at high  $\alpha$ , but then deviate from the full result at smaller  $\alpha$ , when the interactions become more important. Overall, the stochastic approximations tend to work slightly better than the linear ones. Therefore, out of all the approximation schemes we have presented, the stochastic-Gaussian approximation is the most accurate. However, even it cannot be considered to be a good approximation away from the non-interacting limit. Therefore, the clear conclusion is that the full stochastic spectral expansion method is far superior to all of these approximation schemes.



**Figure 4.** The full result for the leading spectral coefficient (left) and the spectral index (right) for the density contrast (4.5) along with the approximations (4.21), (4.26), (4.29) and (4.30). Note that  $(n_\delta^x - 1)/(\sqrt{\lambda} + \frac{m^2}{H^2})$  is a function of only a single parameter,  $\alpha$ .

## 5 Conclusions

The stochastic spectral expansion is a very powerful tool for computing scalar correlation functions at long distances in de Sitter space. We have calculated them explicitly for a scalar field with a mass term and a self-interaction term, by solving the Schrödinger-like eigenvalue equation numerically to high precision. The resulting power spectra for the field and its density contrast are of the form (4.6) with the parameters given in Figs. 1 and 3. We also considered the two limits of a free massive scalar ( $\alpha \rightarrow \infty$ ) and a massless self-interacting scalar ( $\alpha \rightarrow 0$ ) separately, with the power spectra given in Eqs. (4.9) and (4.12), respectively.

Comparison with different approximative schemes used in the literature shows that, except in the non-interacting limit, they fail to describe the correlators accurately at long distances. In particular, they all predict too high an amplitude and too low a spectral index for the density contrast. Therefore they tend to significantly overestimate its amplitude on large scales, which are relevant for any cosmological observations. This was found to be important, for example, in the case of spectator dark matter [9], where the higher spectral index makes the isocurvature amplitude low enough to satisfy observational constraints.

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