Robust Control of Uncertain Systems: H_2 and H_∞ Control and Computation of Invariant Sets

Cheng Hu

Department of Electrical and Electronic Engineering
Imperial College London

This dissertation is submitted for the degree of
Doctor of Philosophy

March 2021

I would like to dedicate this thesis to my loving families
for their encouragement and support.

Declaration of Originality

I hereby confirm that this thesis is the result of my own research work. Any ideas and results of other people have been properly referenced.

Cheng Hu
March 2021

Copyright Declaration

Cheng Hu
March 2021

Acknowledgements

I would like to express my sincere gratitude to my supervisor Dr. Imad M. Jaimoukha for providing me with this opportunity to learn deeply in robust control, for the continuous support of my PhD research. His guidance always inspired me in all the time of the research and writing of publications. It is really my pleasure working with him.

I would also like to thank Dr. Argyrios Zolotas and Dr. David Angeli for kindly accepting to be examiners for my VIVA. Their valuable comments and suggestions have helped me to improve my thesis.

I am very grateful to the Department of Electrical Engineering for providing me with support of the scholarship that made my PhD study a reality.

I would like to thank my colleagues at Control and Power Group (CAP) for their valuable comments on my research work, and for all the fun we have had in the past four years.

Finally, a special thanks to my dear family for their love, patience, and support. Without them, I cannot reach this stage. Words cannot express my gratitude for all the love and encouragement I obtained from my parents and my sister.

Abstract

This thesis is mainly concerned with robust analysis and control synthesis of linear time-invariant systems with polytopic uncertainties. This topic has received considerable attention during the past decades since it offers the possibility to analyze and design controllers to cope with uncertainties. The most common and simplest approach to establish convex optimization procedures for robust analysis and synthesis problems is based on quadratic stability results, which use a single (parameter-independent) Lyapunov function for the entire uncertainty polytope. In recent years, many researchers have used parameter-dependent Lyapunov functions to provide less conservative results than the quadratic stability condition by working with parameterized Linear Matrix Inequalities (LMIs), where auxiliary scalar parameters are introduced. However, treating the scalar parameters as optimization variables leads to large computational complexity since the scalar parameters belong to an unbounded domain in general.

To address this problem, we propose three distinct iterative procedures for H_2 and H_∞ state feedback control, which are all based on true LMIs (without any scalar parameter). The first and second procedures are proposed for continuous-time and discrete-time uncertain systems, respectively. In particular, quadratic stability results can be used as a starting point for these two iterative procedures. This property ensures that the solutions obtained by our iterative procedures with one step update are no more conservative than the quadratic stability results. It is important to emphasize that, to date, for continuous-time systems, all existing methods have to introduce extra scalar parameters into their conditions in order to include the quadratic stability conditions as a special case, while our proposed iterative procedure solves a convex/LMI problem at each update. The third approach deals with the design of robust controllers for both continuous-time and discrete-time cases. It is proved that the proposed conditions contain the many existing conditions as special cases. Therefore, the third iterative procedure can compute a solution, in one step, which is at least as good as the optimal solution obtained using existing methods. All three iterative procedures can compute a sequence of non-increasing upper bounds for H_2-norm and H_∞-norm. In addition, if no feasible initial solution for the iterative procedures is found for some uncertain systems, we

also propose two algorithms based on iterative procedures that offer the possibility of obtaining a feasible initial solution for continuous-time and discrete-time systems, respectively.

Furthermore, to address the problem of analysis of H_∞-norm guaranteed cost computation, a generalized problem is firstly proposed that includes both the continuous-time and discrete-time problems as special cases. A novel description of polytopic uncertainties is then derived and used to develop a relaxation approach based on the S-procedure to lift the uncertainties, which yields an LMI approach to compute H_∞-norm guaranteed cost by incorporating slack variables.

In this thesis, one of the main contributions is to develop convex iterative procedures for the original non-convex H_2 and H_∞ synthesis problems based on the novel separation result. Nonlinear and non-convex problems are general in nature and occur in other control problems; for example, the computation of tightened invariant tubes for output feedback Model Predictive Control (MPC). We consider discrete-time linear time-invariant systems with bounded state and input constraints and subject to bounded disturbances. In contrast to existing approaches which either use pre-defined control and observer gains or optimize the volume of the invariant sets for the estimation and control errors separately, we consider the problem of optimizing the volume of these two sets simultaneously to give a less conservative design.

Table of contents

List of figures **x**

List of tables **xii**

Notation **xiv**

1 Introduction **1**

1.1 Robust analysis and control synthesis of linear systems with polytopic uncertainties 1
1.2 Computation of invariant tubes for robust output feedback MPC 5
1.3 Outline and contributions 6
1.4 Publications 8
1.5 Techniques 9

2 H_2 and H_∞ state feedback control of continuous-time polytopic systems **11**

2.1 Problem description 12
2.2 Extensions of current results for initial computation 17
2.3 Linearization and update computation algorithm 19
2.4 Robust stabilization when no known initial solution exists 23
2.5 Numerical examples 28
2.6 Summary 37

3 H_2 and H_∞ state feedback control of discrete-time polytopic systems **38**

3.1 Problem description 39
3.2 Linearization and initial computation 41
3.3 Update computation algorithm 46
3.4 Robust stabilization when no known initial solution exists 48
3.5 Improvement on update computation 51

3.6 Numerical examples 53
3.7 Summary 58

4 H_2 and H_∞ state feedback control by means of parameter-dependent slack variables 60
4.1 Summary of current results and initial computation 61
4.2 Update computation algorithm 69
4.3 Alternative way of update computation 73
4.4 Numerical examples 75
4.5 Summary 81

5 H_∞-norm guaranteed cost computation by means of S-procedure 82
5.1 Problem description 83
5.2 Reviews and extensions 84
5.3 Main results 87
5.4 Numerical examples 91
5.5 Summary 94

6 Computation of invariant tubes for robust output feedback Model Predictive Control 95
6.1 Problem description 96
6.2 Initial computation for the invariant set of the estimation error 99
6.3 Initial computation for the invariant set of the control error 103
6.4 Update computation algorithm 105
6.5 Numerical examples 109
6.6 Summary 112

7 Conclusion and Future Work 113
7.1 Conclusion 113
7.2 Future work 115

References 116

List of figures

2.1 Bound on the H_2-norm against the number of iterations computed using Algorithm 2.1 for Example 1. 29
2.2 Bound on the H_∞-norm against the number of iterations computed using Algorithm 2.1 for Example 1. 30
2.3 Bound on the H_2-norm against the number of iterations computed using Algorithm 2.1 for Example 2. 33
2.4 Bound on the H_∞-norm against the number of iterations computed using Algorithm 2.1 for Example 2. 33

3.1 Bound on the H_2-norm against the number of iterations computed using Algorithm 3.1 for Example 1. 56
3.2 Bound on the H_∞-norm against the number of iterations computed using Algorithm 3.1 for Example 1. 56
3.3 Bound on the H_2-norm against the number of iterations computed using Algorithm 3.1 and Algorithm 3.3 for Example 2. 58
3.4 Bound on the H_∞-norm against the number of iterations computed using Algorithm 3.1 and and Algorithm 3.3 for Example 2. 59

4.1 Bound on the H_∞-norm against the number of iterations obtained by Algorithm 4.1 for Example 1. 78
4.2 Bound on the H_2-norm against the number of iterations obtained by Algorithm 4.1 for Example 2. 80
4.3 Bound on the H_∞-norm against the number of iterations obtained by Algorithm 4.1 for Example 2. 80

6.1 Comparisons of tube calculated using our Algorithm 6.1 and the existing approach for Example 2 111

6.2 The objective value (-log det(Q)) against the number of iterations computed using Algorithm 6.1 for Example 2 111

List of tables

2.1 The minimum upper bound on H_2-norm computed using some existing methods and Algorithm 2.1 for Example 1. 28
2.2 The minimum upper bound H_∞-norm computed using some existing methods and Algorithm 2.1 for Example 1. 29
2.3 The minimum upper bound on H_2-norm achieved with some existing methods and Algorithm 2.1, as well as the associated solution time for Example 2. 31
2.4 The minimum upper bound on H_∞-norm achieved with some existing methods and Algorithm 2.1, as well as the associated numerical complexity for Example 2 (V: scalar variables, R: LMI rows, T: total solution time). . . . 32
2.5 The maximum values of d within stability domain achieved with some existing methods and Algorithm 2.2, as well as the associated numerical complexity for Example 3 (V: scalar variables, R: LMI rows, T: total solution time). 34
2.6 Number of uncertain systems (among 100) stabilized by some existing methods and Algorithm 2.2, as well as the associated solution time per system demanded for Example 4. 36

3.1 The minimum upper bound on H_2-norm achieved with some existing methods and Algorithm 3.1, as well as the associated solution time for Example 1 54
3.2 The minimum upper bound on H_∞-norm achieved with some existing methods and Algorithm 3.1, as well as the associated solution time for Example 1 54
3.3 The minimum upper bounds on H_2-norm and H_∞-norm computed using Algorithm 3.1 and Algorithm 3.3 for Example 2. 57

4.1 The minimum upper bound on H_2-norm obtained by some existing methods and Theorem 4.1 for Example 1. 76
4.2 The minimum upper bound on H_2-norm obtained by Algorithm 4.1 for Example 1. 76

4.3 The minimum upper bound on H_∞-norm obtained some existing methods, Theorem 4.1, and Algorithm 4.1, as well as the associated solution time for Example 1. 77
4.4 The minimum upper bound on H_2-norm obtained some existing methods, Theorem 4.1, and Algorithm 4.1, as well as the associated solution time for Example 2. 79
4.5 The minimum upper bound on H_∞-norm obtained some existing methods, Theorem 4.1, and Algorithm 4.1, as well as the associated solution time for Example 2. 79

5.1 The minimum guaranteed cost on H_∞-norm comparisons for Example 1. . 91
5.2 The minimum guaranteed cost on H_∞-norm comparisons for Example 2. . 92
5.3 The maximum value of ϵ for which feasibility is achieved for Example 3. . 93
5.4 The minimum guaranteed cost on H_∞-norm comparisons for each ϵ of Example 3. 94

6.1 The comparison of calculated invariant set boundaries for Example 1 109

Notation

Symbols

$\mathbb{R}$	The set of all real numbers
$\mathbb{R}^n$	The set of all real valued n-dimensional (column) vectors
$\mathbb{R}^{n\times m}$	The set of all $m \times n$ matrices with real entries
$\mathbb{S}^m$	The set of symmetric matrices of dimension $m \times m$
$\mathbb{S}^m_+$	The set of symmetric positive definite matrices of dimension $m \times m$
$\mathbb{D}^m_+$	The set of positive semidefinite diagonal matrices of dimension $m \times m$
I_n	The $n \times n$ identity matrix
$0_{n\times m}$	The $n \times m$ null matrix, with the dimensions omitted when defined by the context
$A \succ 0$	The symmetric matrix A is positive definite
$A \prec 0$	The symmetric matrix A is negative definite
A^T	The transpose of a matrix A
$\mathcal{H}(A)$	$\mathcal{H}(A) := A + A^T$
$\mathcal{N}$	$\mathcal{N} := \{1, \cdots, N\}$ for integer $N \geq 1$
$\star$	$\star$ refers to a term readily inferred from symmetry
$\mathcal{P}(P, b)$	The polytope $\{x \in \mathbb{R}^n : -b \leq Px \leq b\}$, $P \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$
$\mathcal{Q}(Q)$	The ellipsoid $\left\{x \in \mathbb{R}^n : x^T Qx \leq 1\right\}$, $Q \in \mathbb{S}^n_+$

$\mathcal{N}_m$ — $\mathcal{N}_m := \{1, \cdots, m\}$ for integer $m \geq 1$

e_i — The ith column of I_m, where m is defined by the context

Operators

$\oplus$ — Minkowski sum of two sets, that is $X \oplus Y := \{x+y \in \mathbb{R}^n : x \in X, y \in Y\}$

$\ominus$ — Minkowski difference of two sets, that is $X \ominus Y := \{z \in \mathbb{R}^n : z + y \in X, \forall y \in Y\}$

Acronyms / Abbreviations

LMI	Linear Matrix Inequality
BMI	Bilinear Matrix Inequality
NLMI	Non-linear Matrix Inequality
LTI	Linear Time-Invariant
MPC	Model Predictive Control
RCI	Robust Control Invariant
RPI	Robust Positively Invariant

Chapter 1

Introduction

1.1 Robust analysis and control synthesis of linear systems with polytopic uncertainties

1.1.1 Analysis of H_∞ guaranteed cost computation

H_∞ theory is an integral part of many robust control problems and H_∞-norm computation for robustness analysis of linear time-invariant (LTI) systems has received a considerable amount of studies over the past years. The well-known Bounded Real Lemma (BRL) allows the computation of the H_∞-norm for nominal LTI systems (system matrices do not include any parameter uncertainty) based on a standard convex optimization problem in terms of LMI conditions, which include the product between the Lyapunov function matrix and system matrices [1]. As to uncertain linear systems with polytopic uncertainties, where system matrices depend affinely on an unknown parameter belonging to a unit simplex, H_∞-norm computation has turned out to be a challenging problem. Based on the concept of quadratic stability [2–4], many studies transformed the original problem into the LMI problem by using a single (parameter-independent) Lyapunov matrix for all of the possible plants within the uncertainty polytope [5].

To overcome the conservatism resulting from the use of parameter-independent Lyapunov functions, many works have tried to characterize a convex procedure for assuring robust stability and computing H_∞-norm guaranteed cost through parameter-dependent Lyapunov functions [6–8]. Robust stability conditions for the existence of a parameter-dependent Lyapunov function have been developed in [9] for continuous-time systems and [10] for discrete-time systems. [11] extended these stability conditions to H_2 and H_∞-norm characterizations for discrete-time systems. Sufficient conditions for H_∞-norm cost for uncertain continuous-time systems can be found in [12]; the author made a simple modification to the BRL and then extended the results to a system with polytopic uncertainties. A sufficient parameterized LMI condition to compute H_∞-norm guaranteed cost was presented by introducing a slack matrix variable and a scalar parameter. However, the method of [12] is applicable for systems where the polytopic uncertainty occurs in the system matrices A, B, C only, and D has to be assumed parameter-independent. [13] provided an equivalent representation of the BRL and a sufficient condition for the analysis of H_∞-norm performance for linear continuous-time polytopic systems; the proposed conditions were expressed by a set of LMI conditions involving a scalar parameter. Compared to the work of [12], the conditions in [13] enable all of the system matrices to vary within an uncertainty polytope. In contrast, [14] presented sufficient LMI conditions that introduced two slack variables and that did not incorporate any scalar parameter. Their results can be considered as an extension of robust stability conditions that appeared in [9]. Furthermore, the conditions of [14] contain the conditions of [13] as a special case by imposing a restriction on the two slack variables.

The common idea of the aforementioned approaches using parameter-dependent Lyapunov function is to separate the Lyapunov matrix from the system matrices and allow the system matrices to be multiplied by extra slack variables, and then formulate a convex problem based on a finite set of LMIs. Good reviews can be found in the survey papers [15–17]. [18] proposed a different approach to provide robust analysis stability conditions for discrete–time polytopic systems. With respect to the aforementioned approaches that require N (the number of the vertices of the uncertain system) inequalities, the conditions of [18] require N^2 inequalities that contain the product of vertex system matrices and affine parameter-dependent Lyapunov matrix as well as N^2 full symmetric matrix variables. In the case of the state feedback synthesis design problem, the multiplicative conditions between the feedback gain and Lyapunov matrices lead to non-convexity.

1.1.2 H_2 and H_∞ state feedback control

The design of robust H_2 and H_∞ state feedback controllers for linear time-invariant systems with polytopic uncertainties has received considerable attention in the past decades. Many studies devoted to investigating robust stability and state feedback controller synthesis are based on LMIs as they can be solved efficiently [1]. Several methods were presented to achieve quadratic stability for such systems using a single, parameter-independent Lyapunov function [1, 2, 4]. To overcome the conservatism due to the use of a single Lyapunov function, approaches using parameter-dependent Lyapunov functions have been widely addressed more recently [6–8, 19]. However, the use of a parameter-dependent Lyapunov function leads to a non-convex robust control design problem due to the multiplication between the Lyapunov and system dynamic matrices.

To address this non-convexity, a significant breakthrough in robust stability analysis for continuous-time systems was made by [9], where an extended LMI condition was presented which separates the system matrices from the Lyapunov matrix but with the system matrices multiplied by two auxiliary slack variables instead. Then, imposing the condition that these slack variables are parameter-independent allows the formulation of a convex problem with a parameter-dependent Lyapunov matrix. The corresponding stability conditions for discrete-time systems appeared in [10]. A great deal of work has been devoted lately to attempt to achieve more relaxed robust control conditions for discrete-time systems by resorting to the use of affine parameter-dependent Lyapunov functions, see [10, 20, 21] for robust stabilization applications and [11, 22] for robust H_2 and H_∞ control synthesis applications. For discrete-time systems, these new extended LMI conditions contain the quadratic stability based conditions as special cases through with a simple choice of the slack variables, thus they generally lead to less conservative designs [11]. More recently, [23] proposed parameterized LMI based conditions for H_2 and H_∞ state feedback control of discrete-time systems, where an additional scalar parameter is introduced. It has been shown in [23] that the conditions of [11] can be reproduced by their results if the scalar parameter is selected to be zero. Therefore, the solutions obtained by this parameterized LMI based method are always no more conservative than the ones computed by [11]. However, parameterized LMIs become linear only if this scalar parameter is fixed. Hence, exhaustive searches on the scalar parameter have to be implemented in [23] to obtain the best possible solution, resulting in large computational complexity.

The synthesis problem for continuous-time systems turned out to be much more challenging. [24] proposed dilated LMI based conditions to compute H_2 upper bound through the use of a slack variable. However, their conditions do not contain quadratic stability conditions as a special case. To reduce this conservatism, some recent work concentrated on parameterized LMIs involving a slack matrix variable as well as an extra scalar parameter to cope with H_2-norm [25–27] and H_∞-norm [12–14] control synthesis problems. The major disadvantage of these approaches is that the scalar parameter belongs to an unbounded set; this results in a large computational burden because exhaustive searches on the scalar parameter need to be performed. In analogy with the work for the discrete-time case [23, 28] whose corresponding scalar parameter belongs to a bounded domain, [29] proposed new extended parameterized LMI characterizations for continuous-time systems with two scalar parameters based on a change of variables [30] and the Elimination Lemma [31]. One parameter belongs to the bounded set (-1,1), and another parameter, though belonging to an unbounded set, is restricted in a bounded subset through numerical experimentation. Although the search domain is limited considerably, the results may still be conservative.

The common idea of the above-mentioned class of methods is to decouple the system matrices and the Lyapunov matrix through an application of the Elimination Lemma [31, 32] or Finsler's Lemma [33] firstly, and then the corresponding infinite number of conditions are converted into N parameterized LMIs based on affine parameter-dependent Lyapunov matrix. Good reviews of the main existing approaches for robust stability analysis and controller synthesis problems can be found in [8, 34]. Another type of existing approaches working with robust stabilization for continuous-time polytopic systems was addressed in [35], which is to convert the original infinite number of conditions into a set of N^2 bilinear matrix inequality (BMI) conditions, and then transform these into parameterized LMIs by separating the feedback gain and the affine Lyapunov matrix. This work was further improved by [34] Again, an exhaustive search procedure for a scalar parameter has to be implemented in their results in order to capture the quadratic stability conditions as a particular case.

1.1.3 Summary of the current results and motivations

The most common and simplest method to deal with H_2 and H_∞ state feedback control for linear polytopic systems is the well-known quadratic stability-based method. However, the results obtained by the quadratic based conditions are very conservative in general.

In recent years, many studies were devoted to working with parameter-dependent Lyapunov functions to give a less conservative design than quadratic stability based results. To the best of our knowledge, to include the quadratic stability conditions as a special case for continuous-time systems, all existing methods introduce unbounded auxiliary scalar parameters in their design conditions. Therefore, these existing methods generally have a large computation burden stemming from a line search procedure for the unbounded scalar parameters. Motivated by this issue, in Chapter 2, we propose new conditions for designing H_2 and H_∞ controller in terms of true LMIs (without any scalar parameter). The proposed conditions can contain the optimal quadratic stability solution as a special case. Similar ideas are also applied to discrete-time polytopic systems and will be discussed in Chapter 3.

Apart from the introduction of scalar parameters, another major source of conservatism in current works available in the literature is that these conditions impose common (parameter-independent) slack variables in the entire polytope to establish a convex optimization problem. Motivated by this conservatism, we propose novel design conditions using parameter-dependent slack variables in Chapter 4. It is proved that the current approaches are special cases of the proposed conditions.

As for the analysis problem of H_∞- guaranteed cost, the common characteristic of the existing conditions is to convert the original parameter-dependent conditions into a finite set of LMIs defined at the vertices of the polytopic system. Chapter 5 gives a novel characterization of polytopic uncertainties and develops the S-Procedure to lift the uncertainties.

1.2 Computation of invariant tubes for robust output feedback MPC

Robust control invariant (RCI) sets are fundamental tools in robust control synthesis for uncertain systems subject to disturbances. RCI sets play an integral part in establishing stability of Robust Model Predictive Control (RMPC) schemes [36] and are also suitable for robust time-optimal control [37, 38]. Invariant set computation has been discussed widely in the past several decades [39], and important results are included in [40, 41]. In [40], the authors showed that the exact computation of polytopic RCI sets for systems subject to uncertainty is an intractable problem in general since it includes infinite Minkowski's sum terms. Therefore, most of the literature has been concerned with the efficient computation of inner/outer approximations to the maximal/minimal RCI sets, see [42–45]. More recently, an appealing approach is to consider both RCI set and feedback gain as decision variables. [46] presented an algorithm to compute low complexity RCI sets for linear discrete-time systems involving additive disturbances and norm bounded uncertainty. Nevertheless, low-complexity polytopic RCI sets restrict the number of faces of the polytope. In the work of [47], the authors advocated a method to compute full-complexity polytopic RCI sets for linear systems subject to additive disturbances, which allows us to compute less conservative invariant approximations of RCI sets. This work has been extended to linear systems subject to additive disturbances and structured norm-bounded or polytopic uncertainties in [48].

Due to the large computational burden of conventional on-line optimizations for RMPC, [49] proposed the concept of tube MPC, which uses a piecewise affine control law to maintain the controlled trajectories in the tube even in the presence of uncertainty. In addition, in many practical control problems, not all states are measurable and an observer is required to estimate the states. [50] proposed output MPC design by using a Luenberger observer, the difference between the actual and nominal states is the sum of the estimation and control errors bounded by two separate invariant sets, which are pre-computed along with pre-defined observer and feedback gains. [51] proposed an idea to compute less conservative results on tighter constraints with respect to [50], they adopt a single tube to describe the sum of the estimation and control errors, but their observer and feedback gains still need to be pre-defined. In the work of [52], the author provided an algorithm to optimize the volume of the invariant set of the estimation error by treating the observer gain as a variable firstly, and then use this given set as an artificial disturbance and the associated observer gain L to optimize the volume of the invariant set of the control error along with the feedback gain K. However, this method is still somewhat conservative due to the fact that it takes L and K as variables separately.

In summary, the existing approaches either use pre-defined control and observer gains or compute the control and observer gains that optimize the volume of the invariant sets for the estimation and control errors separately. We consider the problem of optimizing the volume of these sets simultaneously and will discuss details in Chapter 6.

1.3 Outline and contributions

In this section, we specify in more detail the contributions of each of the following chapters.

Chapter 2

Chapter 2 considers robust H_2 and H_∞ state feedback control synthesis problems for continuous-time systems. We first provide a generalized robust synthesis problem for linear continuous-time polytopic systems. This generalized problem includes robust H_2-norm and H_∞-norm problems as special cases. We then extend the approach of [34] to propose an initial computation method based on parameterized LMI based conditions for the generalized problem. Using a novel general separation result, which separates the state feedback gain from the Lyapunov matrix but with the state feedback gain synthesized from the slack variable, allows the formulation of LMI sufficient conditions for the generalized problem. Compared to existing parameterized LMI based conditions, where auxiliary scalar parameters are introduced to include the quadratic stability conditions as a special case, the proposed new conditions are true LMIs and they contain as a particular case the optimal quadratic stability solution. Utilizing any initial solution derived by the quadratic or the proposed initial computation method as a starting solution, we propose an algorithm based on an iterative procedure to compute a sequence of non-increasing upper bounds for the H_2-norm and H_∞-norm. In addition, if no feasible initial solution can be obtained by the initial computation method for some uncertain systems, another algorithm is presented that offers the possibility of obtaining a feasible initial solution.

Chapter 3

Chapter 3 considers the robust H_2 and H_∞ state feedback control for discrete-time systems. We first define the problem for designing H_2 and H_∞ controllers in terms of BMI conditions. Then new sufficient LMI based conditions for the BMI conditions are proposed by separating the feedback gain and Lyapunov matrices with the introduction of slack variables. Through a particular choice of the slack variables, the proposed LMI conditions can compute an initial solution, including the quadratic conditions as a special case. By considering another choice on the slack variables, it is demonstrated that any known solution to the BMI conditions can be included in the proposed LMI conditions as a particular case. Therefore, we propose an algorithm based on an iterative LMI procedure to compute upper bounds on the H_2 and H_∞-norms. When the initial computation method gives infeasibility, we propose an iterative procedure that offers the possibility of finding a feasible initial solution. Based on the separation result proposed in Chapter 2, we slightly modify our method to improve the update computation for computing less conservative H_2 and H_∞ performance.

Chapter 4

Chapter 4 considers the problem of robust H_2 and H_∞ state feedback control design for linear systems with polytopic uncertainties, in both the continuous-time and discrete-time cases. A unified generalized problem in terms of BMI conditions for designing H_2 and H_∞ controllers is firstly proposed. The proposed BMI conditions are proved to include the existing conditions as special cases. Then, new LMI based sufficient conditions for the BMI conditions are derived using a novel general separation result. It is also shown that the proposed LMI conditions contain any known solution to the BMI conditions as a particular case. Based on this property, starting with an initial solution provided by the existing methods, an algorithm based on an iterative procedure that guarantees recursive feasibility in each update is presented to iteratively reducing the upper bounds on H_2-norm and H_∞-norm.

Moreover, it is shown that the solutions to the design BMI conditions presented in Chapter 2 and Chapter 3 can also be iteratively updated by the novel separation result proposed in this chapter.

Chapter 5

Chapter 5 considers the problem of H_∞-norm guaranteed cost computation for linear time-invariant polytopic systems, in both the continuous-time and discrete-time cases. Firstly, the conditions of H_∞-norm guaranteed cost for both continuous-time and discrete-time systems are introduced in terms of a unified parameter-dependent inequality. We then give a novel characterization of polytopic uncertainties and develop the S-Procedure to lift the uncertainties and provide a sufficient LMI condition, without any extra scalar parameter, for the aforementioned parameter-dependent inequality. The proposed condition can provide H_∞-norm guaranteed cost that is less conservative than (at least equal to) the ones provided by the existing methods, as illustrated by numerical comparisons.

Chapter 6

Chapter 6 considers the computation of tightened invariant tubes for tube-based robust output MPC of discrete-time linear time-invariant systems. Two initial invariant sets for the estimation and control errors are computed separately. The volume of these two sets is then iteratively optimized by considering both the observer and feedback gains as variables simultaneously. Compared with the approach that considers the observer and feedback gains as variables separately, our approach allows considering the interaction between the estimation and control errors and give a less conservative design.

1.4 Publications

Most of the results in this thesis are based on the following papers which have been published, accepted, or in preparation.

Conference papers

- C. Hu and I. M. Jaimoukha. "Robust H_2 and H_∞ State Feedback Control for Discrete–time Polytopic Systems Using an Iterative LMI Based Procedure," in *2020 European Control Conference (ECC)*, Saint Petersburg, Russia, pp. 621-626, 2020.

- C. Hu, C. Liu, and I. M. Jaimoukha. "Computation of Invariant Tubes for Robust Output Feedback Model Predictive Control," *IFAC-PapersOnLine*, Accepted for publication, 2020.
- C. Hu and I. M. Jaimoukha. "New LMI Characterizations for H_∞-norm Guaranteed Cost Computation of Linear Systems with Polytopic Uncertainties," in *2020 59th IEEE Conference on Decision and Control (CDC)*, Jeju, Korea (South), pp. 3957-3962, 2020.

Journal papers

- C. Hu and I. M. Jaimoukha. "New iterative linear matrix inequality based procedure for H_2 and H_∞ state feedback control of continuous-time polytopic systems," *International Journal of Robust and Nonlinear Control*, 31(1), pp. 51-68, 2021.
- C. Hu and I. M. Jaimoukha. "Robust state feedback H_2 and H_∞ control synthesis for linear polytopic systems using parameter-dependent slack variables," Prepared to submit.

1.5 Techniques

Throughout this thesis, some techniques will be used extensively in the development of the main results, which are repeated here for convenience.

Lemma 1.1. *(Schur complement) [1] Define matrices $A = A^T$, $C = C^T$ and B of appropriate dimensions. A Schur complement argument refers to the result:*

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succ 0 \Leftrightarrow A \succ 0, C - B^T A^{-1} B \succ 0 \Leftrightarrow C \succ 0, A - BC^{-1} B^T \succ 0,$$

Lemma 1.2. *(Elimination Lemma) [31] Given $Q = Q^T \in \mathbb{R}^{n \times n}$, $R \in \mathbb{R}^{n \times m}$ and $S \in \mathbb{R}^{n \times p}$, there exists $H \in \mathbb{R}^{m \times p}$ such that*

$$Q + RHS^T + SH^T R^T \prec 0, \tag{1.1}$$

if and only if

$$R_\perp^T Q R_\perp \prec 0 \quad and \quad S_\perp^T Q S_\perp \prec 0, \tag{1.2}$$

where $R_\perp$ and $S_\perp$ are arbitrary matrices whose columns form a basis for the null space of R^T and S^T, respectively, (i.e. $R^T R_\perp = 0$, $S^T S_\perp = 0$).

Lemma 1.3. *[35] If the matrices* $V_{ij} \in \mathbb{S}^m$ *are such that*

$$V_{ij} + V_{ji} \succeq 0,\ 1 \le i < j \le N, \qquad \sum_{i=1}^{N}(V_{ij} + V_{ji}) \preceq 0,\ j = 1, \ldots, N, \tag{1.3}$$

then

$$\sum_{i,j=1}^{N} \alpha_i \alpha_j V_{ij} \preceq 0\ \forall \alpha \in \Omega, \tag{1.4}$$

where $\Omega = \{\alpha \in \mathbb{R}^N : \alpha_i \ge 0, \forall i \in \mathcal{N}; \sum_{i=1}^{N} \alpha_i = 1\}$.

Chapter 2

H_2 and H_∞ state feedback control of continuous-time polytopic systems

This chapter is organized as follows. We first provide a unified parameter-dependent BMI condition for H_2-norm and H_∞-norm state feedback control problems in Section 2.1. We then extend the approach of [35] in Section 2.2 to derive a finite set of BMI sufficient conditions. We also outline and extend the approach of [34] to separate the terms in the bilinear product by introducing parameterized LMIs, which are shown to include the quadratic conditions as a special case. In Section 2.3, we propose new sufficient conditions for the BMI conditions in terms of LMI conditions, without any scalar parameter, by using a novel separation result. Our proposed LMI conditions contain one known solution to the BMI conditions as a particular case. Therefore, starting with a feasible solution to the BMI conditions found by an existing method (e.g., quadratic stability conditions or [34]), the upper bounds on the H_2-norm and H_∞-norm are then iteratively reduced based on an update procedure. In the case that no feasible solution can be found using an existing method, we modify our method in Section 2.4 to provide the possibility of finding a feasible solution. We illustrate the effectiveness of our two algorithms through four examples from the literature in Section 2.5 and summarize this chapter in Section 2.6.

The results presented in this chapter are based on our paper [53] and the associated contributions are highlighted as below:

- Propose novel (true) LMI based conditions for H_2-norm and H_∞-norm design problems. The proposed conditions include the optimal quadratic stability solution as a special case.
- Propose a novel algorithm to iteratively reduce upper bounds for the H_2 and H_∞-norms.
- Propose a novel algorithm via an iterative procedure to find a feedback gain that can stabilize the system when other existing approaches fail.

2.1 Problem description

In this section, we define the robust state feedback H_2-norm and H_∞-norm design problems and embed them in a unified generalized robust design problem involving a parameter-dependent Lyapunov matrix variable. We give sufficient conditions for the solution of this problem in the form of a finite set of BMIs. We also show that in the quadratic case, when the Lyapunov function is restricted to be parameter-independent, the BMIs reduce to LMIs.

2.1.1 Robust Design Problem (RDP)

Consider the uncertain continuous-time linear system

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} A(\alpha) & B(\alpha) & B_w(\alpha) \\ C(\alpha) & D(\alpha) & D_w(\alpha) \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \\ w(t) \end{bmatrix}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^{n_u}$, $w(t) \in \mathbb{R}^{n_w}$ and $z(t) \in \mathbb{R}^{n_z}$ are the state, input, exogenous disturbance and cost signals, respectively, and where the system distribution matrices of appropriate dimensions lie in an uncertainty polytope spanned by the convex combination of N given vertices

$$\begin{bmatrix} A(\alpha) & B(\alpha) & B_w(\alpha) \\ C(\alpha) & D(\alpha) & D_w(\alpha) \end{bmatrix} = \sum_{i=1}^{N} \alpha_i \begin{bmatrix} A_i & B_i & B_{wi} \\ C_i & D_i & D_{wi} \end{bmatrix},$$

where α is a time-invariant parameter belonging to the unit simplex

$$\Omega = \{\alpha \in \mathbb{R}^N : \alpha_i \geq 0, \forall i \in \mathcal{N}; \sum_{i=1}^{N} \alpha_i = 1\}.$$

With single state feedback controller $u(t) = Kx(t)$, where $K \in \mathbb{R}^{n_u \times n}$, is to be designed, then the closed-loop system G is described by

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} A(\alpha) + B(\alpha)K & B_w(\alpha) \\ C(\alpha) + D(\alpha)K & D_w(\alpha) \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}. \tag{2.1}$$

Suppose G is Hurwitz stable and strictly proper, i.e., the eigenvalues of the closed-loop matrix $(A(\alpha) + B(\alpha)K)$ lie in the open left half plane for all $\alpha \in \Omega$ and $D_w(\alpha) = 0$. Then, the H_2-norm of the system G is defined by

$$\|G\|_2 := \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} Trace\left(G(jw) * G(jw)\right) dw}$$

The controllability gramian $P_c(\alpha)$ and the observability gramian $P_o(\alpha)$ are defined by

$$P_c(\alpha) := \int_0^{\infty} exp(A_{cl}(\alpha)t) B_w(\alpha) B_w(\alpha)^T exp(A_{cl}(\alpha)^T t) dt,$$
$$P_o(\alpha) := \int_0^{\infty} exp(A_{cl}(\alpha)^T t) C_{cl}(\alpha)^T C_{cl}(\alpha) exp(A_{cl}(\alpha)t) dt,$$

where $A_{cl}(\alpha) = A(\alpha) + B(\alpha)K$ and $C_{cl}(\alpha) = C(\alpha) + D(\alpha)K$. Note that the controllability gramian $P_c(\alpha)$ and the observability gramian $P_o(\alpha)$ are the solutions to the Lyapunov equations

$$A_{cl}(\alpha)P_c(\alpha) + P_c(\alpha)A_{cl}(\alpha)^T + B_w(\alpha)B_w(\alpha)^T = 0,$$
$$A_{cl}(\alpha)^T P_o(\alpha) + P_o(\alpha)A_{cl}(\alpha) + C_{cl}(\alpha)^T C_{cl}(\alpha) = 0,$$

respectively. Hence,

$$\|G\|_2^2 = Trace\left(C_{cl}(\alpha)P_c(\alpha)C_{cl}(\alpha)^T\right) = Trace\left(B_w(\alpha)^T P_o(\alpha) B_w(\alpha)\right).$$

Moreover, suppose that the system G is stable, then the H_∞-norm of the system G is defined by

$$\|G\|_\infty := \sup_{w(t) \in \mathbb{R}} \|G(jw)\| = \sup_{w(t) \in L_2, \|w(t)\|_2 \neq 0} \frac{\|z(t)\|_2}{\|w(t)\|_2}.$$

In principle, we would like to find an optimal controller gain K such that achieves stabilization and minimizes the H_2-norm or the H_∞-norm of the system G. However, computing such an optimal controller is numerically challenging, a better approach in practice is to design a robust gain K to stabilize the system with guaranteed performance, such that

$$\|G\|_2 < \mu \;\; or \;\; \|G\|_\infty < \gamma,$$

where μ and γ are an upper bound (guaranteed cost) for the H_2-norm and the H_∞-norm of the system G, respectively.
We next characterize the conditions for assuring the upper bounds of the H_2-norm and the H_∞-norm in terms of parameter-dependent BMIs.

Lemma 2.1. *Consider the closed-loop system in* (2.1).

1. *(H_2-norm) [27] System (2.1) with $D_w(\alpha) = 0$ is Hurwitz stable and its H_2-norm is less than μ if there exist parameter-dependent matrices $P(\alpha) \in \mathbb{S}^n_+$ and $W(\alpha) \in \mathbb{S}^{n_z}$ such that, for all $\alpha \in \Omega$,*
$$Trace\big(W(\alpha)\big) < \mu^2, \tag{2.2}$$
$$\begin{bmatrix} \text{-}P(\alpha) & \star \\ (C(\alpha) + D(\alpha)K)P(\alpha) & \text{-}W(\alpha) \end{bmatrix} \prec 0, \tag{2.3}$$
$$\begin{bmatrix} \mathcal{H}\big((A(\alpha) + B(\alpha)K)P(\alpha)\big) & \star \\ B_w(\alpha)^T & \text{-}I_{n_w} \end{bmatrix} \prec 0. \tag{2.4}$$

2. *(H_∞-norm) [29] System (2.1) is Hurwitz stable and its H_∞-norm is less than γ if there exists a parameter-dependent matrix $P(\alpha) \in \mathbb{S}^n_+$ such that, for all $\alpha \in \Omega$,*
$$\begin{bmatrix} \mathcal{H}\big((A(\alpha) + B(\alpha)K)P(\alpha)\big) & \star & \star \\ (C(\alpha) + D(\alpha)K)P(\alpha) & \text{-}I_{n_z} & \star \\ B_w(\alpha)^T & D_w(\alpha)^T & \text{-}\gamma^2 I_{n_w} \end{bmatrix} \prec 0. \tag{2.5}$$

Remark 2.1. *The conditions in Lemma 2.1 are a simple extension of well-known standard results in the literature. For example, the H_2-norm and H_∞-norm conditions follow from controllability gramian and the bounded real lemma applied to the closed-loop system in* (2.1)*, respectively, by effecting a congruence transformation and using $V(x) = x^T P(\alpha)^{\text{-}1} x$ as the Lyapunov function.*

The objective of H_2 or H_∞ control design is to find a controller gain K such that the system G is stabilized and the upper bound μ or γ is minimized. Therefore, H_2 or H_∞ control design can be expressed as the following optimization problems, respectively,

$$\begin{aligned} \min_{P(\alpha),\ W(\alpha),\ \mu,\ K} \quad & \mu \\ s.t. \quad & (2.2), (2.3), (2.4). \end{aligned} \tag{2.6}$$

$$\begin{aligned} \min_{P(\alpha),\ \gamma,\ K} \quad & \gamma \\ s.t. \quad & (2.5). \end{aligned} \tag{2.7}$$

Note that the above optimization problems are non-convex due to the bilinear term $KP(\alpha)$ in (2.3)-(2.5), we next propose new results to translate the original nonlinear H_2 and H_∞ control design problems into convex problems in terms of LMIs.

2.1.2 Parameter-dependent BMI formulation for a Generalized Robust Design Problem (GRDP)

An inspection of the conditions (2.3)-(2.5) and $P(\alpha) \succ 0$ verifies that they are special cases of the following more general problem considered in this paper.

Problem 2.1. *(GRDP) Let $\mathcal{F} \in \mathbb{R}^{m,n}$ and for all $i \in \mathcal{N}$, let $\mathcal{A}_i \in \mathbb{R}^{m\times n}, \mathcal{B}_i \in \mathbb{R}^{m\times n_u}$ and $\mathcal{T}_i \in \mathbb{S}^m$ be given and, for any $\alpha \in \Omega$ let*

$$\begin{bmatrix} \mathcal{A}(\alpha) & \mathcal{B}(\alpha) & \mathcal{T}(\alpha) \end{bmatrix} := \sum_{i=1}^{N} \alpha_i \begin{bmatrix} \mathcal{A}_i & \mathcal{B}_i & \mathcal{T}_i \end{bmatrix}.$$

Find a state feedback controller $K \in \mathbb{R}^{n_u\times n}$ and a parameter-dependent Lyapunov matrix $P(\alpha) \in \mathbb{S}^n_+$ such that for all $\alpha \in \Omega$,

$$\mathcal{T}(\alpha) + \mathcal{H}\Big((\mathcal{A}(\alpha) + \mathcal{B}(\alpha)K)P(\alpha)\mathcal{F}^T\Big) \prec 0. \tag{2.8}$$

Remark 2.2. *Note that $\mathcal{T}(\alpha), \mathcal{A}(\alpha)$ and $\mathcal{B}(\alpha)$ are in general augmented parameter-dependent system matrices, that may also depend affinely on other variables, and $\mathcal{F}$ is a constant matrix:*

- *For the H_2 case*
  - *For the first condition in* (2.3)*:*
    $$\mathcal{T}(\alpha) = \begin{bmatrix} -P(\alpha) & 0 \\ 0 & -W(\alpha) \end{bmatrix},\ \mathcal{A}(\alpha) = \begin{bmatrix} 0 \\ C(\alpha) \end{bmatrix}, \mathcal{B}(\alpha) = \begin{bmatrix} 0 \\ D(\alpha) \end{bmatrix},\ \mathcal{F} = \begin{bmatrix} I_n \\ 0 \end{bmatrix}.$$

– *For the second condition in* (2.4)*:*
$\mathcal{T}(\alpha) = \begin{bmatrix} 0 & B_w(\alpha) \\ B_w(\alpha)^T & \text{-}I \end{bmatrix}$, $\mathcal{A}(\alpha) = \begin{bmatrix} A(\alpha) \\ 0 \end{bmatrix}$, $\mathcal{B}(\alpha) = \begin{bmatrix} B(\alpha) \\ 0 \end{bmatrix}$, $\mathcal{F} = \begin{bmatrix} I_n \\ 0 \end{bmatrix}$.

- *For the H_∞ case in* (2.5)*:* $\mathcal{T}(\alpha) = \begin{bmatrix} 0 & 0 & B_w(\alpha) \\ 0 & \text{-}I & D_w(\alpha) \\ B_w(\alpha)^T & D_w(\alpha)^T & \text{-}\gamma^2 I \end{bmatrix}$,
$\mathcal{A}(\alpha) = \begin{bmatrix} A(\alpha) \\ C(\alpha) \\ 0 \end{bmatrix}$, $\mathcal{B}(\alpha) = \begin{bmatrix} B(\alpha) \\ D(\alpha) \\ 0 \end{bmatrix}$, $\mathcal{F} = \begin{bmatrix} I_n \\ 0 \\ 0 \end{bmatrix}$.

Remark 2.3.

- *Note that $P(\alpha)$ in* (2.8) *is required to be a general function of α for* (2.3)-(2.5)*. For a practical implementation, we follow the standard practice [13, 24, 29, 34, 35] and restrict the Lyapunov function to be affine in the parameters:*

$$P(\alpha) = \sum_{j=1}^{N} \alpha_j P_j, \quad P_j \in \mathbb{S}_+^n, \forall j \in \mathcal{N}. \tag{2.9}$$

This will introduce some conservatism.

- *The inequality in* (2.8) *is parameter-dependent which leads to an infinite number of conditions.*

- *The parameter-dependent inequality* (2.8) *is bilinear due to the product terms $KP(\alpha)$ (unless $P(\alpha)$ is independent of α).*

2.1.3 The Quadratic GRDP

The most common, simplest, though generally conservative, approach to deal with the first issue in Remark 2.3, which also resolves the other issues, is to assume that the Lyapunov matrix is independent of α so that $P(\alpha) = P$ for all α. In this case, Problem 2.1 reduces to the following simple LMI problem: Find $M \in \mathbb{R}^{n_u \times n}$ and $P \in \mathbb{S}_+^n$ such that the following LMIs

$$\mathcal{T}_i + \mathcal{H}\Big(\mathcal{A}_i P \mathcal{F}^T + \mathcal{B}_i M \mathcal{F}^T\Big) \prec 0, \tag{2.10}$$

are satisfied for all $i \in \mathcal{N}$, with $K = MP^{\text{-}1}$.

2.2 Extensions of current results for initial computation

In this section, we first extend the approach of [35] to derive a finite set of N^2 BMI sufficient conditions for the parameter-dependent BMI conditions (2.8) under the assumption of an affine parameter-dependent Lyapunov matrix, where the bilinearity is due to the multiplication between the Lyapunov and state feedback gain matrices. We then outline and extend the work of [34] to transform the BMI generalized problem into a parameterized LMI problem by introducing constrained slack variables to separate the bilinear terms.

2.2.1 Finite set of BMI sufficient conditions for GRDP

In order to address the second issue in Remark 2.3, we next extend the approach of [35] and convert the infinite-dimensional conditions (2.8) of Problem 2.1 into a finite number of BMIs by introducing additional N^2 symmetric matrix variables.

Lemma 2.2. *Let all variables be as given in Problem 2.1 and assume that $P(\alpha)$ has the form* (2.9)*. Then there exists a feasible solution to Problem 2.1 if, for all $i, j \in \mathcal{N}$, there exist $P_j \in \mathbb{S}^n_+$ and $V_{ij} \in \mathbb{S}^m$ such that*

$$\sum_{i,j=1}^{N} \alpha_i \alpha_j V_{ij} \preceq 0 \;\forall \alpha \in \Omega, \tag{2.11}$$

$$\mathcal{T}_i + \mathcal{H}\Big(\mathcal{A}_i P_j \mathcal{F}^T + \mathcal{B}_i K P_j \mathcal{F}^T\Big) \prec V_{ij}. \tag{2.12}$$

Proof. Multiplying (2.12) by $\alpha_i \alpha_j$, for all $i, j \in \mathcal{N}$ and summing gives

$$\mathcal{T}(\alpha) + \mathcal{H}\Big((\mathcal{A}(\alpha) + \mathcal{B}(\alpha)K)P(\alpha)\mathcal{F}^T\Big) \prec \sum_{i,j=1}^{N} \alpha_i \alpha_j V_{ij} \preceq 0.$$

□

Since characterizing (2.11) is intractable, we follow Lemma 1.3 and replace it by tractable constraints, at the expense of introducing further conservatism, to define the following problem which requires a finite number of sufficient conditions for the solution of Problem 2.1.

Problem 2.2. *Let all variables be as given in Problem 2.1. Find $K \in \mathbb{R}^{n_u \times n}$ and for all $i, j \in \mathcal{N}$, find $P_j \in \mathbb{S}^n_+$ and $V_{ij} \in \mathbb{S}^m$ such that* (2.12) *and*

$$V_{ij} + V_{ji} \succeq 0,\; 1 \le i < j \le N, \quad \sum_{i=1}^{N} (V_{ij} + V_{ji}) \preceq 0,\; j = 1, \ldots, N, \tag{2.13}$$

are satisfied.

Remark 2.4. *Note that the LMI constraints in* (2.13) *are sufficient for the nonlinear constraints in* (2.11) *[35]. Note also that for control applications, the parameter $\mathcal{T}_i$ and $\mathcal{H}\Big(\mathcal{A}_i P_j \mathcal{F}^T\Big)$ are typically linear in the system matrices and other variables, e.g. P_j. This makes Problem 2.2 tractable (since the inequality* (2.12) *linear) for system analysis ($K = 0$ or K is given), while for controller synthesis,* (2.12) *is bilinear due to the product terms in KP_j; this results in non-convexity of Problem 2.2. Note finally that when $P_j = P$ and $V_{ij} = 0$ for all $i, j \in \mathcal{N}$,* (2.12) *reduces to* (2.10).

2.2.2 Parameterized LMI sufficient conditions for GRDP

At this stage, almost all other work in the literature uses the result of [9] which, by introducing two slack variables F and G, separates K and P_j in the terms KP_j in (2.12) and replaces them with the terms KF and KG. Then to enforce linearity and at the same time ensure that the solution includes the quadratic case, one of the slack variables is restricted and a scalar parameter is introduced, such that $(F, G) \rightarrow (G, rG)$. One of the least conservative approaches to deal with robust stability synthesis design in the form of (2.12) is the work in [34]. The following result is a simple extension of Lemma 9 in [34] from the robust stabilization problem to the more general Problem 2.2 (which includes the robust H_2-norm and H_∞-norm control problems).

Theorem 2.1. *Let all variables be as given in Problem 2.1. Then Problem 2.2 has a feasible solution if there exist $Y \in \mathbb{R}^{n\times n}$, $M \in \mathbb{R}^{n_u \times n}$ and, for all $i, j \in \mathcal{N}$, there exist $P_j \in \mathbb{S}^n_+$, $V_{ij} \in \mathbb{S}^m$ and a non-zero scalar $r \in \mathbb{R}$ such that* (2.13) *and*

$$\begin{bmatrix} \mathcal{T}_i + \mathcal{H}\Big(\mathcal{A}_i P_j \mathcal{F}^T + \frac{1}{r}\mathcal{B}_i M \mathcal{F}^T\Big) - V_{ij} & \star \\ \Big(\mathcal{F}(rP_j - Y^T) + \frac{1}{r}\mathcal{B}_i M\Big)^T & -\mathcal{H}(Y) \end{bmatrix} \prec 0, \tag{2.14}$$

are satisfied, in which case the state feedback gain K is given by $K = MY^{-1}$. Furthermore, if the quadratic condition (2.10) *holds, then the condition* (2.14) *holds for a sufficiently large r.*

Proof. Effecting the congruence transformation $\begin{bmatrix} I_m & 0 \\ \frac{1}{r}K^T\mathcal{B}_i^T & I_n \end{bmatrix}$ on (2.14) and using the fact that $M = KY$ shows that (2.14) is equivalent to

$$\begin{bmatrix} \mathcal{T}_i + \mathcal{H}\Big(\mathcal{A}_i P_j \mathcal{F}^T + \mathcal{B}_i K P_j \mathcal{F}^T\Big) - V_{ij} & \star \\ \Big(\mathcal{F}(rP_j - Y^T) - \frac{1}{r}\mathcal{B}_i K Y^T\Big)^T & -\mathcal{H}(Y) \end{bmatrix} \prec 0.$$

This shows that (2.14) $\Rightarrow$ (2.12), so the fist part is proved. For the second, setting $P_j = P$, $Y = rP$, and $V_{ij} = 0$, (2.14) reduces to

$$\begin{bmatrix} \mathcal{T}_i + \mathcal{H}\Big(\mathcal{A}_i P \mathcal{F}^T + \mathcal{B}_i K P \mathcal{F}^T\Big) & \star \\ (\mathcal{B}_i K P)^T & -2rP \end{bmatrix} \prec 0.$$

Using a Schur complement argument, the above condition is equivalent to

$$\mathcal{T}_i + \mathcal{H}\Big(\mathcal{A}_i P \mathcal{F}^T + \mathcal{B}_i K P \mathcal{F}^T\Big) + \frac{1}{2r}\mathcal{B}_i K P K^T \mathcal{B}_i^T \prec 0,$$

which is equivalent to (2.10) for a sufficiently large r. □

Remark 2.5. *Note that if* (2.14) *is satisfied, then* Y *is non-singular since* $\mathcal{H}(Y) \succ 0$ *and so* K *can be obtained from* $M = KY$*. Additionally, Theorem 2.1 shows that (2.14) contains the quadratic condition* (2.10) *as a special case. Hence, Theorem 2.1 can be guaranteed to provide no more conservative results than quadratic conditions. However, (2.14) contains an unbounded tuning scalar parameter, thus exhaustive scalar searches of this parameter have to be implemented.*

2.3 Linearization and update computation algorithm

In this section, we propose a general separation result to provide sufficient conditions for Problem 2.2 by removing the associated bilinearity with the help of slack variables. Our conditions are attractive from a computational point of view since they are expressed as true LMIs. Moreover, the proposed conditions contain one known solution to Problem 2.2 as a particular case. By adopting this attractive property, we then present an algorithm to reduce the upper bound on the H_2-norm or H_∞-norm via an iterative procedure if one solution to Problem 2.2 is known.

2.3.1 A general separation result

The next theorem is a general result which allows us to separate the product of two variables of the form EX in a BMI without any conservatism by replacing the term EX by the bilinear terms EY and EZ, where Y and Z are slack variables. It also suggests a procedure for restricting these two slack variables to allow a linear solution that captures a given feasible solution of the BMI.

Theorem 2.2. *Let $T \in \mathbb{S}^m$, $E, F \in \mathbb{R}^{m\times n}$ and $X \in \mathbb{S}^n$. Then*

$$T + \mathcal{H}\Big(EXF^T\Big) \prec 0, \qquad X \succ 0, \tag{2.15}$$

if and only if there exist $Z \in \mathbb{R}^{n\times n}$ and $Y \in \mathbb{R}^{n\times m}$ such that

$$\begin{bmatrix} T\text{-}\mathcal{H}\Big((E\text{ - }\tfrac{1}{2}F)\,Y\Big) & \star & \star \\ Y & \text{-}X & \star \\ ((E+\tfrac{1}{2}F)Z)^T & 0 & X\text{-}\mathcal{H}(Z) \end{bmatrix} \prec 0. \tag{2.16}$$

Proof. Effecting the congruence $\begin{bmatrix} I_m & 0 & 0 \\ (E\text{ - }\tfrac{1}{2}F)^T & I_n & 0 \\ (E+\tfrac{1}{2}F)^T & 0 & I_n \end{bmatrix}$ on (2.16) shows that it is equivalent to

$$\begin{bmatrix} T\text{ - }(E\text{ - }\tfrac{1}{2}F)X(E\text{ - }\tfrac{1}{2}F)^T + (E+\tfrac{1}{2}F)X(E+\tfrac{1}{2}F)^T & \star & \star \\ Y\text{ - }X(E\text{ - }\tfrac{1}{2}F)^T & \text{-}X & \star \\ (X\text{ - }Z)(E+\tfrac{1}{2}F)^T & 0 & X\text{ - }\mathcal{H}(Z) \end{bmatrix} \prec 0. \tag{2.17}$$

This shows that (2.16) $\Rightarrow$ (2.15) since $\mathcal{H}\Big(EXF^T\Big) = \text{-}(E\text{ - }\tfrac{1}{2}F)X(E\text{ - }\tfrac{1}{2}F)^T + (E+\tfrac{1}{2}F)X(E+\tfrac{1}{2}F)^T$. Furthermore, if (2.15) is satisfied, then an inspection of (2.17) shows that by defining

$$Y = X(E\text{ - }\frac{1}{2}F)^T, \qquad Z = X, \tag{2.18}$$

then (2.15) $\Rightarrow$ (2.17) which is equivalent to (2.16) and completes the proof. □

Remark 2.6. *Note that, provided that F is constant, then both* (2.15) *and* (2.16) *are bilinear. However,* (2.16) *is linear in X and, provided that the slack variables Y and Z are suitably restricted e.g. $Y = WY_0$ and $Z = WZ_0$ where W is a variable while Y_0 and Z_0 are constant, then* (2.16) *becomes linear by considering $M := EW$ and W as decision variables, although it will be only sufficient for* (2.15) *because of this restriction.*

2.3.2 LMI sufficient conditions that include one known solution

The following corollary is a direct application of Theorem 2.2 to Problem 2.2, and with a choice of the slack variables Y and Z suggested by (2.17)-(2.18) that will ensure that the solution includes at least one known feasible solution to Problem 2.2.

Corollary 2.1. *Suppose that $\tilde{K} \in \mathbb{R}^{n_u \times n}$ and for all $i, j \in \mathcal{N}$, $\tilde{\mathcal{T}}_i \in \mathbb{S}^m$, $\tilde{P}_j \in \mathbb{S}^n_+$ and $\tilde{V}_{ij} \in \mathbb{S}^m$ solve Problem 2.2 so that*

$$\begin{aligned}
&\tilde{V}_{ij} + \tilde{V}_{ji} \succeq 0,\ 1 \leq i < j \leq N; \\
&\sum_{i=1}^{N} (\tilde{V}_{ij} + \tilde{V}_{ji}) \preceq 0,\ j = 1, \dots, N; \\
&\tilde{\mathcal{T}}_i + \mathcal{H}\Big(\mathcal{A}_i \tilde{P}_j \mathcal{F}^T + \mathcal{B}_i \tilde{K} \tilde{P}_j \mathcal{F}^T\Big) \prec \tilde{V}_{ij}.
\end{aligned}$$

If there exist $Y \in \mathbb{R}^{n \times n}$ and $M \in \mathbb{R}^{n_u \times n}$ and, for all $i, j \in \mathcal{N}$, there exist $P_j \in \mathbb{S}^n_+$ and $V_{ij} \in \mathbb{S}^m$ such that (2.13) *and*

$$\begin{bmatrix}
\mathcal{T}_i + \mathcal{H}\Big(\mathcal{A}_i P_j \mathcal{F}^T - (\mathcal{B}_i M - \tfrac{1}{2}\mathcal{F} Y)\tilde{P}_j(\mathcal{B}_i \tilde{K} - \tfrac{1}{2}\mathcal{F})^T\Big) - V_{ij} & \star & \star \\
Y \tilde{P}_j (\mathcal{B}_i \tilde{K} - \tfrac{1}{2}\mathcal{F})^T & -P_j & \star \\
((\mathcal{B}_i M + \tfrac{1}{2}\mathcal{F} Y)\tilde{P}_j)^T & 0 & P_j - \mathcal{H}\Big(Y \tilde{P}_j\Big)
\end{bmatrix} \prec 0. \tag{2.19}$$

are satisfied, then with $K = MY^{-1}$, and for all $i, j \in \mathcal{N}$, $\mathcal{T}_i$, P_j and V_{ij} also solve Problem 2.2. Furthermore, condition (2.19) *is satisfied by*

$$(\mathcal{T}_i, P_j, V_{ij}, Y, M) := (\tilde{\mathcal{T}}_i, \tilde{P}_j, \tilde{V}_{ij}, I_n, \tilde{K}).$$

Proof. For the first part, inequality (2.12) can be rewritten as the first inequality in (2.15) with $T = \mathcal{T}_i + \mathcal{H}\Big(\mathcal{A}_i P_j \mathcal{F}^T\Big) - V_{ij}$, $E = \mathcal{B}_i K$, $F = \mathcal{F}$, and $X = P_j$, so that it follows from Theorem 2.2 that (2.12) is satisfied if and only if there exist matrices Y_{ij} and $Z_{ij}, \forall i, j \in \mathcal{N}$ such that

$$\begin{bmatrix}
\mathcal{T}_i + \mathcal{H}\Big(\mathcal{A}_i P_j \mathcal{F}^T - (\mathcal{B}_i K - \tfrac{1}{2}\mathcal{F}) Y_{ij}\Big) - V_{ij} & \star & \star \\
Y_{ij} & -P_j & \star \\
((\mathcal{B}_i K + \tfrac{1}{2}\mathcal{F}) Z_{ij})^T & 0 & P_j - \mathcal{H}\Big(Z_{ij}\Big)
\end{bmatrix} \prec 0. \tag{2.20}$$

Setting $Y_{ij} = Y\tilde{P}_j(\mathcal{B}_i\tilde{K} - \frac{1}{2}\mathcal{F})^T$ and $Z_{ij} = Y\tilde{P}_j$ give (2.19). Furthermore, When $(\mathcal{T}_i, P_j, V_{ij}, Y, M) := (\tilde{\mathcal{T}}_i, \tilde{P}_j, \tilde{V}_{ij}, I, \tilde{K})$, inequality (2.19) becomes

$$\begin{bmatrix} \tilde{\mathcal{T}}_i + \mathcal{H}\Big(\mathcal{A}_i\tilde{P}_j\mathcal{F}^T - (\mathcal{B}_i\tilde{K} - \frac{1}{2}\mathcal{F})\tilde{P}_j(\mathcal{B}_i\tilde{K} - \frac{1}{2}\mathcal{F})^T\Big) - \tilde{V}_{ij} & \star & \star \\ \tilde{P}_j(\mathcal{B}_i\tilde{K} - \frac{1}{2}\mathcal{F})^T & -\tilde{P}_j & \star \\ ((\mathcal{B}_i\tilde{K} + \frac{1}{2}\mathcal{F})\tilde{P}_j)^T & 0 & -\tilde{P}_j \end{bmatrix} \prec 0.$$

Using a Schur complement argument shows that the above inequality is equivalent to $\tilde{\mathcal{T}}_i + \mathcal{H}\Big(\mathcal{A}_i\tilde{P}_j\mathcal{F}^T\Big) + \mathcal{H}\Big(\mathcal{B}_i\tilde{K}\tilde{P}_j\mathcal{F}^T\Big) \prec \tilde{V}_{ij}$ and proves the second part. □

Remark 2.7. *Note that $\tilde{K}$ and $\tilde{P}_j$ are given by a known feasible solution to Problem 2.2, so* (2.19) *is a true LMI and therefore it can be efficiently implemented by an LMI solver. Corollary 2.1 illustrates that if a feasible solution to Problem 2.2 can be found, then there exist solutions to Corollary 2.1. Furthermore, these solutions to Corollary 2.1 also solve Problem 2.2. Note also that the slack variable Y remains unconstrained since it is not restricted to have any definiteness or symmetry property, so it provides extra degrees of freedom to search for a better solution. Hence, we conclude that the new solution to Corollary 2.1 is at least as good as the previous known solution to Problem 2.2. Note also that for achieving linearity of the terms KY_{ij} and KZ_{ij} in (2.20), we impose the additional constraints $Y_{ij} = Y\tilde{P}_j(\mathcal{B}_i\tilde{K}-\frac{1}{2}\mathcal{F})^T$ and $Z_{ij} = Y\tilde{P}_j$ so that $M := KY$ and Y are decision variables.*

Remark 2.8. *Suppose that (2.19) is satisfied. Then $\mathcal{H}\Big(Y\tilde{P}_j\Big) - P_j \succ 0$, and this, together with the fact that $\tilde{P}_j$ and P_j are positive definite implies that Y is nonsingular. Thus the feedback gain K can always be recovered from $K = MY^{-1}$.*

As illustrated in Corollary 2.1, there must exist a feasible solution to Corollary 2.1 if a feasible solution to Problem 2.2 is available. In an optimization problem, this solution to Problem 2.2 may be chosen as the optimal solution obtained by an existing method (e.g., quadratic conditions, or indeed some other appropriate methods proposed in Theorem 2.1. This ensures that our solution is no more conservative than the optimal solution computed by any of these existing methods with one step computation. Furthermore, this solution may be chosen as the current solution, which defines an iterative procedure if better solutions are required. We next take advantage of this useful property to present an algorithm to iteratively compute potentially less conservative bounds on the H_2- or the H_∞-norm problems by utilizing the optimal solution computed by any existing method as a starting point.

Algorithm 2.1. *Given $\mathcal{F}$, $\mathcal{T}_i$, $\mathcal{A}_i$, $\mathcal{B}_i$ for all $i \in \mathcal{N}$, tolerance level tol, and it_{max} (maximum number of iterations).*

1. ***Initial solution:*** *Find a solution to Problem 2.2 by using the quadratic method of Section 2.1.3, other appropriate methods available in the literature (e.g. Theorem 2.1) or the methods of Section 2.4 below. Set* $\tilde{\mu} = \mu$ *(or* $\tilde{\gamma} = \gamma$*),* $\tilde{K} = K$ *and* $\tilde{P}_j = P_j$ *for all* $j \in \mathcal{N}$*, and set* $k = 0$*.*

2. ***Update:*** *Minimize* μ *(or* γ*) over the related variables in Corollary 2.1. Record* K*,* P_j*,* μ *(or* γ*).*

3. ***Stopping condition:*** *If* $(\tilde{\mu} - \mu)/\tilde{\mu} \leq tol$ *(or* $(\tilde{\gamma} - \gamma)/\tilde{\gamma} \leq tol$*) or* $k > it_{max}$ *stop. Else set* $\tilde{K} = K$ *and* $\tilde{P}_j = P_j$*,* $\tilde{\mu} = \mu$ *(or* $\tilde{\gamma} = \gamma$*),* $k = k + 1$*, and go to step 2.*

2.4 Robust stabilization when no known initial solution exists

As mentioned in the last section, the proposed Algorithm 2.1 requires a feasible solution to Problem 2.2 provided by an existing method as a starting point; then it can iteratively update the upper bounds of H_2-norm or H_∞-norm. However, as illustrated in the work of [34] and [54], these existing methods may fail to compute a feasible solution for some open-loop unstable uncertain system even when the system is known to be robustly closed-loop stabilizable. Hence, we next propose a modified method via an iterative procedure, based on the results in Corollary 2.1, to allow the possibility of finding a feedback gain that can stabilize the system when all other methods fail.

Our approach for finding a feasible solution to (2.15) uses the following general result. The result is based on Finsler's Lemma and shows that a perturbed version of the BMI in (2.15) is always feasible and easily solvable.

Lemma 2.3. *Let* $T \in \mathbb{S}^m$*,* $E, F \in \mathbb{R}^{m \times n}$ *and* $X \in \mathbb{S}^n_+$ *be given.*

1. *If rank*$(F) = m$*, then there exists* $\beta \in \mathbb{R}$ *such that*

$$T + \mathcal{H}\Big((E - \beta F)XF^T\Big) \prec 0. \tag{2.21}$$

2. *If rank*$(F) < m$*, let* $F_\perp$ *denotes an arbitrary matrix whose columns form a basis for the null space of* F^T *(i.e.* $F^T F_\perp = 0$*). Then there exists* $\beta \in \mathbb{R}$ *such that* (2.21) *is satisfied if and only if* $F_\perp^T T F_\perp \prec 0$*.*

Proof.

1. Suppose that $\text{rank}(F) = m$ and let $Z := T + \mathcal{H}\Big(EXF^T\Big) \in \mathbb{S}^m$ and $Y := 2FXF^T \in \mathbb{S}^m$ so that (2.21) can be written as $Z - \beta Y \prec 0$. Since $\text{rank}(F) = m$ then $FXF^T \succ 0$ since $X \succ 0$. Therefore $Y \succ 0$ and there always exists a $\beta \in \mathbb{R}$ such that $Z - \beta Y \prec 0$, e.g. any β larger than the largest eigenvalue of $Y^{-1}Z$.

2. Suppose that $\text{rank}(F) < m$. Since $X \succ 0$, then it has a Cholesky factorization $X = RR^T$, where $R \in \mathbb{R}^{n\times n}$ is nonsingular. Now, (2.21) can be rewritten as $Q - \mu B^T B \prec 0$ where $Q := T + \mathcal{H}\Big(EXF^T\Big) \in \mathbb{S}^m$, $B := R^T F^T \in \mathbb{R}^{n\times m}$ and $\mu := 2\beta \in \mathbb{R}$ with $\text{rank}(B) = \text{rank}(F) < m$. It follows from Finsler's Lemma (see e.g., [33] for details) that (2.21) is satisfied if and only if $F_\perp^T T F_\perp \prec 0$ since R is nonsingular.

□

Thus, if we cannot find a feasible solution to (2.15) using any of the current approaches, then Lemma 2.3 implies there always exists a solution to (2.21) for some $\beta \in \mathbb{R}$. If $\beta \leq 0$, we are done, otherwise we proceed as follows, where we carry out the analysis for robust stability since we only need a feasible stabilizing solution for the H_2-norm and H_∞-norm problems. We next introduce a slightly modified characterization of Hurwitz stability for the continuous-time closed-loop systems.

Lemma 2.4. *The closed-loop system in* (2.1) *is Hurwitz stable if, for all* $i, j \in \mathcal{N}$, *there exist* $K \in \mathbb{R}^{n_u \times n}$, $\beta \leq 0$, $P_j \in \mathbb{S}_+^n$ *and* $V_{ij} \in \mathbb{S}^m$ *such that* (2.13) *and*

$$\mathcal{H}\Big((A_i - \beta I)P_j + B_i K P_j\Big) - V_{ij} \prec 0, \tag{2.22}$$

are satisfied.

Proof. Multiplying (2.22) by $\alpha_i\alpha_j$, for all $i, j \in \mathcal{N}$, summing and using (2.13) yields

$$\mathcal{H}\big((A(\alpha) + B(\alpha)K - \beta I)P(\alpha)\big) \prec \sum_{i,j=1}^{N} \alpha_i\alpha_j V_{ij} \preceq 0,$$

which shows that $(A(\alpha) + B(\alpha)K - \beta I)$ is Hurwitz stable. It follows that the closed-loop system in (2.1) is Hurwitz stable for $\beta \leq 0$. □

We next relax this characterization by removing the sign requirement on β and consider the following problem:

Problem 2.3. *Let all variables be as in Lemma 2.4. Find*

$$\begin{aligned}\min\{\beta : K \in \mathbb{R}^{n_u \times n}, P_j \in \mathbb{S}^n_+, V_{ij} \in \mathbb{S}^m \text{ s.t.}\\ (2.13) \textit{ and } \mathcal{H}\Big(A_i P_j\Big) + \mathcal{H}\Big(B_i K P_j\Big) - V_{ij} \prec 2\beta P_j \text{ are satisfied}, \forall i, j \in \mathcal{N}\}.\end{aligned} \tag{2.23}$$

Problem 2.3 is bilinear because of the product terms KP_j. Note that if we set $P_j = P$ and $V_{ij} = 0$ for all $i, j \in \mathcal{N}$, then it follows that the computation of the initial solution to Problem 2.3 can now be formulated by the following generalized eigenvalue problem (GEVP) [1].

Problem 2.4. *Let all variables be as in Lemma 2.4. Find*

$$\begin{aligned}\min\{\beta : M \in \mathbb{R}^{n_u \times n}, P \in \mathbb{S}^n_+ \text{ s.t.}\\ \mathcal{H}(A_i P) + \mathcal{H}(B_i M) \prec 2\beta P \text{ are satisfied}, \forall i \in \mathcal{N}, \text{with } K = MP^{-1}\}.\end{aligned} \tag{2.24}$$

We will call this the relaxed quadratic stabilization problem since it follows from Lemma 2.3 (since $F = I$ for Problem 2.4) that it is always feasible. Furthermore, it is a GEVP since $P \succ 0$ and is therefore easily solvable. The key idea of our method is to find a feasible initial solution to Problem 2.3 by solving Problem 2.4, which means that the closed-loop system has all its eigenvalues to the left of the line $Re(s) = \beta$ in the complex plane (in general $\beta > 0$) instead of the open left half-plane required by Lemma 2.4. Subsequently, we try to use the degrees of freedom in the slack variable to obtain a solution for Problem 2.3 with a smaller value of β through the following corollary, until $\beta \leq 0$.

Corollary 2.2. *Let $\tilde{K} \in \mathbb{R}^{n_u \times n}$, $\tilde{\beta} \in \mathbb{R}$, and for all $i, j \in \mathcal{N}$, $\tilde{P}_j \in \mathbb{S}^n_+$ and $\tilde{V}_{ij} \in \mathbb{S}^m$ solve Problem 2.3 so that*

$$\begin{aligned}&\tilde{V}_{ij} + \tilde{V}_{ji} \succeq 0,\ 1 \leq i < j \leq N;\\ &\sum_{i=1}^{N} (\tilde{V}_{ij} + \tilde{V}_{ji}) \preceq 0,\ j = 1, \ldots, N;\\ &\mathcal{H}\Big((A_i - \tilde{\beta} I)\tilde{P}_j + B_i \tilde{K} \tilde{P}_j\Big) - \tilde{V}_{ij} \prec 0.\end{aligned} \tag{2.25}$$

If there exist $Y \in \mathbb{R}^{n \times n}$, $M \in \mathbb{R}^{n_u \times n}$, $\beta \in \mathbb{R}$, and, for all $i, j \in \mathcal{N}$, $P_j \in \mathbb{S}^n_+$ and $V_{ij} \in \mathbb{S}^m$ such that (2.13) *and*

$$\begin{bmatrix} \mathcal{H}\Big((A_i - \beta I)P_j - (B_i M - \frac{1}{2}Y)\tilde{P}_j(B_i\tilde{K} - \frac{1}{2}I)^T\Big) - V_{ij} & \star & \star \\ Y\tilde{P}_j(B_i\tilde{K} - \frac{1}{2}I)^T & -P_j & \star \\ ((B_i M + \frac{1}{2}Y)\tilde{P}_j)^T & 0 & P_j - \mathcal{H}\Big(Y\tilde{P}_j\Big) \end{bmatrix} \prec 0, \tag{2.26}$$

are satisfied, then with $K = MY^{-1}$*,* β*, and for all* $i, j \in \mathcal{N}$*,* P_j *and* V_{ij} *also solve Problem 2.3. Furthermore, condition* (2.26) *is satisfied by*

$$(P_j, V_{ij}, Y, M, \beta) := (\tilde{P}_j, \tilde{V}_{ij}, I_n, \tilde{K}, \tilde{\beta}).$$

.

Proof. The inequality involving β in (2.23) can be rewritten as (2.12) with $\mathcal{T}_i = 0$, $\mathcal{A}_i = A_i - \beta I$, $\mathcal{B}_i = B_i$, and $\mathcal{F} = I_n$, and so the result follows from Corollary 2.1. □

Remark 2.9. *Note that the inequalities in* (2.26) *can be written as*

$$\overbrace{\begin{bmatrix} A_{11}^{ij}(x) & A_{12}^{ij}(x) \\ \star & A_{22}^{ij}(x) \end{bmatrix}}^{A^{ij}(x)} \prec \beta \overbrace{\begin{bmatrix} B_{11}^{ij}(x) & 0 \\ 0 & 0 \end{bmatrix}}^{B^{ij}(x)}$$

where x *denotes all the variables. Technically this cannot be posed as a GEVP since* $B^{ij}(x)$ *is positive semidefinite rather than positive definite (see Section 2.2.3 of [1], where we have defined* $A^{ij}(x)$ *and* $B^{ij}(x)$ *to agree with their notation, with* $C(x) \prec 0$ *denoting all the other LMI conditions in Corollary 2). However, replacing the constraints* $A^{ij}(x) \prec \beta B^{ij}(x)$ *by*

$$A^{ij}(x) \prec \begin{bmatrix} Y^{ij} & 0 \\ 0 & 0 \end{bmatrix}, \; Y^{ij} < \beta B_{11}^{ij}(x), \; B_{11}^{ij}(x) \succ 0,$$

where Y^{ij} *are additional symmetric variables of appropriate dimensions, shows that minimizing* β *subject to the conditions of Corollary 2.2 can be reformulated as a GEVP (see See Section 4.39 of [55] for details), although in this work we use a binary search algorithm as shown in Algorithm 2.2 below.*

As demonstrated in Corollary 2.2, the solution provided by Corollary 2.2 would be no more conservative than the solution to Problem 2.3 since (2.26) and (2.13) contain (2.25) as a special case. Therefore, by solving Corollary 2.2 it may be possible to provide a feasible solution to Problem 2.3 with a smaller value of β. Based on this property, we next propose an update procedure to obtain feasible solutions for Problem 2.3 for a non-increasing sequence of β. If $\beta \leq 0$, then the robust stabilization of closed-loop system is achieved.

Algorithm 2.2. *Given* A_i*,* B_i *for all* $i \in \mathcal{N}$*, tolerance level* tol*, and* it_{max} *(maximum number of iterations).*

1. **Initial solution:** *Solve Problem 2.4 via gevp solver to find the smallest β. If $\beta \leq 0$, then stop, a stabilizing gain is found. Else set $\tilde{K} = K$, $\tilde{\beta} = \beta$, and $\tilde{P}_j = P$ for all $j \in \mathcal{N}$, and set $k = 0$.*

2. **Update:** *Given $\tilde{K}$, $\tilde{\beta}$, and $\tilde{P}_j$ for all $j \in \mathcal{N}$, solve the conditions of Corollary 2.2 for $\beta = 0$. If feasible, record K, β, and P_j for all $j \in \mathcal{N}$ and stop, a stabilizing gain is found. If infeasible, use a bisection algorithm to find a β in the interval $(0 \;\; \tilde{\beta}]$ for which the conditions of Corollary 2.2 are feasible and record K, β, and P_j for all $j \in \mathcal{N}$.*

3. **Stopping condition:** *If $(\tilde{\beta}$ - $\beta)/\tilde{\beta} \leq tol$ or $k > it_{max}$ then stop, the algorithm has failed to find a stabilizing gain. Else set $\tilde{K} = K$, $\tilde{\beta} = \beta$, $\tilde{P}_j = P_j$, $k = k + 1$, and go to Step 2.*

Remark 2.10. *It is worth mentioning that the traditional methods in the literature are based on parameterized LMIs, where an additional exhaustive search procedure on scalar parameters needs to be performed. Though the conditions of Corollary 2.2 also contain a scalar variable β, which are quasiconvex and therefore can be solved efficiently via CVX toolbox using the bisection algorithm (see section 4.2.5 of Boyd and Vandenberghe[56] for details). Moreover, in order to reduce ill-conditioning of Lyapunov matrices, in the practical implementation for Problem 2.4 and the conditions of Corollary 2.2 (which are homogeneous in the variables), we impose the constraint $P \succ I_n$ for Problem 2.4 and $I_n \prec P_j \prec \zeta I_n$ and take ζ as the cost function to be minimized for Corollary 2.2 in the numerical examples later, see [1] for more details.*

Remark 2.11. *Note that although Algorithm 2.2 can guarantee that the computed sequence of β is non-increasing, the final converged value of β cannot be guaranteed to be non-positive, even if Problem 2.3 is known to have a feasible solution for $\beta \leq 0$. Hence the algorithm has two stopping outcomes: (a) a stabilizing gain has been found in Step 1 or Step 2; (b) Algorithm 2.2 fails to find a stabilizing gain in Step 3. In addition, since β is only a upper bound on the maximum real part of eigenvalues of the closed-loop system ($\lambda_{max}(A(\alpha)+B(\alpha)K)$, [57] remarked that the actual value of $\lambda_{max}(A(\alpha)+B(\alpha)K)$ could be negative and therefore the closed-loop system is Hurwitz stable even if $\beta > 0$. Hence, once a state feedback gain K is computed in Step 2, the actual value of $\lambda_{max}(A(\alpha)+B(\alpha)K)$ can be verified, a stabilizing gain is found and therefore Algorithm 2.2 terminates if it is negative. However, checking the actual value of $\lambda_{max}(A(\alpha)+B(\alpha)K)$ in each iteration increases the complexity of Algorithm 2.2; this will be left to a future work.*

2.5 Numerical examples

In this section, we give four examples to illustrate the efficiency of our algorithms. The benefit of Algorithm 2.1 for the computation of less conservative upper bounds on the H_2-norm and H_∞-norm for state feedback control design is illustrated by Example 1 and Example 2. Subsequently, Example 3 and Example 4 are presented to demonstrate the effectiveness of the proposed Algorithm 2.2 for robust stabilization. All algorithms are implemented in Matlab 9.0.0 (R2016a) using the CVX toolbox with MOSEK as solver [58, 59] and running on an Intel(R) Xeon (R) CPU E5-1650, 3.5 GHz, Windows 7 Professional.

2.5.1 Example 1

The problem of controlling a satellite system from [12, 13, 27] is considered in this example. The state space representation is given as follow:

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \text{-}k & k & \text{-}f & f \\ k & \text{-}k & f & \text{-}f \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} w(t),$$
$$z(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.01 \end{bmatrix} u(t),$$

where k and f denote the uncertain parameters of the system, whose uncertainty ranges are [0.09 0.4] and [0.0038 0.04], respectively. This system can be described as in Section 2.1.1 using $N = 4$ vertices. Exhaustive searches on the scalar parameters are performed for current parameterized LMI based methods and Theorem 2.1. The comparisons of the minimum upper bound μ on the H_2-norm and γ on the H_∞-norm obtained by Algorithm 2.1 and some methods in the literature are given in Table 2.1 and Table 2.2.

Method	μ	Scalars
Quadratic [1]	2.5923	-
[24]	1.5526	-
[26]	1.7025	$r = 0.13$
[27]	1.3564	$r = 1.8$
Theorem 2.1	2.1209	$r = 7.7$
Algorithm 2.1	1.2185	tol=0.1%

Table 2.1 The minimum upper bound on H_2-norm computed using some existing methods and Algorithm 2.1 for Example 1.

Method	γ	Scalars
Quadratic [1]	1.5776	-
[12]	1.4782	$r = 0.01$
[13]	1.2416	$r = 0.07$
[29]	1.2414	$\epsilon = 0.1, \xi = 0.54$
Theorem 2.1	1.3062	$r = 28.5$
Algorithm 2.1	1.0418	tol=0.1%

Table 2.2 The minimum upper bound H_∞-norm computed using some existing methods and Algorithm 2.1 for Example 1.

Starting with the solution provided by Theorem 2.1 and setting the tolerance level to be 0.1%, it can be noted from Table 2.1 that Algorithm 2.1 provides less conservative upper bounds on the H_2-norm and H_∞-norm compared to the ones obtained from all other methods. In particular, [27] gives a μ level of 1.3564 with optimized $r = 1.8$, but Algorithm 2.1 obtains a final converged value of 1.2185 for the H_2-norm performance μ with state feedback gain $K = [\text{-}16.3676\ \ \text{-}\ 95.9845\ \ \text{-}\ 8.8811\ \ \text{-}\ 194.6665]$. Compared with the work of [29], which yields the value of 1.2414 for the H_∞-norm performance γ, an upper bound of $\gamma = 1.0418$ can be achieved using Algorithm 2.1 with $K = \text{-}10^4 \times [0.1195\ \ 1.2424\ \ 0.0274\ \ 1.6604]$.

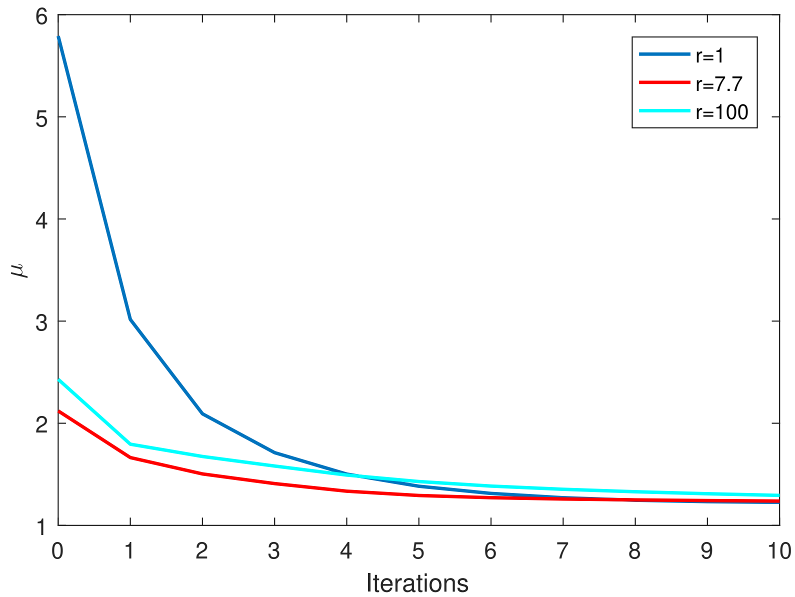

Fig. 2.1 Bound on the H_2-norm against the number of iterations computed using Algorithm 2.1 for Example 1.

Figure 2.1 and Figure 2.2 display the relation between the computed values of μ and γ through Algorithm 2.1 and the number of iterations for different values of r as the initial

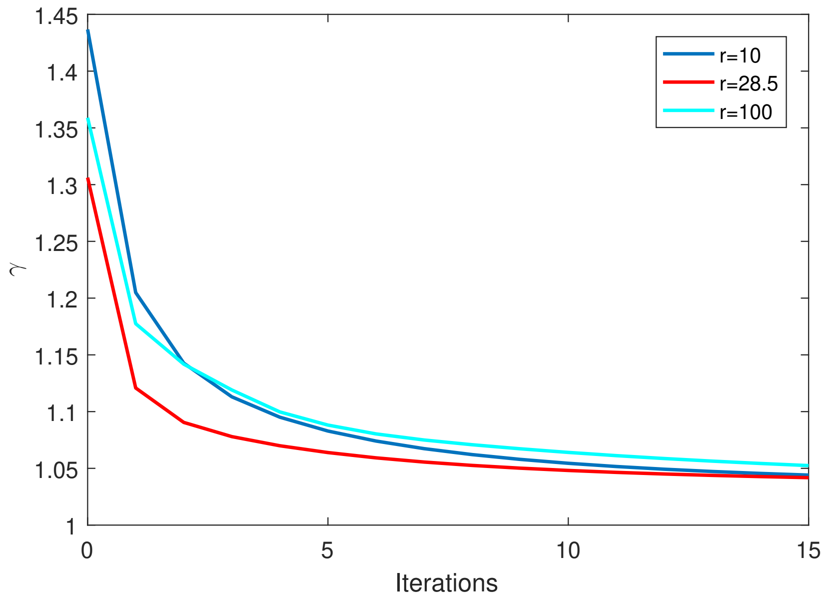

Fig. 2.2 Bound on the H_∞-norm against the number of iterations computed using Algorithm 2.1 for Example 1.

point. It can be seen that the bounds μ and γ are non-increasing with the number of iterations and both converge to nearly the same final values independently of which value of r is used as the initial point. This observed quadratic speed of convergence seems to be an interesting property of our iterative procedure, although a rigorous theoretical proof of this is beyond the scope of this paper.

2.5.2 Example 2

Consider the following uncertain coupled spring-mass system taken from [29]:

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k}{m_1} & \frac{k}{m_1} & 0 & 0 \\ \frac{k}{m_2} & -\frac{k}{m_2} & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_1} \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix} w(t),$$
$$z(t) = \begin{bmatrix} 0.5k & 0.5k & -2k & -k \end{bmatrix} x(t) - 0.1u(t).$$

Here $x(t) = [x_1(t) \;\; x_2(t) \;\; x_3(t) \;\; x_4(t)]^T$ where the states $x_1(t)$ and $x_2(t)$ are the displacements of body 1 and body 2, respectively, while their respective velocities are represented by the states $x_3(t)$ and $x_4(t)$. The parameters m_1 and m_2 denote the masses of body 1 and body

2, respectively, and whose values are given as $m_1 = m_2 = 1$. The stiffness parameter k is uncertain but is known to lie in the interval [0.5 2].

It has been suggested in [29], that in order to avoid the large computational burden caused by an exhaustive search procedure on scalar parameters, the search range of scalar variables is constrained to the following values (total of 12 searches): $\epsilon \in \left\{10^{-1}, 10^{0}\right\}$ and $\xi \in \{-0.9, -0.54, -0.18, 0.18, 0.54, 0.9\}$. For the other parameterized LMI based methods, the scalar is limited to thirteen logarithmically spaced values (total of 13 searches): $r \in \left\{10^{-6}, 10^{-5}, \cdots, 10^{0}, 10^{1}, \cdots 10^{6}\right\}$. For the scalar parameter r of Theorem 2.1, we select $r = 1$ instead of performing searches on r, to obtain a feasible solution used for the starting point of Algorithm 2.1.

Table 2.3 compares the results of μ achieved using our Algorithm 2.1 with the other existing approaches available in the literature as well as the associated solution time.

Method	μ	Scalars	T: solution time
Quadratic [1]	10.2217	-	0.19 s
[26]	5.2704	$r = 0.1$	2.89 s
[24]	3.2638	-	0.20 s
[27]	1.4855	$r = 0.1$	2.65 s
Theorem 2.1	6.8438	$r = 1$	0.34 s
Algorithm 2.1	1.1511	$it = 7$	3.05 s
Algorithm 2.1	0.5044	$it = 15$	6.14 s

Table 2.3 The minimum upper bound on H_2-norm achieved with some existing methods and Algorithm 2.1, as well as the associated solution time for Example 2.

Among the current methods, [27] gives the minimum upper bound on the H_2-norm as $\mu = 1.4855$ with the mean solution time 2.65 s. Applying Theorem 2.1 gives the initial stabilizing state feedback gain $K_0 = [-0.8560\ \ 0.2864\ \ -1.7048\ \ -0.7639]$ in 0.34 s. Algorithm 2.1 shows improvement on H_2-norm performance with respect to [27] after 7 iterations and gives a converged value of $\mu = 0.5044$ after 15 iterations by setting $tol = 0.5\%$; the corresponding state feedback gain is $K = [-5.8340\ \ 0.1948\ \ -29.0254\ \ -15.4930]$. The mean times for Algorithm 2.1 to take 7 iterations and 15 iterations are 3.05 s and 6.14 s, respectively.
As shown in the second column in Table 2.4, the H_∞-norm upper bound provided by [29] is 1.9745. Starting with the initial stabilizing state feedback gain $K_0 = [-0.6036\ \ 0.0001\ \ -1.0105\ \ -0.4056]$ provided by Theorem 2.1, our Algorithm 2.1 can outperform [29] after 6 iterations and finally gives $\gamma = 1.1153$ after 17 iterations (with $tol = 0.5\%$). The final resulting feedback gain is $K = [-4.6271\ \ -0.7696\ \ -17.4697\ \ -9.3864]$. Table 2.4 also gives

Method	γ	Scalars	V	R	T
Quadratic [1]	8.5569	-	15	15	0.18 s
[12]	7.9236	$r = 0.01$	41	23	2.52 s
[13]	2.3847	$r = 10$	41	31	2.65 s
[29]	1.9745	$\epsilon = 1, \xi = 0.18$	41	23	2.33 s
Theorem 2.1	9.4149	$r = 1$	153	74	0.28 s
Algorithm 2.1	1.6275	$it = 6$	153	90	1.99 s
Algorithm 2.1	1.1153	$it = 17$	153	90	5.19 s

Table 2.4 The minimum upper bound on H_∞-norm achieved with some existing methods and Algorithm 2.1, as well as the associated numerical complexity for Example 2 (V: scalar variables, R: LMI rows, T: total solution time).

the associated numerical complexity of the existing methods and Algorithm 2.1 for H_∞ control. For each LMI test, in contrast to the current approaches, the greater number of scalar variables and LMI rows in Corollary 2.1 is due to the additional matrix variables V_{ij}. Note that parameterized LMI based methods require exhaustive search procedures on the scalar parameters while Algorithm 2.1 requires the repeated use of Corollary 2.1. The last column in Table 2.4 gives the total solution time for all methods to evaluate their computational burden.

Algorithm 2.1, which implements the computation of initial solution by Theorem 2.1 and 17 iterations to convergence, demands a longer solution time than the methods from the previous studies. However, it is worth mentioning that the number of LMI tests for the parameterized LMI methods is limited to at most 13 since the search range on scalar parameters is substantially constrained in this example, this leads to less computational time but the obtained results are in general suboptimal. Furthermore, the second row from the bottom in Table 2.4 indicates that Algorithm 2.1 is able to provide less conservative results than the ones in the literature after only 6 iterations, which takes a shorter computational time than these parameterized LMI methods.

Finally, the relation between the computed values of μ and γ through Algorithm 2.1 and the number of iterations for different initial points ($r = 0.5$, 1, and 1.5) can be observed by Figure 2.3 and Figure 2.4.

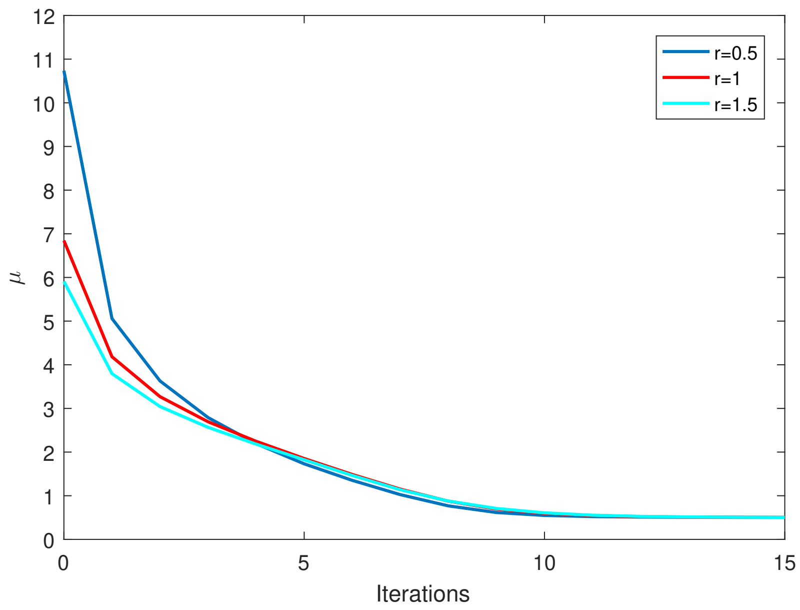

Fig. 2.3 Bound on the H_2-norm against the number of iterations computed using Algorithm 2.1 for Example 2.

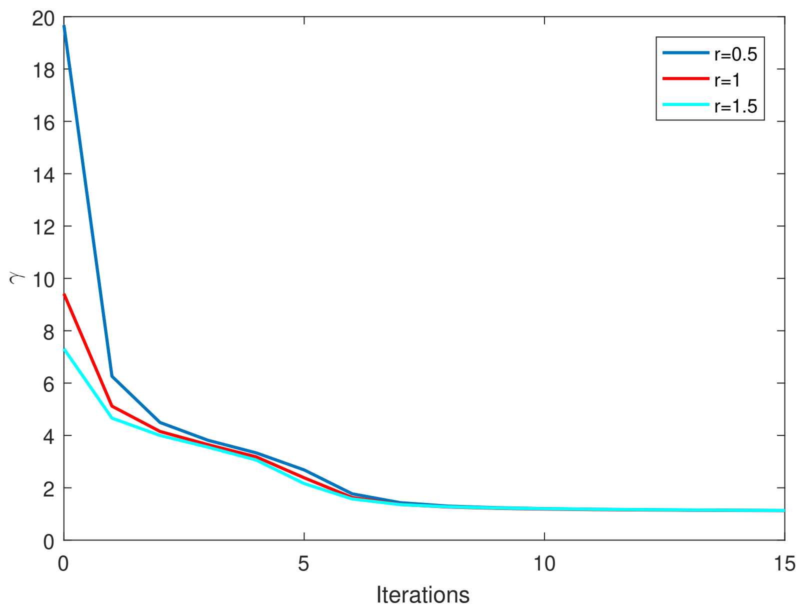

Fig. 2.4 Bound on the H_∞-norm against the number of iterations computed using Algorithm 2.1 for Example 2.

2.5.3 Example 3

Consider the following example for $n = 4, n_u = 1$ and $N = 2$ presented in [35] and [54] The vertex system matrices of the continuous-time polytopic system are given by

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \text{-}12 - 3d & -12 - 3d & -25 & -1 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0 \\ 0 \\ \text{-}6 \\ 6 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \text{-}12 + 3d & -12 + 3d & -25 & -1 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 6 \end{bmatrix},$$

where $d \in \mathbb{R}$ is a free parameter. For each value of $d \geq 0$, the robust stability conditions available in the literature have been solved to yield a state feedback gain K to guarantee that $\mathcal{A}(K) := co \{A_i + B_i K : i \in \mathcal{N}\}$ is Hurwitz stable.

The scalar searches performed by the parameterized LMI based methods described in previous works follow from the values suggested in Example 2, so that a total of 13 searches for Sha01 [12], EH04 [25], GK06 [35], and OdOP11 [34] (Lemma 9) are conducted while a total of 12 searches are conducted for ROC18 [29, 54]. The maximum value of d assuring that the existing robust stability conditions and Algorithm 2.2 are feasible as well as the associated numerical complexity can be seen in the Table 2.5 below.

	ATB01[24]	Sha01	EH04	GK06	OdOP11	ROC18	Algorithm 2.2
d_{max}	5.44	10.37	13.30	11.62	12.60	14.29	19.42
V	40	40	40	120	80	40	80
R	24	16	24	84	52	16	68
T	0.20 s	2.57 s	2.59 s	3.59 s	3.48 s	2.45 s	22.04 s

Table 2.5 The maximum values of d within stability domain achieved with some existing methods and Algorithm 2.2, as well as the associated numerical complexity for Example 3 (V: scalar variables, R: LMI rows, T: total solution time).

The first row in Table 2.5 demonstrates that ROC18 [29, 54] provides the best performance of among the existing approaches in the literature; their robust stability conditions are always feasible for all $0 \leq d \leq 14.29$. However, when d is slightly larger than 14.29 all the above methods give infeasibility for finding a stabilizing state feedback gain. Next, applying the proposed Algorithm 2.2 to this system for $d > 14.29$, setting $it_{max} = 30$, $tol = 0.1\%$, and $\epsilon = 10^{-3}$, where ϵ is the tolerance of the bisection algorithm.[56] It can be verified that Algorithm 2.2 can stabilize the uncertain system up to $d \leq 19.42$. For $d = 19.42$, Algorithm 2.2 gives a feasible state feedback gain $K = [-13.0706\ \ -23.9313\ \ -0.0693\ \ -1.3499]$. This controller gives the maximum real part of eigenvalues of $\mathcal{A}(K)$ as -0.0536 which verifies the stability of the closed-loop system.

Regarding the associated computational burden for each LMI test, Sha01[12] and ROC18[29, 54] demand a lower number of scalar variables and LMI rows followed by ATB01[24], EH04[25], OdOP11 [34], Algorithm 2.2, and GK06[35], respectively. Moreover, the last row in Table 2.5 gives the solution time for each method. It can be observed that the solution time for ROC18[29, 54] (total of 12 LMI tests) is 2.45 s. For $d = 19.42$, it takes 22.04 s for Algorithm 2.2 to find the stabilizing gain, where a total of 80 LMI tests are conducted. Moreover, exhaustive numerical experiments show that the uncertain system becomes increasingly harder to be stabilized as d increases, this leads to the number of LMI tests and therefore the solution time by Algorithm 2.2 growing accordingly. When d is smaller, for example, Algorithm 2.2 can stabilize the uncertain system for $d = 16.5$ in 5.79 s (total of 21 LMI tests) and for $d = 18.5$ in 13.82 s (total of 50 LMI tests), respectively. In particular, when $d = 14.29$, Algorithm 2.2 solves 10 LMI tests in 2.75 s to find a stabilizing gain, the solution time is comparable to ROC18[29, 54]. It is also important to mention that even for $d = 16.5$, the current parameterized LMI based methods cannot provide feasible solutions even if an exhaustive search procedure on scalar parameters is implemented. In conclusion, Algorithm 2.2 can be used as an alternative way to find a feasible solution for robust stabilization when all other methods fail but at the price of a potentially larger computational effort.

2.5.4 Example 4

In this example we compare the performance of Algorithm 2.2 with the previous methods in robust stabilization for uncertain systems through statistical analysis. The database of open-loop unstable uncertain systems from [34] (available for download) is used in this example. We only consider uncertain systems that are guaranteed to be robustly stabilizable by a state feedback gain, but not quadratically stabilizable. In what follows, 100 systems for each combination of the dimension $n = 2, \cdots, 5$ and $N = 2, \cdots, 5$ and $n_u = 1$ are tested. We use the values of scalar parameters suggested in Example 3 for the current methods [34, 54] while Algorithm 2.2 sets $tol = 0.1\%$, $\epsilon = 10^{-3}$, and $it_{max} = 100$.

(n, N)	ATB01	Sha01	EH04	GK06	OdOP11	ROC18	Algorithm 2.2
$(2, 2)$	64	9	100	4	13	100	68
$(2, 3)$	33	18	58	11	13	57	75
$(2, 4)$	2	16	50	6	8	52	79
$(2, 5)$	2	24	60	13	19	63	89
$(3, 2)$	44	11	82	6	11	81	75
$(3, 3)$	2	11	49	8	10	52	83
$(3, 4)$	0	19	38	10	13	39	80
$(3, 5)$	0	21	33	11	15	37	94
$(4, 2)$	32	17	75	9	12	74	81
$(4, 3)$	1	13	49	9	12	54	87
$(4, 4)$	0	21	41	5	5	42	84
$(4, 5)$	0	13	28	5	6	32	93
$(5, 2)$	18	17	77	13	15	78	78
$(5, 3)$	0	24	54	12	14	61	87
$(5, 4)$	0	17	39	10	11	45	96
$(5, 5)$	0	17	26	5	8	30	93
success	12.38%	16.75%	53.69%	8.56%	11.56%	56.06%	83.88%
time	0.21 s	2.71 s	2.73 s	7.04 s	6.68 s	2.55 s	31.11 s

Table 2.6 Number of uncertain systems (among 100) stabilized by some existing methods and Algorithm 2.2, as well as the associated solution time per system demanded for Example 4.

As can be observed from Table 2.6, for the overall success rate of systems stabilized by all methods for all 1600 systems, ROC18 [29, 54] provides the highest success rate of 56.06% among the methods from the previous works, but Algorithm 2.2 shows a clear improvement over ROC18 [29, 54] with 83.88% success rate. Additionally, compared with the robust stabilization conditions of [30], which use polynomial Lyapunov matrices, and which give 81.3% success rate, Algorithm 2.2 still has a higher success rate even though we only use affine Lyapunov matrices. Moreover, the last row in Table 2.6 gives the mean solution time per system spent by each method. Compared with ROC18 [29, 54] which demands 2.55 s per system, the mean solution time per system taken by Algorithm 2.2 is 31.11 s per system. Note finally that Algorithm 2.2 can stabilize 87.44% of the systems in 60.21 s per system if $tol = 0.01\%$, $\epsilon = 10^{-4}$, and $it_{max} = 200$. The statistical results of this example corroborate our expectation that Algorithm 2.2 can provide better robust stabilization performance than the methods from the previous works but at the expense of more computational effort.

2.6 Summary

In conclusion, this chapter has investigated the problem of robust H_2 and H_∞ state feedback control of continuous-time polytopic systems. We proposed an iterative procedure (Algorithm 2.1) to compute a sequence of non-increasing upper bounds for the H_2-norm and H_∞-norm by utilizing the initial solution obtained by the quadratic method or the proposed Theorem 2.1 as a starting point. Example 1 and Example 2 from the literature were presented to show that the proposed Algorithm 2.1 can provide much less conservative upper bounds on H_2-norm and H_∞-norm than the ones obtained by the methods from previous works. Moreover, we also presented Algorithm 2.2 as an alternative way to find a stabilizing gain when the initial computation fails. Example 3 and Example 4 from the literature were provided to illustrate that Algorithm 2.2 can find a stabilizing gain for some uncertain systems when all other methods are infeasible at the expense of an increased computational effort.

Chapter 3

H_2 and H_∞ state feedback control of discrete-time polytopic systems

This chapter is organized as follows. We first provide parameter-dependent BMI conditions for H_2-norm and H_∞-norm state feedback control synthesis problems in Section 3.1. In Section 3.2, we extend the approach of [18] to derive a finite set of BMI sufficient conditions. We then propose new sufficient LMI conditions for the BMI conditions by introducing slack variables. We also propose a method by imposing a particular choice on the slack variables to compute the initial solution, which includes the quadratic conditions as a special case. By considering another special choice for the slack variables, it is proved that the proposed LMI conditions contain one known solution to the BMI conditions as a particular case. Therefore, in Section 3.3, we propose an iterative procedure to iteratively reduce the upper bounds on the H_2-norm and H_∞-norm once a feasible initial solution is found. If the proposed initial method cannot find a feasible solution, we modify our method in Section 3.4 to provide the possibility of finding a feasible initial solution. Improvement on the update computation for computing less conservative upper bounds is discussed in Section 3.5. We give numerical examples in Section 3.6 to compare our proposed results with existing approaches and summarize this chapter in Section 3.7.

The results presented in this chapter are based on our paper [60] and the associated contributions are highlighted as below:

- Propose a new initial computation method for H_2-norm and H_∞-norm design problems. The proposed conditions include the quadratic conditions as a particular case.

- Propose a new iterative procedure to iteratively reduce the upper bounds for both the H_2 and H_∞-norms.
- Propose a new iterative procedure to find a feasible initial solution when the initial computation method fails.
- Propose less conservative conditions for update computation by using the general separation result proposed in Chapter 2.

3.1 Problem description

Consider the discrete-time linear time-invariant system with polytopic uncertainties

$$\begin{bmatrix} x(k+1) \\ z(k) \end{bmatrix} = \begin{bmatrix} A(\alpha) & B(\alpha) & B_w(\alpha) \\ C(\alpha) & D(\alpha) & D_w(\alpha) \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \\ w(k) \end{bmatrix}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}^{n_u}$ is the control signal, $w(k) \in \mathbb{R}^{n_w}$ is the arbitrary noisy input and $z(k) \in \mathbb{R}^{n_z}$ is the cost signal. The uncertain system matrices $A(\alpha) \in \mathbb{R}^{n\times n}$, $B(\alpha) \in \mathbb{R}^{n\times n_u}$, $B_w(\alpha) \in \mathbb{R}^{n\times n_w}$, $C(\alpha) \in \mathbb{R}^{n_z\times n}$, $D(\alpha) \in \mathbb{R}^{n_z\times n_u}$, $D_w(\alpha) \in \mathbb{R}^{n_z\times n_w}$ belong to the polytopic domain

$$\begin{bmatrix} A(\alpha) & B(\alpha) & B_w(\alpha) \\ C(\alpha) & D(\alpha) & D_w(\alpha) \end{bmatrix} = \sum_{i=1}^{N} \alpha_i \begin{bmatrix} A_i & B_i & B_{wi} \\ C_i & D_i & D_{wi} \end{bmatrix},$$

where the uncertain parameter α lies in the unit simplex given by

$$\Omega = \left\{ \alpha \in \mathbb{R}^N : \alpha_i \geq 0, \forall i \in \mathcal{N}; \sum_{i=1}^{N} \alpha_i = 1 \right\},$$

with $\mathcal{N} = \{1, \cdots, N\}$. A linear parameter-independent constant state feedback control law $u(k) = Kx(k)$, where $K \in \mathbb{R}^{n_u\times n}$ is considered. Then the closed-loop system G is given by

$$\begin{bmatrix} x(k+1) \\ z(k) \end{bmatrix} = \begin{bmatrix} A_{cl}(\alpha) & B_w(\alpha) \\ C_{cl}(\alpha) & D_w(\alpha) \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}, \tag{3.1}$$

where $A_{cl}(\alpha) = A(\alpha) + B(\alpha)K$ and $C_{cl}(\alpha) = C(\alpha) + D(\alpha)K$.

Suppose G is Schur stable, i.e., the eigenvalues of $A_{cl}(\alpha)$ for all $\alpha \in \Omega$ lie in the open unit circle centered around the origin of the complex plane. Then, the H_2-norm and H_∞-norm of the system G are defined by

$$\|G\|_2 := \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} Trace\left(G(e^{jw}) * G(e^{jw})\right) dw},$$

$$\|G\|_\infty := \sup_{w(k)\in[0\ 2\pi]} \|G(e^{jw})\| = \sup_{w(k)\in L_2, \|w(k)\|_2\neq 0} \frac{\|z(k)\|_2}{\|w(k)\|_2}.$$

The problem investigated in this Chapter is to design a gain matrix K such that the closed-loop system in (3.1) is Schur stable and also guarantees that its H_2 or H_∞-norm is less than an upper bound, i.e., $\|G\|_2 < \mu$ or $\|G\|_\infty < \gamma$. The following is a simple extension of standard results of H_2 and H_∞-norms available in the literature [11] to the closed-loop system in (3.1), which follows from discrete controllability gramian and the bounded real lemma by using $V(x) = x^T P(\alpha)^{-1} x$ as the Lyapunov function.

Lemma 3.1. *Consider the closed-loop system in (3.1).*

1. *(H_2-norm) System (3.1) (assuming that $D_w(\alpha) = 0$) is Schur stable and its H_2-norm is less than μ if there exist $P(\alpha) \in \mathbb{S}^n_+$ and $W(\alpha) \in \mathbb{S}^{n_z}$, such that for all $\alpha \in \Omega$, $Trace\left(W(\alpha)\right) < \mu^2$,*

$$\begin{bmatrix} -W(\alpha) & C_{cl}(\alpha)P(\alpha) \\ \star & -P(\alpha) \end{bmatrix} \prec 0 \tag{3.2}$$

and

$$\begin{bmatrix} -P(\alpha) & B_w(\alpha) & A_{cl}(\alpha)P(\alpha) \\ \star & -I_{n_w} & 0 \\ \star & \star & -P(\alpha) \end{bmatrix} \prec 0. \tag{3.3}$$

2. *(H_∞-norm) System (3.1) is Schur stable and its H_∞-norm is less than γ if there exist $P(\alpha) \in \mathbb{S}^n_+$, such that for all $\alpha \in \Omega$,*

$$\begin{bmatrix} -P(\alpha) & B_w(\alpha) & 0 & A_{cl}(\alpha)P(\alpha) \\ \star & -\gamma^2 I_{n_w} & D_w(\alpha)^T & 0 \\ \star & \star & -I_{n_z} & C_{cl}(\alpha)P(\alpha) \\ \star & \star & \star & -P(\alpha) \end{bmatrix} \prec 0. \tag{3.4}$$

Note that the matrix inequalities (3.2)-(3.4) are nonlinear because of the product terms $A_{cl}(\alpha)P(\alpha)$ and $C_{cl}(\alpha)P(\alpha)$. This causes the optimization of minimizing μ (or γ) subject to the H_2-norm (or H_∞-norm) conditions in Lemma 3.1 to be non-convex. In [61] and [62], a simple procedure based on parameter-independent Lyapunov function for linearizing these inequalities is to fix $P(\alpha) = P$, and subsequently to consider P and M ($M = KP$) as the decision variables. However, this often leads to excessive conservativenes, which forms our motivation to propose new results to better cope with the problem.

3.2 Linearization and initial computation

Based on the work of [18] that provides robust stability analysis conditions for discrete-time polytopic systems, we extend their idea to robust H_2 and H_∞ state feedback control design and derive sufficient conditions in the form of BMIs

Theorem 3.1. *Define*

$$A_i^K := A_i + B_i K, \qquad C_i^K := C_i + D_i K.$$

Let $V_{ij} \in \mathbb{S}^m$ *satisfying*

$$V_{ij} + V_{ji} \succeq 0,\ 1 \le i < j \le N,\ \sum_{i=1}^{N} (V_{ij} + V_{ji}) \preceq 0,\ j = 1, \dots, N. \tag{3.5}$$

1. *(H_2-norm) System (3.1) is Schur stable and its H_2-norm is less than μ if, for all $i, j \in \mathcal{N}$, there exist $P_j \in \mathbb{S}_+^n$, $W_i \in \mathbb{S}^{n_z}$, $\hat{V}_{1ij} \in \mathbb{S}^{(n+n_z)}$ satisfying* (3.5)*, and $\hat{V}_{2ij} \in \mathbb{S}^{(2n+n_w)}$ satisfying* (3.5)*, such that $\mathrm{Trace}\,(W_i) < \mu^2$,*

$$\begin{bmatrix} -W_i & C_i^K P_j \\ \star & -P_j \end{bmatrix} - \hat{V}_{1ij} \prec 0, \tag{3.6}$$

$$\begin{bmatrix} -P_j & B_{wi} & A_i^K P_j \\ \star & -I_{n_w} & 0 \\ \star & \star & -P_j \end{bmatrix} - \hat{V}_{2ij} \prec 0. \tag{3.7}$$

2. *(H_∞-norm) System (3.1) is Schur stable and its H_∞-norm is less than γ if, for all $i,j \in \mathcal{N}$, there exist $P_j \in \mathbb{S}^n_+$ and $\hat{V}_{ij} \in \mathbb{S}^{(2n+n_z+n_w)}$ satisfying* (3.5)*, such that*

$$\begin{bmatrix} \text{-}P_j & B_{wi} & 0 & A_i^K P_j \\ \star & \text{-}\gamma^2 I_{n_w} & D_{wi}^T & 0 \\ \star & \star & \text{-}I_{n_z} & C_i^K P_j \\ \star & \star & \star & \text{-}P_j \end{bmatrix} \text{-}\ \hat{V}_{ij} \prec 0. \tag{3.8}$$

Proof. First, when inequality (3.5) is satisfied, it yields that $\sum_{i,j=1}^{N} \alpha_i \alpha_j V_{ij} \preceq 0, \forall \alpha \in \Omega$ (see Lemma 1.3). Then multiplying $\mathit{Trace}\,(W_i) < \mu^2$ and the inequalities in (3.6)-(3.7) by $\alpha_i\alpha_j$, for all $i,j \in \mathcal{N}$ and taking their sum, we get $\mathit{Trace}\,\big(W(\alpha)\big) < \mu^2$,

$$\begin{bmatrix} \text{-}W(\alpha) & C_{cl}(\alpha)P(\alpha) \\ \star & \text{-}P(\alpha) \end{bmatrix} \prec \sum_{i,j=1}^{N} \alpha_i\alpha_j \hat{V}_{1ij} \preceq 0,$$

$$\begin{bmatrix} \text{-}P(\alpha) & B_w(\alpha) & A_{cl}(\alpha)P(\alpha) \\ \star & \text{-}I_{n_w} & 0 \\ \star & \star & \text{-}P(\alpha) \end{bmatrix} \prec \sum_{i,j=1}^{N} \alpha_i\alpha_j \hat{V}_{2ij} \preceq 0,$$

respectively. Then the H_2-norm conditions of Lemma 3.1 are immediately satisfied.
The H_∞-norm condition follows from following a similar procedure on (3.8) which gives

$$\begin{bmatrix} \text{-}P(\alpha) & B_w(\alpha) & 0 & A_{cl}(\alpha)P(\alpha) \\ \star & \text{-}\gamma^2 I_{n_w} & D_w(\alpha)^T & 0 \\ \star & \star & \text{-}I_{n_z} & C_{cl}(\alpha)P(\alpha) \\ \star & \star & \star & \text{-}P(\alpha) \end{bmatrix} \prec \sum_{i,j=1}^{N} \alpha_i\alpha_j \hat{V}_{ij} \preceq 0,$$

then the H_∞-norm condition of Lemma 3.1 is immediately satisfied. □

Note that the bilinearity of the conditions of Theorem 3.1 comes from the multiplicative terms KP_j. We next propose sufficient conditions for conditions of Theorem 3.1 based on the Elimination Lemma and slack variables, in the form of LMIs. Furthermore, by imposing a special structure for the slack variables, it will be shown that, through the computation of an initial solution, the proposed conditions can contain quadratic stability based conditions as special cases.

3.2.1 Initial computation for robust H_2 control

Theorem 3.2. *Given pre-defined nonsingular matrices* $\tilde{Y}_j \in \mathbb{R}^{n\times n}$ *for all* $j \in \mathcal{N}$*. Suppose there exist* $Y \in \mathbb{R}^{n\times n}$, $M \in \mathbb{R}^{n_u\times n}$*, and, for all* $i, j \in \mathcal{N}$*, there exist* $P_j \in \mathbb{S}^n_+$, $W_i \in \mathbb{S}^{n_z}$, $V_{1ij} \in \mathbb{S}^{n_z}$ *satisfying* (3.5)*, and* $V_{2ij} \in \mathbb{S}^{(n+n_w)}$ *satisfying* (3.5)*, such that* $\mathrm{Trace}\,(W_i) < \mu^2$,

$$\begin{bmatrix} -W_i - V_{1ij} & C_i^{M,Y}\tilde{Y}_j \\ \star & -\mathcal{H}\left(Y\tilde{Y}_j\right) + P_j \end{bmatrix} \prec 0, \tag{3.9}$$

$$\left[\begin{array}{c|c} \begin{bmatrix} -P_j & B_{wi} \\ B_{wi}^T & -I_{n_w} \end{bmatrix} - V_{2ij} & \begin{bmatrix} A_i^{M,Y}\tilde{Y}_j \\ 0 \end{bmatrix} \\ \hline \star & -\mathcal{H}\left(Y\tilde{Y}_j\right) + P_j \end{array}\right] \prec 0, \tag{3.10}$$

where $A_i^{M,Y} = A_iY + B_iM$ *and* $C_i^{M,Y} = C_iY + D_iM$*. Then the* H_2*-norm conditions of Theorem 3.1 are satisfied with* $K := MY^{-1}$ *and therefore system (3.1) is Schur stable and its* H_2*-norm is less than* μ*.*

Proof. Inequality (3.9) can be rewritten as (1.1) with

$$\left[\begin{array}{c|c} Q & R \\ \hline S^T & H \end{array}\right] := \left[\begin{array}{cc|c} -W_i - V_{1ij} & 0 & C_i^K \\ 0 & P_j & -I_n \\ \hline 0 & \tilde{Y}_j & Y \end{array}\right],$$

and bases of the null spaces of R^T and S^T are given by

$$R_\perp = \begin{bmatrix} I_n \\ (C_i^K)^T \end{bmatrix} \quad \text{and} \quad S_\perp = \begin{bmatrix} I_n \\ 0 \end{bmatrix},$$

respectively. Based on the result of the Elimination Lemma, inequality (3.9) is satisfied, which implies, for all $i, j \in \mathcal{N}$,

$$R_\perp^T Q R_\perp = -W_i - V_{1ij} + C_i^K P_j C_i^{K\,T} \prec 0, \tag{3.11}$$

$$S_\perp^T Q S_\perp = -W_i - V_{1ij} \prec 0. \tag{3.12}$$

It is clear that (3.11) implies (3.6) directly by effecting a Schur complement and imposing a particular structure $\hat{V}_{1ij} = \begin{bmatrix} V_{1ij} & 0 \\ 0 & 0 \end{bmatrix}$. (3.12) is immediately satisfied when (3.9) is satisfied.

Similarly, condition (3.10) can be rewritten as (1.1) with $H = Y$,

$$Q = \left[\begin{array}{c|c} \begin{bmatrix} -P_j & B_{wi} \\ B_{wi}^T & I_{n_w} \end{bmatrix} - V_{2ij} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \hline \star & -P_j \end{array}\right], R = \begin{bmatrix} A_i^K \\ 0 \\ -I_n \end{bmatrix}, S = \begin{bmatrix} 0 \\ 0 \\ \tilde{Y}_j^{\,T} \end{bmatrix},$$

respectively. The bases of the null spaces of R and S are given by

$$R_\perp = \begin{bmatrix} I_n & 0 \\ 0 & I_{n_w} \\ (A_i^K)^T & 0 \end{bmatrix} \quad and \quad S_\perp = \begin{bmatrix} I_n & 0 \\ 0 & I_{n_w} \\ 0 & 0 \end{bmatrix},$$

respectively. From the Elimination Lemma, inequality (3.10) implies the following two conditions for all $i, j \in \mathcal{N}$,

$$R_\perp^T Q R_\perp = \begin{bmatrix} -P_j + A_i^K P_j (A_i^K)^T & B_{wi} \\ \star & -I \end{bmatrix} - V_{2ij} \prec 0, \tag{3.13}$$

$$S_\perp^T Q S_\perp = \begin{bmatrix} -P_j & B_{wi} \\ \star & -I \end{bmatrix} - V_{2ij} \prec 0. \tag{3.14}$$

Note that (3.14) is obtained from (3.10) by excluding the third row and column, and (3.13) is equivalent to (3.7) from a direct application of a Schur complement and $\hat{V}_{2ij} = \begin{bmatrix} V_{2ij} & 0 \\ 0 & 0 \end{bmatrix}$.

□

Note that $\tilde{Y}_j$ is not a matrix variable; it can be pre-defined in several ways. We next demonstrate the benefit for a particular choices of $\tilde{Y}_j = I_n$ for all $j \in \mathcal{N}$.

Corollary 3.1. *If quadratic stability based conditions for H_2-norm (see Lemma 4 of [23]) hold, then the conditions of Theorem 3.2 also hold when $\tilde{Y}_j = I_n$.*

Proof. When $P_j = P$ and $\tilde{Y}_j = I_n$ for all $j \in \mathcal{N}$, the conditions of Theorem 3.2 reduce to $Trace\,(W_i) < \mu^2$,

$$\begin{bmatrix} -W_i - V_{1ij} & C_i^{M,Y} \\ \star & -\mathcal{H}(Y) + P \end{bmatrix} \prec 0,$$

$$\left[\begin{array}{c|c} \begin{bmatrix} -P & B_{wi} \\ B_{wi}^T & -I_{n_w} \end{bmatrix} - V_{2ij} & \begin{bmatrix} A_i^{M,Y} \\ 0 \end{bmatrix} \\ \hline \star & -\mathcal{H}(Y) + P \end{array}\right] \prec 0.$$

The above conditions are equivalent to quadratic stability based conditions for H_2-norm with the choices $Y = P$, $V_{1ij} = 0$ and $V_{2ij} = 0$. □

3.2.2 Initial computation for robust H_∞ control

Theorem 3.3. *Given pre-defined nonsingular matrices $\tilde{Y}_j \in \mathbb{R}^{n\times n}$ for all $j \in \mathcal{N}$. Suppose there exist $Y \in \mathbb{R}^{n\times n}$, $M \in \mathbb{R}^{n_u\times n}$, and, for all $i, j \in \mathcal{N}$, there exist $P_j \in \mathbb{S}^n_+$ and $\hat{V}_{ij} \in \mathbb{S}^{(2n+n_z+n_w)}$ satisfying* (3.5)*, such that*

$$\left[\begin{array}{c|c} \begin{bmatrix} -P_j & B_{wi} & 0 \\ B_{wi}^T & -\gamma^2 I_{n_w} & D_{wi}^T \\ 0 & D_{wi} & -I_{n_z} \end{bmatrix} - V_{ij} & \begin{bmatrix} A_i^{M,Y}\tilde{Y}_j \\ 0 \\ C_i^{M,Y}\tilde{Y}_j \end{bmatrix} \\ \hline \star & -\mathcal{H}\left(Y\tilde{Y}_j\right) + P_j \end{array}\right] \prec 0. \tag{3.15}$$

Then the H_∞-norm conditions Theorem 3.1 are satisfied with $K := MY^{-1}$ and therefore system (3.1) is Schur stable and its H_∞-norm is less than γ.

Proof. Condition (3.15) can be reformulated as (1.1) with $H = Y$,

$$Q = \left[\begin{array}{c|c} \begin{bmatrix} -P_j & B_{wi} & 0 \\ B_{wi}^T & -\gamma^2 I_{n_w} & D_{wi}^T \\ 0 & D_{wi} & -I_{n_z} \end{bmatrix} - V_{ij} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \hline \star & -P_j \end{array}\right], R = \begin{bmatrix} A_i^K \\ 0 \\ C_i^K \\ -I_n \end{bmatrix}, S = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \tilde{Y}_j^T \end{bmatrix},$$

respectively. The bases of the null spaces of R and S are given by

$$R_\perp = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_{n_w} & 0 \\ 0 & 0 & I_{n_z} \\ (A_i^K)^T & 0 & (C_i^K)^T \end{bmatrix} \quad and \quad S_\perp = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_{n_w} & 0 \\ 0 & 0 & I_{n_z} \\ 0 & 0 & 0 \end{bmatrix},$$

respectively. By the Elimination Lemma, if inequality (3.15) holds, which implies, for all $i, j \in \mathcal{N}$

$$R_\perp^T Q R_\perp = \begin{bmatrix} \text{-}P_j + A_i^K P_j (A_i^K)^T & B_{wi} & A_i^K P_j (C_i^K)^T \\ \star & \text{-}\gamma^2 I_{n_w} & D_{wi}^T \\ \star & \star & \text{-} I_{n_z} + C_i^K P_j (C_i^K)^T \end{bmatrix} \text{-} V_{ij} \prec 0, \quad (3.16)$$

$$S_\perp^T Q S_\perp = \begin{bmatrix} \text{-}P_j & B_{wi} & 0 \\ \star & \text{-}\gamma^2 I_{n_w} & D_{wi}^T \\ \star & \star & \text{-}I_{n_z} \end{bmatrix} \text{-} V_{ij} \prec 0. \quad (3.17)$$

Note that (3.17) is included in (3.15) after excluding its fourth row and column, and (3.16) is equivalent to (3.8) by applying a Schur complement and $\hat{V}_{ij} = \begin{bmatrix} V_{ij} & 0 \\ 0 & 0 \end{bmatrix}$. □

Corollary 3.2. *If quadratic stability based conditions for H_∞-norm (see Lemma 2 of [23]) hold, then the conditions of Theorem 3.3 also hold when $\tilde{Y}_j = I_n$.*

Proof. The proof is similar to that of Corollary 3.1 with $P_j = P = Y$, $V_{ij} = 0$ and is therefore omitted. □

Remark 3.1. *The computation of K from the equality $K = MY^{\text{-}1}$ is feasible if and only if Y is nonsingular. This is satisfied since $Y\tilde{Y}_j + \tilde{Y}_j^T Y^T \succ P_j \succ 0$ (which follows from* (3.9), (3.10), (3.15)*) implies that $Y\tilde{Y}_j$ is nonsingular, which in turn shows that Y is nonsingular since $\tilde{Y}_j$ is presupposed to be nonsingular.*

3.3 Update computation algorithm

In last section, it is shown that a feasible initial solution for H_2 and H_∞ design can be obtained by Theorem 3.2 and 3.3 with $\tilde{Y}_j = I_n$, respectively. This section will show that any known solution to Theorem 3.1 is a special case of Theorem 3.2 and 3.3 by means of another particular choice of $\tilde{Y}_j$. Then, an iterative algorithm is proposed to update the initial solution.

3.3.1 Update computation for robust H_2 control

Corollary 3.3. *Suppose there exist a feasible initial solution $\tilde{P}_j$, $\tilde{W}_i$, $\tilde{K}$, $\tilde{\mu}$, $\tilde{V}_{1ij} \in \mathbb{S}^{n_z}$ satisfying* (3.5)*, and $\tilde{V}_{2ij} \in \mathbb{S}^{(n+n_w)}$ satisfying* (3.5)*, for all $i, j \in \mathcal{N}$, such that the H_2-norm* *conditions of Theorem 3.1 hold, so that* $Tr(\tilde{W}_i) < \tilde{\mu}^2$

$$\begin{bmatrix} -\tilde{W}_i - \tilde{V}_{1ij} & C_i^{\tilde{K}}\tilde{P}_j \\ \star & -\tilde{P}_j \end{bmatrix} \prec 0,$$

$$\left[\begin{array}{c|c} \begin{bmatrix} -\tilde{P}_j & B_{wi} \\ B_{wi}^T & -I_{n_w} \end{bmatrix} - \tilde{V}_{2ij} & \begin{bmatrix} A_i^{\tilde{K}}\tilde{P}_j \\ 0 \end{bmatrix} \\ \hline \star & -\tilde{P}_j \end{array}\right] \prec 0.$$

Then the conditions of Theorem 3.2 also hold when $\tilde{Y}_j = \tilde{P}_j$.

Proof. When $\tilde{Y}_j = \tilde{P}_j$, with the choices $P_j = \tilde{P}_j$, $W_i = \tilde{W}_i$, $\mu = \tilde{\mu}$, $V_{1ij} = \tilde{V}_{1ij}$, $V_{2ij} = \tilde{V}_{2ij}$, $Y = I_n$ and $M = \tilde{K}$, it is direct to see the conditions in Theorem 3.2 become the initial conditions as above. □

3.3.2 Update computation for robust H_∞ control

Corollary 3.4. *Suppose there exist a feasible initial solution* $\tilde{P}_j$, $\tilde{K}$, *and* $\tilde{\gamma}$, *and* $\tilde{V}_{ij} \in \mathbb{S}^{(n+n_z+n_w)}$ *satisfying* (3.5), *for all* $i, j \in \mathcal{N}$, *such that the* H_∞*-norm conditions of Theorem 3.1 hold, so that*

$$\left[\begin{array}{c|c} \begin{bmatrix} -\tilde{P}_j & B_{wi} & 0 \\ B_{wi}^T & -\tilde{\gamma}^2 I_{n_w} & D_{wi}^T \\ 0 & D_{wi} & -I_{n_z} \end{bmatrix} - \tilde{V}_{ij} & \begin{bmatrix} A_i^{\tilde{K}}\tilde{P}_j \\ 0 \\ C_i^{\tilde{K}}\tilde{P}_j \end{bmatrix} \\ \hline \star & -\tilde{P}_j \end{array}\right] \prec 0.$$

Then the conditions of Theorem 3.3 also hold when $\tilde{Y}_j = \tilde{P}_j$.

Proof. When $\tilde{Y}_j = \tilde{P}_j$, with the choices $P_j = \tilde{P}_j$, $\gamma = \tilde{\gamma}$, $V_{ij} = \tilde{V}_{ij}$, $Y = I_n$ and $M = \tilde{K}$, it is clear to see that the conditions of Theorem 3.3 become the initial conditions. □

3.3.3 Iterative algorithm

It is necessary to mention that the conservatism of the method of [11] and [23] comes from the fact that they use a common slack variable Y for all $i \in \mathcal{N}$. We pursue a different approach to overcome the conservatism by inserting the pre-defined matrix $\tilde{Y}_j$ into the expression for the slack variables such that $Y_j = Y\tilde{Y}_j, \forall j \in \mathcal{N}$. By doing this the slack variables are allowed to vary with the index j but at the same time allowing the optimization problem to be still tractable since $\tilde{Y}_j$ is not a variable.

The degrees of freedom provided by these extra pre-defined matrices $\tilde{Y}_j$ have been exploited. When $\tilde{Y}_j = I_n$, the initial solution obtained by Theorem 3.2 and Theorem 3.3 can reproduce quadratic stability based results for H_2 and H_∞-norm as special cases. Once an feasible initial solution is found, then solving the conditions of Theorem 3.2 and Theorem 3.3 with $\tilde{Y}_j = \tilde{P}_j$ will generally provide a less conservative solution than the initial ones; at least it is capable of reproducing the initial solution. Therefore, it is possible to search for better solutions via an iterative procedure. The next algorithm presents such an iterative procedure based on our results to compute sequences of non-increasing H_2 and H_∞ guaranteed costs.

Algorithm 3.1. *Given tolerance level tol and it_{max} (maximum number of iterations)*

1. ***Initial solution:*** *Apply Theorem 3.2 and Theorem 3.3 with $\tilde{Y}_j = I_n$ to compute an feasible initial solution for the H_2 or H_∞ problem. Set $\tilde{\mu} = \mu$ (or $\tilde{\gamma} = \gamma$), $\tilde{P}_j = P_j$ for all $j \in \mathcal{N}$, and $k = 0$.*

2. ***Update:*** *Minimize μ (or γ) over the related variables Theorem 3.2 (or Theorem 3.3) with $\tilde{Y}_j = \tilde{P}_j$. Record P_j, μ (or γ).*

3. ***Stopping condition:*** *If $(\tilde{\mu}$ - $\mu)/\tilde{\mu} \leq tol$ (or $(\tilde{\gamma}$ - $\gamma)/\tilde{\gamma} \leq tol$) or $k > it_{max}$ stop. Else set $\tilde{P}_j = P_j$, $\tilde{\mu} = \mu$ (or $\tilde{\gamma} = \gamma$), $k = k + 1$, and go to step 2.*

3.4 Robust stabilization when no known initial solution exists

As illustrated in the last section, getting a feasible initial solution to the H_2-norm and H_∞-norm conditions of Theorem 3.1 through the initial computation method proposed in Section 3.2 is an essential step to execute Algorithm 3.1. However, there exist some open-loop unstable uncertain systems that are known to be robustly stabilizable by some robust state feedback gain, but the initial computation method may fail to find a feasible solution. Hence, in this section, we propose an algorithm based on an iterative procedure to allow the possibility of finding a stabilizing gain when the initial computation fails.

We next carry out robust stabilization analysis since we only need a feasible stabilizing solution for the H_2-norm and H_∞-norm problems. First, we need the following slightly modified characterization of Schur stability for the discrete-time closed-loop systems.

Lemma 3.2. *The closed-loop system in* (3.1) *is Schur stable if, for all* $i, j \in \mathcal{N}$*, there exist* $K \in \mathbb{R}^{n_u \times n}$, $r \leq 1$, $P_j \in \mathbb{S}^n_+$*, and* $V_{ij} \in \mathbb{S}^n$ *satisfying* (3.5)*, such that*

$$\begin{bmatrix} -rP_j - V_{ij} & (A_i + B_i K)P_j \\ \star & -rP_j \end{bmatrix} \prec 0 \tag{3.18}$$

are satisfied.

Proof. First, let $V_{ij} \in \mathbb{S}^n$ satisfying (3.5), which implies $\sum_{i,j=1}^{N} \alpha_i \alpha_j V_{ij} \preceq 0, \forall \alpha \in \Omega$. Then multiplying (3.18) by $\alpha_i \alpha_j$, for all $i, j \in \mathcal{N}$, summing them, we have

$$\begin{bmatrix} -rP(\alpha) & (A(\alpha) + B(\alpha)K)P(\alpha) \\ \star & -rP(\alpha) \end{bmatrix} \prec \begin{bmatrix} \sum_{i,j=1}^{N} \alpha_i \alpha_j V_{ij} & 0 \\ 0 & 0 \end{bmatrix} \preceq 0.$$

Using a Schur complement argument, the above inequality implies

$$-P(\alpha) - (\frac{1}{r} \left(A(\alpha) + B(\alpha)K\right) P(\alpha)(\frac{1}{r} \left(A(\alpha) + B(\alpha)K\right)\right)^T \prec 0,$$

which shows that the eigenvalues of $A(\alpha)+B(\alpha)K$ for all $\alpha \in \Omega$ lie in a open disk centered at the origin with radius r. It follows that the closed-loop system in (3.1) is stabilized by K when $r \leq 1$. □

We next relax this characterization by removing the inequality constraint on r and consider the following relaxed problem:

Problem 3.1. *Find* $K \in \mathbb{R}^{n_u \times n}$*, a scalar* $r \in \mathbb{R}$*, and for all* $i, j \in \mathcal{N}$*,* $P_j \in \mathbb{S}^n_+$*,* $V_{ij} \in \mathbb{S}^n$ *satisfying* (3.5)*, such that*

$$\begin{bmatrix} -V_{ij} & (A_i + B_i K)P_j \\ \star & 0 \end{bmatrix} \prec r \begin{bmatrix} P_j & 0 \\ \star & P_j \end{bmatrix} \tag{3.19}$$

The above Problem 3.1 is non-convex since the bilinear term KP_j in (3.19). Note that if we set $P_j = P$ and $V_{ij} = 0$ for all $i, j \in \mathcal{N}$, then it follows the following relaxed quadratic stabilization problem that can be used to compute an initil solution to Problem 3.1.

Problem 3.2. *Find* $M \in \mathbb{R}^{n_u \times n}$*, a scalar* $r \in \mathbb{R}$*, and* $P \in \mathbb{S}^n_+$*,* $i \in \mathcal{N}$*, such that*

$$\begin{bmatrix} 0 & A_i P + B_i M \\ \star & 0 \end{bmatrix} \prec r \begin{bmatrix} P & 0 \\ \star & P \end{bmatrix}, \tag{3.20}$$

then with $K := MP^{-1}$*,* r*, and* P *solve Problem 3.1.*

Note that let

$$Z := \begin{bmatrix} 0 & (A_i + B_i K)P \\ \star & 0 \end{bmatrix}, \; Y = \begin{bmatrix} P & 0 \\ \star & P \end{bmatrix}$$

so that (3.20) can be written as Z - $rY \prec 0$. Since $P \succ 0$ then $Y \succ 0$, there always exists a $r \in \mathbb{R}$ such that Z - $rY \prec 0$, e.g. any r larger than the largest eigenvalue of $Y^{-1}Z$. Therefore, there always exists a solution to Problem 3.2. Furthermore, Problem 3.2 is a standard generalized eigenvalue problem (GEVP) [1] since $Y \succ 0$ and therefore it is easily solvable via gevp solver. Once a feasible initial solution to Problem 3.1 is found by solving Problem 3.2, which means that the closed-loop system has all its eigenvalues to lie in a open disk centered at the origin with radius r (in general $r > 1$), then we can update the solution to Problem 3.1 with a smaller value of r through the following result, until $r \leq 1$.

Theorem 3.4. *Let $\tilde{K} \in \mathbb{R}^{n_u \times n}$, $\tilde{r} \in \mathbb{R}$, and for all $i, j \in \mathcal{N}$, $\tilde{P}_j \in \mathbb{S}^n_+$, and $\tilde{V}_{ij} \in \mathbb{S}^n$ satisfying* (3.5)*, solve Problem 3.1 so that*

$$\begin{bmatrix} -\tilde{r}\tilde{P}_j - \tilde{V}_{ij} & (A_i + B_i\tilde{K})\tilde{P}_j \\ \star & -\tilde{r}\tilde{P}_j \end{bmatrix} \prec 0 \tag{3.21}$$

If there exist $Y \in \mathbb{R}^{n\times n}$, $M \in \mathbb{R}^{n_u \times n}$, $r \in \mathbb{R}$, and, for all $i, j \in \mathcal{N}$, $P_j \in \mathbb{S}^n_+$ and $V_{ij} \in \mathbb{S}^m$ satisfying (3.5) *, such that*

$$\begin{bmatrix} -rP_j - V_{ij} & \frac{\tilde{r}}{r}(A_i Y + B_i M)\tilde{P}_j \\ \star & -\mathcal{H}\left(\tilde{r}\, Y\tilde{P}_j\right) + rP_j \end{bmatrix} \prec 0, \tag{3.22}$$

then with $K = MY^{-1}$, r, and for all $i, j \in \mathcal{N}$, P_j and V_{ij} also solve Problem 3.1. Furthermore, condition (3.22) *is satisfied by*

$$(P_j, V_{ij}, Y, M, r) := (\tilde{P}_j, \tilde{V}_{ij}, I_n, \tilde{K}, \tilde{r}).$$

.

Proof. For the first part, effecting the congruence $diag\left(I_n, (rP_j)^{-1}Y(\tilde{r}\tilde{P}_j)\right)$ on (3.19) shows that it implies

$$\begin{bmatrix} -rP_j - V_{ij} & \frac{\tilde{r}}{r}(A_i Y + B_i M)\tilde{P}_j \\ \star & -(\tilde{r}\, Y\tilde{P}_j)^T (rP_j)^{-1}(\tilde{r}\, Y\tilde{P}_j) \end{bmatrix} \prec 0,$$

the above inequality gives (3.22) since $-(\tilde{r}Y\tilde{P}_j)^T(rP_j)^{-1}(\tilde{r}Y\tilde{P}_j) \preceq -\mathcal{H}\Big(\tilde{r}Y\tilde{P}_j\Big) + rP_j$. For the second part, it is readily to see (3.22) becomes the initial one of (3.21) when $(P_j, V_{ij}, Y, M, r) := (\tilde{P}_j, \tilde{V}_{ij}, I_n, \tilde{K}, \tilde{r})$. □

As demonstrated in Theorem 3.4, the update solution obtained by Theorem 3.4 would be no more conservative than any known solution to Problem 3.1 since (3.21) is included by (3.22) as a special case. Hence, an iterative procedure can be employed to find a feasible solution to Problem 3.1 for a non-increasing sequence of r. If $r \leq 1$, the stabilizing gain for the closed-loop system is found.

Algorithm 3.2. *Given tolerance level tol and it_{max} (maximum number of iterations).*

1. **Initial solution:** *Solve Problem 3.2 via gevp solver to find the smallest r. If $r \leq 1$, then stop, a stabilizing gain is found. Else set $\tilde{K} = K$, $\tilde{r} = r$, and $\tilde{P}_j = P$ for all $j \in \mathcal{N}$, and set $k = 0$.*

2. **Update:** *Given $\tilde{K}$, $\tilde{r}$, and $\tilde{P}_j$ for all $j \in \mathcal{N}$, solve Theorem 3.4 for $r = 1$. If feasible, record K, r, and P_j for all $j \in \mathcal{N}$ and stop, a stabilizing gain is found. If infeasible, use a bisection algorithm to find r in the interval $(1 \;\; \tilde{r}]$ for which Theorem 3.4 is feasible and record K, r, and P_j for all $j \in \mathcal{N}$.*

3. **Stopping condition:** *If $(\tilde{r} - r)/\tilde{r} \leq tol$ or $k > it_{max}$ then stop, the algorithm has failed to find a stabilizing gain. Else set $\tilde{K} = K$, $\tilde{r} = r$, $\tilde{P}_j = P_j$,$k = k + 1$, and go to Step 2.*

3.5 Improvement on update computation

Theorem 3.2 and Theorem 3.3 give sufficient LMI conditions for the H_2-norm and H_∞-norm conditions of Theorem 3.1, respectively, by separating the bilinear term KP_j. Moreover, we impose a particular structure on the matrices $\hat{V}_{1ij}$, $\hat{V}_{2ij}$ and $\hat{V}_{ij}$ in (3.9), (3.10) and (3.15) as

$$\hat{V}_{1ij} := \begin{bmatrix} V_{1ij} & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{V}_{2ij} := \begin{bmatrix} V_{2ij} & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{V}_{ij} := \begin{bmatrix} V_{ij} & 0 \\ 0 & 0 \end{bmatrix}, \tag{3.23}$$

respectively. The above equality constraint on $\hat{V}_{ij}$ can simplify the conditions of Theorem 3.1 and allow the bilinear term KP_j to be easily decoupled. However, this constraint has restricted some block submatrices of $\hat{V}_{ij}$ to be zero, which introduces conservatism in general. In this section, based on the work of Chapter 2, we propose a modified update computation without imposing the constraint (3.23) on $\hat{V}_{ij}$.

First, we notice that the parameter-dependent H_2-norm and H_∞-norm conditions of Lemma 3.1 can also be formulated as a generalized robust design problem given in Section 2.1.2.

Problem 3.3. *Let $\mathcal{F} \in \mathbb{R}^{m,n}$ and for all $i \in \mathcal{N}$, let $\mathcal{A}_i \in \mathbb{R}^{m\times n}, \mathcal{B}_i \in \mathbb{R}^{m\times n_u}$ and $\mathcal{T}_i \in \mathbb{S}^m$ be given and, for any $\alpha \in \Omega_N$ let*

$$\begin{bmatrix} \mathcal{A}(\alpha) & \mathcal{B}(\alpha) & \mathcal{T}(\alpha) \end{bmatrix} := \sum_{i=1}^{N} \alpha_i \begin{bmatrix} \mathcal{A}_i & \mathcal{B}_i & \mathcal{T}_i \end{bmatrix}.$$

Find a state feedback controller $K \in \mathbb{R}^{n_u\times n}$ and a parameter-dependent Lyapunov matrix $P(\alpha) \in \mathbb{S}^n_+$ such that for all $\alpha \in \Omega_N$,

$$\mathcal{T}(\alpha) + \mathcal{H}\Big((\mathcal{A}(\alpha) + \mathcal{B}(\alpha)K)P(\alpha)\mathcal{F}^T\Big) \prec 0, \tag{3.24}$$

where

- *For the H_2 case*
  - *For the first condition in* (3.2)*:*
    $\mathcal{T}(\alpha) = \begin{bmatrix} -W(\alpha) & 0 \\ 0 & -P(\alpha) \end{bmatrix}$, $\mathcal{A}(\alpha) = \begin{bmatrix} C(\alpha) \\ 0 \end{bmatrix}$, $\mathcal{B}(\alpha) = \begin{bmatrix} D(\alpha) \\ 0 \end{bmatrix}$, $\mathcal{F} = \begin{bmatrix} 0 \\ I_n \end{bmatrix}$.
  - *For the second condition in* (3.3)*:* $\mathcal{T}(\alpha) = \begin{bmatrix} -P(\alpha) & B_w(\alpha) & 0 \\ B_w(\alpha)^T & -I_{n_w} & 0 \\ 0 & 0 & -P(\alpha) \end{bmatrix}$,

$$\mathcal{A}(\alpha) = \begin{bmatrix} A(\alpha) \\ 0 \\ 0 \end{bmatrix}, \ \mathcal{B}(\alpha) = \begin{bmatrix} B(\alpha) \\ 0 \\ 0 \end{bmatrix}, \ \mathcal{F} = \begin{bmatrix} 0 \\ 0 \\ I_n \end{bmatrix}.$$

- *For the H_∞ case in* (3.4)*:* $\mathcal{T}(\alpha) = \begin{bmatrix} -P(\alpha) & B_w(\alpha) & 0 & 0 \\ B_w(\alpha)^T & -\gamma^2 I_{n_w} & D_w(\alpha)^T & 0 \\ 0 & D_w(\alpha) & -I_{n_z} & 0 \\ 0 & 0 & 0 & -P(\alpha) \end{bmatrix}$,

$$\mathcal{A}(\alpha) = \begin{bmatrix} A(\alpha) \\ 0 \\ C(\alpha) \\ 0 \end{bmatrix}, \ \mathcal{B}(\alpha) = \begin{bmatrix} B(\alpha) \\ 0 \\ D(\alpha) \\ 0 \end{bmatrix}, \ \mathcal{F} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ I_n \end{bmatrix}.$$

Next problem gives a finite set of sufficient conditions for the solution of Problem 3.3.

Problem 3.4. *Let all variables be as given in Problem 3.3. Find* $K \in \mathbb{R}^{n_u \times n}$ *and for all* $i, j \in \mathcal{N}$*, find* $P_j \in \mathbb{S}^n_+$ *and* $\hat{V}_{ij} \in \mathbb{S}^m$ *satisfying* (3.5)*, such that*

$$\mathcal{T}_i + \mathcal{H}\Big(\mathcal{A}_i P_j \mathcal{F}^T + \mathcal{B}_i K P_j \mathcal{F}^T\Big) \prec \hat{V}_{ij}. \tag{3.25}$$

As discussed in section 2.3.2, Corollary 2.1 gives sufficient LMI conditions for the solution of Problem 3.4, and it includes any known solution to Problem 3.4 as a particular case. Notice that Corollary 2.1 does not require imposing the constraint (3.23) on $\hat{V}_{ij}$. Hence, it follows that H_2 and H_∞ state feedback control for discrete-time polytopic systems can now be formulated by the following modified algorithm.

Algorithm 3.3. *Given tolerance level* tol *and* it_{max} *(maximum number of iterations)*

1. ***Initial solution:*** *Apply Theorem 3.2 and Theorem 3.3 with* $\tilde{Y}_j = I_n$ *to compute an feasible initial solution to Problem 3.4. Set* $\tilde{\mu} = \mu$ *(or* $\tilde{\gamma} = \gamma$*),* $\tilde{K} = K$*,* $\tilde{P}_j = P_j$ *for all* $j \in \mathcal{N}$*, and* $k = 0$*.*

2. ***Update:*** *Minimize* μ *(or* γ*) over the related variables in Corollary 2.1 of Section 2.3.2. Record* K*,* P_j*,* μ *(or* γ*).*

3. ***Stopping condition:*** *If* $(\tilde{\mu} - \mu)/\tilde{\mu} \leq tol$ *(or* $(\tilde{\gamma} - \gamma)/\tilde{\gamma} \leq tol$*) or* $k > it_{max}$ *stop. Else set* $\tilde{K} = K$*,* $\tilde{P}_j = P_j$*,* $\tilde{\mu} = \mu$ *(or* $\tilde{\gamma} = \gamma$*),* $k = k + 1$*, and go to step 2.*

3.6 Numerical examples

3.6.1 Example 1

The benefit of Algorithm 3.1 for H_2 and H_∞ state feedback control design is demonstrated by the following example. We consider the satellite system presented in [13]. Its discrete-time equivalent system dynamics is obtained via first-order Euler approximation with a sampling time of $0.1s$, and are given as

$$x(k+1) = \begin{bmatrix} 1 & 0 & 0.1 & 0 \\ 0 & 1 & 0 & 0.1 \\ \text{-}0.1f_1 & 0.1f_1 & 1 - 0.1f_2 & 0.1f_2 \\ 0.1f_1 & \text{-}0.1f_1 & 0.1f_2 & 1 - 0.1f_2 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ 0.1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.1 \end{bmatrix} w(t),$$

$$z(k) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0.01 \end{bmatrix} u(k).$$

The torque constant and viscous damping are given by f_1 and f_2, which are uncertain in the ranges of $[0.09, 0.4]$ and $[0.0038, 0.04]$, respectively. The values of the H_2 and H_∞ guaranteed costs and the associated computation time obtained by Algorithm 3.1 and the methods of [11] and [23] are displayed in Table 3.1 and Table 3.2 below. Note that 199 linearly equally spaced points between -0.99 and 0.99 is considered as the search domain for the scalar ξ of [23], giving a total of 199 tests to search for the minimum values.

Method	μ	Scalars	T: solution time
[61]	1.140	-	0.30 s
[11]	0.7537	-	0.33 s
[23]	0.7464	ξ = -0.13	65.25 s
Initial	0.7808	-	0.91 s
Algorithm 3.1	0.6232	$it_{max} = 1$	1.94 s
Algorithm 3.1	0.4954	$it_{max} = 11$	12.27 s

Table 3.1 The minimum upper bound on H_2-norm achieved with some existing methods and Algorithm 3.1, as well as the associated solution time for Example 1

Method	γ	Scalars	T: solution time
[61]	3.1259	-	0.29 s
[11]	2.1922	-	0.33 s
[23]	2.1311	ξ = -0.34	64.13 s
Initial	2.2752	-	0.69 s
Algorithm 3.1	1.9749	$it_{max} = 1$	1.41 s
Algorithm 3.1	1.8582	$it_{max} = 7$	5.63 s

Table 3.2 The minimum upper bound on H_∞-norm achieved with some existing methods and Algorithm 3.1, as well as the associated solution time for Example 1

Indeed, [23] achieves reduction of conservatism with respect to [11] through scalar parameter search in the computation of guaranteed costs of H_2-norm and H_∞-norm, but at the price of much more computation time. As illustrated in the above tables, the initial computation obtained from the proposed conditions of Theorem 3.2 and Theorem 3.3 with $\tilde{Y}_j = I_n$ provides better results for both the H_2 and H_∞ performance than [61] (quadratic). It is shown in Table 3.1 and Table 3.2 that the H_2 and H_∞ performance computed by our method is better than the one in [23] even after the first iteration. After a few iterations, our method can achieve much better results, the final converged value for H_2 performance is 0.4954 with corresponding obtained feedback gain $K =$ [-20.2481 - 88.7875 - 9.4603 - 172.8005], which reveal a noticeable improvement of 26% compared to the result obtained in [23].

Moreover, Algorithm 3.1 can achieve the best value with 1.8582 for H_∞ performance when $K =$ [-66.6520 - 407.9715 - 17.8399 - 662.9663], with a relative improvement of 19% with respect to [23]. Regarding the computational burden, the results show that although the proposed Algorithm 3.1 requires a larger computational effort than [61] and [11], but the overall computation time is still acceptable and it is much shorter than the one spent by [23].

In order to demonstrate the fast convergence speed of our Algorithm 3.1, the relation between the number of iterations and H_2 and H_∞ performance is shown in Figure 3.1 and Figure 3.2 below. With tol is set to be 0.1%, we notice that the H_2 and H_∞ performance can be iteratively reduced, and converge to their lowest values within only 11 steps for the H_2 performance and 7 steps for the H_∞ performance, respectively.

As a final comparison, Algorithm 3.3 yields a converged value of 0.4938 for H_2 performance after 11 iterations and a converged value of 1.8525 for H_∞ performance after 7 iterations. The computation time spent by Algorithm 3.3 for H_2 control and H_∞ control are 16.56 s and 6.92 s, respectively. Compared to the results obtained by Algorithm 3.1, it is shown that Algorithm 3.3 provides slightly smaller H_2 and H_∞ guaranteed costs than Algorithm 3.1 but at the expense of more computational burden.

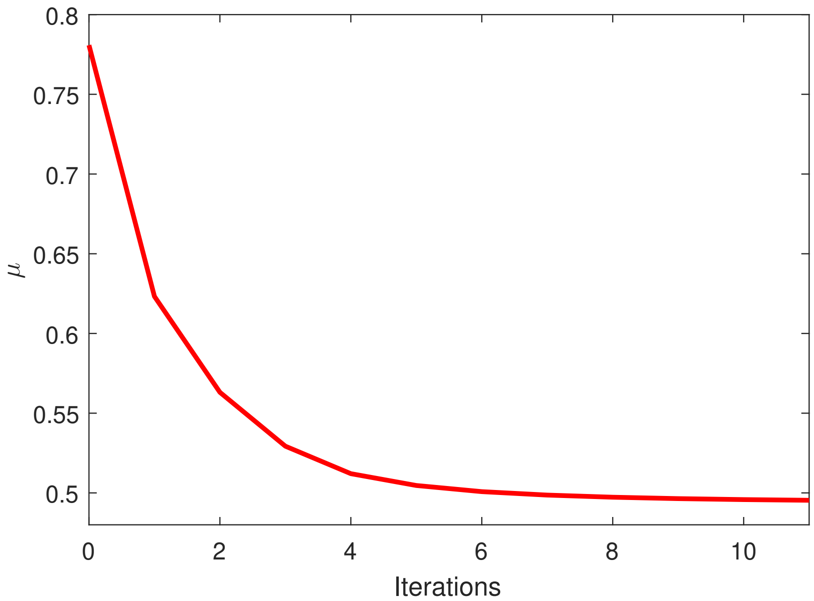

Fig. 3.1 Bound on the H_2-norm against the number of iterations computed using Algorithm 3.1 for Example 1.

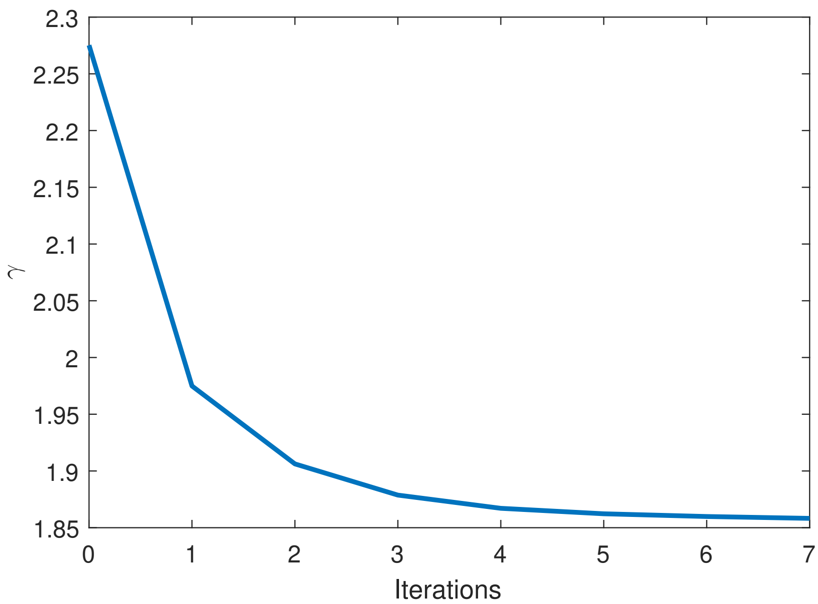

Fig. 3.2 Bound on the H_∞-norm against the number of iterations computed using Algorithm 3.1 for Example 1.

3.6.2 Example 2

Consider a discrete-time polytopic system from [29] with the following vertex system matrices:

$$A_1 = \begin{bmatrix} -0.9113 & 0.5904 \\ 1.2798 & -1.1808 \end{bmatrix},\ A_2 = \begin{bmatrix} 0.7855 & -2.4260 \\ 0.8241 & -1.2928 \end{bmatrix}, B_1 = \begin{bmatrix} -2 \\ 4 \end{bmatrix},\ B_2 = \begin{bmatrix} -5 \\ 3 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} 1.2699 & -0.7024 \\ 1.0372 & 1.2452 \end{bmatrix},\ A_4 = \begin{bmatrix} 0.3685 & -1.1333 \\ -0.5527 & 1.5642 \end{bmatrix},\ B_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ B_4 = \begin{bmatrix} 2 \\ -3 \end{bmatrix},$$
$$B_{w1} = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix},\ B_{w2} = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix},\ B_{w3} = B_{w4} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, D_{w1} = D_{w2} = D_{w3} = D_{w4} = 0,$$
$$C_1^T = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ C_2^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ C_3^T = C_4^T = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ D_1 = D_3 = 1,\ D_2 = D_4 = \text{-}1.$$

First of all, applying the proposed initial computation method gives infeasibility for computing initial H_2 and H_∞ solutions. Now applying the proposed Algorithm 3.2 gives a feasible solution to the H_2-norm and H_∞-norm conditions of Theorem 3.1, the initial stabilizing gain and initial Lyapunov matrices are given by $\tilde{K} = [\text{-}0.3568\ 0.4838]$ and

$$\tilde{P}_1 = \begin{bmatrix} 1.8860 & -0.1312 \\ -0.1312 & 2.8370 \end{bmatrix},\ \tilde{P}_2 = \begin{bmatrix} 1.0468 & 0.1958 \\ 0.1958 & 2.5472 \end{bmatrix},$$
$$\tilde{P}_3 = \begin{bmatrix} 1.0228 & 0.3549 \\ 0.3549 & 6.5219 \end{bmatrix},\quad \tilde{P}_4 = \begin{bmatrix} 1.4767 & -0.1305 \\ -0.1305 & 4.0468 \end{bmatrix}.$$

Using the given $\tilde{K}$ and $\tilde{P}_j$, we next applying both the proposed iterative procedures-as given by Algorithm 3.1 and Algorithm 3.3, giving the minimum values of the H_2 and H_∞ guaranteed costs shown in Table 3.3.

Method	μ	it_{max}	γ	it_{max}
Algorithm 3.1	0.5124	10	3.9977	25
Algorithm 3.3	0.2706	10	0.7705	8

Table 3.3 The minimum upper bounds on H_2-norm and H_∞-norm computed using Algorithm 3.1 and Algorithm 3.3 for Example 2.

We note that although both Algorithm 3.1 and Algorithm 3.3 give final converged values for H_2 and H_∞ guaranteed costs after a finite number of iterations, the minimum values of μ and γ obtained by Algorithm 3.3 are much superior to those provided by Algorithm 3.1. This example shows that the advantages of Algorithm 3.3 over Algorithm 3.1, by removing the equality constraint (3.23) on $\hat{V}_{ij}$ as discussed in section 3.5.

For comparison, Figure 3.3 and Figure 3.4 show that the relation between the computed results through Algorithm 3.1 and Algorithm 3.3 for H_2 and H_∞ cases, respectively.

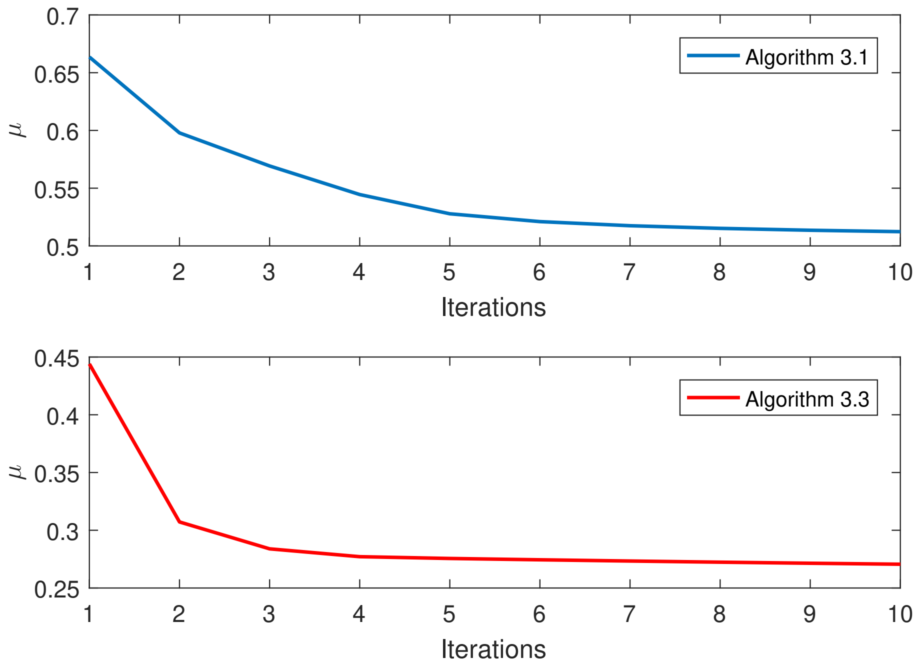

Fig. 3.3 Bound on the H_2-norm against the number of iterations computed using Algorithm 3.1 and Algorithm 3.3 for Example 2.

3.7 Summary

In conclusion, this chapter has investigated the problem of robust H_2 and H_∞ state feedback control of discrete-time polytopic systems. We proposed an iterative procedure (Algorithm 3.1) to compute a sequence of non-increasing upper bounds for the H_2-norm and H_∞-norm by utilizing the initial solution obtained by the proposed initial computation method as a starting point. Based on the general separation results of Chapter 2, we also proposed an improved algorithm (Algorithm 3.3) that can reduce the conservatism for the update computation for Algorithm 3.1. Example 1 from the literature showed that the H_2 and H_∞ performances derived by Algorithm 3.1 and Algorithm 3.3 were nearly the same, and both were much superior to the results computed by other existing methods. Moreover, we also presented Algorithm 3.2 as an alternative approach for finding a stabilizing gain when

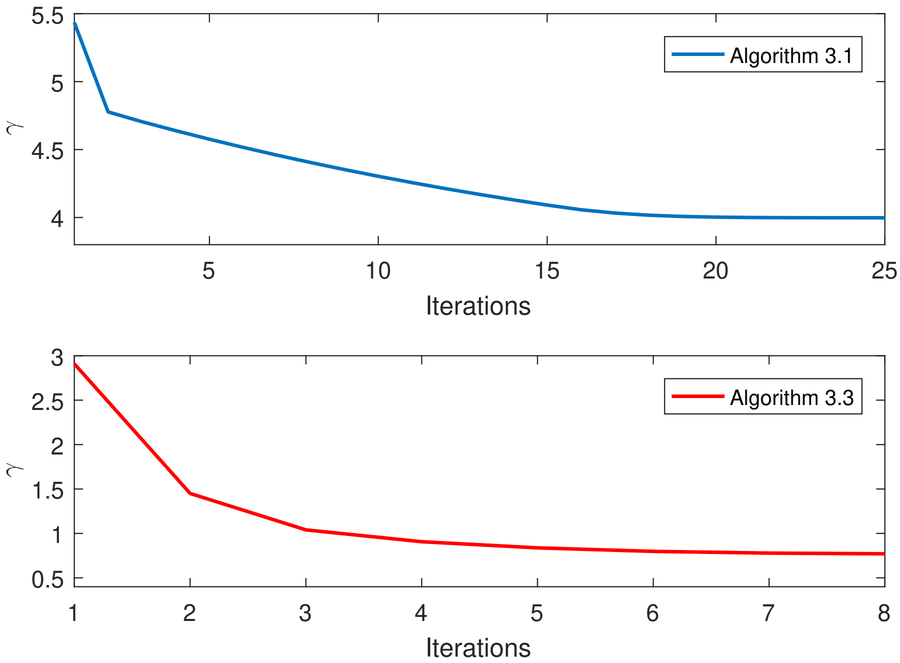

Fig. 3.4 Bound on the H_∞-norm against the number of iterations computed using Algorithm 3.1 and and Algorithm 3.3 for Example 2.

the initial computation fails. Example 2 was provided to illustrate that Algorithm 3.2 can find a stabilizing gain when the initial computation is infeasible. Using the feasible initial solution provided by Algorithm 3.2, then Algorithm 3.3 provided much smaller final converged values of the H_2 and H_∞-norms than Algorithm 3.3. We conclude that it is advisable to implement both Algorithm 3.1 and Algorithm 3.3 for the computation of H_2 and H_∞ guaranteed costs and select the best results among them, especially when the proposed initial computation method is infeasible, although Algorithm 3.3 requires a larger computational effort.

Chapter 4

H_2 and H_∞ state feedback control by means of parameter-dependent slack variables

New iterative procedures for H_2 and H_∞ state feedback control of continuous-time and discrete-time polytopic systems have been proposed in Chapter 2 and Chapter 3, respectively. We have shown that the proposed iterative procedures can compute less conservative upper bounds for H_2 and H_∞-norms than the existing methods through numerical examples. However, we cannot verify that our proposed conditions can give a less conservative design than the existing methods by rigorous theoretical proof. In this chapter, we will propose new design conditions that can include the conditions of the existing methods as special cases.

This chapter is organized as follows. In Section 4.1, we first review the current results and propose BMI conditions for designing H_2 and H_∞ controller with the introduction of the affine parameter-dependent slack variables. Using a given controller gain provided by the existing methods, we propose a method to compute the initial solution, which includes the current methods as special cases. In Section 4.2, we present a novel separation result to give sufficient LMI conditions for the BMI conditions. By theoretical proof, it is proved that any known solution to the BMI conditions can be included by the LMI conditions as a particular case. Therefore, an algorithm is presented to iteratively update the solutions using the initial solution obtained from Section 4.1 as a starting point. In Section 4.3, we show that the solutions to the proposed BMI conditions for H_2 and H_∞ synthesis problems in Chapter 2 and Chapter 3 can also be iteratively updated through the separation result presented in Section 4.2. We illustrate the effectiveness of the proposed algorithms through two examples from the literature in Section 4.4 and summarize this chapter in Section 4.5.

The results presented in this chapter are based on our prepared paper [63] and the associated contributions are highlighted as below:

- Propose an initial computation method that can compute a solution, in one step, which is proved to be no more conservative than the optimal solution obtained by the approaches available in the literature.
- Propose a new separation result that is more general than the separation result proposed in Chapter 2. It is shown that this new separation result can iteratively solve the BMI design conditions presented in Chapter 2 and Chapter 3.

4.1 Summary of current results and initial computation

In this section, we give a summary of the existing methods based on parameter-dependent Lyapunov matrices for designing H_2 and H_∞ state feedback controller for both continuous-time and discrete-time systems and highlight the associated conservatism. We also derive new unified BMI based conditions for this design problem. It is proved that the conservatism of conditions from previous studies has been avoided in our results and the proposed BMI conditions contain the existing methods as special cases.

4.1.1 Current results based on the Elimination Lemma

The quadratic stability based method is generally very conservative since it uses a parameter-independent Lyapunov function in the entire uncertainty polytope. Therefore, many recent works have widely introduced slack variables to separate the system matrices from the Lyapunov matrix through an application of the Elimination Lemma, thus allowing the Lyapunov matrix to be parameter-dependent and rendering a less conservative design. We next reproduce the conditions of the existing methods in a unified form involving a parameter-independent slack variable.

Lemma 4.1. *The continuous-time closed-loop system in* (2.1) *with* $D_w(\alpha) = 0$ *is Hurwitz stable and its* H_2*-norm is less than* μ

1. *[24] if there exist* $P(\alpha) \in \mathbb{S}^n_+$, $W(\alpha) \in \mathbb{S}^{n_w}$, *and* $G \in \mathbb{R}^{n\times n}$, *such that*

$$\begin{bmatrix} -W(\alpha) & B_w(\alpha)^T \\ B_w(\alpha) & -P(\alpha) \end{bmatrix} \prec 0, \;\; Trace\left(W(\alpha)\right) < \mu^2, \tag{4.1}$$

$$\underbrace{\begin{bmatrix} 0 & P(\alpha) & 0 & 0 \\ P(\alpha) & -P(\alpha) & 0 & 0 \\ 0 & 0 & I_{n_z} & 0 \\ 0 & 0 & 0 & -P(\alpha) \end{bmatrix}}_{\mathcal{T}(\alpha)} + \mathcal{H}\left(\underbrace{\left(\overbrace{\begin{bmatrix} -I_n \\ A(\alpha) \\ C(\alpha) \\ I_n \end{bmatrix}}^{\mathcal{A}(\alpha)} + \overbrace{\begin{bmatrix} 0 \\ B(\alpha) \\ D(\alpha) \\ 0 \end{bmatrix}}^{\mathcal{B}(\alpha)} K \right)}_{\mathcal{A}_{cl}(\alpha)} G \underbrace{\begin{bmatrix} I_n & 0 & 0 & 0 \end{bmatrix}}_{\mathcal{F}^T} \right) \prec 0 \tag{4.2}$$

are satisfied for all $\alpha \in \Omega$.

2. *[25] if there exist* $P(\alpha) \in \mathbb{S}^n_+$, $W(\alpha) \in \mathbb{S}^{n_w}$, $G \in \mathbb{R}^{n\times n}$, *and an arbitrarily prescribed scalar* $r > 0$, *such that* (4.1) *and*

$$\underbrace{\begin{bmatrix} 0 & -P(\alpha) & 0 \\ -P(\alpha) & 0 & 0 \\ 0 & 0 & -I_{n_z} \end{bmatrix}}_{\mathcal{T}(\alpha)} + \mathcal{H}\left(\underbrace{\left(\overbrace{\begin{bmatrix} A(\alpha) \\ I_n \\ C(\alpha) \end{bmatrix}}^{\mathcal{A}(\alpha)} + \overbrace{\begin{bmatrix} B(\alpha) \\ 0 \\ D(\alpha) \end{bmatrix}}^{\mathcal{B}(\alpha)} K \right)}_{\mathcal{A}_{cl}(\alpha)} G \underbrace{\begin{bmatrix} I_n & -rI_n & 0 \end{bmatrix}}_{\mathcal{F}^T} \right) \prec 0 \tag{4.3}$$

are satisfied for all $\alpha \in \Omega$.

3. *[29] if there exist* $P(\alpha) \in \mathbb{S}_{+}^{n}$, $W(\alpha) \in \mathbb{S}^{n_w}$, $G \in \mathbb{R}^{n\times n}$, *an arbitrarily prescribed scalar* $\xi \in (\text{-}1, 1)$ *and a nonzero scalar* ϵ, *such that* (4.1) *and*

$$\underbrace{\begin{bmatrix} P(\alpha) & 0 & 0 \\ 0 & \text{-}P(\alpha) & 0 \\ 0 & 0 & \text{-}I_{n_z} \end{bmatrix}}_{\mathcal{T}(\alpha)} + \mathcal{H}\left(\underbrace{\left(\overbrace{\begin{bmatrix} \epsilon A(\alpha) \text{-} \frac{I}{2\epsilon} \\ \text{-}\epsilon A(\alpha) \text{-} \frac{I}{2\epsilon} \\ C(\alpha) \end{bmatrix}}^{\mathcal{A}(\alpha)} + \overbrace{\begin{bmatrix} \epsilon B(\alpha) \\ \text{-}\epsilon B(\alpha) \\ D(\alpha) \end{bmatrix}}^{\mathcal{B}(\alpha)} K\right)}_{\mathcal{A}_{cl}(\alpha)} G \underbrace{\begin{bmatrix} I_n & \xi I_n & 0 \end{bmatrix}}_{\mathcal{F}^T}\right) \prec 0 \tag{4.4}$$

are satisfied for all $\alpha \in \Omega$.

Lemma 4.2. *The continuous-time closed-loop system in* (2.1) *is Hurwitz stable and its* H_∞*-norm is less than* γ

1. *[13] if there exist* $P(\alpha) \in \mathbb{S}_{+}^{n}$, $G \in \mathbb{R}^{n\times n}$, *and an arbitrarily prescribed scalar* $r > 0$, *for all* $\alpha \in \Omega$, *such that*

$$\underbrace{\begin{bmatrix} 0 & P(\alpha) & 0 & B_w(\alpha) \\ P(\alpha) & 0 & 0 & 0 \\ 0 & 0 & \text{-}I_{n_z} & D_w(\alpha) \\ B_w(\alpha)^T & 0 & D_w(\alpha)^T & \text{-}\gamma^2 I_{n_w} \end{bmatrix}}_{\mathcal{T}(\alpha)} + \mathcal{H}\left(\underbrace{\left(\overbrace{\begin{bmatrix} A(\alpha) \\ \text{-}I_n \\ C(\alpha) \\ 0 \end{bmatrix}}^{\mathcal{A}(\alpha)} + \overbrace{\begin{bmatrix} B(\alpha) \\ 0 \\ D(\alpha) \\ 0 \end{bmatrix}}^{\mathcal{B}(\alpha)} K\right)}_{\mathcal{A}_{cl}(\alpha)} G \underbrace{\begin{bmatrix} I_n & rI_n & 0 & 0 \end{bmatrix}}_{\mathcal{F}^T}\right) \prec 0 \tag{4.5}$$

2. *[29] if there exist $P(\alpha) \in \mathbb{S}_+^n$, $G \in \mathbb{R}^{n\times n}$, an arbitrarily prescribed scalar $\xi \in (\text{-}1,1)$ and a nonzero scalar ϵ, for all $\alpha \in \Omega$, such that*

$$\underbrace{\begin{bmatrix} P(\alpha) & 0 & \epsilon B_w(\alpha) & 0 \\ 0 & \text{-}P(\alpha) & \text{-}\epsilon B_w(\alpha) & 0 \\ \epsilon B_w(\alpha)^T & \text{-}\epsilon B_w(\alpha)^T & \text{-}\gamma^2 I_{n_w} & D_w(\alpha)^T \\ 0 & 0 & D_w(\alpha) & \text{-}I_{n_z} \end{bmatrix}}_{\mathcal{T}(\alpha)} +$$

$$\mathcal{H}\left(\underbrace{\left(\overbrace{\begin{bmatrix} \epsilon A(\alpha)\text{-}\frac{I}{2\epsilon} \\ \text{-}\epsilon A(\alpha)\text{-}\frac{I}{2\epsilon} \\ 0 \\ C(\alpha) \end{bmatrix}}^{\mathcal{A}(\alpha)} + \overbrace{\begin{bmatrix} \epsilon B(\alpha) \\ \text{-}\epsilon B(\alpha) \\ 0 \\ D(\alpha) \end{bmatrix}}^{\mathcal{B}(\alpha)} K\right)}_{\mathcal{A}_{cl}(\alpha)} G \underbrace{\begin{bmatrix} I_n & \xi I_n & 0 & 0 \end{bmatrix}}_{\mathcal{F}^T}\right) \prec 0 \tag{4.6}$$

Remark 4.1. *The H_2-norm design conditions of Lemma 4.1 follow from the observability gramian. Note that the H_2-norm synthesis problem discussed in Chapter 2 relies on the controllability gramian. These two characterizations for the H_2 performance are directly related through the concept of duality, and they generally provide different upper bounds (better or worse) for H_2-norm.*

Remark 4.2. *Although condition* (4.2) *of [24] is a true LMI (without any scalar parameter), it does not necessarily reduce to quadratic stability based conditions. To overcome this weakness, [25], [13], and [29] introduce scalar parameters to allow their proposed conditions include quadratic conditions as special cases, by means of a special choice for the slack variable such that $P(\alpha) = P = G = G^T$. However, a line search procedure over the scalar parameters has to be performed to search less conservative results, which increases the computational effort.*

Lemma 4.3. *The discrete-time closed-loop system in* (3.1) *(assuming that $D_w(\alpha) = 0$) is Schur stable and its H_2-norm is less than μ*

1. *[11] if there exist* $P(\alpha) \in \mathbb{S}_{+}^{n}$, $W(\alpha) \in \mathbb{S}^{n_z}$, *and* $G \in \mathbb{R}^{n\times n}$, *such that*

$$Trace\left(W(\alpha)\right) < \mu^2, \tag{4.7}$$

$$\underbrace{\begin{bmatrix} \text{-}W(\alpha) & 0 \\ 0 & P(\alpha) \end{bmatrix}}_{\mathcal{T}(\alpha)} + \mathcal{H}\left(\underbrace{\left(\overbrace{\begin{bmatrix} C(\alpha) \\ \text{-}I_n \end{bmatrix}}^{\mathcal{A}(\alpha)} + \overbrace{\begin{bmatrix} D(\alpha) \\ 0 \end{bmatrix}}^{\mathcal{B}(\alpha)} K \right)}_{\mathcal{A}_{cl}(\alpha)} G \underbrace{\begin{bmatrix} 0 & I_n \end{bmatrix}}_{\mathcal{F}^T} \right) \prec 0, \tag{4.8}$$

$$\underbrace{\begin{bmatrix} \text{-}P(\alpha) & 0 & B_w(\alpha) \\ 0 & P(\alpha) & 0 \\ B_w(\alpha)^T & 0 & \text{-}I_{n_w} \end{bmatrix}}_{\mathcal{T}(\alpha)} + \mathcal{H}\left(\underbrace{\left(\overbrace{\begin{bmatrix} A(\alpha) \\ \text{-}I_n \\ 0 \end{bmatrix}}^{\mathcal{A}(\alpha)} + \overbrace{\begin{bmatrix} B(\alpha) \\ 0 \\ 0 \end{bmatrix}}^{\mathcal{B}(\alpha)} K \right)}_{\mathcal{A}_{cl}(\alpha)} G \underbrace{\begin{bmatrix} 0 & I_n & 0 \end{bmatrix}}_{\mathcal{F}^T} \right) \prec 0, \tag{4.9}$$

are satisfied for all $\alpha \in \Omega$.

2. *[23] if there exist* $P(\alpha) \in \mathbb{S}_{+}^{n}$, $W(\alpha) \in \mathbb{S}^{n_z}$, $G \in \mathbb{R}^{n\times n}$, *and an arbitrarily prescribed scalar* $\xi \in (\text{-}1, 1)$, *such that* (4.7), (4.8), *and*

$$\underbrace{\begin{bmatrix} \text{-}P(\alpha) & 0 & B_w(\alpha) \\ 0 & P(\alpha) & 0 \\ B_w(\alpha)^T & 0 & \text{-}I_{n_w} \end{bmatrix}}_{\mathcal{T}(\alpha)} + \mathcal{H}\left(\underbrace{\left(\overbrace{\begin{bmatrix} A(\alpha) \\ \text{-}I_n \\ 0 \end{bmatrix}}^{\mathcal{A}(\alpha)} + \overbrace{\begin{bmatrix} B(\alpha) \\ 0 \\ 0 \end{bmatrix}}^{\mathcal{B}(\alpha)} K \right)}_{\mathcal{A}_{cl}(\alpha)} G \underbrace{\begin{bmatrix} \xi I_n & I_n & 0 \end{bmatrix}}_{\mathcal{F}^T} \right) \prec 0, \tag{4.10}$$

are satisfied for all $\alpha \in \Omega$.

Lemma 4.4. *The discrete-time closed-loop system in* (3.1) *is Schur stable and its H_∞-norm is less than γ*

1. *[11] if there exist $P(\alpha) \in \mathbb{S}_+^n$, and $G \in \mathbb{R}^{n\times n}$, such that*

$$\underbrace{\begin{bmatrix} -P(\alpha) & 0 & 0 & B_w(\alpha) \\ 0 & P(\alpha) & 0 & 0 \\ 0 & 0 & -I_{n_z} & D_w(\alpha) \\ B_w(\alpha)^T & 0 & D_w(\alpha)^T & -\gamma^2 I_{n_w} \end{bmatrix}}_{\mathcal{T}(\alpha)} + \mathcal{H}\left(\underbrace{\left(\overbrace{\begin{bmatrix} A(\alpha) \\ -I_n \\ C(\alpha) \\ 0 \end{bmatrix}}^{\mathcal{A}(\alpha)} + \overbrace{\begin{bmatrix} B(\alpha) \\ 0 \\ D(\alpha) \\ 0 \end{bmatrix}}^{\mathcal{B}(\alpha)} K \right)}_{\mathcal{A}_{cl}(\alpha)} G \underbrace{\begin{bmatrix} 0 & I_n & 0 & 0 \end{bmatrix}}_{\mathcal{F}^T} \right) \prec 0 \tag{4.11}$$

   are satisfied for all $\alpha \in \Omega$.

2. *[23] if there exist $P(\alpha) \in \mathbb{S}_+^n$, $G \in \mathbb{R}^{n\times n}$, and an arbitrarily prescribed scalar $\xi \in (-1, 1)$, such that*

$$\underbrace{\begin{bmatrix} -P(\alpha) & 0 & 0 & B_w(\alpha) \\ 0 & P(\alpha) & 0 & 0 \\ 0 & 0 & -I_{n_z} & D_w(\alpha) \\ B_w(\alpha)^T & 0 & D_w(\alpha)^T & -\gamma^2 I_{n_w} \end{bmatrix}}_{\mathcal{T}(\alpha)} + \mathcal{H}\left(\underbrace{\left(\overbrace{\begin{bmatrix} A(\alpha) \\ -I_n \\ C(\alpha) \\ 0 \end{bmatrix}}^{\mathcal{A}(\alpha)} + \overbrace{\begin{bmatrix} B(\alpha) \\ 0 \\ D(\alpha) \\ 0 \end{bmatrix}}^{\mathcal{B}(\alpha)} K \right)}_{\mathcal{A}_{cl}(\alpha)} G \underbrace{\begin{bmatrix} \xi I_n & I_n & 0 & 0 \end{bmatrix}}_{\mathcal{F}^T} \right) \prec 0 \tag{4.12}$$

   are satisfied for all $\alpha \in \Omega$.

Remark 4.3. *Although [11] does not incorporate any scalar parameter, it contains the quadratic stability conditions for discrete-time systems, by fixing $P(\alpha) = P = G = G^T$. The parameterized LMI conditions of [23] recover the conditions given in [11] when the scalar ξ is set to zero, so [23] can provide better results than [11] but at the expense of more computational burden due to scalar parameter searches.*

As shown in lemmas above, the design conditions of H_2 and H_∞ state feedback controller from the previous studies for both continuous-time and discrete-time systems can be represented as the following unified parameter-dependent LMI conditions:

$$\exists\ G \in \mathbb{R}^{n\times n} : \mathcal{T}(\alpha) + \mathcal{H}\Big((\mathcal{A}(\alpha) + \mathcal{B}(\alpha)K)G\mathcal{F}^T\Big) \prec 0, \tag{4.13}$$

where only $\mathcal{T}(\alpha) \in \mathbb{S}^m$ depends on the Lyapunov matrix $P(\alpha)$, while $\mathcal{A}(\alpha) \in \mathbb{R}^{m\times n}$ and $\mathcal{B}(\alpha) \in \mathbb{R}^{m\times n_u}$ are augmented parameter-dependent matrices consisting of system matrices, $G \in \mathbb{R}^{n\times n}$ is a additional parameter-independent slack variable, and $\mathcal{F} \in \mathbb{R}^{m\times n}$ is a constant matrix consisting of identity and zero matrices that may also depend on a scalar parameter. Note that inequality (4.13) can be rewritten as inequality (2.1) of the Elimination Lemma with $Q = \mathcal{T}(\alpha)$, $R = \mathcal{A}_{cl}(\alpha)$, $H = G$, and $S = \mathcal{F}$. The common idea of the existing methods is to apply the Elimination Lemma to inequality (4.13) that implies $\mathcal{A}_{cl}(\alpha)_\perp^T\mathcal{T}(\alpha)\mathcal{A}_{cl}(\alpha)_\perp \prec 0$, which is the the standard synthesis conditions of H_2 and H_∞ state feedback control for uncertain systems, see proof of the references for more details.
By considering affine parameter-dependency for $P(\alpha)$ and $W(\alpha)$, the augmented parameter-dependent matrices $\mathcal{T}(\alpha), \mathcal{A}(\alpha)$, and $\mathcal{B}(\alpha)$ can be rewritten as:

$$\begin{bmatrix} \mathcal{T}(\alpha) & \mathcal{A}(\alpha) & \mathcal{B}(\alpha) \end{bmatrix} = \sum_{i=1}^{N} \alpha_i \begin{bmatrix} \mathcal{T}_i & \mathcal{A}_i & \mathcal{B}_i \end{bmatrix}. \tag{4.14}$$

Then the parameter-dependent condition (4.13) can be converted into the following a finite set of sufficient LMI conditions:

$$\exists\ G \in \mathbb{R}^{n\times n} : \mathcal{T}_i + \mathcal{H}\Big((\mathcal{A}_i G + \mathcal{B}_i M)\mathcal{F}^T\Big) \prec 0, \tag{4.15}$$

where $M := KG$ and G are considered to be decision variables.

Remark 4.4. *It is important to emphasize that* (4.13) *is only sufficient for H_2 and H_∞ synthesis conditions due to the fixed structure of the introduced slack variable G, this restriction is applied to obtain linearity in* (4.13) *but is not required by the Elimination Lemma. Additionally, according to the Elimination Lemma,* (4.13) *also imply $\mathcal{F}_\perp^T\mathcal{T}(\alpha)\mathcal{F}_\perp \prec 0$, which introduces conservatism in general.*

4.1.2 Initial computation for H_2 and H_∞ performance

To overcome the conservatism of the existing methods that use a parameter-independent slack variable G in (4.13), we now propose the following generalized condition that introduces a parameter-dependent slack variable to design the robust H_2 and H_∞ state feedback controller

$$\exists\, X(\alpha)\in\mathbb{R}^{n\times p}:\mathcal{T}(\alpha)+\mathcal{H}\Big((\mathcal{A}(\alpha)+\mathcal{B}(\alpha)K)X(\alpha)S^T\Big)\prec 0. \tag{4.16}$$

Remark 4.5. *In contrast to condition* (4.13) *from previous studies, the additional slack variable $X(\alpha)$ of* (4.16) *is not restricted to be parameter-independent and square, whose column dimension is determined by the column dimension of S. $S\in\mathbb{R}^{m\times p}$ is a designed constant matrix that can be constructed in four strategies such that $S_\perp^T\mathcal{T}(\alpha)S_\perp\prec 0$ does not introduce conservatism. The simplest and least conservative design of S is to set $S=I_m$ and in this way $S_\perp^T\mathcal{T}(\alpha)S_\perp\prec 0$ vanishes since the nullspace of S does not exist, see [8] for details. In this note, we opt to design $S=I_m$ for* (4.16)*, and therefore S will be omitted in the sequel.*

The next theorem presents a finite set of BMI based sufficient conditions based on affine parameter-dependent Lyapunov matrix for (4.16).

Theorem 4.1. *For all $i\in\mathcal{N}$, let $\mathcal{T}_i\in\mathbb{S}^m$, $\mathcal{A}_i\in\mathbb{R}^{m\times n}$, and $\mathcal{B}_i\in\mathbb{R}^{m\times n_u}$ be given as in* (4.14)*. For all $i,j\in\mathcal{N}$, let $V_{ij}\in\mathbb{S}^m$ satisfying the following linear inequalities:*

$$V_{ij}+V_{ji}\succeq 0,\quad 1\le i<j\le N,\ \sum_{i=1}^{N}(V_{ij}+V_{ji})\preceq 0,\quad j=1,\cdots,N. \tag{4.17}$$

Then (4.16) is satisfied if there exist $K\in\mathbb{R}^{n_u\times n}$ and for all $i,j\in\mathcal{N}$, there exist $X_j\in\mathbb{R}^{n\times m}$ and $V_{ij}\in\mathbb{S}^m$ such that (4.17) and

$$\mathcal{T}_i+\mathcal{H}\Big(\mathcal{A}_iX_j+\mathcal{B}_iKX_j\Big)\prec V_{ij}. \tag{4.18}$$

are satisfied.

Proof. First, when (4.17) is satisfied, it yields that $\sum_{i,j=1}^{N}\alpha_i\alpha_j V_{ij}\preceq 0,\forall\alpha\in\Omega$ (see Lemma 1.3 for details). Then multiplying (4.18) by $\alpha_i\alpha_j$, for all $i,j\in\mathcal{N}$ and taking the sum, we have $\mathcal{T}(\alpha)+\mathcal{H}\big((\mathcal{A}(\alpha)+\mathcal{B}(\alpha)K)X(\alpha)\big)\prec\sum_{i,j=1}^{N}\alpha_i\alpha_j V_{ij}\preceq 0$, which implies (4.16) for $S=I_m$. □

Remark 4.6. *The nonlinearity of condition* (4.18) *only occurs in the bilinear term KX_j and inequality* (4.17) *is linear in V_{ij}, thus the conditions of Theorem 4.1 become LMIs for a* *given K. Note also that when we set $X_j = G\mathcal{F}^T$ and $V_{ij} = 0$ for all $i, j \in \mathcal{N}$,* (4.18) *reduces to* (4.15). *This property shows that the solutions to* (4.15) *are also feasible for* (4.18). *Hence, the initial computation for the upper bounds on H_2-norm and H_∞ can be characterized by solving Theorem 4.1 with a given controller gain K provided by any of the existing methods. Due to the extra degrees of freedom provided by X_j and V_{ij}, the obtained H_2/H_∞ performance by the initial computation is no more conservative than the ones from the previous studies.*

4.2 Update computation algorithm

In this section, we propose a novel separation result to provide new LMI based sufficient conditions for (4.18) by decoupling K and X_j with the introduction of new additional slack variables. We will show that the new LMI conditions contain any known solution to (4.18) as a particular case by imposing a constraint on the slack variables. Finally, an algorithm based on LMIs is presented to iteratively update the solutions to Theorem 4.1 by using the solutions provided by the existing methods as a starting point.

4.2.1 A general separation result

First, we need the following general separation theorem.

Theorem 4.2. *Let $T \in \mathbb{S}^m$, $E \in \mathbb{R}^{m\times n}$, $X \in \mathbb{R}^{n\times p}$, and $F \in \mathbb{R}^{m\times p}$ be general matrix variables. Then the following two statements are equivalent:*

$$i) \quad T + \mathcal{H}\Big(EXF^T\Big) \prec 0. \tag{4.19}$$

$$ii) \quad \exists\, Y \in \mathbb{R}^{n\times p}, Z \in \mathbb{R}^{n\times n} : \begin{bmatrix} T + \mathcal{H}(EY) & \star \\ XF^T + Z^T E^T - Y & -\mathcal{H}(Z) \end{bmatrix} \prec 0. \tag{4.20}$$

Proof. (4.20)→(4.19): Effecting the congruence $\begin{bmatrix} I_m & 0 \\ E^T & I_n \end{bmatrix}$ on (4.20), then we have the following condition that is equivalent to (4.20)

$$\begin{bmatrix} T + \mathcal{H}\Big(EXF^T\Big) & \star \\ XF^T - ZE^T - Y & -\mathcal{H}(Z) \end{bmatrix} \prec 0. \tag{4.21}$$

This shows that $T + EXF^T + FX^T E^T \prec 0$ if (4.20)/(4.21) is satisfied.

(4.19)$\rightarrow$(4.20): If (4.19) is satisfied, it is readily to see (4.21) is satisfied by defining $Y = XF^T - ZE^T$ where Z is any matrix that satisfies $\mathcal{H}(Z) \succ 0$. This completes the proof since (4.21) $\leftrightarrow$ (4.20). □

Remark 4.7. *Note that provided that F is constant matrix, then both* (4.19) *and* (4.20) *are bilinear. Note also that* (4.20) *separates the bilinear term EX in* (4.19) *without any conservatism but using two slack variables Y and Z with the bilinear terms EY and EZ instead. However,* (4.20) *becomes linear provided that we impose further restrictions on these two slack variables, e.g. $Y = WY_0$ and $Z = WZ_0$, where W is a variable but $Y_0 \in \mathbb{R}^{m\times n}$ and $Z_0 \in \mathbb{R}^{n\times n}$ are not variables. This restriction leads to* (4.20) *being linear, although it will be a sufficient condition for* (4.19) *only.*

Remark 4.8. *Notice that the variable X in* (4.19) *is a full general matrix without imposing any structural constraint, e.g. square, symmetry or definiteness. Theorem 2.2 in Section 2.3.1 also allows us to separate the variables E and X without any conservatism. However, Theorem 2.2 requires X (in* (2.15)*) to be positive definite. Therefore, the separation result in Theorem 4.2 in more general than the one proposed in Theorem 2.2. It will be shown later that Theorem 4.2 can deal with the BMI conditions in* (4.18) *while Theorem 2.2 cannot do it since X_j is not positive definite.*

4.2.2 Application

The next result is a direct application of Theorem 4.2 on (4.18) to give sufficient LMI conditions for Theorem 4.1.

Theorem 4.3. *Let all variables be as defined in Theorem 4.1. For all $i, j \in \mathcal{N}$, given constant matrices $\tilde{Y}_{ij} \in \mathbb{R}^{n\times p}$ and $\tilde{Z}_{ij} \in \mathbb{R}^{n\times n}$, suppose there exist $Y \in \mathbb{R}^{n\times n}$ and $M \in \mathbb{R}^{n_u\times n}$ and, for all $i, j \in \mathcal{N}$, there exist, $X_j \in \mathbb{R}^{n\times m}$, and $V_{ij} \in \mathbb{S}^m$ such that* (4.17) *and the following LMIs hold:*

$$\begin{bmatrix} \mathcal{T}_i + \mathcal{H}\Big(\mathcal{A}_i X_j + \mathcal{B}_i M \tilde{Y}_{ij}\Big) - V_{ij} & \star \\ X_j + (\mathcal{B}_i M \tilde{Z}_{ij})^T - Y\tilde{Y}_{ij} & -\mathcal{H}\Big(Y\tilde{Z}_{ij}\Big) \end{bmatrix} \prec 0. \tag{4.22}$$

Then with $K = MY^{-1}$, and for all $i, j \in \mathcal{N}$, $\mathcal{T}_i$, X_j and V_{ij} are feasible for Theorem 4.1.

Proof. (4.18) can be reformulated as (4.19) with

$$\left[\begin{array}{c|c} T & E \\ \hline X & F \end{array}\right] = \left[\begin{array}{c|c} \mathcal{T}_i + \mathcal{H}\Big(\mathcal{A}_i X_j\Big) - V_{ij} & \mathcal{B}_i K \\ \hline X_j & I_m \end{array}\right]$$

It follows from Theorem 4.2 that (4.18) is satisfied if and only if there exist Y_{ij} and Z_{ij}, for all $i, j \in \mathcal{N}$, satisfying the following inequalities:

$$\begin{bmatrix} \mathcal{T}_i + \mathcal{H}\Big(\mathcal{A}_i X_j + \mathcal{B}_i K Y_{ij}\Big) - V_{ij} & \star \\ X_j + (\mathcal{B}_i K Z_{ij})^T - Y_{ij} & -\mathcal{H}\Big(Z_{ij}\Big) \end{bmatrix} \prec 0. \tag{4.23}$$

Then setting $Y_{ij} = Y\tilde{Y}_{ij}$ and $Z_{ij} = Y\tilde{Z}_{ij}$ gives (4.22). □

Remark 4.9. *Note that* (4.23) *is a sufficient and necessary condition to* (4.18)*, but* (4.23) *is bilinear due to the term KY_{ij} and KZ_{ij}. In order to obtain convex conditions, we impose the additional equality constrains $Y_{ij} = Y\tilde{Y}_{ij}$ and $Y_{ij} = Y\tilde{Z}_{ij}$ where Y is a parameter-independent variable while $\tilde{Y}_{ij}$ and $\tilde{Z}_{ij}$ are parameter-dependent matrices but given, so that the corresponding condition* (4.22) *become linear by considering $M := KY$ and Y as decision variables, although* (4.22) *is only sufficient to* (4.18) *because of the equality constraints. Note also that $\mathcal{A}_i$ contained in* (4.18) *and* (4.22) *depends on a scalar parameter ϵ when $\mathcal{A}_i$ is defined as the conditions of [29], ϵ is optimized in [29] but it is fixed in* (4.18) *and* (4.22)*, so our proposed conditions do not include any scalar parameter.*

The following result uses the proof of Theorem 4.2 to show that $\tilde{Y}_{ij}$ and $\tilde{Z}_{ij}$ can be chosen so that the solution provided by Theorem 4.3 includes at least one feasible solution to Theorem 4.1.

Theorem 4.4. *Suppose that $\tilde{K} \in \mathbb{R}^{n_u \times n}$ and for all $i, j \in \mathcal{N}$, $\tilde{\mathcal{T}}_i \in \mathbb{S}^m$, $\tilde{X}_j \in \mathbb{R}^{n \times m}$ and $\tilde{V}_{ij} \in \mathbb{S}^m$ solve Theorem 4.1 so that*

$$\begin{aligned} &\tilde{V}_{ij} + \tilde{V}_{ji} \succeq 0, \quad 1 \le i < j \le N, \\ &\sum_{i=1}^{N} (\tilde{V}_{ij} + \tilde{V}_{ji}) \preceq 0, \quad j = 1, \cdots, N, \\ &\tilde{\mathcal{T}}_i + \mathcal{H}\Big(\mathcal{A}_i \tilde{X}_j + \mathcal{B}_i \tilde{K} \tilde{X}_j\Big) \prec \tilde{V}_{ij}. \end{aligned}$$

If $\tilde{Y}_{ij}$ and $\tilde{Z}_{ij}$ are set to be

$$\tilde{Y}_{ij} = \tilde{X}_j - \tilde{Z}_{ij}(\mathcal{B}_i \tilde{K})^T, \; \tilde{Z}_{ij} = \tilde{P}_i, \tag{4.24}$$

then Theorem 4.3 also has a feasible solution.

Proof. Using the congruence $\begin{bmatrix} I_m & 0 \\ (\mathcal{B}_i K)^T & I_n \end{bmatrix}$ on the matrix inequality in (4.22) yields the following equivalent condition:

$$\begin{bmatrix} \mathcal{T}_i + \mathcal{H}\Big(\mathcal{A}_i X_j + \mathcal{B}_i K X_j\Big) - V_{ij} & \star \\ X_j - Y\tilde{Z}_{ij}(\mathcal{B}_i K)^T - Y\tilde{Y}_{ij} & -\mathcal{H}\Big(Y\tilde{Z}_{ij}\Big) \end{bmatrix} \prec 0.$$

When $Y = I_n$, $T_i = \tilde{\mathcal{T}}_i$, $K = \tilde{K}$, $X_j = \tilde{X}_j$, $V_{ij} = \tilde{V}_{ij}$, $\tilde{Y}_{ij}$ and $\tilde{Z}_{ij}$ are defined as (4.24), the above condition reduces to

$$\begin{bmatrix} \tilde{\mathcal{T}}_i + \mathcal{H}\Big(\mathcal{A}_i \tilde{X}_j + \mathcal{B}_i \tilde{K}\tilde{X}_j\Big) - \tilde{V}_{ij} & \star \\ 0 & -2\tilde{P}_i \end{bmatrix} \prec 0$$

and proves the result. □

Remark 4.10. *Theorem 4.4 ensures recursive feasibility since it shows that the initial/previous solutions to Theorem 4.1 are also feasible for Theorem 4.3 by setting $Y = I_n$ and the other variables equal to previous solutions if $\tilde{Y}_{ij}$ and $\tilde{Z}_{ij}$ are defined as* (4.24)*. Therefore, the current solution obtained by Theorem 4.3 would be at least as good as the previous solution. Note also that when* (4.22) *is satisfied with $\tilde{Z}_{ij} = \tilde{P}_i$, it implies Y is non-singular since $\mathcal{H}(Y\tilde{P}_i) \succ 0$ and together with $\tilde{P}_i$ is positive definite, so the feedback gain K can always be recovered from $K = MY^{-1}$. Finally, $\tilde{Z}_{ij}$ can be designed in various ways as long as $\mathcal{H}(\tilde{Z}_{ij}) \succ 0$. In this note, we opt to set $\tilde{Z}_{ij} = \tilde{P}_i$, where $\tilde{P}_i$ is implicitly included in $\tilde{\mathcal{T}}_i$, other possible choices of $\tilde{Z}_{ij}$ are left to future work.*

The overall algorithm to iteratively compute less conservative upper bounds for H_2-norm and H_∞-norm of the continuous-time closed-loop system in (2.1) or the discrete-time closed-loop system in (3.1) can now be summarized as follows.

Algorithm 4.1. *Given tolerance level tol and it_{max} (maximum number of iterations)*

1. ***Initial data:*** *Choose one of the existing methods from Lemma 4.1-Lemma 4.4 and give the corresponding formulation of $\mathcal{T}_i$, $\mathcal{A}_i$, and $\mathcal{B}_i$ defined as* (4.14)*. Solve* (4.15) *and record the obtained stabilizing gain as K_0.*

2. ***Initial solution:*** *Given $K = K_0$, compute the initial solution by solving Theorem 4.1. Set $\tilde{\mu} = \mu$ (or $\tilde{\gamma} = \gamma$), $\tilde{K} = K$, $\tilde{P}_i = P_i$, $\tilde{X}_j = X_j$, and set $k = 0$.*

3. ***Update solution:*** *Given $\tilde{K}$, $\tilde{P}_i$, $\tilde{X}_j = X_j$, and substitute them into the expression of $\tilde{Y}_{ij}$ and $\tilde{Z}_{ij}$ defined as* (4.24). *Then compute K, P_i, X_j, μ (or γ) by solving* (4.22) *as given in Theorem 4.3.*

4. ***Stopping condition*** *Stop the loop if $(\tilde{\mu}\text{ - }\mu)/\tilde{\mu} \leq tol$ (or $(\tilde{\gamma}\text{ - }\gamma)/\tilde{\gamma} \leq tol$) or $k > it_{max}$. Else set $\tilde{\mu} = \mu$ (or $\tilde{\gamma} = \gamma$), $\tilde{K} = K$, $\tilde{P}_i = P_i$, $\tilde{X}_j = X_j$, $k = k + 1$, and go to step 3).*

4.3 Alternative way of update computation

Getting an initial stabilizing gain is an essential step since the iterative procedure proposed in Algorithm 4.1 requires a feasible initial solution as a starting point. Due to the conservatism of the current approaches, their feasibility cannot be guaranteed for some uncertain systems even when the system is known to be robustly stabilizable by a robust state feedback gain. Hence, we next propose a method as an alternative way to compute a sequence of solutions for the H_2/H_∞ performance when all the current approaches in Lemma 4.1-Lemma 4.4 fail to compute a stabilizing gain.

Based on our preliminary works of Chapter 2 and Chapter 3, we first recall the following lemmas that give a finite set of BMI conditions for H_2 and H_∞ state feedback control for continuous-time and discrete-time polytopic systems, respectively.

Lemma 4.5. *Consider the continuous-time closed-loop system in* (2.1).

1. *(H_2-norm) System* (2.1) *with $D_w(\alpha) = 0$ is Hurwitz stable and its H_2-norm is less than μ if, for all $i, j \in \mathcal{N}$, there exist $W_i \in \mathbb{S}^{n_w}$, $P_j \in \mathbb{S}^n_+$, and $V_{ij} \in \mathbb{S}^{(n+n_z)}$ such that* (4.17) *and*

$$\begin{bmatrix} \text{-}W_i & B_{wi}^T \\ B_{wi} & \text{-}P_j \end{bmatrix} \prec 0, \;\; Trace\,(W_i) < \mu^2, \tag{4.25}$$

$$\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & \text{-}I_{n_z} \end{bmatrix}}_{\mathcal{T}_i} + \mathcal{H}\left(\left(\underbrace{\begin{bmatrix} A_i \\ C_i \end{bmatrix}}_{\mathcal{A}_i} + \underbrace{\begin{bmatrix} B_i \\ D_i \end{bmatrix}}_{\mathcal{B}_i} K\right) P_j \underbrace{\begin{bmatrix} I_n & 0 \end{bmatrix}}_{\mathcal{F}^T}\right) \prec V_{ij}. \tag{4.26}$$

2. *(H_∞-norm) System* (2.1) *is Hurwitz stable and its H_∞-norm is less than γ if, for all $i, j \in \mathcal{N}$, there exist $P_j \in \mathbb{S}^n_+$ and $V_{ij} \in \mathbb{S}^{(n+n_z+n_w)}$ such that* (4.17) *and*

$$\underbrace{\begin{bmatrix} 0 & 0 & B_{wi} \\ 0 & \text{-}I_{n_z} & D_{wi} \\ B_{wi}^T & D_{wi}^T & \text{-}\gamma^2 I_{n_w} \end{bmatrix}}_{\mathcal{T}_i} + \mathcal{H}\left(\left(\underbrace{\begin{bmatrix} A_i \\ C_i \\ 0 \end{bmatrix}}_{\mathcal{A}_i} + \underbrace{\begin{bmatrix} B_i \\ D_i \\ 0 \end{bmatrix}}_{\mathcal{B}_i} K\right) P_j \underbrace{\begin{bmatrix} I_n & 0 & 0 \end{bmatrix}}_{\mathcal{F}^T}\right) \prec V_{ij}. \tag{4.27}$$

Lemma 4.6. *Consider the discrete-time closed-loop system in* (3.1).

1. *(H_2-norm) System* (3.1) *(assuming that $D_w(\alpha) = 0$) is Schur stable and its H_2-norm is less than μ if, for all $i, j \in \mathcal{N}$, there exist $W_i \in \mathbb{S}^{n_z}$, $P_j \in \mathbb{S}^n_+$, $V_{1ij} \in \mathbb{S}^{(n+n_z)}$ satisfying* (4.17)*, and $V_{2ij} \in \mathbb{S}^{(2n+n_w)}$ satisfying* (4.17)*, such that $\mathit{Trace}\,(W_i) < \mu^2$ and*

$$\underbrace{\begin{bmatrix} \text{-}W_i & 0 \\ 0 & \text{-}P_j \end{bmatrix}}_{\mathcal{T}_i} + \mathcal{H}\left(\left(\underbrace{\begin{bmatrix} C_i \\ 0 \end{bmatrix}}_{\mathcal{A}_i} + \underbrace{\begin{bmatrix} D_i \\ 0 \end{bmatrix}}_{\mathcal{B}_i} K\right) P_j \underbrace{\begin{bmatrix} 0 & I_n \end{bmatrix}}_{\mathcal{F}^T}\right) \prec V_{1ij}, \tag{4.28}$$

$$\underbrace{\begin{bmatrix} \text{-}P_j & B_{wi} & 0 \\ B_{wi}^T & \text{-}\gamma^2 I_{n_w} & 0 \\ 0 & 0 & \text{-}P_j \end{bmatrix}}_{\mathcal{T}_i} + \mathcal{H}\left(\left(\underbrace{\begin{bmatrix} A_i \\ 0 \\ 0 \end{bmatrix}}_{\mathcal{A}_i} + \underbrace{\begin{bmatrix} B_i \\ 0 \\ 0 \end{bmatrix}}_{\mathcal{B}_i} K\right) P_j \underbrace{\begin{bmatrix} 0 & 0 & I_n \end{bmatrix}}_{\mathcal{F}^T}\right) \prec V_{2ij}. \tag{4.29}$$

2. *(H_∞-norm) System* (3.1) *is Schur stable and its H_∞-norm is less than γ if, for all $i, j \in \mathcal{N}$, there exist $P_j \in \mathbb{S}_+^n$ and $V_{ij} \in \mathbb{S}^{(2n+n_z+n_w)}$ such that* (4.17) *and*

$$\underbrace{\begin{bmatrix} -P_j & B_{wi} & 0 & 0 \\ B_{wi}^T & -\gamma^2 I_{n_w} & D_{wi}^T & 0 \\ 0 & D_{wi} & -I_{n_z} & 0 \\ 0 & 0 & 0 & -P_j \end{bmatrix}}_{\mathcal{T}_i} + \mathcal{H}\left(\left(\underbrace{\begin{bmatrix} A_i \\ 0 \\ C_i \\ 0 \end{bmatrix}}_{\mathcal{A}_i} + \underbrace{\begin{bmatrix} B_i \\ 0 \\ D_i \\ 0 \end{bmatrix}}_{\mathcal{B}_i} K\right) P_j \underbrace{\begin{bmatrix} 0 & 0 & 0 & I_n \end{bmatrix}}_{\mathcal{F}^T}\right) \prec V_{ij}. \tag{4.30}$$

The BMI conditions of Lemma 4.5 and Lemma 4.6 are presented in a unified form as follows:

$$\mathcal{T}_i + \mathcal{H}\Big(\mathcal{A}_i P_j \mathcal{F}^T + \mathcal{B}_i K P_j \mathcal{F}^T\Big) \prec V_{ij}. \tag{4.31}$$

It can readily be seen that (4.31) is included in (4.18) as a special case by imposing an extra equality constraint $X_j = P_j \mathcal{F}^T$ on (4.18). Hence, once a feasible initial solution to (4.31) is found by the proposed initial computation methods (see Section 2.2.2 for the continuous-time case or Section 3.2 for the discrete-time case), then the upper bounds on H_2-norm and H_∞-norm can be iteratively updated through Algorithm 4.1 with $X_j = P_j \mathcal{F}^T$. In the case of no feasible solution found by these initial computation methods, we have provided iterative procedures to present the possibility of finding a stabilizing gain; see more details in Algorithm 2.2 and Algorithm 3.2 for continuous-time and discrete-time systems, respectively.

4.4 Numerical examples

In this section, two examples are given to illustrate the effectiveness of the proposed Algorithm 4.1. The comparisons between our results and the existing methods are presented for both continuous-time and discrete-time polytopic systems, respectively.

4.4.1 Example 1

Consider a continuous-time uncertain spring-mass system presented in Section 2.5.2.

Robust H_2 control

The comparison of the minimum upper bound on H_2-norm achieved by Theorem 4.1 and the conditions given in Lemma 4.1 is given by the following Table 4.1.

Method	[24]	T1$_A$	[25]	T1$_{EH}$	[29]	T1$_R$
μ	2.6871	1.5074	0.7073	0.5107	0.7073	0.5107

Table 4.1 The minimum upper bound on H_2-norm obtained by some existing methods and Theorem 4.1 for Example 1.

As shown in Table 4.1, Apakarian et al. [24] yields a minimum value of 2.6871 for the H_2-norm performance μ. Exhaustive searches on the scalar parameters are performed for the parameterized LMI conditions of Ebihara and Hagiwara [25] and Rodrigues et al. [29] to attain less conservative results. [25] gives a minimum value of 0.7073 with the optimized $r = 2.5$ while [29] yields the same value of μ as [25] with the optimized values of $\epsilon = 1$ and $\xi = 0.11$. Moreover, T1$_A$, T1$_{EH}$, and T1$_R$ represents the upper bound μ obtained by our Theorem 4.1 using the given stabilizing controller gain provided by [24], [25], and [29], respectively. Table 4.1 indicates Theorem 4.1 using parameter-dependent slack variables is more relaxed than the current methods using the fixed slack variable for the computation of the upper bound on H_2-norm.

If less conservative results are required, Algorithm 4.1 is carried out to iteratively update the solutions by utilizing the solution of Theorem 4.1 as a starting point.

Method	Alg1-T1$_A$	Alg1-T1$_{EH}$	Alg-T1$_R$
μ	0.4993	0.4999	0.4964

Table 4.2 The minimum upper bound on H_2-norm obtained by Algorithm 4.1 for Example 1.

In Table 4.2, Alg1-T1$_A$, Alg1-T1$_{EH}$, and Alg-T1$_R$ denote the final converged values of μ achieved by Algorithm 4.1 for different initial solutions given by T1$_A$, T1$_{EH}$, and T1$_R$, respectively, where $it_{max} = 10$. It can be noted that although the initial solution of T1$_A$ is more conservative than the ones of Alg1-T1$_{EH}$ and Alg-T1$_R$, Algorithm 4.1 converges to nearly the same final values for different initial solutions. The final resulting state feedback gain for Alg1-T1$_A$, , Alg1-T1$_{EH}$, and Alg-T1$_R$ are K = [-8.3780 - 1.7243 - 24.9122 -

13.4383], K = [-10.0862 - 2.2877 - 26.6148 - 14.3759], and K = [-9.3917 - 2.1988 - 25.0670 - 13.5264], respectively.

Robust H_∞ control

To avoid excessive computational burden for the parameterized LMI based conditions given in Lemma 4.2, we follow the work of [29] to select the constrained set: $\epsilon \in \left\{10^{-1}, 10^{0}\right\}$ and $\xi \in \{-0.9, -0.54, -0.18, 0.18, 0.54, 0.9\}$ (total of 12 searches) for [29] and thirteen logarithmically spaced values: $r \in \left\{10^{-6}, 10^{-5}, \cdots, 10^{0}, 10^{1}, \cdots 10^{6}\right\}$ (total of 13 searches) for [13].

Table 4.3 shows the minimum upper bound γ on H_∞-norm and the associated solution time obtained by the methods from previous studies, Theorem 4.1, and Algorithm 4.1.

Method	γ	Scalars	T: solution time
[13]	2.3847	$r = 10$	2.65 s
T1$_X$	1.4982	-	2.90 s
Alg1-T1$_X$	0.9870	$it_{max} = 5$	4.57 s
[29]	1.9745	$\epsilon = 1$, $\xi = 0.18$	2.33 s
T1$_R$	1.1390	-	2.58 s
Alg1-T1$_R$	0.9801	$it_{max} = 5$	4.26 s

Table 4.3 The minimum upper bound on H_∞-norm obtained some existing methods, Theorem 4.1, and Algorithm 4.1, as well as the associated solution time for Example 1.

As can be observed in Table 4.3, Xie [13] gives a minimum value of 2.3747 for H_∞ performance with $r = 10$ while Rodrigues et al. [29] yields a minimum of 1.9745 by using $\epsilon = 1$ and $\xi = 0.18$. The mean total solution time spent by [13] and [29] are 2.65 s and 2.33 s, respectively. Using the initial stabilizing gain K computed by [13], Theorem 4.1 (T1$_X$) gives an initial value of 1.4982 for γ in 2.90 s. Then starting with the initial solution of T1$_X$, Algorithm 4.1 (Alg1-T1$_X$) yields a final converged value of $\gamma = 0.9870$ in 4.57 s after 5 iterations, where the final resulting gain is given by K = [-8.5635 - 0.9270 - 29.7820 - 16.1670]. Table 4.3 also shows that the results obtained with Theorem 4.1 and Algorithm 4.1 by considering the state feedback gain of [29] as initial data. Theorem 4.1 (T1$_R$) yields an initial value of 1.1390 and Algorithm 4.1 (Alg1-T1$_R$) yields a final converged value of 0.9801 with K = [-9.9176 - 1.3292 - 31.4827 - 17.1449] after 5 iterations, the total solution time of T1$_R$ and Alg1-T1$_R$ are 2.58 s and 4.26 s, respectively. Note that the total solution time of Theorem 4.1 and Algorithm 4.1 have included the computation efforts spent by [13] or [29] to obtain the initial value for the state feedback gain.

In summary, it should be noted that both T1$_X$ and T1$_R$ can provide less conservative H_∞ bounds than those obtained with [13] and [29] after only one-step computation. Note also that Alg1-T1$_X$ and Alg1-T1$_R$ obtain nearly the same final converged values that are much superior to [13] and [29] at the expense of more computational time. It is also important to emphasize that even if exhaustive searches on scalar parameters are implemented for [13] and [29], the minimum value of γ obtained by [13] is 1.9116 with optimized $r = 3.99$, whereas [29] yields the same value of γ as [13] by using the optimized values of $\epsilon = 1$ and $\xi = 0.33$. Finally, the relation between H_∞ performance and the number of iterations for both Alg1-T1$_X$ and Alg1-T1$_R$ can be observed from the following Figure 4.1.

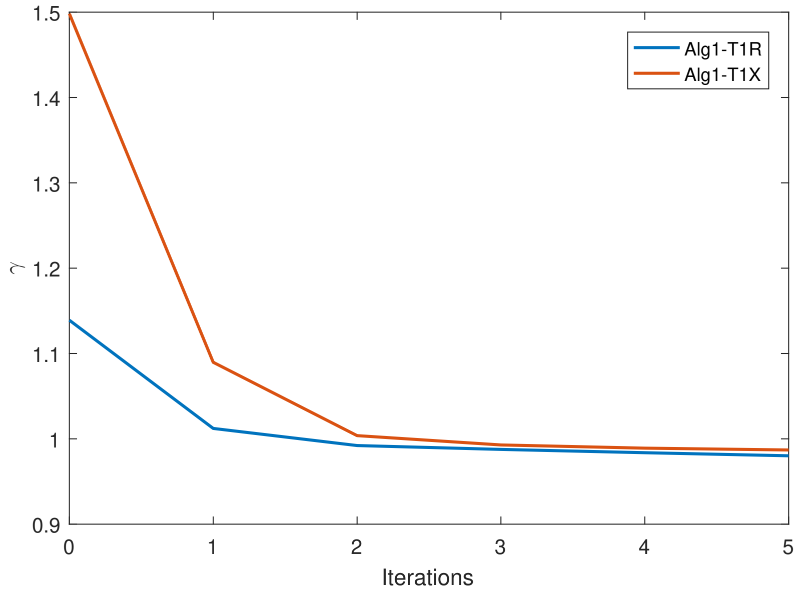

Fig. 4.1 Bound on the H_∞-norm against the number of iterations obtained by Algorithm 4.1 for Example 1.

4.4.2 Example 2

Consider the discrete-time version of the satellite system given in Section 3.6.1.

While [11] gives less conservative results than [23] in general, we next use [11] to compute the initial stabilizing gain as the starting point of Theorem 4.1 and Algorithm 4.1 since [23] requires much greater computational effort due to exhaustive searches on the scalar parameter. We consider 199 linearly equally spaced points between -0.99 and 0.99 as the search domain for the scalar ξ of [23]. By applying those approaches, the minimum values of the H_2 performance μ and the H_∞ performance γ and the associated computational time are given in the following tables.

Method	μ	Scalars	T
[11]	0.7537	-	0.33 s
[23]	0.7464	$\xi =$ -0.13	65.25 s
Theorem 4.1	0.6020	-	1.40 s
Algorithm 4.1	0.5063	$it_{max} = 10$	17.19 s

Table 4.4 The minimum upper bound on H_2-norm obtained some existing methods, Theorem 4.1, and Algorithm 4.1, as well as the associated solution time for Example 2.

Method	γ	Scalars	T
[11]	2.1922	-	0.33 s
[23]	2.1311	$\xi =$ -0.34	64.13 s
Theorem 4.1	1.9328	-	1.39 s
Algorithm 4.1	1.8061	$it_{max} = 6$	10.89 s

Table 4.5 The minimum upper bound on H_∞-norm obtained some existing methods, Theorem 4.1, and Algorithm 4.1, as well as the associated solution time for Example 2.

The computation results in Table 4.4 and Table 4.5 show that [23] can reduce the conservatism for computation of the upper bounds on H_2-norm and H_∞-norm with respect to [11], at the price of greatly increasing the computation time. Theorem 4.1 yields less conservative bounds and requires much shorter computation time compared with [23] for both H_2 and H_∞ cases. The final converged minimum values of μ and γ obtained with Algorithm 4.1 are 0.5063 and 1.8061, respectively, with the corresponding H_2 controller gain $K = [-21.8501\ -71.8163\ -10.6147\ -211.5043]$ and H_∞ controller gain $K = [-70.0308\ -404.5961\ -18.6397\ -724.8660]$. Figure 4.2 and Figure 4.3 displays the relation of H_2 and H_∞ performance against the number of iterations.

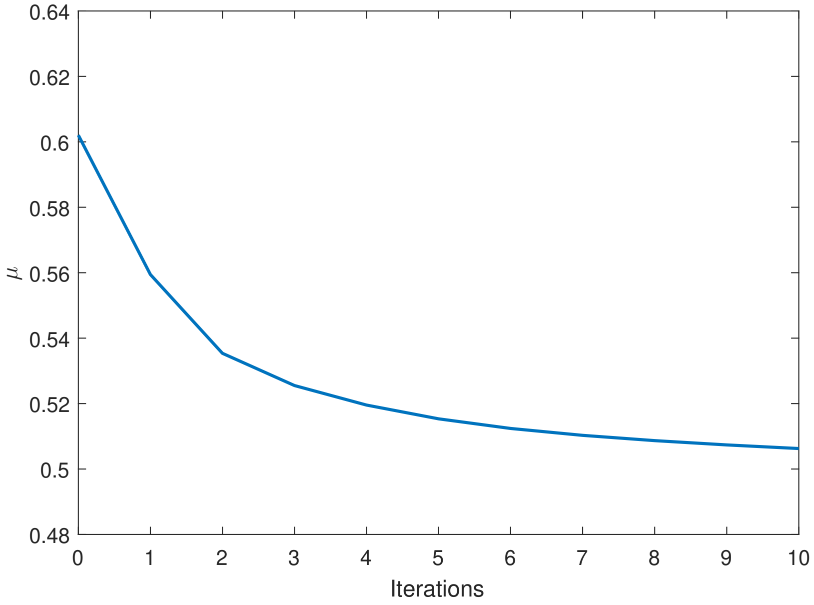

Fig. 4.2 Bound on the H_2-norm against the number of iterations obtained by Algorithm 4.1 for Example 2.

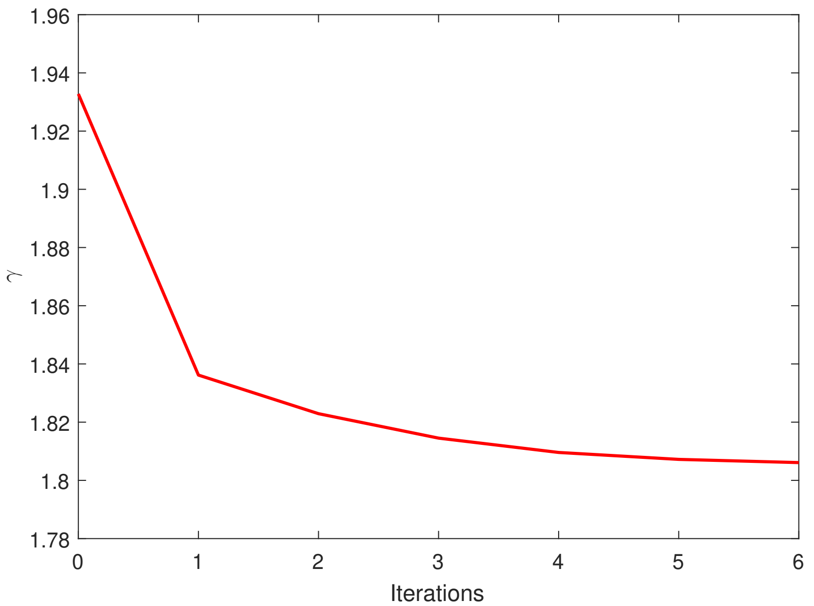

Fig. 4.3 Bound on the H_∞-norm against the number of iterations obtained by Algorithm 4.1 for Example 2.

4.5 Summary

In conclusion, this chapter has investigated the problem of robust H_2 and H_∞ state feedback control of linear systems with polytopic uncertainties in both the continuous-time and discrete-time cases. We proposed Theorem 4.1 to compute the initial solution, which has been proved to include the existing methods as special cases. Then an iterative procedure (Algorithm 4.1) was proposed to compute a sequence of non-increasing upper bounds for the H_2-norm and H_∞-norm by utilizing the obtained initial solution as a starting point. Example 1 and Example 2 from the literature were presented to show that the proposed Theorem 4.1 can provide less conservative upper bounds on the H_2 and H_∞ norms than those obtained by the existing methods, and the bounds can be further reduced by Algorithm 4.1 after a few iterations.

Moreover, compared with the results obtained by the algorithms (Algorithm 2.1, Algorithm 3.1, Algorithm 3.3) proposed in previous chapters, Algorithm 4.1 does not show clear advantages on minimization of the upper bound μ or γ. However, it is worth mentioning that Algorithm 4.1 can utilize the optimal solution of any of the existing methods as a starting point while previous proposed algorithms cannot. Therefore, with one step computation only, Algorithm 4.1 can be guaranteed to give no more conservative results than any of the existing methods.

Chapter 5

H_∞-norm guaranteed cost computation by means of S-procedure

The novel results proposed in previous chapters follow the idea of [35] to deal with the polytopic uncertainty, which is to convert the original parameter-dependent (infinite-dimensional) conditions into a finite set of conditions. In this chapter, we will pursue a novel approach based on S procedure to lift the uncertainty.

This chapter is organized as follows. Section 5.1 details a description of the problem, formulates the conditions for H_∞-norm guaranteed cost computation and highlights the associated difficulties. In Section 5.2, we review and extend some available approaches in the literature. We propose a novel result based on the S-procedure to compute H_∞-norm guaranteed cost in Section 5.3. We give numerical examples in Section 5.4 to compare our proposed results with existing approaches and summarize this chapter in Section 5.5.

The results presented in this chapter are based on our paper [64] and the associated contributions are highlighted as below:

- Extend the approach of [35] from the robust stabilization to H_∞ performance analysis.
- Develop a relaxation approach in terms of one LMI to compute H_∞-norm guaranteed cost.

5.1 Problem description

Consider the following uncertain linear continuous-time system

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} A(\alpha) & B(\alpha) \\ C(\alpha) & D(\alpha) \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}, \tag{5.1}$$

and also the discrete-time system

$$\begin{bmatrix} x(k+1) \\ z(k) \end{bmatrix} = \begin{bmatrix} A(\alpha) & B(\alpha) \\ C(\alpha) & D(\alpha) \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}, \tag{5.2}$$

where $x(\cdot) \in \mathbb{R}^n, w(\cdot) \in \mathbb{R}^{n_w}$ and $z(\cdot) \in \mathbb{R}^{n_z}$ are the system state vector, exogenous disturbance signal, objective function signal, respectively. The symbol $(\cdot)$ denotes (t) for continuous-time systems and (k) for discrete-time systems. All system matrices of appropriate dimensions are not precisely known, but depend affinely on uncertain parameter α, that is represented by the convex combination of the vertices of system matrices (A_i, B_i, C_i, D_i):

$$\begin{bmatrix} A(\alpha) & B(\alpha) \\ C(\alpha) & D(\alpha) \end{bmatrix} = \sum_{i=1}^{N} \alpha_i \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix},$$

where α belongs to the unit simplex given by

$$\Omega = \left\{ \alpha \in \mathbb{R}^N : \alpha_i \geq 0, \forall i \in \mathcal{N}; \sum_{i=1}^{N} \alpha_i = 1 \right\}.$$

For any possible value of α, $A(\alpha)$ is assumed to be Hurwitz stable for the continuous-time case (that is, all its eigenvalues have strictly negative real parts), and Schur stable for the discrete-time case (that is, all its eigenvalues have modulus less than one).
We next present an extension of the BRL representation in [1] to polytopic systems (5.1) and (5.2) involving the existence of a parameter-dependent Lyapunov function.

Lemma 5.1. *[29]: System (5.1) is Hurwitz stable and its H_∞-norm is less than γ if and only if, there exist a parameter-dependent Laypunov matrix $P(\alpha) \in \mathbb{S}^n_+$, for all $\alpha \in \Omega$, such that the following inequality holds:*

$$\begin{bmatrix} A(\alpha)P(\alpha) + P(\alpha)A(\alpha)^T & \star & \star \\ C(\alpha)P(\alpha) & -I_{n_z} & \star \\ B(\alpha)^T & D(\alpha)^T & -\gamma^2 I_{n_w} \end{bmatrix} \prec 0. \tag{5.3}$$

Lemma 5.2. *[11]: System (5.2) is Schur stable and its H_∞-norm is less than γ if and only if, there exist a parameter-dependent Laypunov matrix $P(\alpha) \in \mathbb{S}_+^n$, for all $\alpha \in \Omega$, such that the following inequality holds:*

$$\begin{bmatrix} -P(\alpha) & B(\alpha) & 0 & A(\alpha)P(\alpha) \\ \star & -\gamma^2 I_{n_w} & D(\alpha)^T & 0 \\ \star & \star & -I_{n_z} & C(\alpha)P(\alpha) \\ \star & \star & \star & -P(\alpha) \end{bmatrix} \prec 0. \tag{5.4}$$

Note that minimizing γ (guaranteed cost for the H_∞-norm) subject to condition (5.3)/(5.4) is numerically intractable since (5.3)/(5.4) is parameter-dependent which results in infinite number of conditions. It follows from [1], a simple procedure to deal with the aforementioned difficulty, that is when the Lyapunov matrix $P(\alpha)$ is restricted to be independent of α, i.e. when $P(\alpha) = P$ for all $\alpha \in \Omega$. This idea is the well-known quadratic stability based results [5]. However, the use of a parameter-independent Lyapunov function generally gives conservative bounds. Therefore, LMI based approaches in terms of parameter-dependent Lyapunov functions to obtain less conservative guaranteed costs to H_∞-norm are proposed in the sequel.

5.2 Reviews and extensions

In this section, we outline and extend some existing methods in the literature to state a finite set of sufficient LMI conditions for Lemma 5.1 and Lemma 5.2 by using a parameter-dependent Lyapunov function matrix given by $P(\alpha) = \sum_{i=1}^{N} \alpha_i P_i$.

5.2.1 Reviews

The mainstream of existing methods is to decouple the Lyapunov matrix from the system matrices and allow a product term between the system matrices and slack variables instead. The following result, presented in Lemma 3 of [15], includes both Lemma 1 and Lemma 2 of [15] and some other existing methods as particular cases, and gives less conservative sufficient LMI conditions than those available in the literature for Lemma 5.1.

Lemma 5.3. *Consider system (5.1), for all $i, j \in \mathcal{N}$, if there exist $P_i \in \mathbb{S}_+^n$, and matrices $Y_j \in \mathbb{R}^{n\times n}$, $Z_j \in \mathbb{R}^{n\times n}$, such that the following inequalities hold:*

$$\Gamma_{ij} + \Gamma_{ji} \prec 0, \quad 1 \le i \le j \le N, \tag{5.5}$$

where

$$\Gamma_{ij} = \begin{bmatrix} A_i Y_j + Y_j^T A_i^T & \star & \star & \star \\ P_i - Y_j + Z_j^T A_i^T & -(Z_j + Z_j^T) & \star & \star \\ C_i Y_j & C_i Z_j & -I_{n_z} & \star \\ B_i^T & 0 & D_i^T & -\gamma^2 I_{n_w} \end{bmatrix},$$

then system (5.1) is Hurwitz stable and its H_∞-norm is less than γ.

Remark 5.1. *Lemma 5.3 introduces slack variables Y_j and Z_j to separate Lyapunov and system matrices and replace it with the terms $A_i Y_j$ ($C_i Y_j$) and $A_i Z_j$ ($C_i Z_j$) which are linear for robust analysis of H_∞-norm performance. Note that Lemma 5.3 contains quadratic stability based conditions and some other existing results as special cases. For example, Theorem 2 of [14] and Lemma 3.3 of [13] can be viewed as derived from Lemma 5.3 by imposing the restriction $(Y_j, Z_j) \to (Y, Z)$ and $(Y_j, Z_j) \to (Y, rY)$, respectively, where r is a scalar parameter.*

5.2.2 Extensions

Another type of existing approaches using parameter-dependent Lyapunov function is based on the idea of converting the infinite-dimensional matrix inequalities (5.3)-(5.4) into a finite set of LMIs with the help of N^2 additional symmetric matrix variables, without the need of separation between system and Lyapunov matrices. The following result is a simple extension of robust stability LMI conditions for continuous-time and discrete-time polytopic systems that appeared in the work of [18].

Theorem 5.1. *Let V_{ij} be symmetric matrices satisfying*

$$V_{ij} + V_{ji} \succeq 0,\ 1 \le i < j \le N, \qquad \sum_{i=1}^{N} (V_{ij} + V_{ji}) \preceq 0,\ j = 1, \ldots, N, \tag{5.6}$$

Consider system (5.1), for all $i, j \in \mathcal{N}$, if there exist $P_j \in \mathbb{S}_+^n$ and $V_{ij} \in \mathbb{S}^{(n_z+n_w)}$ satisfying (5.6)*, such that the following LMIs hold:*

$$\Lambda_{ii} \prec 0, \quad 1 \le i \le N, \tag{5.7}$$

$$\Lambda_{ij} + \Lambda_{ji} \prec 0, \quad 1 \le i < j \le N, \tag{5.8}$$

where

$$\Lambda_{ij} = \begin{bmatrix} A_iP_j + P_jA_i^T & \star & \star \\ C_iP_j & \text{-}I_{n_z} & \star \\ B_i^T & D_i^T & \text{-}\gamma^2 I_{n_w} \end{bmatrix} \text{ - } V_{ij},$$

then system (5.1) is Hurwitz stable and its H_∞-norm is less than γ.

Proof. First, when inequality (5.6) is satisfied, it yields that $\sum_{i,j=1}^{N}\alpha_i\alpha_j V_{ij} \preceq 0, \forall\alpha \in \Omega$ (see Lemma 1.3). Then multiplying (5.7) by α_i^2 for all $i \in \mathcal{N}$ and (5.8) by $\alpha_i\alpha_j$ for $1 \le i < j \le N$ and taking the sum gives

$$\sum_{i=1}^{N} \alpha_i^2\Lambda_{ii} + \sum_{i=1}^{N\text{-}1}\sum_{j=i+1}^{N} \alpha_i\alpha_j(\Lambda_{ij} + \Lambda_{ji}) = \sum_{i,j=1}^{N} \alpha_i\alpha_j\Lambda_{ij}$$

$$= \begin{bmatrix} \mathcal{H}\big(A(\alpha)P(\alpha)\big) & \star & \star \\ C(\alpha)P(\alpha) & \text{-}I_{n_z} & \star \\ B(\alpha)^T & D(\alpha)^T & \text{-}\gamma^2 I_{n_w} \end{bmatrix} \text{ - } \sum_{i,j=1}^{N} \alpha_i\alpha_j V_{ij} \prec 0,$$

which implies (5.3) since $\sum_{i,j=1}^{N}\alpha_i\alpha_j V_{ij} \preceq 0$. □

Remark 5.2. *It can be noted that Theorem 5.1 reduces to Lemma 1 of [15] when $V_{ij} = 0$ for all $i, j \in \mathcal{N}$, so the extra degree of freedom provided by V_{ij} can guarantee that the results obtained by Theorem 5.1 are no more conservative than Lemma 1 of [15].*

The corresponding results for discrete-time systems are also presented below.

Theorem 5.2. *Consider system (5.2), for all $i, j \in \mathcal{N}$, if there exist $P_j \in \mathbb{S}_+^n$ and $V_{ij} \in \mathbb{S}^{(2n+n_z+n_w)}$ satisfying* (5.6)*, such that the following LMIs hold:*

$$\Phi_{ii} \prec 0, \quad 1 \le i \le N, \tag{5.9}$$

$$\Phi_{ij} + \Phi_{ji} \prec 0, \quad 1 \le i < j \le N, \tag{5.10}$$

where

$$\Phi_{ij} = \begin{bmatrix} \text{-}P_j & B_i & 0 & A_iP_j \\ \star & \text{-}\gamma^2 I_{n_w} & D_i^T & 0 \\ \star & \star & \text{-}I_{n_z} & C_iP_j \\ \star & \star & \star & \text{-}P_j \end{bmatrix} \text{ - } V_{ij},$$

then system (5.2) is Schur stable and its H_∞-norm is less than γ.

Proof. The discrete-time case follows from a similar procedure on (5.9) and (5.10) which yields

$$\begin{bmatrix} -P(\alpha) & B(\alpha) & 0 & A(\alpha)P(\alpha) \\ \star & -\gamma^2 I_{n_w} & D(\alpha)^T & 0 \\ \star & \star & -I_{n_z} & C(\alpha)P(\alpha) \\ \star & \star & \star & -P(\alpha) \end{bmatrix} - \sum_{i,j=1}^{N} \alpha_i \alpha_j V_{ij} \prec 0,$$

which implies (5.4) since $\sum_{i,j=1}^{N} \alpha_i \alpha_j V_{ij} \preceq 0$. □

5.3 Main results

In this section, we develop the S-Procedure for lifting α by introducing slack variables to derive sufficient conditions for BRL of continuous-time and discrete-time polytopic systems in terms of one LMI.

5.3.1 A generalized form of BRL

First, we pose the BRL for continuous-time and discrete-time cases in an equivalent unified generalized problem by adding constant and linear terms in α and generalizing the dimensions.

Problem 5.1. *For all $i \in \mathcal{N}$, let $T_1, L_i \in \mathbb{R}^{m\times m}$, $F, E_i \in \mathbb{R}^{m\times n}$ be given and, for any $\alpha \in \Omega$ define*

$$L(\alpha) = \sum_{i=1}^{N} \alpha_i L_i, \quad E(\alpha) = \sum_{i=1}^{N} \alpha_i E_i.$$

Find a parameter–dependent Lyapunov matrix $P(\alpha) \in \mathbb{S}_+^n$ such that

$$T_1 + \mathcal{H}\Big(L(\alpha) + E(\alpha)P(\alpha)F^T\Big) \prec 0 \;\; \forall \alpha \in \Omega. \tag{5.11}$$

Remark 5.3. *Note that T_1 is typically a general augmented parameter–independent matrix variable, $L(\alpha)$ and $E(\alpha)$ are linear in general augmented system matrices and may also depend on other matrix variables, while F is a constant matrix. It can be verified that the BRL conditions* (5.3) *and* (5.4) *are special cases of* (5.11)*:*

For continuous-time case,

$$T_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -I_{n_z} & 0 \\ 0 & 0 & -\gamma^2 I_{n_w} \end{bmatrix}, \; L(\alpha) = \begin{bmatrix} 0 & 0 & B(\alpha) \\ 0 & 0 & D(\alpha) \\ 0 & 0 & 0 \end{bmatrix}, E(\alpha) = \begin{bmatrix} A(\alpha) \\ C(\alpha) \\ 0 \end{bmatrix}, F = \begin{bmatrix} I_n \\ 0 \\ 0 \end{bmatrix}.$$

For discrete-time case,

$$L(\alpha) = \begin{bmatrix} -\frac{1}{2}P(\alpha) & B(\alpha) & 0 & 0 \\ 0 & 0 & D(\alpha)^T & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}P(\alpha) \end{bmatrix},$$
$$T_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\gamma^2 I_{n_w} & 0 & 0 \\ 0 & 0 & -I_{n_z} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, E(\alpha) = \begin{bmatrix} A(\alpha) \\ 0 \\ C(\alpha) \\ 0 \end{bmatrix}, F = \begin{bmatrix} 0 \\ 0 \\ 0 \\ I_n \end{bmatrix}.$$

Note also that Problem 5.1 includes robust stabilization and H_2-norm performance analysis as special cases, this will not be discussed here.

We next pose Problem 5.1 in terms of an uncertainty description reminiscent of norm-bounded structured uncertainty and the Lyapunov function matrix given by $P(\alpha) = \sum_{i=1}^{N} \alpha_i P_i$.

Problem 5.2. *Let all variables be as described in Problem 5.1. Define*

$$\begin{aligned} \mathbf{\Delta} &= \{\text{diag}(\alpha_1 I_m, \ldots, \alpha_N I_m) : \alpha \in \Omega\} \subset \mathbb{R}^{Nm \times Nm}, \qquad (5.12) \\ \hat{F} &= \text{diag}(F, \ldots, F) \in \mathbb{R}^{Nm \times Nn}, \\ T_2 &= \begin{bmatrix} I_m & \cdots & I_m \end{bmatrix} \in \mathbb{R}^{m \times Nm}, \\ T_3 &= \begin{bmatrix} L_1 \\ \vdots \\ L_N \end{bmatrix} \in \mathbb{R}^{Nm \times m}, \; E = \begin{bmatrix} E_1 \\ \vdots \\ E_N \end{bmatrix} \in \mathbb{R}^{Nm \times n}. \end{aligned}$$

Find $X = \begin{bmatrix} P_1 \\ \vdots \\ P_N \end{bmatrix} \in \mathbb{R}^{Nn\times n}$*, with* $P_i \in \mathbb{S}^n_+$ *for all* $i \in \mathcal{N}$ *such that*

$$\underbrace{T_1 + \mathcal{H}\Big((T_2\Delta)T_3 + (T_2\Delta)EX^T\hat{F}^T(T_2\Delta)^T\Big)}_{T(\Delta)} \prec 0\ \forall \Delta \in \boldsymbol{\Delta}.$$

5.3.2 A relaxation of the uncertainty set

The following result uses the definition of Ω to give a relaxation of the uncertainty set $\boldsymbol{\Delta}$ in (5.12).

Theorem 5.3. *Let all variables be as defined in Problem 5.2 and define the sets* $\mathbb{S}, \mathbb{G} \subset \mathbb{R}^{Nm\times Nm}$ *respectively, as*

$$\begin{aligned} \mathbb{S} &= \{\mathrm{diag}(S_1, \dots, S_N)\colon S_i = S_i^T \succeq 0 \forall i \in \mathcal{N}\}, \\ \mathbb{G} &= \{\mathrm{diag}(G_1, \dots, G_N)\colon \mathcal{H}(G_i) = 0 \forall i \in \mathcal{N}\}. \end{aligned}$$

Then

$$\begin{aligned} \boldsymbol{\Delta} \subset \{\Delta \in \mathbb{R}^{Nm\times Nm}: \ & T_2\mathcal{H}(\text{-}\Delta G)\, T_2^T = 0 \forall G \in \mathbb{G}; \\ & T_2\mathcal{H}\Big(\Delta S \Delta^T \text{-} \Delta S\Big)\, T_2^T \preceq 0 \forall S \in \mathbb{S}; \\ & \mathcal{H}\Big(T_2\Delta T_2^T R \text{-} R\Big) = 0 \forall R \in \mathbb{R}^{m\times m}; \\ & \mathcal{H}\Big(T_2\Delta \hat{M} \Delta^T T_2^T \text{-} M\Big) = 0 \forall M \in \mathbb{R}^{m\times m}\}, \end{aligned}$$

where $\hat{M} = \mathbf{1}_{N,N} \otimes M$*, where* $\mathbf{1}_{N,N}$ *is the* $N \times N$ *matrix of ones and* $\otimes$ *denotes the Kronecker product.*

Proof. Let $\Delta = \mathrm{diag}(\alpha_1 I, \dots, \alpha_N I) \in \boldsymbol{\Delta}$ so that $0 \leq \alpha_i \leq 1 \forall i \in \mathcal{N}$ and $\sum_{i=1}^N \alpha_i = 1$. Then $T_2\mathcal{H}(\text{-}\Delta G)\, T_2^T = 0$ for all $G \in \mathbb{G}$ from the structure of Δ. Next, $\mathcal{H}\Big(T_2\Delta T_2^T R \text{-} R\Big) = 0$ for all $R \in \mathbb{R}^{m\times m}$ since $\sum_{i=1}^N \alpha_i = 1$. Furthermore, $T_2\mathcal{H}\Big(\Delta S \Delta^T \text{-} \Delta S\Big)\, T_2^T \preceq 0 \forall S \in \mathbb{S}$ since $\alpha_i(1 \text{-} \alpha_i) \geq 0$ for all $i \in \mathcal{N}$ which follows from the fact that $0 \leq \alpha_i \leq 1 \forall i \in \mathcal{N}$. Finally, $T_2\Delta\hat{M}\Delta^T T_2^T = M$ for all $M \in \mathbb{R}^{m\times m}$ since $(\sum_{i=1}^N \alpha_i)^2 = 1$. This proves the result. □

5.3.3 The S-Procedure for lifting polytopic uncertainty

Since the polytopic uncertainty in Problem 5.2 is non-convex, we develop a relaxation approach based on the S-Procedure to lift uncertainty as this has proved successful for the norm-bounded structured uncertainty problem.

Theorem 5.4. *Let all variables be as defined in Theorem 5.3. Suppose there exist $G\in\mathbb{G}$, $S\in\mathbb{S}$, $R\in\mathbb{R}^{m\times m}$, $M\in\mathbb{R}^{m\times m}$ and $X=\begin{bmatrix} P_1 & \dots & P_N \end{bmatrix}^T$ with $P_i\succ 0$ for all $i\in\mathcal{N}$ such that*

$$\mathcal{L}:=\begin{bmatrix} T_1\text{-}\,\mathcal{H}(R\text{-}M) & (T_3+(S+G)T_2^T+T_2^TR)^T \\ \star & \mathcal{H}\Big(EX^T\hat{F}^T\text{-}S\text{-}\,\hat{M}\Big) \end{bmatrix}\prec 0. \tag{5.13}$$

Then X is feasible for Problem 5.2.

Proof. For any $G\in\mathbb{G}, S\in\mathbb{S}, R\in\mathbb{R}^{m\times m}, M\in\mathbb{R}^{m\times m}$, it can be verified that

$$\begin{aligned} T(\Delta) &= T_1+\mathcal{H}\Big((T_2\Delta)T_3+(T_2\Delta)EX^T\hat{F}^T(T_2\Delta)^T\Big) \\ &= \text{-}\Big((T_2\Delta)GT_2^T + T_2G^T(T_2\Delta)^T\Big) \\ &\quad +\Big((T_2\Delta)2S(T_2\Delta)^T\text{-}(T_2\Delta)ST_2^T\text{-}\,T_2S(T_2\Delta)^T\Big) \\ &\quad \text{-}\Big((T_2\Delta)T_2^TR+R^TT_2(T_2\Delta)^T\text{-}(R+R^T)\Big) \\ &\quad +(T_2\Delta)(\hat{M}+\hat{M}^T)(T_2\Delta)^T\text{ - }(M+M^T) \\ &\quad +\begin{bmatrix} I & (T_2\Delta) \end{bmatrix}\mathcal{L}\begin{bmatrix} I \\ (T_2\Delta)^T \end{bmatrix}. \end{aligned} \tag{5.14}$$

where $\mathcal{L}$ is the matrix in (5.13). Furthermore, it follows from the characterization of $\boldsymbol{\Delta}$ in (5.12) and Theorem 5.3 that each of the first four terms on the right-hand-side of the second equality in (5.14) are negative semidefinite or zero for all $\Delta\in\boldsymbol{\Delta}$, $G\in\mathbb{G}, S\in\mathbb{S}, R\in\mathbb{R}^{m\times m}$ and $M\in\mathbb{R}^{m\times m}$. This shows that $T(\Delta)\prec 0$ if $\mathcal{L}\prec 0$ and proves the result. □

Remark 5.4. *Note that in Theorem 5.4:*

- *the slack variable G captures the structure constraint that $\Delta_i=\alpha_i I$.*
- *the slack variable R captures the constraint $\sum_{i=1}^{N}\alpha_i=1$.*
- *the slack variable S captures the constraint $\alpha_i^2\le\alpha_i$.*
- *the slack variable M captures the redundant constraint $(\sum_{i=1}^N\alpha_i)^2=\sum_{i,j=1}^N\alpha_i\alpha_j=1$.*

Remark 5.5. *Since α has been lifted, Theorem 5.4 gives LMI sufficient conditions for Problem 5.2, and therefore Problem 5.1 because T_2, E, and $\hat{F}$ are not variables. We can also add more slack variables to capture other redundant properties. This would generally give a less conservative sufficient condition at the expense of introducing more variables in the problem; however, this will not be pursued here.*

5.4 Numerical examples

In this section, we give examples to demonstrate the effectiveness of our proposed results for the computation of the guaranteed cost on the H_∞-norm for both continuous-time and discrete-time systems.

5.4.1 Example 1

Consider a continuous-time uncertain spring-mass system presented in Section 2.5.2.

With a computed stabilizing state feedback law $u(t) = Kx(t)$, where K = [-9.9420 -0.9935 -29.5539 - 16.0522] from [29], the closed-loop system is Hurwitz stable. The comparison of the guaranteed cost γ on the closed-loop H_∞-norm obtained with different methods are given in Table 5.1

Method	γ
Quadratic method [5]	infeasible
[14]	1.2739
Lemma 1 of [15]	1.2251
Lemma 5.3 [15]	1.1390
Theorem 5.1	1.2210
Theorem 5.4	1.1390

Table 5.1 The minimum guaranteed cost on H_∞-norm comparisons for Example 1.

The computation results show that Theorem 5.1 provides a relative improvement with respect to Lemma 1 of [15], and Theorem 5.4 gives the same γ as Lemma 5.3 of [15], which is better than the ones obtained by other methods.

5.4.2 Example 2

We consider a randomly generated Hurwitz stable uncertain continuous-time system with a larger dimension of the state ($n = 6$) and all system matrices parameter-varying with the corresponding vertex matrices given by

$$A_1 = \begin{bmatrix} -1.05 & -0.63 & -9.96 & -3.44 & -1.57 & 3.28 \\ -1.48 & -5.51 & 11.68 & 5.05 & 9.08 & -7.54 \\ 2.56 & -2.06 & -17.35 & -5.21 & -4.34 & 5.72 \\ -6.14 & 3.32 & 2.89 & -3.00 & 3.88 & -5.21 \\ 7.52 & 3.05 & -6.85 & -1.51 & -6.71 & -0.35 \\ 3.46 & -0.12 & -6.18 & -1.79 & 3.42 & -7.99 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -4.73 & 5.55 & 25.93 & -9.37 & -19.70 & -17.93 \\ 3.64 & -4.42 & 4.95 & 0.32 & -14.06 & -2.36 \\ -1.06 & 2.74 & 8.18 & -2.20 & -9.53 & -7.60 \\ -2.62 & 5.21 & 11.29 & -6.25 & -8.46 & -9.20 \\ -1.40 & 2.58 & 7.39 & -3.23 & -7.88 & -5.26 \\ -0.81 & 2.29 & 12.85 & -1.14 & -12.78 & -10.73 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.51 & -1.53 & 0.36 & 1.65 & 0.27 & 0.38 \end{bmatrix}^T,$$

$$B_2 = \begin{bmatrix} 0.26 & -1.35 & -0.73 & -2.24 & 0.03 & 0.11 \end{bmatrix}^T,$$

$$C_1 = \begin{bmatrix} 1.69 & -1.13 & -0.27 & -0.92 & 0.60 & -0.89 \\ -1.15 & -0.26 & 0.31 & 1.05 & -1.14 & -0.39 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0.28 & -0.89 & -1.65 & 0.95 & -0.45 & -0.51 \\ 1.06 & 0.51 & 0.69 & -0.44 & -1.48 & -0.85 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 0.46 \\ 0.78 \end{bmatrix}, \ D_2 = \begin{bmatrix} 0.17 \\ 0.36 \end{bmatrix}.$$

Method	γ
Quadratic method [5]	infeasible
[14]	32.6566
Lemma 1 of [15]	16.0967
Lemma 5.3 [15]	12.8433
Theorem 5.1	14.1262
Theorem 5.4	9.6864

Table 5.2 The minimum guaranteed cost on H_∞-norm comparisons for Example 2.

Again, the quadratic method fails to compute a feasible solution due to its conservatism. [14] give the minimum γ of 32.6566. Theorem 5.1 clearly present a reduction of conservatism with respect to Lemma 1 of [15]. Moreover, Lemma 5.3 [15] provides the second minimum γ of 12.8433 in this example, but the best value of 9.6864 is obtained by Theorem 5.4, which reveals a noticeable improvement compared to all other available methods.

5.4.3 Example 3

A discrete-time system with four states and two vertices is investigated in this example. An uncertain system state matrix $A(\alpha)$ is given by the convex hull of $(\epsilon A_1, \epsilon A_2)$, where ϵ is a scalar parameter and A_1, A_2 are randomly generated with the spectral radius of each equal to 0.8, that is

$$A_1 = \begin{bmatrix} 0.0732 & -0.3980 & 0.6317 & 0.1239 \\ 0.9868 & -0.4464 & 0.2246 & 0.0055 \\ 0.1627 & -0.3778 & -0.0022 & 0.8056 \\ -0.5713 & 0.4413 & 0.3572 & 0.3896 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.3636 & 0.4750 & -0.5380 & -0.3809 \\ 0.3228 & -0.4738 & 0.3318 & -0.0934 \\ 0.5862 & 0.3416 & 0.1713 & 0.0476 \\ 1.0538 & 0.2286 & 0.2414 & -0.2216 \end{bmatrix},$$

and other system matrices are given by $B_1 = B_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$, $C_1 = C_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$, $D_1 = D_2 = 0$. Firstly, we determine the maximum value of ϵ such that H_∞-norm guaranteed cost computation is feasible by quadratic method [5], Oliveira et al. [11], Theorem 5.2, and Theorem 5.4 for discrete-time case.

Method	Quadratic [5]	[11]	Theorem 5.2	Theorem 5.4
ϵ_{max}	1.02	1.20	1.23	1.24

Table 5.3 The maximum value of ϵ for which feasibility is achieved for Example 3.

From Table 5.3, we can find that quadratic method gives the most conservative robust stability margin with $\epsilon_{max} = 1.02$, the best result is provided by Theorem 5.4 with $\epsilon_{max} = 1.24$. Note that the exact value of the robust stability margin is $\epsilon = 1.25$ since the spectral radius of A_1 and A_2 is exactly one in that case.

We next compare the H_∞-norm guaranteed cost obtained by these four methods for each $0.01 \leq \epsilon \leq 1.24$ with steps of 0.01. The results for a few values of ϵ are displayed in Table 5.4, where the symbol - denotes no feasible solution. Through exhaustive testings, it

ϵ	Quadratic [5]	[11]	Theorem 5.2	Theorem 5.4
0.19	0.2103	0.2103	0.2103	0.2103
0.71	1.3393	1.1066	1.1066	1.1066
1.15	–	11.5136	7.6799	7.6799
1.20	–	195.6283	18.0938	15.6493
1.23	–	–	111.0862	39.5474
1.24	–	–	–	79.4004

Table 5.4 The minimum guaranteed cost on H_∞-norm comparisons for each ϵ of Example 3.

can be verified that all four methods give the same γ for $0.01 \leq \epsilon \leq 0.19$, e.g. $\gamma = 0.2103$ for $\epsilon = 0.19$. For $0.20 \leq \epsilon \leq 0.71$, [11], Theorem 5.2, and Theorem 5.4 yields the same γ that is less conservative than quadratic results. Theorem 5.2 and Theorem 5.4 start to outperform [11] for $\epsilon \geq 0.72$, and they still give the same results until $\epsilon = 1.15$. Finally, when ϵ belongs to the interval $[1.16\ 1.24]$, Theorem 5.4 provides the best results with guaranteed cost always smaller than all other three methods, and it can also find feasible solutions when other methods fail to do so.

5.5 Summary

In conclusion, this chapter has investigated the problem of H_∞-norm guaranteed cost computation for linear time-invariant polytopic systems. We proposed a generalized problem (Problem 5.1) based on a unified parameter-dependent inequality to compute H_∞-norm guaranteed cost for both continuous-time and discrete-time systems. We then developed the S-Procedure to lift the uncertainties and presented Theorem 5.4 that gives sufficient LMI conditions for the solution of Problem 5.1. Numerical examples have demonstrated that Theorem 5.4 can provide no more conservative results than the ones obtained by the previous methods from the literature.

Chapter 6

Computation of invariant tubes for robust output feedback Model Predictive Control

Model predictive control (MPC) has been widely used in the process industry due mainly to its ability for handling hard constraints compared to other conventional control algorithms. MPC is a form of control scheme which solves an on-line optimization problem to yield a sequence of control inputs at each sampling instant, and only the first control element is implemented. At the next sampling instant, a new sequence of control inputs is computed again. However, in real life, processes often involve additive disturbances and/or model dynamics uncertainties. MPC algorithms that deal with such disturbances/uncertainties within their optimization are called Robust MPC schemes. Most of the Robust MPC algorithms available in the literature can be classified into two categories: open-loop MPC and feedback MPC. Open-loop MPC considers the future control inputs as a function of the current state only, which is computationally efficient but very conservative in general. Feedback MPC gives a less conservative design, in which the future inputs are considered as a function of the future predicted states. However, the computational burden of feedback MPC is excessive and therefore, it is not suitable for fast dynamic systems. Many authors mitigate this problem by working with fast MPC algorithms, e.g., explicit MPC based on the lookup table [65], the primal barrier method [66] [67], and the tube-based MPC algorithm [49] [68] [69].

Tube-based MPC algorithm yields a tube and an associated controller that ensure the controlled state trajectories lie in a tube in the presence of uncertainty. The center of the tube is obtained by solving the on-line (disturbance-free) nominal MPC problem. The computational complexity of tube MPC is linear in horizon length, rather than exponential increase as in conventional RMPC algorithms. Accurate off-line calculation of the tube is an essential step for implementing tube MPC, which is the main focus of this chapter. Details are given in the sequel.

This chapter is is organized as follows. Section 6.1 details a description of the problem, formulates the conditions for the computation of the invariant sets of the estimation and control errors. Section 6.2 and Section 6.3 give initial computation for the invariant sets of the estimation and control errors, respectively. We propose a novel result based on the Newton-like update to optimize the volume of these two sets simultaneously in Section 6.4. We give numerical examples in Section 6.5 to compare our proposed results with [52] and summarize this chapter in Section 6.6.

The results presented in this chapter are based on our paper [70] and the associated contributions are highlighted as below:

- Propose a Newton-like approach to iteratively optimize the volume of invariant sets for the estimation and control errors simultaneously. Our approach is less conservative than the method from previous work that optimizes these two invariant sets separately.

6.1 Problem description

We consider the following linear discrete-time system with additive disturbance:

$$
\begin{aligned}
x^+ &= Ax + Bu + B_d d, \\
y &= Cx + Du + D_v v,
\end{aligned}
$$

where x, $x^+ \in \mathbb{R}^n$, $u \in \mathbb{R}^{n_u}$, $d \in \mathbb{R}^{n_d}$, $v \in \mathbb{R}^{n_v}$, $y \in \mathbb{R}^{n_y}$ are the current state, successor state, control input, process noise, measurement noise and current output, respectively; all other symbols denote the appropriate distribution matrices. We combine the input and output noises as one augmented variable w, yielding the following dynamics with some redefinitions:

$$
\begin{aligned}
x^+ &= Ax + Bu + B_w w, \\
y &= Cx + Du + D_w w, \\
B_w := \begin{bmatrix} B_d & 0 \end{bmatrix}, \quad D_w &:= \begin{bmatrix} 0 & D_v \end{bmatrix}, \quad w := \begin{bmatrix} d \\ v \end{bmatrix}.
\end{aligned}
\tag{6.1}
$$

We assume that (A, B) is controllable and (A, C) is observable. The state and input constraint sets are assumed to have the form:

$$
\begin{aligned}
\mathcal{X} &= \left\{ x \in \mathbb{R}^n \mid \underline{x} \leq V_x x \leq \overline{x} \right\}, \; V_x \in \mathbb{R}^{m \times n}, \, \underline{x}, \, \overline{x} \in \mathbb{R}^m, \\
\mathcal{U} &= \left\{ u \in \mathbb{R}^{n_u} \mid \underline{u} \leq V_u u \leq \overline{u} \right\}, \; V_u \in \mathbb{R}^{m_u \times n_u}, \, \underline{u}, \, \overline{u} \in \mathbb{R}^{m_u}.
\end{aligned}
$$

The augmented disturbance w belongs to the bounded and symmetric polytope:

$$
\mathcal{W} = \left\{ w \in \mathbb{R}^{n_w} \mid - \overline{w} \leq V_w w \leq \overline{w} \right\}, \; V_w \in \mathbb{R}^{m_w \times n_w}, \overline{w} \in \mathbb{R}^{m_w}.
$$

Furthermore, a simple Luenberger observer is employed to estimate the state:

$$
\begin{bmatrix} \hat{x}^+ \\ \hat{y} \end{bmatrix} = \begin{bmatrix} A & B & L \\ C & D & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ u \\ y - \hat{y} \end{bmatrix}, \tag{6.2}
$$

where $\hat{x} \in \mathbb{R}^n$ is the current observer state, $\hat{x}^+ \in \mathbb{R}^n$ is the successor state of the estimated system, $\hat{y} \in \mathbb{R}^{n_y}$ is the current observer output, and $L \in \mathbb{R}^{n \times n_y}$ is the Luenberger observer gain. We define the state estimation error $\tilde{x} := x - \hat{x}$, whose dynamics from (6.1) and (6.2) are given by:

$$
\tilde{x}^+ = (A - LC)\tilde{x} + (B_w - LD_w)w,
$$

where L satisfies $\rho(A - LC) < 1$ and $\rho(\cdot)$ denotes the spectral radius. The tube based MPC controller is implemented on the associated nominal system ([50]), which is obtained from (6.1) by neglecting the disturbance w:

$$
\overline{x}^+ = A\overline{x} + B\overline{u},
$$

where $\overline{x}$, $\overline{x}^+ \in \mathbb{R}^n$, $\overline{u} \in \mathbb{R}^{n_u}$ are the current state, successor state, and the control input of the nominal system, respectively. The control input is given by:

$$u = \overline{u} + K(\hat{x} - \overline{x}),$$

where $K \in \mathbb{R}^{n_u \times n}$ is the feedback gain, which satisfies $\rho(A + BK) < 1$. The error between the observer and nominal states, called the control error, is defined as $\xi := \hat{x} - \overline{x}$; its dynamics are given by:

$$\xi^+ = (A + BK)\xi + LC\tilde{x} + LD_w w. \tag{6.3}$$

We follow the standard definitions ([71];[40]) for robust positively invariant set.

Definition 6.1. *A set* $\Omega \subset \mathbb{R}^n$ *is robust positively invariant for the system* $x^+ = f(x, w)$ *and the constraint set* $(\mathcal{X}, \mathcal{W})$ *if* $\Omega \subseteq \mathcal{X}$ *and* $x^+ = f(x, w) \in \Omega$, $\forall w \in \mathcal{W}, \forall x \in \Omega$.

Then the polytopic invariant sets for the estimation error $\mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}})$ and the control error $\mathcal{P}(P_\xi, b_\xi)$ can be defined by:

$$\left.\begin{matrix} \tilde{x} \in \mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}}) \\ w \in \mathcal{W} \end{matrix}\right\} \Rightarrow \tilde{x}^+ \in \mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}}), \tag{6.4}$$

$$\left.\begin{matrix} \xi \in \mathcal{P}(P_\xi, b_\xi) \\ w \in \mathcal{W} \\ \tilde{x} \in \mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}}) \end{matrix}\right\} \Rightarrow \xi^+ \in \mathcal{P}(P_\xi, b_\xi), \tag{6.5}$$

where

$$\mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}}) = \left\{\tilde{x} \in \mathbb{R}^n : -b_{\tilde{x}} \le P_{\tilde{x}}\tilde{x} \le b_{\tilde{x}}\right\}, \tag{6.6}$$

$$\mathcal{P}(P_\xi, b_\xi) = \left\{\xi \in \mathbb{R}^n : -b_\xi \le P_\xi \xi \le b_\xi\right\}, \tag{6.7}$$

and $P_{\tilde{x}}, P_\xi \in \mathbb{R}^{m \times n}$ and $b_{\tilde{x}}, b_\xi \in \mathbb{R}^m$ are decision variables for the structure of the invariant set. By definition, the actual state differs from the nominal state by the estimation error $\tilde{x}$ and control error ξ, so that:

$$x = \overline{x} + \xi + \tilde{x}.$$

Similarly, the difference between the actual control input and nominal input is given by $K\xi$:

$$u = \overline{u} + K\xi.$$

We assume that the initial values of estimation and control errors belong to their respective RCI sets, $\xi(0) \in \mathcal{P}(P_\xi, b_\xi)$ and $\tilde{x}(0) \in \mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}})$. The original state and control input constraints are satisfied for all $w \in \mathcal{W}$ if

$$\overline{x} \in \overline{\mathcal{X}} := \mathcal{X} \ominus \mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}}) \ominus \mathcal{P}(P_{\xi}, b_{\xi}),$$
$$\overline{u} \in \overline{\mathcal{U}} := \mathcal{U} \ominus K\mathcal{P}(P_{\xi}, b_{\xi}).$$

Therefore, we can choose the initial nominal state $\overline{x}(0)$ and the nominal control input $\overline{u}$ to ensure that the actual (unknown) state and control input always satisfy the original constraints. In this way, the original constraints $\mathcal{X}$ and $\mathcal{U}$ are tightened by $\mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}})$ and $\mathcal{P}(P_{\xi}, b_{\xi})$. We next establish the conditions such that the sets $\mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}})$ and $\mathcal{P}(P_{\xi}, b_{\xi})$ are circumscribed by outer bounding ellipsoids $\mathcal{Q}(Q_{\tilde{x}})$ and $\mathcal{Q}(Q_{\xi})$, respectively,

$$\exists Q_{\tilde{x}} \in \mathcal{S}_{+}^{n} : \mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}}) \subseteq \mathcal{Q}(Q_{\tilde{x}}), \tag{6.8}$$

$$\exists Q_{\xi} \in \mathcal{S}_{+}^{n} : \mathcal{P}(P_{\xi}, b_{\xi}) \subseteq \mathcal{Q}(Q_{\xi}). \tag{6.9}$$

Since the volume of $\mathcal{Q}(Q_{\tilde{x}})$ is proportional to the determinant of the matrix $Q_{\tilde{x}}^{-\frac{1}{2}}$, the term $log\ detQ_{\tilde{x}}^{-1}$ is adopted as the objective function to minimize the volume of the set $\mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}})$; similarly, we use $log\ detQ_{\xi}^{-1}$ as the objective function for $\mathcal{P}(P_{\xi}, b_{\xi})$. Combining the invariance and outer bounding conditions for the invariant sets of the estimation and control errors, respectively, we can present the following problems to optimize the volume of $\mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}})$ and $\mathcal{P}(P_{\xi}, b_{\xi})$, respectively:

$$\begin{aligned} \min_{P_{\tilde{x}}, b_{\tilde{x}}, L, Q_{\tilde{x}}} \quad & log\ detQ_{\tilde{x}}^{-1} \\ s.t. \quad & (6.4), (6.8). \end{aligned} \tag{6.10}$$

$$\begin{aligned} \min_{P_{\xi}, b_{\xi}, K, Q_{\xi}} \quad & log\ detQ_{\xi}^{-1} \\ s.t. \quad & (6.5), (6.9). \end{aligned} \tag{6.11}$$

6.2 Initial computation for the invariant set of the estimation error

In this section, we first derive necessary and sufficient conditions, in the form of nonlinear matrix inequalities (NLMIs), for the existence of an admissible triple $(P_{\tilde{x}}, b_{\tilde{x}}, L)$ for problem (6.10) by using Farkas' Theorem ([72]). Subsequently, the corresponding sufficient conditions in the form of LMIs are given by the use of the following result, which is deduced from the Elimination Lemma.

Lemma 6.1. *[47]: Let $R \in \mathcal{S}^n$, $E \in \mathbb{R}^{n\times p}$, $F \in \mathbb{R}^{p\times m}$, and $Z \in \mathcal{S}^m$. Consider the following two statements:*

$$(i) \quad \begin{bmatrix} R & EF \\ \star & Z \end{bmatrix} \succ 0, \tag{6.12}$$

$$(ii) \quad \exists Y \in \mathcal{Y}: \quad \begin{bmatrix} R & EY & 0 \\ \star & Y + Y^T & F \\ \star & \star & Z \end{bmatrix} \succ 0. \tag{6.13}$$

Then $(ii) \Rightarrow (i)$ if $\mathcal{Y} \subseteq \mathbb{R}^{p\times p}$ and $(ii) \Leftrightarrow (i)$ if $\mathcal{Y} = \mathbb{R}^{p\times p}$.

Theorem 6.1. *The invariance and outer bounding conditions for the invariant set of the estimation error are satisfied if, $\forall i \in \mathcal{N}_m$, there exist $D_i \in \mathcal{D}_+^m$, $W_i \in \mathcal{D}_+^{m_w}$, $\overline{D}_{\tilde{x}} \in \mathcal{D}_+^m$ and $Q_{\tilde{x}} \in \mathcal{S}_+^n$ such that*

$$L_{\tilde{x}} := \begin{bmatrix} \Delta_{11}^i & e_i^T P_{\tilde{x}} B_w^L & e_i^T P_{\tilde{x}} A^L \\ \star & V_w^T W_i V_w & 0 \\ \star & \star & P_{\tilde{x}}^T D_i P_{\tilde{x}} \end{bmatrix} \succ 0, \tag{6.14}$$

$$P_{\tilde{x}}^T \overline{D}_{\tilde{x}} P_{\tilde{x}} - Q_{\tilde{x}} \succ 0, \quad 1 - b_{\tilde{x}}^T \overline{D}_{\tilde{x}} b_{\tilde{x}} > 0, \tag{6.15}$$

where $\Delta_{11}^i = 2e_i^T b_{\tilde{x}} - b_{\tilde{x}}^T D_i b_{\tilde{x}} - \overline{w}^T W_i \overline{w}$, $A^L = A - LC$, and $B_w^L = B_w - LD_w$.

Proof. The proof of (6.14) is an application of Farkas' Theorem. Follow the definition of $\mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}})$ in (6.6), the invariance condition (6.4) is equivalent to

$$\left.\begin{array}{l} -b_{\tilde{x}} \le P_{\tilde{x}}\tilde{x} \le b_{\tilde{x}} \\ -\overline{w} \le V_w w \le \overline{w} \end{array}\right\} \Rightarrow -b_{\tilde{x}} \le P_{\tilde{x}}(A^L\tilde{x} + B_w^L w) \le b_{\tilde{x}}. \tag{6.16}$$

Considering the symmetry of the sets $\mathcal{W}$ and $\mathcal{P}$, the last inequality in (6.16) can be written as

$$2e_i^T(P_{\tilde{x}}(A^L\tilde{x} + B_w^L w) - b_{\tilde{x}}) \le 0, \forall i \in \mathcal{N}_m.$$

For any $D_i \in \mathcal{D}_+^m$ and $W_i \in \mathcal{D}_+^{m_w}$, $\forall i \in \mathcal{N}_m$, it can be verified that

$$\begin{aligned} 2e_i^T(P_{\tilde{x}}(A^L\tilde{x} + B_w^L w) - b_{\tilde{x}}) = & - (V_w w + \overline{w})^T W_i(\overline{w} - V_w w) \\ & - (b_{\tilde{x}} - P_{\tilde{x}}\tilde{x})^T D_i (P_{\tilde{x}}\tilde{x} + b_{\tilde{x}}) \\ & - g^T L_{\tilde{x}} g, \end{aligned} \tag{6.17}$$

where $L_{\tilde{x}}$ is defined in (6.14) and $g^T := \begin{bmatrix} \text{-}1 & w^T & \tilde{x}^T \end{bmatrix}$. Since the first and second terms on the RHS of (6.17) are nonpositive for all $\tilde{x} \in \mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}})$ and $w \in \mathcal{W}$, the invariance condition is satisfied if $L_{\tilde{x}} \succ 0$, which gives (6.14).

Similarly, the outer bounding condition (6.8) is equivalent to

$$\text{-}\, b_{\tilde{x}} \leq P_{\tilde{x}}\tilde{x} \leq b_{\tilde{x}} \Rightarrow \tilde{x} Q_{\tilde{x}} \tilde{x} \leq 1. \tag{6.18}$$

For any $\overline{D}_{\tilde{x}} \in \mathcal{D}_+^m$ and $Q_{\tilde{x}} \in \mathcal{S}_+^n$, we have

$$\begin{aligned} \tilde{x} Q_{\tilde{x}} \tilde{x}\text{-}1 &= \text{-}(b_{\tilde{x}}\text{ - }P_{\tilde{x}}\tilde{x})^T \overline{D}_i (P_{\tilde{x}}\tilde{x} + b_{\tilde{x}}) \\ &\quad \text{-}\begin{bmatrix} \text{-}1 & \tilde{x}^T \end{bmatrix} \underbrace{\begin{bmatrix} 1\text{-}b_{\tilde{x}}^T \overline{D}_{\tilde{x}} b_{\tilde{x}} & 0 \\ 0 & P_{\tilde{x}}^T \overline{D}_{\tilde{x}} P_{\tilde{x}}\text{-}Q_{\tilde{x}} \end{bmatrix}}_{\overline{L}_{\tilde{x}}} \begin{bmatrix} \text{-}1 \\ \tilde{x} \end{bmatrix}. \end{aligned}$$

It is clear that since the first term on the RHS of the above equality is nonpositive for all $\tilde{x} \in \mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}})$, the outer bounding condition is satisfied if $\overline{L}_{\tilde{x}} \succ 0$, which gives (6.15). □

As can be seen from (6.14) and (6.15), the nonlinearity terms include $P_{\tilde{x}} B_w^L$, $P_{\tilde{x}} A^L$, $P_{\tilde{x}}^T D_i P_{\tilde{x}}$, $b_{\tilde{x}}^T D_i b_{\tilde{x}}$, $P_{\tilde{x}}^T \overline{D}_{\tilde{x}} P_{\tilde{x}}$, $b_{\tilde{x}}^T \overline{D}_{\tilde{x}} b_{\tilde{x}}$. In order to deal with these nonlinearities, we next propose an initial full-complexity outer approximation to the minimal RCI set, such that

$$\mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}}) = \mathcal{P}(P_r X_{\tilde{x}}, b_r) = \left\{ x \in \mathbb{R}^n \colon \text{-}b_r \leq P_r X_{\tilde{x}} x \leq b_r \right\},$$

where P_r and b_r are given, and $X_{\tilde{x}} \in \mathbb{R}^{n \times n}$ is a variable to rotate and scale the polyhedral set defined by P_r (see the work of [47] for details). The next result uses Lemma 6.1 and a congruence transformation to derive sufficient conditions, in the form of LMIs, for computing an admissible triple $(P_{\tilde{x}}, b_{\tilde{x}}, L)$.

Theorem 6.2. *With all variables as defined in Theorem 6.1, let $P_{\tilde{x}} = P_r X_{\tilde{x}}$ and $b_{\tilde{x}} = b_r$ and define $\hat{L} = X_{\tilde{x}} L$. The NLMIs of (6.14) and (6.15) are satisfied if, $\forall i \in \mathcal{N}_m$, there exist $\hat{D}_i \in \mathcal{D}_+^m$, $\hat{W}_i \in \mathcal{D}_+^{m_w}$, $\overline{D}_{\tilde{x}} \in \mathcal{D}_+^m$, $Q_{\tilde{x}} \in \mathcal{S}_+^n$ and $\lambda_i > 0$, such that*

$$\begin{bmatrix} \Gamma_{11}^i & e_i^T P_r \hat{B} & e_i^T P_r \hat{A} & 0 & 0 \\ \star & 2I_{n_w} & 0 & \lambda_i I_{n_w} & 0 \\ \star & \star & X_{\tilde{x}} + X_{\tilde{x}}^T & 0 & \lambda_i I_n \\ \star & \star & \star & V_w^T \hat{W}_i V_w & 0 \\ \star & \star & \star & \star & P_r^T \hat{D}_i P_r \end{bmatrix} \succ 0, \tag{6.19}$$

$$\begin{bmatrix} X_{\tilde{x}} + X_{\tilde{x}}^T - Q_{\tilde{x}} & I_n \\ \star & P_r^T \overline{D}_{\tilde{x}} P_r \end{bmatrix} \succ 0, \; 1 - b_r^T \overline{D}_{\tilde{x}} b_r > 0, \tag{6.20}$$

where $\Gamma_{11}^i = 2\lambda_i e_i^T b_r - b_r^T \hat{D}_i b_r - \overline{w}^T \hat{W}_i \overline{w}$*,* $\hat{A} = X_{\tilde{x}} A - \hat{L} C$*, and* $\hat{B} = X_{\tilde{x}} B_w - \hat{L} D_w$*.*

Proof. Substituting $P_{\tilde{x}} = P_r X_{\tilde{x}}$ and $b_{\tilde{x}} = b_r$ shows that (6.14) can be rewritten as (6.12) with

$$\left[\begin{array}{c|c} R & E \\ \hline F & Z \end{array}\right] =$$

$$\left[\begin{array}{c|c} 2e_i^T b_r - b_r^T D_i b_r - \overline{w}^T W_i \overline{w} & e_i^T P_r X_{\tilde{x}} \begin{bmatrix} B_w^L & A^L X_{\tilde{x}}^{-1} \end{bmatrix} \\ \hline \begin{bmatrix} I_{n_w} & 0 \\ 0 & X_{\tilde{x}} \end{bmatrix} & \begin{bmatrix} V_w^T W_i V_w & 0 \\ 0 & X_{\tilde{x}}^T P_r^T D_i P_r X_{\tilde{x}} \end{bmatrix} \end{array}\right].$$

Applying Lemma 6.1 with $Y = \lambda_i^{-1} \begin{bmatrix} I_{n_w} & 0 \\ 0 & X_{\tilde{x}} \end{bmatrix}$, then effecting the congruence $diag(\lambda_i^{\frac{1}{2}}, \lambda_i^{\frac{1}{2}} I_{n_w}, \lambda_i^{\frac{1}{2}} I_n, \lambda_i^{\frac{1}{2}} I_{n_w}, \lambda_i^{\frac{1}{2}} X_{\tilde{x}}^{-T})$ implies that (6.19) is a sufficient condition of (6.14), with the following redefinitions:

$$\hat{D}_i = \lambda_i D_i, \quad \hat{W}_i = \lambda_i W_i.$$

For the first inequality of (6.15), substituting $P_{\tilde{x}}$ with $P_r X_{\tilde{x}}$, followed by applying the congruence $X_{\tilde{x}}^{-T}$ and applying a Schur complement argument gives the following equivalent inequality

$$\begin{bmatrix} X_{\tilde{x}} Q_{\tilde{x}}^{-1} X_{\tilde{x}}^T & I_n \\ \star & P_r^T \overline{D}_{\tilde{x}} P_r \end{bmatrix} \succ 0. \tag{6.21}$$

Using the identity

$$X_{\tilde{x}} Q_{\tilde{x}}^{-1} X_{\tilde{x}}^T = X_{\tilde{x}} + X_{\tilde{x}}^T - Q_{\tilde{x}} + (X_{\tilde{x}} - Q_{\tilde{x}})^T Q_{\tilde{x}}^{-1} (X_{\tilde{x}} - Q_{\tilde{x}}),$$

the (1,1) block of (6.21) can be replaced with the first three terms on the right of the above identity since its last term is nonnegative. This gives the first inequality of (6.20). For the second inequality in (6.15), replacing $b_{\tilde{x}}$ by b_r gives (6.20) directly. □

Remark 6.1. *While the feasibility of the LMI problem is not guaranteed by using arbitrary* P_r *and* b_r*, in practice, we found that using the vector of ones for* b_r *and the regular polytope with* $2m$ *faces for* P_r *can usually result in a feasible solution, although this may introduce some conservatism to Theorem 6.2. Note also that the degree of freedom in the choice of* m *provides flexibility in the shape of the RCI set, which provides additional accuracy of* *expressing the set. In general, guaranteeing the existence of an initial feasible RCI set is difficult ([71]). However, Theorem 4 in [48] provides a choice of the initial RCI set that is guaranteed to be feasible under certain conditions.*

In conclusion, the initial computation for the invariant set of the estimation error can be posed as the convex semidefinite program

$$\begin{aligned} \min_{X_{\tilde{x}},\hat{L},\hat{D}_i,\hat{W}_i,\overline{D}_{\tilde{x}},Q_{\tilde{x}},\lambda_i} \quad & log\ detQ_{\tilde{x}}^{-1} \\ s.t. \quad & (6.19),(6.20). \end{aligned} \tag{6.22}$$

6.3 Initial computation for the invariant set of the control error

In the last section, an initial admissible triple $(P_{\tilde{x}}, b_{\tilde{x}}, L)$ for the invariant set of the estimation error has been obtained. For the given invariant set $\mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}})$, we propose to re-parameterize the estimation error $\tilde{x}$ as an artificial disturbance by augmenting the dynamics of control error ξ in (6.3), such that

$$\xi^+ = \underbrace{(A+BK)}_{A^K}\xi + \underbrace{\begin{bmatrix} LD_w & LC \end{bmatrix}}_{B_\eta}\underbrace{\begin{bmatrix} w \\ \tilde{x} \end{bmatrix}}_{\eta}. \tag{6.23}$$

The new disturbance η belongs to be an extended polytope,

$$\eta \in \mathcal{W}^\eta := \left\{\eta \in \mathbb{R}^{n_w+n} \mid -\overline{\eta} \leq V_\eta \eta \leq \overline{\eta}\right\},$$

with the following redefinitions:

$$V_\eta = \begin{bmatrix} V_w & 0 \\ 0 & P_{\tilde{x}} \end{bmatrix}, \quad \overline{\eta} = \begin{bmatrix} \overline{w} \\ b_{\tilde{x}} \end{bmatrix}.$$

We next propose the corresponding conditions for the initial computation of the admissible triple (P_ξ, b_ξ, K) by using Farkas' Theorem.

Theorem 6.3. *The invariance and outer bounding conditions for the invariant set of the control error are satisfied if and only if,* $\forall i \in \mathcal{N}_m$*, there exist* $D_\xi^i \in \mathcal{D}_+^m$*,* $W_\eta^i \in \mathcal{D}_+^{m_w+m}$*,*

$\overline{D}_\xi \in \mathcal{D}_+^m$ and $Q_\xi \in \mathcal{S}_+^n$, such that

$$\begin{bmatrix} 2e_i^T b_\xi - b_\xi^T D_\xi^i b_\xi - \overline{\eta}^T W_\eta^i \overline{\eta} & e_i^T P_\xi B_\eta & e_i^T P_\xi A^K \\ \star & V_\eta^T W_\eta^i V_\eta & 0 \\ \star & \star & P_\xi^T D_\xi^i P_\xi \end{bmatrix} \succ 0, \tag{6.24}$$

$$P_\xi^T \overline{D}_\xi P_\xi - Q_\xi \succ 0, \quad 1 - b_\xi^T \overline{D}_\xi b_\xi > 0. \tag{6.25}$$

Proof. The proof is also an application of Farkas' Theorem that is similar to the proof in Theorem 6.1, thus it is omitted here for brevity. □

We also use an initial outer approximation to the minimal RCI set to convert the NLMIs of (6.24) and (6.25) into LMIs using Lemma 6.1 and a congruence transformation.

Theorem 6.4. *With all variables as in Theorem 6.3 let $P_\xi = P_r X_\xi$ and $b_\xi = b_r$ and define $\tilde{X} = X_\xi^{-1}$ and $\hat{K} = K X_\xi^{-1}$. The NLMIs of (6.24) and (6.25) are satisfied if, $\forall i \in \mathcal{N}_m$, there exist $\hat{D}_\xi^i \in \mathcal{D}_+^m$, $\hat{W}_\eta^i \in \mathcal{D}_+^{m_w+m}$, $\overline{D}_\xi \in \mathcal{D}_+^m$, $Q_\xi^{-1} \in \mathcal{S}_+^n$ and $\gamma_i > 0$, such that*

$$\begin{bmatrix} \Lambda_{11}^i & \gamma_i e_i^T P_r & 0 & 0 \\ \star & \tilde{X} + \tilde{X}^T & B_\eta & A\tilde{X} + B\hat{K} \\ \star & \star & V_\eta^T \hat{W}_\eta^i V_\eta & 0 \\ \star & \star & \star & P_r^T \hat{D}_\xi^i P_r \end{bmatrix} \succ 0, \tag{6.26}$$

$$\begin{bmatrix} Q_\xi^{-1} & \tilde{X} \\ \star & P_r^T \overline{D}_\xi P_r \end{bmatrix} \succ 0, \quad 1 - b_r^T \overline{D}_\xi b_r > 0, \tag{6.27}$$

where $\Lambda_{11}^i = 2\gamma_i e_i^T b_r - b_r^T \hat{D}_\xi^i b_r - \overline{\eta}^T \hat{W}_\eta^i \overline{\eta}$.

Proof. Substituting $P_\xi = P_r X_\xi$ and $b_\xi = b_r$ shows that (6.24) can be rewritten as (6.12) with

$$\left[\begin{array}{c|c} R & E \\ \hline F & Z \end{array}\right] = \left[\begin{array}{c|c} 2e_i^T b_r - b_r^T D_\xi^i b_r - \overline{\eta}^T W_\eta^i \overline{\eta} & e_i^T P_r X_\xi \\ \hline \begin{bmatrix} B_\eta & A^K \end{bmatrix} & \begin{bmatrix} V_\eta^T W_\eta^i V_\eta & 0 \\ 0 & X_\xi^T P_r^T D_\xi^i P_r X_\xi \end{bmatrix} \end{array}\right].$$

Applying Lemma 6.1 with $Y = \gamma_i X_\xi^{-1}$ where $0 < \gamma_i \in \mathbb{R}$, then effecting the congruence $diag(\gamma_i^{\frac{1}{2}}, \gamma_i^{-\frac{1}{2}} I_n, \gamma_i^{\frac{1}{2}} I_{n_w}, \gamma_i^{\frac{1}{2}} X_\xi^{-T})$ implies that (6.26) is a sufficient condition for (6.24) with the redefinitions:

$$\hat{D}_\xi^i = \gamma_i D_\xi^i, \quad \hat{W}_\eta^i = \gamma_i W_\eta^i.$$

For the first inequality in (6.25), substituting P_ξ with $P_r X_\xi$ followed by applying the congruence X_ξ^{-T} and applying a Schur complement argument gives the first inequality in (6.27). For the second inequality in (6.25), replacing b_ξ by b_r gives the second term in (6.27) directly. □ □

To summarize, the initial computation for the invariant set of the control error can be posed as the convex semidefinite program

$$\begin{aligned} \min_{\tilde{X}, \hat{K}, \hat{D}_\xi^i, \hat{W}_\eta^i, \overline{D}_\xi, Q_\xi^{-1}, \gamma_i} \quad & trace\ (Q_\xi^{-1}) \\ s.t. \quad & (6.26), (6.27). \end{aligned} \tag{6.28}$$

Remark 6.2. *Since the function $log\ det(Q_\xi^{-1})$ is concave, we minimize an upper bound on $log\ det(Q_\xi^{-1})$ by replacing it with trace(Q_ξ^{-1}).*

Remark 6.3. *Theorems 6.2 and 6.4 give sufficient condition only; the conservatism comes from restricting the structure of Y in Lemma 6.1 to obtain a tractable solution. Necessary and sufficient conditions could be obtained if the structure of Y is free, however, this will result in an intractable solution.*

6.4 Update computation algorithm

In the previous two sections, we proposed the initial computations of the invariant sets of the estimation and control errors by considering L and K as variables separately. Since the linearization algorithm resulting from using Lemma 6.1 gives sufficient condition only, this conservatism leads to the RCI sets being unlikely to be minimal. Therefore, in this section, we propose an update computation algorithm based on the following Newton-like update to obtain approximate minimal RCI sets.

Lemma 6.2. *[48]: Let L, $L_0 \in \mathbb{R}^{m \times n}$ and D, $D_0 \in \mathcal{S}_+^m$. Denote*

$$\begin{aligned} \mathcal{L}_{L,D}^{L_0,D_0} &:= L^T D_0^{-1} L_0 + L_0^T D_0^{-1} L - L_0^T D_0^{-1} D D_0^{-1} L_0 \\ \mathcal{N}_{L,D} &:= L^T D^{-1} L \end{aligned}$$

Then $\mathcal{N}_{L,D} \succeq \mathcal{L}_{L,D}^{L_0,D_0}$ and $\mathcal{N}_{L_0,D_0} = \mathcal{L}_{L_0,D_0}^{L_0,D_0}$. Therefore,

$$\left\{\exists L_0 \in \mathbb{R}^{m\times n}, D_0 \in \mathcal{S}_+^m : \mathcal{N}_{L_0,D_0} \succ 0\right\} \Rightarrow$$
$$\left\{\exists L \in \mathbb{R}^{m\times n}, D \in \mathcal{S}_+^m : \mathcal{N}_{L,D} \succeq \mathcal{L}_{L,D}^{L_0,D_0} \succ 0\right\}.$$

Theorem 6.5. *Let the initial solutions of the invariant sets of the estimation and control errors be denoted as* $(P_{\tilde{x}}^0, b_{\tilde{x}}^0, L_0, D_{\tilde{x}}^{i0}, W_{\tilde{x}}^{i0}, Q_{\tilde{x}}^0, \overline{D}_{\tilde{x}0})$ *and* $(P_{\xi}^0, b_{\xi}^0, K_0, D_{\xi}^{i0}, D_{\tilde{x}\xi}^{i0}, W_{\xi}^{i0}, Q_{\xi}^0, \overline{D}_{\xi 0})$*, which satisfy conditions (6.14), (6.15), (6.25) and (6.34). Then these solutions can be updated if there exist* $P_{\tilde{x}} \in \mathbb{R}^{m\times n}$*,* $b_{\tilde{x}} \in \mathbb{R}^m$*,* $L \in \mathbb{R}^{n\times n_y}$*,* $(D_{\tilde{x}}^i)^{-1} \in \mathcal{D}_+^m$*,* $W_{\tilde{x}}^i \in \mathcal{D}_+^{m_w}$*,* $Q_{\tilde{x}} \in \mathcal{S}_+^n$*,* $(\overline{D}_{\tilde{x}})^{-1} \in \mathcal{D}_+^m$*,* $P_\xi \in \mathbb{R}^{m\times n}$*,* $b_\xi \in \mathbb{R}^m$*,* $K \in \mathbb{R}^{n_u\times n}$*,* $(D_{\tilde{x}\xi}^i)^{-1} \in \mathcal{D}_+^m$*,* $(D_\xi^i)^{-1} \in \mathcal{D}_+^m$*,* $W_\xi^i \in \mathcal{D}_+^{m_w}$*,* $Q_\xi \in \mathcal{S}_+^n$ *and* $(\overline{D}_\xi)^{-1} \in \mathcal{D}_+^m$*,* $\forall i \in \mathcal{N}_m$ *such that*

$$\begin{bmatrix} \mathcal{M}_{\tilde{x}} + \mathcal{L}_{L_{\tilde{x}}^i, F_{\tilde{x}}^i}^{L_{\tilde{x}}^{i0}, F_{\tilde{x}}^{i0}} & \star \\ E_{\tilde{x}} L_{\tilde{x}}^i & I_n \end{bmatrix} \succ 0, \tag{6.29}$$

$$\mathcal{L}_{P_{\tilde{x}}, \overline{D}_{\tilde{x}}^{-1}}^{P_{\tilde{x}}^0, \overline{D}_{\tilde{x}0}^{-1}} - Q_{\tilde{x}} \succ 0, \qquad \begin{bmatrix} \overline{D}_{\tilde{x}}^{-1} & b_{\tilde{x}} \\ \star & 1 \end{bmatrix} \succ 0, \tag{6.30}$$

$$\begin{bmatrix} \mathcal{M}_{\xi} + \mathcal{L}_{L_{\xi}^i, F_{\xi}^i}^{L_{\xi}^{i0}, F_{\xi}^{i0}} & \star \\ E_{\xi} L_{\xi}^i & I_n \end{bmatrix} \succ 0, \tag{6.31}$$

$$\mathcal{L}_{P_{\xi}, \overline{D}_{\xi}^{-1}}^{P_{\xi}^0, \overline{D}_{\xi 0}^{-1}} - Q_{\xi} \succ 0, \qquad \begin{bmatrix} \overline{D}_{\xi}^{-1} & b_{\xi} \\ \star & 1 \end{bmatrix} \succ 0, \tag{6.32}$$

where,

$$E_{\tilde{x}} = \begin{bmatrix} -I_n & I_n & 0 \end{bmatrix}, \quad F_\xi^i = diag(I_n, I_n, (D_{\tilde{x}\xi}^i)^{-1}; (D_\xi^i)^{-1}),$$
$$E_{\xi} = \begin{bmatrix} -I_n & I_n & 0 & 0 \end{bmatrix}, F_{\tilde{x}}^i = diag(I_n, I_n, (D_{\tilde{x}}^i)^{-1}),$$
$$\mathcal{M}_\xi = \begin{bmatrix} (D_{\tilde{x}\xi}^i)^{-1} & 0 & b_{\tilde{x}} & 0 & 0 & 0 \\ \star & (D_\xi^i)^{-1} & b_\xi & 0 & 0 & 0 \\ \star & \star & 2e_i^T b_\xi - \overline{w}^T W_\xi^i \overline{w} & 0 & 0 & 0 \\ \star & \star & \star & V_w^T W_\xi^i V_w & 0 & 0 \\ \star & \star & \star & \star & 0 & 0 \\ \star & \star & \star & \star & \star & 0 \end{bmatrix},$$

$$\mathcal{M}_{\tilde{x}} = \begin{bmatrix} (D^i_{\tilde{x}})^{-1} & b_{\tilde{x}} & 0 & 0 \\ \star & 2e_i^T b_{\tilde{x}} - \overline{w}^T W^i_{\tilde{x}} \overline{w} & 0 & 0 \\ \star & \star & V_w^T W^i_{\tilde{x}} V_w & 0 \\ \star & \star & \star & 0 \end{bmatrix},$$

$$L^i_{\tilde{x}} = \begin{bmatrix} 0 & P_{\tilde{x}}^T e_i & 0 & 0 \\ 0 & 0 & B_w^L & A^L \\ 0 & 0 & 0 & P_{\tilde{x}} \end{bmatrix}, L^i_{\xi} = \begin{bmatrix} 0 & 0 & P_{\xi}^T e_i & 0 & 0 & 0 \\ 0 & 0 & 0 & LD_w & LC & A^K \\ 0 & 0 & 0 & 0 & P_{\tilde{x}} & 0 \\ 0 & 0 & 0 & 0 & 0 & P_{\xi} \end{bmatrix}.$$

Proof. Applying an upper Schur complement on $b_{\tilde{x}}^T D^i_{\tilde{x}} b_{\tilde{x}}$ in (6.14), the following identity can be verified

$$(6.14) \Leftrightarrow \mathcal{M}_{\tilde{x}} + \mathcal{N}_{L^i_{\tilde{x}}, F^i_{\tilde{x}}} - (E_{\tilde{x}} L^i_{\tilde{x}})^T (E_{\tilde{x}} L^i_{\tilde{x}}) \succ 0. \tag{6.33}$$

A subsequent application of Lemma 6.2 on $\mathcal{N}_{L^i_{\tilde{x}}, F^i_{\tilde{x}}}$ in (6.33), followed by a Schur complement on the third term gives (6.29). For the first inequality in (6.15), it can be noted that $\mathcal{N}_{P_{\tilde{x}}, \overline{D}^{-1}_{\tilde{x}}} = P_{\tilde{x}}^T \overline{D}_{\tilde{x}} P_{\tilde{x}}$. Then using Lemma 6.2 on this equality gives the first inequality in (6.30). The second inequality in (6.15) and (6.30) are equivalent by effecting Schur complement directly.

The invariant set of the estimation error is unknown if we want to update these two sets simultaneously. Hence, the invariance condition (6.24) for the invariant set of the control error in Theorem 6.3 needs to be modified by Farkas' Theorem. The invariance condition in (6.5) is equivalent to

$$\left.\begin{array}{l} -b_{\tilde{x}} \le P_{\tilde{x}} \tilde{x} \le b_{\tilde{x}} \\ -b_{\xi} \le P_{\xi} \xi \le b_{\xi} \\ -\overline{w} \le V_w w \le \overline{w} \end{array}\right\} \Rightarrow -b_{\xi} \le P_{\xi}(A^K \xi + LC\tilde{x} + LD_w w) \le b_{\xi}.$$

For any $D^i_{\tilde{x}\xi} \in \mathcal{D}^m_+$, $D^i_{\xi} \in \mathcal{D}^m_+$, $W^i_{\xi} \in \mathcal{D}^{m_w}_+$, $\forall i \in \mathcal{N}_m$,

$$\begin{aligned} & 2e_i^T (P_{\xi}(A^K \xi + LC\tilde{x} + LD_w w) - b_{\xi}) \\ = & -(V_w w + \overline{w})^T W^i_{\xi} (\overline{w} - V_w w) - (b_{\tilde{x}} - P_{\tilde{x}} \tilde{x})^T D^i_{\tilde{x}\xi} (P_{\tilde{x}} \tilde{x} + b_{\tilde{x}}) \\ & - (b_{\xi} - P_{\xi} \xi)^T D^i_{\xi} (P_{\xi} \xi + b_{\xi}) \\ & - \begin{bmatrix} -1 & w^T & \tilde{x}^T & \xi^T \end{bmatrix} L_{\xi} \begin{bmatrix} -1 & w^T & \tilde{x}^T & \xi^T \end{bmatrix}^T \le 0 \end{aligned}$$

if

$$L_\xi := \begin{bmatrix} \Phi_{11}^i & e_i^T P_\xi L D_w & e_i^T P_\xi L C & e_i^T P_\xi A^K \\ \star & V_w^T W_\xi^i V_w & 0 & 0 \\ \star & \star & P_{\tilde{x}}^T D_{\tilde{x}\xi}^i P_{\tilde{x}} & 0 \\ \star & \star & \star & P_\xi^T D_\xi^i P_\xi \end{bmatrix} \succ 0, \tag{6.34}$$

where $\Phi_{11}^i = 2e_i^T b_\xi - b_\xi^T D_\xi^i b_\xi - b_{\tilde{x}}^T D_{\tilde{x}\xi}^i b_{\tilde{x}} - \overline{w}^T W_\xi^i \overline{w}$. Note that (6.34) is equivalent to (6.24) with the definition $W_\eta^i = diag(W_\xi^i, D_{\tilde{x}\xi}^i)$. Subsequently, applying a Schur complement on $b_{\tilde{x}}^T D_{\tilde{x}\xi}^i b_{\tilde{x}}$ and $b_\xi^T D_\xi^i b_\xi$ of (6.34) successively shows that it is equivalent to the following inequality

$$\mathcal{M}_\xi + \mathcal{N}_{L_\xi^i, F_\xi^i} - (E_\xi L_\xi^i)^T (E_\xi L_\xi^i) \succ 0. \tag{6.35}$$

Using similar procedures to the previous proof for (6.29)/(6.30) on (6.35)/(6.25), giving (6.31) and (6.32), respectively. □

To summarize, the problem of updating the RCI sets of the estimation and control errors simultaneously can be posed as the convex semidefinite program

$$\begin{aligned} \underset{P_{\tilde{x}}, b_{\tilde{x}}, L, (D_{\tilde{x}}^i)^{-1}, W_{\tilde{x}}^i, Q_{\tilde{x}}, \overline{D}_{\tilde{x}}^{-1}, P_\xi, b_\xi, K, (D_{\tilde{x}\xi}^i)^{-1}, (D_\xi^i)^{-1}, W_\xi^i, Q_\xi, \overline{D}_\xi^{-1}}{min} \quad & log\ det Q_{\tilde{x}}^{-1} \\ s.t. \quad & (6.29), (6.30), (6.31), (6.32),\ Q_{\tilde{x}} = Q_\xi. \end{aligned} \tag{6.36}$$

Remark 6.4. *Since the identity* $\mathcal{N}_{L_0,D_0} = \mathcal{L}_{L_0,D_0}^{L_0,D_0}$ *in Lemma 6.2 ensures that the constraints* (6.29)-(6.32) *are also feasible by setting the corresponding optimized variables equal to their initial value, then problem* (6.36) *results in a no more conservative solution than the initial one, namely the volume of the RCI set defined by* $Q_{\tilde{x}}$ *would be smaller or at least equal to the initial set defined by* $Q_{\tilde{x}}^0$.

Remark 6.5. *Note that the constraints in the optimization problem (6.36) includes the equality constraint* $Q_{\tilde{x}} = Q_\xi$*. This means that only one ellipse is used to circumscribe the two polytopes simultaneously. This leads to some conservatism in the updating algorithm, the best approach is to consider two ellipses circumscribing two polytopes separately, and then to optimize the total volume of two ellipses; however, this will be a direction for future work.*

Finally, the complete computation algorithm for the RCI sets of the estimation and control errors based on successive iterations is summarized as follows.

Algorithm 6.1. *Given tolerance level* tol

1. ***Initial data:*** *Given system (1) and disturbance set $\mathcal{W}$, choose an initial polytope $\mathcal{P}(P_r, b_r)$ and tolerance level tol.*

2. ***Initial solution:*** *Compute the initial RCI sets of the estimation and control errors by the optimizations in (6.22) and (6.28) separately.*

3. ***Update:*** *Update the two sets simultaneously by the optimization in (6.36).*

4. ***Stopping condition:*** *Stop if the absolute value of the difference between the current and previous values of log $detQ_{\tilde{x}}^{-1}$ is less than tol.*

6.5 Numerical examples

6.5.1 Example 1

We consider a scalar system:

$$x^+ = 1.1x + u + d,$$
$$y = x + v,$$

with $d \in \mathcal{W}^d := \{d \in \mathbb{R} \mid |d|_\infty \leq 0.5\}$ and $v \in \mathcal{V} := \{v \in \mathbb{R} \mid |v|_\infty \leq 1\}$. The invariant sets of the estimation and control errors obtained with the proposed Algorithm 6.1 and [52] are shown in Table 6.1 below.

Methods	$\mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}})$	$\mathcal{P}(P_\xi, b_\xi)$	$\mathcal{X}$ - $\overline{\mathcal{X}}$	$\mathcal{U}$ - $\overline{\mathcal{U}}$
[52]	1.6000	2.8600	4.4600	3.1460
Algorithm 6.1	2.0490	2.0490	4.0980	2.2539

Table 6.1 The comparison of calculated invariant set boundaries for Example 1

Note that $\mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}})$ and $\mathcal{P}(P_\xi, b_\xi)$ denote the invariant sets of the estimation and control errors, respectively. $\mathcal{X}$ - $\overline{\mathcal{X}} = \mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}}) \oplus \mathcal{P}(P_\xi, b_\xi)$ and $\mathcal{U}$ - $\overline{\mathcal{U}} = K\mathcal{P}(P_\xi, b_\xi)$ represent the tightened invariant tube on state and tightened constraint on input, respectively. As shown in Table 6.1, [52] obtains a smaller invariant set of $\mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}})$ with the computed $K =$-1.1 and $L = 1.1$ while the proposed Algorithm 6.1 achieves less conservative results for total volumes of $\mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}}) \oplus \mathcal{P}(P_\xi, b_\xi)$ with $K =$-1.1 and $L = 0.6720$, this leads to less tightened constraints on the nominal system state by using our Algorithm 6.1. Note also that the tightened constraint on input obtained by [52] is $\mathcal{U}$ - $\overline{\mathcal{U}} =$ [-3.1460, 3.1460] while we achieve a smaller interval of [-2.2539, 2.2539].

The above results confirm our expectation, because [52] optimizes the two sets separately and can make sure that the invariant set of the estimation error is minimal only, but it might lead to a larger disturbance set for the control error $\mathcal{P}(P_\xi, b_\xi)$. Our algorithm uses a common set to optimize these two sets simultaneously, it is possible to achieve better trade-off between $\mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}})$ and $\mathcal{P}(P_\xi, b_\xi)$ and therefore a smaller total volume.

6.5.2 Example 2

A double integrator system from [73] is considered in this example

$$\begin{aligned} x^+ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0.2 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} d, \\ y &= \begin{bmatrix} 1 & 1 \end{bmatrix} x + v, \end{aligned}$$

with $d \in \mathcal{W}^d := \left\{ d \in \mathbb{R}^2 \| d|_\infty \leq 0.1 \right\}$ and $v \in \mathcal{V} := \left\{ v \in \mathbb{R} \mid |v|_\infty \leq 0.1 \right\}$. State and input constraints are $\mathcal{X} := \left\{ x \in \mathbb{R}^2 \mid -25 \leq x_i \leq 3 \right\}$ and $\mathcal{U} := \left\{ u \in \mathbb{R} \mid |u| \leq 5 \right\}$, respectively, where x_i denotes the ith element of x. We set $m = 3$ and produce the same (randomly generated) initial polytope $\mathcal{P}(P_r, b_r)$ for $\mathcal{P}(P_{\tilde{x}}, b_{\tilde{x}})$ and $\mathcal{P}(P_\xi, b_\xi)$, where

$$P_r = \begin{bmatrix} -0.5817 & 0.9493 \\ -1.8301 & 0.7174 \\ -0.4491 & 2.2878 \end{bmatrix}, b_r = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Figure 6.1 shows the the invariant tube $\mathcal{X}$ - $\overline{\mathcal{X}}$ obtained by Algorithm 6.1 (yellow) and [52] (pink). We observe that our invariant tube is smaller, which could provide a larger admissible domain on the nominal system state. The state feedback and observer gains computed by the method in [52] are $L = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ and $K = \begin{bmatrix} -1 & -1.8 \end{bmatrix}$, the constraint on nominal input is $\overline{\mathcal{U}} = [-1.5230,\ 1.5230]$. In contrast, the corresponding results obtained by our algorithm are $L = [1\ \ 0.3279]^T$ and $K = \begin{bmatrix} -1 & -1.8 \end{bmatrix}$, and $\overline{\mathcal{U}} = [-2.6149,\ 2.6149]$. Note that our obtained $\overline{\mathcal{U}}$ is significantly larger compared to the method in [52].

The relation between the objective value and the number of iterations for the update of Algorithm 6.1 is shown as the following Figure 6.2. We note that the objective value are non-increasing with the number of iterations and it converge to its final value with an observed quadratic speed of convergence.

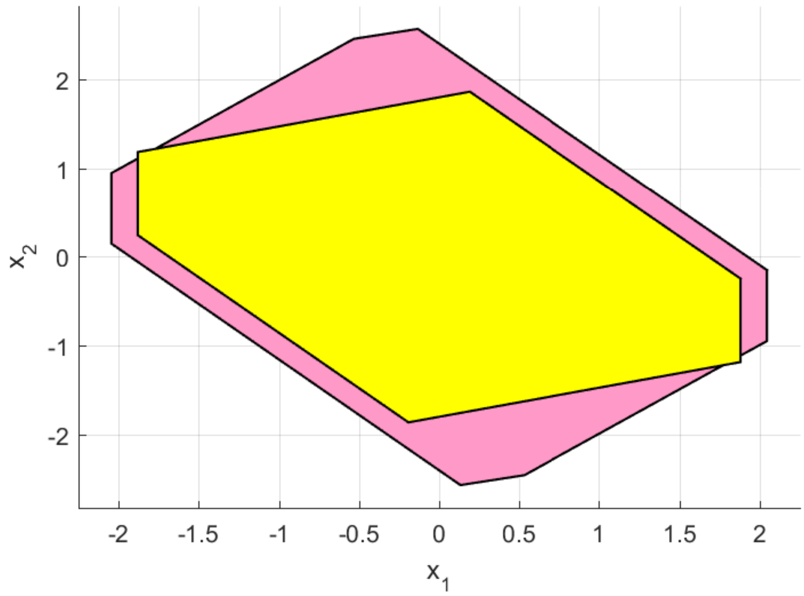

Fig. 6.1 Comparisons of tube calculated using our Algorithm 6.1 and the existing approach for Example 2

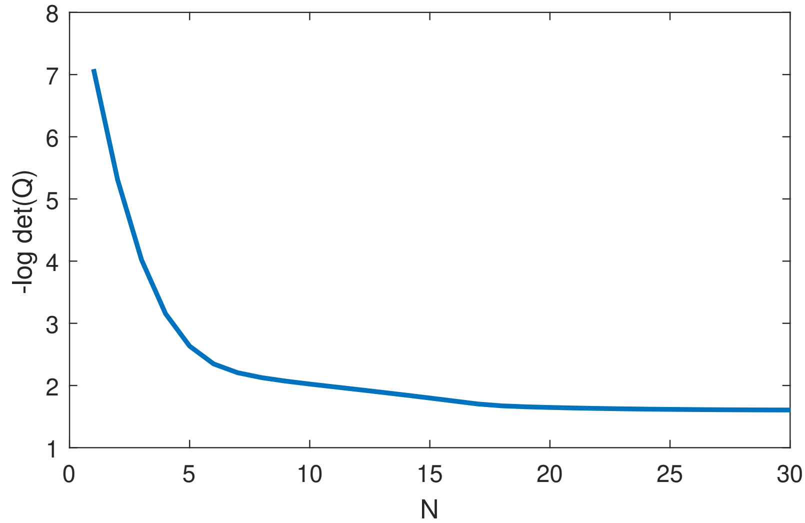

Fig. 6.2 The objective value (-log det(Q)) against the number of iterations computed using Algorithm 6.1 for Example 2

6.6 Summary

In conclusion, this chapter has investigated the computation of tightened invariant tubes for tube-based robust output MPC of discrete-time linear time-invariant systems. Nonlinear conditions for the existence of admissible invariant sets of the estimation and control errors were first provided, respectively. An extended Elimination Lemma was then used to derive LMI based sufficient conditions for the nonlinear conditions, thus rendering the tractable LMI optimizations to compute the initial invariant sets of the estimation and control errors, respectively. An update algorithm was then proposed to reduce the volume of these two invariant sets simultaneously. Two numerical examples were presented to illustrate the effectiveness of the proposed Algorithm 6.1.

Chapter 7

Conclusion and Future Work

7.1 Conclusion

In Chapter 2, new LMI conditions for robust H_2 and H_∞ state feedback control synthesis of continuous-time systems with polytopic uncertainty have been presented. We firstly expressed the design conditions for H_2 and H_∞ state feedback controllers in terms of a unified BMI formulation. Next, we provided new sufficient conditions in terms of LMIs to this BMI problem using a new separation result. It was shown that any known solution to the BMI problem could be included as a particular case for the proposed LMI conditions. Therefore, the proposed conditions can compute a new solution that would be at least as good as the previous known solution. We then proposed a new algorithm based on an iterative procedure, which ensures recursive feasibility and computes non-increasing upper bounds for both the H_2-norm and H_∞-norm. In addition, we also proposed another algorithm that can potentially find a robustly stabilizing gain when all other existing methods fail.

In Chapter 3, A new LMI based framework to design robust H_2 and H_∞ state feedback controllers for discrete-time polytopic systems have been proposed. A finite set of BMI conditions for H_2 and H_∞ controller synthesis were first derived. The BMI conditions were relaxed through an application of Elimination Lemma to obtain LMI based sufficient conditions. The proposed LMI conditions can contain any known solution to the BMI conditions as a particular case. Therefore, using the initial solution provided by the proposed initial computation method, we proposed an iterative procedure to compute non-increasing upper bounds of the H_2 and H_∞-norms. If the initial computation method gives infeasibility, we also proposed another iterative procedure to offer the possibility of finding a feasible initial solution. Moreover, based on the separation result of Chapter 2, improved update computation has also been discussed.

In Chapter 4, a novel method for designing robust state feedback H_2 and H_∞ controller of uncertain linear continuous and discrete systems has been studied. The first step of this method is to obtain an initial solution using a given controller provided by any of the existing methods. It was shown that the initial solution could include the current methods as special cases. The second step is to design the H_2 and H_∞ controllers based on an iterative LMI approach, which is recursively feasible for each update. This iterative procedure can iteratively reduce the upper bounds on H_2-norm and H_∞-norm. Based on the BMI design condition in Chapeter 2 and Chapter 3, we also presented an alternative way of update computation.

In Chapter 5, a novel LMI approach based on affine parameter-dependent Lyapunov functions has been proposed to compute H_∞-norm guaranteed cost of linear systems with polytopic uncertainty. We firstly expressed the BRL conditions for both continuous-time and discrete-time systems using a unified generalized problem. We then presented an alternative characterization of polytopic uncertainties and developed a relaxation approach based on the S-procedure to deal with the uncertainty.

In Chapter 6, a numerically efficient algorithm based on LMIs to compute invariant tubes for robust output MPC of DLTI systems with additive state and output disturbances has been presented. Instead of using pre-defined observer and control gains methods or optimizing the invariant sets of the estimation and control errors separately as the current approaches, we proposed an algorithm that optimizes the volumes of these two sets simultaneously. Therefore, our algorithm considers the effect of estimation error on the dynamics of the control error rather than treat them as decoupled problems, which can provide a more relaxed design.

In summary, this thesis aims to develop numerically efficient algorithms for robust control problems of uncertain systems, e.g., robust H2 and H infinity control and computation of invariant sets. These problems considered in this thesis are nonlinear and non-convex in general. It is generally challenging for such optimization problems to compute the actual (global) optimal solution. In this thesis, the non-convex problem is relaxed by novel separation results to obtain sufficient LMI conditions, which provides an approximate convex optimization problem for the original non-convex problem. The initial solution of the approximate convex problem is first obtained and then iteratively optimized, which provides a sequence of solutions and non-increasing criteria. The numerical examples have shown that the non-increasing sequence converges to a local minimum and it appears to promote quadratic convergence speed. The theoretical investigations of the convergence of the iterative procedure are left to future work.

7.2 Future work

Chapter 2-4 proposed three different iterative procedures for computing less conservative upper bounds on H_2-norm and H_∞-norm of continuous-time and discrete-time polytopic systems, respectively. These proposed iterative procedures are based on three types of separation results, which are applicable to a wider range of problems than the ones considered in this work, such as the problem of robust (static, dynamic, observed based) output feedback controller design for both continuous and discrete-time systems. We are currently investigating the extension of our method to handle the output feedback problem and compare with the methods available in the literature, e.g., two-step methods[74–78] and parameterized LMI based methods [79, 80]. Other future research directions include modifying our approaches to allow the use of polynomial parameter-dependent Lyapunov matrices and improving the convergence speed to reduce the computational effort.

Chapter 5 investigated the analysis of H_∞ guaranteed cost computation only, we are currently investigating the extension of the approach to the design of H_2 and H_∞ state feedback controllers. Other future research directions include investigating whether the conditions of existing methods can be proved to be special cases of the proposed approach and extending the approach to incorporate extra slack variables to provide less conservative results than [31] for Example 1 in Section 5.4.

Chapter 6 presented an algorithm to optimize the volumes of the invariant sets of the estimation error and control error simultaneously, but we used one outer ellipse to circumscribe these two invariant sets simultaneously to obtain linearity. However, this leads to some conservatism in the update computation. Considering two different outer ellipses to circumscribe two invariant sets, respectively, and then optimizing the total volume of two ellipses will be less conservative, which is under investigation.

References

[1] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear matrix inequalities in system and control theory*. SIAM Studies in Applied Mathematics, Philadelphia, PA, 1994.

[2] B. R. Barmish, "Necessary and sufficient conditions for quadratic stabilizability of an uncertain system," *Journal of Optimization theory and applications*, vol. 46, no. 4, pp. 399–408, 1985.

[3] J. Bernussou, P. L. D. Peres, and J. C. Geromel, "A linear programming oriented procedure for quadratic stabilization of uncertain systems," *Systems & Control Letters*, vol. 13, no. 1, pp. 65–72, 1989.

[4] G. Becker, A. Packard, D. Philbrick, and G. Balas, "Control of parametrically-dependent linear systems: A single quadratic Lyapunov approach," in *Proceedings of 1993 American Control Conference*, (San Francisco, CA, USA), pp. 2795–2799, IEEE, 1993.

[5] R. M. Palhares, R. H. C. Takahashi, and P. L. D. Peres, "H_∞ and H_2 guaranteed costs computation for uncertain linear systems," *International Journal of Systems Science*, vol. 28(2), 183–188, 1997.

[6] P. Gahinet, P. Apkarian, and M. Chilali, "Affine parameter-dependent Lyapunov functions and real parametric uncertainty," *IEEE Transactions on Automatic control*, vol. 41, no. 3, pp. 436–442, 1996.

[7] E. Feron, P. Apkarian, and P. Gahinet, "Analysis and synthesis of robust control systems via parameter-dependent Lyapunov functions," *IEEE Transactions on Automatic Control*, vol. 41, no. 7, pp. 1041–1046, 1996.

[8] G. Pipeleers, B. Demeulenaere, J. Swevers, and L. Vandenberghe, "Extended LMI characterizations for stability and performance of linear systems," *Systems & Control Letters*, vol. 58, no. 7, pp. 510–518, 2009.

[9] J. C. Geromel, M. C. de Oliveira, and L. Hsu, "LMI characterization of structural and robust stability," *Linear Algebra and its Applications*, vol. 285, no. 1-3, pp. 69–80, 1998.

[10] M. C. de Oliveira, J. C. Geromel, and L. Hsu, "LMI characterization of structural and robust stability: the discrete-time case," *Linear Algebra and its Applications*, vol. 296, no. 1-3, pp. 27–38, 1999.

[11] M. C. de Oliveira, J. C. Geromel, and J. Bernussou, “Extended H_2 and H_∞ norm characterizations and controller parametrizations for discrete-time systems,” *International Journal of Control*, vol. 75, no. 9, pp. 666–679, 2002.

[12] U. Shaked, “Improved LMI representations for the analysis and the design of continuous-time systems with polytopic type uncertainty,” *IEEE Transactions on Automatic Control*, vol. 46, no. 4, pp. 652–656, 2001.

[13] W. Xie, “An equivalent LMI representation of bounded real lemma for continuous-time systems,” *Journal of Inequalities and Applications*, vol. 2008, pp. 1–8, 2008.

[14] Y. He, M. Wu, and J. H. She, “Improved bounded real lemma representation and H_∞ control of systems with polytopic uncertainties,” *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol. 52, no. 7, pp. 380–383, 2005.

[15] P. J. de Oliveira, R. C. L. F. Oliveira, V. J. S. Leite, V. F. Montagner, and P. L. D. Peres, “H_∞ guaranteed cost computation by means of parameter-dependent Lyapunov functions,” *Automatica*, vol. 40, no. 6, pp. 1053–1061, 2004.

[16] C. W. Scherer, “LMI relaxations in robust control,” *European Journal of Control*, vol. 12, no. 1, pp. 3–29, 2006.

[17] Y. Ebihara, D. Peaucelle, and D. Arzelier, *S-variable approach to LMI-based robust control*. Springer, 2015.

[18] J. C. Geromel and R. H. Korogui, “Matrix quadratic polynomials with application to robust stability analysis,” *IFAC Proceedings Volumes*, vol. 39, no. 9, pp. 549–554, 2006.

[19] Y. Altun and K. Gulez, “Linear parameter varying feedforward control synthesis using parameter-dependent Lyapunov function,” *Nonlinear Dynamics*, vol. 78, no. 4, pp. 2293–2307, 2014.

[20] M. C. de Oliveira, J. Bernussou, and J. C. Geromel, “A new discrete-time robust stability condition,” *Systems & control letters*, vol. 37, no. 4, pp. 261–265, 1999.

[21] D. C. W. Ramos and P. L. D. Peres, “A less conservative LMI condition for the robust stability of discrete-time uncertain systems,” *Systems and Control Letters*, vol. 43, no. 5, pp. 371–378, 2001.

[22] G. Hilhorst, G. Pipeleers, R. C. L. F. Oliveira, P. L. D. Peres, and J. Swevers, “On extended LMI conditions for H_2/H_∞ control of discrete-time linear systems,” *IFAC Proceedings Volumes*, vol. 47, no. 3, pp. 9307 – 9312, 2014.

[23] C. F. Morais, M. F. Braga, R. C. L. F. Oliveira, and P. L. D. Peres, “Robust state feedback control for discrete-time linear systems via LMIs with a scalar parameter,” in *Proceedings of 2013 American Control Conference*, (Washington, DC, USA), pp. 3870–3875, IEEE, 2013.

[24] P. Apkarian, H. D. Tuan, and J. Bernussou, “Continuous-time analysis, eigenstructure assignment, and H_2 synthesis with enhanced linear matrix inequalities (LMI) characterizations,” *IEEE Transactions on Automatic Control*, vol. 46, no. 12, pp. 1941–1946, 2001.

[25] Y. Ebihara and T. Hagiwara, "New dilated LMI characterizations for continuous-time multiobjective controller synthesis," *Automatica*, vol. 40, no. 11, pp. 2003–2009, 2004.

[26] W. Xie, "H_2 gain scheduled state feedback for LPV system with new LMI formulation," *IEE Proceedings-Control Theory and Applications*, vol. 152, no. 6, pp. 693–697, 2005.

[27] Y. Altun, "A new extended LMI-based robust gain scheduled state feedback H_2 controller design," *International Journal of Control, Automation and Systems*, vol. 15, no. 3, pp. 967–975, 2017.

[28] S. Salhi and G. Garcia, "H_2 and H_∞ robust control for continuous–time linear systems," *Technical Report 08146, LAAS-CNRS*, 2008.

[29] L. A. Rodrigues, R. C. L. F. Oliveira, and J. F. Camino, "Parameterized LMIs for robust H_2 and H_∞ state feedback control of continuous-time polytopic systems," *International Journal of Robust and Nonlinear Control*, vol. 28, no. 3, pp. 940–952, 2018.

[30] A. Trofino, D. F. Coutinho, and K. A. Barbosa, "Improved H_2 and H_∞ conditions for robust 2004 and control synthesis of linear systems," *SBA: Controle & Automação*, vol. 16, no. 4, pp. 427–434, 2005.

[31] P. Gahinet and P. Apkarian, "A linear matrix inequality approach to H_∞ control," *International journal of robust and nonlinear control*, vol. 4, no. 4, pp. 421–448, 1994.

[32] T. Iwasaki and R. E. Skelton, "All controllers for the general H_∞ control problem: LMI existence conditions and state space formulas," *Automatica*, vol. 30, no. 8, pp. 1307–1317, 1994.

[33] M. C. de Oliveira and R. E. Skelton, "Stability tests for constrained linear systems," in *Perspectives in robust control (Lecture Notes in Control and Information Science)*, pp. 241–257, Springer, 2001.

[34] R. C. L. F. Oliveira, M. C. de Oliveira, and P. L. D. Peres, "Robust state feedback LMI methods for continuous-time linear systems: Discussions, extensions and numerical comparisons," in *2011 IEEE International Symposium on Computer-Aided Control System Design (CACSD)*, (Denver, CO), pp. 1038–1043, IEEE, 2011.

[35] J. C. Geromel and R. H. Korogui, "Analysis and synthesis of robust control systems using linear parameter dependent Lyapunov functions," *IEEE Transactions on Automatic Control*, vol. 51, no. 12, pp. 1984–1989, 2006.

[36] F. Tahir and I. M. Jaimoukha, "Causal state-feedback parameterizations in robust model predictive control," *Automatica*, vol. 49, no. 9, pp. 2675–2682, 2013.

[37] F. Blanchini, "Minimum-time control for uncertain discrete-time linear systems," in *Proceedings of the 31st IEEE Conference on Decision and Control*, pp. 2629–2634, 1992.

[38] D. Q. Mayne and W. R. Schroeder, "Robust time-optimal control of constrained linear systems," *Automatica*, vol. 33, no. 12, pp. 2103–2118, 1997.

[39] F. Blanchini and S. Miani, *Set-theoretic methods in control*. Springer, 2008.

[40] I. Kolmanovsky and E. G. Gilbert, "Theory and computation of disturbance invariant sets for discrete-time linear systems," *Mathematical problems in engineering*, vol. 4, no. 4, pp. 317–367, 1998.

[41] C. E. T. Dorea and J. C. Hennet, "(A, B)-invariant polyhedral sets of linear discrete-time systems," *Journal of Optimization Theory and Applications*, vol. 103, no. 3, pp. 521–542, 1999.

[42] S. V. Raković, D. Q. Mayne, E. C. Kerrigan, and K. I. Kouramas, "Optimized robust control invariant sets for constrained linear discrete-time systems," *IFAC Proceedings Volumes*, vol. 38, no. 1, pp. 584–589, 2005.

[43] S. V. Raković, E. C. Kerrigan, K. I. Kouramas, and D. Q. Mayne, "Invariant approximations of the minimal robust positively invariant set," *IEEE Transactions on automatic control*, vol. 50, no. 3, pp. 406–410, 2005.

[44] S. V. Raković and M. Baric, "Parameterized robust control invariant sets for linear systems: Theoretical advances and computational remarks," *IEEE Transactions on Automatic Control*, vol. 55, no. 7, pp. 1599–1614, 2010.

[45] P. Trodden, "A one-step approach to computing a polytopic robust positively invariant set," *IEEE Transactions on Automatic Control*, vol. 61, no. 12, pp. 4100–4105, 2016.

[46] F. Tahir and I. M. Jaimoukha, "Low-complexity polytopic invariant sets for linear systems subject to norm-bounded uncertainty," *IEEE Transactions on Automatic Control*, vol. 60, no. 5, pp. 1416–1421, 2014.

[47] C. Liu and I. M. Jaimoukha, "The computation of full-complexity polytopic robust control invariant sets," in *2015 54th IEEE Conference on Decision and Control (CDC)*, pp. 6233–6238, IEEE, 2015.

[48] C. Liu, F. Tahir, and I. M. Jaimoukha, "Full-complexity polytopic robust control invariant sets for uncertain linear discrete-time systems," *International Journal of Robust and Nonlinear Control*, vol. 29, no. 11, pp. 3587–3605, 2019.

[49] W. Langson, I. Chryssochoos, S. V. Raković, and D. Q. Mayne, "Robust model predictive control using tubes," *Automatica*, vol. 40, no. 1, pp. 125–133, 2004.

[50] D. Q. Mayne, S. V. Raković, R. Findeisen, and F. Allgöwer, "Robust output feedback model predictive control of constrained linear systems," *Automatica*, vol. 42, no. 7, pp. 1217–1222, 2006.

[51] M. Kögel and R. Findeisen, "Robust output feedback MPC for uncertain linear systems with reduced conservatism," *IFAC-PapersOnLine*, vol. 50, no. 1, pp. 10685–10690, 2017.

[52] C. Liu, "Robust Model Predictive Control: Robust Control Invariant Sets and Efficient Implementation," Ph.D. thesis, Imperial College London, UK, 2017.

[53] C. Hu and I. M. Jaimoukha, "New iterative linear matrix inequality based procedure for H_2 and H_∞ state feedback control of continuous-time polytopic systems," *International Journal of Robust and Nonlinear Control*, vol. 31, no. 1, pp. 51–68, 2021.

[54] L. A. Rodrigues, R. C. L. F. Oliveira, and J. F. Camino, "New extended LMI characterization for state feedback control of continuous-time uncertain linear systems," in *Proceedings of 2015 European Control Conference (ECC)*, (Linz, Austria), pp. 1992–1997, IEEE, 2015.

[55] G. Balas, R. Chiang, A. Packard, and M. Safonov, "Robust control toolbox user's guide," *The Math Works, Inc., Tech. Rep*, 2007.

[56] S. Boyd and L. Vandenberghe, *Convex optimization*. Cambridge university press, 2004.

[57] A. Felipe, R. C. L. F. Oliveira, and P. L. D. Peres, "An iterative LMI based procedure for robust stabilization of continuous-time polytopic systems," in *2016 American Control Conference (ACC)*, (Boston, MA, USA), pp. 3826–3831, IEEE, 2016.

[58] M. Grant and S. Boyd, "CVX: Matlab software for disciplined convex programming, version 2.1." http://cvxr.com/cvx, Mar. 2014.

[59] M. Grant and S. Boyd, "Graph implementations for nonsmooth convex programs. *Recent Advances in Learning and Control*," 2008. http://stanford.edu/~boyd/graph_dcp.html.

[60] C. Hu and I. M. Jaimoukha, "Robust H_2 and H_∞ state feedback control for discrete–time polytopic systems using an iterative LMI based procedure," in *2020 European Control Conference (ECC)*, pp. 621–626, 2020.

[61] J. C. Geromel, P. L. D. Peres, and S. R. Souza, "A convex approach to the mixed H_2/H_∞ control problem for discrete-time uncertain systems," *SIAM Journal on Control and Optimization*, vol. 33, no. 6, pp. 1816–1833, 1995.

[62] K. Zhou, J. C. Doyle, and K. Glover, *Robust and optimal control*. Prentice hall New Jersey, 1996.

[63] C. Hu and I. M. Jaimoukha, "Robust state feedback H_2 and H_∞ control synthesis for linear polytopic systems using parameter-dependent slack variables," *Prepared to submit*, 2020.

[64] C. Hu and I. M. Jaimoukha, "New LMI characterizations for H_∞-norm guaranteed cost computation of linear systems with polytopic uncertainties," in *2020 59th IEEE Conference on Decision and Control (CDC)*, pp. 3957–3962, 2020.

[65] A. Bemporad, M. Morari, V. Dua, and E. N. Pistikopoulos, "The explicit linear quadratic regulator for constrained systems," *Automatica*, vol. 38, no. 1, pp. 3–20, 2002.

[66] J. Nocedal and S. Wright, *Numerical optimization*. Springer Science & Business Media, 2006.

[67] Y. Wang and S. Boyd, "Fast model predictive control using online optimization," *IEEE Transactions on control systems technology*, vol. 18, no. 2, pp. 267–278, 2009.

[68] D. Q. Mayne and W. Langson, "Robustifying model predictive control of constrained linear systems," *Electronics Letters*, vol. 37, no. 23, pp. 1422–1423, 2001.

[69] D. Q. Mayne, M. M. Seron, and S. Raković, "Robust model predictive control of constrained linear systems with bounded disturbances," *Automatica*, vol. 41, no. 2, pp. 219–224, 2005.

[70] C. Hu, C. Liu, and I. M. Jaimoukha, "Computation of invariant tubes for robust output feedback model predictive control," in *IFAC-PapersOnLine*, 2020.

[71] F. Blanchini, "Set invariance in control," *Automatica*, vol. 35, no. 11, pp. 1747–1767, 1999.

[72] I. Pólik and T. Terlaky, "A survey of the S-lemma," *SIAM review*, vol. 49, no. 3, pp. 371–418, 2007.

[73] P. J. Goulart and E. C. Kerrigan, "Output feedback receding horizon control of constrained systems," *International Journal of Control*, vol. 80, no. 1, pp. 8–20, 2007.

[74] D. Peaucelle and D. Arzelier, "An efficient numerical solution for H_2 static output feedback synthesis," in *Proceedings of 2001 European Control Conference (ECC)*, (Porto, Portugal), pp. 3800–3805, IEEE, 2001.

[75] D. Arzelier, D. Peaucelle, and S. Salhi, "Robust static output feedback stabilization for polytopic uncertain systems: Improving the guaranteed performance bound," in *Proceedings of 4th IFAC Symposium on Robust Control Design (ROCOND 2003)*, (Milan, Italy), pp. 425–430, Elsevier, 2003.

[76] C. M. Agulhari, R. C. L. F. Oliveira, and P. L. D. Peres, "LMI relaxations for reduced-order robust H_∞ control of continuous-time uncertain linear systems," *IEEE Transactions on Automatic Control*, vol. 57, no. 6, pp. 1532–1537, 2011.

[77] G. Hilhorst, G. Pipeleers, and J. Swevers, "An LMI approach for reduced-order H_2 LTI controller synthesis," in *Proceedings of 2013 American Control Conference*, (Washington, DC, USA), pp. 2392–2396, IEEE, 2013.

[78] G. Hilhorst, G. Pipeleers, W. Michiels, and J. Swevers, "Sufficient LMI conditions for reduced-order multi-objective H_2/H_∞ control of LTI systems," *European Journal of Control*, vol. 23, pp. 17–25, 2015.

[79] J. Dong and G. H. Yang, "Robust static output feedback control synthesis for linear continuous systems with polytopic uncertainties," *Automatica*, vol. 49, no. 6, pp. 1821–1829, 2013.

[80] X. H. Chang, J. H. Park, and J. Zhou, "Robust static output feedback H_∞ control design for linear systems with polytopic uncertainties," *Systems & Control Letters*, vol. 85, pp. 23–32, 2015.