Strange nonchaotic attractors in a family of quasiperiodically forced piecewise linear maps

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Abstract

In this paper, a family of quasiperiodically forced piecewise linear maps is considered. We prove that there exists a unique strange nonchaotic attractor for some set of parameter values. It is the graph of an upper semi-continuous function which is invariant, discontinuous almost everywhere and attracts almost all orbits. Moreover, both Lyapunov exponents are nonpositive, a necessary condition for the existence of a strange nonchaotic attractor.

Keywords: Strange nonchaotic attractor; skew product map; invariant graph;

1. Introduction

There has been considerable interest in recent years in quasiperiodically forced systems, partly due to the fact that these commonly exhibit strange nonchaotic attractors (SNAs). For instance, damped pendulum with quasiperiodic forcing [1, 2], Chua's circuit with two-frequency quasiperiodic excitation [3], Harper map, which is intimately related to certain discrete Schrödinger operators with quasiperiodic potential [4], see also [5]. This type of attractors were uncovered by Grebogi et al. [6]. In dynamical systems, the types of attractors usually include periodic attractors, quasiperiodic attractors and chaotic attractors. However, SNA is considered as the fourth type of attractor. In 2015, strange nonchaotic star dynamics has been demonstrated in the RR Lyra Constellation, which further validates the presence of strange nonchaotic phenomena in nature [7]. An SNA has fractal structure [8, 9], but is nonchaotic in the dynamical sense. Pikovsky and Feudel [10] introduced the methods of phase sensitivity and rational approximations to characterize the strange property of SNAs.

Precisely, what constitutes a distinct mechanism for the formation of an SNA is somewhat nebulous since the bifurcations of quasiperiodically driven systems have not been studied in formal mathematical detail. However, several routes to SNAs have been described in the literature, such as torus collisions route [11, 12], fractalization route [13], intermittency route [14, 15], quasiperiodic route [8], blowout bifurcation route [16], grazing bifurcation route [17] and so on. Prasad et al [18] gave a good overview and further reference for this.

The theoretical results of SNAs are mainly limited to skew product maps. Keller [19] studied a class of quasiperiodically forced interval maps which are monotonically increasing and is strictly

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concave, and proved the existence of attracting invariant graphs that are discontinuous almost everywhere. Jäeger [20] studied quasiperiodically forced interval maps which are monotonically increasing and have negative Schwarzian derivative, and gave a classification, with respect to the number and to the Lyapunov exponents of invariant graphs, for this class of systems. It turns out that the possibilities for the invariant graphs are exactly analogous to those for the fixed points of the unperturbed fibre maps. Alsedà et al. [21] generalized the results of [19] to quasiperiodically forced unimodal maps; the strictly concavity of fibre maps plays a basic role in the proof.

In this paper we consider a skew product system $F : \mathbb{S}^1 \times \mathbb{R}^+ \to \mathbb{S}^1 \times \mathbb{R}^+$ defined by

$$(\theta, x) \mapsto (R(\theta), f(x)g(\theta)),$$

where $R(\theta) = \theta + \omega (\mathrm{mod}\, 1)$, $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ denotes the unit circle, and ω is an irrational number. We take $g(\theta) = \sin(\pi\theta)$ and

$$f(x) = \begin{cases} ax & \text{if } x \leq \frac{1}{2}, \\ bx + \frac{a-b}{2} & \text{if } x \geq \frac{1}{2}, \end{cases} \qquad a, b > 0.$$

The map F is similar to the one considered by Keller in [19], except that the function f in his case is strictly concave. Since in our case f is piecewise linear, the results in [19] cannot be applied directly here. On the other hand, if we use the known results of general quasiperiodically forced monotonic maps, then we can merely show that the Lyapunov exponents on the SNAs in the x direction is nonpositive. The purpose of this paper is to prove that, when $a > 2$ and $0 \leq b < 1$, the map F possesses a unique SNA. It is the graph of a semi-continuous function which is invariant, discontinuous almost everywhere and attracts almost all orbits.

2. Invariant graph and Lyapunov exponent in the x direction

Due to the aperiodicity of the quasiperiodic forcing, there cannot be any fixed points or periodic points for such system. Therefore, invariant graphs are the most simple invariant objects that can occur.

Definition 2.1. *A function $\eta : \mathbb{S}^1 \to [0, \infty)$ is said to have an invariant graph with respect to the map F, if for all $\theta \in \mathbb{S}^1$:*

$$F(\theta, \eta(\theta)) = (R(\theta), \eta(R(\theta))).$$

Obviously, $\mathbb{S}^1 \times \{0\}$ is an invariant graph of F.

Definition 2.2. *If $\eta : \mathbb{S}^1 \to [0, \infty)$ has an invariant graph $Gr(\eta) := \{(\theta, \eta(\theta)) : \theta \in \mathbb{S}^1\}$ for F, then the induced measure μ_η on $Gr(\eta)$ is defined by*

$$\mu_\eta(U) = m(\{\theta \in \mathbb{S}^1 : (\theta, \eta(\theta)) \in U\})$$

for each measurable set $U \in Gr(\eta)$, where $m(\cdot)$ is the Lebesgue measure on $\mathbb{S}^1$.

It turns out that any such measure is F-invariant and ergodic.

The stability of an invariant graph $\mathrm{Gr}(\eta)$ is determined by its Lyapunov exponents. For a.e. $(\theta, \eta(\theta)) \in \mathrm{Gr}(\eta)$ with respect to μ_η, the Lyapunov exponent at $(\theta, \eta(\theta))$ in the x direction

$$\lambda(\theta, \eta(\theta)) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \log |Df(\eta(\theta)_k)g(\theta_k)|$$

is well defined, where $(\theta_k, \eta(\theta)_k) = F^k(\theta, \eta(\theta))$. Since the map R preserves the Lebesgue measure on $\mathbb{S}^1$ and is ergodic, by Birkhoff's ergodic theorem,

$$\lambda(\theta, \eta(\theta)) = \int_{\mathbb{S}^1} \log(Df(\eta(\theta))g(\theta))d\theta \tag{1}$$

for a.e. $\theta \in \mathbb{S}^1$.

By Oseledec's multiplicative ergodic theorem (see e.g. [22]), for μ_η-almost every $(\theta, \eta(\theta))$ there are two Lyapunov exponents, the first one is $\lambda(\theta, \eta(\theta))$ and the second one is denoted by $\varrho(\theta, \eta(\theta))$. Moreover, for μ_η-almost every $(\theta, \eta(\theta))$ we have

$$\begin{aligned}
\lambda(\theta, \eta(\theta)) + \varrho(\theta, \eta(\theta)) &= \lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{1}{n} \log|\det(DF(\theta_k, \eta(\theta)_k))| \\
&= \lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{1}{n} \log\left|\det\begin{pmatrix} 1 & 0 \\ f(\eta(\theta)_k)Dg(\theta_k) & Df(\eta(\theta)_k)g(\theta_k) \end{pmatrix}\right| \\
&= \lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{1}{n} \log|Df(\eta(\theta)_k)g(\theta_k)| = \lambda(\theta, \eta(\theta)).
\end{aligned}$$

Therefore, $\varrho(\theta, \eta(\theta)) = 0$. This shows that the stability of the graph of η is completely determined by the Lyapunov exponent in the x direction $\lambda(\eta) := \int_{\mathbb{S}^1} \log(Df(\eta(\theta))g(\theta))d\theta$.

3. Existence of SNAs

In this section we will prove the existence of an SNA for the map F for certain parameter set. First, we list some notations and definitions.

Definition 3.1. *A function $\eta : \mathbb{S}^1 \to \mathbb{R}$ is upper semi-continuous at θ_0 if*

$$\limsup_{\theta\to\theta_0} \eta(\theta) \leq \eta(\theta_0).$$

The function η is called upper semi-continuous if it is upper semi-continuous at every point of its domain.

For any set $\mathcal{A} \subseteq \mathbb{S}^1 \times [0, \infty)$, denote by $\mathcal{A}_\theta$ its intersection with the θ-fibre, i.e., $\mathcal{A}_\theta := (\{\theta\} \times [0, \infty)) \cap \mathcal{A}$. The following concept turned out to be very important in the study of quasiperiodically forced maps (see [23, 24]).

Definition 3.2. *A set $\mathcal{A} \subseteq \mathbb{S}^1 \times [0, \infty)$ is called pinched, if for some $\theta \in \mathbb{S}^1$ the set $\mathcal{A}_\theta$ consists only of a single point. In this case we call $\mathcal{A}$ is pinched at θ.*

Since the map R preserves the Lebesgue measure on $\mathbb{S}^1$ and is ergodic, if $\mathcal{A}$ is invariant and pinched, then it is pinched on a whole dense set, namely on the forward orbit of a pinched fibre. If in addition $\mathcal{A}$ is compact then the set of θ at which $\mathcal{A}$ is pinched is even residual. This follows from a Baire argument, as in this case all sets $\{\theta \in \mathbb{S}^1 : \text{diam}(\mathcal{A}_\theta) < 1/n\}$ are open and dense, and their intersection gives exactly the set of θ where $\mathcal{A}$ is pinched.

Denote by $\mathcal{P}$ the space of functions from $\mathbb{S}^1$ to $[0, \infty)$. Define the transfer operator $T : \mathcal{P} \to \mathcal{P}$ as

$$(T\eta)(\theta) = f(\eta(R^{-1}(\theta)))g(R^{-1}(\theta)).$$

Remark 3.1. *Observe that $\varphi \in \mathcal{P}$ has an invariant graph if and only if $T\varphi = \varphi$.*

Denote by $\mathcal{P}_T$ the set of fixed points of T.

Remark 3.2. *When f is monotone increasing, in view of the definition of T, for $\varphi, \phi \in \mathcal{P}$ if $\varphi \le \phi$ then $T\varphi \le T\phi$.*

Remark 3.3. *If $\eta \in \mathcal{P}_T$ then for all $\theta \in \mathbb{S}^1$ we have*

$$\eta(R(\theta)) = f(\eta(\theta))g(\theta).$$

Therefore, if $\eta(\theta) = 0$ then $\eta(R(\theta)) = 0$. Together with the ergodicity of R, this implies that either η is zero a.e. or it is positive a.e. Here and in what follows, a.e. means almost everywhere with respect to the Lebesgue measure on $\mathbb{S}^1$.

Note that the fixed point of the map f are 0 and $\frac{a-b}{2(1-b)}$. Set $I = [0, M]$ for some $M \ge \frac{a-b}{2(1-b)}$. Denote by π_1 and π_2 the projections from $\mathbb{S}^1 \times [0, \infty)$ to $\mathbb{S}^1$ and $[0, \infty)$, respectively.

Theorem 3.1. *Suppose that $a > 2$ and $0 \le b < 1$. There exists a function $\phi : \mathbb{S}^1 \to I$ such that*
(i) ϕ is an upper semi-continuous function and has an invariant graph;
(ii) ϕ is positive a.e.;
(iii) ϕ is discontinuous a.e.

Proof. Let

$$\phi_n : \mathbb{S}^1 \to [0, \infty), \qquad \phi_n(\theta) = T^n(M) = \pi_2 \circ F^n(R^{-n}(\theta), M) \tag{2}$$

for all $n \in \mathbb{Z}^+$. Then

$$\phi_1(\theta) = \quad \pi_2 \circ F(R^{-1}(\theta), M) = f(M)\sin \pi(R^{-1}(\theta)) \le M. \tag{3}$$

Hence, by (3) and Remark 3.2 we get

$$M \ge TM \ge T^2M \ge T^3M \ge \cdots . \tag{4}$$

This, together with the fact that $T^nM \ge 0$ for all $n \in \mathbb{Z}^+$, shows that there is a function $\phi \in \mathcal{P}$ such that

$$\phi(\theta) = \lim_{n\to\infty} \phi_n(\theta) = \inf_n \phi_n(\theta).$$

As the infimum of a decreasing sequence of continuous function ϕ is upper semi-continuous. Observe that

$$\begin{aligned} f(\phi(\theta))g(\theta) &= \lim_{n\to\infty} f(\phi_n(\theta))g(\theta) = \lim_{n\to\infty} \pi_2 \circ F(\theta, \phi_n(\theta)) \\ &= \lim_{n\to\infty} \pi_2 \circ F(F^n(R^{-n}(\theta), M)) = \lim_{n\to\infty} \phi_{n+1}(R(\theta)) \\ &= \phi(R(\theta)). \end{aligned}$$

Therefore, ϕ has an invariant graph.

To prove (ii), note that by the formula in (1) we have

$$\lambda_x(\theta, 0) = \int_{\mathbb{S}^1} \log Df(0)d\theta + \int_{\mathbb{S}^1} \log \sin(\pi\theta)d\theta = \log a - \log 2 > 0 \tag{5}$$

for a.e. $\theta \in \mathbb{S}^1$. Due to $f(0) = 0$, the set $\{\theta \in \mathbb{S}^1 : \phi(\theta) = 0\}$ is invariant under rotation by ω. Since R preserve the Lebesgue measure on $\mathbb{S}^1$ and is ergodic, $\{\theta \in \mathbb{S}^1 : \phi(\theta) > 0\}$ has Lebesgue measure 1 or 0.

Assume that $\phi(\theta) = 0$ for a.e. $\theta \in \mathbb{S}^1$. By (2) and (4), we have

$$\frac{f \circ (T^n M)}{T^n M} = \frac{(T^{n+1}M) \circ R}{(T^n M) \cdot g} \leq \frac{(T^n M) \circ R}{(T^n M) \cdot g}.$$

$\frac{f \circ (T^n M)}{T^n M}$ is bounded and away from 0 as a function of θ. Therefore its logarithm is integrable. Hence,

$$\log \frac{f \circ (T^n M)}{T^n M} + \log g \leq \log \frac{(T^n M) \circ R}{(T^n M) \cdot g}$$

and the left-hand side is itegrable. Therefore, by Lemma 3.4, the integral of the right-hand side is 0. We need this argument because a priori the right-hand side could be not integrable. Thus,

$$\int_{\mathbb{S}^1} \log \frac{f \circ (T^n M)}{T^n M} + \int_{\mathbb{S}^1} \log g \leq 0. \tag{6}$$

Under the above assumption, $(T^n)_{n\geq 0}$ is a monotone sequence of function converging a.e. to $\phi = 0$. Then the inequality in (6) can pass to the limit with n and we get $\lambda(\theta, 0) \leq 0$ for a.e. $\theta \in \mathbb{S}^1$. This contrasts with (5). Therefore $\phi(\theta)$ is positive for a.e. $\theta \in \mathbb{S}^1$.

For each $\theta \in \mathbb{S}^1$ with $\phi(\theta) > 0$, we can take a sequence $\{\theta_k\}$ such that $\phi(\theta_k) = 0$ for all k and $\lim_{k\to\infty} \theta_k = \theta$, then $\lim_{k\to\infty} \phi(\theta_k) = 0$. Therefore, ϕ is almost everywhere discontinuous. □

We will show that ϕ is the unique function which has an invariant graph (with respect to F) and is positive a.e. There is a subtle issue in the definition of invariant graphs that has to be addressed. Any invariant graph η can be modified on a set of measure zero to yield another invariant graph $\hat{\eta}$, equal to η a.e. We usually do not distinguish between such graphs. We make the following convention. We will consider two invariant graphs as equivalent if they are a.e. equal and implicitly convey about equivalence classes of invariant graphs. If any further assumptions about invariant graphs are made, such as semi-continuity, measurability or inequalities between invariant graphs, we will understand it in the way that there is at least one representative in each of the respective equivalence classes such that the assumptions are met. These representatives will then be used in the proofs, and all conclusions which are drawn from the assumed properties will be true for all such representative.

Lemma 3.1. *Each element of $\mathcal{P}_T$ is a measurable function.*

Proof. Assume that $\eta_1, \eta_2 \in \mathcal{P}_T$ with $\eta_1 \leq \eta_2$ and that there is no other element of $\mathcal{P}_T$ in between. Take a measurable function η such that $\eta_1 \leq \eta \leq \eta_2$. By Remark 3.2, we get $T^k \eta \leq \eta_2$ for all $k \geq 0$. This together with the assumption show that $T^k \eta \to \phi$ as $k \to \infty$. Therefore, η_2 is a measurable function. □

Lemma 3.2. *Let a and b be as in Theorem 3.1. For each $\eta \in \mathcal{P}_T$ with $\eta(\theta) > 0$ a.e., the essential supremum of η is $> 1/2$.*

Proof. Since η is positive a.e., assume that $\eta \leq 1/2$ a.e. Then

$$\begin{aligned} \log \eta(\theta) &= \log T(\eta(\theta)) = \log f(\eta(R^{-1}(\theta))) + \log g(R^{-1}(\theta)) \\ &\geq \log a + \log \eta(R^{-1}(\theta)) + \log g(R^{-1}(\theta)) \end{aligned}$$

a.e. By induction we get

$$\log \eta(\theta) - \log \eta(R^{-n}(\theta)) \geq n \log a + \sum_{k=1}^{n} \log g(R^{-k}(\theta))$$

a.e. By replacing θ by $R^n(\theta)$ this can be rewritten as

$$\log \eta(R^n(\theta)) - \log \eta(\theta) \geq n \log 2 + \sum_{k=0}^{n-1} \log g(R^k(\theta))$$

a.e. By the ergodic theorem the right hand side of above goes to infinity as $n \to \infty$. This contrasts with the fact that η is positive a.e. and is bounded. Therefore, the essential supremum of η is $> 1/2$. $\square$

Theorem 3.2. *Let a and b be as in Theorem 3.1. ϕ is the unique element of $\mathcal{P}_T$ with $\phi > 0$ almost everywhere.*

To prove Theorem 3.2, we need some preliminaries. Given points $x, y \in I$ with $x \neq y$ (note that $f(x) \neq f(y)$), set

$$\kappa(x, y) = \frac{|x - y|}{\max\{x, y\}}$$

and

$$\tau(x, y) = \frac{\kappa(f(x), f(y))}{\kappa(x, y)}.$$

Observe that $|Df(x)| = \frac{f(x)}{x}$ for $x \in (0, 1/2)$ and $|Df(x)| < \frac{f(x)}{x}$ for $x \in (1/2, M]$.

Lemma 3.3. *Let a and b be as in Theorem 3.1. Fix $n \in \mathbb{Z}^+$, $\theta_0 \in \mathbb{S}^1$ and $x_0, y_0 \in I$. Denote $(\theta_k, x_k) = F^k(\theta_0, x_0)$ and $(\theta_k, y_k) = F^k(\theta_0, y_0)$ for $k \in 1, 2, \ldots, n-1$. Then $|x_n - y_n| \leq M\mu^{m(n)}$, where $m(n)$ is the number of indices $k \in \{0, 1, \ldots, n-1\}$ such that $x_k, y_k > 1/2$.*

Proof. Note that if $0 < x < y < \frac{1}{2}$ then $\tau(x, y) = 1$. If $0 < x < y \leq M$ and $y > \frac{1}{2}$ then

$$\tau(x, y) = \frac{f(y) - f(x)}{f(y)} \cdot \frac{y}{y - x} = \frac{f(y) - f(x)}{y - x} \cdot \frac{y}{f(y)}$$

which is a strictly decreasing function of x. Therefore, $\tau(x, y) < 1$. In particular, there exist a constant $\mu < 1$ such that if $x, y > 1/2$ then $\tau(x, y) \leq \mu$.

If $x_n = y_n$ then there is nothing to prove. Assume that $x_n \neq y_n$. Then also $x_k \neq y_k$ for $k = 0, 1, \ldots, n-1$. We have

$$\begin{aligned}\kappa(x_{k+1}, y_{k+1}) &= \frac{|x_{k+1} - y_{k+1}|}{\max\{x_{k+1}, y_{k+1}\}} = \frac{|f(x_k)g(\theta_k) - f(y_k)g(\theta_k)|}{\max\{f(x_k)g(\theta_k), f(y_k)g(\theta_k)\}} \\ &= \frac{|f(x_k) - f(y_k)|}{\max\{f(x_k), f(y_k)\}} = \kappa(f(x_k), f(y_k)).\end{aligned}$$

Therefore,

$$\begin{aligned}|x_n - y_n| &= y_n\kappa(x_n, y_n) = y_n\kappa(x_0, y_0)\prod_{k=0}^{n-1}\frac{\kappa(x_{k+1}, y_{k+1})}{\kappa(x_k, y_k)}\\ &= y_n\kappa(x_0, y_0)\prod_{k=0}^{n-1}\tau(x_k, y_k) \le M\prod_{k=0}^{n-1}\tau(x_k, y_k).\end{aligned}$$

Thus, we get $|x_n - y_n| \le M\mu^{m(n)}$. □

The proof of Theorem 3.2. Due to the monotonicity of the transfer operator T, there is a natural order for the elements of $\mathcal{P}_T$. Let $\hat{\phi}$ be the smallest element of $\mathcal{P}_T$ with $\hat{\phi} > 0$ a.e. By Lemma 3.2, there exists a set $A \in \mathbb{S}^1$ with positive Lebesgue measure such that $\hat{\phi}(\theta) > 1/2$ for $\theta \in A$. By the ergodicity of R, for almost every $\theta \in \mathbb{S}^1$ we have $R^{-k} \in A$ for infinitely many positive integer k. We choose such θ.

Fix $n > 0$. Denote $\theta_0 = R^{-n}(\theta)$ and choose any $x_0, y_0 \in [\hat{\phi}(\theta_0), \phi(\theta_0)]$. We also denote $(\theta_k, x_k) = F(\theta_0, x_0)$ for $k = 1, 2, \ldots$. Whenever $\theta_k \in A$, we have $x_k, y_k > 1/2$. By Lemma 3.3, we get $|x_n - y_n| \le M\mu^{m(n)}$ for some $\mu < 1$, where $m(n)$ denotes the number of indices $k \in \{0, 1, \ldots, n-1\}$ such that $\theta_k \in A$. By our choice of θ the length of the segment $F^n(\{R^{-n}(\theta)\} \times [\hat{\phi}(R^{-n}(\theta)), \phi(R^{-n}(\theta))])$ goes to 0 as $n \to \infty$. Therefore their intersection consists of a one point. □

It remains to prove that $\lambda(\phi) < 0$. To this end we need the following lemma.

Lemma 3.4. *[19, Lemma 2]. Let $(X, \mathcal{F}, \mu)$ be a probability space, $T : X \to X$ a measurable transformation leaving the measure μ invariant, and $f : X \to \mathbb{R}$ a measurable function. If the function $f \circ T - f$ has a minorant $g \in \mathrm{L}^1_\mu$, then $f \circ T - f \in \mathrm{L}^1_\mu$ and*

$$\int (f \circ T - f)d\mu = 0.$$

Now we can prove the attracting property of the invariant graph $\{(\theta, \phi(\theta)) : \theta \in \mathbb{S}^1\}$.

Theorem 3.3. *The Lyapunov exponent on the graph of ϕ in the x direction $\lambda(\phi(\theta), \theta) < 0$ for a.e. $\theta \in \mathbb{S}^1$.*

Proof. Recall that

$$\lambda(\theta, \phi(\theta)) = \int_{\mathbb{S}^1}\log|Df(\phi(\theta))|d\theta + \int_{\mathbb{S}^1}\log g(\theta)d\theta.$$

for a.e. $\theta \in \mathbb{S}^1$. Since $f(0) = 0$, $f(\phi(\theta))g(\theta) = \phi(R(\theta))$ and $Df(x) < \frac{f(x)}{x}$ for $x > 1/2$, by Lemma 3.2 there exists a set $A \in \mathbb{S}^1$ with positive Lebesgue measure such that

$$Df(\phi(\theta)) < \frac{f(\phi(\theta))}{\phi(\theta)} = \frac{\phi(R(\theta))}{\phi(\theta)g(\theta)}$$

for $\theta \in A$ and that

$$Df(\phi(\theta)) \le \frac{f(\phi(\theta))}{\phi(\theta)} = \frac{\phi(R(\theta))}{\phi(\theta)g(\theta)}$$

for $\theta \in \mathbb{S}^1 - A$. Thus, $\log \frac{\phi(R(\theta))}{\phi(\theta)}$ has the integrable minorant $\log Df(\phi(\theta)) + \log g(\theta)$. Using Lemma 3.4, it follows that $\log \frac{\phi(R(\theta))}{\phi(\theta)}$ is integrable and $\int_{\mathbb{S}^1} \log \frac{\phi(R(\theta))}{\phi(\theta)} d\theta = 0$. Hence

$$\int_{\mathbb{S}^1} \log Df(\phi(\theta))d\theta + \int_{\mathbb{S}^1} \log g(\theta)d\theta < \int_{\mathbb{S}^1} \log \frac{\phi(R(\theta))}{\phi(\theta)} d\theta = 0.$$

This proves $\lambda(\theta, \phi(\theta)) < 0$ for a.e. $\theta \in \mathbb{S}^1$. $\square$

Remark 3.4. *Since $\lambda(\phi(\theta), \theta) < 0$ for a.e. $\theta \in \mathbb{S}^1$, there exists a $\delta_\theta > 0$ such that for every $(\theta, x) \in B_{\delta_\theta}(\theta, \phi(\theta)) := \{(\theta, x) \in \{\theta\} \times I : |x - \phi(\theta)| < \delta_\theta\}$ we have*

$$|\pi_2 \circ F^n(\theta, x) - \phi(R^n(\theta))| \to 0$$

as $n \to \infty$. See [20] for a proof.

By Remark 3.4, μ_ϕ is an SRB measure of the map F.

4. Topological properties of the invariant graph

Denote by $\overline{A}$ the topological closure of a set A.

Theorem 4.1. *$\overline{Gr(\phi)}$ contains $\mathbb{S}^1 \times \{0\}$. For a.e. $\theta \in \mathbb{S}^1$ and all $x \in I$, we have $\omega(x, \theta) = \overline{\mathcal{B}^+}$, where $\omega(x, \theta)$ is the ω-limit set of the point (θ, x).*

Proof. $\mathbb{S}^1 \times \{0\} \subset \overline{\mathrm{Gr}(\phi)}$ follows from the fact that $\phi(\theta) = 0$ for a dense set of $\mathbb{S}^1$.

Denote by A the set $\{\theta \in \mathbb{S}^1 : \phi(\theta) > 0\}$. For $\theta \in A$ and $x \in I$, $\omega(\theta, x)$ is a non-empty compact invariant set of F. Then $\pi_1(\omega(\theta, x))$ is a compact subset of $\mathbb{S}^1$, which is invariant under the irrational rotation R. As it is non-empty, it must be the whole circle (minimality of R). Let

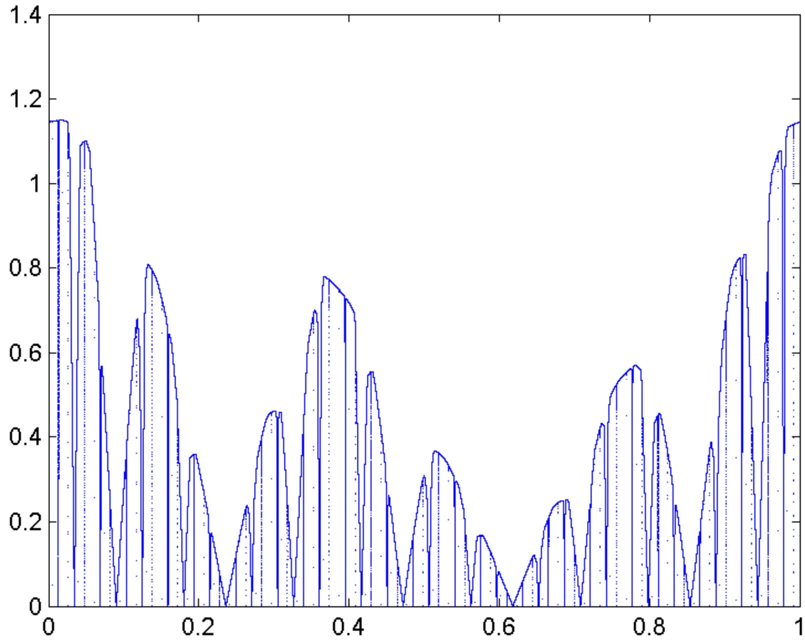
$$\gamma^+(\theta) := \sup\{x \in [0, M] : (\theta, x) \in \omega(\theta, x)\},$$
$$\gamma^-(\theta) := \inf\{x \in [0, M] : (\theta, x) \in \omega(\theta, x)\}.$$

Then by the monotonicity and continuity of the fibre maps, γ^+ and γ^- are invariant graph of F. As $\omega(\theta, x)$ is closed we get $\limsup_{\theta' \to \theta} \gamma^+(\theta') \leq \gamma^+(\theta)$, which means that γ^+ is upper semi-continuous. In the same way γ^- is lower semi-continuous. This also gives the measurability of the two graphs. By Theorem 3.2 there is no semi-continuous function in between 0 and ϕ. Therefore, we get $\gamma^+ = \phi$ and $\gamma^- = 0$, and hence $\overline{\mathrm{Gr}(\phi)} \subset \omega(\theta, x)$. On the other hand, by Remark 3.4 for each $\theta \in \mathbb{S}^1$ with $\lambda(\phi(\theta), \theta) < 0$ there is an open neighborhood B_{δ_θ} of $(\theta, \phi(\theta))$ in the fibre $\{\theta\} \times I$ such that $|x_n - \phi(\theta_n)| \to 0$ as as $n \to \infty$ for all $(\theta, x) \in B_{\delta_\theta}$. This together with the fact that $\overline{\mathrm{Gr}(\phi)} \subset \omega(\theta, x)$ imply that $\omega(x, \theta) = \overline{\mathrm{Gr}(\phi)}$. $\square$

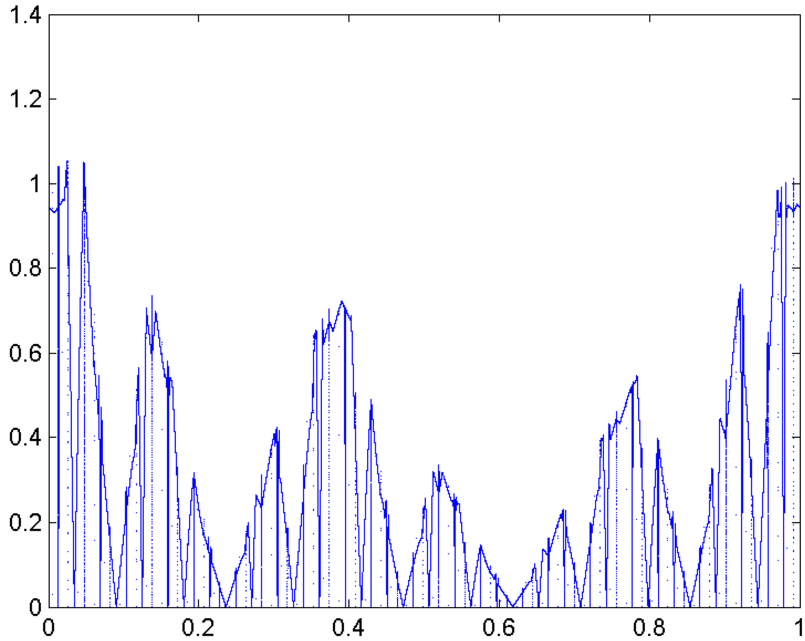
Remark 4.1. *Since $\mathcal{K} := \cap_{n \geq 0} F^n(\mathbb{S}^1 \times [0, M])$ is pinched and $\sup \mathcal{K}_\theta = \phi(\theta)$ is a.e. discontinuous and $\inf \mathcal{K}_\theta = 0$, by Corollary 4.5 of [25] F has sensitive dependence on initial conditions on $\mathbb{S}^1 \times [0, M]$.*

5. Numerical results

Take $\omega = \frac{\sqrt{5}-1}{2}$ and the initial condition $(\theta, x) = (0.01, 0.01)$. In the plot, we discard the first 10^5 images and plot the next 10^6 ones. For $a = 2.2$ and $b = 0.5$, the phase portrait of the map F is shown in Fig 1(a). This agrees fully with the mathematical results. In the previous



(a) $a = 2.2, b = 0.5$



(b) $a = 2.2, b = -0.5$

Figure 1: The attractors of the map F.

proofs the hypothesis $Df(x) < \frac{f(x)}{x}$ for $x > \frac{1}{2}$ plays an important role. When $a > 2$ and $b < 0$,

f is a unimodal map. In such case, showing a complete agreement we conjecture that F has an SNA provided $|Df(x)| < \frac{f(x)}{x}$ for $\frac{1}{2} < x \leq f(1/2)$. See Fig 1(b).

6. Conclusions

Quasiperiodically driven dynamical systems are, on general grounds, expected to display regimes wherein the dynamics is on SNAs. However, there are relatively few rigorous results available. The most extensively studied cases all have a skew-product dynamical structure. In this paper we consider a family of quasiperiodically forced piecewise linear maps. We prove that, for some set of parameter values, there exists a unique graph of a semi-continuous function which is invariant, discontinuous almost everywhere and attracts almost all orbits. Moreover, the Lyapunov exponent in the x direction is negative. Therefore, the invariant graph is an SNA. Though the proof is made for a concrete family, many of the arguments can be applied to other families.

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