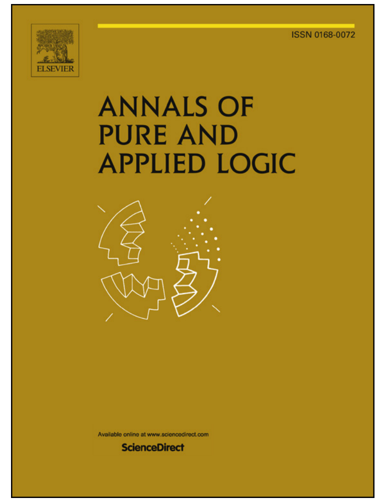
Article:

Madarász, J.X., Stannett, M. orcid.org/0000-0002-2794-8614 and Székely, G. (2022) Investigations of isotropy and homogeneity of spacetime in first-order logic. Annals of Pure and Applied Logic. 103153. ISSN 0168-0072

https://doi.org/10.1016/j.apal.2022.103153

Journal Pre-proof



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PII: S0168-0072(22)00068-9
DOI: https://doi.org/10.1016/j.apal.2022.103153
Reference: APAL 103153

To appear in: *Annals of Pure and Applied Logic*

Received date: 22 February 2021
Revised date: 22 March 2022
Accepted date: 1 June 2022

Please cite this article as: J.X. Madarász et al., Investigations of isotropy and homogeneity of spacetime in first-order logic, *Ann. Pure Appl. Logic* (2022), 103153, doi: https://doi.org/10.1016/j.apal.2022.103153.

Investigations of Isotropy and Homogeneity of Spacetime in First-Order Logic

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Abstract

We investigate the logical connection between (spatial) isotropy, homogeneity of space, and homogeneity of time within a general axiomatic framework. We show that isotropy not only entails homogeneity of space, but also, in certain cases, *homogeneity of time*. In turn, homogeneity of time implies homogeneity of space in general, and the converse also holds true in certain cases.

An important innovation in our approach is that formulations of physical properties are simultaneously empirical and axiomatic (in the sense of first-order mathematical logic). In this case, for example, rather than presuppose the existence of spacetime metrics – together with all the continuity and smoothness apparatus that would entail – the basic logical formulas underpinning our work refer instead to the sets of (idealised) experiments that support the properties in question, e.g., isotropy is axiomatized by considering a set of experiments whose outcomes remain unchanged under spatial rotation. Higher-order constructs are not needed.

Keywords: first-order logic, relativity theory, classical spacetime, homogeneity, isotropy, axiomatization
2020 MSC: 03B30, 83A05, 03B10, 03B80

1. Introduction

In this paper, we investigate the logical connections between the following properties of space and time, (spatial) isotropy (ISO), homogeneity of space ($\mathsf{HOM}_{\mathsf{space}}$), and homogeneity of time ($\mathsf{HOM}_{\mathsf{time}}$) within a general axiomatic framework for physics assuming some minimal kinematical axioms. Among

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other things, we show formally that isotropy implies not just homogeneity of space, but also *homogeneity of time* if there are clocks that loose synchrony.[1] In turn, homogeneity of time implies homogeneity of space in general, and the converse implication also holds if there are clocks that loose synchrony, see Figure 1.

The novelty in our approach is, accordingly, that formulations of homogeneity and isotropy are simultaneously empirical and axiomatic. Rather than presuppose the existence of, e.g., metrics satisfying appropriate differentiability constraints, the basic logical formulas underpinning our work refer instead to the sets of experiments that support the properties in question. So, for example, the idea that space is isotropic is directly connected to a set of (idealised) physical experiments whose outcomes should remain unchanged under spatial rotation. Higher-order mathematical constructs are not needed, because in descriptions of experiments, one typically does not quantify over sets or other higher-order logic objects.[2]

This work is part of the Andréka–Németi school's long-running project to axiomatize and analyse relativity theories within first-order logic, see e.g., [1, 2, 4, 7, 20, 24, 25]. It is also related to Hilbert's Sixth Problem:

> "The investigations on the foundations of geometry suggest the problem: *To treat in the same manner, by means of axioms, those physical sciences in which mathematics plays an important part*" [19, p. 454, (Hilbert's emphasis)].

For discussion of the methodological and epistemological significance of this project, see Friend and Molinini [13, 14]. For a comprehensive comparison between Hilbert's project on the foundations of physics and the Andréka–Németi project, see Formica and Friend [12].

An important feature of the axiomatic approach is that it helps avoid hidden assumptions, which is fundamental in foundational analyses of this nature. It also allows us to verify our results computationally using interactive theorem provers like Isabelle/HOL [18, 33]. For a fuller discussion of the benefits of first-order logic see e.g., [6, §Appendix: Why FOL?], [36, §11].

Another important feature of the axiomatic approach is that it helps explain why some apparently contradicting results do not lead to real contradictions. This is so because the same informal idea can have several slightly different formalisations which may lead to contradicting consequences especially if one uses different hidden assumptions in deriving these consequences. Using the axiomatic approach, formalisation forces us to make every detail explicit and to reveal not just the hidden assumptions but also these slight differences in our

[1] That spatial isotropy entails spatial homogeneity is to be expected (but nonetheless requires formal confirmation within the restricted framework used here) because every spatial translation can be obtained as a composition of spatial rotations. However, the fact that *temporal* homogeneity is also entailed, in certain cases, was more surprising.

[2] Nevertheless, if we decide for some reason that including sets, functions, etc., would be useful, we can include them via new sorts by enriching our first-order language.

understandings of certain key assumptions. Hence the easily ignorable details that led to the apparent contradictions become highlighted and unignorable.

1.1. Informal explanations of the main results

By *homogeneity of time*, we mean that no matter when we initiate a certain experimental configuration the progress and outcome of the experiment will be the same. Analogously, the intuitive meaning of *homogeneity of space* is that the progress and outcome of an experiment does not change if we simply translate its configuration to another location. Likewise, the intuitive meaning of *isotropy* is that rotating an experimental configuration in space does not change the outcome.

In order to define these concepts formally, we need to describe what we mean by an experiment. Since we generally adopt a static representation of spacetime, the most natural approach is to represent any given experiment as a 4-dimensional spacetime "scenario," capturing its initial and final states, together with its progress from one to the other and any related feasibility constraints. For example, an elastic collision between two point particles would be described by giving a point with two incoming and two outgoing worldlines. We then ask which experiments remain feasible under various types of spacetime transformation. For example, isotropy can be captured by the claim that, if an experiment is realisable, then all spatially-rotated versions of that experiment are also realisable. These ideas will be formulated using formula schemas in our first-order logic framework, see p.21.

In this framework, we prove the connections between isotropy (ISO), homogeneity of time ($\mathsf{HOM}_{\mathsf{time}}$) and homogeneity of space ($\mathsf{HOM}_{\mathsf{space}}$) illustrated in the left-hand side of Figure 1 using just the following simple and natural kinematical assumptions as auxiliary axioms:

- The structure of physical quantities satisfies some of the most fundamental properties of real numbers, i.e., they form an ordered field ($\mathsf{AxOField}$, p. 24).
- Inertial coordinate systems remain inertial if they are rotated or translated ($\mathsf{AxReloc}^{\mathsf{Tran}}$, $\mathsf{AxReloc}^{\mathsf{Rot}}$, p. 24).
- Coordinate transformations between inertial observers are affine ($\mathsf{AxAffTr}$, p. 24).
- There is an inertial coordinate system relative to which there are three inertial coordinate systems moving in linearly independent spatial directions ($\mathsf{Ax3Dir\exists Motion}$, p. 25).

To show the extra logical connections illustrated in the right-hand side of Figure 1, we only need one extra assumption implied by relativity theory, viz.

- There are clocks that lose synchrony (ASync, p. 25).

1.2. Structure of the paper

We present the main results of this paper in two levels of sophistication. First, in Section 3, we use a simple language where we put the experimental

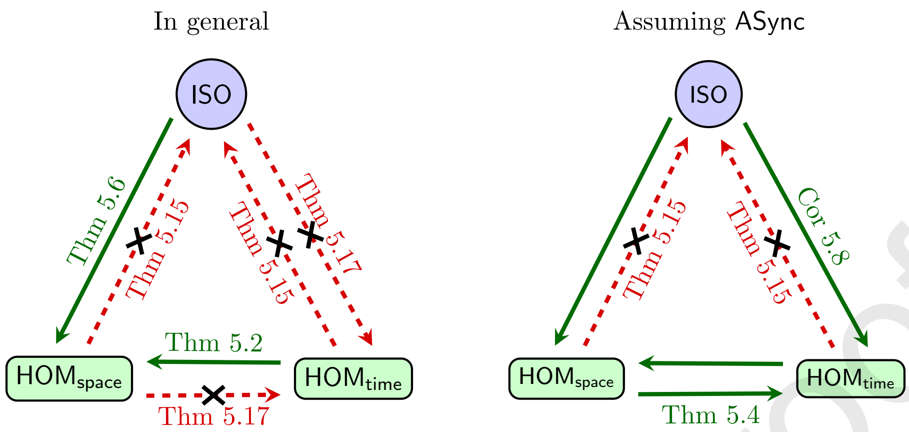

Figure 1: The relationships we establish in this paper between isotropy, homogeneity of space and homogeneity of time.

indistinguishability of inertial observers into a "black box" represented by an abstract equivalence relation. Later in Section 4, we replace this black box with a concrete notion explicitly talking about experiments. Then in Section 5, we revisit the main results in this more complex language.

There are three possible ways to read this paper: one can read every section in order; one can jump immediately from Section 2 to Section 4; or one may read the discussion in Section 6 after reading Section 3. The structure of the paper is depicted in Figure 2.

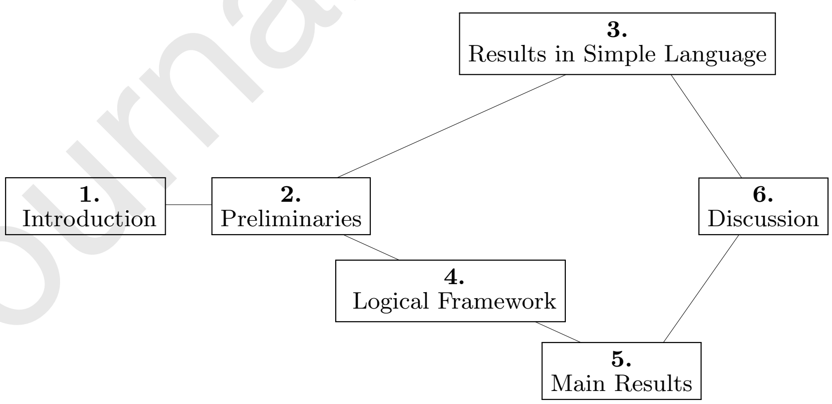

Figure 2: Structure of the paper.

2. Preliminaries: Structure of the quantities, coordinate systems, translations and spatial rotations

With some sporadic exceptions, most of the non-axiomatic literature assumes that the structure of physical quantities is isomorphic to the field of real numbers. This is quite a strong assumption, which cannot be justified experimentally. Therefore, we prefer assuming only a small fragment of this, namely that the structure of physical quantities forms an ordered field, i.e., we can add, multiply and compare physical quantities and these operations satisfy some basic properties valid in the field of real numbers.

Throughout the paper Q is a nonempty set of *Quantities* which are used to specify coordinates, lengths and related quantities, and we assume that Q is equipped with the usual binary operations, $\cdot$ (multiplication) and $+$ (addition); constants, 0 and 1 (additive and multiplicative identities); and a binary relation, $\leqslant$. We assume that $(Q, +, \cdot, 0, 1, \leqslant)$ satisfies the most fundamental algebraic properties of the real numbers, so that calculations can be performed and results compared with one another:

$\mathsf{AxOField}$ $\quad (Q, +, \cdot, 0, 1, \leqslant)$ is an ordered field.[3]

Elements of the coordinate system Q^4 representing spacetime locations are denoted using over-arrows, e.g., $\vec{p}$, $\vec{q}$, $\vec{x}$, $\vec{y}$, $\vec{v}$, ... We define the time-axis, $\mathbf{t}$, to be the set $\mathbf{t} \stackrel{\text{def}}{=} \{(t, 0, 0, 0) : t \in Q\}$. Alongside the time axis, we define the *simultaneity*, S, to be the set $\mathsf{S} \stackrel{\text{def}}{=} \{(0, x, y, z) : x, y, z \in Q\}$. If $\vec{p} = (t, x, y, z) \in Q^4$, then $\vec{p}_{\mathsf{t}} \stackrel{\text{def}}{=} t$ is the *time component* and $\vec{p}_{\mathsf{s}} \stackrel{\text{def}}{=} (x, y, z)$ is the *space component* of $\vec{p}$. For simplicity, we write $\vec{\mathsf{o}} \stackrel{\text{def}}{=} (0, 0, 0, 0)$ for the *origin* of Q^4. We will often omit the multiplication symbol "$\cdot$". The *squared Euclidean length* of $\bar{x} = (x_1, \ldots, x_n) \in Q^n$ is defined as $|\bar{x}|^2 \stackrel{\text{def}}{=} x_1^2 + \ldots + x_n^2$. We will use the usual vector space operations on Q^n. The unit vectors of Q^4 are denoted as $\vec{\mathsf{e}}_1 \stackrel{\text{def}}{=} (1, 0, 0, 0)$, $\vec{\mathsf{e}}_2 \stackrel{\text{def}}{=} (0, 1, 0, 0)$, $\vec{\mathsf{e}}_3 \stackrel{\text{def}}{=} (0, 0, 1, 0)$ and $\vec{\mathsf{e}}_4 \stackrel{\text{def}}{=} (0, 0, 0, 1)$.

A function $T : Q^4 \to Q^4$ is a *translation* iff there is $\vec{v} \in Q^4$ such that $T(\vec{p}) = \vec{p} + \vec{v}$ for every $\vec{p} \in Q^4$. It is a *spatial translation* when $\vec{v} \in \mathsf{S}$ and a *temporal translation* when $\vec{v} \in \mathbf{t}$. We denote the set of all translations, spatial translations, and temporal translations, respectively, by Tran, $\mathsf{Tran}_{\mathsf{space}}$, and $\mathsf{Tran}_{\mathsf{time}}$. A map $L : Q^4 \to Q^4$ is a *linear transformation* iff it is a bijection and $L(\lambda\vec{p} + \vec{q}) = \lambda L(\vec{p}) + L(\vec{q})$ for every $\vec{p}, \vec{q} \in Q^4$ and $\lambda \in Q$. A map $A : Q^4 \to Q^4$ is an *affine transformation* iff it is a composition of a linear transformation and a translation. A linear transformation $R : Q^4 \to Q^4$ is a *spatial rotation* iff it leaves the time axis pointwise fixed, preserves the simultaneity setwise (as well as squared Euclidean lengths measured within it), and preserves the orientation of space, i.e., if $\vec{p} \in \mathbf{t}$, then $R(\vec{p}) = \vec{p}$, if $\vec{p} \in \mathsf{S}$, then $R(\vec{p}) \in \mathsf{S}$ and $|R(\vec{p})|^2 = |\vec{p}|^2$,

[3] That $(Q, +, \cdot, 0, 1, \leqslant)$ is an ordered field means that $(Q, +, \cdot, 0, 1)$ is a field which is totally ordered by $\leqslant$, and we have the following two properties for all $x, y, z \in Q$: (1) $x + z \leqslant y + z$ if $x \leqslant y$, and (2) $0 \leqslant xy$ if $0 \leqslant x$ and $0 \leqslant y$.

and the determinant of 3×3 matrix $[R(\vec{e}_2)_s, R(\vec{e}_3)_s, R(\vec{e}_4)_s]$ is positive[4]. Id denotes the identity transformation from Q^4 to Q^4.

3. Main Results Presented in a Simple Language

To highlight the group theoretic and geometric ideas behind the proofs of our main results, first we formulate and prove them using a very simple language in which we use an abstract equivalence relation to capture the idea of experimental indistinguishability between inertial observers. Later, using a more complex language, we are going to introduce an explicit notion of observers agreeing on certain experiments and use that notion in place of this abstract one.

3.1. Our simple language

We are concerned in this section with two sorts of objects, *(inertial) observers* and *quantities*, which we represent as elements of nonempty sets IOb and Q, respectively.

Observers are interpreted to be labels for inertial coordinate systems and as we explained in Section 2, we assume that Q is equipped with the usual binary operations, $\cdot$ (multiplication) and $+$ (addition); binary relation, $\leqslant$ (ordering) and constants 0 and 1.

For $o, o' \in IOb$, we assume the existence of a function $w_{oo'} : Q^4 \to Q^4$, called the *worldview transformation* from the worldview of o' to the worldview of o, which we interpret as saying that the event seen (coordinatized) by o' at $\vec{p}$ is seen (coordinatized) by o at $w_{oo'}(\vec{p})$. In later sections, in richer languages, these events and worldview transformations will be defined concepts.

3.2. Formulations of isotropy, homogeneity in our simple language

Homogeneity of time/space simply means that the outcome of an experiment does not depend on when/where the experiment is performed and isotropy of space means that the spatial orientation of an experiment does not affect its outcome. We can capture the translations/rotations of experiments by using the coordinate transformations between observers. For example, consider a certain experiment initiated at time t_0 in the coordinate system of observer o and suppose we are interested in whether the outcome of the experiment would be the same or not if it was initiated at time t_1. Then instead of translating the experiment to t_1 in the coordinate system of o, we can consider observer o' such that the worldview transformation from o to o' is the temporal translation that maps t_1 to t_0, and we can ask o' if it sees the same outcome when initiating

[4] This can be expressed without any assumption about the structure of quantities as: $R(\vec{e}_2)_2R(\vec{e}_3)_3R(\vec{e}_4)_4 + R(\vec{e}_2)_4R(\vec{e}_3)_2R(\vec{e}_4)_3 + R(\vec{e}_2)_3R(\vec{e}_3)_4R(\vec{e}_4)_2 > R(\vec{e}_2)_4R(\vec{e}_3)_3R(\vec{e}_4)_2 + R(\vec{e}_2)_2R(\vec{e}_3)_4R(\vec{e}_4)_3 + R(\vec{e}_2)_3R(\vec{e}_3)_2R(\vec{e}_4)_4$, here $R(\vec{p})_2$, $R(\vec{p})_3$, and $R(\vec{p})_4$ denote the second, third and fourth component of $R(\vec{p}) \in Q^4$, i.e., if $R(\vec{p}) = (t, x, y, z)$, then $R(\vec{p})_2 = x$, $R(\vec{p})_3 = y$, and $R(\vec{p})_4 = z$.

the experiment with the same spatial orientation at t_0. This story works for spatial translations and rotations in exactly the same way. (The trick of moving observers instead of the experiments is also used in informal approaches, see e.g., [11, pp. 21–22].)

The simple language of Section 3.1 is not suitable to formulate explicitly what we mean when we say that two observers agree on the outcomes of experiments. In the present section, we substitute this notion with an abstract equivalence relation, which we interpret as saying that two observers are equivalent exactly if they agree on the outcomes of experiments performed at the same coordinate point and with same spatial orientation. So let $\sim$ be a binary relation on IOb. In our theorems we will assume that $\sim$ is an equivalence relation.

We formulate the homogeneity of time/space by the statement that observers whose coordinate systems differ only by temporal/spatial translation are $\sim$-equivalent. Isotropy is formulated by the statement that observers whose coordinate systems differ only by a spatial rotation are $\sim$-equivalent. Formally:

$\mathscr{HOM}^{\sim}_{\mathsf{time}}$ Homogeneity of time:

For every $o, o' \in IOb$, $o \sim o'$ whenever $w_{oo'} \in \mathsf{Tran}_{\mathsf{time}}$.

$\mathscr{HOM}^{\sim}_{\mathsf{space}}$ Homogeneity of space:

For every $o, o' \in IOb$, $o \sim o'$ whenever $w_{oo'} \in \mathsf{Tran}_{\mathsf{space}}$.

$\mathscr{HOM}^{\sim}$ Homogeneity of spacetime:

For every $o, o' \in IOb$, $o \sim o'$ whenever $w_{oo'} \in \mathsf{Tran}$.

$\mathscr{ISO}^{\sim}$ Isotropy of space:

For every $o, o' \in IOb$, $o \sim o'$ whenever $w_{oo'} \in \mathsf{Rot}_{\mathsf{space}}$.

In these definitions, $\mathsf{Tran}_{\mathsf{time}}$, $\mathsf{Tran}_{\mathsf{space}}$ and Tran are the usual sets of temporal, spatial and spacetime translations of Q^4, respectively; and $\mathsf{Rot}_{\mathsf{space}}$ is the set of spatial rotations of Q^4 introduced in Section 2.

3.3. Assumptions

The *worldline of observer* o' according to another observer o is the $w_{oo'}$-image of the time-axis:

$$w\ell_o(o') \stackrel{\text{def}}{=} w_{oo'}[\mathbf{t}] = \{w_{oo'}(\vec{p}) \,:\, \vec{p} \in \mathbf{t}\}.$$

Below we list the assumptions that we use in the present section.

$\mathsf{AxOField}$ $(Q, +, \cdot, 0, 1, \leqslant)$ is an ordered field.

Wvt For every $o, o', o'' \in IOb$, $w_{oo'} \circ w_{o'o''} = w_{oo''}$ and w_{oo} is the identity transformation Id.

$\mathsf{Reloc}^{\mathsf{Tran}}$ Translations of inertial coordinate systems are inertial:

For every $o \in IOb$ and every $T \in \mathsf{Tran}$ there is o' such that $w_{oo'} = T$.

$\mathsf{Reloc}^{\mathsf{Rot}}$ Rotations of inertial coordinate systems are inertial:

For every $o \in IOb$ and every $R \in \mathsf{Rot}_{\mathsf{space}}$ there is o' such that $w_{oo'} = R$.

AffTr Worldview transformations are affine transformations.

$\mathsf{3Dir\exists Motion}$ There is an inertial observer o according to which there are three inertial observers o_1, o_2 and o_3 moving in linearly independent spatial directions (see Figure 3):

There are $o, o_1, o_2, o_3 \in IOb$ with $\vec{\mathsf{o}} \in w\ell_o(o_1) \cap w\ell_o(o_2) \cap w\ell_o(o_3)$ and there are $\vec{p}^{\,1} \in w\ell_o(o_1)$, $\vec{p}^{\,2} \in w\ell_o(o_2)$ and $\vec{p}^{\,3} \in w\ell_o(o_3)$ such that $\vec{p}^{\,1}_{\mathsf{s}}$, $\vec{p}^{\,2}_{\mathsf{s}}$ and $\vec{p}^{\,3}_{\mathsf{s}}$ are linearly independent vectors.

async There are clocks that get out of sync, i.e., there are events which are simultaneous for one observer but not for another one (see Figure 3):

There are $o, o' \in IOb$ and $\vec{p}, \vec{q} \in Q^4$ such that $\vec{p}_{\mathsf{t}} \neq \vec{q}_{\mathsf{t}}$ and $w_{o'o}(\vec{p})_{\mathsf{t}} = w_{o'o}(\vec{q})_{\mathsf{t}}$.

In Section 4.4, the assumptions above will be reformulated in a first-order language. These reformulations will all be taken as axioms apart for the reformulation ASync of async. We treat async differently, because it is not a natural basic assumption for kinematics.

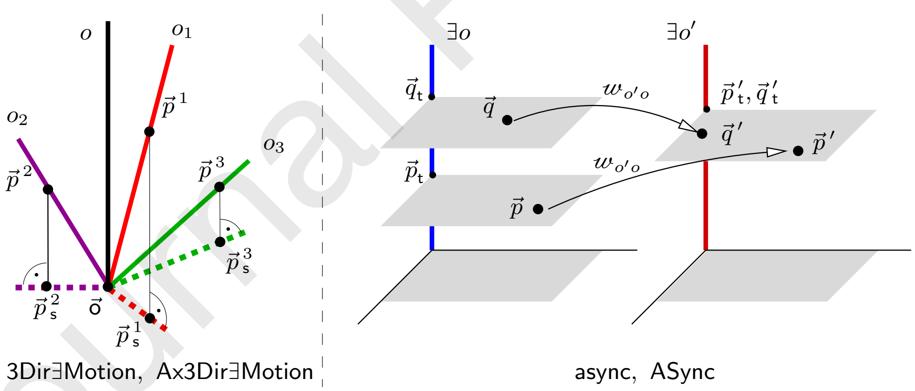

Figure 3: Illustration of 3Dir∃Motion, Ax3Dir∃Motion, async and ASync.

Let $\sim$ be a binary relation on IOb. We say that $\sim$ has the *transformation property* iff for every $o, u, o', u' \in IOb$, if $w_{ou} = w_{o'u'}$ and $o \sim o'$, then $u \sim u'$, see Figure 4.

The intuitive meaning of transformation property is the following: Assume o and o' agree on the outcomes of experiments performed with the same initialisation (i.e., at the same coordinate point and with the same orientation) and the

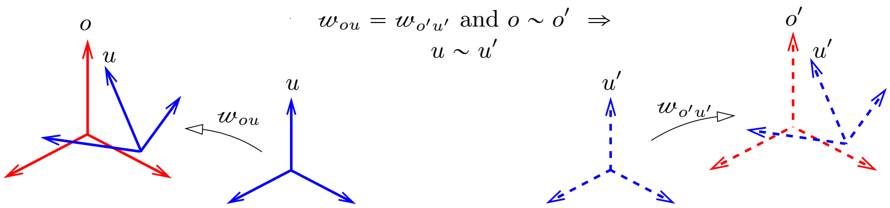

Figure 4: Transformation property.

worldview of u is related to the worldview of o the same way as the worldview of u' is related to the worldview of o'. Then u and u' also agree on the outcomes of experiments performed with the same initialisation. The intuitive motivation for this conclusion is the following. Let $\mathscr{e}$ denote an experiment together with its initialisation that is performed by both u and u'. In the worldviews of o and o', the manifestations of $\mathscr{e}$ are respectively $\breve{\mathscr{e}}$ and $\breve{\mathscr{e}}'$ (which are experiments together with their initialisations). Since u is related to o the same way as u' is related to o', $\breve{\mathscr{e}}$ and $\breve{\mathscr{e}}'$ are the same. Observers o and o' agree on the outcome of $\breve{\mathscr{e}} = \breve{\mathscr{e}}'$ by assumption. Therefore, it is natural to assume that u and u' agree on the outcome of $\mathscr{e}$. Later when we will be able to talk about experiments explicitly, we will make this intuitive derivation of the transformation property precise. Specifically, in Section 4.5, we are going to define a unary map $\breve{}$ from formulas-describing-experiments to formulas, and instead of the transformation property, we will assume that this unary operation maps descriptions of experiments to descriptions of experiments. Then we will prove the transformation property in Lemma 5.27.

3.4. Main results in the simple language

Since the intuitive meaning of relation $o \sim o'$ is that observers o and o' get the same outcomes when performing the same experiments with the same initial configuration, it is natural to assume that $\sim$ is an equivalence relation.

We prove the results of the present section in Section 3.6.

3.4.1. Homogeneity of time implies homogeneity of space

Theorem 3.1. *Assume* $\mathsf{AxOField}$, Wvt, $\mathsf{Reloc^{Tran}}$, AffTr *and* $\mathsf{3Dir\exists Motion}$. *Let* $\sim$ *be an equivalence relation on IOb that has the transformation property. Then*

$$\mathscr{HOM}^{\sim}_{\mathsf{time}} \quad \Rightarrow \quad \mathscr{HOM}^{\sim}_{\mathsf{space}}.$$

By Theorem 3.5, homogeneity of space does not imply homogeneity of time in general, cf. Theorems 5.3 and 5.17. However, if we assume that there are clocks that lose synchrony, homogeneity of space implies homogeneity of time:

Theorem 3.2. *Assume* AxOField*,* Wvt*,* $\mathsf{Reloc}^{\mathsf{Tran}}$*,* AffTr *and that* $\sim$ *is an equivalence relation on IOb that has the transformation property. Then*

$$(\mathscr{HOM}^{\sim}_{\mathsf{space}} + \mathsf{async}) \quad \Rightarrow \quad \mathscr{HOM}^{\sim}_{\mathsf{time}}.$$

While homogeneity of spacetime clearly entails homogeneity of both time and space, the converse does not hold in general, i.e., homogeneities of time and space together do not imply homogeneity of spacetime, cf. Proposition 5.1(ii). However, if we make some mild additional assumptions, the converse becomes true.

Proposition 3.3. *Assume* AxOField*,* Wvt*,* $\mathsf{Reloc}^{\mathsf{Tran}}$ *and that* $\sim$ *is an equivalence relation on IOb. Then*

$$(\mathscr{HOM}^{\sim}_{\mathsf{time}} + \mathscr{HOM}^{\sim}_{\mathsf{space}}) \quad \Rightarrow \quad \mathscr{HOM}^{\sim}.$$

3.4.2. Isotropy of space implies homogeneity of space

Theorem 3.4. *Assume* AxOField*,* Wvt*,* $\mathsf{Reloc}^{\mathsf{Tran}}$ *and* $\mathsf{Reloc}^{\mathsf{Rot}}$*. Let* $\sim$ *be an equivalence relation on IOb that has the transformation property. Then*

$$\mathscr{ISO}^{\sim} \quad \Rightarrow \quad \mathscr{HOM}^{\sim}_{\mathsf{space}}.$$

In the case of positive results, the formulations and proofs in the simple language give some insights and explanations even though $\sim$ is just an abstract equivalence relation. We can formulate and prove also our negative results using the simple language; however, the counterexamples are less explanatory than in the language of Section 4 because they do not contain any physical phenomena showing why some property, say homogeneity of time, does not hold. Nevertheless, for illustration, we formulate and prove one of our negative results, namely Theorem 5.17, in the simple language. The other counterexamples of Section 5.5 can also be transformed into the simple language.

3.4.3. Isotropy and homogeneity of space together do not imply homogeneity of time

Theorem 3.5. *Assume that* $\sim$ *is an equivalence relation on IOb. Then*

$$(\mathscr{ISO}^{\sim} + \mathscr{HOM}^{\sim}_{\mathsf{space}}) \quad \not\Rightarrow \quad \mathscr{HOM}^{\sim}_{\mathsf{time}}$$

even if we assume AxOField*,* Wvt*,* $\mathsf{Reloc}^{\mathsf{Tran}}$*,* $\mathsf{Reloc}^{\mathsf{Rot}}$*,* AffTr*,* 3Dir∃Motion *and that* $\sim$ *has the transformation property.*

We note that homogeneity of spacetime does not imply isotropy, cf. Theorems 5.9, 5.15.

3.5. Connections to groups and group actions

The set of worldview transformations is defined as

$$\mathscr{W} \stackrel{\text{def}}{=} \{w_{ou} : o, u \in IOb\}.$$

Notice that $\mathscr{W}$ does not necessarily form a group under composition. For example, if $(Q, +, \cdot, 0, 1, \leqslant)$ is the field of reals and IOb consists of only two observers o and u such that $w_{ou} \neq \mathsf{Id}$ is a Lorentz boost, $w_{oo} := w_{uu} := \mathsf{Id}$, and $w_{uo} := w_{ou}^{-1}$, then $w_{ou} \circ w_{ou} \notin \mathscr{W} = \{\mathsf{Id}, w_{ou}, w_{ou}^{-1}\}$. This counterexample satisfies AxOField, Wvt, AffTr and async. It is not difficult to construct a counterexample that satisfies all the assumptions listed in Section 3.3.

We will see that if we assume Spr below in addition to Wvt, $\mathscr{W}$ forms a group under composition.

Spr For every $o, o', u \in IOb$, there is $u' \in IOb$ such that $w_{ou} = w_{o'u'}$.

Assumption Spr is a small slice of the special principle of relativity; it captures the idea that inertial observers cannot be distinguished by how they can be related to other inertial observers. Spr is an axiom in [22] and [23].

Proposition 3.6. *Assume* Wvt*. Then:*

(i) Worldview transformations are bijections and $w_{uo} = w_{ou}^{-1}$ for every $o, u \in IOb$.
(ii) $\mathscr{W}$ forms a group under composition if Spr *holds.*

Proof. (i): It follows from $w_{ou} \circ w_{uo} = w_{oo} = \mathsf{Id}$ and $w_{uo} \circ w_{ou} = w_{uu} = \mathsf{Id}$ that w_{ou} and w_{uo} are mutual inverses, and hence that they are both bijections.

(ii): By (i), it is enough to prove that $\mathscr{W}$ is closed under composition. To see this, let $w_{jo'}, w_{ou} \in \mathscr{W}$. Let $u' \in IOb$ be such that $w_{ou} = w_{o'u'}$. Such u' exists by Spr. Now, by Wvt, $w_{jo'} \circ w_{ou} = w_{jo'} \circ w_{o'u'} = w_{ju'} \in \mathscr{W}$.

For the next definition, we also need the following extensionality assumption:

Ext If $w_{ou} = \mathsf{Id}$, then $o = u$, for every $o, u \in IOb$.

Intuitively, Ext states if observers o and u have the same worldviews, then they have to be the same observer; or in other words, different observers have to have different worldviews.

Proposition 3.7. *Assume* Wvt*,* Spr *and* Ext*. Let $w \in \mathscr{W}$ and $o \in IOb$. Then there is a unique $u \in IOb$ such that $w_{ou} = w$.*

Proof. Let $w = w_{o'u'} \in \mathscr{W}$. Let $u \in IOb$ such that $w = w_{o'u'} = w_{ou}$. Such a u exists by Spr. To prove the uniqueness assume that $h \in IOb$ is such that $w_{oh} = w$. Then $w_{hu} = w_{ho} \circ w_{ou} = w^{-1} \circ w = \mathsf{Id}$ by Wvt and Proposition 3.6. Then $h = u$ by Ext.

Definition 3.8. Assume Wvt, Spr and Ext. We define a function

$$\alpha : \mathscr{W} \times IOb \to IOb$$

as follows. Let $w \in \mathscr{W}$ and $o \in IOb$. Then, by Proposition 3.7, there is a unique $u \in IOb$ such that $w_{ou} = w$. We define $\alpha(w, o)$ to be this unique u. Instead of $\alpha(w, o)$ we will write wo.

The following proposition gives group theoretic reformulations of our assumptions if Wvt, Spr and Ext hold. It is straightforward to prove its items, hence we omit their proofs.

Proposition 3.9. *Assume* Wvt*,* Spr *and* Ext*. Let* $\sim$ *be an equivalence relation on* IOb*. Then*

(i) α *is a regular*[5] *group action of* $\mathscr{W}$ *on* IOb*.*
(ii) $\sim$ *has the transformation property iff, for all* $o, o' \in IOb$ *and* $w \in \mathscr{W}$

$$o \sim o' \;\Rightarrow\; wo \sim wo'.$$

(iii) $\mathsf{Reloc}^{\mathsf{Tran}}$ *and* $\mathsf{Reloc}^{\mathsf{Rot}}$ *are equivalent to* $\mathsf{Tran} \subseteq \mathscr{W}$ *and* $\mathsf{Rot}_{\mathsf{space}} \subseteq \mathscr{W}$*, respectively.*
(iv) Assuming $\mathsf{Reloc}^{\mathsf{Tran}}$*,* $\mathscr{HOM}^{\sim}_{\mathsf{time}}$ *holds iff the equivalence classes of* $\sim$ *are closed under the actions of temporal translations, i.e.,*

$$w \in \mathsf{Tran}_{\mathsf{time}},\; o \in IOb \;\Rightarrow\; o \sim wo.$$

(v) Assuming $\mathsf{Reloc}^{\mathsf{Tran}}$*,* $\mathscr{HOM}^{\sim}_{\mathsf{space}}$ *holds iff the equivalence classes of* $\sim$ *are closed under the actions of spatial translations, i.e.,*

$$w \in \mathsf{Tran}_{\mathsf{space}},\; o \in IOb \;\Rightarrow\; o \sim wo.$$

(vi) Assuming $\mathsf{Reloc}^{\mathsf{Tran}}$*,* $\mathscr{HOM}^{\sim}$ *holds iff the equivalence classes of* $\sim$ *are closed under the actions of translations, i.e.,*

$$w \in \mathsf{Tran},\; o \in IOb \;\Rightarrow\; o \sim wo.$$

(vii) Assuming $\mathsf{Reloc}^{\mathsf{Rot}}$*,* $\mathscr{ISO}^{\sim}$ *holds iff the equivalence classes of* $\sim$ *are closed under the actions of spatial rotations, i.e.,*

$$w \in \mathsf{Rot}_{\mathsf{space}},\; o \in IOb \;\Rightarrow\; o \sim wo.$$

3.6. Proofs of Theorems 3.1, 3.2, 3.4, 3.5, and Proposition 3.3

Let $T_{\vec{v}} : Q^4 \to Q^4$ denote the *translation* by vector $\vec{v} \in Q^4$, i.e., $T_{\vec{v}}(\vec{p}) = \vec{p} + \vec{v}$ for all $\vec{p} \in Q^4$. Assuming $\mathsf{AxOField}$, we recall that: $T_{\vec{v}}$ is a bijection, ${T_{\vec{v}}}^{-1} = T_{-\vec{v}}$, $T_{\vec{o}} = \mathsf{Id}$ and $T_{\vec{v}+\vec{v}\,'} = T_{\vec{v}} \circ T_{\vec{v}\,'} = T_{\vec{v}\,'} \circ T_{\vec{v}}$ for every $\vec{v}, \vec{v}\,' \in Q^4$.

[5] A group action is *regular* if it is transitive and only the action of the identity element has fixed points.

Lemma 3.10. *Assume* AxOField. *Let A be an affine transformation, and suppose that $\vec{p}, \vec{q} \in Q^4$ and $\lambda \in Q$. Then:*

(i) $A \circ T_{\vec{p}-\vec{q}} = T_{A(\vec{p})-A(\vec{q})} \circ A$.
(ii) $A(\lambda\vec{p}) - A(\lambda\vec{q}) = \lambda\big(A(\vec{p}) - A(\vec{q})\big)$.

Proof. This follows easily from the fact that $A = L \circ T$ for some linear transformation L and translation T, and that $A(\vec{p}\,') - A(\vec{q}\,') = L(\vec{p}\,') - L(\vec{q}\,')$ for all $\vec{p}\,', \vec{q}\,' \in Q^4$.

Let us recall that, by Proposition 3.6(i), from Wvt it follows that worldview transformations are bijections and $w_{hk}^{-1} = w_{kh}$ for every $k, h \in IOb$. We will use this fact in the definitions and proofs.

Definition 3.11. Let $k, h, m \in IOb$ and $\vec{v} \in Q^4$. We say that h is a *$\vec{v}$-translated version* of k according to m, and write $k \xrightarrow{\vec{v}}_m h$, if $w_{mh} = T_{\vec{v}} \circ w_{mk}$, see the left-hand side of Figure 5.

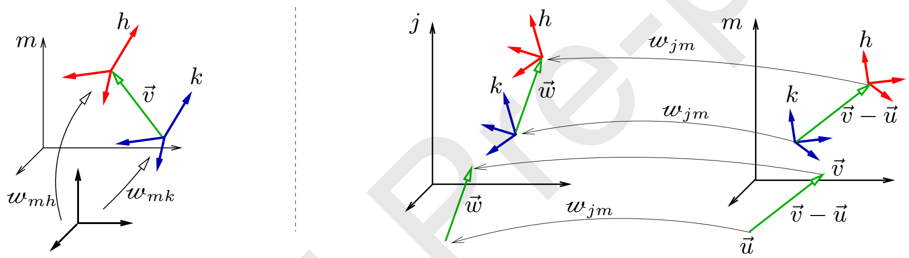

Figure 5: Illustrations for Definition 3.11 and Lemma 3.12(iv).

Lemma 3.12. *Assume* AxOField *and* Wvt. *Then for all $k, h, j, m \in IOb$ and $\vec{v}, \vec{u} \in Q^4$, we have:*

(i) $k \xrightarrow{\vec{o}}_m k$.
(ii) If $k \xrightarrow{\vec{v}}_m h$, *then* $h \xrightarrow{-\vec{v}}_m k$.
(iii) If $k \xrightarrow{\vec{v}}_m h$ *and* $h \xrightarrow{\vec{u}}_m j$, *then* $k \xrightarrow{\vec{v}+\vec{u}}_m j$.
(iv) Suppose w_{jm} is an affine transformation, and $k \xrightarrow{\vec{v}-\vec{u}}_m h$. *Then* $k \xrightarrow{\vec{w}}_j h$ *for $\vec{w} := w_{jm}(\vec{v}) - w_{jm}(\vec{u})$, see the right-hand side of Figure 5.*
(v) If $w_{kh} = T_{\vec{v}}$, then $k \xrightarrow{\vec{v}}_k h$.

Proof. (i), (ii), (iii) are straightforward. To prove (iv), let $\vec{w} := w_{jm}(\vec{v}) - w_{jm}(\vec{u})$. Then, by Lemma 3.10(i), $w_{jm} \circ T_{\vec{v}-\vec{u}} = T_{\vec{w}} \circ w_{jm}$. By $k \xrightarrow{\vec{v}-\vec{u}}_m h$, we have that $w_{mh} = T_{\vec{v}-\vec{u}} \circ w_{mk}$. By these and Wvt, we have $w_{jh} = w_{jm} \circ w_{mh} = w_{jm} \circ T_{\vec{v}-\vec{u}} \circ w_{mk} = T_{\vec{w}} \circ w_{jm} \circ w_{mk} = T_{\vec{w}} \circ w_{jk}$, as required. To prove (v), assume that $w_{kh} = T_{\vec{v}}$. Then, $w_{kh} = w_{kh} \circ \mathsf{Id} = T_{\vec{v}} \circ w_{kk}$, as claimed.

Lemma 3.13. *Assume* AxOField, Wvt, $\mathsf{Reloc}^{\mathsf{Tran}}$ *and* AffTr. *Then, for every* $k, m \in IOb$ *and* $\vec{v} \in Q^4$, *there exists* $h \in IOb$ *such that* $k \xrightarrow{\vec{v}}_m h$. *This* h *is unique if* Ext *is assumed.*

Proof. Define $\vec{u} := w_{km}(\vec{v}) - w_{km}(\vec{o})$. Let $h \in IOb$ be such that $w_{kh} = T_{\vec{u}}$. Such an h exists by $\mathsf{AxReloc}^{\mathsf{Tran}}$. Then, by Lemma 3.12(v), $k \xrightarrow{\vec{u}}_k h$. By Lemma 3.12(iv), it now follows that $k \xrightarrow{\vec{v}}_m h$ because $w_{mk}(w_{km}(\vec{v})) - w_{mk}(w_{km}(\vec{o})) = \vec{v} - \vec{o} = \vec{v}$. To prove the uniqueness assume Ext and that $h' \in IOb$ is such that $k \xrightarrow{\vec{v}}_m h'$. Then $w_{mh} = w_{mh'}$ $(= T_{\vec{v}} \circ w_{mk})$. By Wvt, we have $w_{hh'} = w_{hm} \circ w_{mh'} = w_{hm} \circ w_{mh} = w_{hh} = \mathsf{Id}$. Thus $h = h'$ by Ext.

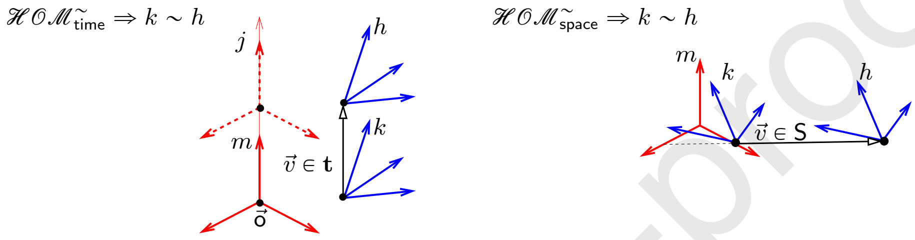

Figure 6: Illustration for Lemma 3.14

Lemma 3.14. *Assume* AxOField, Wvt, $\mathsf{Reloc}^{\mathsf{Tran}}$. *Let* $\sim$ *be an equivalence relation on* IOb *that has the transformation property. Let* $\vec{v} \in Q^4$ *and* $m, k, h \in IOb$ *be such that* $k \xrightarrow{\vec{v}}_m h$. *Then items (i) and (ii) below hold, see Figure 6.*

(i) If $\mathscr{HOM}^{\sim}_{\mathsf{time}}$ *holds and* $\vec{v} \in \mathbf{t}$, *then* $k \sim h$.
(ii) If $\mathscr{HOM}^{\sim}_{\mathsf{space}}$ *holds and* $\vec{v} \in \mathsf{S}$, *then* $k \sim h$.

Proof. By $k \xrightarrow{\vec{v}}_m h$, we have $w_{mh} = T_{\vec{v}} \circ w_{mk}$. Let $j \in IOb$ be such that $w_{mj} = T_{\vec{v}}$. Such a j exists by $\mathsf{Reloc}^{\mathsf{Tran}}$. By Wvt and Proposition 3.6(i), $w_{jh} = w_{jm} \circ w_{mh} = T_{\vec{v}}^{-1} \circ T_{\vec{v}} \circ w_{mk} = \mathsf{Id} \circ w_{mk} = w_{mk}$. Thus $w_{mk} = w_{jh}$.

(i) Suppose $\mathscr{HOM}^{\sim}_{\mathsf{time}}$ holds and that $\vec{v} \in \mathbf{t}$. Then $m \sim j$ because $w_{mj} = T_{\vec{v}} \in \mathsf{Tran}_{\mathsf{time}}$. But then $k \sim h$ because $w_{mk} = w_{jh}$, $m \sim j$ and $\sim$ has the transformation property.

(ii) Analogously, if $\mathscr{HOM}^{\sim}_{\mathsf{space}}$ holds and $\vec{v} \in \mathsf{S}$, then $k \sim h$.

Remark 3.15. Assume AxOField, Wvt, $\mathsf{Reloc}^{\mathsf{Tran}}$, AffTr, Ext and that $\sim$ is an equivalence relation on IOb that has the transformation property. By Lemmas 3.12 and 3.13, for every $m \in IOb$, $\longrightarrow_m$ determines a group action[6] of Tran on IOb as follows: Let $m \in IOb$. Then the action of $T_{\vec{v}} \in \mathsf{Tran}$ on $k \in IOb$ is defined to be the unique h for which $k \xrightarrow{\vec{v}}_m h$. By Lemma 3.14 it can be proven that

[6]We consider Tran as a group under composition.

(i) $\mathscr{HOM}^{\sim}_{\mathsf{time}}$ holds iff for every $m \in IOb$, the equivalence classes of $\sim$ are closed under the actions of temporal translations, and

(ii) $\mathscr{HOM}^{\sim}_{\mathsf{space}}$ holds iff for every $m \in IOb$, the equivalence classes of $\sim$ are closed under the actions of spatial translations.

Proof of Theorem 3.1. Assume $\mathscr{HOM}^{\sim}_{\mathsf{time}}$. We will prove that for every $\vec{v} \in Q^4$ and $k, h \in IOb$ if $w_{kh} = T_{\vec{v}}$, then $k \sim h$, which is a stronger statement than the required $\mathscr{HOM}^{\sim}_{\mathsf{space}}$. So, let $\vec{v} \in Q^4$ and $k, h \in IOb$ satisfy $w_{kh} = T_{\vec{v}}$. We want to prove that $k \sim h$.

By $\mathsf{3Dir\exists Motion}$, we can fix $m, j_1, j_2, j_3 \in IOb$ and $\vec{p}^{\,1}, \vec{p}^{\,2}, \vec{p}^{\,3} \in Q^4$ such that $\vec{p}^{\,1}_{\mathsf{s}}, \vec{p}^{\,2}_{\mathsf{s}}, \vec{p}^{\,3}_{\mathsf{s}}$ are linearly independent and $\vec{o}, \vec{p}^{\,i} \in w\ell_m(j_i)$ for every $i \in \{1, 2, 3\}$, see Figure 7. By Lemma 3.12(v) and $w_{kh} = T_{\vec{v}}$, we have that $k \xrightarrow{\vec{v}}_k h$. Let $\vec{u} \in Q^4$ be such that $k \xrightarrow{\vec{u}}_m h$, see Figure 7. Such $\vec{u}$ exists by Lemma 3.12(iv). Let $\lambda_1, \lambda_2, \lambda_3 \in Q$ be such that $\vec{u}_{\mathsf{s}} = \lambda_1 \vec{p}^{\,1}_{\mathsf{s}} + \lambda_2 \vec{p}^{\,2}_{\mathsf{s}} + \lambda_3 \vec{p}^{\,3}_{\mathsf{s}}$.

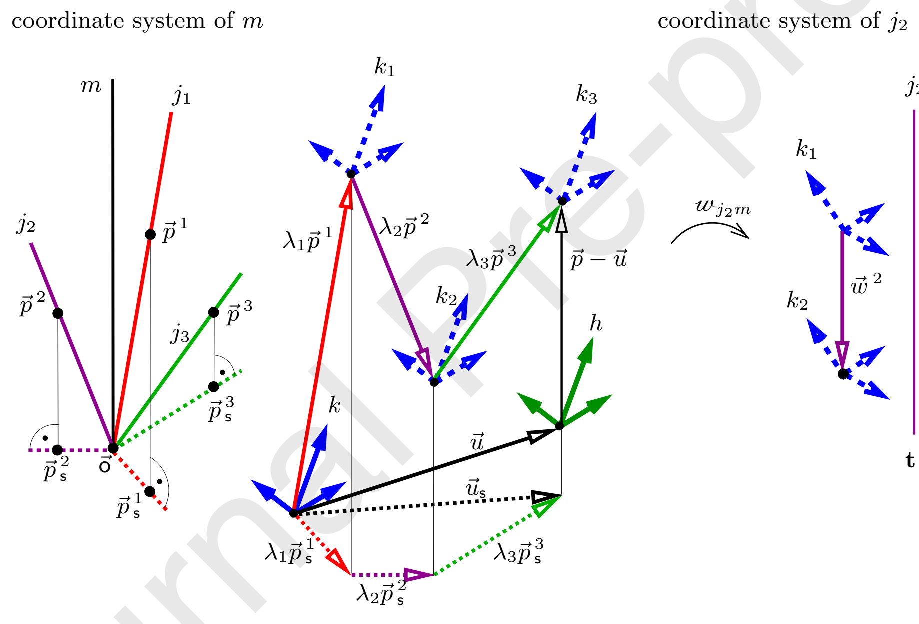

Figure 7: Illustration for the proof of Theorem 3.1.

By Lemma 3.13, we can fix k_1, k_2, k_3 such that

$$k \xrightarrow{\lambda_1\vec{p}^{\,1}}_m k_1, \quad k_1 \xrightarrow{\lambda_2\vec{p}^{\,2}}_m k_2 \quad \text{and} \quad k_2 \xrightarrow{\lambda_3\vec{p}^{\,3}}_m k_3. \tag{1}$$

By Lemma 3.12(iv), it now follows that $k \xrightarrow{\vec{w}^{\,1}}_{j_1} k_1$, $k_1 \xrightarrow{\vec{w}^{\,2}}_{j_2} k_2$ and $k_2 \xrightarrow{\vec{w}^{\,3}}_{j_3} k_3$ for $\vec{w}^{\,i} := w_{j_i m}(\lambda_i \vec{p}^{\,i}) - w_{j_i m}(\vec{\mathsf{o}})$ ($i \in \{1, 2, 3\}$). Since $w_{j_i m}$ is an affine transformation, $\vec{w}^{\,i} = \lambda\big(w_{j_i m}(\vec{p}^{\,i}) - w_{j_i m}(\vec{\mathsf{o}})\big)$. We have that $w_{j_i m}(\vec{\mathsf{o}}), w_{j_i m}(\vec{p}^{\,i}) \in \mathbf{t}$, because $\vec{\mathsf{o}}, \vec{p}^{\,i} \in w\ell_m(j_i) := w_{m j_i}[\mathbf{t}]$ and $w^{-1}_{m j_i} = w_{j_i m}$. Therefore $\vec{w}^{\,i} \in \mathbf{t}$ for every $i \in \{1, 2, 3\}$. Now, by Lemma 3.14(i),

$$k \sim k_1, \;\; k_1 \sim k_2 \text{ and } k_2 \sim k_3.$$

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Let $\vec{p} := \lambda_1\vec{p}^{\,1} + \lambda_2\vec{p}^{\,2} + \lambda_3\vec{p}^{\,3}$ and note that $\vec{p}_{\mathsf{s}} = \vec{u}_{\mathsf{s}}$. By Lemma 3.12(ii) and $k \xrightarrow{\vec{u}}_m h$, we have that $h \xrightarrow{-\vec{u}}_m k$. By Lemma 3.12(iii) and (1), it follows that $k \xrightarrow{\vec{p}}_m k_3$. Then by Lemma 3.12(iii) again, we have that $h \xrightarrow{\vec{p}-\vec{u}}_m k_3$. Since $\vec{p}_{\mathsf{s}} = \vec{u}_{\mathsf{s}}$, we have $\vec{p} - \vec{u} \in \mathbf{t}$, and hence by Lemma 3.14(i),

$$h \sim k_3.$$

Since $\sim$ is an equivalence relation, we conclude that $k \sim h$, as required.

Proof of Theorem 3.2. Assume $\mathscr{HOM}^{\sim}_{\mathsf{space}}$. We will prove that for every $\vec{v} \in Q^4$ and $k, h \in IOb$, if $w_{kh} = T_{\vec{v}}$, then $k \sim h$, which is a stronger statement than the required $\mathscr{HOM}^{\sim}_{\mathsf{time}}$.

Let $\vec{v} \in Q^4$ and $k, h \in IOb$ be such that $w_{kh} = T_{\vec{v}}$. We want to prove that $k \sim h$. By async we can fix $m, j \in IOb$ and $\vec{p}, \vec{q} \in Q^4$ (see Figure 8) such that $\vec{p}_{\mathsf{t}} \neq \vec{q}_{\mathsf{t}}$ and $w_{jm}(\vec{p})_{\mathsf{t}} = w_{jm}(\vec{q})_{\mathsf{t}}$.

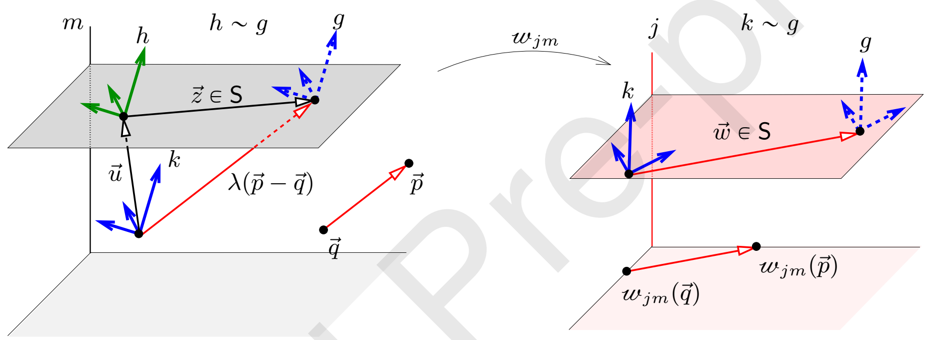


Figure 8: Illustration for the proof of Theorem 3.2.

By Lemma 3.12(v) and $\mathsf{w}_{kh} = T_{\vec{v}}$, we have that $k \xrightarrow{\vec{v}}_k h$. By (iv) of Lemma 3.12, we can fix $\vec{u} \in Q^4$ such that $k \xrightarrow{\vec{u}}_m h$. Let $\lambda \in Q$ be such that $\lambda(\vec{p}_{\mathsf{t}} - \vec{q}_{\mathsf{t}}) = \vec{u}_{\mathsf{t}}$. By Lemma 3.13, we can fix $g \in IOb$ such that $k \xrightarrow{\lambda\vec{p}-\lambda\vec{q}}_m g$. Then, by Lemma 3.12(iv), $k \xrightarrow{\vec{w}}_j g$ for $\vec{w} := w_{jm}(\lambda\vec{p}) - w_{jm}(\lambda\vec{q})$. By AffTr and Lemma 3.10(ii), we have that $\vec{w} = \lambda\left(w_{jm}(\vec{p}) - w_{jm}(\vec{q})\right)$. By this and $w_{jm}(\vec{p})_{\mathsf{t}} = w_{jm}(\vec{q})_{\mathsf{t}}$, we have that $\vec{w}_{\mathsf{t}} = 0$, i.e., $\vec{w} \in \mathsf{S}$. But then by Lemma 3.14(ii),

$$k \sim g.$$

By $k \xrightarrow{\lambda\vec{p}-\lambda\vec{q}}_m g$, $k \xrightarrow{\vec{u}}_m h$ and items (i) and (ii) of Lemma 3.12, we have that $h \xrightarrow{-\vec{u}}_m k$ and $h \xrightarrow{\vec{z}}_m g$ for $\vec{z} := \lambda\vec{p} - \lambda\vec{q} - \vec{u}$. By $\lambda(\vec{p}_{\mathsf{t}} - \vec{q}_{\mathsf{t}}) = \vec{u}_{\mathsf{t}}$, we have that $\vec{z}_{\mathsf{t}} = 0$, i.e., $\vec{z} \in \mathsf{S}$. Therefore, by Lemma 3.14(ii), we have that

$$h \sim g.$$

Since $\sim$ is an equivalence relation, we conclude that $k \sim h$, as required.

Proof of Proposition 3.3. Assume $\mathscr{HOM}^{\sim}_{\mathsf{time}}$ and $\mathscr{HOM}^{\sim}_{\mathsf{space}}$. To prove $\mathscr{HOM}^{\sim}$, let $k, h \in IOb$, and $\vec{v} \in Q^4$ be such that $w_{kh} = T_{\vec{v}}$. We have to prove that $k \sim h$. Let $\vec{t} \in \mathbf{t}$ and $\vec{s} \in \mathsf{S}$ be such that $\vec{v} = \vec{t} + \vec{s}$. Such $\vec{t}$ and $\vec{s}$ exist by $\mathsf{AxOField}$. Then $T_{\vec{t}} \in \mathsf{Tran}_{\mathsf{time}}$ and $T_{\vec{s}} \in \mathsf{Tran}_{\mathsf{space}}$. Let $m \in IOb$ be such that $w_{km} = T_{\vec{t}}$. Such an m exists by $\mathsf{Reloc}^{\mathsf{Tran}}$. Then, by $\mathscr{HOM}^{\sim}_{\mathsf{time}}$ and $T_{\vec{t}} \in \mathsf{Tran}_{\mathsf{time}}$,

$$k \sim m.$$

By Wvt and $\mathsf{AxOField}$, $w_{mh} = w_{mk} \circ w_{kh} = T_{-\vec{t}} \circ T_{\vec{v}} = T_{\vec{v}-\vec{t}} = T_{\vec{s}} \in \mathsf{Tran}_{\mathsf{space}}$. By this, and $\mathscr{HOM}^{\sim}_{\mathsf{space}}$,

$$m \sim h.$$

Since $\sim$ is an equivalence relation, the result now follows.

Proof of Theorem 3.4. Assume $\mathscr{ISO}^{\sim}$. To prove $\mathscr{HOM}^{\sim}_{\mathsf{space}}$, let $k, h \in IOb$ satisfy $w_{kh} \in \mathsf{Tran}_{\mathsf{space}}$. We have to prove that $k \sim h$. Let $\vec{v} \in \mathsf{S}$ be such that $w_{kh} = T_{2\vec{v}}$, see Figure 9, and let m satisfy $w_{km} = T_{\vec{v}}$ (such an m exists by $\mathsf{Reloc}^{\mathsf{Tran}}$). Then, by Wvt and Proposition 3.6(i), $w_{mh} = w_{mk} \circ w_{kh} = T_{-\vec{v}} \circ T_{2\vec{v}} = T_{\vec{v}}$. We have to prove that $k \sim h$.

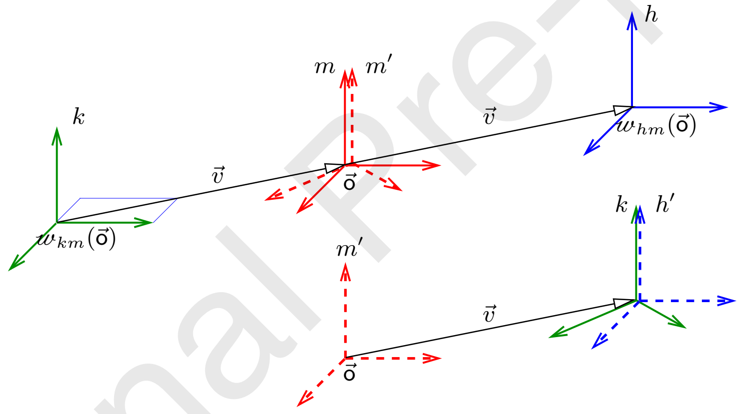


Figure 9: Illustration for the proof of Theorem 3.4.

Let $R \in \mathsf{Rot}_{\mathsf{space}}$ be such that $R(\vec{v}) = -\vec{v}$. Such an R exists because of the following: Consider the linear map that takes $\alpha\vec{\mathsf{e}}_1 + \beta\vec{v}$ to $\alpha\vec{\mathsf{e}}_1 - \beta\vec{v}$. By $\vec{v} \in \mathsf{S}$, it is easy to see that this map is a linear bijection preserving Euclidean scalar product on subspace generated by $\vec{\mathsf{e}}_1$ and $\vec{v}$. Hence, by the refinement of Witt's theorem [32, Thm 234.1, p.234] there is an extension $R : Q^4 \to Q^4$ which is a linear transformation preserving the Euclidean scalar product with determinant 1. It is easy to see that this R is a spatial rotation, because $R(\vec{\mathsf{e}}_1) = \vec{\mathsf{e}}_1$.

Let $m' \in IOb$ be such that $w_{mm'} = R$. Such an m' exists by $\mathsf{Reloc}^{\mathsf{Rot}}$. Then, by $\mathscr{ISO}^{\sim}$, we have that $m \sim m'$. Let h' be such that $w_{m'h'} = T_{\vec{v}}$. Such an h' exists by $\mathsf{Reloc}^{\mathsf{Tran}}$. Then

$$h \sim h'$$

because $w_{mh} = w_{m'h'} = T_{\vec{v}}$, $m \sim m'$ and $\sim$ has the transformation property. Next we are going to prove that $w_{kh'} = R$. By Wvt, we have that $w_{kh'} = w_{km} \circ w_{mm'} \circ w_{m'h'} = T_{\vec{v}} \circ R \circ T_{\vec{v}}$. Thus, for every $\vec{p} \in Q^4$, $w_{kh'}(\vec{p}) = (T_{\vec{v}} \circ R \circ T_{\vec{v}})(\vec{p}) = (T_{\vec{v}} \circ R)(\vec{p} + \vec{v}) = T_{\vec{v}}\big(R(\vec{p}) + R(\vec{v})\big) = T_{\vec{v}}\big(R(\vec{p}) - \vec{v}\big) = R(\vec{p})$. Therefore, $w_{kh'} = R \in \mathsf{Rot}_{\mathsf{space}}$ as claimed. Then, by $\mathsf{ISO}^{\mathscr{S}}$, we have that

$$k \sim h'.$$

Since $\sim$ is an equivalence relation, we conclude that $k \sim h$, as required.

Proof of Theorem 3.5. Let $(Q, +, \cdot, 0, 1, \leqslant)$ be the ordered field of reals. Let G and H be respectively the sets of affine transformations that map simultaneous coordinate points to simultaneous coordinate points and the set of affine transformations that fix the time components of the coordinate points, i.e.,

$$\begin{aligned} \mathsf{G} &:= \{A \in \mathsf{AffTr} \,:\, \vec{p}_{\mathsf{t}} = \vec{q}_{\mathsf{t}} \Rightarrow A(\vec{p})_{\mathsf{t}} = A(\vec{q})_{\mathsf{t}}\} \text{ and} \\ \mathsf{H} &:= \{A \in \mathsf{AffTr} \,:\, A(\vec{p})_{\mathsf{t}} = \vec{p}_{\mathsf{t}}\}. \end{aligned}$$

Both G and H form groups under composition and H is a proper subgroup of G. Let $IOb := \mathsf{G}$, $w_{kh} := k^{-1} \circ h$ and $k \sim h :\Leftrightarrow w_{kh} \in \mathsf{H}$ for every $k, h \in IOb$.

Obviously $\mathsf{AxOField}$ holds, and it is easy to check that Wvt holds, too. $\mathsf{Reloc}^{\mathsf{Tran}}$ and $\mathsf{Reloc}^{\mathsf{Rot}}$ hold because $\mathsf{Tran}, \mathsf{Rot}_{\mathsf{space}} \subseteq \mathsf{G}$ and G forms a group. To prove $\mathsf{3Dir\exists Motion}$, let g_1, g_2 and g_3 be the linear transformations that map $\vec{\mathsf{e}}_1$ to $(1,1,0,0)$, $(1,0,1,0)$ and $(1,0,0,1)$ respectively, and leave $\vec{\mathsf{e}}_2$, $\vec{\mathsf{e}}_3$ and $\vec{\mathsf{e}}_4$ fixed. Then $g_1, g_2, g_3 \in \mathsf{G} = IOb$, $\vec{\mathsf{o}} \in w\ell_{\mathsf{Id}}(g_1) \cap w\ell_{\mathsf{Id}}(g_2) \cap w\ell_{\mathsf{Id}}(g_3)$, $(1,1,0,0) \in w\ell_{\mathsf{Id}}(g_1)$, $(1,0,1,0) \in w\ell_{\mathsf{Id}}(g_2)$, $(1,0,0,1) \in w\ell_{\mathsf{Id}}(g_3)$ and $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ are linearly independent. Thus $\mathsf{3Dir\exists Motion}$ holds.

By Wvt it is easy to check that $\sim$ is an equivalence relation because H is a group. To prove that $\sim$ has the transformation property let $k, k', h, h' \in IOb$ be such that $k \sim k'$ and $w_{kh} = w_{k'h'}$. Then $w_{hk} = w_{h'k'}$ by Proposition 3.6. We want to prove that $h \sim h'$, i.e., $w_{hh'}(\vec{p})_{\mathsf{t}} = \vec{p}_{\mathsf{t}}$ for every $\vec{p} \in Q^4$. Let $\vec{p} \in Q^4$ be an arbitrary coordinate point. By Wvt,

$$w_{hh'}(\vec{p})_{\mathsf{t}} = w_{hk}(w_{kk'}(w_{k'h'}(\vec{p})))_{\mathsf{t}}.$$

Because $\mathsf{w}_{kk'} \in \mathsf{H}$ by $k \sim k'$,

$$w_{kk'}(w_{k'h'}(\vec{p}))_{\mathsf{t}} = w_{k'h'}(\vec{p})_{\mathsf{t}}.$$

From this and $w_{hk} = w_{h'k'} \in \mathsf{G}$, we have that

$$w_{hk}(w_{kk'}(w_{k'h'}(\vec{p}))_{\mathsf{t}} = w_{h'k'}(w_{k'h'}(\vec{p}))_{\mathsf{t}} = w_{h'h'}(\vec{p})_{\mathsf{t}} = \vec{p}_{\mathsf{t}}.$$

Thus $w_{hh'}(\vec{p})_{\mathsf{t}} = \vec{p}_{\mathsf{t}}$; and hence $w_{hh'} \in \mathsf{H}$, which is equivalent to $h \sim h'$ by definition. Hence $\sim$ has the transformation property.

$\mathscr{HOM}^{\sim}_{\mathsf{space}}$ and $\mathscr{ISO}^{\sim}$ hold by $\mathsf{Tran}_{\mathsf{space}}, \mathsf{Rot}_{\mathsf{space}} \subseteq \mathsf{H}$ and the definition of $\sim$. Homogeneity of time $\mathscr{HOM}^{\sim}_{\mathsf{time}}$ does not hold because of the following: By $\mathsf{Reloc}^{\mathsf{Tran}}$ there are $k, h \in IOb$ for which $\mathsf{Id} \neq w_{kh} \in \mathsf{Tran}_{\mathsf{time}}$. For such k and h, we have that $w_{kh} \notin \mathsf{H}$ and therefore $k \nsim h$.

4. Logical Framework

To get a more refined understanding of the key notion of observers agreeing on experiments, now we dig deeper and make its meaning explicit. This notion clearly depends on the language on which the experimental descriptions are formulated. So we give a minimal core language that is needed to capture this notion, and we formulate our theorems in a way that makes them applicable to any language containing this core language. For methodological reasons, we work within the framework of first-order logic.

4.1. Language

Following [25], we use the 3-sorted first-order language

$$\mathscr{L}_{\mathsf{core}} = \{IOb, B, Q, +, \cdot, 0, 1, \leqslant, \mathsf{W}\}$$

as our core language for kinematics, where

- IOb is the sort of *inertial observers* – we use observers to label *coordinate systems*.
- B is the sort of *bodies* – these represent *things that can move*.
- Q is the sort of physical *quantities*, with constants 0 and 1, the usual binary operations $\cdot$ and $+$, and ordering relation $\leqslant$. We use Q to specify coordinates, lengths, etc.
- W is the *worldview relation*, a 6-ary relation of type $IOb \times B \times Q^4$. We interpret the atomic formula $\mathsf{W}(o, b, t, x, y, z)$ to mean "inertial observer o coordinatizes body b to be at spatial location (x, y, z) at time t."

As usual, by a *model* of $\mathscr{L}_{\mathsf{core}}$ we mean a collection of three non-empty sets corresponding to sorts IOb, B and Q, as well as constants, operations and relations defined on them.

Later, we will talk about extensions of $\mathscr{L}_{\mathsf{core}}$, and distinguish between "mathematical" and "non-mathematical" sorts (for example, we consider Q to be the *mathematical sort* of $\mathscr{L}_{\mathsf{core}}$, while IOb and B are non-mathematical sorts). "Mathematical" sorts will occur as parameters in scenarios, while non-mathematical ones are typically used to capture important physical attributes, e.g., a representation of quantum electrodynamics might extend $\mathscr{L}_{\mathsf{core}}$ by introducing new sorts representing photons, electrons and positrons. This approach means our results are not restricted to pure kinematics, but generalise automatically to a wide range of physically relevant theories. For examples of languages extending $\mathscr{L}_{\mathsf{core}}$, see e.g., [3, 21, 24].

4.2. Basic Notation and Definitions

Let $\mathscr{L}$ be any language extending $\mathscr{L}_{\mathsf{core}}$, and suppose that some of the sorts of $\mathscr{L}$ are distinguished as mathematical sorts, for use as mathematical parameters of the experiments. We assume, in particular, that Q is a mathematical sort of $\mathscr{L}$, while B and IOb are non-mathematical. The sorts of variables are usually clear from context, but to help the reader we regularly use variables o, k, h, m, j, g and their variants for inertial observers, b for bodies; x, y, z, t and λ for quantities, and $\bar{x}$ for sequences of variables of mathematical sorts. Elements of Q^4 representing spacetime locations are denoted using over-arrows, e.g., $\vec{p}$, $\vec{q}$, $\vec{x}$, $\vec{y}$, $\vec{v}$, ...

Let us recall here some key concepts easily definable in $\mathscr{L}$. The *event* observed by observer o at coordinate point $\vec{p}$ is the set of bodies that o coordinatizes there:

$$\mathsf{ev}_o(\vec{p}) \stackrel{\text{def}}{=} \{b \in B : \mathsf{W}(o, b, \vec{p})\}\,.$$

The *worldview transformation* $\mathsf{w}_{oo'}$ is a binary relation on Q^4 connecting the coordinate systems of observers o' and o by mapping coordinate point $\vec{p}$ to $\vec{q}$ if the event that o' coordinatizes at $\vec{p}$ is the same event that o coordinatizes at $\vec{q}$:

$$\mathsf{w}_{oo'}(\vec{p}, \vec{q}) \stackrel{\text{def}}{\iff} \mathsf{ev}_{o'}(\vec{p}) = \mathsf{ev}_o(\vec{q}).$$

Under the mild background assumptions made here, $\mathsf{w}_{oo'}$ is a bijection for all observers o and o', and it maps the coordinate system of o' to that of o.[7]

Because bodies and observers are of different formal sorts, we provide two distinct definitions of what we mean by a *worldline*. The *worldline of body* b according to observer o is the set of coordinate points at which o coordinatizes b:

$$\mathsf{wl}_o(b) \stackrel{\text{def}}{=} \left\{\vec{p} \in Q^4 : \mathsf{W}(o, b, \vec{p})\right\}.$$

From an observer's standpoint, their own worldline maps out the *time-axis* because they consider themselves to be at rest spatially. So the *worldline of observer* o' according to another observer o is simply the $\mathsf{w}_{oo'}$-image of the time-axis:[8]

$$\mathsf{wl}_o(o') \stackrel{\text{def}}{=} \left\{\vec{q} \in Q^4 : (\exists \vec{p} \in \mathbf{t})\mathsf{w}_{oo'}(\vec{p}, \vec{q})\right\}.$$

4.3. Formalisation of Homogeneity and Isotropy

Now we are going to formalise Homogeneity and Isotropy making the meaning of observers agreeing on the outcome of experiments explicit. Let us recall that isotropy means intuitively that rotating an experimental configuration in space should not change the overall outcome of the experiment, and homogeneity has an analogous intuitive meaning using translations in place of rotations. Let us also recall that we adopt a static representation of experiments where their configuration, progress and outcome are all contained in the description of the experiment – the representation provides a complete record of the experiment containing every relevant detail from beginning to end.

[7] The order of observers in the subscript of w is chosen to fit functional composition, i.e., so that $\mathsf{w}_{oo'}\big(\mathsf{w}_{o'o''}(\vec{p})\big) = \mathsf{w}_{oo''}(\vec{p})$ for all observers o, o', o'' and coordinate points $\vec{p}$ (assuming our background axioms ensuring that worldview transformations are indeed bijections).

[8] This approach differs from that in [2, 4], where we instead represented the motion of observers by considering co-moving bodies.

To formalise isotropy and homogeneity, we first capture this notion of static experiments by considering experimental scenarios expressible in any formal language $\mathscr{L}$ extending our base language $\mathscr{L}_{\mathsf{core}}$. We assume that certain sorts of $\mathscr{L}$ are distinguished as being mathematical, that Q is a mathematical sort, and that B and IOb are non-mathematical. We will consider statements, φ, which describe experiments statically. For example, φ might say "if this device has the following specified spatial configuration at time t, then it also produces some definite reading (an observable in the resulting spatial configuration) at time $t+1$".

Recalling the notation of [25], a formula $\varphi(o,\bar{x})$ of language $\mathscr{L}$ is called a *scenario* if o is the only free variable of φ of sort IOb and all the other free variables $\bar{x}$ of φ are of mathematical sorts (these correspond to experimental parameters in the coordinate system of o). For concrete choices of the parameters, we get concrete experiments. For example, $\varphi(o,v)$ might capture the statement "o can send out a body b moving with speed v from the origin". In this case, in a concrete model, the truth of $\varphi(o,1)$ means the realizability of the concrete experiment sending out a body moving with (coordinate) speed $v=1$ from the origin by observer o, while $\varphi(o,\sqrt{2})$ means the same but corresponding to the similar experiment where the speed $v=\sqrt{2}$.

See Figure 10 for examples of scenarios expressed in formal terms; for further motivation of this concept, and for further examples, see [25]. Notice the difference between a *scenario* and an *experiment*: in every model, each scenario describes a *family* of experiments, each determined by a specific evaluation of the mathematical variables, $\bar{x}$.

Even though $\varphi(o,\bar{x})$ is a first-order formula, this is not a real limitation on the formalisability of experiments because the language $\mathscr{L}$ extending $\mathscr{L}_{\mathsf{core}}$ can be chosen arbitrarily. Typically, one does not use higher-order constructions in describing experiments (e.g., no quantification over sets is used in such descriptions). However, even that would not be a problem with an appropriately chosen language $\mathscr{L}$ because $\mathscr{L}$ can contain sorts for sets, functions, etc.

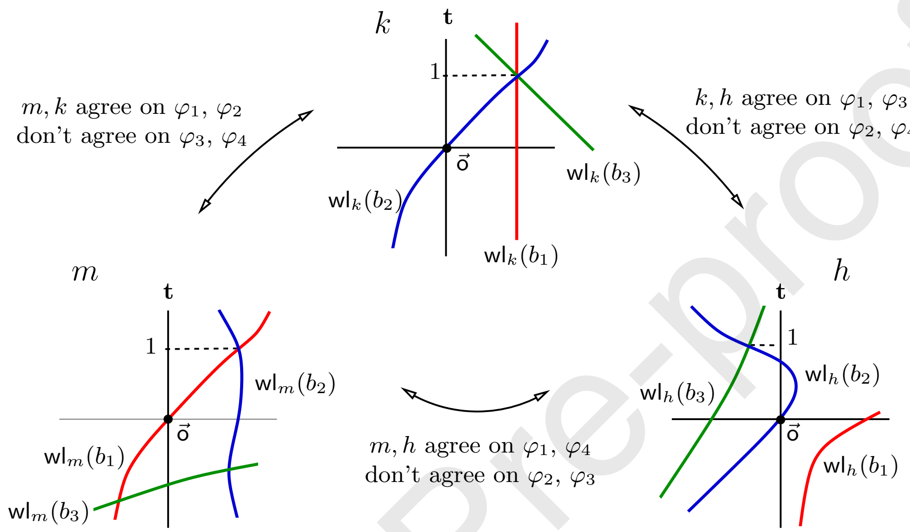
We write $\mathscr{L}$-$\mathsf{Scenarios}$ for the set of all scenarios of language $\mathscr{L}$. To capture the idea that all inertial observers agree on the truth value of formula $\varphi(o,\bar{x})$ for every evaluation of the variables $\bar{x}$, we introduce the following formula[9]

$$\mathsf{Agree}_{\varphi}\left(o,o'\right) \stackrel{\text{def}}{=} (\forall \bar{x})\big(\varphi(o,\bar{x}) \leftrightarrow \varphi(o',\bar{x})\big).$$

The informal meaning of $\mathsf{Agree}_{\varphi}(o,o')$ is that observers o and o' agree as to the realisability of scenario φ, i.e., they are in experimental agreement for all experiments described by φ. For examples illustrating the use of formula $\mathsf{Agree}_{\varphi}(o,o')$ in a simple model, see Figure 10.

Now we can formalise homogeneity and isotropy by moving the observers instead of the experiments, cf. e.g., Section 3.2 and [11, pp. 21–22]. Let $\mathscr{S} \subseteq$

[9]Here, $\varphi(o',\bar{x})$ is the formula obtained from $\varphi(o,\bar{x})$ by replacing all free occurrences of o with o' while avoiding collision of variables using any (fixed) method of changing bound variables if needed.



In these examples we assume that $B = \{b_1, b_2, b_3\}$ and take $t = 1$ for illustrative purposes. Scenarios $\varphi(1)$–$\varphi(4)$ are:

$\varphi(1)$ Observer o can see a body at the origin: $\varphi_1(o) \equiv (\exists b)\mathsf{W}(o, b, \vec{\mathsf{o}})$.

$\varphi(2)$ Observer o can send a body from the origin to (t, x, y, z):
$\varphi_2(o, t, x, y, z) \equiv (\exists b)\big(\mathsf{W}(o, b, \vec{\mathsf{o}}) \wedge \mathsf{W}(o, b, t, x, y, z)\big)$.

$\varphi(3)$ Observer o can observe (at least) two distinct bodies meeting at time instant t:
$\varphi_3(o, t) \equiv (\exists b, b')(\exists x, y, z)\big(b \neq b' \wedge \mathsf{W}(o, b, t, x, y, z) \wedge \mathsf{W}(o, b', t, x, y, z)\big)$.

$\varphi(4)$ Observer o can observe a stationary body at space location (x, y, z):
$\varphi_4(o, x, y, z) \equiv (\exists b)(\forall \vec{p})\big(\mathsf{W}(o, b, \vec{p}) \leftrightarrow \vec{p}_{\mathsf{s}} = (x, y, z)\big)$.

Figure 10: Some basic experimental scenarios expressed in formal terms, and whether observers m, k, h agree as to their realisability.

$\mathscr{L}$-$\mathsf{Scenarios}$ be some set of physically relevant scenarios, i.e., the ones that correspond to experiments we are interested in a certain situation. Notice that we are not concerned as to what those experiments might be. They will differ according to the underlying physical properties they are designed to investigate, but even so, our results cover all eventualities.

By *homogeneity of time*, we understand the following formula schema stating that inertial observers agree on the realisability of scenarios in $\mathscr{S}$ if their coordinate systems differ only by a temporal translation:

$$\mathsf{HOM}^{\mathscr{S}}_{\mathsf{time}} \stackrel{\text{def}}{=} \left\{(\forall o, o')\big(\mathsf{w}_{oo'} \in \mathsf{Tran}_{\mathsf{time}} \;\rightarrow\; \mathsf{Agree}_{\varphi}\left(o, o'\right)\big) \;:\; \varphi \in \mathscr{S}\right\}.$$

Homogeneity of space is defined analogously using spatial translations instead of temporal ones:

$$\mathsf{HOM}^{\mathscr{S}}_{\mathsf{space}} \stackrel{\text{def}}{=} \left\{(\forall o, o')\big(\mathsf{w}_{oo'} \in \mathsf{Tran}_{\mathsf{space}} \;\rightarrow\; \mathsf{Agree}_{\varphi}\left(o, o'\right)\big) \;:\; \varphi \in \mathscr{S}\right\}.$$

In the same way, *homogeneity (of spacetime)* is characterised by translations in spacetime:

$$\mathsf{HOM}^{\mathscr{S}} \stackrel{\text{def}}{=} \left\{(\forall o, o')\big(\mathsf{w}_{oo'} \in \mathsf{Tran} \;\rightarrow\; \mathsf{Agree}_{\varphi}\left(o, o'\right)\big) \;:\; \varphi \in \mathscr{S}\right\},$$

while *isotropy* is defined analogously using rotations instead of translations:

$$\mathsf{ISO}^{\mathscr{S}} \stackrel{\text{def}}{=} \left\{(\forall o, o')\big(\mathsf{w}_{oo'} \in \mathsf{Rot}_{\mathsf{space}} \;\rightarrow\; \mathsf{Agree}_{\varphi}\left(o, o'\right)\big) \;:\; \varphi \in \mathscr{S}\right\}.$$

Evidently, the logical strength of these schemas depends on the set $\mathscr{S}$ of 'relevant' experiments: increasing $\mathscr{S}$ extends the set of associated constraints. If we assume that observers in question must agree on realisability of *all* scenarios (not only the physically relevant ones) we get the strongest forms of the isotropy (ISO^+) and homogeneity schemas ($\mathsf{HOM}^+_{\mathsf{time}}$, $\mathsf{HOM}^+_{\mathsf{space}}$ and HOM^+):

$$\begin{aligned}
\mathsf{HOM}^+_{\mathsf{time}} &\stackrel{\text{def}}{=} \left\{(\forall o, o')\big(\mathsf{w}_{oo'} \in \mathsf{Tran}_{\mathsf{time}} \;\rightarrow\; \mathsf{Agree}_{\varphi}\left(o, o'\right)\big) \;:\; \varphi \in \mathscr{L}\text{-}\mathsf{Scenarios}\right\},\\
\mathsf{HOM}^+_{\mathsf{space}} &\stackrel{\text{def}}{=} \left\{(\forall o, o')\big(\mathsf{w}_{oo'} \in \mathsf{Tran}_{\mathsf{space}} \;\rightarrow\; \mathsf{Agree}_{\varphi}\left(o, o'\right)\big) \;:\; \varphi \in \mathscr{L}\text{-}\mathsf{Scenarios}\right\},\\
\mathsf{HOM}^+ &\stackrel{\text{def}}{=} \left\{(\forall o, o')\big(\mathsf{w}_{oo'} \in \mathsf{Tran} \;\rightarrow\; \mathsf{Agree}_{\varphi}\left(o, o'\right)\big) \;:\; \varphi \in \mathscr{L}\text{-}\mathsf{Scenarios}\right\},\\
\mathsf{ISO}^+ &\stackrel{\text{def}}{=} \left\{(\forall o, o')\big(\mathsf{w}_{oo'} \in \mathsf{Rot}_{\mathsf{space}} \;\rightarrow\; \mathsf{Agree}_{\varphi}\left(o, o'\right)\big) \;:\; \varphi \in \mathscr{L}\text{-}\mathsf{Scenarios}\right\}.
\end{aligned}$$

While homogeneity clearly entails homogeneity of both time and space, the converse does not hold in general, i.e., homogeneities of time and space together do not imply homogeneity. However, if we make some mild additional assumptions, this converse becomes true. See Proposition 5.1.

Throughout the paper, we assume that $\mathscr{L}$ is an arbitrary language extending $\mathscr{L}_{\mathsf{core}}$, that some of the sorts of $\mathscr{L}$ are distinguished as mathematical sorts, and that $\mathscr{S} \subseteq \mathscr{L}$-$\mathsf{Scenarios}$.

4.3.1. Connections between the formulations of isotropy and homogeneity in Section 3.2 and their formalisations in the present section

Definition 4.1. Let $\mathfrak{M}$ be a model of $\mathscr{L}$ and suppose $h, h' \in IOb$. We say that h and h' *agree on* $\mathscr{S}$ in $\mathfrak{M}$, in symbols $h \sim^{\mathscr{S}}_{\mathfrak{M}} h'$, iff $\mathsf{Agree}_{\varphi}(h, h')$ holds in $\mathfrak{M}$ for every $\varphi \in \mathscr{S}$.

Remark 4.2. Relation $\sim_{\mathfrak{M}}^{\mathscr{S}}$ is an equivalence relation on IOb in every model $\mathfrak{M}$ of $\mathscr{L}$ because, by the properties of biconditional $\leftrightarrow$, formula Agree_φ defines an equivalence relation on IOb for every $\varphi \in \mathscr{S}$ and relation $\sim_{\mathfrak{M}}^{\mathscr{S}}$ is the intersection of these equivalence relations.

Proposition 4.3. *Let $\mathfrak{M}$ be a model of $\mathscr{L}$. Assume that the worldview transformations are functions from Q^4 to Q^4. Let $w_{kh} := \mathsf{w}_{kh}$ for every $k, h \in IOb$. Let $\sim$ be $\sim_{\mathfrak{M}}^{\mathscr{S}}$. Then, $\mathsf{HOM}^{\mathscr{S}}_{\mathsf{time}} \Leftrightarrow \mathscr{HOM}^{\sim}_{\mathsf{time}}$, $\mathsf{HOM}^{\mathscr{S}}_{\mathsf{space}} \Leftrightarrow \mathscr{HOM}^{\sim}_{\mathsf{space}}$, $\mathsf{HOM}^{\mathscr{S}} \Leftrightarrow \mathscr{HOM}^{\sim}$ and $\mathsf{ISO}^{\mathscr{S}} \Leftrightarrow \mathscr{ISO}^{\sim}$ hold in $\mathfrak{M}$.*

PROOF. These statements follow trivially from the definition of $\sim_{\mathfrak{M}}^{\mathscr{S}}$.

4.4. Axioms

In formalising the axioms below, we adopt the usual convention that whenever we write $R(a)$ (where R is a relation), then there exists a unique b such that $R(a, b)$ holds and $R(a)$ denotes this unique b. In particular, when we write $\mathsf{w}_{oo'}(\vec{r})$ we are implicitly stating that there is exactly one $\vec{s} \in Q^4$ such that the relation $\mathsf{w}_{oo'}(\vec{r}, \vec{s})$ holds and that $\mathsf{w}_{oo'}(\vec{r})$ denotes this unique $\vec{s}$. By this convention, both $\mathsf{AxReloc}^{\mathsf{Tran}}$ and $\mathsf{AxAffTr}$ below imply that worldview transformations are bijections because then, for all observers o and o', both $\mathsf{w}_{oo'}$ and its inverse $\mathsf{w}_{o'o}$ are functions defined everywhere.

The axiom system used in this paper is

$$\mathsf{FRAME} \stackrel{\text{def}}{=} \{\mathsf{AxOField},\ \mathsf{AxReloc}^{\mathsf{Tran}},\ \mathsf{AxReloc}^{\mathsf{Rot}},\ \mathsf{AxAffTr},\ \mathsf{Ax3Dir\exists Motion}\},$$

where the five basic axioms say that:

$\mathsf{AxOField}$ $(Q, +, \cdot, 0, 1, \leqslant)$ is an ordered field.

$\mathsf{AxReloc}^{\mathsf{Tran}}$ Translations of inertial coordinate systems are inertial:

$$(\forall o)(\forall \vec{v})(\exists o')(\forall \vec{p})(\mathsf{w}_{oo'}(\vec{p}) = \vec{p} + \vec{v}).$$

$\mathsf{AxReloc}^{\mathsf{Rot}}$ Rotations of inertial coordinate systems are inertial:[10]

$$(\forall o)(\forall R \in \mathsf{Rot}_{\mathsf{space}})(\exists o')(\mathsf{w}_{oo'} = R).$$

$\mathsf{AxAffTr}$ Worldview transformations are affine:

$$(\forall o, o')(\exists \vec{v})(\forall \vec{p}, \vec{q}, \lambda)\Big(\mathsf{w}_{oo'}(\lambda \cdot \vec{p}) + \vec{v} = \lambda \cdot \big(\mathsf{w}_{oo'}(\vec{p}) + \vec{v}\big) \ \wedge \\ \mathsf{w}_{oo'}(\vec{p} + \vec{q}) = \mathsf{w}_{oo'}(\vec{p}) + \mathsf{w}_{oo'}(\vec{q}) + \vec{v}\Big).$$

[10]Quantification over $\mathsf{Rot}_{\mathsf{space}}$ appears to be second-order. However, because spatial rotations are determined by the images of the three spatial unit vectors, this axiom can be formalised in our first-order language by quantifying over the 12 parameters representing the images of the three spatial unit vectors.

$\mathsf{Ax3Dir\exists Motion}$ There is an inertial observer according to which there are three inertial observers moving in linearly independent spatial directions (see Figure 3):

$$(\exists o, o_1, o_2, o_3)(\exists \vec{p}^{\,1}, \vec{p}^{\,2}, \vec{p}^{\,3}) \Big(\mathsf{Indep}(\vec{p}^{\,1}_{\mathsf{s}}, \vec{p}^{\,2}_{\mathsf{s}}, \vec{p}^{\,3}_{\mathsf{s}}) \ \wedge \bigwedge_{i=1}^{3} \big(\vec{o}, \vec{p}^{\,i} \in \mathsf{wl}_o(o_i)\big)\Big),$$

where $\mathsf{Indep}(\vec{p}^{\,1}_{\mathsf{s}}, \vec{p}^{\,2}_{\mathsf{s}}, \vec{p}^{\,3}_{\mathsf{s}})$ abbreviates the following formula:

$$(\forall \lambda_1, \lambda_2, \lambda_3)\big(\lambda_1 \vec{p}^{\,1}_{\mathsf{s}} + \lambda_2 \vec{p}^{\,2}_{\mathsf{s}} + \lambda_3 \vec{p}^{\,3}_{\mathsf{s}} = (0,0,0) \ \rightarrow \ \lambda_1 = \lambda_2 = \lambda_3 = 0\big).$$

Models for language $\mathscr{L}$ will be called $\mathscr{L}$-models. If $\mathfrak{M}$ is some $\mathscr{L}$-model, and Σ is a collection of formulas in $\mathscr{L}$, we write $\mathfrak{M} \models \Sigma$ to mean that every $\sigma \in \Sigma$ is valid when interpreted in $\mathfrak{M}$. In this case, we will also say that σ holds in $\mathfrak{M}$. We write $\Sigma_1 \models \Sigma_2$ to mean that $\mathfrak{M} \models \Sigma_2$ whenever $\mathfrak{M} \models \Sigma_1$. If language $\mathscr{L}$ contains $\mathscr{L}_{\mathsf{core}}$, we define

$$\Sigma_1 \models_{\mathsf{frm}} \Sigma_2 \overset{\text{def}}{\iff} \Sigma_1 + \mathsf{FRAME} \models \Sigma_2,$$
$$\Sigma_1 \mathrel{=\!\!|\!\models}_{\mathsf{frm}} \Sigma_2 \overset{\text{def}}{\iff} (\Sigma_1 \models_{\mathsf{frm}} \Sigma_2 \ \text{ and } \ \Sigma_2 \models_{\mathsf{frm}} \Sigma_1).$$

A number of our results refer to the notion of clocks “losing synchrony”. This is captured by the formula ASync, which is not treated as an axiom because it is not a natural basic assumption for kinematics.

ASync There are clocks that get out of sync, i.e., there are events which are simultaneous for one observer but not for another one (see Figure 3):

$$(\exists o, o')(\exists \vec{p}, \vec{p}^{\,\prime}, \vec{q}, \vec{q}^{\,\prime})\big(\mathsf{w}_{o'o}(\vec{p}, \vec{p}^{\,\prime}) \wedge \mathsf{w}_{o'o}(\vec{q}, \vec{q}^{\,\prime}) \wedge \vec{p}_{\mathsf{t}} \neq \vec{q}_{\mathsf{t}} \wedge \vec{p}^{\,\prime}_{\mathsf{t}} = \vec{q}^{\,\prime}_{\mathsf{t}}\big).$$

Proposition 4.4. *Both*

$$\mathsf{FRAME} + \mathsf{ASync} + \mathsf{HOM}^{+} + \mathsf{ISO}^{+} \ \textit{and} \ \mathsf{FRAME} + \neg\mathsf{ASync} + \mathsf{HOM}^{+} + \mathsf{ISO}^{+}$$

are consistent.

This follows by Propositions 5.13 and 5.12.

In most of our negative results we will refer to the formula

$$\mathscr{m}(o, \vec{x}) \overset{\text{def}}{=} (\exists b)\big(\mathsf{W}(o, b, \vec{\mathsf{o}}) \wedge \mathsf{W}(o, b, \vec{x})\big),$$

which describes the simple experimental scenario testing whether a body b can move from $\vec{\mathsf{o}}$ to $\vec{x}$.

In Proposition 5.20 we connect ASync and the axioms of the present section to the assumptions formulated in the simple language of Section 3.3.

4.5. Defining New Scenarios

Recall that we assume that $\mathscr{L}$ is an arbitrary language extending $\mathscr{L}_{\mathsf{core}}$, that some of the sorts of $\mathscr{L}$ are distinguished as mathematical sorts, and that $\mathcal{S} \subseteq \mathscr{L}$-Scenarios.

For each $\varphi \in \mathscr{L}$-Scenarios, we will define a new scenario $\breve{\varphi} \in \mathscr{L}$-Scenarios (in some of our theorems, we will assume that $\varphi \in \mathcal{S} \Rightarrow \breve{\varphi} \in \mathcal{S}$). To define $\breve{\varphi}$, let $\vec{\mathsf{e}}_0 \stackrel{\text{def}}{=} \vec{\mathsf{o}}$ (the origin) and recall that $\vec{\mathsf{e}}_1 \stackrel{\text{def}}{=} (1,0,0,0)$, $\vec{\mathsf{e}}_2 \stackrel{\text{def}}{=} (0,1,0,0)$, $\vec{\mathsf{e}}_3 \stackrel{\text{def}}{=} (0,0,1,0)$, $\vec{\mathsf{e}}_4 \stackrel{\text{def}}{=} (0,0,0,1)$. Let $\vec{y}_0, \ldots, \vec{y}_4$ be 4-tuples of variables of type Q, which are all distinct and which do not occur in φ. Given $\varphi(o, \bar{x}) \in \mathscr{L}$-Scenarios we define formula $\breve{\varphi}$ (see Figure 11), in which these variables occur free, by

$$\breve{\varphi}(o, \bar{x}, \vec{y}_0, \ldots, \vec{y}_4) \stackrel{\text{def}}{=} (\exists \breve{o}) \left(\bigwedge_{i=0}^{4} \left(\mathsf{w}_{o\breve{o}}(\vec{\mathsf{e}}_i) = \vec{y}_i\right) \wedge \varphi(\breve{o}, \bar{x}) \right).$$

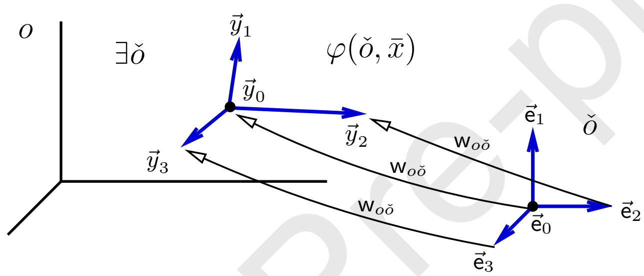


Figure 11: In $\breve{\varphi}$, rather than perform the experiment itself, observer o asks $\breve{o}$ to perform it.

The definition of $\breve{\varphi}$ is unique apart from the freedom we have in choosing the new variables. Clearly, $\breve{\varphi} \in \mathscr{L}$-Scenarios for all allowed choices of $\vec{y}_0, \ldots, \vec{y}_4$. There are several ways to make this choice deterministic, so let us choose one of them for use in all situations. Then for each φ we can treat $\breve{\varphi}$ as a uniquely and well-defined formula of our language $\mathscr{L}$.

Experimental scenario $\breve{\varphi}$ together with the choice of the parameters in a model describes the following situation from the point of view of observer o: o asks a colleague $\breve{o}$ related to o by the transformation determined by the parameters corresponding to $\vec{y}_0, \ldots, \vec{y}_4$ to perform the experiment described by scenario φ. If more such colleagues exist, the worldview transformation between them is the identity function by AxAffTr, hence, it is natural to assume that they all agree on the realisability of φ.[11] In this case, o can ask any of them to perform the experiment. Therefore, if φ describes an experiment, $\breve{\varphi}$ also describes an experiment. If no such colleague exists, then the experiment corresponding to $\breve{\varphi}$ is not realisable and so fails. For example, if $\vec{y}_0, \ldots, \vec{y}_4$ all evaluate to $\vec{\mathsf{o}}$, then the

[11] We do not include this assumption into our axiom system FRAME, because in our theorems this will follow from other assumptions.

experiment corresponding to $\breve{\varphi}$ should fail (because worldview transformations are bijections by $\mathsf{AxAffTr}$).

5. Main Results

5.1. Connections Between Homogeneities

In this section, we collect together the various connections we will establish between versions of homogeneity defined over any language $\mathscr{L}$ containing $\mathscr{L}_{\mathsf{core}}$.

Proposition 5.1.

(i) $\mathsf{HOM}^{\mathcal{S}} \models \mathsf{HOM}^{\mathcal{S}}_{\mathsf{time}} + \mathsf{HOM}^{\mathcal{S}}_{\mathsf{space}}$ *and* $\mathsf{HOM}^{+} \models \mathsf{HOM}^{+}_{\mathsf{time}} + \mathsf{HOM}^{+}_{\mathsf{space}}$.

(ii) $\mathsf{HOM}^{+}_{\mathsf{time}} + \mathsf{HOM}^{+}_{\mathsf{space}} \not\models \mathsf{HOM}^{\mathcal{S}}$ *if* $\mathscr{L} = \mathscr{L}_{\mathsf{core}}$ *and* $(\exists b)\mathsf{W}(o, b, \vec{o}) \in \mathcal{S}$, *thus* $\mathsf{HOM}^{+}_{\mathsf{time}} + \mathsf{HOM}^{+}_{\mathsf{space}} \not\models \mathsf{HOM}^{+}$ *if* $\mathscr{L} = \mathscr{L}_{\mathsf{core}}$.

(iii) $\mathsf{HOM}^{\mathcal{S}}_{\mathsf{time}} + \mathsf{HOM}^{\mathcal{S}}_{\mathsf{space}} \mathrel{=\!\!|\!\models}_{\mathsf{frm}} \mathsf{HOM}^{\mathcal{S}}$, *and hence,* $\mathsf{HOM}^{+}_{\mathsf{time}} + \mathsf{HOM}^{+}_{\mathsf{space}} \mathrel{=\!\!|\!\models}_{\mathsf{frm}} \mathsf{HOM}^{+}$. *Moreover, this remains true even if we omit assumptions* $\mathsf{AxReloc}^{\mathsf{Rot}}$, $\mathsf{AxAffTr}$ *and* $\mathsf{Ax3Dir\exists Motion}$.

Here, (i) follows by definition since temporal and spatial translations are also spacetime translations; (ii) follows from Proposition 5.24 below; (iii) follows from Lemma 5.23 and (i).

Theorem 5.2. *Homogeneity of time implies homogeneity of space:*

$$\mathsf{HOM}^{\mathcal{S}}_{\mathsf{time}} \models_{\mathsf{frm}} \mathsf{HOM}^{\mathcal{S}}_{\mathsf{space}} \quad \text{if } \varphi \in \mathcal{S} \Rightarrow \breve{\varphi} \in \mathcal{S},$$

and hence, $\mathsf{HOM}^{+}_{\mathsf{time}} \models_{\mathsf{frm}} \mathsf{HOM}^{+}_{\mathsf{space}}$.

This follows from Lemma 5.21, Lemma 5.27 and Remark 5.28 below. Moreover, by Lemma 5.21 and Lemma 5.27, it remains true even if we omit assumption $\mathsf{AxReloc}^{\mathsf{Rot}}$.

Theorem 5.3. *Strong homogeneity of space does not imply homogeneity of time: Assume* $\mathscr{L} = \mathscr{L}_{\mathsf{core}}$. *Then*

$$\mathsf{HOM}^{+}_{\mathsf{space}} \not\models_{\mathsf{frm}} \mathsf{HOM}^{\mathcal{S}}_{\mathsf{time}} \quad \text{if } m(o, \vec{x}) \in \mathcal{S},$$

and hence, $\mathsf{HOM}^{+}_{\mathsf{space}} \not\models_{\mathsf{frm}} \mathsf{HOM}^{+}_{\mathsf{time}}$.

This follows by Proposition 5.13 and Theorem 5.17 below.

Theorem 5.4. *Homogeneity of space implies homogeneity of time if there are clocks that get out of sync:*

$$\mathsf{HOM}^{\mathcal{S}}_{\mathsf{space}} + \mathsf{ASync} \models_{\mathsf{frm}} \mathsf{HOM}^{\mathcal{S}}_{\mathsf{time}} \quad \text{if } \varphi \in \mathcal{S} \Rightarrow \breve{\varphi} \in \mathcal{S},$$

and hence, $\mathsf{HOM}^{+}_{\mathsf{space}} + \mathsf{ASync} \models_{\mathsf{frm}} \mathsf{HOM}^{+}_{\mathsf{time}}$.

This follows by Lemma 5.22, Lemma 5.27 and Remark 5.28 below. Moreover, by Lemma 5.22 and Lemma 5.27, it remains true even if we omit assumptions $\mathsf{AxReloc}^{\mathsf{Rot}}$ and $\mathsf{Ax3Dir\exists Motion}$.

The following is a corollary of Theorem 5.2, Theorem 5.4 and Proposition 5.1:

Corollary 5.5. *Homogeneities of time, space and spacetime are equivalent if there are clocks that get out of sync: If $\varphi \in \mathcal{S} \Rightarrow \breve{\varphi} \in \mathcal{S}$,*

$$\mathsf{HOM}^{\mathcal{S}}_{\mathsf{time}} + \mathsf{ASync} \quad \Dashv\models_{\mathsf{frm}} \quad \mathsf{HOM}^{\mathcal{S}}_{\mathsf{space}} + \mathsf{ASync} \quad \Dashv\models_{\mathsf{frm}} \quad \mathsf{HOM}^{\mathcal{S}} + \mathsf{ASync}.$$

Hence,

$$\mathsf{HOM}^{+}_{\mathsf{time}} + \mathsf{ASync} \quad \Dashv\models_{\mathsf{frm}} \quad \mathsf{HOM}^{+}_{\mathsf{space}} + \mathsf{ASync} \quad \Dashv\models_{\mathsf{frm}} \quad \mathsf{HOM}^{+} + \mathsf{ASync}.$$

5.2. Connections Between Isotropy and Homogeneity

We continue with the connections between isotropy and the various homogeneities of time, space and spacetime.

Theorem 5.6. *Isotropy of space implies homogeneity of space:*

$$\mathsf{ISO}^{\mathcal{S}} \models_{\mathsf{frm}} \mathsf{HOM}^{\mathcal{S}}_{\mathsf{space}} \qquad \textit{if } \varphi \in \mathcal{S} \Rightarrow \breve{\varphi} \in \mathcal{S},$$

and hence, $\qquad \mathsf{ISO}^{+} \models_{\mathsf{frm}} \mathsf{HOM}^{+}_{\mathsf{space}}.$

This follows by Lemma 5.25, Lemma 5.27 and Remark 5.28 below. Moreover, by Lemma 5.25 and Lemma 5.27, it remains true even if we omit assumption $\mathsf{Ax3Dir\exists Motion}$.

Theorem 5.7. *Strong isotropy does not imply homogeneity of time: Assume $\mathscr{L} = \mathscr{L}_{\mathsf{core}}$. Then*

$$\mathsf{ISO}^{+} \not\models_{\mathsf{frm}} \mathsf{HOM}^{\mathcal{S}}_{\mathsf{time}} \qquad \textit{if } m(o,\vec{x}) \in \mathcal{S},$$

and hence, $\qquad \mathsf{ISO}^{+} \not\models_{\mathsf{frm}} \mathsf{HOM}^{+}_{\mathsf{time}}.$

This follows by Proposition 5.13 and Theorem 5.17 below.

The following is a corollary of Theorem 5.6, Theorem 5.4 and Proposition 5.1(iii):

Corollary 5.8. *Isotropy implies homogeneity if there are clocks that get out of sync:*

$$\mathsf{ISO}^{\mathcal{S}} + \mathsf{ASync} \models_{\mathsf{frm}} \mathsf{HOM}^{\mathcal{S}} \quad \textit{if } \varphi \in \mathcal{S} \Rightarrow \breve{\varphi} \in \mathcal{S},$$

and hence, $\qquad \mathsf{ISO}^{+} + \mathsf{ASync} \models_{\mathsf{frm}} \mathsf{HOM}^{+}.$

Theorem 5.9. *Strong homogeneity does not imply isotropy (whether or not there are clocks that get out of sync). Let $\mathscr{L} = \mathscr{L}_{\mathsf{core}}$. Then if $m(o,\vec{x}) \in \mathcal{S}$,*

$$\mathsf{HOM}^{+} + \mathsf{ASync} \not\models_{\mathsf{frm}} \mathsf{ISO}^{\mathcal{S}} \quad \textit{and} \quad \mathsf{HOM}^{+} + \neg\mathsf{ASync} \not\models_{\mathsf{frm}} \mathsf{ISO}^{\mathcal{S}}.$$

Hence,

$$\mathsf{HOM}^{+} + \mathsf{ASync} \not\models_{\mathsf{frm}} \mathsf{ISO}^{+} \quad \textit{and} \quad \mathsf{HOM}^{+} + \neg\mathsf{ASync} \not\models_{\mathsf{frm}} \mathsf{ISO}^{+}.$$

This follows by Proposition 5.13 and Theorem 5.15 below.

5.3. Some Models

We have looked, so far, at general logical connections between homogeneity and isotropy. We now turn our attention towards more specific models. To introduce these models precisely, we need some definitions. Throughout this section, we assume $\mathsf{AxOField}$. This allows us to use the derived operation of unary negation $(-)$.

We say that an affine transformation $A : Q^4 \to Q^4$ is *proper orthochronous* iff $A(\vec{\mathsf{e}}_1)_\mathsf{t} > A(\vec{\mathsf{o}})_\mathsf{t}$ and the determinant representing the linear part of A is positive. Intuitively, proper orthochronous affine transformations preserve both the direction of time and the orientation of space. A map $P : Q^4 \to Q^4$ is a *Poincaré transformation* iff it is an affine transformation that preserves squared Minkowski distances, i.e., for every $\vec{p}, \vec{q} \in Q^4$,

$$|P(\vec{p})_\mathsf{t} - P(\vec{q})_\mathsf{t}|^2 - |P(\vec{p})_\mathsf{s} - P(\vec{q})_\mathsf{s}|^2 = |\vec{p}_\mathsf{t} - \vec{q}_\mathsf{t}|^2 - |\vec{p}_\mathsf{s} - \vec{q}_\mathsf{s}|^2.$$

The set of Poincaré transformations is denoted Poi and the set of proper orthochronous Poincaré transformations is denoted $\mathsf{Poi}^\uparrow_+$. A map $G : Q^4 \to Q^4$ is a *Galilean transformation* iff it is an affine transformation and, for every $\vec{p}, \vec{q} \in Q^4$,

$$|G(\vec{p})_\mathsf{t} - G(\vec{q})_\mathsf{t}|^2 = |\vec{p}_\mathsf{t} - \vec{q}_\mathsf{t}|^2 \ \text{ and } \ \vec{p}_\mathsf{t} = \vec{q}_\mathsf{t} \Rightarrow |G(\vec{p})_\mathsf{s} - G(\vec{q})_\mathsf{s}|^2 = |\vec{p}_\mathsf{s} - \vec{q}_\mathsf{s}|^2.$$

The set of Galilean transformations is denoted Gal and the set of proper orthochronous Galilean transformation is denoted $\mathsf{Gal}^\uparrow_+$.

Let $\mathfrak{M}$ be an $\mathscr{L}$-model. The set $\mathbb{W}_k$ of worldview transformations associated with a specific observer $k \in IOb$ is defined by

$$\mathbb{W}_k \stackrel{\text{def}}{=} \{\mathsf{w}_{kh} : h \in IOb\}.$$

Definition 5.10. We call an $\mathscr{L}$-model $\mathfrak{M}$ a Poi*-based $\mathscr{L}$-model* iff for every $k \in IOb$, $\mathsf{Poi}^\uparrow_+ \subseteq \mathbb{W}_k \subseteq \mathsf{Poi}$.

Definition 5.11. We call an $\mathscr{L}$-model $\mathfrak{M}$ a Gal*-based $\mathscr{L}$-model* iff for every $k \in IOb$, $\mathsf{Gal}^\uparrow_+ \subseteq \mathbb{W}_k \subseteq \mathsf{Gal}$.

It is worth noting that both Gal-based and Poi-based models exist which are homogeneous and isotropic (whence the results shown in this paper are non-vacuous).

Proposition 5.12. *Over any ordered field, there exists a* Poi*-based* $\mathscr{L}_\mathsf{core}$*-model* $\mathfrak{M}_P$ *and a* Gal*-based* $\mathscr{L}_\mathsf{core}$*-model* $\mathfrak{M}_G$ *such that* $\mathfrak{M}_P, \mathfrak{M}_G \models \mathsf{HOM}^+ \cup \mathsf{ISO}^+$.

This is proven on page 41.

Proposition 5.13. *For every* Poi*-based $\mathscr{L}$-model* $\mathfrak{M}_P$ *and* Gal*-based $\mathscr{L}$-model* $\mathfrak{M}_G$*,*

$$\mathfrak{M}_P \models \mathsf{FRAME} + \mathsf{ASync} \ \text{ and } \ \mathfrak{M}_G \models \mathsf{FRAME} + \neg\mathsf{ASync}.$$

Proof. $\mathsf{AxOField}$ and $\mathsf{AxAffTr}$ hold in both models by Definitions 5.10 and 5.11. A trivial reformulation of $\mathsf{AxReloc}^{\mathsf{Tran}}$ is that $\mathsf{Tran} \subseteq \mathbb{W}_k$ for all $k \in IOb$, and $\mathsf{AxReloc}^{\mathsf{Rot}}$ can similarly be reformulated as saying that $\mathsf{Rot}_{\mathsf{space}} \subseteq \mathbb{W}_k$ for all $k \in IOb$. It is also easy to see that $\mathsf{Tran} \cup \mathsf{Rot}_{\mathsf{space}} \subseteq \mathsf{Poi}_+^{\uparrow} \cap \mathsf{Gal}_+^{\uparrow}$. Therefore, $\mathsf{AxReloc}^{\mathsf{Tran}}$ and $\mathsf{AxReloc}^{\mathsf{Rot}}$ hold in both models. To prove that $\mathsf{Ax3Dir\exists Motion}$ holds, let o be an arbitrary observer. It is enough to show that there are $o_1, o_2, o_3 \in IOb$ and $\vec{p}^{\,1}, \vec{p}^{\,2}, \vec{p}^{\,3} \in Q^4$ such that

$$\vec{p}^{\,1}_{\mathsf{s}}, \vec{p}^{\,2}_{\mathsf{s}}, \vec{p}^{\,3}_{\mathsf{s}} \text{ are lin. independent and } \vec{\mathsf{o}}, \vec{p}^{\,i} \in \mathsf{wl}_o(o_i) \text{ for every } i \in \{1,2,3\}. \quad (2)$$

We will first prove this for the Poi-based model $\mathfrak{M}_P$. It is not difficult to see that there are linear proper orthochronous Poincaré transformations f_1, f_2 and f_3 taking the time unit vector $\vec{\mathsf{e}}_1$, respectively, to $\vec{p}^{\,1} := (5/3, 4/3, 0, 0)$, $\vec{p}^{\,2} := (5/3, 0, 4/3, 0)$ and $\vec{p}^{\,3} := (5/3, 0, 0, 4/3)$.[12] By $\mathsf{Poi}_+^{\uparrow} \subseteq \mathbb{W}_o$, we can choose o_1, o_2, o_3 such that $\mathsf{w}_{oo_1} = f_1$, $\mathsf{w}_{oo_2} = f_2$ and $\mathsf{w}_{oo_3} = f_3$. By definition of worldline, for the choice of o_1, o_2, o_3 and $\vec{p}^{\,1}, \vec{p}^{\,2}, \vec{p}^{\,3}$ above, (2) holds. Therefore, $\mathsf{Ax3Dir\exists Motion}$ holds in $\mathfrak{M}_P$. For the case of Gal-based model $\mathfrak{M}_G$ the proof is analogous: First we choose linear proper orthochronous Galilean transformations f_1, f_2 and f_3 which take the time unit vector $\vec{\mathsf{e}}_1$, respectively, to $\vec{p}^{\,1} := (1,1,0,0)$, $\vec{p}^{\,2} := (1,0,1,0)$ and $\vec{p}^{\,3} := (1,0,0,1)$; and then choose o_1, o_2, o_3 such that $\mathsf{w}_{oo_1} = f_1$, $\mathsf{w}_{oo_2} = f_2$ and $\mathsf{w}_{oo_3} = f_3$. This completes the proof that FRAME holds in both models.

If worldview transformations are bijections from Q^4 to Q^4 (which is true in both $\mathfrak{M}_P$ and $\mathfrak{M}_G$ because $\mathbb{W}_k \subseteq \mathsf{Poi}$ and $\mathbb{W}_k \subseteq \mathsf{Gal}$, respectively), then ASync is equivalent to the statement that there exists a worldview transformation $f = \mathsf{w}_{kh}$ and points $\vec{p}, \vec{q} \in Q^4$ such that

$$\vec{p}_{\mathsf{t}} \neq \vec{q}_{\mathsf{t}} \quad \text{and} \quad f(\vec{p})_{\mathsf{t}} = f(\vec{q})_{\mathsf{t}}, \quad (3)$$

see Figure 3. It is not difficult to see that there exist $f \in \mathsf{Poi}_+^{\uparrow}$ and $\vec{p}, \vec{q} \in Q^4$ such that (3) holds. Therefore, $\mathfrak{M}_P \models \mathsf{ASync}$ because every transformation in $\mathsf{Poi}_+^{\uparrow}$ is a worldview transformation of $\mathfrak{M}_P$. On the other hand, there is no transformation $f \in \mathsf{Gal}$ (and points $\vec{p}, \vec{q} \in Q^4$) such that (3) holds. Therefore, $\mathfrak{M}_G \models \neg\mathsf{ASync}$ because every worldview transformation of $\mathfrak{M}_G$ is an element of Gal.

The following statements are corollaries of Theorem 5.2, Theorem 5.6 and Proposition 5.13.

Corollary 5.14. *Let $\mathfrak{M}$ be a Poi-based or a Gal-based $\mathscr{L}$-model. Then*

- $\mathfrak{M} \models \mathsf{HOM}^{\mathscr{S}}_{\mathsf{time}} \Rightarrow \mathfrak{M} \models \mathsf{HOM}^{\mathscr{S}}_{\mathsf{space}}$ *if $\varphi \in \mathscr{S} \Rightarrow \breve{\varphi} \in \mathscr{S}$,*

[12] Linear transformations given by matrices $\begin{bmatrix} 5/3 & 4/3 & 0 & 0 \\ 4/3 & 5/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 5/3 & 0 & 4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 4/3 & 0 & 5/3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, and $\begin{bmatrix} 5/3 & 0 & 0 & 4/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 4/3 & 0 & 0 & 5/3 \end{bmatrix}$ are such.

- $\mathfrak{M} \models \mathsf{HOM}^{+}_{\mathsf{time}} \;\Rightarrow\; \mathfrak{M} \models \mathsf{HOM}^{+}_{\mathsf{space}}$,
- $\mathfrak{M} \models \mathsf{ISO}^{\mathcal{S}} \;\Rightarrow\; \mathfrak{M} \models \mathsf{HOM}^{\mathcal{S}}_{\mathsf{space}}$ *if* $\varphi \in \mathcal{S} \Rightarrow \breve{\varphi} \in \mathcal{S}$, *and*
- $\mathfrak{M} \models \mathsf{ISO}^{+} \;\Rightarrow\; \mathfrak{M} \models \mathsf{HOM}^{+}_{\mathsf{space}}$.

Theorem 5.15. *Strong homogeneity does not imply isotropy in* Gal*-based and* Poi*-based models: Over any ordered field, there exists a* Poi*-based* $\mathscr{L}_{\mathsf{core}}$*-model* $\mathfrak{M}_P$ *and a* Gal*-based* $\mathscr{L}_{\mathsf{core}}$*-model* $\mathfrak{M}_G$ *such that*

$\mathfrak{M}_G, \mathfrak{M}_P \models \mathsf{HOM}^{+}$,

$\mathfrak{M}_G, \mathfrak{M}_P \not\models \mathsf{ISO}^{\mathcal{S}}$ *if* $m(o, \vec{x}) \in \mathcal{S}$, *and hence,* $\mathfrak{M}_G, \mathfrak{M}_P \not\models \mathsf{ISO}^{+}$.

This is proven on page 37.

The following is a corollary of Corollary 5.5, Corollary 5.8 and Proposition 5.13:

Corollary 5.16. *Let* $\mathfrak{M}$ *be a* Poi*-based* $\mathscr{L}$*-model. Then*

- $\mathfrak{M} \models \mathsf{HOM}^{\mathcal{S}}_{\mathsf{time}} \;\Leftrightarrow\; \mathfrak{M} \models \mathsf{HOM}^{\mathcal{S}}_{\mathsf{space}} \;\Leftrightarrow\; \mathfrak{M} \models \mathsf{HOM}^{\mathcal{S}}$ *if* $\varphi \in \mathcal{S} \Rightarrow \breve{\varphi} \in \mathcal{S}$,
- $\mathfrak{M} \models \mathsf{HOM}^{+}_{\mathsf{time}} \;\Leftrightarrow\; \mathfrak{M} \models \mathsf{HOM}^{+}_{\mathsf{space}} \;\Leftrightarrow\; \mathfrak{M} \models \mathsf{HOM}^{+}$,
- $\mathfrak{M} \models \mathsf{ISO}^{\mathcal{S}} \;\Rightarrow\; \mathfrak{M} \models \mathsf{HOM}^{\mathcal{S}}$ *if* $\varphi \in \mathcal{S} \Rightarrow \breve{\varphi} \in \mathcal{S}$, *and*
- $\mathfrak{M} \models \mathsf{ISO}^{+} \;\Rightarrow\; \mathfrak{M} \models \mathsf{HOM}^{+}$.

Theorem 5.17. *Strong isotropy and homogeneity of space do not imply homogeneity of time in* Gal*-based models: Over any ordered field, there is a* Gal*-based* $\mathscr{L}_{\mathsf{core}}$*-model* $\mathfrak{M}$ *such that*

$\mathfrak{M} \models \mathsf{ISO}^{+} + \mathsf{HOM}^{+}_{\mathsf{space}}$,

$\mathfrak{M} \not\models \mathsf{HOM}^{\mathcal{S}}_{\mathsf{time}}$ *if* $m(o, \vec{x}) \in \mathcal{S}$, *and hence* $\mathfrak{M} \not\models \mathsf{HOM}^{+}_{\mathsf{time}}$.

This is proven on page 39.

5.4. Lemmas

In this section, we state and prove some underpinning lemmas and propositions on which many of the proofs in Section 5.1 and Section 5.2 are based.

We assume the standard compositional properties of (binary) relations and functions: we define the composition of relations R and S by

$$R \circ S \stackrel{\text{def}}{=} \{(a,b) : \exists c\big((a,c) \in S \wedge (c,b) \in R\big)\},$$

so that $(f \circ g)(x) = f\big(g(x)\big)$ when f and g are functions, and recall that we write Id for the *identity* relation on Q^4.

Proposition 5.18. *For all* $k, h, m \in IOb$*, we have:*

(i) $\mathsf{w}_{kh}^{-1} = \mathsf{w}_{hk}$.
(ii) $\mathsf{Id} \subseteq \mathsf{w}_{kk}$.
(iii) $\mathsf{w}_{kk} = \mathsf{Id}$ *if* w_{kk} *is a function.*

(iv) $\mathsf{w}_{km} \circ \mathsf{w}_{mh} \subseteq \mathsf{w}_{kh}$.
(v) $\mathsf{w}_{km} \circ \mathsf{w}_{mh} = \mathsf{w}_{kh}$ *if* w_{mh} *is a function from* Q^4 *to* Q^4.

Proof. (i), (ii), (iii) and (iv) follow immediately from the definition of the worldview transformation.

(v) By (iv), it is enough to show that $\mathsf{w}_{kh} \subseteq \mathsf{w}_{km} \circ \mathsf{w}_{mh}$. To prove this, let $(\vec{p}, \vec{q}) \in \mathsf{w}_{kh}$. Then $\mathsf{ev}_h(\vec{p}) = \mathsf{ev}_k(\vec{q})$. Let $\vec{r}$ be such that $(\vec{p}, \vec{r}) \in \mathsf{w}_{mh}$. Then $\mathsf{ev}_m(\vec{r}) = \mathsf{ev}_h(\vec{p}) = \mathsf{ev}_k(\vec{q})$. Thus $(\vec{r}, \vec{q}) \in \mathsf{w}_{km}$, which together with $(\vec{p}, \vec{r}) \in \mathsf{w}_{mh}$ imply that $(\vec{p}, \vec{q}) \in \mathsf{w}_{km} \circ \mathsf{w}_{mh}$.

Recall that $\sim^{\mathscr{S}}_{\mathfrak{M}}$ was defined in Definition 4.1.

Definition 5.19. We say that $\mathscr{L}$-model $\mathfrak{M}$ has the $\mathscr{S}$-*transformation* property iff for every $m, k, m', k' \in IOb$, if $\mathsf{w}_{mk} = \mathsf{w}_{m'k'}$ and whenever $m \sim^{\mathscr{S}}_{\mathfrak{M}} m'$, then $k \sim^{\mathscr{S}}_{\mathfrak{M}} k'$, see Figure 4. Intuitively if m and m' agree on every scenario in $\mathscr{S}$ and k is related to m the same way as k' is related to m', then k and k' must agree on every scenario in $\mathscr{S}$, too.

The following proposition connects ASync, the axioms of Section 4.4, and the $\mathscr{S}$-transformation property to the assumptions of Section 3.3.

Proposition 5.20. *Let* $\mathfrak{M}$ *be an* $\mathscr{L}$*-model for which* $\mathsf{AxOField}$ *holds and the worldview transformations are functions from* Q^4 *to* Q^4*. For every* $k, h \in IOb$*, let* $w_{kh} := \mathsf{w}_{kh}$*. Then:*

(i) $\mathsf{AxReloc}^{\mathsf{Tran}} \Leftrightarrow \mathsf{Reloc}^{\mathsf{Tran}}$, $\mathsf{AxReloc}^{\mathsf{Rot}} \Leftrightarrow \mathsf{Reloc}^{\mathsf{Rot}}$, $\mathsf{AxAffTr} \Leftrightarrow \mathsf{AffTr}$, $\mathsf{Ax3Dir\exists Motion} \Leftrightarrow \mathsf{3Dir\exists Motion}$, $\mathsf{ASync} \Leftrightarrow \mathsf{async}$ *and* Wvt *all hold in* $\mathfrak{M}$.

(ii) Model $\mathfrak{M}$ *has the* $\mathscr{S}$*-transformation property iff relation* $\sim^{\mathscr{S}}_{\mathfrak{M}}$ *has the transformation property.*

Proof. Wvt holds by Proposition 5.18. $\mathsf{ASync} \Leftrightarrow \mathsf{async}$ holds because $\mathsf{w}_{o'o}(\vec{p}, \vec{p}\,')$ and $\mathsf{w}_{o'o}(\vec{q}, \vec{q}\,')$ in ASync are equivalent to $\vec{p}\,' = \mathsf{w}_{o'o}(\vec{p})$ and $\vec{q}\,' = \mathsf{w}_{o'o}(\vec{q})$ since $\mathsf{w}_{o'o}$ is a function. The rest of the proof is straightforward.[13]

Lemma 5.21. *Let* $\mathfrak{M}$ *be an* $\mathscr{L}$*-model which has the* $\mathscr{S}$*-transformation property. Then*

$$\mathfrak{M} \models \mathsf{HOM}^{\mathscr{S}}_{\mathsf{time}} \cup \mathsf{FRAME}\setminus\{\mathsf{AxReloc}^{\mathsf{Rot}}\} \quad \Rightarrow \quad \mathfrak{M} \models \mathsf{HOM}^{\mathscr{S}}_{\mathsf{space}}.$$

Proof. The lemma easily follows from Theorem 3.1 by Propositions 4.3, 5.20 and Remark 4.2. In more detail: Assume $\mathsf{HOM}^{\mathscr{S}}_{\mathsf{time}} \cup \mathsf{FRAME}\setminus\{\mathsf{AxReloc}^{\mathsf{Rot}}\}$. Then the worldview transformations are functions by $\mathsf{AxAffTr}$. For every $o, o' \in IOb$, let $w_{oo'} := \mathsf{w}_{oo'}$ and let $\sim$ be $\sim^{\mathscr{S}}_{\mathfrak{M}}$. By Propositions 4.3, 5.20 and Remark 4.2, $\mathscr{HOM}^{\sim}_{\mathsf{time}}$, $\mathsf{AxOField}$, Wvt, $\mathsf{Reloc}^{\mathsf{Tran}}$, AffTr and $\mathsf{3Dir\exists Motion}$ hold. Furthermore,

[13] For proving $\mathsf{Ax3Dir\exists Motion} \Leftrightarrow \mathsf{3Dir\exists Motion}$ we have to notice that $w\ell_k(h) = \mathsf{wl}_k(h)$.

$\sim$ is an equivalence relation on *IOb* that has the transformation property. Then, by Theorem 3.1, $\mathscr{HOM}^{\sim}_{\mathsf{space}}$ holds, which is equivalent to $\mathsf{HOM}^{\mathscr{S}}_{\mathsf{space}}$ by Proposition 4.3.

Lemma 5.22. *Let $\mathfrak{M}$ be an $\mathscr{L}$-model which has the $\mathscr{S}$-transformation property. Then*

$$\mathfrak{M} \models \mathsf{HOM}^{\mathscr{S}}_{\mathsf{space}} \cup \{\mathsf{AxOField}, \mathsf{AxReloc}^{\mathsf{Tran}}, \mathsf{AxAffTr}, \mathsf{ASync}\} \ \Rightarrow\ \mathfrak{M} \models \mathsf{HOM}^{\mathscr{S}}_{\mathsf{time}}.$$

Proof. The lemma easily follows from Theorem 3.2 by Propositions 4.3, 5.20 and Remark 4.2: Assume $\mathsf{HOM}^{\mathscr{S}}_{\mathsf{space}} \cup \{\mathsf{AxOField}, \mathsf{AxReloc}^{\mathsf{Tran}}, \mathsf{AxAffTr}, \mathsf{ASync}\}$. Then the worldview transformations are functions by $\mathsf{AxAffTr}$. For every o, o' let $w_{oo'} := \mathsf{w}_{oo'}$ and let $\sim$ be $\sim^{\mathscr{S}}_{\mathfrak{M}}$. By Propositions 4.3, 5.20 and Remark 4.2, $\mathscr{HOM}^{\sim}_{\mathsf{space}}$, $\mathsf{AxOField}$, Wvt, $\mathsf{Reloc}^{\mathsf{Tran}}$, AffTr and async hold. Furthermore, $\sim$ is an equivalence relation on *IOb* that has the transformation property. Then by Theorem 3.2, $\mathscr{HOM}^{\sim}_{\mathsf{time}}$ holds, which is equivalent to $\mathsf{HOM}^{\mathscr{S}}_{\mathsf{time}}$ by Proposition 4.3.

Lemma 5.23. *Let $\mathfrak{M}$ be an $\mathscr{L}$-model. Then*

$$\mathfrak{M} \models \mathsf{HOM}^{\mathscr{S}}_{\mathsf{time}} \cup \mathsf{HOM}^{\mathscr{S}}_{\mathsf{space}} \cup \{\mathsf{AxOField}, \mathsf{AxReloc}^{\mathsf{Tran}}\} \quad \Rightarrow \quad \mathfrak{M} \models \mathsf{HOM}^{\mathscr{S}}.$$

Proof. We cannot apply Proposition 3.3 directly because the worldview transformations are not necessarily functions. However, the proof of Proposition 3.3 goes through for Lemma 5.23 with the obvious modifications. The only nontrivial modification is when we have to replace the reference to Wvt with reference to Proposition 5.18.

By Proposition 5.24, Lemma 5.23 does not remain true if we omit assumption $\mathsf{AxReloc}^{\mathsf{Tran}}$.

Proposition 5.24. *Suppose $\mathscr{S} \subseteq \mathscr{L}_{\mathsf{core}}$-Scenarios. If $(\exists b)\mathsf{W}(o, b, \vec{\mathsf{o}}) \in \mathscr{S}$, there is an $\mathscr{L}_{\mathsf{core}}$-model $\mathfrak{M}$ such that*

$$\mathfrak{M} \models \mathsf{HOM}^{+}_{\mathsf{time}} \cup \mathsf{HOM}^{+}_{\mathsf{space}} \cup \{\mathsf{AxOField}\} \qquad \textit{but} \qquad \mathfrak{M} \not\models \mathsf{HOM}^{\mathscr{S}}.$$

We will prove Proposition 5.24 in Section 5.5 (Constructions of Counterexamples).

Lemma 5.25. *Let $\mathfrak{M}$ be an $\mathscr{L}$-model which has the $\mathscr{S}$-transformation property. Then*

$$\mathfrak{M} \models \mathsf{ISO}^{\mathscr{S}} \cup \{\mathsf{AxOField}, \mathsf{AxReloc}^{\mathsf{Tran}}, \mathsf{AxReloc}^{\mathsf{Rot}}, \mathsf{AxAffTr}\} \ \Rightarrow\ \mathfrak{M} \models \mathsf{HOM}^{\mathscr{S}}_{\mathsf{space}}.$$

Proof. The lemma easily follows from Theorem 3.4 by Propositions 4.3, 5.20 and Remark 4.2.

Lemma 5.26. *Let $\mathfrak{M}$ be an $\mathscr{L}$-model. Assume $\varphi(o, \bar{x}) \in \mathscr{L}$-Scenarios, and $h, h' \in IOb$. Then $\mathsf{Agree}_{\varphi}(h, h')$ holds in $\mathfrak{M}$ if and only if for every evaluation $\bar{q}$ of variables $\bar{x}$ in $\mathfrak{M}$, $\varphi(h, \bar{q})$ holds in $\mathfrak{M}$ iff $\varphi(h', \bar{q})$ holds in $\mathfrak{M}$.*

Proof. The proof is straightforward.

Lemma 5.27. *Let $\mathfrak{M}$ be a model of language $\mathscr{L}$. Assume $\varphi \in \mathscr{S} \Rightarrow \breve{\varphi} \in \mathscr{S}$, $\mathfrak{M} \models \{\mathsf{AxOField}, \mathsf{AxAffTr}\}$, and for every $h, h' \in IOb$,*

$$\mathsf{w}_{hh'} = \mathsf{Id} \quad \Rightarrow \quad h \sim^{\mathscr{S}}_{\mathfrak{M}} h'. \tag{4}$$

Then $\mathfrak{M}$ has the $\mathscr{S}$-transformation property.

Remark 5.28. Let $\mathfrak{M}$ be an $\mathscr{L}$-model such that $\mathfrak{M} \models \mathsf{AxOField}$ and

$$\mathfrak{M} \models \mathsf{HOM}^{\mathscr{S}}_{\mathsf{time}} \ \text{ or } \ \mathfrak{M} \models \mathsf{HOM}^{\mathscr{S}}_{\mathsf{space}} \ \text{ or } \ \mathfrak{M} \models \mathsf{HOM}^{\mathscr{S}} \ \text{ or } \ \mathfrak{M} \models \mathsf{ISO}^{\mathscr{S}}.$$

Then, by Proposition 4.3 and the fact that $\mathsf{Id} \in \mathsf{Tran}_{\mathsf{time}} \cap \mathsf{Tran}_{\mathsf{space}} \cap \mathsf{Rot}_{\mathsf{space}}$, assumption (4) in Lemma 5.27 holds in $\mathfrak{M}$.

Proof of Lemma 5.27. Let $m, m', k, k' \in IOb$ such that $m \sim^{\mathscr{S}}_{\mathfrak{M}} m'$ and $\mathsf{w}_{mk} = \mathsf{w}_{m'k'}$. We have to prove that $k \sim^{\mathscr{S}}_{\mathfrak{M}} k'$, so by Lemma 5.26 and symmetry of $\sim^{\mathscr{S}}_{\mathfrak{M}}$, it is enough to prove that for every $\varphi(o, \bar{x}) \in \mathscr{S}$ and evaluation $\bar{q}$ of variables $\bar{x}$ in $\mathfrak{M}$, if $\varphi(k, \bar{q})$ holds in $\mathfrak{M}$, then $\varphi(k', \bar{q})$ holds in $\mathfrak{M}$. To prove this, let $\varphi(o, \bar{x}) \in \mathscr{S}$ and $\bar{q}$ be an evaluation of variables $\bar{x}$ in $\mathfrak{M}$, and assume that $\varphi(k, \bar{q})$ holds in $\mathfrak{M}$. We have to prove that $\varphi(k', \bar{q})$ holds in $\mathfrak{M}$, too. Recall that $\breve{\varphi}(o, \bar{x}, \vec{y}_0, \ldots, \vec{y}_4) \stackrel{\text{def}}{=} (\exists \breve{o}) \left(\bigwedge_{i=0}^{4} \left(\mathsf{w}_{o\breve{o}}(\vec{\mathsf{e}}_i) = \vec{y}_i\right) \wedge \varphi(\breve{o}, \bar{x}) \right)$. Since $\varphi(k, \bar{q})$ holds in $\mathfrak{M}$, $\bigwedge_{i=0}^{4} \left(\mathsf{w}_{mk}(\vec{\mathsf{e}}_i) = \mathsf{w}_{mk}(\vec{\mathsf{e}}_i)\right) \wedge \varphi(k, \bar{q})$ holds in $\mathfrak{M}$. Taking $\breve{o} = k$, it follows that $(\exists \breve{o}) \left(\bigwedge_{i=0}^{4} \left(\mathsf{w}_{m\breve{o}}(\vec{\mathsf{e}}_i) = \mathsf{w}_{mk}(\vec{\mathsf{e}}_i)\right) \wedge \varphi(\breve{o}, \bar{q}) \right)$ holds in $\mathfrak{M}$, i.e., $\breve{\varphi}\big(m, \bar{q}, \mathsf{w}_{mk}(\vec{\mathsf{e}}_0), \ldots, \mathsf{w}_{mk}(\vec{\mathsf{e}}_4)\big)$ holds in $\mathfrak{M}$. We have $m \sim^{\mathscr{S}}_{\mathfrak{M}} m'$ by assumption, and $\breve{\varphi} \in \mathscr{S}$ since $\varphi \in \mathscr{S}$. Therefore, by definition of $\sim^{\mathscr{S}}_{\mathfrak{M}}$, $\mathsf{Agree}_{\breve{\varphi}}(m, m')$. Therefore, by Lemma 5.26, $\breve{\varphi}\big(m', \bar{q}, \mathsf{w}_{mk}(\vec{\mathsf{e}}_0), \ldots, \mathsf{w}_{mk}(\vec{\mathsf{e}}_4)\big)$ holds in $\mathfrak{M}$, i.e., $(\exists \breve{o}) \left(\bigwedge_{i=0}^{4} \left(\mathsf{w}_{m'\breve{o}}(\vec{\mathsf{e}}_i) = \mathsf{w}_{mk}(\vec{\mathsf{e}}_i)\right) \wedge \varphi(\breve{o}, \bar{q}) \right)$ holds in $\mathfrak{M}$. Let $\breve{k} \in IOb$ be such that $\bigwedge_{i=0}^{4} \left(\mathsf{w}_{m'\breve{k}}(\vec{\mathsf{e}}_i) = \mathsf{w}_{mk}(\vec{\mathsf{e}}_i)\right) \wedge \varphi(\breve{k}, \bar{q})$ holds in $\mathfrak{M}$. Since $\mathsf{w}_{mk} = \mathsf{w}_{m'k'}$, it follows that $\bigwedge_{i=0}^{4} \left(\mathsf{w}_{m'\breve{k}}(\vec{\mathsf{e}}_i) = \mathsf{w}_{m'k'}(\vec{\mathsf{e}}_i)\right)$ and so, by Proposition 5.18 and $\mathsf{AxAffTr}$, we have $\mathsf{w}_{\breve{k}k'}(\vec{\mathsf{e}}_i) = \mathsf{w}_{\breve{k}m'} \circ \mathsf{w}_{m'k'}(\vec{\mathsf{e}}_i) = \mathsf{w}^{-1}_{m'\breve{k}} \circ \mathsf{w}_{m'k'}(\vec{\mathsf{e}}_i) = \vec{\mathsf{e}}_i$ for every $i \in \{0, 1, 2, 3, 4\}$. Since $\mathsf{w}_{\breve{k}k'}$ fixes every $\vec{\mathsf{e}}_i$ and (by $\mathsf{AxAffTr}$) is an affine transformation on a vector space (by $\mathsf{AxOField}$), we have $\mathsf{w}_{\breve{k}k'} = \mathsf{Id}$. But then $\breve{k} \sim^{\mathscr{S}}_{\mathfrak{M}} k'$ holds by assumption. Hence, $\mathsf{Agree}_{\varphi}\left(\breve{k}, k'\right)$ because $\varphi \in \mathscr{S}$. Then, by Lemma 5.26 and the fact that $\varphi(\breve{k}, \bar{q})$ holds in $\mathfrak{M}$, we conclude that $\varphi(k', \bar{q})$ holds in $\mathfrak{M}$, as required.

5.5. Constructions of Counterexamples

In this section, we prove Theorem 5.15, Theorem 5.17, and Proposition 5.24 by constructing appropriate counterexamples. Using a similar construction, we also prove Proposition 5.12.

As usual, if f is a function and H is a set, then the f-image of H is defined by $f[H] \stackrel{\text{def}}{=} \{f(x) : x \in H\}$. Recall that $\mathrm{Sym}(Q^4)$, the set of bijections of Q^4 onto itself, forms a group under composition (with identity Id).

Definition 5.29. Let $\mathscr{Q} = (Q, +, \cdot, 0, 1, \leqslant)$ be an arbitrary structure in the language of ordered fields, $IOb \subseteq \mathrm{Sym}(Q^4)$ and $B \subseteq \mathscr{P}(Q^4)$. We define model $\mathfrak{M}(IOb, B, \mathscr{Q})$ of $\mathscr{L}_{\mathsf{core}}$ (see Figure 12) by:

$$\mathfrak{M}(IOb, B, \mathscr{Q}) \stackrel{\text{def}}{=} (IOb, B, Q, +, \cdot, 0, 1, \leqslant, \mathsf{W}), \text{ where } \mathsf{W}(m, b, \vec{p}) \stackrel{\text{def}}{\iff} \vec{p} \in m[b].$$

It is easy to see that in model $\mathfrak{M}(IOb, B, \mathscr{Q})$, we have $\mathsf{wl}_m(b) = m[b]$ for every $m \in IOb$ and $b \in B$. Furthermore, $k \circ h^{-1} \subseteq \mathsf{w}_{kh}$ for every $k, h \in IOb$, cf. the proof of Lemma 5.31 below.

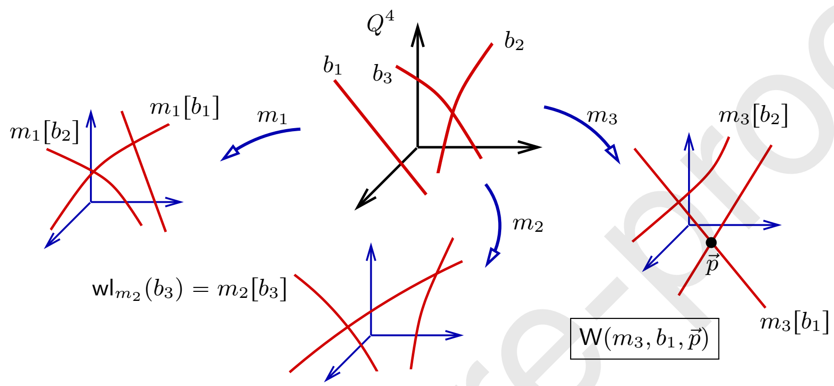


Figure 12: Illustration for Definition 5.29.

Lemma 5.30. *Let $\mathfrak{M}$ be an $\mathscr{L}$-model. Let S_{mat} denote the union of mathematical sorts in $\mathfrak{M}$. Let $k, h \in IOb$. Assume that there is an automorphism of $\mathfrak{M}$ which maps k to h and leaves all the elements of S_{mat} fixed. Then for every $\varphi \in \mathscr{L}$-Scenarios, $\mathsf{Agree}_\varphi(k, h)$ holds in $\mathfrak{M}$.*

Proof. Let α be an automorphism of $\mathfrak{M}$ such that $\alpha(k) = h$ and $\alpha(q) = q$ for every $q \in S_{\mathsf{mat}}$. Let $\varphi(o, x_1, \ldots, x_n) \in \mathscr{L}$-Scenarios. Since α is an automorphism of $\mathfrak{M}$, for every $q_1, \ldots, q_n \in S_{\mathsf{mat}}$, $\varphi(k, q_1, \ldots, q_n)$ holds in $\mathfrak{M}$ iff $\varphi(h, q_1, \ldots, q_n) = \varphi(\alpha(k), \alpha(q_1), \ldots, \alpha(q_n))$ holds in $\mathfrak{M}$. Then, by Lemma 5.26, $\mathsf{Agree}_\varphi(k, h)$ holds in $\mathfrak{M}$.

Lemma 5.31. *Let $\mathscr{Q} = (Q, +, \cdot, 0, 1, \leqslant)$ be an arbitrary structure, $B \subseteq \mathscr{P}(Q^4)$, and $IOb \subseteq \mathrm{Sym}(Q^4)$. Assume that for every distinct $\vec{p}, \vec{q} \in Q^4$ there is $b \in B$ such that $\vec{p} \in b$, $\vec{q} \notin b$ or $\vec{p} \notin b$, $\vec{q} \in b$. Then in model $\mathfrak{M}(IOb, B, \mathscr{Q})$, we have:*

(i) $\mathsf{w}_{kh} = k \circ h^{-1}$ *for every $k, h \in IOb$.*

(ii) Assume that IOb forms a group under operation of composition. Then: (a) $\mathbb{W}_k = IOb$ for every $k \in IOb$. (b) If $k, h \in IOb$ agree on the set of worldlines of bodies, i.e., $\{k[b] : b \in B\} = \{h[b] : b \in B\}$, then $\mathsf{Agree}_\varphi(k, h)$ holds for every $\varphi \in \mathscr{L}_{\mathsf{core}}$-Scenarios.

Proof. (i) First we prove that every observer sees distinct events at distinct coordinate points. To prove this, choose $m \in IOb$ and let $\vec{p}, \vec{q}$ be distinct elements of Q^4. We show that $\mathsf{ev}_m(\vec{p}) \neq \mathsf{ev}_m(\vec{q})$. By definition of events and the worldview relation in $\mathfrak{M}(IOb, B, \mathcal{Q})$, we have

$$\begin{aligned}\mathsf{ev}_m(\vec{p}) = \{b \in B \,:\, \mathsf{W}(m, b, \vec{p})\} = \{b \in B \,:\, \vec{p} \in m[b]\} \\ = \left\{b \in B \,:\, m^{-1}(\vec{p}) \in b\right\}.\end{aligned}$$

Analogously, $\mathsf{ev}_m(\vec{q}) = \left\{b \in B \,:\, m^{-1}(\vec{q}) \in b\right\}$. Since $\vec{p} \neq \vec{q}$ and m^{-1} is a bijection, we have $m^{-1}(\vec{p}) \neq m^{-1}(\vec{q})$. Then, by assumption, there is $b \in B$ such that $m^{-1}(\vec{p}) \in b$, $m^{-1}(\vec{q}) \notin b$ or $m^{-1}(\vec{p}) \notin b$, $m^{-1}(\vec{q}) \in b$. Then, for such b we have that $b \in \mathsf{ev}_m(\vec{p})$, $b \notin \mathsf{ev}_m(\vec{q})$ or $b \notin \mathsf{ev}_m(\vec{p})$, $b \in \mathsf{ev}_m(\vec{q})$. Hence $\mathsf{ev}_m(\vec{p}) \neq \mathsf{ev}_m(\vec{q})$, i.e., every observer sees distinct events at distinct coordinate points, as claimed.

Next we prove that $\mathsf{w}_{kh} = k \circ h^{-1}$ for every $k, h \in IOb$. Let us notice that

$$k \circ h^{-1} = \left\{\big(h(\vec{p}), k(\vec{p})\big) \,:\, p \in Q^4\right\} \tag{5}$$

and that $k \circ h^{-1}$ is a bijection from Q^4 to Q^4 because k and h are bijections.

We claim that $k \circ h^{-1} \subseteq \mathsf{w}_{kh}$. By (5) and the definition of w_{kh}, it is enough to prove that for every $\vec{p} \in Q^4$, $\mathsf{ev}_k(k(\vec{p})) = \mathsf{ev}_h(h(\vec{p}))$. Let $\vec{p} \in Q^4$. Then by the definitions of events and W in $\mathfrak{M}(IOb, B, \mathcal{Q})$, and the fact that k, h are bijections, we have:

$$\begin{aligned}\mathsf{ev}_k\big(k(\vec{p})\big) = \{b \in B \,:\, \mathsf{W}(k, b, k(\vec{p}))\} = \{b \in B \,:\, k(\vec{p}) \in k[b]\} = \\ \{b \in B \,:\, \vec{p} \in b\} = \left\{b \in B \,:\, \mathsf{W}\big(h, b, h(\vec{p})\big)\right\} = \mathsf{ev}_h\big(h(\vec{p})\big).\end{aligned}$$

Therefore, $k \circ h^{-1} \subseteq \mathsf{w}_{kh}$ as claimed.

Recall that w_{kh} is a binary relation on Q^4 and that $k \circ h^{-1}$ is a bijection from Q^4 to Q^4. Assume on the contrary that $\mathsf{w}_{kh} \neq k \circ h^{-1}$, and hence w_{kh} is not bijection from Q^4 to Q^4. Then there are $\vec{r} \in Q^4$ and distinct $\vec{p}, \vec{q} \in Q^4$ such that $\mathsf{w}_{kh}(\vec{p}, \vec{r})$, $\mathsf{w}_{kh}(\vec{q}, \vec{r})$ or $\mathsf{w}_{kh}(\vec{r}, \vec{p})$, $\mathsf{w}_{kh}(\vec{r}, \vec{q})$. Therefore, by definition of worldview transformation, there are distinct $\vec{p}, \vec{q} \in Q^4$ such that $\mathsf{ev}_h(\vec{p}) = \mathsf{ev}_h(\vec{q})$ or $\mathsf{ev}_k(\vec{p}) = \mathsf{ev}_k(\vec{q})$, which cannot hold because every observer sees distinct events at distinct coordinate points. Therefore, w_{kh} is a bijection and $\mathsf{w}_{kh} = k \circ h^{-1}$.

(ii)(a) Let $k \in IOb$. By definition of $\mathbb{W}_k$, by Lemma 5.31(i) and properties of groups, we have

$$\begin{aligned}\mathbb{W}_k = \{\mathsf{w}_{kh} : h \in IOb\} = \left\{k \circ h^{-1} : h \in IOb\right\} = \\ k \circ \left\{h^{-1} : h \in IOb\right\} = k \circ IOb = IOb.\end{aligned}$$

(ii)(b) Assume $k, h \in IOb$ are such that

$$\{k[b] \,:\, b \in B\} = \{h[b] \,:\, b \in B\}. \tag{6}$$

By Lemma 5.30, it is enough to find an automorphism α of our model that takes k to h and fixes all the elements of Q. To this end we define

$$\alpha(x) \stackrel{\text{def}}{=} \begin{cases} x \circ k^{-1} \circ h & \text{if } x \in IOb, \\ (h^{-1} \circ k)[x] & \text{if } x \in B, \text{ and} \\ x & \text{if } x \in Q. \end{cases}$$

If $x \in IOb$, then $\alpha(x) = x \circ k^{-1} \circ h \in IOb$, because IOb forms a group under composition. If $x \in B$, then choose $x' \in B$ such that $k[x] = h[x']$ (this exists by assumption (6)). Then $\alpha(x) = (h^{-1} \circ k)[x] = x' \in B$ as well. This α also takes k to h because $\alpha(k) = k \circ k^{-1} \circ h = h$, and by definition, α fixes the elements of Q.

To prove that α is an automorphism, it remains to show that for every $m \in IOb$, $b \in B$ and $\vec{p} \in Q^4$, $\mathsf{W}\big(\alpha(m), \alpha(b), \vec{p}\big) \Leftrightarrow \mathsf{W}(m, b, \vec{p})$. To prove this, let $m \in IOb$, $b \in B$ and $\vec{p} \in Q^4$. Now, by definition of α, we have $\alpha(m)[\alpha(b)] = (m \circ k^{-1} \circ h)\big[(h^{-1} \circ k)[b]\big] = m[b]$. By this and the definition of W, we have $\mathsf{W}(\alpha(m), \alpha(b), \vec{p}) \Leftrightarrow \vec{p} \in \alpha(m)[\alpha(b)] \Leftrightarrow \vec{p} \in m[b] \Leftrightarrow \mathsf{W}(m, b, \vec{p})$.

Thus, α is an automorphism, and the claim follows.

Proof of Theorem 5.15. Assume $\mathcal{Q} = (Q, +, \cdot, 0, 1, \leqslant)$ is an ordered field. For every (straight) line[14] ℓ, the set of lines parallel[15] to line ℓ is denoted by PL_ℓ. We note that parallelism is an equivalence relation on the set of lines because $\mathcal{Q}$ is assumed to be a field. Hence $PL_\ell = PL_{\ell'}$ iff lines ℓ and ℓ' are parallel.

Let l be the line containing $\vec{\mathsf{o}}$ and $(1, 1, 0, 0)$, i.e., $l := \{\lambda(1, 1, 0, 0) : \lambda \in Q\}$. Let $B := PL_l \cup \big\{\{\vec{q}\} : \vec{q} \in Q^4\big\}$. Let $\mathfrak{M}_G := \mathfrak{M}(\mathsf{Gal}^{\uparrow}_{+}, B, \mathcal{Q})$ and $\mathfrak{M}_P := \mathfrak{M}(\mathsf{Poi}^{\uparrow}_{+}, B, \mathcal{Q})$, see Figure 13. Since both $\mathsf{Gal}^{\uparrow}_{+}$ and $\mathsf{Poi}^{\uparrow}_{+}$ form groups under composition, by Lemma 5.31(ii)(a) we have $\mathbb{W}_k = \mathsf{Gal}^{\uparrow}_{+}$ in $\mathfrak{M}_G$ and $\mathbb{W}_k = \mathsf{Poi}^{\uparrow}_{+}$ in $\mathfrak{M}_P$ for every inertial observer k. Thus $\mathfrak{M}_G$ is a Gal-based model and $\mathfrak{M}_P$ is a Poi-based model.

Next we prove that $\mathfrak{M}_G, \mathfrak{M}_P \models \mathsf{HOM}^+$. We will prove this for both models simultaneously. We have to prove, for all inertial observers k and h, that if $\mathsf{w}_{kh} \in \mathsf{Tran}$, then for every $\varphi \in \mathscr{L}_{\mathsf{core}}$-$\mathsf{Scenarios}$, $\mathsf{Agree}_\varphi(k, h)$ holds. To prove this, let k and h be inertial observers such that $\mathsf{w}_{kh} \in \mathsf{Tran}$. By Lemma 5.31(ii)(b), to prove that for every $\varphi \in \mathscr{L}_{\mathsf{core}}$-$\mathsf{Scenarios}$, $\mathsf{Agree}_\varphi(k, h)$ holds it is enough to show that

$$\{k[b] : b \in B\} = \{h[b] : b \in B\}. \tag{7}$$

By Lemma 5.31(i), $\mathsf{w}_{kh} = k \circ h^{-1}$. Transformation w_{kh} takes $h[l]$ to $k[l]$ because $\mathsf{w}_{kh}[h[l]] = k \circ h^{-1} \circ h[l] = k[l]$. Since w_{kh} is a translation, $h[l]$ and $k[l]$ are parallel. Therefore, $PL_{h[l]} = PL_{k[l]}$. Let us note that if A is an affine transformation, then $\{A[\ell] : \ell \in PL_l\} = PL_{A[l]}$ and $\big\{A[\{\vec{q}\}] : \vec{q} \in Q^4\big\} = \big\{\{\vec{q}\} : \vec{q} \in Q^4\big\}$, the latter holding because A is a bijection. Since k and h are affine transformations, unfolding the definition of B and using $PL_{h[l]} = PL_{k[l]}$

[14] We call $\ell \subseteq Q^4$ a *line* iff there are distinct $\vec{p}, \vec{q} \in Q^4$ such that $\ell = \{\vec{p} + \lambda(\vec{q} - \vec{p}) : \lambda \in Q\}$.

[15] Lines ℓ and ℓ' are *parallel* iff there are distinct $\vec{p}, \vec{q} \in \ell$ and distinct $\vec{p}\,', \vec{q}\,' \in \ell'$ such that $\vec{q} - \vec{p} = \vec{q}\,' - \vec{p}\,'$.

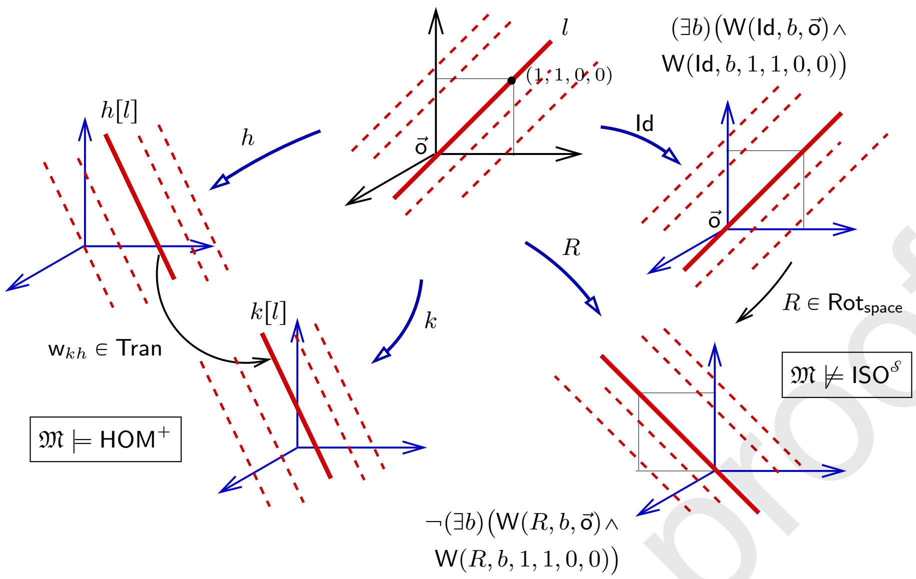


Figure 13: Illustration for the proof of Theorem 5.15.

gives

$$\{k[b] \,:\, b \in B\} = \{k[\ell] \,:\, \ell \in PL_l\} \cup \big\{k[\{\vec{q}\}] \,:\, \vec{q} \in Q^4\big\} = PL_{k[l]} \cup \big\{\{\vec{q}\} \,:\, \vec{q} \in Q^4\big\} = PL_{h[l]} \cup \big\{\{\vec{q}\} \,:\, \vec{q} \in Q^4\big\} = \{h[\ell] \,:\, \ell \in PL_l\} \cup \big\{h[\{\vec{p}\}] \,:\, \vec{p} \in Q^4\big\} = \{h[b] \,:\, b \in B\}.$$

Thus (7) holds, and hence by Lemma 5.31(ii)(b), for every $\varphi \in \mathscr{L}_{\mathsf{core}}$-$\mathsf{Scenarios}$, $\mathsf{Agree}_{\varphi}(k, h)$ holds. Therefore, $\mathfrak{M}_G, \mathfrak{M}_P \models \mathsf{HOM}^{+}$.

Finally we prove that $\mathfrak{M}_G, \mathfrak{M}_P \not\models \mathsf{ISO}^{\mathscr{S}}$ if

$$m(o, \vec{x}) \stackrel{\text{def}}{=} (\exists b)\big(\mathsf{W}(o, b, \vec{\mathsf{o}}) \wedge \mathsf{W}(o, b, \vec{x})\big) \in \mathscr{S}.$$

We will again prove this for both models simultaneously. Assume $m \in \mathscr{S}$ and define $R : Q^4 \to Q^4$ by $R(t, x, y, z) := (t, -x, -y, z)$. Then $R \in \mathsf{Rot_{space}} \cap \mathsf{Gal}_{+}^{\uparrow} \cap \mathsf{Poi}_{+}^{\uparrow}$, so R is an inertial observer in both models. By definition of W, $m\big(\mathsf{Id}, (1, 1, 0, 0)\big)$ is equivalent to $(\exists b)\big(\vec{\mathsf{o}} \in b \wedge (1, 1, 0, 0) \in b\big)$, which holds because $\vec{\mathsf{o}}, (1, 1, 0, 0) \in l$. On the other hand, $m\big(R, (1, 1, 0, 0)\big)$ is equivalent to $(\exists b)\big(\vec{\mathsf{o}} \in R[b] \wedge (1, 1, 0, 0) \in R[b]\big)$, which by definition of R is equivalent to $(\exists b)\big(\vec{\mathsf{o}} \in b \wedge (1, -1, 0, 0) \in b\big)$, which cannot hold in either model because there is no line parallel with l containing $\vec{\mathsf{o}}$ and $(1, -1, 0, 0)$. Thus $\mathsf{Agree}_{m}(R, \mathsf{Id})$ does not hold in either model. But by Lemma 5.31(i), $\mathsf{w}_{R\,\mathsf{Id}} = R \circ \mathsf{Id}^{-1} = R \in \mathsf{Rot_{space}}$. Therefore, $\mathfrak{M}_G, \mathfrak{M}_P \not\models \mathsf{ISO}^{\mathscr{S}}$.

Proof of Theorem 5.17. Assume $\mathcal{Q} = (Q, +, \cdot, 0, 1, \leqslant)$ is an ordered field. For every time instant $t \in Q$, the simultaneity at t is defined as:

$$\mathsf{S}_t := \left\{ \vec{p} \in Q^4 \ : \ \vec{p}_{\mathsf{t}} = t \right\}.$$

Clearly $\mathsf{S}_0 = \mathsf{S}$. Let $\vec{\mathsf{e}} := (1, 0, 0, 0)$. We will prove that for every $G \in \mathsf{Gal}$,

$$G[\mathsf{S}_0] = \mathsf{S}_{G(\vec{\mathsf{o}})_{\mathsf{t}}} \text{ and } G[\mathsf{S}_1] = \mathsf{S}_{G(\vec{\mathsf{e}})_{\mathsf{t}}}. \tag{8}$$

To prove this, first we will prove $G[\mathsf{S}_0] \subseteq \mathsf{S}_{G(\vec{\mathsf{o}})_{\mathsf{t}}}$ and $G[\mathsf{S}_1] \subseteq \mathsf{S}_{G(\vec{\mathsf{e}})_{\mathsf{t}}}$. Let $\vec{p} \in \mathsf{S}_0$ and $\vec{q} \in \mathsf{S}_1$. Then $\vec{p}_{\mathsf{t}} = \vec{\mathsf{o}}_{\mathsf{t}} = 0$ and $\vec{q}_{\mathsf{t}} = \vec{\mathsf{e}}_{\mathsf{t}} = 1$. Hence, $G(\vec{p})_{\mathsf{t}} = G(\vec{\mathsf{o}})_{\mathsf{t}}$ and $G(\vec{q})_{\mathsf{t}} = G(\vec{\mathsf{e}})_{\mathsf{t}}$. So $G(\vec{p}) \in \mathsf{S}_{G(\vec{\mathsf{o}})_{\mathsf{t}}}$ and $G(\vec{q}) \in \mathsf{S}_{G(\vec{\mathsf{e}})_{\mathsf{t}}}$, whence $G[\mathsf{S}_0] \subseteq \mathsf{S}_{G(\vec{\mathsf{o}})_{\mathsf{t}}}$ and $G[\mathsf{S}_1] \subseteq \mathsf{S}_{G(\vec{\mathsf{e}})_{\mathsf{t}}}$, as claimed. Now, since G is an affine transformation, it maps hyperplanes onto hyperplanes, so because S_t is a hyperplane for every $t \in Q$, we conclude that (8) holds.

For any two distinct coordinate points $\vec{p}, \vec{q} \in Q^4$, let $\vec{p}\vec{q}$ denote the half-line with initial point $\vec{p}$ containing $\vec{q}$, i.e., $\vec{p}\vec{q} := \{\vec{p} + \lambda(\vec{q} - \vec{p}) \ : \ 0 \leqslant \lambda \in Q\}$.

Now let $IOb := \mathsf{Gal}_+^{\uparrow}$ and $B := \{\vec{p}\vec{q} \ : \ \vec{p} \in \mathsf{S}_0, \ \vec{q} \in \mathsf{S}_1\} \cup \left\{\{\vec{q}\} \ : \ \vec{q} \in Q^4\right\}$, and consider the model $\mathfrak{M} := \mathfrak{M}(IOb, B, \mathcal{Q})$, see Figure 14. By Lemma 5.31(ii)(a), $\mathfrak{M}$ is Gal-based because $IOb = \mathsf{Gal}_+^{\uparrow}$ forms a group under composition.

Notice that if A is an affine transformation, then $A[\vec{p}\vec{q}] = A(\vec{p})A(\vec{q})$. Since the elements of $\mathsf{Gal}_+^{\uparrow}$ are affine transformations, we have by (8) that for every $m \in IOb = \mathsf{Gal}_+^{\uparrow}$, $\{m[\vec{p}\vec{q}] \ : \ \vec{p} \in \mathsf{S}_0, \ \vec{q} \in \mathsf{S}_1\} = \{m(\vec{p})m(\vec{q}) \ : \ \vec{p} \in \mathsf{S}_0, \ \vec{q} \in \mathsf{S}_1\} = \left\{\vec{p}\vec{q} \ : \ \vec{p} \in \mathsf{S}_{m(\vec{\mathsf{o}})_{\mathsf{t}}}, \ \vec{q} \in \mathsf{S}_{m(\vec{\mathsf{e}})_{\mathsf{t}}}\right\}$ and $\left\{m[\{\vec{q}\}] : \vec{q} \in Q^4\right\} = \left\{\{\vec{q}\} \ : \ \vec{q} \in Q^4\right\}$. Therefore, for every $m \in IOb$,

$$\{m[b] \ : \ b \in B\} = \left\{\vec{p}\vec{q} \ : \ \vec{p} \in \mathsf{S}_{m(\vec{\mathsf{o}})_{\mathsf{t}}}, \ \vec{q} \in \mathsf{S}_{m(\vec{\mathsf{e}})_{\mathsf{t}}}\right\} \cup \left\{\{\vec{q}\} \ : \ \vec{q} \in Q^4\right\}. \tag{9}$$

We now prove that $\mathfrak{M} \models \mathsf{HOM}^+_{\mathsf{space}} + \mathsf{ISO}^+$ by showing that for every $k, h \in IOb$ if $\mathsf{w}_{kh} \in \mathsf{Tran}_{\mathsf{space}} \cup \mathsf{Rot}_{\mathsf{space}}$, then for every $\varphi \in \mathscr{L}_{\mathsf{core}}$-$\mathsf{Scenarios}$, $\mathsf{Agree}_{\varphi}(k, h)$ holds. To prove this, let $k, h \in IOb$ be such that $\mathsf{w}_{kh} \in \mathsf{Tran}_{\mathsf{time}} \cup \mathsf{Rot}_{\mathsf{space}}$. Worldview transformation w_{kh} maps $h(\vec{\mathsf{o}})$ to $k(\vec{\mathsf{o}})$ by Lemma 5.31(i): $\mathsf{w}_{kh}(h(\vec{\mathsf{o}})) = k \circ h^{-1}(h(\vec{\mathsf{o}})) = k(\vec{\mathsf{o}})$. Analogously, w_{kh} maps $h(\vec{\mathsf{e}})$ to $k(\vec{\mathsf{e}})$. Then $h(\vec{\mathsf{o}})_{\mathsf{t}} = k(\vec{\mathsf{o}})_{\mathsf{t}}$ and $h(\vec{\mathsf{e}})_{\mathsf{t}} = k(\vec{\mathsf{e}})_{\mathsf{t}}$ because spatial rotations and spatial translations preserve the time components of coordinate points. By $h(\vec{\mathsf{o}})_{\mathsf{t}} = k(\vec{\mathsf{o}})_{\mathsf{t}}$, $h(\vec{\mathsf{e}})_{\mathsf{t}} = k(\vec{\mathsf{e}})_{\mathsf{t}}$ and (9), $\{k[b] \ : \ b \in B\} = \{h[b] \ : \ b \in B\}$. But then, by Lemma 5.31(ii)(b), $\mathsf{Agree}_{\varphi}(k, h)$ holds for every $\varphi \in \mathscr{L}_{\mathsf{core}}$-$\mathsf{Scenarios}$. Therefore, $\mathfrak{M} \models \mathsf{HOM}^+_{\mathsf{space}} + \mathsf{ISO}^+$.

Finally, we prove that if $m(o, \vec{x}) \stackrel{\text{def}}{=} (\exists b)\big(\mathsf{W}(o, b, \vec{\mathsf{o}}) \wedge \mathsf{W}(o, b, \vec{x})\big) \in \mathscr{S}$, then $\mathfrak{M} \not\models \mathsf{HOM}^{\mathscr{S}}_{\mathsf{time}}$. To this end, define $T : Q^4 \to Q^4$ by $T(t, x, y, z) := (t+1, x, y, z)$, and note that $T \in \mathsf{Tran}_{\mathsf{time}} \subseteq \mathsf{Gal}_+^{\uparrow}$, i.e., $T \in IOb$. Given the definition of W, $m(\mathsf{Id}, (1, 1, 0, 0))$ is equivalent to $(\exists b)\big(\vec{\mathsf{o}} \in b \wedge (1, 1, 0, 0) \in b\big)$ which holds in $\mathfrak{M}$ because $\vec{\mathsf{o}}(1, 1, 0, 0) \in B$. Likewise, $m(T, (1, 1, 0, 0))$ is equivalent to $\exists b\big(\vec{\mathsf{o}} \in T[b] \wedge (1, 1, 0, 0) \in T[b]\big)$, which is equivalent to $(\exists b)\big((-1, 0, 0, 0) \in b \wedge (0, 1, 0, 0) \in b\big)$ which does not hold in $\mathfrak{M}$. Thus $m(\mathsf{Id}, (1, 1, 0, 0))$ holds in $\mathfrak{M}$ but $m(T, (1, 1, 0, 0))$ does not, so $\mathsf{Agree}_{m}(T, \mathsf{Id})$ does not hold in $\mathfrak{M}$. But by Lemma 5.31(i), $\mathsf{w}_{T\,\mathsf{Id}} = T \in \mathsf{Tran}_{\mathsf{time}}$. Therefore, $\mathfrak{M} \not\models \mathsf{HOM}^{\mathscr{S}}_{\mathsf{time}}$.

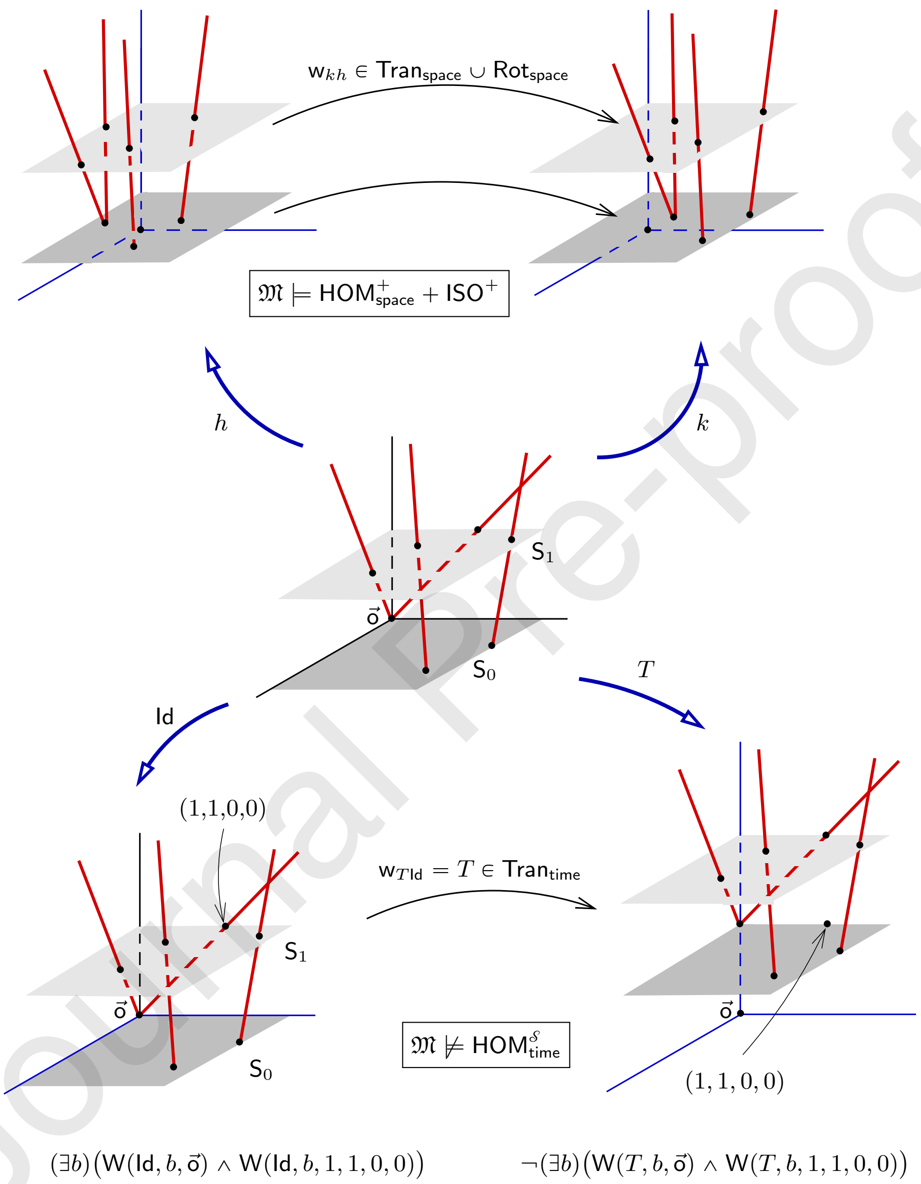


Figure 14: Illustration for the proof of Theorem 5.17.

Proof of Proposition 5.24. Let $\mathbb{Q} = (Q, +, \cdot, 0, 1, \leqslant)$ be an arbitrary ordered field, and define $T : Q^4 \to Q^4$ by $T(t, x, y, z) = (t+1, x+1, y, z)$. Clearly $T \in \mathsf{Tran}$ but $T \notin \mathsf{Tran}_{\mathsf{time}} \cup \mathsf{Tran}_{\mathsf{space}}$. Let $B := \{\{\vec{q}\} : \vec{q} \in Q^4 \setminus \{\vec{\mathsf{o}}\}\}$ and $IOb := \{\mathsf{Id}, T\}$, and consider the model $\mathfrak{M} := \mathfrak{M}(B, IOb, \mathbb{Q})$. Then $\mathfrak{M} \models \mathsf{AxOField}$ by construction. By Lemma 5.31(i), the only worldview transformations in $\mathfrak{M}$ are T, T^{-1} and Id. So there are no worldview transformations between distinct observers in $\mathsf{Tran}_{\mathsf{time}} \cup \mathsf{Tran}_{\mathsf{space}}$, and $\mathfrak{M} \models \mathsf{HOM}^+_{\mathsf{time}} \cup \mathsf{HOM}^+_{\mathsf{space}}$ holds vacuously. Assume $\psi(o) \stackrel{\text{def}}{=} (\exists b)\mathsf{W}(o, b, \vec{\mathsf{o}}) \in \mathscr{S}$. We will show that $\mathfrak{M} \not\models \mathsf{HOM}^{\mathscr{S}}$. By definition of W, $\mathsf{W}(\mathsf{Id}, b, \vec{\mathsf{o}}) \Leftrightarrow \vec{\mathsf{o}} \in \mathsf{Id}[b] \Leftrightarrow \vec{\mathsf{o}} \in b$ and $\mathsf{W}(T, b, \vec{\mathsf{o}}) \Leftrightarrow \vec{\mathsf{o}} \in T[b] \Leftrightarrow T^{-1}(\vec{\mathsf{o}}) \in b \Leftrightarrow (-1, -1, 0, 0) \in b$. Therefore, $\psi(\mathsf{Id})$ does not hold in $\mathfrak{M}$ but $\psi(T)$ does, whence $\mathsf{Agree}_\psi(T, \mathsf{Id})$ does not hold in $\mathfrak{M}$. By Lemma 5.31(i), $\mathsf{w}_{T\,\mathsf{Id}} = T \in \mathsf{Tran}$. Therefore, $\mathfrak{M} \not\models \mathsf{HOM}^{\mathscr{S}}$.

Proof of Proposition 5.12. Assume $\mathbb{Q} = (Q, +, \cdot, 0, 1, \leqslant)$ is an ordered field. For any distinct $\vec{p}, \vec{q} \in Q^4$, $\ell_{\vec{p}\vec{q}}$ denotes the straight line containing $\vec{p}$ and $\vec{q}$. Let

$$B_G := \{\ell_{\vec{p}\vec{q}} : \vec{p}, \vec{q} \in Q^4,\ \vec{p}_{\mathsf{t}} \neq \vec{q}_{\mathsf{t}}\} \text{ and } B_P := \{\ell_{\vec{p}\vec{q}} : \vec{p}, \vec{q} \in Q^4,\ |\vec{p}_{\mathsf{s}} - \vec{q}_{\mathsf{s}}| < |\vec{p}_{\mathsf{t}} - \vec{q}_{\mathsf{t}}|\}$$

and define $\mathfrak{M}_G := \mathfrak{M}(\mathsf{Gal}^{\uparrow}_{+}, B_G, \mathbb{Q})$ and $\mathfrak{M}_P := \mathfrak{M}(\mathsf{Poi}^{\uparrow}_{+}, B_P, \mathbb{Q})$. Since both $\mathsf{Gal}^{\uparrow}_{+}$ and $\mathsf{Poi}^{\uparrow}_{+}$ are groups under composition, Lemma 5.31(ii)(a) tells us that $\mathbb{W}_k = \mathsf{Gal}^{\uparrow}_{+}$ in $\mathfrak{M}_G$ and $\mathbb{W}_k = \mathsf{Poi}^{\uparrow}_{+}$ in $\mathfrak{M}_P$ for every inertial observer k. Thus $\mathfrak{M}_G$ is a Gal-based model and $\mathfrak{M}_P$ is a Poi-based model. It is not difficult to see that $m \in \mathsf{Poi}^{\uparrow}_{+} \Rightarrow \{m[b] : b \in B_P\} = B_P$ and $m \in \mathsf{Gal}^{\uparrow}_{+} \Rightarrow \{m[b] : b \in B_G\} = B_G$, whence, by Lemma 5.31(ii)(b), $\mathsf{Agree}_\varphi(k, h)$ holds for all observers k, h and all $\varphi \in \mathscr{L}_{\mathsf{core}}$-$\mathsf{Scenarios}$ in both models. This proves that $\mathfrak{M}_P, \mathfrak{M}_G \models \mathsf{HOM}^+ \cup \mathsf{ISO}^+$.

6. Discussion

A number of different axiomatizations of relativity theory have appeared in the literature (e.g., [2, 4, 8, 10, 15, 28, 29, 34, 31, 35]), and Andréka and Németi have recently pioneered work investigating ways to connect axiom systems using interpretations (translation functions between logics) [7]. Various recent studies discuss axiomatizing the principle of relativity itself [16, 17], and this can be done in many different, non-equivalent, ways [25]. There are also many ways to formulate isotropy and homogeneity. One key factor is what we understand by "inertial coordinate systems", and whether this concept should be defined a priori or derived from the axioms. In our work, we introduce them as a basic concept and then define axioms which capture the idea that they coordinatize some kind of non-accelerated laboratory. For us, it is natural that spatial rotation should not influence the types of experiments that can be carried out, since looking at an experiment from another fixed vantage point involves no acceleration. Nonetheless, other approaches also exist, and these may lead to apparently contradictory findings when investigating such basic questions as whether the principle of relativity implies isotropy [11, 30]. For example, Budden [9] and Mamone Capria [27] have considered systems which are inherently *anisotropic*

– the question obviously arises whether their work therefore contradicts our own findings. Close examination of their axiom systems shows, however, that this is not the case – for example, Mamone Capria bans certain worldview transformations a priori by declaring various rotations to be unrealisable, which limits the set of viewpoints from which experiments can be observed; this in turn leads to a different interpretation of what it means for two frames to be inertially related to one another. These apparent contradictions nonetheless serve to highlight how important it is to approach matters formally, for without the ability to study the axioms underpinning different authors' findings, one is left with no opportunity to decide whether different approaches are compatible or contradictory, nor any way to decide which approach is best suited to the problem at hand.

By axiom $\mathsf{Ax3Dir\exists Motion}$, we have only assumed the possibility that observers can be seen moving in 3 directions according to *one* observer. It can be shown that axioms $\mathsf{Ax3Dir\exists Motion}$, $\mathsf{AxAffTr}$ and $\mathsf{AxOField}$ imply that observers can move in at least 3 independent directions according to *every* observer. Related to this, one may wonder whether it is possible for observers to move in every direction with every speed slower than the speed of light (i.e., slower than 1) in every Poi-based model no matter which ordered field is the structure of the quantities. In fact, this is not true if the structure of quantities is the field $\mathbb{Q}$ of rational numbers, for example, because no observer can move with speed $\frac{1}{2}$ in direction $(1,1,0)$. This is so because the time unit vector of such an observer should be mapped to $(2\sqrt{2},1,1,0) \notin \mathbb{Q}^4$. There are observers moving in direction $(1,1,0)$ over the field $\mathbb{Q}$ moving with some speed, but not with speed $\frac{1}{2}$. Related to this subject, the following natural question is still open:

Question 6.1. Can observers move in every direction with *some* speed in every Poi-based model over every ordered field?

It is worth mentioning some results from the literature related to this open question. Let us start this with the following observation: if observers can move in every direction with every speed slower than light, then every positive quantity has to have a square root. This is so because the time unit vector of an observer moving with speed $v < 1$ according to an another observer seeing the same event at the origin should be mapped by the worldview transformation between these two observers to a vector having time component $\frac{1}{\sqrt{1-v^2}}$. Therefore, $1 - v^2$ has to have a square root for all positive $v \in Q$ for which $v < 1$. From this, since every positive $x \in Q$ can be expressed as

$$x = \left(\frac{x+1}{2}\right)^2 \left(1 - \left(\frac{x-1}{x+1}\right)^2\right),$$

it follows that every positive x has to have a square root in Q because $-1 < (x-1)/(x+1) < 1$ for all $x > 0$. By [6, Thm.3.6.17(i)], even more is true if the dimension of spacetime is 3. In the *scale free* Poi*-based models*, where the transformations between inertial observers are compositions of dilations and

Poincaré transformations,[16] if the spacetime dimension is 3 and if observers can move in every direction with every speed slower than light, then every positive quantity in Q has to have a square root. In general, this implication is not true. If the dimension of spacetime is an even number, then there are scale free Poi-based models over certain fields in which some square roots are missing yet observers can move in every direction with every speed slower than light [5, Thm.3.9]. The question "What happens in odd dimensions higher than 3?" is open, as is the question "Over which fields is it possible to construct scale free Poi-based models in which observers can move in every direction with every speed slower than light?", see [5, Question 3.10]. By the results in [26], it follows that in Poi-based models over the field of rational numbers observers can move *approximately* in every direction with *approximately* any speed slower than light.

In Poi-based models over the field $\mathbb{Q}$, the fact that there are directions in which observers cannot move with certain speeds while in other directions they can move with those speeds is a kind of anisotropy. However, this anisotropy is different from what is captured by axiom scheme ISO because over every ordered field there is a Poi-based model satisfying ISO by Proposition 5.12. This kind of quantity-induced anisotropy disappears if we assume that every positive quantity in Q has a square root, because in that case it can be shown that observers can move with the same speeds in every direction in every Poi-based model.

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