

Coalgebraic Reasoning with Global Assumptions in Arithmetic Modal Logics

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We establish a generic upper bound EXPTIME for reasoning with global assumptions (also known as TBoxes) in coalgebraic modal logics. Unlike earlier results of this kind, our bound does not require a tractable set of tableau rules for the instance logics, so that the result applies to wider classes of logics. Examples are Presburger modal logic, which extends graded modal logic with linear inequalities over numbers of successors, and probabilistic modal logic with polynomial inequalities over probabilities. We establish the theoretical upper bound using a type elimination algorithm. We also provide a global caching algorithm that potentially avoids building the entire exponential-sized space of candidate states, and thus offers a basis for practical reasoning. This algorithm still involves frequent fixpoint computations; we show how these can be handled efficiently in a concrete algorithm modelled on Liu and Smolka’s linear-time fixpoint algorithm. Finally, we show that the upper complexity bound is preserved under adding nominals to the logic, i.e. in coalgebraic hybrid logic.

1 INTRODUCTION

While modal logic is classically concerned with purely relational systems (e.g. [7]), there is, nowadays, widespread interest in flavours of modal logic interpreted over state-based structures in a wider sense, e.g. featuring probabilistic or, more generally, weighted branching. Under the term *arithmetic modal logics*, we subsume logics that feature arithmetical constraints on the number or combined weight of successors. The simplest logics of this type compare weights to constants, such as graded modal logic [18] or some variants of probabilistic modal logic [28, 32]. More involved examples are *Presburger modal logic* [13], which allows Presburger constraints on numbers of successors, and probabilistic modal logic with linear [16] or polynomial [17] inequalities over probabilities. Presburger modal logic allows for statements like ‘the majority of university students are female’, or ‘dance classes have even numbers of participants’, while probabilistic modal logic with polynomial inequalities can assert, for example, independence of events.

These logics are the main examples we address in a more general coalgebraic framework in this paper. Our main observation is that satisfiability for coalgebraic logics can be decided in a step-by-step fashion, peeling off one layer of operators at a time. We thus reduce the overall satisfiability problem to satisfiability in a *one-step logic* involving only immediate successor states, and hence no nesting of modalities [36, 46]. We define a *strict* variant of this *one-step satisfiability problem*, distinguished by a judicious redefinition of its input size; if strict one-step satisfiability is in EXPTIME , we obtain a (typically optimal) EXPTIME upper bound for satisfiability under global assumptions in the full logic. For our two main examples, the requisite complexity bounds (in fact, even PSPACE) on strict one-step satisfiability follow in essence directly from known complexity results in integer programming and the existential theory of the reals, respectively; in other words, even in fairly involved examples the complexity bound for the full logic is obtained with comparatively little effort once the generic result is in place.

Applied to Presburger constraints, our results complement previous work showing that the complexity of Presburger modal logic without global assumptions is PSPACE [12, 13], the same as for the modal logic K (or equivalently the description logic \mathcal{ALC}). For polynomial inequalities on

probabilities, our syntax generalizes propositional *polynomial weight* formulae [17] to a full modal logic allowing nesting of weights (and global assumptions).

In more detail, our first contribution is to show via a type elimination algorithm [40] that also in presence of global assumptions (and, hence, in presence of the universal modality [21]), the satisfiability problem for coalgebraic modal logics is no harder than for K , i.e. in EXPTIME , provided that strict one-step satisfiability is in EXPTIME . Additionally, we show that this result can be extended to cover nominals, i.e. to coalgebraic hybrid logic [36, 49]. In the Presburger example, we thus obtain that reasoning with global assumptions in Presburger hybrid logic, equivalently reasoning with general TBoxes in the extension of the description logic \mathcal{ALCO} with Presburger constraints (which subsumes \mathcal{ALCOQ}), remains in EXPTIME .

We subsequently refine the algorithm to use global caching in the spirit of Goré and Nguyen [24], i.e. bottom-up expansion of a tableau-like graph and propagation of satisfiability and unsatisfiability through the graph. We thus potentially avoid constructing the whole exponential-sized tableau, and provide maneuvering space for heuristic optimization. Global caching algorithms have been demonstrated to perform well in practice [25]. Moreover, we go on to present a concrete algorithm, in which the fixpoint computations featuring in the propagation step of the global caching algorithm are implemented efficiently in the style of Liu and Smolka [34].

Organization. We discuss some preliminaries on fixpoints in Section 2, and recall the generic framework of coalgebraic logic in Section 3. In Section 4, we discuss the concepts of one-step logic and one-step satisfiability that underlie our generic algorithms. We establish the generic EXPTIME upper bound for reasoning with global assumptions in coalgebraic modal logics via type elimination in Section 5. In Sections 6 and 7, we present the global caching algorithm and its concretization. We extend the EXPTIME complexity result to coalgebraic hybrid logics in Section 8.

Related Work. Our algorithms use a semantic method, and as such complement earlier results on global caching in coalgebraic description logics that rely on tractable sets of tableau rules [22], which are not currently available for our leading examples. (In fact, tableau-style axiomatizations of various logics of linear inequalities over the reals and over the integers have been given in earlier work [29]; however, over the integers the rules appear to be incomplete: if $\#p$ denotes the integer weight of successors satisfying p , then the formula $2\#T < 1 \vee 2\#T > 1$ is clearly valid, but cannot be derived.)

Demri and Lugiez' proof that Presburger modal logic *without* global assumptions is in PSPACE [12, 13] can be viewed as showing that strict one-step satisfiability in Presburger modal logic is in PSPACE (as we discuss below, more recent results in integer programming simplify this proof). Generally, our coalgebraic treatment of Presburger modal logic and related logics relies on an equivalence of the standard Kripke semantics of these logics and an alternative semantics in terms of non-negative-integer-weighted systems called *multigraphs* [11], the point being that the latter, unlike the former, is subsumed by the semantic framework of coalgebraic logic (we explain details in Section 3).

Work related to XML query languages has shown that reasoning in Presburger fixpoint logic is EXPTIME complete [51], and that a logic with Presburger constraints and nominals is in EXPTIME [6], when these logics are interpreted *over finite trees*, thus not subsuming our EXPTIME upper bound for Presburger modal logic with global assumptions. It may be possible to obtain the latter bound alternatively via looping tree automata like for graded modal logic [52]. The description logic \mathcal{ALCN} (featuring the basic \mathcal{ALC} operators and number restrictions $\geq n$. \top) has been extended with explicit quantification over integer variables and number restrictions mentioning integer variables [5], in formulae such as $\downarrow n. ((=n R. \top) \wedge (=n S. \top))$ with n an integer variable, and \downarrow read as existential quantification, so the example formula says that there are as

many R -successors as S -successors. This logic remains decidable if quantification is restricted to be existential. It appears to be incomparable to Presburger modal logic in that it does not support general linear inequalities or qualified number restrictions, but on the other hand allows the same integer variable to be used at different modal depths.

Reasoning with polynomial inequalities over probabilities has been studied in propositional logics [17] and in many-dimensional modal logics [26], which work with a single distribution on worlds rather than with world-dependent probability distributions as in [16, 28, 32].

This paper is a revised and extended version of a previous conference publication [30]; besides including full proofs and additional examples, it contains new material on the concretized version of the global caching algorithm (Section 7) and on EXPTIME reasoning with global assumptions in coalgebraic hybrid logics (Section 8).

2 PRELIMINARIES

Our reasoning algorithms will centrally involve fixpoint computations on powersets of finite sets; we recall some notation. Let X be a finite set, and let $F: \mathcal{P}X \rightarrow \mathcal{P}X$ be a function that is monotone with respect to set inclusion. A set $Y \in \mathcal{P}X$ is a *prefixpoint* of F if $F(Y) \subseteq Y$; a *postfixpoint* of F if $Y \subseteq F(Y)$; and a *fixpoint* of F if $Y = F(Y)$. By the Knaster-Tarski fixpoint theorem, F has a least fixpoint μF and a greatest fixpoint νF . Moreover, μF is even the least prefixpoint of F , and νF the greatest postfixpoint. We alternatively use a μ -calculus-like notation, writing $\mu S. E(S)$ and $\nu S. E(S)$ for the least and greatest fixpoints, respectively, of the function on $\mathcal{P}X$ that maps $S \in \mathcal{P}X$ to $E(S)$, where E is an expression (in an informal sense) depending on S . Since X is finite, we can compute least and greatest fixpoints by *fixpoint iteration* according to Kleene's fixpoint theorem: Given a monotone F as above, the sets $F^n(\emptyset)$ (where F^n denotes n -fold application of F) form an ascending chain

$$\emptyset = F^0(\emptyset) \subseteq F(\emptyset) \subseteq F^2(\emptyset) \subseteq \dots,$$

which must stabilize at some $F^k(\emptyset)$ (i.e. $F^{k+1}(\emptyset) = F^k(\emptyset)$), and then $\mu F = F^k(\emptyset)$. Similarly, the sets $F^n(X)$ form a descending chain, which must stabilize at some $F^k(X)$, and then $\nu F = F^k(X)$.

3 COALGEBRAIC LOGIC

As indicated above, we cast our results in the generic framework of *coalgebraic logic* [10], which allows us to treat structurally different modal logics, such as Presburger and probabilistic modal logics, in a uniform way. We briefly recall the main concepts needed. Familiarity with basic concepts of category theory (e.g. [3]) will be helpful, but we will explain the requisite definitions as far as necessary for the present purposes. Overall, coalgebraic logic is concerned with the specification of state-based systems in a general sense by means of *modalities*, which are logical connectives that traverse the transition structure in specific ways. The basic example of such a modal logic is what for our present purposes we shall term *relational modal logic* (e.g. [7]). Here, states are connected by a successor relation, and modalities \Box, \Diamond talk about the successors of a state: a formula of the form $\Box\phi$ holds for a state if *all* its successors satisfy ϕ , and a formula of the form $\Diamond\phi$ holds for a state if it has *some* successor that satisfies ϕ . Our main interest, however, is in logics where the transition structure of states goes beyond a simple successor relation, with correspondingly adapted, and often more complex, modalities.

We parametrize modal logics in terms of their syntax and their coalgebraic semantics. In the **syntax**, we work with a modal similarity type Λ of modal operators with given finite arities. The set $\mathcal{F}(\Lambda)$ of Λ -formulae is then given by the grammar

$$\mathcal{F}(\Lambda) \ni \phi, \psi ::= \perp \mid \phi \wedge \psi \mid \neg\phi \mid \heartsuit(\phi_1, \dots, \phi_n) \quad (\heartsuit \in \Lambda \text{ } n\text{-ary}). \quad (1)$$

We omit explicit propositional atoms; these can be regarded as nullary modalities. The operators \top , \rightarrow , \vee , \leftrightarrow are assumed to be defined in the standard way. Standard examples of modal operators include the mentioned (unary) box and diamond operators \square , \diamond of relational modal logic; as indicated above, in the present setting, our main interest is in more complex examples introduced in Sections 3.1 and 3.2. For the complexity analysis of reasoning problems, we assume a suitable encoding of the modal operators in Λ as strings over some alphabet. The *size* $|\phi|$ of a formula ϕ is then defined by counting 1 for each Boolean operation (\perp , \neg , \wedge), and for each modality $\heartsuit \in \Lambda$ the length of the encoding of \heartsuit . We assume that numbers occurring in the description of modal operators are coded in binary. To ease notation, we generally let $\epsilon\phi$, for $\epsilon \in \{-1, 1\}$, denote ϕ if $\epsilon = 1$ and $\neg\phi$ if $\epsilon = -1$.

The **semantics** of the logic is formulated in the paradigm of *universal coalgebra* [41], in which a wide range of state-based system types, e.g. relational, neighbourhood-based, probabilistic, weighted, or game-based systems, is subsumed under the notion of functor coalgebra. Here, a *functor* T on the category of sets assigns to each set X a set TX , thought of as a type of structured collections over X , and to each map $f: X \rightarrow Y$ a map $Tf: TX \rightarrow TY$, preserving identities and composition. A standard example is the (covariant) *powerset functor* \mathcal{P} , which maps a set X to its powerset $\mathcal{P}X$ and a map $f: X \rightarrow Y$ to the direct image map $\mathcal{P}f: \mathcal{P}X \rightarrow \mathcal{P}Y$, i.e. $(\mathcal{P}f)(A) = f[A]$ for $A \in \mathcal{P}X$. In this case, structured collections are thus just sets. A further example, more relevant to our present purposes, and to be taken up again in Section 3.2, is the (discrete) *distribution functor* \mathcal{D} . This functor assigns to a set X the set of discrete probability distributions on X , which thus play the role of structured collections, and to a map $f: X \rightarrow Y$ the map $\mathcal{D}f: \mathcal{D}X \rightarrow \mathcal{D}Y$ that takes image measures; i.e. $(\mathcal{D}f)(\mu)(B) = \mu(f^{-1}[B])$ for $B \subseteq Y$. We recall here that a probability distribution μ on X is *discrete* if $\mu(A) = \sum_{x \in A} \mu(\{x\})$ for every $A \subseteq X$, i.e. we can equivalently regard μ as being given by its *probability mass function* $x \mapsto \mu(\{x\})$. Note that the support $\{x \mid \mu(\{x\}) \neq 0\}$ of μ is then necessarily countable. A functor T defines a system type in the shape of its class of *T-coalgebras*, which are pairs $C = (X, \gamma)$ consisting of a set X of *states* and a *transition map*

$$\gamma: X \rightarrow TX, \quad (2)$$

thought of as assigning to each state x a structured collection $\gamma(x) \in TX$ of successors. For instance, \mathcal{P} -coalgebras are just transition systems or Kripke frames, as they assign to each state a set of successors (i.e. they capture precisely the semantic structures that underlie relational modal logic as recalled at the beginning of the section), and \mathcal{D} -coalgebras are Markov chains, as they assign to each state a distribution over successors.

We further parametrize the semantics over an interpretation of modalities as predicate liftings, as follows. Recall [39, 44] that an *n-ary predicate lifting* for T is a natural transformation

$$\lambda: Q^n \rightarrow Q \circ T^{op}$$

where Q denotes the *contravariant powerset functor*. We shall use predicate liftings in connection with the transition map (2) to let modalities look one step ahead in the transition structure of a coalgebra. The definition of predicate liftings unfolds as follows. Recall that every category C has a *dual* category C^{op} , which has the same objects as C and the same morphisms, but with the direction of morphisms reversed. In particular, Set^{op} , the dual category of the category Set of sets and maps, has sets as objects, and maps $Y \rightarrow X$ as morphisms $X \rightarrow Y$. Then the contravariant powerset functor $Q: \text{Set}^{op} \rightarrow \text{Set}$ assigns to a set X its powerset $QX = \mathcal{P}X$, and to a map $f: X \rightarrow Y$ the preimage map $Qf: QY \rightarrow QX$, given by $(Qf)(B) = f^{-1}[B]$ for $B \subseteq Y$. By Q^n , we denote the pointwise n -th Cartesian power of Q , i.e. $Q^n X = (QX)^n$. The functor $T^{op}: \text{Set}^{op} \rightarrow \text{Set}^{op}$ acts like T . Thus, λ is a family of maps $\lambda_X: (QX)^n \rightarrow Q(TX)$ indexed over all sets X , satisfying the *naturality* equation $\lambda_X \circ (Qf)^n = Q(T^{op}f) \circ \lambda_Y$ for $f: X \rightarrow Y$. That is, λ_X takes n subsets of X

as arguments, and returns a subset of TX . The naturality condition amounts to commutation of λ with preimage, i.e.

$$\lambda_X(f^{-1}[B_1], \dots, f^{-1}[B_n]) = Tf^{-1}[\lambda_Y(B_1, \dots, B_n)] \quad (3)$$

for $B_1, \dots, B_n \subseteq Y$. We assign an n -ary predicate lifting $\llbracket \heartsuit \rrbracket$ to each modality $\heartsuit \in \Lambda$, of arity n , thus determining the semantics of \heartsuit . For $t \in TX$ and $A_1, \dots, A_n \subseteq TX$, we write

$$t \models \heartsuit(A_1, \dots, A_n) \quad (4)$$

to abbreviate $t \in \llbracket \heartsuit \rrbracket_X(A_1, \dots, A_n)$.

Predicate liftings thus turn predicates on the set X of states into predicates on the set TX of structured collections of successors. A basic example is the predicate lifting for the usual diamond modality \diamond , given by $\llbracket \diamond \rrbracket_X(A) = \{B \in \mathcal{P}X \mid B \cap A \neq \emptyset\}$. We will see more examples in Sections 3.1 and 3.2. For purposes of the generic technical development, we fix the data Λ, T , and $\llbracket \heartsuit \rrbracket$ throughout, and by abuse of notation sometimes refer to them jointly as (the logic) Λ .

Satisfaction $x \models_C \phi$ (or just $x \models \phi$ when C is clear from the context) of formulae $\phi \in \mathcal{F}(\Lambda)$ in states x of a coalgebra $C = (X, \gamma)$ is defined inductively by

$$\begin{aligned} x \not\models_C \perp & \\ x \models_C \phi \wedge \psi & \quad \text{iff } x \models_C \phi \text{ and } x \models_C \psi \\ x \models_C \neg\phi & \quad \text{iff } x \not\models_C \phi \\ x \models_C \heartsuit(\phi_1, \dots, \phi_n) & \quad \text{iff } \gamma(x) \models \heartsuit(\llbracket \phi_1 \rrbracket_C, \dots, \llbracket \phi_n \rrbracket_C) \end{aligned}$$

where we write $\llbracket \phi \rrbracket_C = \{x \in X \mid x \models_C \phi\}$ (and use notation as per (4)). Continuing the above example, the predicate lifting $\llbracket \diamond \rrbracket$ thus induces exactly the usual semantics of \diamond : Given a \mathcal{P} -coalgebra, i.e. Kripke frame, $(X, \gamma: X \rightarrow \mathcal{P}X)$, we have $x \models_C \diamond\phi$ iff the set $\gamma(x)$ of successors of x intersects with $\llbracket \phi \rrbracket_C$, i.e. iff x has a successor that satisfies ϕ .

We will be interested in satisfiability under global assumptions, or, in description logic terminology, reasoning with general TBoxes [4], that is, under background axioms that are required to hold in every state of a model:

Definition 3.1 (Global assumptions). Given a formula ψ , the *global assumption*, a coalgebra $C = (X, \gamma)$ is a ψ -*model* if $\llbracket \psi \rrbracket_C = X$; and a formula ϕ is ψ -*satisfiable* if there exists a ψ -model C such that $\llbracket \phi \rrbracket_C \neq \emptyset$. The *satisfiability problem under global assumptions* is to decide, given ψ and ϕ , whether ϕ is ψ -satisfiable. We extend these notions to sets Γ of formulae: We write $x \models_C \Gamma$ if $x \models \phi$ for all $\phi \in \Gamma$, and we say that Γ is ψ -*satisfiable* if there exists a state x in a ψ -model C such that $x \models_C \Gamma$. For distinction, we will occasionally refer to satisfiability in the absence of global assumptions, i.e. \top -satisfiability, as *plain satisfiability*.

Remark 3.2. While the typical complexity of plain satisfiability is PSPACE, that of satisfiability under global assumptions is EXPTIME. In particular, this holds for the basic example of relational modal logic [19, 31].

As indicated above, global assumptions are referred to as *TBox axioms* in description logic parlance, in honour of the fact that they capture what is, in that context, called *terminological knowledge*: They record facts that hold about the world at large, such as ‘every car has a motor’ (formalized, e.g., in relational modal logic as $\psi := (\text{Car} \rightarrow \diamond \text{Motor})$ if the relation that underlies \diamond is understood as parthood). Contrastingly, a formula ϕ is satisfiable under the global assumption ψ as soon as ϕ holds in *some* state of some ψ -model, so ϕ is thought of as describing some states (*individuals* in description logic terminology) but not as being universally true. Correspondingly, the reasoning task of checking satisfiability (under global assumptions) is called *concept satisfiability*

(under general T Boxes) in description logic. For instance, the atomic proposition (‘concept’) Car is ψ -satisfiable in the above example, but not of course necessarily true in every state of a ψ -model.

Global consequence, i.e. entailment between global assumptions, reduces to satisfiability under global assumptions: We say that a formula ϕ is a *global consequence* of a formula ψ if every ψ -model is also a ϕ -model. Then ϕ is a global consequence of ψ iff $\neg\phi$ is not ψ -satisfiable. For instance, in relational modal logic, $\Box\psi$ is always a global consequence of ψ , i.e. $\neg\Box\psi$ is not ψ -satisfiable; this fact corresponds to the well-known necessitation rule of relational modal logic [7].

Remark 3.3. As indicated in the introduction, for purposes of the complexity analysis, global assumptions are equivalent to the universal modality. We make this claim more precise as follows. We define *coalgebraic modal logic with the universal modality* by extending the grammar (1) with an additional alternative

$$\dots \mid [\forall] \phi,$$

and the semantics with the clause

$$x \models_C [\forall] \phi \quad \text{iff} \quad y \models_C \phi \text{ for all } y \in X$$

for a coalgebra $C = (X, \gamma)$. In this logic, we restrict attention to plain satisfiability checking, asking whether, for a given formula ϕ , there exists a state x in a coalgebra C such that $x \models_C \phi$. Then satisfiability under global assumptions clearly reduces in logarithmic space to plain satisfiability in coalgebraic modal logic with the universal modality – a formula ϕ is satisfiable under the global assumption ψ iff $\phi \wedge [\forall] \psi$ is satisfiable.

Conversely, satisfiability of a formula ϕ in coalgebraic modal logic with the universal modality is reducible in nondeterministic polynomial time to satisfiability under global assumptions in coalgebraic modal logic, as follows. Call a subformula of ϕ a $[\forall]$ -*subformula* if it is of the shape $[\forall] \psi$, and let $[\forall] \psi_1, \dots, [\forall] \psi_n$ be the $[\forall]$ -subformulae of ϕ . Given a subset $U \subseteq \{1, \dots, n\}$ and a subformula χ of ϕ , denote by $\chi[U]$ the $[\forall]$ -free formula obtained from χ by replacing every $[\forall]$ -subformula $[\forall] \psi_k$ that is not in scope of a further $[\forall]$ -operator by \top if $k \in U$, and by \perp otherwise. We claim that

(*) ϕ is satisfiable (in coalgebraic modal logic with the universal modality) iff there is $U \subseteq \{1, \dots, n\}$ such that $\phi[U]$, as well as each formula $\neg\psi_k[U]$ for $k \in \{1, \dots, n\} \setminus U$, are (separately) satisfiable under the global assumption ψ_U given by $\psi_U = \bigwedge_{k \in U} \psi_k[U]$.

Using (*), we can clearly reduce satisfiability in coalgebraic modal logic with the universal modality to satisfiability under global assumptions in coalgebraic modal logic as claimed by just guessing U . It remains to prove (*). For the ‘only if’ direction, suppose that $x \models_C \phi$ for some state x in a T -coalgebra $C = (X, \gamma)$. Put $U = \{k \mid x \models_C [\forall] \psi_k\}$. It is readily checked that, in the above notation, C is a ψ_U -model, $x \models_C \phi[U]$, and for each $k \in \{1, \dots, n\} \setminus U$, $\neg\psi_k[U]$ is satisfied in some state of C . For the converse implication, let $U \subseteq \{1, \dots, n\}$, let C and C_k , for $k \in \{1, \dots, n\} \setminus U$, be ψ_U -models, let $x \models_C \phi[U]$, and let $x_k \models_{C_k} \neg\psi_k[U]$ for $k \in \{1, \dots, n\} \setminus U$. Let D be the disjoint union of C and the C_k ; it is straightforward to check that $x \models_D \phi$.

It follows that from the exponential-time upper bound for satisfiability checking under global assumptions proved in Section 5, we obtain an exponential-time upper bound for satisfiability checking in coalgebraic modal logic with the universal modality. On the other hand, the non-deterministic reduction described above of course does not allow for inheriting practical reasoning algorithms. The design of tableau-based algorithms in presence of the universal modality is faced with the challenge that instances of $[\forall]$ uncovered deep in the formula by the rule-based decomposition will subsequently influence the entire tableau built so far. Our global caching algorithm (Section 6) is meant for reasoning under global assumptions; we leave the design of a

practical generic reasoning algorithm for coalgebraic modal logic with the universal modality to future work.

Generic algorithms in coalgebraic logic frequently rely on complete rule sets for the given modal operators [47] (an overview of the relevant concepts is given in Remark 4.16); in particular, such a rule set is assumed by our previous algorithm for satisfiability checking under global assumptions in coalgebraic hybrid logic [49]. In the present paper, our interest is in cases for which suitable rule sets are not (currently) available. We proceed to present our leading examples of this kind, Presburger modal logic and a probabilistic modal logic with polynomial inequalities. For the sake of readability, we focus on the case with a single (weighted) transition relation, and omit propositional atoms. Both propositional atoms and indexed transition relations are easily added, e.g. using compositionality results in coalgebraic logic [48], and in fact we use them freely in the examples; more details on this point will be provided in Remark 3.7.

3.1 Presburger Modal Logic

Presburger modal logic [13] admits statements in Presburger arithmetic over numbers $\# \phi$ of successors satisfying a formula ϕ . Throughout, we let Rels denote the set $\{<, >, =\} \cup \{\equiv_k \mid k \in \mathbb{N}\}$ of *arithmetic relations*, with \equiv_k read as congruence modulo k . Syntactically, Presburger modal logic is then defined in our syntactic framework by taking the modal similarity type

$$\Lambda = \{L_{u_1, \dots, u_n; \sim v} \mid \sim \in \text{Rels}, n \in \mathbb{N}, u_1, \dots, u_n, v \in \mathbb{Z}\}$$

where $L_{u_1, \dots, u_n; \sim v}$ has arity n . The application of a modal operator $L_{u_1, \dots, u_n; \sim v}$ to argument formulae ϕ_1, \dots, ϕ_n is written

$$\sum_{i=1}^n u_i \cdot \# \phi_i \sim v.$$

We refer to these modalities as *Presburger constraints*. Weak inequalities can be coded as strict ones, replacing, e.g., $\geq k$ with $> k - 1$. The numbers u_i and v , as well as the modulus k in \equiv_k , are referred to as the *coefficients* of a Presburger constraint. We also apply this terminology (Presburger constraint, coefficient) to constraints of the form $\sum_{i=1}^n u_i \cdot x_i \sim v$ in general, interpreted over the non-negative integers.

The semantics of Presburger modal logic was originally defined over standard Kripke frames; in order to make sense of sums with arbitrary (possibly negative) integer coefficients, one needs to restrict to finitely branching frames. We consider an alternative semantics in terms of *multigraphs*, which have some key technical advantages [11]. Informally, a multigraph is like a Kripke frame but with every transition edge annotated with a non-negative-integer-valued multiplicity; ordinary finitely branching Kripke frames can be viewed as multigraphs by just taking edges to be transitions with multiplicity 1. Formally, a multigraph can be seen as a coalgebra for the *finite multiset functor* \mathcal{B} : For a set X , $\mathcal{B}X$ consists of the *finite multisets over X* , which are maps $\mu: X \rightarrow \mathbb{N}$ with finite support, i.e. $\mu(x) > 0$ for only finitely many x . We view μ as an \mathbb{N} -valued measure, and write $\mu(Y) = \sum_{x \in Y} \mu(x)$ for $Y \subseteq X$. Then, $\mathcal{B}f$, for maps $f: X \rightarrow Y$, acts as image measure formation in the same way as the distribution functor \mathcal{D} described above, i.e. $(\mathcal{B}f)(\mu)(B) = \mu(f^{-1}[B])$ for $\mu \in \mathcal{B}X$ and $B \subseteq Y$. A coalgebra $\gamma: X \rightarrow \mathcal{B}X$ assigns to each state x a multiset $\gamma(x)$ of successor states, i.e. each successor state is assigned a transition multiplicity.

The semantics of the modal operators is then given by the predicate liftings

$$\llbracket L_{u_1, \dots, u_n; \sim v} \rrbracket_X(A_1, \dots, A_n) = \{\mu \in \mathcal{B}X \mid \sum_{i=1}^n u_i \cdot \mu(A_i) \sim v\},$$

that is, a state x in a \mathcal{B} -coalgebra $C = (X, \gamma)$ satisfies $\sum_{i=1}^n u_i \cdot \# \phi_i \sim v$ iff $\sum_{i=1}^n u_i \cdot \gamma(x)(\llbracket \phi_i \rrbracket_C) \sim v$.

Remark 3.4. Graded modal logic [18] is interpreted over the same systems (originally Kripke frames, equivalently multigraphs) as Presburger modal logic. It combines a Boolean propositional

base with modalities \diamond_k ‘in more than k successors’; these have made their way into modern expressive description logics in the shape of *qualified number restrictions* [4]. The multigraph semantics of graded modal logic is captured coalgebraically by assigning to \diamond_k the predicate lifting for \mathcal{B} given by $\llbracket \diamond_k \rrbracket_X(A) = \{\mu \in \mathcal{B}(X) \mid \mu(A) > k\}$. Presburger modal logic subsumes graded modal logic, via a translation t of graded modal logic into Presburger modal logic that is defined by commutation with all Boolean connectives and $t(\diamond_k \phi) = (\sharp(t(\phi)) > k)$.

We note that satisfiability is the same over Kripke frames and over multigraphs:

LEMMA 3.5. [43, Remark 6] [50, Lemma 2.4] *A formula ϕ is ψ -satisfiable over multigraphs iff ϕ is ψ -satisfiable over Kripke frames.*

(The proof of the non-trivial direction is by making copies of states to accommodate multiplicities.)

Remark 3.6. From the point of view of the present work, the technical reason to work with multigraphs rather than Kripke frames in the semantics of Presburger modal logic is that the key naturality condition (3) fails over Kripke semantics, i.e. for the powerset functor. Beyond the mere fact that for this reason, our methods do not apply to the Kripke semantics of Presburger or graded modal logic, we note that indeed key results of coalgebraic modal logic fail to hold for this semantics. For instance, we shall prove later (Lemma 4.7) that coalgebraic modal logic has the exponential model property, i.e. every satisfiable formula ϕ has a model with at most exponentially many states in the number of subformulae of ϕ . Over Kripke semantics, this clearly fails already for simple formulae such as $\sharp(\top) > k$.

Remark 3.7. As indicated above, the overall setup generalizes effortlessly to allow for both propositional atoms and multiple (weighted) transition relations: Let A be a set of *propositional atoms* and R a set of *relation names* (*atomic concepts* and *roles*, respectively, in description logic terminology). We then take the modal operators to be the propositional atoms and all operators

$$L_{u_1^{r_1}, \dots, u_n^{r_n}; \sim v}$$

where $\sim \in \text{Rels}$, $n \in \mathbb{N}$, $u_1, \dots, u_n, v \in \mathbb{Z}$, and $r_1, \dots, r_n \in R$. The arity of $L_{u_1^{r_1}, \dots, u_n^{r_n}; \sim v}$ is n , and the application of $L_{u_1^{r_1}, \dots, u_n^{r_n}; \sim v}$ to argument formulae ϕ_1, \dots, ϕ_n is written

$$\sum_{i=1}^n u_i \cdot \sharp_{r_i} \phi_i \sim v$$

where $\sharp_r(\cdot)$ is meant to represent the number of successors along the (weighted) transition relation r . The logic is then interpreted over structures that assign to each state x a subset of A (of propositional atoms that hold at x) and R -many multisets of successors. Such structures as coalgebras for the functor that maps a set X to $\mathcal{P}A \times \mathcal{B}X^R$; the associated predicate liftings are given by

$$\begin{aligned} \llbracket L_{u_1^{r_1}, \dots, u_n^{r_n}; \sim v} \rrbracket_X(A_1, \dots, A_n) &= \{(U, f) \in \mathcal{P}A \times (\mathcal{B}X)^R \mid \sum_{i=1}^n u_i \cdot f(r_i)(A_i) \sim v\} \\ \llbracket p \rrbracket_X &= \{(U, f) \in \mathcal{P}A \times (\mathcal{B}X)^R \mid p \in U\}. \end{aligned}$$

The effect of these extensions on the technical development does not go beyond heavier notation, so as announced above we restrict the exposition to only a single transition relation and no propositional atoms, for readability.

Remark 3.8. Two of us (Kupke and Pattinson) have exhibited modal sequent rules for various modal logics of linear inequalities, both over the non-negative reals (e.g. probabilistic and stochastic logics) and over the non-negative integers [29]. One of these logics can be seen as the fragment of Presburger modal logic obtained by removing modular congruence \equiv_k . Soundness and completeness of the rules for this logic would imply our upper complexity bounds by instantiating our own previous generic results in coalgebraic logic [49], which rely on precisely such rules. However,

while the rules given for logics with real-valued multiplicities appear to be sound and complete as claimed, the rule system given for the integer-valued case is sound but clearly not complete, as indicated already in Section 1. For instance, the formula $\phi := (2\sharp\top < 1 \vee 2\sharp\top > 1)$ is valid for integer multiplicities (ϕ says that the integer total weight of all successors of a state cannot be $1/2$) but not provable in the given rule system. The latter fact is most easily seen by comparing the rule for integer multiplicities [29, Section 4] with the rule given for the case of real-valued multiplicities [29, Section 5]: The rule instances applying to ϕ are the same in both cases, and as the rules are easily seen to be sound in the real-valued case, ϕ is not provable (as it fails to be valid in the real-valued case). There does not seem to be an easy fix for this, so for the time being there is no known sound and complete set of modal sequent rules (equivalently, modal tableau rules) for Presburger modal logic.

Expressiveness and Examples. As mentioned above, Presburger modal logic subsumes graded modal logic [18]. Moreover, Presburger modal logic subsumes majority logic [37] (more precisely, the version of majority logic interpreted over finitely branching systems): The *weak majority* formula $W\phi$ ('at least half the successors satisfy ϕ ') is expressed in Presburger modal logic as $\sharp(\phi) - \sharp(\neg\phi) \geq 0$. Using propositional atoms, incorporated in the way discussed above, we express the examples given in the introduction ('the majority of university students are female', 'dance classes have even numbers of participants') by the formulae

$$\begin{aligned} \text{University} &\rightarrow (\sharp_{\text{hasStudent}}\text{Female} - \sharp_{\text{hasStudent}}\text{Male} > 0) \\ \text{DanceCourse} &\rightarrow (\sharp_{\text{hasParticipant}}\top \equiv_2 0) \end{aligned}$$

where indices informally indicate the understanding of the successor relation. In the extension with multiple successor relations (Remark 3.7), one may also impose inequalities between numbers of successors under different roles as in the introduction, e.g. in the formula

$$\text{Workaholic} \rightarrow (\sharp_{\text{hasColleague}}\top - \sharp_{\text{hasFriend}}\top > 0)$$

('workaholics have more colleagues than friends'). As an example involving non-unit coefficients, a chamber of parliament in which a motion requiring a $2/3$ majority has sufficient support is described by the formula

$$\sharp_{\text{hasMember}}(\text{SupportsMotion}) - 2\sharp_{\text{hasMember}}(\neg\text{SupportsMotion}) \geq 0.$$

3.2 Probabilistic Modal Logic with Polynomial Inequalities

Probabilistic logics of various forms have been studied in different contexts such as reactive systems [32] and uncertain knowledge [16, 28]. A typical feature of such logics is that they talk about probabilities $w(\phi)$ of formulae ϕ holding for the successors of a state; the concrete syntax then variously includes only inequalities of the form $w(\phi) \sim p$ for $\sim \in \{>, \geq, =, <, \leq\}$ and $p \in \mathbb{Q} \cap [0, 1]$ [28, 32], linear inequalities over terms $w(\phi)$ [16], or polynomial inequalities, with the latter so far treated only in either purely propositional settings [17] or in many-dimensional logics such as the probabilistic description logic Prob- \mathcal{ALC} [26], which use a single global distribution over worlds. An important use of polynomial inequalities over probabilities is to express independence constraints [26]. For instance, two properties ϕ and ψ (of successors) are independent if $w(\phi \wedge \psi) = w(\phi)w(\psi)$, and we can express that the probability that the first of two independently sampled successors satisfies ϕ and the second satisfies ψ is at least p by a formula such as $w(\phi)w(\psi) \geq p$; the latter is similar to the *independent product* of real-valued probabilistic modal logic [35].

We thus define the following *probabilistic modal logic with polynomial inequalities*: The system type is given by a variant of the distribution functor \mathcal{D} as described above, viz. the *subdistribution*

functor \mathcal{S} , in which we require for $\mu \in \mathcal{S}X$ that the measure of the whole set X satisfies $\mu(X) \leq 1$ rather than $\mu(X) = 1$. Then \mathcal{S} -coalgebras $\gamma : X \rightarrow \mathcal{S}X$ are like Markov chains (where $\gamma(x)$ is interpreted as a distribution over possible future evolutions of the system), or (single-agent) type spaces in the sense of epistemic logic [28] (where $\gamma(x)$ is interpreted as the subjective probabilities assigned by the agent to possible alternative worlds in world x), with the difference that each state x has a probability $1 - \gamma(x)(X)$ of being deadlocked. We use the modal similarity type

$$\Lambda = \{L_p \mid p \in \mathbb{Q}[X_1, \dots, X_n], n \geq 0\};$$

for $p \in \mathbb{Q}[X_1, \dots, X_n]$, the modality L_p has arity n . We denote the application of L_p to formulae ϕ_1, \dots, ϕ_n by substituting each variable X_i in p with $w(\phi_i)$ and postulating the result to be non-negative, i.e. as

$$p(w(\phi_1), \dots, w(\phi_n)) \geq 0.$$

For instance, $L_{X_1 - X_2 X_3}(\phi \wedge \psi, \phi, \psi)$ is written more readably as $w(\phi \wedge \psi) - w(\phi)w(\psi) \geq 0$, and thus expresses one half of the above-mentioned independence constraint (the other half, of course, being $w(\phi)w(\psi) - w(\phi \wedge \psi) \geq 0$). We correspondingly interpret L_p by the predicate lifting

$$[[L_p]]_X(A_1, \dots, A_n) = \{\mu \in \mathcal{S}X \mid p(\mu(A_1), \dots, \mu(A_n)) \geq 0\}.$$

We will use Presburger modal logic and probabilistic modal logic as running examples in the sequel.

Remark 3.9. The use of \mathcal{S} in place of \mathcal{D} serves only to avoid triviality of the logic in the absence of propositional atoms: Since $|\mathcal{D}(1)| = 1$ for any singleton set 1 , all states in \mathcal{D} -coalgebras (i.e. Markov chains) are bisimilar, and thus satisfy the same formulae of any coalgebraic modal logic on \mathcal{D} -coalgebras [39, 44], so any formula in such a logic is either valid or unsatisfiable. This phenomenon disappears as soon as we add propositional atoms as per Remark 3.7. All our results otherwise apply to \mathcal{D} in the same way as to \mathcal{S} .

4 ONE-STEP SATISFIABILITY

The key ingredient of our algorithmic approach is to deal with modal operators (i.e., in our running examples, arithmetic statements about numbers or weights of successors) level by level; the core concepts of the arising notion of one-step satisfiability checking go back to work on plain satisfiability in coalgebraic logics [36, 43, 46]. From now on, we restrict the technical treatment to unary modal operators to avoid cumbersome notation, although our central examples all do have modal operators with higher arities; a fully general treatment requires no more than additional indexing. Considering only one level of modal operators and abstracting from their arguments amounts to working in a *one-step logic*, whose syntax and semantics are defined as follows (subsequent to fixing some notation).

Definition 4.1 (Notation for propositional variables and propositional logic). We fix a countably infinite set \mathcal{V} of (propositional) variables. We denote the set of Boolean formulae (presented in terms of \perp , \wedge , and \neg) over a set $V \subseteq \mathcal{V}$ of propositional variables by $\text{Prop}(V)$; that is, formulae $\eta, \rho \in \text{Prop}(V)$ are defined by the grammar

$$\eta, \rho ::= \perp \mid \neg\eta \mid \eta \wedge \rho \mid a \quad (a \in V).$$

We write $\mathbb{2}$ for the set $\{\perp, \top\}$ of truth values, and then have a standard notion of satisfaction of propositional formulae over V by valuations $\kappa : V \rightarrow \mathbb{2}$. As usual, a *literal* over V is a propositional variable $a \in V$ or a negated variable $\neg a$ for $a \in V$, often written ϵa with $\epsilon \in \{-1, 1\}$ as per the previous convention (Section 3), and a *conjunctive clause* over V is a finite conjunction $\epsilon_1 a_1 \wedge \dots \wedge \epsilon_n a_n$ of literals over V , represented as a finite set of literals. We write $\Phi \vdash_{PL} \eta$ to indicate that a

set $\Phi \subseteq \text{Prop}(V)$ *propositionally entails* $\eta \in \text{Prop}(V)$, meaning that there exist $\rho_1, \dots, \rho_n \in \Phi$ such that $\rho_1 \wedge \dots \wedge \rho_n \rightarrow \eta$ is a propositional tautology. For $\{\rho\} \vdash_{PL} \eta$, we briefly write $\rho \vdash_{PL} \eta$.

By a *substitution*, we will mean a map σ from (some subset of) \mathcal{V} into another set Z , typically a set of formulae of some kind. In case $Z = \mathcal{V}$, we will also refer to σ as a *renaming*. We write application of a substitution σ to formulae ϕ containing propositional variables (either propositional formulae or formulae of the one-step logic as introduced in the next definition) in postfix notation $\phi\sigma$ as usual (i.e. $\phi\sigma$ is obtained from ϕ by replacing all occurrences of propositional variables a in ϕ with $\sigma(a)$). We extend the propositional entailment relation to formulae beyond $\text{Prop}(\mathcal{V})$ by substitution, i.e. for a formula ψ and a set Φ of formulae (in the one-step logic or in coalgebraic modal logic), we write $\Phi \vdash_{PL} \psi$ if Φ, ψ can be written in the form $\Phi = \Phi'\sigma, \psi = \psi'\sigma$ for a substitution σ and $\Phi' \subseteq \text{Prop}(\mathcal{V}), \psi' \in \text{Prop}(\mathcal{V})$ such that $\Phi' \vdash_{PL} \psi'$ in the sense defined above (that is, if there are $\phi_1, \dots, \phi_n \in \Phi'$ such that $\phi_1 \wedge \dots \wedge \phi_n \rightarrow \psi'$ is a substitution instance of a propositional tautology).

The **syntax** of the one-step logic is given in the following terms:

Definition 4.2 (One-step pairs). Given a set $V \subseteq \mathcal{V}$ of propositional variables, we denote by

$$\Lambda(V) = \{\heartsuit a \mid \heartsuit \in \Lambda, a \in V\}$$

the set of *modal atoms over V*. A *modal literal over V* is a modal atom over V or a negation thereof, i.e. has the form either $\heartsuit a$ or $\neg\heartsuit a$ for $\heartsuit \in \Lambda, a \in V$. A *modal conjunctive clause* ϕ is a finite conjunction $\epsilon_1 \heartsuit_1 a_1 \wedge \dots \wedge \epsilon_n \heartsuit_n a_n$ of modal literals over V , represented as a finite set of modal literals. We write $\text{Var}(\phi) = \{a_1, \dots, a_n\}$ for the set of variables occurring in ϕ . We say that ϕ is *clean* if ϕ mentions each variable in V at most once. A *one-step pair* (ϕ, η) over V consists of

- a clean modal conjunctive clause ϕ over V and
- a Boolean formula $\eta \in \text{Prop}(\text{Var}(\phi))$.

We measure the size $|\phi|$ of a modal conjunctive clause ϕ by counting 1 for each variable and each propositional operator, and for each modality the size of its encoding (in the same way as in the definition of the size of modal formulae in Section 3). The propositional component η is assumed to be given as a DNF consisting of conjunctive clauses each mentioning every variable occurring in ϕ (such conjunctive clauses are effectively truth valuations for the variables in ϕ), and the size $|\eta|$ of η is the size of this DNF.

In a one-step pair (ϕ, η) , the modal component ϕ effectively specifies what happens one transition step ahead from the (implicit) current state; as indicated above, in the actual satisfiability checking algorithm, ϕ will arise by peeling off the top layer of modalities of a given modal formula, with the propositional variables in V abstracting the argument formulae of the modalities. The propositional component η then records the propositional dependencies among the argument formulae. Formally, the **semantics** of the one-step logic is given as follows:

Definition 4.3 (One-step models, one-step satisfiability). A *one-step model* $M = (X, \tau, t)$ over V consists of

- a set X together with a $\mathcal{P}X$ -valuation $\tau: V \rightarrow \mathcal{P}X$; and
- an element $t \in TX$ (thought of as the structured collection of successors of an anonymous state).

For $\eta \in \text{Prop}(V)$, we write $\tau(\eta)$ for the interpretation of η in the Boolean algebra $\mathcal{P}X$ under the valuation τ ; explicitly, $\tau(\perp) = \emptyset$, $\tau(\neg\eta) = X \setminus \tau(\eta)$, and $\tau(\eta \wedge \rho) = \tau(\eta) \cap \tau(\rho)$. For a modal atom $\heartsuit a \in \Lambda(V)$, we put

$$\tau(\heartsuit a) = \llbracket \heartsuit \rrbracket_X(\tau(a)) \subseteq TX.$$

We extend this assignment to modal atoms and modal conjunctive clauses using the Boolean algebra structure of $\mathcal{P}(TX)$; explicitly,

$$\begin{aligned}\tau(\neg\heartsuit a) &= TX \setminus \tau(\heartsuit a) \\ \tau(\epsilon_1\heartsuit_1 a_1 \wedge \dots \wedge \epsilon_n\heartsuit_n a_n) &= \tau(\epsilon_1\heartsuit_1 a_1) \cap \dots \cap \tau(\epsilon_n\heartsuit_n a_n).\end{aligned}$$

We say that the one-step model $M = (X, \tau, t)$ *satisfies* the one-step pair (ϕ, η) , and write $M \models (\phi, \eta)$, if

$$\tau(\eta) = X \quad \text{and} \quad t \in \tau(\phi).$$

(That is, η is a global propositional constraint on the values of τ while ϕ specifies a property of the collection t of successors.) Then, (ϕ, η) is *(one-step) satisfiable* if there exists a one-step model M such that $M \models (\phi, \eta)$. The *lax one-step satisfiability problem* (of Λ) is to decide whether a given one-step pair (ϕ, η) is one-step satisfiable; the size of the input is measured as $|\phi| + |\eta|$ with $|\phi|$ and $|\eta|$ defined as above. The *strict one-step satisfiability problem* (of Λ) is the same problem but with the input size defined to be just $|\phi|$. For purposes of space complexity, we thus assume in the strict one-step satisfiability problem that η is stored on an input tape that does not count towards space consumption. It will be technically convenient to assume moreover that in the strict one-step satisfiability problem, η is given as a bit vector indicating which conjunctive clauses (mentioning every variable occurring in ϕ , in some fixed order) are contained in the DNF η ; contrastingly, we assume that in the lax one-step satisfiability problem, η is given as a list of conjunctive clauses as indicated in Definition 4.2 (hence need not have exponential size in all cases). For time complexity, we assume that the input tape is random access (i.e. accessed via a dedicated address tape, in the model of random access Turing machines [20]; this is necessary to enable subexponential time bounds for the strict one-step satisfiability problem since otherwise it takes exponential time just to move the head to the last bits of the input). We say that Λ has the *(weak) one-step small model property* if there is a polynomial p such that every one-step satisfiable (ϕ, η) has a one-step model (X, τ, t) with $|X| \leq p(|\text{Var}(\phi)|)$ (respectively $|X| \leq p(|\phi|)$). (Note that no bound is assumed on the representation of t .)

As indicated above, the intuition behind these definitions is that the propositional variables in V are placeholders for argument formulae of modalities; their valuation τ in a one-step model (X, τ, t) over V represents the extensions of these argument formulae in a model; and the second component η of a one-step pair (ϕ, η) captures the Boolean constraints on the argument formulae that are globally satisfied in a given model. The component $t \in TX$ of (X, τ, t) represents the structured collection of successors of an implicit current state, so the modal component ϕ of the one-step pair is evaluated on t . We will later construct full models of modal formulae using one-step models according to this intuition. One may think of a one-step model (X, τ, μ) of a one-step pair (ϕ, η) as a counterexample to soundness of $\eta/\neg\phi$ as a proof rule: ϕ is satisfiable despite η being globally valid in the model.

Example 4.4. (1) In the basic example of relational modal logic ($\Lambda = \{\diamond\}$, $T = \mathcal{P}$, see Section 3), consider the one-step pair $(\phi, \eta) := (\neg\diamond a \wedge \neg\diamond b \wedge \diamond c, c \rightarrow a \vee b)$. The propositional component η is represented as a DNF $\eta = (c \wedge a \wedge b) \vee (\neg c \wedge \neg a \wedge \neg b) \vee \dots$. A one-step model (X, τ, t) of (ϕ, η) (where $t \in \mathcal{P}(X)$) would need to satisfy $\tau(c) \subseteq \tau(a) \cup \tau(b)$ to ensure $\tau(\eta) = X$, as well as $t \cap \tau(c) \neq \emptyset$, $t \cap \tau(a) = \emptyset$, and $t \cap \tau(b) = \emptyset$ to ensure $t \in \tau(\phi)$. As this is clearly impossible, (ϕ, η) is unsatisfiable. In fact, it is easy to see that the strict one-step satisfiability problem of relational modal logic in this sense is in NP: To check whether a one-step pair (ψ, χ) is satisfiable, guess a conjunctive clause ρ in χ for each positive modal literal $\diamond a$ in ψ , and check that ρ contains on the one hand a , and on the other hand $\neg b$ for every negative modal literal $\neg\diamond b$ in ψ .

(2) In Presburger modal logic, let $\phi := (\#(a) + \#(b) - \#(c) > 0)$ (a conjunctive clause consisting of a single modal literal). Then a one-step pair of the form (ϕ, η) is one-step satisfiable iff η is consistent with $\rho := (a \wedge b) \vee (a \wedge \neg c) \vee (b \wedge \neg c)$: For the ‘if’ direction, note that η is consistent with some disjunct ρ' of ρ ; we distinguish cases over ρ' , and build a one-step model (X, τ, μ) of (ϕ, η) . In each case, we take X to consist of a single point 1; since $\eta \wedge \rho'$ is consistent, we can pick τ such that $\tau(\eta \wedge \rho') = X$ (and hence $\tau(\eta) = X$). Moreover, we always take μ to be the multiset given by $\mu(1) = 1$. If $\rho' = (a \wedge \neg c)$, then $\mu(\tau(a)) + \mu(\tau(b)) - \mu(\tau(c)) = 1 + \mu(\tau(b)) - 0 > 0$, so $\mu \in \tau(\phi)$, and we are done. The case $\rho' = (b \wedge \neg c)$ is analogous. Finally, if $\rho' = (a \wedge b)$, then $\mu(\tau(a)) + \mu(\tau(b)) - \mu(\tau(c)) = 2 - \mu(\tau(c)) > 0$. For the ‘only if’ direction, assume that $\eta \wedge \rho$ is inconsistent, so η propositionally entails $a \rightarrow c$, $b \rightarrow c$, and $\neg(a \wedge b)$, and let (X, τ, μ) be a one-step model such that $\tau(\eta) = X$; we have to show that $\mu \notin \tau(\psi)$. Indeed, since $\tau(\eta) = X$ we have $\tau(a) \subseteq \tau(c)$, $\tau(b) \subseteq \tau(c)$, and $\tau(a) \cap \tau(b) = \emptyset$, so $\mu(\tau(a)) + \mu(\tau(b)) - \mu(\tau(c)) \leq 0$.

(3) The reasoning in the previous example applies in the same way to one-step pairs of the form $(w(a) + w(b) - w(c) > 0, \eta)$ in probabilistic modal logic.

(4) The example formula given in Remark 3.8 translates into a one-step pair $(2\#(a) < 1 \wedge 2\#(a) > 0, a)$ in Presburger modal logic whose unsatisfiability does depend on multiplicities being integers; that is, the corresponding one-step pair $(2w(a) < 1 \wedge 2w(a) > 0, a)$ in probabilistic modal logic is satisfiable.

Remark 4.5. For purposes of upper complexity bounds PSPACE and above for the strict one-step satisfiability problem, it does not matter whether the propositional component η of a one-step pair (ψ, η) is represented as a list or as a bit vector, as we have obvious mutual conversions between these formats that can be implemented using only polynomial space in $|\text{Var}(\psi)|$. For subexponential time bounds, on the other hand, the distinction between the formats does appear to matter, as the mentioned conversions do take exponential time in $|\text{Var}(\psi)|$.

Note that most of a one-step pair (ϕ, η) is disregarded for purposes of determining the input size of the *strict* one-step satisfiability problem, as η can be exponentially larger than ϕ . Indeed, we have the following relationship between the respective complexities of the lax one-step satisfiability problem and the strict one-step satisfiability problem.

LEMMA 4.6. *The strict one-step satisfiability problem of Λ is in EXPTIME iff the lax one-step satisfiability problem of Λ can be solved on one-step pairs (ϕ, η) in time $2^{O((\log |\eta| + |\phi|)^k)}$ for some k .*

(Recent work on the coalgebraic μ -calculus uses essentially the second formulation [27].)

PROOF. ‘Only if’ is trivial, since the time bound allows converting η from the list representation assumed in the lax version of the problem to the bit vector representation assumed in the strict version. ‘If’: Since we require that all variables mentioned by η occur also in ϕ , and assume that η is given in DNF, we have $|\eta| = 2^{O(|\phi|)}$, so $\log |\eta| = O(|\phi|)$, and hence $2^{O((\log |\eta| + |\phi|)^k)} = 2^{O(|\phi|^k)}$. \square

We note that the one-step logic has an exponential-model property (which in slightly disguised form has appeared first as [45, Proposition 3.10]):

LEMMA 4.7. *A one-step pair (ϕ, η) over V is satisfiable iff it is satisfiable by a one-step model of the form (X, τ, t) where X is the set of valuations $V \rightarrow 2$ satisfying η (where $2 = \{\top, \perp\}$ is the set of Booleans) and $\tau(a) = \{\kappa \in X \mid \kappa(a) = \top\}$ for $a \in V$.*

PROOF. ‘If’ is trivial; we prove ‘only if’. Let $M = (Y, \vartheta, s)$ be a one-step model of (ϕ, η) . Take X and τ as in the claim; it is clear that $\tau(\eta) = X$. Define a map $f: Y \rightarrow X$ by $f(y)(a) = \top$ iff $y \in \vartheta(a)$ for $y \in Y$, $a \in V$. Then put $t = Tf(s) \in TX$. By construction, we have $f^{-1}[\tau(a)] = \vartheta(a)$ for all $a \in V$. By naturality of predicate liftings and commutation of preimage with Boolean operators,

this implies that $(Tf)^{-1}[\tau(\phi)] = \vartheta(\phi)$, so $s \in \vartheta(\phi)$ implies $t = Tf(s) \in \tau(\phi)$; i.e. (X, τ, t) is a one-step model of (ϕ, η) . \square

From the construction in the above lemma, we obtain the following equivalent characterization of the one-step small model property:

LEMMA 4.8. *The logic Λ has the (weak) one-step small model property iff there exists a polynomial p such that the following condition holds: Whenever a one-step pair (ϕ, η) is one-step satisfiable, then there exists η' such that*

- (1) (ϕ, η') is one-step satisfiable;
- (2) the list representation of η' according to Definition 4.2 has size at most $p(|\text{Var}(\phi)|)$ (respectively at most $p(|\phi|)$); and
- (3) $\eta' \vdash_{PL} \eta$.

PROOF. ‘Only if’: Take the conjunctive clauses of the DNF η' to be the ones realized in a polynomial-sized one-step model (X, τ, t) of (ϕ, η) ; that is, η' is the disjunction of all conjunctive clauses ρ mentioning all variables occurring in ϕ such that $\tau(\rho) \neq \emptyset$.

‘If’: Take X as in Lemma 4.7 and note that $|X|$ is the number of conjunctive clauses in the representation of η' as per Definition 4.2. \square

Under the one-step small model property, the two versions of the one-step satisfiability problem coincide for our purposes, as detailed next. Recall that a multivalued function f is *NPMV* [8] if the representation length of values of f on x is polynomially bounded in that of x and moreover the graph of f is in NP; we generalize this notion slightly to allow for size measures of x other than representation length (such as the input size measure used in the strict one-step satisfiability problem). Most reasonable complexity classes containing NP are closed under NPMV reductions; in particular this holds for PSPACE, EXPTIME, and all levels of the polynomial hierarchy.

LEMMA 4.9. *Let Λ have the weak one-step small model property (Definition 4.2). Then the strict one-step satisfiability problem of Λ is NPMV-reducible to lax one-step satisfiability. In particular, if lax one-step satisfiability is in NP (PSPACE/EXPTIME), then strict one-step satisfiability is in NP (PSPACE/EXPTIME).*

PROOF. By Lemma 4.8, and in the notation of its statement, the NPMV function that maps (ϕ, η) (with η in bit vector representation) to all (ϕ, η') with η' of (list) representation size at most $p(|\phi|)$ and $\eta' \vdash_{PL} \eta$ reduces strict one-step satisfiability to lax one-step satisfiability. \square

Of the two versions of the one-step small model property, the stronger version (polynomial in $|\text{Var}(\phi)|$) turns out to be prevalent in the examples. The weak version (polynomial in $|\phi|$) is of interest mainly due to the following equivalent characterization:

THEOREM 4.10. *Suppose that the lax one-step satisfiability problem of Λ is in NP. Then the weak one-step small model property holds for Λ iff the strict one-step satisfiability problem of Λ is in NP.*

PROOF. ‘Only if’ is immediate by Lemma 4.9; we prove ‘if’. Let M be a non-deterministic (random access) Turing machine that solves the strict one-step satisfiability problem in polynomial time, and let the one-step pair (ϕ, η) be one-step satisfiable. Then M has a successful run on (ϕ, η) . Since this run takes polynomial time in $|\phi|$, it accesses only polynomially many bits in the bit vector representation of η . We can therefore set all other bits to 0, obtaining a polynomial-sized DNF η' such that $\eta' \vdash_{PL} \eta$ and (ϕ, η') is still one-step satisfiable, as witnessed by otherwise the same run of M . By Lemma 4.8, this proves the weak one-step small model property. \square

Although not phrased in these terms, the complexity analysis of Presburger modal logic (without global assumptions) by Demri and Lugiez [13] is based on showing that the strict one-step satisfiability problem is in PSPACE [46], without using the one-step small model property for Presburger modal logic – in fact, our proof of the latter is based on more recent results from integer programming: We recall that the classical *Carathéodory theorem* (e.g. [42]) may be phrased as saying that every system of d linear equations that has a solution over the non-negative reals has such a solution with at most d non-zero components. Eisenbrand and Shmonin [14] prove an analogue over the integers, which we correspondingly rephrase as follows.

LEMMA 4.11 (INTEGER CARATHÉODORY THEOREM [14]). *Every system of d linear equations $\sum u_i x_i = v$ with integer coefficients u_i of binary length at most s that has a solution over the non-negative integers has such a solution with at most polynomially many non-zero components in d and s (specifically, $O(sd \log d)$).*

To deal with lax one-step satisfiability, we will moreover need the well-known result by Papadimitriou that establishes a polynomial bound on the size of components of solutions of systems of integer linear equations:

LEMMA 4.12. [38] *Every system of integer linear equations in variables x_1, \dots, x_n that has a solution over the non-negative integers has such a solution $(\hat{x}_1, \dots, \hat{x}_n)$ with the binary length of each component \hat{x}_i bounded polynomially in the overall binary representation size of the equation system.*

COROLLARY 4.13. *Solvability of systems of Presburger constraints is in NP.*

PROOF. It suffices to show that we can generalize Lemma 4.12 to systems of Presburger constraints. Indeed, we can reduce Presburger constraints to equations involving additional variables. Specifically, we replace an inequality $\sum u_i \cdot x_i > v$ with the equation $\sum u_i \cdot x_i - y = v + 1$ and a modular constraint $\sum u_i \cdot x_i \equiv_k v$ with either $\sum u_i \cdot x_i - k \cdot y = v$ or $\sum u_i \cdot x_i + k \cdot y = v$, depending on whether the given solution satisfies $\sum u_i \cdot x_i \geq v$ or $\sum u_i \cdot x_i \leq v$; in every such replacement, choose y as a fresh variable. \square

From these observations, we obtain sufficient tractability of strict one-step satisfiability in our key examples:

Example 4.14. (1) *Presburger modal logic has the one-step small model property.* To see this, let a one-step pair (ϕ, η) over $V = \{a_1, \dots, a_n\}$ be satisfied by a one-step model $M = (X, \tau, \mu)$, where by Lemma 4.7 we can assume that X consists of satisfying valuations of η , hence has at most exponential size in $|\phi|$. Put $q_i = \mu(\tau(a_i))$. Now all we need to know about μ to guarantee that M satisfies ϕ is that the (non-negative integer) numbers $y_x := \mu(x)$, for $x \in X$, satisfy

$$\sum_{x \in \tau(a_i)} y_x = q_i \quad \text{for } i = 1, \dots, n.$$

We can see this as a system of n linear equations in the y_x , which by the integer Carathéodory theorem (Lemma 4.11) has a non-negative integer solution $(y'_x)_{x \in X}$ with only m nonzero components where m is polynomially bounded in n (the coefficients of the y_x all being 1), and hence in $|\phi|$; from this solution, we immediately obtain a one-step model (X', τ', μ') of (ϕ, η) with m states. Specifically, take $X' = \{x \in X \mid y'_x > 0\}$, $\tau'(a_i) = \tau(a_i) \cap X'$ for $i = 1, \dots, n$, and $\mu'(x) = y'_x$ for $x \in X'$.

Moreover, again using Lemma 4.7, lax one-step satisfiability in Presburger modal logic reduces straightforwardly to checking solvability of Presburger constraints over the non-negative integers, which by Corollary 4.13 can be done in NP. Specifically, given a one-step pair (ϕ, η) , with η represented as per Definition 4.2, introduce a variable x_ρ for every conjunctive clause ρ of η (i.e. for

every valuation satisfying η), and translate every constraint $\sum_i u_i \cdot \sharp(a_i) \sim v$ in ϕ into

$$\sum_i u_i \cdot \sum_{\rho \vDash_{PL} \eta \wedge a_i} x_\rho \sim v.$$

Thus, the lax one-step satisfiability problem of Presburger modal logic is in NP, and by Lemma 4.9, we obtain that *strict one-step satisfiability in Presburger modal logic is in NP*.

(2) By a completely analogous argument as for Presburger modal logic (using the standard Carathéodory theorem), *probabilistic modal logic with polynomial inequalities has the one-step small model property*. Moreover, lax one-step satisfiability reduces, analogously as in the previous item, to solvability of systems of polynomial inequalities over the reals, which can be checked in PSPACE [9] (this argument can essentially be found in [17]). Again, we obtain that *strict one-step satisfiability in probabilistic modal logic with polynomial inequalities is in PSPACE*.

Remark 4.15 (Variants of the running examples). The proof of the one-step small model property for Presburger modal logic and probabilistic modal logic with polynomial inequalities will in both cases work for any modal logic over integer- or real-weighted systems, respectively, whose modalities depend only on the measures of their arguments; call such modalities *fully explicit*. There are quite sensible operators that violate this restriction; e.g. an operator $I(\phi, \psi)$ ‘ ϕ is independent of ψ ’ would depend on the probabilities of ϕ and ψ but also on that of $\phi \wedge \psi$. Indeed, in this vein we easily obtain a natural logic over probabilistic systems that fails to have the one-step small model property: If we generalize the independence modality I to several arguments and combine it with operators $w(-) > 0$ stating that their arguments have positive probability, then every one-step model of the one-step pair

$$(I(a_1, \dots, a_n) \wedge \bigwedge_{i=1}^n w(a_i) > 0 \wedge \bigwedge_{i=1}^n w(\neg a_i) > 0, \top)$$

has at least 2^n states.

However, a completely analogous argument as in the proof of Lemma 4.7 shows that every predicate lifting for functors such as \mathcal{D} , \mathcal{S} , or \mathcal{B} depends only on the measures of Boolean combinations of its arguments, which can equally well be expressed using the propositional operators of the logic. That is, every coalgebraic modal logic over weighted systems translates (possibly with exponential blowup) into one that has only fully explicit modalities and hence has the one-step small model property, as exemplified for the case of I in Section 3.2.

Incidentally, a similar example as the above produces a natural example of a logic that does not have the one-step small model property but whose lax one-step satisfiability problem is nevertheless in EXPTIME. Consider a variant of probabilistic modal logic (Section 3.2) featuring *linear* (rather than polynomial) inequalities over probabilities $w(\phi)$, and additionally *fixed-probability conditional independence* operators I_{p_1, \dots, p_n} of arity $n+1$ for $n \geq 1$ and $p_1, \dots, p_n \in \mathbb{Q} \cap [0, 1]$. The application of I_{p_1, \dots, p_n} to formulae $\phi_1, \dots, \phi_n, \psi$ is written $I_{p_1, \dots, p_n}(\phi_1, \dots, \phi_n \mid \psi)$, and read ‘ ϕ_1, \dots, ϕ_n are conditionally independent given ψ , and each ϕ_i has conditional probability p_i given ψ ’. A one-step modal literal $I_{p_1, \dots, p_n}(a_1, \dots, a_n \mid b)$ translates, by definition, into linear equalities

$$w(\bigwedge_{i \in I} a_i) - (\prod_{i \in I} p_i)w(\psi) = 0 \quad \text{for all } I \subseteq \{1, \dots, n\}.$$

Thus, a given one-step clause ψ generates, in the same way as previously, a system of linear inequalities, now of exponential size in $|\psi|$. Since solvability of systems of linear inequalities can, by standard results in linear programming [42], be checked in polynomial time, we obtain that the strict one-step satisfiability problem is in EXPTIME as claimed. On the other hand, the one-step small model property fails for the same reasons as for the I operator described above.

By previous results in coalgebraic logic [46], the observations in Example 4.14.1 imply decidability in PSPACE of the respective *plain* satisfiability problems, reproducing a previous result by Demri

and Lugiez [13] for the case of Presburger modal logic; we show in Section 5 that the same observations yield an optimal upper bound EXPTIME for satisfiability under global assumptions.

Remark 4.16 (Comparison with tractable modal rule sets). Most previous generic complexity results in coalgebraic logic have relied on complete sets of modal tableau rules that are sufficiently tractable for purposes of the respective complexity bound, e.g. [22, 47, 49]. We briefly discuss how these assumptions imply the ones used in the present paper.

The rules in question (*one-step tableau rules*) are of the shape ϕ/ρ where ϕ is a modal conjunctive clause over V and $\rho \in \text{Prop}(V)$, subject to the same syntactic restrictions as one-step pairs, i.e. ϕ must be clean and ρ can only mention variables occurring in ϕ . Such rules form part of a tableau system that includes also the standard propositional rules. As usual in tableau systems, algorithms for satisfiability checking based on the tableau rules proceed roughly according to the principle ‘in order to establish that ψ is satisfiable, show that the conclusions of all rule matches to ψ are satisfiable’ (this is dual to validity checking via formal proof rules, where to show that ψ is valid one needs to find some proof rule whose conclusion matches ψ and whose premiss is valid). More precisely, the (one-step) soundness and completeness requirement on a rule set \mathcal{R} demands that a one-step pair (ψ, η) is satisfiable iff for every rule ϕ/ρ in \mathcal{R} and every injective variable renaming σ such that $\psi \vdash_{PL} \phi\sigma$ (see Definition 4.1 for the notation \vdash_{PL}), the propositional formula $\eta \wedge \rho\sigma$ is satisfiable. Since ψ and ϕ are modal conjunctive clauses (and ψ , being clean, cannot contain clashing modal literals), $\psi \vdash_{PL} \phi\sigma$ means that ψ contains every modal literal of $\phi\sigma$.

The exact requirements on tractability of a rule set vary with the intended complexity bound for the full logic. In connection with EXPTIME bounds, one uses *exponential tractability* of the rule set (e.g. [10]). This condition requires that rules have an encoding as strings such that every rule ϕ/ρ in \mathcal{R} that *matches* a given modal conjunctive clause ψ over V under a given injective renaming σ , i.e. $\psi \vdash_{PL} \phi\sigma$, has an encoding of polynomial size in $|\psi|$, and moreover given a modal conjunctive clause ψ over V , it can be decided in exponential time in $|\psi|$ whether (i) an encoded rule ϕ/ρ matches ψ under a given renaming σ , and (ii) whether a given conjunctive clause χ over $\text{Var}(\psi)$ propositionally entails the conclusion $\rho\sigma$ the instance $\phi\sigma/\rho\sigma$ of an encoded rule ϕ/ρ under a given renaming σ .

Now suppose that a set \mathcal{R} of modal tableau rules satisfies all these requirements, i.e. is one-step sound and complete for the given logic and exponentially tractable, with polynomial bound p on the size of rule codes. Then one sees easily that the strict one-step satisfiability problem is in EXPTIME : Given a one-step pair (ψ, η) to be checked for one-step satisfiability, we can go through all rules ϕ/ρ represented by codes of length at most $p(|\psi|)$ and all injective renamings σ of the variables of ϕ into the variables of ψ such that ϕ/ρ matches ψ under σ , and then for each such match go through all conjunctive clauses χ over $\text{Var}(\psi)$ that propositionally entail $\rho\sigma$, checking for each such χ that $\eta \wedge \chi$ is propositionally satisfiable. Both loops go through exponentially many iterations, and all computations involved take at most exponential time. Summing up, complexity bounds obtained by our current semantic approach subsume earlier tableau-based ones.

5 TYPE ELIMINATION

We now describe a type elimination algorithm that realizes an EXPTIME upper bound for reasoning with global assumptions in coalgebraic logics. Like all type elimination algorithms, it is not suited for practical use, as it begins by constructing the full exponential-sized set of types (in the initialization phase of the computation of a greatest fixpoint). We therefore refine the algorithm to a global caching algorithm in Section 6.

As usual, we rely on defining a scope of relevant formulae:

Definition 5.1. We define *normalized negation* \sim by taking $\sim\phi = \phi'$ if a formula ϕ has the form $\phi = \neg\phi'$, and $\sim\phi = \neg\phi$ otherwise. A set Σ of formulae is *closed* if Σ is closed under subformulae and normalized negation. The *closure* of a set Γ of formulae is the least closed set containing Γ .

We fix from now on a global assumption ψ and a formula ϕ_0 to be checked for ψ -satisfiability. We denote the closure of $\{\psi, \phi_0\}$ in the above sense by Σ . Next, we approximate the ψ -satisfiable subsets of Σ from above via a notion of type that takes into account only propositional reasoning and the global assumption ψ :

Definition 5.2. A ψ -type is a subset $\Gamma \subseteq \Sigma$ such that

- $\psi \in \Gamma \not\equiv \perp$;
- whenever $\neg\phi \in \Sigma$, then $\neg\phi \in \Gamma$ iff $\phi \notin \Gamma$;
- whenever $\phi \wedge \chi \in \Sigma$, then $\phi \wedge \chi \in \Gamma$ iff $\phi, \chi \in \Gamma$.

The design of the algorithm relies on one-step satisfiability as an abstraction: We denote the set of all ψ -types by $\mathcal{T}(\psi)$. For a formula $\phi \in \Sigma$, we put

$$\hat{\phi} = \{\Gamma \in \mathcal{T}(\psi) \mid \phi \in \Gamma\},$$

intending to construct a model on a suitable subset $S \subseteq \mathcal{T}(\psi)$ in such a way that $\hat{\phi} \cap S$ becomes the extension of ϕ . We take V_Σ to be the set of propositional variables $a_{\heartsuit\rho}$ for all modal atoms $\heartsuit\rho \in \Sigma$; we then define a substitution σ_Σ by $\sigma_\Sigma(a_{\heartsuit\rho}) = \rho$ for $a_{\heartsuit\rho} \in V_\Sigma$. For $S \subseteq \mathcal{T}(\psi)$ and $\Gamma \in S$, we construct a one-step pair

$$(\phi_\Gamma, \eta_S)$$

over V_Σ by taking ϕ_Γ to be the conjunction of all modal literals $\epsilon\heartsuit a_{\heartsuit\rho}$ over V_Σ such that $\epsilon\heartsuit\rho \in \Gamma$ (note that indexing the propositional variables $a_{\heartsuit\rho}$ over $\heartsuit\rho$ instead of just ρ ensures that ψ_Γ is clean as required), and η_S to be the DNF (for definiteness, in bit vector representation as per Definition 4.2) containing for each $\Delta \in S$ a conjunctive clause

$$\bigwedge_{\heartsuit\rho \in \Sigma \mid \rho \in \Delta} a_{\heartsuit\rho} \wedge \bigwedge_{\heartsuit\rho \in \Sigma \mid \sim\rho \in \Delta} \neg a_{\heartsuit\rho}.$$

That is, ϕ_Γ arises from Γ by abstracting the arguments ρ of modalized formulae $\heartsuit\rho \in \Gamma$ as propositional variables $a_{\heartsuit\rho}$, and η captures the propositional dependencies that will hold in S among these arguments if the construction works as intended. We define a functional

$$\begin{aligned} \mathcal{E}: \mathcal{P}(\mathcal{T}(\psi)) &\rightarrow \mathcal{P}(\mathcal{T}(\psi)) \\ S &\mapsto \{\Gamma \in S \mid (\phi_\Gamma, \eta_S) \text{ is one-step satisfiable}\}, \end{aligned} \quad (5)$$

whose greatest fixpoint $\nu\mathcal{E}$ will turn out to contain precisely the satisfiable types. Existence of $\nu\mathcal{E}$ is guaranteed by the Knaster-Tarski fixpoint theorem and the following lemma:

LEMMA 5.3. *The functional \mathcal{E} is monotone w.r.t. set inclusion.*

PROOF. For $S \subseteq S'$, the DNF $\eta_{S'}$ is weaker than η_S , as it contains more disjuncts. □

By Kleene's fixpoint theorem, we can compute $\nu\mathcal{E}$ by just iterating \mathcal{E} :

Algorithm 5.4. (Decide by type elimination whether ϕ_0 is satisfiable over ψ)

- (1) Set $S := \mathcal{T}(\psi)$.
- (2) Compute $S' = \mathcal{E}(S)$; if $S' \neq S$ then put $S := S'$ and repeat.
- (3) Return 'yes' if $\phi_0 \in \Gamma$ for some $\Gamma \in S$, and 'no' otherwise.

The run time analysis is straightforward:

LEMMA 5.5. *If the strict one-step satisfiability problem of Λ is in EXPTIME, then Algorithm 5.4 has at most exponential run time.*

PROOF. Since $\mathcal{T}(\psi)$ has at most exponential size, the algorithm runs through at most exponentially many iterations. In a single iteration, we have to compute $\mathcal{E}(S)$, checking for each of the at most exponentially many $\Gamma \in S$ whether (ϕ_Γ, η_S) is one-step satisfiable. The assumption of the lemma guarantees that each one-step satisfiability check takes only exponential time, as ϕ_Γ is of linear size. \square

It remains to prove correctness of the algorithm; that is, we show that, as announced above, $\nu\mathcal{E}$ consists precisely of the ψ -satisfiable types. We split this claim into two inclusions, corresponding to soundness and completeness, respectively:

LEMMA 5.6. *The set of ψ -satisfiable types is a postfixpoint of \mathcal{E} .*

(Since $\nu\mathcal{E}$ is also the greatest postfixpoint of \mathcal{E} , this implies that $\nu\mathcal{E}$ contains all ψ -satisfiable types. This means that Algorithm 5.4 is *sound*, i.e. answers ‘yes’ on ψ -satisfiable formulae.)

PROOF. Let R be the set of ψ -satisfiable types; we have to show that $R \subseteq \mathcal{E}(R)$. So let $\Gamma \in R$; then we have a state x in a ψ -model $C = (X, \gamma)$ such that $x \models_C \Gamma$. By definition of \mathcal{E} , we have to show that the one-step pair (ϕ_Γ, η_R) is one-step satisfiable. We claim that the one-step model $M = (X, \tau, \xi(x))$, where τ is defined by

$$\tau(a_{\heartsuit\rho}) := \llbracket \sigma_\Sigma(a_{\heartsuit\rho}) \rrbracket_C = \llbracket \rho \rrbracket_C$$

for $a_{\heartsuit\rho} \in V_\Sigma$, satisfies (ϕ_Γ, η_R) . For the propositional part η_R , let $y \in X$; we have to show $y \in \tau(\eta_R)$. Put $\Delta = \{\rho \in \Sigma \mid y \models \rho\}$. Then $\Delta \in R$, so that η_R contains the conjunctive clause

$$\vartheta := \bigwedge_{\heartsuit\rho \in \Sigma \mid \rho \in \Delta} a_{\heartsuit\rho} \wedge \bigwedge_{\heartsuit\rho \in \Sigma \mid \rho \notin \Delta} \neg a_{\heartsuit\rho}.$$

By the definitions of τ and ϑ , we have $y \in \tau(\vartheta) \subseteq \tau(\eta_R)$, as required (e.g. if $\heartsuit\rho \in \Sigma$ and $\rho \in \Delta$, then $y \models \rho$, i.e. $y \in \llbracket \rho \rrbracket_C = \tau(a_{\heartsuit\rho})$; the negative case is similar). Finally, for ψ_Γ , let $\heartsuit\rho \in \Sigma$; we have to show that $\heartsuit\rho \in \Gamma$ iff $\xi(x) \in \llbracket \heartsuit \rrbracket(\tau(a_{\heartsuit\rho})) = \llbracket \heartsuit \rrbracket(\llbracket \rho \rrbracket_C)$. But the latter just means that $x \models \heartsuit\rho$, so the equivalence holds because $x \models \Gamma$. \square

For the converse inclusion, i.e. completeness, we show the following (combining the usual existence and truth lemmas):

LEMMA 5.7. *Let S be a postfixpoint of \mathcal{E} . Then there exists a T -coalgebra $C = (S, \gamma)$ such that for each $\rho \in \Sigma$, $\llbracket \rho \rrbracket_C = \hat{\rho} \cap S$.*

PROOF. To construct the transition structure γ , let $\Gamma \in S$. Since S is a postfixpoint of \mathcal{E} , the one-step pair (ϕ_Γ, η_S) is satisfiable; let (X, τ, t) be a one-step model of (ϕ_Γ, η_S) . By construction of η_S , we then have a map $f : X \rightarrow S$ such that for all $\heartsuit\rho \in \Sigma$,

$$x \in \tau(a_{\heartsuit\rho}) \quad \text{iff} \quad \rho \in f(x) \quad \text{iff} \quad f(x) \in \hat{\rho}. \quad (6)$$

We put $\gamma(\Gamma) = Tf(t) \in TS$. For the T -coalgebra $C = (S, \gamma)$ thus obtained, we show the claim $\llbracket \rho \rrbracket_C = \hat{\rho} \cap S$ by induction over $\rho \in \Sigma$. The propositional cases are by the defining properties of

types (Definition 5.2). For the modal case, we have (for Γ and associated data f, t as above)

$$\begin{aligned}
\Gamma \models \heartsuit \rho &\iff \gamma(\Gamma) = Tf(t) \in \llbracket \heartsuit \rrbracket_S(\llbracket \rho \rrbracket_C) \\
&\iff t \in \llbracket \heartsuit \rrbracket_X(f^{-1}[\llbracket \rho \rrbracket_C]) && \text{(naturality)} \\
&= \llbracket \heartsuit \rrbracket_X(f^{-1}[\hat{\rho} \cap S]) && \text{(induction)} \\
&= \llbracket \heartsuit \rrbracket_X(\tau(a_{\heartsuit \rho})) && (6) \\
&\iff \heartsuit \rho \in \Gamma && \text{(definition of } \phi_\Gamma) \quad \square
\end{aligned}$$

A T -coalgebra as in Lemma 5.7 is clearly a ψ -model, so the above lemma implies that every postfix-point of \mathcal{E} , including $\nu\mathcal{E}$, consists only of ψ -satisfiable types. That is, that Algorithm 5.4 is indeed complete, i.e. answers ‘yes’ *only* on ψ -satisfiable formulae. This completes the correctness proof of Algorithm 5.4; in combination with the run time analysis (Lemma 5.5) we thus obtain

THEOREM 5.8 (COMPLEXITY OF SATISFIABILITY UNDER GLOBAL ASSUMPTIONS). *If the strict one-step satisfiability problem of the logic Λ is in EXPTIME, then satisfiability under global assumptions in Λ is in EXPTIME.*

Example 5.9. By the results of the previous section (Example 4.14) and by inheriting lower bounds from reasoning with global assumptions in K [19], we obtain that reasoning with global assumptions in Presburger modal logic and in probabilistic modal logic with polynomial inequalities is EXPTIME-complete. We note additionally that the same holds also for our separating example, probabilistic modal logic with linear inequalities and fixed-probability independence operators (which does not have the one-step small model property but whose strict one-step satisfiability problem is nevertheless in EXPTIME).

6 GLOBAL CACHING

We now develop the type elimination algorithm from the preceding section into a global caching algorithm. Roughly speaking, global caching algorithms perform *expansion* steps, in which new nodes to be explored are added to the tableau, and *propagation* steps, in which the satisfiability (or unsatisfiability) is determined for those nodes for which the tableau already contains enough information to allow this. The practical efficiency of global caching algorithms is based on the fact that the algorithm can stop as soon as the root node is marked satisfiable or unsatisfiable in a propagation step, thus potentially avoiding generation of all (exponentially many) possible nodes. Existing global caching algorithms work with systems of tableau rules (satisfiability is guaranteed if every applicable rule has at least one satisfiable conclusion) [22]. The fact that we work with a semantics-based decision procedure impacts on the design of the algorithm in two ways:

- In a tableaux setting, node generation in the expansion steps is driven by the tableau rules, and a global caching algorithm generates modal successor nodes by applying tableau rules. In principle, however, modal successor nodes can be generated at will, with the rules just pointing to relevant nodes. In our setting, we make the relevant nodes explicit using the concept of *children*.
- The rules govern the propagation of satisfiability and unsatisfiability among the nodes. Semantic propagation of satisfiability is straightforward, but propagation of unsatisfiability again needs the concept of children: a (modal) node can only be marked as unsatisfiable once all its children have been generated (and too many of them are unsatisfiable).

We continue to work with a closed set Σ as in Section 5 (generated by the global assumption ψ and the target formula ϕ_0) but replace types with (*tableau*) *sequents*, i.e. arbitrary subsets $\Gamma, \Theta \subseteq \Sigma$, understood conjunctively; in particular, a sequent need not determine the truth of every formula

in Σ . We write $\text{Seqs} = \mathcal{P}\Sigma$, and occasionally refer to sequents as *nodes* in allusion to an implicit graphical structure (made more explicit in Section 7). A *state* is a sequent consisting of modal literals only (recall that we regard propositional atoms as nullary modalities; so if propositional atoms in this sense are part of the logic, then states may also contain propositional atoms or their negations). We denote the set of states by States .

To convert sequents into states, we employ the usual *propositional rules*

$$\frac{\Gamma, \phi_1 \wedge \phi_2}{\Gamma, \phi_1, \phi_2} \quad \frac{\Gamma, \neg(\phi_1 \wedge \phi_2)}{\Gamma, \neg\phi_1 \mid \Gamma, \neg\phi_2} \quad \frac{\Gamma, \neg\neg\phi}{\Gamma, \phi} \quad \frac{\Gamma, \perp}{}$$

where \mid separates alternative conclusions (and the last rule has no conclusion).

Remark 6.1. Completeness of the global caching algorithm will imply that the usual clash rule $\Gamma, \phi, \neg\phi/$ (a rule with no conclusions, like the rule for \perp above) is admissible. Notice that in logics featuring propositional atoms p , i.e. nullary modalities, the atomic clash rule $\Gamma, p, \neg p/$ would be considered a modal rule.

As indicated above, the expansion steps of the algorithm will be driven by the following child relation on tableau sequents:

Definition 6.2. The *children* of a state Γ are the sequents consisting of ψ and, for each modal literal $\epsilon\heartsuit\phi \in \Gamma$, a choice of either ϕ or $\neg\phi$. The *children* of a non-state sequent are its conclusions under the propositional rules. In both cases, we write $\text{ch}(\Gamma)$ for the set of children of Γ .

For purposes of the global caching algorithm, we modify the functional \mathcal{E} defined in Section 5 to work also with sequents (rather than only types) and to depend on a set $G \subseteq \text{Seqs}$ of sequents already generated. To this end, we introduce for each state $\Gamma \in G$ a set V_Γ containing a propositional variable $a_{\epsilon\heartsuit\rho}$ for each modal literal $\epsilon\heartsuit\rho \in \Gamma$, as well as a substitution σ_Γ on V_Γ defined by $\sigma_\Gamma(a_{\epsilon\heartsuit\rho}) = \rho$. Given $S \subseteq G$, we then define a one-step pair (ϕ_Γ, η_S) over V_Γ similarly as in Section 5: We take ϕ_Γ to be the conjunction of all modal literals $\epsilon\heartsuit a_{\epsilon\heartsuit\rho}$ over V_Γ such that $\epsilon\heartsuit\sigma_\Gamma(a_{\epsilon\heartsuit\rho}) = \epsilon\heartsuit\rho \in \Gamma$ (we need to index $a_{\epsilon\heartsuit\rho}$ over $\epsilon\heartsuit\rho$ instead of just $\heartsuit\rho$ to ensure that ϕ_Γ is clean, since sequents, unlike types, may contain clashes), and η_S to be the DNF containing for each $\Delta \in S$ a conjunctive clause

$$\bigwedge_{\epsilon\heartsuit\rho \in \Gamma \mid \rho \in \Delta} a_{\epsilon\heartsuit\rho} \wedge \bigwedge_{\epsilon\heartsuit\rho \in \Gamma \mid \sim\rho \in \Delta} \neg a_{\epsilon\heartsuit\rho}.$$

We now define a functional

$$\mathcal{E}_G: \mathcal{P}G \rightarrow \mathcal{P}G$$

by taking $\mathcal{E}_G(S)$ to consist of

- all non-state sequents $\Gamma \in G \setminus \text{States}$ such that $S \cap \text{ch}(\Gamma) \neq \emptyset$ (i.e. some propositional rule that applies to Γ has a conclusion that is contained in S), and
- all states $\Gamma \in G \cap \text{States}$ such that the one-step pair $(\phi_\Gamma, \eta_{S \cap \text{ch}(\Gamma)})$ is one-step satisfiable.

To propagate unsatisfiability, we introduce a second functional $\mathcal{A}_G: \mathcal{P}G \rightarrow \mathcal{P}G$, where we take $\mathcal{A}_G(S)$ to consist of

- all non-state sequents $\Gamma \in G \setminus \text{States}$ such that there is a propositional rule applying to Γ all whose conclusions are in S , and
- all states $\Gamma \in G \cap \text{States}$ such that $\text{ch}(\Gamma) \subseteq G$ and the one-step pair $(\phi_\Gamma, \eta_{\text{ch}(\Gamma) \setminus S})$ is one-step unsatisfiable.

Both \mathcal{E}_G and \mathcal{A}_G are clearly monotone. We note additionally that they also depend monotonically on G :

LEMMA 6.3. *Let $G \subseteq G' \subseteq \text{Seqs}$. Then*

- (1) $\mathcal{E}_G(S) \subseteq \mathcal{E}_{G'}(S)$ and $\mathcal{A}_G(S) \subseteq \mathcal{A}_{G'}(S)$ for all $S \in \mathcal{P}G$;
- (2) $\nu\mathcal{E}_G \subseteq \nu\mathcal{E}_{G'}$ and $\mu\mathcal{A}_G \subseteq \mu\mathcal{A}_{G'}$.

PROOF. Claim (1) is immediate from the definitions (for \mathcal{A}_G , this hinges on the condition $\text{ch}(\Gamma) \subseteq G$ for states Γ); we show Claim (2). For \mathcal{E}_G , it suffices to show that $\nu\mathcal{E}_G$ is a postfixpoint of $\mathcal{E}_{G'}$. Indeed, by (1), we have $\nu\mathcal{E}_G = \mathcal{E}_G(\nu\mathcal{E}_G) \subseteq \mathcal{E}_{G'}(\nu\mathcal{E}_G)$. For \mathcal{A}_G , we show that $G \cap \mu\mathcal{A}_{G'}$ is a prefixpoint of \mathcal{A}_G . Indeed, by (1), we have $\mathcal{A}_G(\mu\mathcal{A}_{G'} \cap G) \subseteq \mathcal{A}_{G'}(\mu\mathcal{A}_{G'} \cap G) \subseteq \mathcal{A}_{G'}(\mu\mathcal{A}_{G'}) = \mu\mathcal{A}_{G'}$, and $\mathcal{A}_G(\mu\mathcal{A}_{G'} \cap G) \subseteq G$ by the definition of \mathcal{A}_G . \square

Remark 6.4. The reader will note that the functionals \mathcal{A}_G and \mathcal{E}_G fail to be mutually dual, as \mathcal{E}_G quantifies existentially instead of universally over propositional rules. We will show that the well-known commutation of the propositional rules implies that the more permissive use of existential quantification eventually leads to the same answers (see proof of Lemma 6.7.(5)); it allows for more economy in the generation of new nodes in the global caching algorithm, described next.

The global caching algorithm maintains, as global variables, a set G of sequents with subsets E and A of sequents already decided as satisfiable or unsatisfiable, respectively.

Algorithm 6.5. (Decide ψ -satisfiability of ϕ_0 by global caching.)

- (1) Initialize $G = \{\Gamma_0\}$ with $\Gamma_0 = \{\phi_0, \psi\}$, and $E = A = \emptyset$.
- (2) (Expand) Select a sequent $\Gamma \in G$ that has children that are not in G , and add any number of these children to G . If no sequents with missing children are found, go to Step 5
- (3) (Propagate) Optionally recalculate E as the greatest fixed point $\nu S. \mathcal{E}_G(S \cup E)$, and A as $\mu S. \mathcal{A}_G(S \cup A)$. If $\Gamma_0 \in E$, return ‘yes’; if $\Gamma_0 \in A$, return ‘no’.
- (4) Go to Step 2.
- (5) Recalculate E as $\nu S. \mathcal{E}_G(S \cup E)$; return ‘yes’ if $\Gamma_0 \in E$, and ‘no’ otherwise.

Remark 6.6. As explained at the beginning of the section, the key feature of the global caching algorithm is that it potentially avoids generating the full exponential-sized set of tableau sequents by detecting satisfiability or unsatisfiability on the fly in the intermediate optional propagation steps. The non-determinism in the formulation of the algorithm can be resolved arbitrarily, i.e. we will see that any choice (e.g. of which sequents to add in the expansion step and whether or not to trigger propagation) leads to correct results; thus, it affords room for heuristic optimization. Detecting *unsatisfiability* in Step 3 requires previous generation of all, in principle exponentially many, children of a sequent. This is presumably not necessarily prohibitive in practice, as the exponential dependence is only in the number of *top-level* modalities in a sequent. As an extreme example, if we encode the graded modality $\diamond_0\phi$ as $\sharp(\phi) > 0$ in Presburger modal logic, then the sequent $\{\diamond_0^n \top\}$ (n successive diamonds) induces 2^n types but has only two children, $\{\diamond_0^{n-1} \top\}$ and $\{\neg \diamond_0^{n-1} \top\}$.

We next prove correctness of the algorithm. As a first step, we show that a sequent can be added to E (or to A) in the optional Step 3 of the algorithm only if it will at any rate end up in E (or outside E , respectively) in the final step of the algorithm. To this end, let G_f denote the least set of sequents such that $\Gamma_0 \in G_f$ and G_f contains all children of nodes contained in G_f , i.e. $\text{ch}(\Gamma) \subseteq G_f$ for each $\Gamma \in G_f$; that is, at the end of a run of the algorithm without intermediate propagation steps, we have $G = G_f$ and $E = \nu S. \mathcal{E}_{G_f}(S)$. We then formulate the claim in the following invariants:

LEMMA 6.7. *At any stage throughout a run of Algorithm 6.5 we have*

- (1) $E \subseteq \nu S. \mathcal{E}_{G_f}(S)$
- (2) $A \subseteq \mu S. \mathcal{A}_{G_f}(S)$
- (3) $E \subseteq \nu S. \mathcal{E}_{G_f}(S)$

- (4) $A \subseteq \mu S. \mathcal{A}_{G_f}(S)$
 (5) $A \cap \nu S. \mathcal{E}_{G_f}(S) = \mu S. \mathcal{A}_{G_f}(S) \cap \nu S. \mathcal{E}_{G_f}(S) = \emptyset$.

In the proof, we use the following simple fixpoint laws (for which no novelty is claimed):

LEMMA 6.8. *Let X be a set, and let $F : \mathcal{P}X \rightarrow \mathcal{P}X$ be monotone w.r.t. set inclusion. Then*

$$\nu S. F(S \cup \nu S. F(S)) = \nu S. F(S) \quad \text{and} \quad \mu S. F(S \cup \mu S. F(S)) = \mu S. F(S).$$

PROOF. In both claims, ‘ \supseteq ’ is trivial; we show ‘ \subseteq ’. For ν , we show (already using ‘ \supseteq ’) that the left-hand side is a fixpoint of F :

$$\begin{aligned} & \nu S. F(S \cup \nu S. F(S)) \\ &= F((\nu S. F(S \cup \nu S. F(S))) \cup (\nu S. F(S))) && \text{(fixpoint unfolding)} \\ &= F(\nu S. F(S \cup \nu S. F(S))) && (\nu S. F(S \cup \nu S. F(S)) \supseteq \nu S. F(S)). \end{aligned}$$

For μ , we show that the right-hand side is a fixpoint of $S \mapsto F(S \cup \mu S. F(S))$:

$$F(\mu S. F(S) \cup \mu S. F(S)) = F(\mu S. F(S)) = \mu S. F(S). \quad \square$$

PROOF (LEMMA 6.7). (1) and (2): Clearly, these invariants hold *initially*, as E and A are initialized to \emptyset .

In *expansion steps*, the invariants are preserved because by Lemma 6.3, $\nu S. \mathcal{E}_G(S)$ and $\mu S. \mathcal{A}_G(S)$ depend monotonically on G .

Finally, in a *propagation step*, we change E into

$$E' = \nu S. \mathcal{E}_G(S \cup E) \subseteq \nu S. \mathcal{E}_G(S \cup \nu S. \mathcal{E}_G(S)) = \nu S. \mathcal{E}_G(S),$$

where the inclusion is by the invariant for E and the equality is by Lemma 6.8. Thus, the invariant (1) is preserved. Similarly, A is changed into

$$A' = \mu S. \mathcal{A}_G(S \cup A) \subseteq \mu S. \mathcal{A}_G(S \cup \mu S. \mathcal{A}_G(S)) = \mu S. \mathcal{A}_G(S)$$

where the equality is by Lemma 6.8, preserving invariant (2).

(3) and (4): Immediate from (1) and (2) by Lemma 6.3, since $G \subseteq G_f$ at all stages.

(5): Let $\overline{\mathcal{A}_{G_f}}$ denote the dual of \mathcal{A}_{G_f} , i.e. $\overline{\mathcal{A}_{G_f}}(S) = G_f \setminus \mathcal{A}_{G_f}(G_f \setminus S)$; that is, $\overline{\mathcal{A}_{G_f}}$ is defined like \mathcal{E}_{G_f} except that $\overline{\mathcal{A}_{G_f}}(S)$ contains a non-state sequent $\Gamma \in G_f \setminus \text{States}$ if *every* propositional rule that applies to Γ has a conclusion that is contained in S (cf. Remark 6.4). Then $\nu S. \overline{\mathcal{A}_{G_f}}(S)$ is the complement of $\mu S. \mathcal{A}_{G_f}(S)$, so by (4) it suffices to show $\nu S. \mathcal{E}_{G_f}(S) \subseteq \nu S. \overline{\mathcal{A}_{G_f}}(S)$. To this end, we show that $\nu S. \mathcal{E}_{G_f}(S)$ is a postfixpoint of $\overline{\mathcal{A}_{G_f}}$. So let $\Gamma \in \nu S. \mathcal{E}_{G_f}(S) = \mathcal{E}_{G_f}(\nu S. \mathcal{E}_{G_f}(S))$. If Γ is a state, then it follows immediately that $\Gamma \in \overline{\mathcal{A}_{G_f}}(\nu S. \mathcal{E}_{G_f}(S))$, since the definitions of \mathcal{E}_{G_f} and $\overline{\mathcal{A}_{G_f}}$ agree on containment of states (note that by definition of G_f , $\text{ch}(\Gamma) \subseteq G_f$ for every $\Gamma \in G_f$). Otherwise, we proceed by induction on the size of Γ . By definition of \mathcal{E}_{G_f} , there exists a conclusion $\Gamma' \in \nu S. \mathcal{E}_{G_f}(S)$ of a propositional rule R applied to Γ . By induction, $\Gamma' \in \overline{\mathcal{A}_{G_f}}(\nu S. \mathcal{E}_{G_f}(S))$. Now let Δ be the set of conclusions of a propositional rule R' applied to Γ , w.l.o.g. distinct from R . Since the propositional rules commute, there is a rule application to Γ' (corresponding to a postponed application of R') that has a conclusion $\Gamma'' \in \nu S. \mathcal{E}_{G_f}(S)$ such that Γ'' is, via postponed application of R , a conclusion of a propositional rule applied to some $\Gamma''' \in \Delta$. Then, $\Gamma''' \in \mathcal{E}_{G_f}(\nu S. \mathcal{E}_{G_f}(S)) = \nu S. \mathcal{E}_{G_f}(S)$ by definition of \mathcal{E}_{G_f} , showing $\Gamma \in \overline{\mathcal{A}_{G_f}}(\nu S. \mathcal{E}_{G_f}(S))$ as required. \square

Invariants (3) and (5) in Lemma 6.7 imply that once we prove correctness for runs of the algorithm that perform propagation only in the last step 5 (that is, once all children have been added), correctness of the general algorithm follows. That is, it remains to show that $vS. \mathcal{E}_{G_f}(S)$ consists precisely of the satisfiable sequents in G_f . We split this claim into two inclusions respectively corresponding to soundness and completeness in the same way as for the type elimination algorithm (Section 5). The following statement is analogous to Lemma 5.7.

LEMMA 6.9. *Let E be a postfixpoint of \mathcal{E}_{G_f} and denote by $E_s = E \cap \text{States}$ the collection of states contained in E . Then there is a coalgebra $C = (E_s, \gamma)$ such that $E_s \cap \{\Gamma \mid \Gamma \vdash_{PL} \phi\} \subseteq \llbracket \phi \rrbracket_C$ for all $\phi \in \Sigma$ (recall that \vdash_{PL} denotes propositional entailment, see Definition 4.1). Consequently, whenever $\Gamma \in E$ and $\phi \in \Gamma$, then ϕ is ψ -satisfiable.*

PROOF. The proof proceeds similarly to the one of Lemma 5.7: In order to define a suitable γ , let $\Gamma \in E_s$. By the definition of \mathcal{E}_{G_f} , the one-step pair $(\phi_\Gamma, \eta_{E \cap \text{ch}(\Gamma)})$ is satisfiable. Let $M = (X, \tau, t)$ be a one-step model satisfying $(\phi_\Gamma, \eta_{E \cap \text{ch}(\Gamma)})$. By the definition of $\eta_{E \cap \text{ch}(\Gamma)}$, we can then define a function $f: X \rightarrow E \cap \text{ch}(\Gamma)$ such that for all $x \in X$ and all $\epsilon \heartsuit \rho \in \Gamma$ we have $\rho \in f(x)$ iff $x \in \tau(a_{\epsilon \heartsuit \rho})$ (noting that by the definition of children of Γ , $f(x)$ contains either ρ or $\neg \rho$). Now note that since E is a postfixpoint of \mathcal{E}_{G_f} , every non-state sequent $\Delta \in E$ has a child in E that is a conclusion of a propositional rule applied to Δ , and hence propositionally entails Δ . Since every propositional rule removes a propositional connective, this implies that we eventually reach a state in E_s from Δ along the child relation; that is, for every $\Delta \in E$ there is a state $\Delta' \in E_s$ such that Δ' propositionally entails Δ . We can thus prolong f to a function $\bar{f}: X \rightarrow E_s$ such that

$$\bar{f}(x) \vdash_{PL} \rho \quad \text{iff} \quad x \in \tau(a_{\epsilon \heartsuit \rho}) \quad (7)$$

for all $\epsilon \heartsuit \rho \in \Gamma$ and all $x \in X$. We now define $\gamma(\Gamma) := T\bar{f}(t)$, obtaining $\gamma: E_s \rightarrow TE_s$. We will show that

$$\Gamma \vdash_{PL} \chi \quad \text{implies} \quad \Gamma \in \llbracket \chi \rrbracket_C \quad (8)$$

for all $\chi \in \Sigma$ and all $\Gamma \in E_s$, which implies the first claim of the lemma. We proceed by induction on χ ; by soundness of propositional reasoning, we immediately reduce to the case where $\chi \in \Gamma$, in which case χ has the form $\chi = \epsilon \heartsuit \rho$ since Γ is a state. We continue to use the data $M = (X, \tau, t)$, f, \bar{f} featuring in the above construction of $\gamma(\Gamma) = T\bar{f}(t)$. Note again that for every $x \in X$, we have by the defining property of children of Γ that either $f(x) \vdash_{PL} \rho$ or $f(x) \vdash_{PL} \neg \rho$; since the conclusions of propositional rules are propositionally stronger than the premisses, it follows that the same holds for $\bar{f}(x)$. The inductive hypothesis therefore implies that $\bar{f}(x) \in \llbracket \rho \rrbracket_C$ iff $\bar{f}(x) \vdash_{PL} \rho$; combining this with (7), we obtain $f^{-1}[\llbracket \rho \rrbracket_C] = \tau(a_{\epsilon \heartsuit \rho})$. To simplify notation, assume that $\epsilon = 1$ (the case where $\epsilon = -1$ being entirely analogous). We then have to show $\gamma(\Gamma) \in \llbracket \heartsuit \rho \rrbracket_{E_s}(\llbracket \rho \rrbracket_C)$, which by naturality of $\llbracket \heartsuit \rrbracket$ is equivalent to $t \in \llbracket \heartsuit \rrbracket_X(f^{-1}[\llbracket \rho \rrbracket_C]) = \llbracket \heartsuit \rrbracket_X(\tau(a_{\heartsuit \rho}))$, where the equality is by the preceding calculation. But $t \in \llbracket \heartsuit \rrbracket_X(\tau(a_{\heartsuit \rho}))$ follows from $M \models (\phi_\Gamma, \eta_{E \cap \text{ch}(\Gamma)})$ and $\heartsuit \rho \in \Gamma$ by the definition of ϕ_Γ .

The second claim of the lemma is now immediate for states $\Gamma \in E_s$. As indicated above, all other sequents $\Gamma \in E \setminus E_s$ can be transformed into some $\Gamma' \in E_s$ using the propositional rules, in which case Γ' propositionally entails all $\rho \in \Gamma$; thus, satisfiability of Γ' implies satisfiability of all $\rho \in \Gamma$. \square

Lemma 6.9 ensures completeness of the algorithm, i.e. whenever the algorithm terminates with 'yes', then ϕ_0 is ψ -satisfiable. For soundness (i.e. the converse implication, the algorithm answers 'yes' if ϕ_0 is ψ -satisfiable) we proceed similarly as for Lemma 5.6:

LEMMA 6.10. *The set of ψ -satisfiable sequents contained in G_f is a post-fixpoint of \mathcal{E}_{G_f} .*

PROOF. Let S be the set of ψ -satisfiable sequents in G_f . We have to show that $S \subseteq \mathcal{E}_{G_f}(S)$; so let $\Gamma \in S$. If Γ is not a state, then to show $\Gamma \in \mathcal{E}_{G_f}(S)$ we have to check that some propositional rule that applies to Γ has a ψ -satisfiable conclusion that is moreover contained in G_f ; this is easily verified by inspection of the rules, noting that all children of Γ are in G_f . Now suppose that Γ is a state; we then have to show that the one-step pair $(\phi_\Gamma, \eta_{S \cap \text{ch}(\Gamma)})$ is one-step satisfiable. Let x be a state in a ψ -model $C = (X, \gamma)$ such that $x \models_C \Gamma$. We construct a one-step model of $(\phi_\Gamma, \eta_{S \cap \text{ch}(\Gamma)})$ from C in the same way as in the proof of Lemma 5.6. The only point to note additionally is that for every $y \in X$, we have some $\Delta \in S \cap \text{ch}(\Gamma)$ such that $y \models_C \Delta$, namely $\Delta = \{\epsilon\rho \mid \epsilon' \heartsuit \rho \in \Gamma, y \models_C \epsilon\rho\}$ (where ϵ and ϵ' range over $\{-1, 1\}$). \square

Summing up, we have

THEOREM 6.11. *If the strict one-step satisfiability problem of Λ is in EXPTIME, then the global caching algorithm decides satisfiability under global assumptions in exponential time.*

PROOF. Correctness is by Lemma 6.10 and Lemma 6.9, taking into account the reduction to runs without intermediate propagation according to Lemma 6.7. It remains to analyse run time; this point is similar as in Lemma 5.5: There are at only exponentially many sequents, so there can be at most exponentially many expansion steps, and the fixpoint calculations in the propagation steps run through at most exponentially many iterations. The run time analysis of a single fixpoint iteration step is essentially the same as in Lemma 5.5, using that strict one-step satisfiability is in EXPTIME for state sequents; for non-state sequents Γ just note that there are only polynomially many conclusions of propositional rules arising from Γ , which need to be compared with at most exponentially many existing nodes. \square

7 CONCRETE ALGORITHM

In the following we provide a more concrete description of the global caching algorithm, which does not use the computation of least and greatest fixpoints as primitive operators. The algorithm closely follows Liu and Smolka's well-known algorithm for fixpoint computation in what the authors call "dependency graphs" [34]; in our case, these structures are generated by the derivation rules. The main difference between the algorithm described below and Liu and Smolka's is caused by the treatment of "modal" sequents, i.e. states, as the condition that these sequents need to satisfy is not expressible purely as a reachability property.

As in the previous section we work with a closed set Σ (generated by the global assumption ψ and the target formula ϕ_0) and (*tableau*) *sequents*, i.e. arbitrary subsets $\Gamma, \Theta \subseteq \Sigma$, understood conjunctively. We continue to write $\text{Seqs} = \mathcal{P}\Sigma$ for the set of sequents, and States for the set of states, i.e. sequents consisting of modal literals only (recall that we take propositional atoms as nullary operators).

The set Seqs of sequents carries a hypergraph structure $E \subseteq \text{Seqs} \times \mathcal{P}(\text{Seqs})$ that contains

- for each $\Gamma \in \text{States}$ the pair $(\Gamma, \text{ch}(\Gamma))$ (recall that $\text{ch}(\Gamma) \subseteq \text{Seqs}$ denotes the set of children of Γ); and
- for each $\Gamma \in \text{Seqs} \setminus \text{States}$ the set of pairs $\{(\Gamma, \Delta) \mid \Gamma/\Delta \text{ a propositional rule applicable to } \Gamma\}$.

In the following we write E_M for the "modal" part of E induced by the state-child relationships as per the first bullet point, and E_P for the part of E induced by the propositional rules as per the second bullet point (so E is the disjoint union of E_m and E_p).

Our algorithm maintains a *partial* function $\alpha : \text{Seqs} \rightarrow \{0, 1\}$ that maps a sequent to 0 if it is not ψ -satisfiable, to 1 if it is ψ -satisfiable and is undefined in case its satisfiability cannot be determined yet. In the terminology of the previous section α should have the following properties:

- $\alpha(\Gamma) = 1$ iff $\Gamma \in \nu X. \mathcal{E}_G(X)$ and

- $\alpha(\Gamma) = 0$ iff $\Gamma \in \mu X. \mathcal{A}_G(X)$

where G denotes the set of sequents for which α is defined. The idea of computing a partial function is that this allows determining ψ -satisfiability of a given sequent without exploring the full hypergraph. We will now describe an algorithm for computing α that is inspired by Liu and Smolka's *local* algorithm [34, Figures 3,4] and then show its correctness.

Algorithm 7.1. Concrete Global Caching

Initialize α to be undefined everywhere;

$\alpha(\Gamma_0) := 1; D(\Gamma_0) = \emptyset, W := \{(\Gamma_0, \Delta) \mid (\Gamma_0, \Delta) \in E\};$

while $W \neq \emptyset$ **do**

 Pick $e = (\Gamma, \Delta) \in W;$

$W := W - \{e\};$

if $\exists \Gamma' \in \Delta. (\alpha(\Gamma') \text{ is undefined})$ **then**

\triangleright Expansion step

 Pick non-empty $U \subseteq \{\Gamma' \in \Delta \mid \alpha(\Gamma') \text{ undefined}\};$

 For each $\Gamma' \in U$ put $\alpha(\Gamma') := 1, D(\Gamma') := \emptyset, W = W \cup \{(\Gamma', \Delta') \mid (\Gamma', \Delta') \in E\};$

if $e \in E_p$ **then**

\triangleright Propagation step

if $\forall \Gamma' \in \Delta. \alpha(\Gamma') = 0$ **then**

\triangleright Case $\Gamma \notin \text{States}$

$\alpha(\Gamma) := 0; W := W \cup D(\Gamma); D(\Gamma) := \emptyset;$

else if $\exists \Gamma' \in \Delta. \alpha(\Gamma') = 1$ **then**

 pick $\Gamma' \in \Delta$ s.t. $\alpha(\Gamma') = 1$ and put $D(\Gamma') := D(\Gamma') \cup \{(\Gamma, \Delta)\};$

$W := W - \{(\Gamma'', \Delta'') \in W \mid \Gamma'' == \Gamma\};$

else if $e \in E_M$ **then**

\triangleright Propagation step

$S_0 := \{\Gamma' \in \Delta \mid \alpha(\Gamma') == 0\}; S_1 := \{\Gamma' \in \Delta \mid \alpha(\Gamma') == 1\}$

\triangleright Case $\Gamma \in \text{States}$

if $\Delta == S_0 \cup S_1$ and $(\phi_\Gamma, \eta_{S_1})$ is not one-step satisfiable **then**

$\alpha(\Gamma) := 0; W := W \cup D(\Gamma); D(\Gamma) := \emptyset;$

else if $(\phi_\Gamma, \eta_{S_1})$ is one-step satisfiable **then**

for $\Gamma' \in S_1$ **do** $D(\Gamma') := D(\Gamma') \cup \{(\Gamma, \Delta)\};$

else if $\Delta \neq S_0 \cup S_1$ **then** $W := W \cup \{e\};$

Remark 7.2. In Algorithm 7.1, hyperedges should be understood as represented symbolically, i.e. either by describing matches of propositional rules or by marking a hyperedge as modal (which determines the hyperedge uniquely given the source node). This serves in particular to avoid having to create all of the exponentially many children of a state node at once. Target nodes $\Gamma' \in \Delta$ of hyperedges (Γ, Δ) are generated explicitly only once they are picked from Δ in the expansion step (the propagation step only accesses nodes that are already generated).

We proceed to show correctness of Algorithm 7.1 and establish a precise connection to our global caching algorithm. First we need a couple of lemmas that establish key invariants of the algorithm. Note that the current state of a run of the algorithm can be characterized by the triple (α, D, W) where α is the current (partial) labelling of sequents, D assigns to any given sequent Γ a set of hyperedges that need to be investigated if the α -value of Γ changes, and W contains the set of hyperedges that the algorithm still has to check. The algorithm terminates when it reaches a state of the form (α, D, \emptyset) , i.e. when there are no edges left to be checked. Given a state $s = (\alpha, D, W)$ of the algorithm, we put $G_i^s := \{\Gamma \in \text{Seqs} \mid \alpha(\Gamma) = i\}$ for $i = 0, 1$, and $G^s = G_0^s \cup G_1^s$ (so G^s is the domain of definition of α).

LEMMA 7.3. *Let $\Gamma \in \text{Seqs}$ and suppose $s = (\alpha, D, W)$ is a state reached during execution of the algorithm. Then $\alpha(\Gamma) = 0$ implies that $\Gamma \in \mu X. \mathcal{A}_{G^s}(X)$ and therefore, by Lemma 6.7(5) and Lemma 6.10, the sequent Γ is not ψ -satisfiable.*

PROOF. First note that once $\alpha(\Gamma) = 0$ for some sequent Γ , the value $\alpha(\Gamma)$ will not change any more throughout the run of the algorithm, as the only moment when a sequent Γ is assigned value 1 is when Γ is newly added to the domain of α . Since G_s can only grow during a run of the algorithm and by Lemma 6.3, $\Gamma \in \mu X. \mathcal{A}_{G^s}(X)$ depends monotonically on G_s , it suffices to establish the invariant for the point where $\alpha(\Gamma)$ is set to 0. So suppose that this happens while $e = (\Gamma, \Delta)$ is processed, with the state being $s = (\alpha, D, W)$ before and $s' = (\alpha', D', W')$ after processing e . Suppose that s satisfies the claimed invariant; we have to show that s' satisfies it as well. We do this for the case where $e \in E_P$; the case $e \in E_M$ is completely analogous. Since $e \in E_P$, the reason for setting $\alpha'(\Gamma) = 0$ is that for all $\Gamma' \in \Delta$ we have $\alpha(\Gamma') = 0$ – in other words, we have $\Gamma \in \mathcal{A}_{G^s}(G_0^s)$. This implies $\Gamma \in \mathcal{A}_{G^{s'}}(G_0^s)$ by Lemma 6.3 as $G^s \subseteq G^{s'}$. By assumption on s , we have $G_0^s \subseteq \mu X. \mathcal{A}_{G^s}(X) \subseteq \mu X. \mathcal{A}_{G^{s'}}(X)$, again using Lemma 6.3 in the second step. Monotonicity of $\mathcal{A}_{G^{s'}}$ now yields

$$\Gamma \in \mathcal{A}_{G^{s'}}(G_0^s) \subseteq \mathcal{A}_{G^{s'}}(\mu X. \mathcal{A}_{G^{s'}}(X)) = \mu X. \mathcal{A}_{G^{s'}}(X)$$

as required. \square

The following technical lemma follows by inspecting the details of the algorithm:

LEMMA 7.4. *Suppose $s = (\alpha, D, W)$ is a state reached during execution of the algorithm. Then for all $\Gamma \in G_1^s$ and all $(\Gamma, \Delta) \in E$ precisely one of the following holds:*

- $(\Gamma, \Delta) \in W$ or
- $\Gamma \notin \text{States}$ and there is $(\Gamma, \Delta') \in E_P$ with $(\Gamma, \Delta') \in D(\Gamma'')$ for some $\Gamma'' \in \Delta'$ or
- $\Gamma \in \text{States}$ and (ϕ_Γ, η_S) is one-step satisfiable with $S = \{\Gamma' \in \Delta \mid (\Gamma, \Delta) \in D(\Gamma')\}$

We also note that $D(\Gamma) \neq \emptyset$ implies $\alpha(\Gamma) = 1$.

Correctness of the algorithm is established in the following theorem.

THEOREM 7.5. *When Algorithm 7.1 terminates at $s = (\alpha, D, \emptyset)$ then for all $\Gamma \in \text{Seqs}$ we have:*

- (1) $\alpha(\Gamma) = 0$ implies $\Gamma \in \mu X. \mathcal{A}_{G^s}(X)$ and thus Γ is not ψ -satisfiable.
- (2) $\alpha(\Gamma) = 1$ implies $\Gamma \in \nu X. \mathcal{E}_{G^s}(X)$ and thus Γ is ψ -satisfiable.

PROOF. The first claim is immediate by Lemma 7.3. For the second claim it suffices to prove that G_1^s is included in the greatest fixpoint of $\mathcal{E}_{G^s}(X)$ – the claim concerning ψ -satisfiability of Γ then follows from Lemmas 6.3 and 6.9 in the previous section. It suffices to show that G_1^s is a post-fixpoint of \mathcal{E}_{G^s} – but this follows immediately from Lemma 7.4 together with $W = \emptyset$ and $\{\Gamma \mid D(\Gamma) \neq \emptyset\} \subseteq G_1^s$. \square

Algorithm 7.1 is closely related to Algorithm 6.5: Both algorithms explore the collection of sequents that are “reachable” from Γ_0 , making non-deterministic choices concerning which sequents to expand next. A crucial difference to Algorithm 6.5 is that Algorithm 7.1 contains a concrete description of how to compute the fixpoints of \mathcal{E} and \mathcal{A} by successively updating the labelling function; to this end, it imposes a more definite strategy regarding propagation by enforcing a propagation step after every expansion step. We conclude by providing an estimate of the complexity of the algorithm:

PROPOSITION 7.6. *If the strict one-step satisfiability problem of Λ is in EXPTIME, then Algorithm 7.1 decides satisfiability under global assumptions in exponential time.*

PROOF. To get the upper bound, we observe first that each hyperedge $e = (\Gamma, \Delta) \in E_M$ will be checked at most $2 \cdot |\Delta|$ times by the algorithm: after e has been added to W it could be tested up to $|\Delta|$ times (in the worst case, until all of the children in Δ have been added to the domain of α) and then again each time the status of one of the children in Δ changes. Similarly, each hyperedge

$e = (\Gamma, \Delta) \in E_P$ will be checked at most $|\Delta| + 1$ times (each time when the status of one of the children changes). The EXPTIME bound then follows from the observation that (i) the hypergraph is exponential in the size of the input, (ii) for $\Gamma \in \text{States}$ there is exactly one edge $(\Gamma, \Delta) \in E_M$ and (iii) for each $\Gamma \in \text{Seqs} \setminus \text{States}$ the algorithm only verifies one hyperedge of the form $(\Gamma, \Delta) \in E_P$. \square

8 NOMINALS

A key feature of *hybrid logic* [2] as an extension of modal logic are *nominals*, which are special atomic predicates that are semantically restricted to hold in exactly one state, and hence uniquely designate a state. Nominals form part of many relational description logics (recognizable by the letter *O* in the standard naming scheme) [4], where they serve as expressive means to express facts involving specific individuals – for instance, using nominals, concepts over an ontology of music can not only speak about the notion of composer in general, but also concretely about Mozart and Stockhausen. We proceed to discuss how to extend some of the above results to cover coalgebraic hybrid logic, i.e. the extension of coalgebraic modal logic with nominals in the standard sense. Specifically, we show that the generic EXPTIME upper bound for reasoning under global assumptions (Theorem 5.8) remains true in presence of nominals; we leave the design of a global caching algorithm for this setting as an open problem (for the case where a complete set of modal tableau rules in the sense recalled in Remark 4.16 is available, we have presented such an algorithm in previous work [23]).

Syntactically, we introduce a set N of *nominals* i, j, \dots , i.e. names for individual states, and work with an extended set $\mathcal{F}(N, \Lambda)$ of *hybrid* formulae ϕ, ψ , defined by the grammar

$$\mathcal{F}(N, \Lambda) \ni \phi, \psi ::= \perp \mid \phi \wedge \psi \mid \neg\phi \mid \heartsuit(\phi_1, \dots, \phi_n) \mid i \mid @_i\phi \quad (\heartsuit \in \Lambda \text{ } n\text{-ary}, i \in N);$$

that is, nominals may be used as atomic formulae and within *satisfaction operators* $@_i$, with $@_i\phi$ stating that the state denoted by i satisfies ϕ . (We explicitly do not include local binding \downarrow , with formulae $\downarrow i. \phi$ read ‘ ϕ holds if i is changed to denote the present state’, which would lead to undecidability [1].)

Semantically, we work with *hybrid models* $\mathcal{M} = (C, \pi)$ consisting of a T -coalgebra $C = (X, \gamma)$ and an assignment of a singleton set $\pi(i) \subseteq X$ to each nominal $i \in N$. We write $\models_{\mathcal{M}}$ for the satisfaction relation between states x in hybrid models $\mathcal{M} = (C, \pi)$ and hybrid formulae, defined by

$$\begin{aligned} x \models_{\mathcal{M}} i & \quad \text{iff} \quad x \in \pi(i) \\ x \models_{\mathcal{M}} @_i\phi & \quad \text{iff} \quad y \models_{\mathcal{M}} \phi \quad \text{for the unique } y \in \pi(i), \end{aligned}$$

and otherwise the same clauses as \models_C (Section 3). Similarly as for the purely modal logic, we sometimes refer to these data just as the coalgebraic hybrid logic Λ .

Example 8.1. We illustrate how the presence of nominals impacts on logical consequence.

(1) In Presburger modal logic, the formula

$$@_i(\#(i) > \#(p)),$$

with i a nominal and p a propositional atom, says that state i has higher transition weight to itself than to states satisfying p . One consequence of this formula is

$$@_i\neg p.$$

(2) In probabilistic modal logic, the formula

$$@_i(w(j) > w(\neg j) \wedge w(k) \geq w(\neg k)),$$

with nominals i, j, k , says that from state i , we reach state j with probability strictly greater than $1/2$, and state k with probability at least $1/2$. From this, we conclude that $j = k$, i.e.

$$\textcircled{a}_j k.$$

Remark 8.2. In the presence of nominals, the equivalence of the Kripke semantics and multigraph semantics of Presburger modal logic (Lemma 3.5) breaks down: For a nominal i , the formula $\sharp(i) > 1$ is satisfiable in multigraph semantics but not in Kripke semantics. Using global assumptions, we can however encode Kripke semantics into multigraph semantics, by extending the global assumption ψ with additional conjuncts $\sharp(i) \leq 1$ for all nominals i appearing either in ψ or in the target formula ϕ_0 . We therefore continue to use multigraph semantics for Presburger hybrid logic.

Remark 8.3. As in the case of coalgebraic modal logic (Remark 3.3), satisfiability under global assumptions in coalgebraic hybrid logic is mutually reducible with plain satisfiability in an extended logic featuring the universal modality $[\forall]$, with the same syntax and semantics as in Remark 3.3. The non-trivial reduction (from the universal modality to global assumptions) works slightly differently than in the modal case, due to the fact that we cannot just take disjoint unions of hybrid models: Like before, let $[\forall] \psi_1, \dots, [\forall] \psi_n$ be the $[\forall]$ -subformulae of the target formula ϕ (now in coalgebraic hybrid logic with the universal modality), and guess a subset $U \subseteq \{1, \dots, n\}$, inducing a map $\chi \mapsto \chi[U]$ eliminating $[\forall]$ from subformulae χ of ϕ as in Remark 3.3. Then check that $\phi[U]$ is satisfiable under the global assumption

$$\psi_U = \bigwedge_{k \in U} \psi_k[U] \wedge \bigwedge_{k \in \{1, \dots, n\} \setminus U} (i_k \rightarrow \neg \psi_k[U])$$

where the i_k are fresh nominals. It is easy to see that this non-deterministic reduction is correct, i.e. that ϕ is satisfiable iff $\phi[U]$ is ψ_U -satisfiable for some U .

A consequence of Remark 8.3 is that for purposes of estimating the complexity of satisfiability under global assumptions, we can eliminate satisfaction operators: Using the universal modality $[\forall]$, we can express $\textcircled{a}_i \phi$ as $[\forall](i \rightarrow \phi)$. We will thus consider only the language without satisfaction operators in the following. For a further reduction, we say that the global assumption ψ is *globally satisfiable* if \top is ψ -satisfiable, i.e. if there exists a non-empty ψ -model. Then note that ϕ_0 is ψ -satisfiable iff $\psi \wedge (i \rightarrow \phi_0)$ is globally satisfiable for a fresh nominal i ; so we can forget about the target formula and just consider global satisfiability.

We proceed to adapt the type elimination algorithm of Section 5 to this setting. Fix a global assumption ψ to be checked for global satisfiability, and let Σ be the closure of $\{\psi\}$.

Definition 8.4. For $i \in \mathbb{N} \cap \Sigma$ and $\Gamma \in \mathcal{T}(\psi)$, we say that i has type Γ in a hybrid model (C, π) if $y \models \Gamma$ for the unique $y \in \pi(i)$.

A *type assignment* (for Σ) is a map

$$\beta: \mathbb{N} \cap \Sigma \rightarrow \mathcal{T}(\psi).$$

We say that β is *consistent* if for all $i, j \in \mathbb{N} \cap \Sigma$, we have $i \in \beta(j)$ iff $\beta(i) = \beta(j)$ (in particular, $i \in \beta(i)$ for all $i \in \mathbb{N} \cap \Sigma$). A hybrid model \mathcal{M} *satisfies* β if every $i \in \mathbb{N} \cap \Sigma$ has type $\beta(i)$ in \mathcal{M} ; β is *ψ -satisfiable* if there exists a hybrid ψ -model that satisfies β .

(In description logic terminology, we may think of type assignments as complete ABoxes.) We note the following obvious properties:

- FACT 8.5. (1) *The formula ψ is globally satisfiable iff there exists a ψ -satisfiable type assignment for Σ .*
 (2) *There are at most exponentially many type assignments for Σ .*

(3) *All satisfiable type assignments are consistent.*

(4) *Consistency of a type assignment can be checked in polynomial time.*

To obtain an upper bound EXPTIME for global satisfiability of ψ , it thus suffices to show that we can decide in EXPTIME whether a given consistent type assignment β is ψ -satisfiable. To this end, we form the set

$$\mathcal{T}(\beta, \psi) = \beta[\mathbb{N} \cap \Sigma] \cup \{\Gamma \in \mathcal{T}(\psi) \mid \Gamma \cap \mathbb{N} = \emptyset\}$$

of types – that is, $\mathcal{T}(\beta, \psi)$ includes the assigned types $\beta(i)$ for all nominals $i \in \mathbb{N} \cap \Sigma$, and moreover all types that do not specify any nominal to be locally satisfied. To check whether β is ψ -satisfiable, we then run type elimination on $\mathcal{T}(\beta, \psi)$; that is, we compute $\nu\mathcal{E}_\beta$ by fixpoint iteration starting from $\mathcal{T}(\beta, \psi)$, where

$$\begin{aligned} \mathcal{E}_\beta: \quad \mathcal{P}(\mathcal{T}(\beta, \psi)) &\rightarrow \mathcal{P}(\mathcal{T}(\beta, \psi)) \\ S &\mapsto \{\Gamma \in S \mid (\phi_\Gamma, \eta_S) \text{ is one-step satisfiable}\} \end{aligned}$$

(in analogy to the functional \mathcal{E} according to (5) as used in the type elimination algorithm for the purely modal case). We answer ‘yes’ if $\beta[\mathbb{N} \cap \Sigma] \subseteq \nu\mathcal{E}_\beta$, i.e. if no type $\beta(i)$ is eliminated, and ‘no’ otherwise.

By the same analysis as in Lemma 5.5, we see that the computation of $\nu\mathcal{E}_\beta$ runs in exponential time if the strict one-step satisfiability problem of Λ is in EXPTIME . Correctness of the algorithm is immediate from the following fact.

LEMMA 8.6. *Let β be a consistent type assignment. Then β is ψ -satisfiable iff $\beta[\mathbb{N} \cap \Sigma] \subseteq \nu\mathcal{E}_\beta$.*

PROOF. Soundness (‘only if’) follows from

$$R_\beta = \{\Gamma \in \mathcal{T}(\beta, \psi) \mid \Gamma \text{ satisfiable in a hybrid } \psi\text{-model satisfying } \beta\}$$

being a postfixpoint of \mathcal{E}_β ; the proof is completely analogous to that of Lemma 5.6.

To see completeness (‘if’), construct a T -coalgebra $C = (\nu\mathcal{E}_\beta, \gamma)$ in the same way as in the proof of Lemma 5.7. We turn C into a hybrid model $\mathcal{M} = (C, \pi)$ by putting $\pi(i) = \{\Gamma \in \nu\mathcal{E}_\beta \mid i \in \Gamma\}$, noting that $\pi(i)$ is really the singleton $\{\beta(i)\}$ because (i) β is consistent and no type in $\mathcal{T}(\beta, \psi)$ other than the $\beta(j)$ (for $j \in \mathbb{N} \cap \Sigma$) contains a nominal positively, and (ii) $\beta(i) \in \nu\mathcal{E}_\beta$ by assumption. The truth lemma

$$\llbracket \rho \rrbracket_C = \hat{\rho} \cap \nu\mathcal{E}_\beta = \{\Gamma \in \nu\mathcal{E}_\beta \mid \rho \in \Gamma\}$$

is shown by induction on $\rho \in \Sigma$. All cases are as in the proof of Lemma 5.7, except for the new case $\rho = i \in \mathbb{N}$; this case is by construction of π . The truth lemma implies that \mathcal{M} is a ψ -model and satisfies β . \square

In summary, we obtain

THEOREM 8.7. *If the strict one-step satisfiability problem of Λ is in EXPTIME , then satisfiability with global assumptions in the coalgebraic hybrid logic Λ is EXPTIME -complete.*

Remark 8.8. The EXPTIME algorithm described above is not, of course, one that one would wish to use in practice. Specifically, while the computation of $\nu\mathcal{E}_\beta$ for a given consistent type assignment can be made practical along the lines of the global caching algorithm for the nominal-free case discussed in Sections 6 and 7, the initial reductions – elimination of satisfaction operators and, more importantly, going through all consistent type assignments – will consistently incur exponential cost. We leave the design of a more practical algorithm for coalgebraic hybrid logic with global assumptions for future work. In particular, adapting the global caching algorithm described in Section 6 to this setting remains an unsolved challenge: e.g. types such as $\{i, \phi\}$ and $\{i, \neg\phi\}$, where i is a nominal and ϕ is any formula such that both ϕ and $\neg\phi$ are satisfiable, are

clearly both satisfiable but cannot both form part of a model. The generic algorithm we presented in earlier work with Goré [23] solves this problem by gathering up ABoxes along strategies in a tableau game (so that no strategy will win that uses both types mentioned above); however, the algorithm requires a complete set of tableau-style rules, which is not currently available for our two main examples.

We record the instantiation of the generic result to our key examples explicitly:

Example 8.9. Reasoning with global assumptions in *Presburger hybrid logic* and in *probabilistic hybrid logic with polynomial inequalities*, i.e. in the extensions with nominals of the corresponding modal logics as defined in Sections 3.1 and 3.2, is in EXPTIME.

9 CONCLUSIONS

We have proved a generic upper bound EXPTIME for reasoning with global assumptions in coalgebraic modal and hybrid logics, based on a semantic approach centered around *one-step satisfiability checking*. This approach is particularly suitable for logics for which no tractable sets of modal tableau rules are known; our core examples of this type are Presburger modal logic and probabilistic modal logic with polynomial inequalities. The upper complexity bounds that we obtain for these logics by instantiating our generic results appear to be new. The upper bound is based on a type elimination algorithm; additionally, for the purely modal case (i.e. in the absence of nominals), we have designed a global caching algorithm that offers a perspective for efficient reasoning in practice.

In earlier work on upper bounds PSPACE for plain satisfiability checking (i.e. reasoning in the absence of global assumptions) [46], we have used the more general setting of coalgebraic modal logic over *copointed* functors. This has allowed covering logics with frame conditions that are non-iterative [33], i.e. do not nest modal operators but possibly have top-level propositional variables, such as the *T*-axiom $\Box a \rightarrow a$ that defines reflexive relational frames; an important example of this type is Elgesem’s logic of agency [15]. We leave a corresponding generalization of our present results to future work. A further key point that remains for future research is to extend the global caching algorithm to cover nominals and satisfaction operators, combining the methods developed in the present paper with ideas underlying the existing rule-based global caching algorithm for coalgebraic hybrid logic [23].

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