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Citation: Su, Guifu, Wang, Shuai, Du, Junfeng, Gao, Mingjing, Das, Kinkar Chandra and Shang, Yilun (2022) Sufficient Conditions for a Graph to Be l -Connected, l -Deficient, l -Hamiltonian and l -Independent in Terms of the Forgotten Topological Index. *Mathematics*, 10 (11). p. 1802. ISSN 2227-7390

Published by: MDPI

URL: <https://doi.org/10.3390/math10111802> <<https://doi.org/10.3390/math10111802>>

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Sufficient Conditions for a Graph to be ℓ -Connected, ℓ -Deficient, ℓ -Hamiltonian and ℓ^- -Independent in Terms of the Forgotten Topological Index

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Abstract: The forgotten topological index of a (molecule) graph is the sum of cubes of all its vertex degrees, which plays a significant role in measuring the branching of the carbon atom skeleton. It is meaningful and difficult to explore sufficient conditions for a given graph keeping certain properties in graph theory. In this paper, we mainly explore sufficient conditions in terms of the forgotten topological index for a graph to be ℓ -connected, ℓ -deficient, ℓ -Hamiltonian and ℓ^- -independent, respectively. The conditions cannot be dropped.

Keywords: the forgotten topological index; ℓ -connected; ℓ -deficient; ℓ -Hamiltonian; ℓ^- -independent

MSC: 05C07; 05C90; 97K30



Citation: Su, G.; Wang, S.; Du, J.; Gao, M.; Das, K.C.; Shang, Y. Sufficient Conditions for a Graph to Be ℓ -Connected, ℓ -Deficient, ℓ -Hamiltonian and ℓ^- -Independent in Terms of the Forgotten Topological Index. *Mathematics* **2022**, *10*, 1802. <https://doi.org/10.3390/math10111802>

Academic Editor: Janez Žerovnik

Received: 5 April 2022

Accepted: 23 May 2022

Published: 25 May 2022

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1. Introduction

Let $G = (V(G), E(G))$ be a finite, simple, connected graph; we use $n = |V(G)|$ and $m = |E(G)|$ to denote the number of vertices and edges of G , respectively. We call the number of edges in G which are incident to a vertex $v \in V(G)$ degree and are denoted by $d_G(v)$. The minimum value of such numbers is said to be the minimum degree of G , and denoted by $\delta = \delta(G)$. A sequence of non-negative integers $\pi = (d_1, d_2, \dots, d_n)$ is said to be the degree sequence of G if $d_i = d_G(v_i)$ for $i = 1, 2, \dots, n$. In particular, if the vertex degrees are non-decreasing, then we use $\pi = (d_1 \leq d_2 \leq \dots \leq d_n)$ to denote the degree sequence for simplicity. For some integer ℓ , a connected graph G is said to be ℓ -connected if any pair of vertices can be separated by deleting at least ℓ vertices from the graph. The deficiency of G , denoted by $def(G)$, is equal to the number of vertices which are not matched by a maximum matching in G . It is not difficult to find that there exists a 1-factor in G if, and only if, $def(G) = 0$. Meanwhile, for a positive number ℓ , a graph G is called ℓ -deficient if its deficiency does not exceed ℓ . As usual, a cycle (resp. path) passing through every vertex of a graph is called a Hamilton cycle (resp. Hamilton path). The corresponding graph is Hamiltonian (resp. traceable) if there exists a Hamilton cycle (resp. Hamilton path) in it. For any subset $X \subseteq V(G)$ with $|X| \leq \ell$, if the subgraph induced by $V(G) \setminus X$ is Hamiltonian, then the original graph G is called ℓ -Hamiltonian. In the following context, we always omit the subscript G from the notation if there is no confusion. The reader is referred to [1] for standard graph-theoretic notation and terminology.

A topological index is a numeric quantity related to a molecular graph, which can be used to characterize properties of the corresponding graph. A large number of topological indices, such as Wiener index [2] and Harary index [3,4], have been introduced

and found interesting applications. We encourage the reader to find more detail in [5–7] and therein.

It is stated that the authors in [8] proposed an approximate formula for the total π -electron energy (\mathcal{E}). One of the main terms occurring in this expression was an important degree-based graph invariant, the first Zagreb index, of the corresponding molecular graph. Fortunately, this numeric quantity was precisely interpreted to be useful in measuring the extent of branching of the carbon atom skeleton for molecules. From then on, many mathematicians and theoretical chemists concentrated on this topological index and obtained a series of meaningful results; we refer the readers to [9–11] and therein for more details. Another considerable contribution for the expression of \mathcal{E} is the sum of cubes of all vertex degrees in graph G :

$$F(G) = \sum_{u \in V} (d_G(u))^3,$$

which also plays an important role in measuring the branching of the carbon atom skeleton for the corresponding underlying molecular graph. The later researchers call this graph invariant the forgotten topological index of graphs. We encourage interested readers to consult [12–16] for more information.

It is stated in [17] that determining whether a graph is traceable or Hamiltonian is always *NP*-complete. Thus, exploring sufficient conditions for graphs to have some properties attracts a vast number of scientists in graph theory. For a given graph G with n vertices, there must exist a cycle of length at least $n - \ell$ in G if the minimum degree is not less than $n - \ell - 1$, see [1]. In 2013, the authors in [18] studied the traceability of graphs and presented a sufficient condition by using the Harary index. In the same year, the authors in [19] explored a new sufficient condition for a graph to be traceable in terms of the Wiener index. We refer readers to consult [20–25] for more information and details.

To the best of our knowledge, there are absolutely few sufficient conditions for a graph to have some promising properties in terms of degree-based topological indices of graphs. In this paper, we focus on exploring such conditions by using the forgotten topological index for graphs to be ℓ -connected, ℓ -deficient, ℓ -Hamiltonian and ℓ^- -independent (will be introduced in Section 5), respectively. This work will help us to open up a new development direction for these problems.

2. Sufficient Conditions for ℓ -Connected Graphs

It is stated in [26] that if G is a graph with $n \geq \ell + 1$ vertices, and if its minimum degree $\delta(G) \geq \frac{1}{2}(n + \ell - 2)$, then G is ℓ -connected. In 2017, in terms of the well-known Wiener index and Harary index, Feng et al. [27] presented a sufficient condition for a graph to be ℓ -connected. About one year later, another one was found in terms of the first Zagreb index for a graph to be ℓ -connected [25]. In this section, we continue this program to explore such results by using the forgotten topological index.

We begin with the following lemma, which will be used in later proofs.

Lemma 1 (Bondy [28]). *Let $\pi = (d_1 \leq d_2 \leq \dots \leq d_n)$ be a degree sequence with $n \geq 2$ and $1 \leq \ell \leq n - 1$. If*

$$d_i \leq i + \ell - 2 \Rightarrow d_{n-\ell+1} \geq n - i \text{ for } 1 \leq i \leq \frac{n - \ell + 1}{2},$$

then π enforces ℓ -connected.

Let G_1 and G_2 be two vertex-disjoint graphs, denoted by $G_1 \cup G_2$ the union of these two graphs, and $G_1 + G_2$ is the join which is obtained from the disjoint union of G_1 and G_2 by connecting each vertex in G_1 with that in G_2 .

Theorem 1. Let G be a connected graph of order $n \geq \ell + 1$. If

$$F(G) > (\ell - 1)^3 + (n - \ell)(n - 2)^3 + (\ell - 1)(n - 1)^3 \doteq Q_1(n, \ell),$$

then G is ℓ -connected.

Proof. We assume that G is not ℓ -connected, it then follows from Lemma 1 that there must exist an integer i such that $d_i \leq i + \ell - 2$ and $d_{n-\ell+1} \leq n - i - 1$ for $1 \leq i \leq \frac{n-\ell+1}{2}$. Note that $1 \leq \ell \leq n - 1$, then we have

$$\begin{aligned} F(G) &\leq i(i + \ell - 2)^3 + (n - \ell - i + 1)(n - i - 1)^3 + (\ell - 1)(n - 1)^3 \\ &= 2i^4 - 4(n - \ell + 1)i^3 + (6n^2 - (3\ell + 6)n + 3\ell^2 - 9\ell + 12)i^2 \\ &\quad - (4n^3 - (3\ell + 6)n^2 + 6\ell n - \ell^3 + 6\ell^2 - 15\ell + 10)i + n(n - 1)^3. \end{aligned}$$

For simplicity, we define the following function on $x \in [1, \frac{n-\ell+1}{2}]$:

$$\begin{aligned} \mathcal{H}_1(x) &= 2x^4 - 4(n - \ell + 1)x^3 + (6n^2 - (3\ell + 6)n + 3\ell^2 - 9\ell + 12)x^2 \\ &\quad - (4n^3 - (3\ell + 6)n^2 + 6\ell n - \ell^3 + 6\ell^2 - 15\ell + 10)x. \end{aligned}$$

Then the first and second derivative of $\mathcal{H}_1(x)$, respectively, equals to

$$\begin{aligned} \mathcal{H}'_1(x) &= 8x^3 - 12(n - \ell + 1)x^2 + (12n^2 - (6\ell + 12)n + 6\ell^2 - 18\ell + 24)x \\ &\quad - (4n^3 - (3\ell + 6)n^2 + 6\ell n - \ell^3 + 6\ell^2 - 15\ell + 10) \end{aligned}$$

and

$$\mathcal{H}''_1(x) = 24x^2 - 24(n - \ell + 1)x + 12n^2 - (6\ell + 12)n + 6\ell^2 - 18\ell + 24.$$

It is routine to check that the discriminant of $\mathcal{H}''_1(x) = 0$ is $\Delta_1 = 576(-n^2 - n\ell + 4n + \ell - 3) < 0$. Hence, $\mathcal{H}_1(x)$ is a convex function in the interval $[1, \frac{n-\ell+1}{2}]$, and therefore $\mathcal{H}_1(x) \leq \max \left\{ \mathcal{H}_1(1), \mathcal{H}_1(\frac{n-\ell+1}{2}) \right\}$.

We distinguish the following two cases.

Case 1. $n - \ell + 1$ is even.

Now,

$$\begin{aligned} \mathcal{H}_1(1) - \mathcal{H}_1\left(\frac{n - \ell + 1}{2}\right) &= \frac{1}{8}(\ell^4 + (2n - 2)\ell^3 - (12n - 12)\ell^2) \\ &\quad - \frac{1}{8}(10n^3 - 54n^2 + 78n - 34)\ell + \frac{1}{8}(7n^4 - 40n^3 + 78n^2 - 64n + 19). \end{aligned}$$

Let us define a function on $\ell \in [1, n - 1]$:

$$\begin{aligned} \mathcal{A}(\ell) &= \ell^4 + (2n - 2)\ell^3 - (12n - 12)\ell^2 - (10n^3 - 54n^2 + 78n - 34)\ell \\ &\quad + 7n^4 - 40n^3 + 78n^2 - 64n + 19. \end{aligned}$$

Direct calculations yield that the first and second derivative of $\mathcal{A}(x)$, respectively, is

$$\mathcal{A}'(\ell) = 4\ell^3 + (6n - 6)\ell^2 - (24n - 24)\ell - 10n^3 + 54n^2 - 78n + 34$$

and

$$\mathcal{A}''(\ell) = 12\ell^2 - 12\ell + 12n\ell - 24n + 24.$$

To continue the proof, we need consider the following two subcases.

Subcase 1.1. $\ell = 1$.

One can easily check that $\mathcal{A}(1) = 7n^4 - 50n^3 + 132n^2 - 152n + 64 > 0$, implying that $\mathcal{H}_1(x) \leq \mathcal{H}_1(1)$. Hence, $F(G) \leq (\ell - 1)^3 + (n - \ell)(n - 2)^3 + (\ell - 1)(n - 1)^3$, a contradiction. Hence, G is ℓ -connected.

Subcase 1.2. $\ell \in [2, n - 1]$.

By direct calculation, one can find that $\mathcal{A}''(\ell) = 12\ell(\ell - 1) + 12n(\ell - 2) + 24 > 0$. This implies that $\mathcal{A}'(\ell)$ is an increasing function in the interval $[2, n - 1]$. Thus, we have $\mathcal{A}'(\ell) \leq \mathcal{A}'(n - 1) = 0$, which immediately yields that $\mathcal{A}(\ell)$ is a decreasing function for $\ell \in [2, n - 1]$. Hence, $\mathcal{A}(\ell) \geq \mathcal{A}(n - 1) = 0$. As desired we confirm that $\mathcal{H}_1(1) - \mathcal{H}_1\left(\frac{n - \ell + 1}{2}\right) \geq 0$. Hence, $F(G) \leq (\ell - 1)^3 + (n - \ell)(n - 2)^3 + (\ell - 1)(n - 1)^3$, again a contradiction. Hence, G is ℓ -connected.

Case 2. $n - \ell + 1$ is odd.

Now,

$$\begin{aligned} \mathcal{H}_1(1) - \mathcal{H}_1\left(\frac{n - \ell}{2}\right) &= \frac{1}{8}\left(\ell^4 + (2n - 2)\ell^3 - (12n - 12)\ell^2\right) \\ &\quad - \frac{1}{8}\left(10n^3 - 54n^2 + 84n - 40\right)\ell + \frac{1}{8}\left(7n^4 - 40n^3 + 72n^2 - 40n\right). \end{aligned}$$

Let us define a function on $\ell \in [1, n - 2]$:

$$\begin{aligned} \mathcal{B}(\ell) &= \ell^4 + (2n - 2)\ell^3 - (12n - 12)\ell^2 - (10n^3 - 54n^2 + 84n - 40)\ell \\ &\quad + 7n^4 - 40n^3 + 72n^2 - 40n. \end{aligned}$$

By taking the first and second derivatives of $\mathcal{B}(\ell)$, we obtain

$$\mathcal{B}'(\ell) = 4\ell^3 + (6n - 6)\ell^2 - (24n - 24)\ell - 10n^3 + 54n^2 - 84n + 40$$

and

$$\mathcal{B}''(\ell) = 12\ell^2 - 12\ell + 12n\ell - 24n + 24.$$

We now consider the following two possibilities.

Subcase 2.1. $\ell = 1$.

It is not difficult to declare that $\mathcal{B}(1) = 7n^4 - 50n^3 + 126n^2 - 134n + 51 > 0$, implying that $\mathcal{H}_1(x) \leq \mathcal{H}_1(1)$. Hence, $F(G) \leq (\ell - 1)^3 + (n - \ell)(n - 2)^3 + (\ell - 1)(n - 1)^3$, a contradiction. Hence, G is ℓ -connected.

Subcase 2.2. $\ell \in [2, n - 2]$.

Note that the initial condition is $\ell \geq 2$; it is deduced that $\mathcal{B}''(\ell) = 12\ell(\ell - 1) + 12n(\ell - 2) + 24 > 0$, implying that $\mathcal{B}'(\ell)$ is increasing in the interval $[2, n - 2]$. Hence, $\mathcal{B}'(\ell) \leq \mathcal{B}'(n - 2) = -24n^2 + 84n - 64 < 0$ for $n \geq 3$. It immediately yields that $\mathcal{B}(\ell)$ is decreasing in the interval $\ell \in [2, n - 2]$. Hence, $\mathcal{B}(\ell) \geq \mathcal{B}(n - 2) = 0$, and therefore $\mathcal{H}_1(1) - \mathcal{H}_1\left(\frac{n - \ell}{2}\right) \geq 0$. Thus, we have $F(G) \leq (\ell - 1)^3 + (n - \ell)(n - 2)^3 + (\ell - 1)(n - 1)^3$, again a contradiction. Hence, G is ℓ -connected. \square

Remark 1. If $G \cong K_{\ell-1} + (K_1 \cup K_{n-\ell})$, then directly computations yields that $F(G) = Q_1(n, \ell)$. Conversely, let $F(G) = Q_1(n, \ell)$, then all the inequalities in previous proof should be equalities. Hence, $i = 1$; therefore, $d_1 = \ell - 1, d_2 = d_3 = \dots = d_{n-\ell+1} = n - 2$ and $d_{n-\ell+2} = d_{n-\ell+3} = \dots = d_n = n - 1$. This implies that $G \cong K_{\ell-1} + (K_1 \cup K_{n-\ell})$. Hence, the condition in Theorem 1 cannot be dropped.

3. Sufficient Conditions for ℓ -Deficient Graphs

In [27], the authors characterized some structural properties of ℓ -deficient and ℓ -path-coverable graphs, mainly including several sufficient conditions in terms of the Wiener index and Harary index, respectively. Recently, An et al. [25] obtained a sufficient condition in terms of the well-known degree-based graph invariant, the first Zagreb index, for a graph to be ℓ -deficient. The aim of this section is to explore sufficient condition by using the forgotten topological index for a graph to be ℓ -deficient.

The following Lemma will be useful for our later proof, which was first proved by Vergnas in their Ph.D thesis.

Lemma 2 (Las Vergnas [29]). *Let $\pi = (d_1 \leq d_2 \leq \dots \leq d_n)$ be a degree sequence, and also let $0 \leq \ell \leq n$ with $n \equiv \ell \pmod{2}$. If*

$$d_{i+1} \leq i - \ell \Rightarrow d_{n+\ell-i} \geq n - i - 1 \text{ for } 1 \leq i \leq \frac{n + \ell - 2}{2},$$

then π enforces ℓ -deficient.

Now, we shall state the main result:

Theorem 2. *Let G be a connected graph of order $n \geq 52$ with $n \equiv \ell \pmod{2}$ and $0 \leq \ell \leq n$. If*

$$F(G) > n^4 - 11n^3 + (51 - 6\ell)n^2 + (24\ell - 105)n - 2\ell^3 + 6\ell^2 - 32\ell + 82 \doteq Q_2(n, \ell),$$

then G is ℓ -deficient.

Proof. We suppose that G is a graph satisfying the conditions of Theorem 2, which is not ℓ -deficient. Then, by Lemma 2, there must exist an integer, i , such that $d_{i+1} \leq i - \ell$ and $d_{n+\ell-i} \leq n - i - 2$ for $1 \leq i \leq \frac{n+\ell-2}{2}$. Thus

$$\begin{aligned} F(G) &\leq (i + 1)(i - \ell)^3 + (n + \ell - 2i - 1)(n - i - 2)^3 + (i - \ell)(n - 1)^3 \\ &= 3i^4 - (7n + 4\ell - 14)i^3 + \left(9n^2 + (3\ell - 33)n + 3\ell^2 - 9\ell + 30\right)i^2 \\ &\quad - \left(4n^3 + (3\ell - 24)n^2 + (45 - 12\ell)n + \ell^3 - 3\ell^2 + 12\ell - 27\right)i \\ &\quad + n^4 - 7n^3 + (18 - 3\ell)n^2 + (9\ell - 20)n - \ell^3 - 7\ell + 8. \end{aligned}$$

For simplicity, we define the following function on $\left[1, \frac{n+\ell-2}{2}\right]$:

$$\begin{aligned} \mathcal{H}_2(x) &= 3x^4 - (7n + 4\ell - 14)x^3 + \left(9n^2 + (3\ell - 33)n + 3\ell^2 - 9\ell + 30\right)x^2 \\ &\quad - \left(4n^3 + (3\ell - 24)n^2 + (45 - 12\ell)n + \ell^3 - 3\ell^2 + 12\ell - 27\right)x \\ &\quad + n^4 - 7n^3 + (18 - 3\ell)n^2 + (9\ell - 20)n - \ell^3 - 7\ell + 8. \end{aligned}$$

Then, the first and second derivatives of $\mathcal{H}_2(x)$ are

$$\begin{aligned} \mathcal{H}'_2(x) &= 12x^3 - (21n + 12\ell - 42)x^2 + \left(18n^2 + (6\ell - 66)n + 6\ell^2 - 18\ell + 60\right)x \\ &\quad - \left(4n^3 + (3\ell - 24)n^2 + (45 - 12\ell)n - \ell^3 + 3\ell^2 + 12\ell - 27\right) \end{aligned}$$

and

$$\mathcal{H}''_2(x) = 36x^2 - (42n + 24\ell - 84)x + 18n^2 + (6\ell - 66)n + 6\ell^2 - 18\ell + 60,$$

respectively.

The discriminant of $\mathcal{H}_2''(x) = 0$ is $\Delta_2 = -828n^2 + 2448n + 1152n\ell - 288\ell^2 - 1440\ell - 1584 < 0$. It follows that $\mathcal{H}_2''(x) > 0$, implying that $\mathcal{H}_2(x)$ is a convex function in the interval $[1, \frac{n+\ell-2}{2}]$. Hence, $\mathcal{H}_2(x) \leq \max \left\{ \mathcal{H}_2(1), \mathcal{H}_2\left(\frac{n+\ell-2}{2}\right) \right\}$.

By direct calculations, we obtain

$$\mathcal{H}_2(1) = n^4 - 11n^3 + (51 - 6\ell)n^2 + (24\ell - 105)n - 2\ell^3 + 6\ell^2 - 32\ell + 82$$

and

$$\begin{aligned} \mathcal{H}_2\left(\frac{n+\ell-2}{2}\right) &= -\frac{1}{16}\ell^4 + \left(\frac{1}{8}n - \frac{1}{2}\right)\ell^3 + \left(\frac{3}{4}n - \frac{3}{2}\right)\ell^2 \\ &\quad - \left(\frac{5}{8}n^3 - \frac{3}{2}n^2 + \frac{3}{2}\right)\ell + \frac{9}{16}n^4 - \frac{11}{4}n^3 + \frac{9}{2}n^2 - \frac{5}{2}n. \end{aligned}$$

Thus, we have

$$\begin{aligned} \mathcal{H}_2(1) - \mathcal{H}_2\left(\frac{n+\ell-2}{2}\right) &= \frac{1}{16}\ell^4 - \left(\frac{1}{8}n + \frac{3}{2}\right)\ell^3 - \left(\frac{3}{4}n - \frac{15}{2}\right)\ell^2 \\ &\quad + \left(\frac{5}{8}n^3 - \frac{15}{2}n^2 + 24n - \frac{61}{2}\right)\ell + \left(\frac{7}{16}n^4 - \frac{33}{4}n^3\right) \\ &\quad + \left(\frac{93}{2}n^2 - \frac{205}{2}n\right) + 82. \end{aligned}$$

According to the given conditions, one can find that all the five terms in the above expression is positive when $n \geq 52$. Thus, $\mathcal{H}_2(1) - \mathcal{H}_2\left(\frac{n+\ell-2}{2}\right) > 0$. It immediately yields that $\mathcal{H}_2(x) \leq \mathcal{H}_2(1) = n^4 - 11n^3 + (51 - 6\ell)n^2 + (24\ell - 105)n - 2\ell^3 + 6\ell^2 - 32\ell + 82$, a contradiction. So, the result follows easily. \square

Remark 2. If $G \cong K_1 + (2K_1 \cup K_{n-3})$, then direct computation yields that $F(G) = Q_2(n, \ell)$. Conversely, let $F(G) = Q_2(n, \ell)$, implying that $i = 1$. Noticing that $\ell \leq i$, it yields that $\ell = 0$ or $\ell = 1$. If $\ell = 1$, it routine to check that $d_1 = d_2 = 0$. Consequently, we know that G is a disconnected graph, a contradiction with our hypothesis. Hence, $\ell = 0$ and therefore $d_1 = d_2 = 1, d_3 = d_4 = \dots = d_{n-1} = n - 3$ and $d_n = n - 1$. Hence, $G \cong K_1 + (2K_1 \cup K_{n-3})$. Therefore, the condition in Theorem 2 cannot be dropped.

4. Sufficient Conditions for ℓ -Hamiltonian Graphs

Generally speaking, determining whether a given graph possesses certain structural property is one of the most difficult problems in graph theory. For example, whether a fixed graph is Hamiltonian or not is always NP-complete. To the best of our knowledge, a vast number of sufficient or necessary conditions were found by mathematicians for a graph to be ℓ -Hamiltonian and Hamiltonian.

It was proved by Chartrand et al. that if the minimum degree of a graph G does not less than $\frac{1}{2}(n + \ell)$, then it must be ℓ -Hamiltonian, see [30]. A sufficient condition was studied in the random graph setting [31]. In 2013, Hua and Wang investigated the traceability of graphs and showed a reasonable sufficient condition in terms of the Harary index [18]. In the same year, by using the Wiener index, Yang also gave a sufficient condition for a graph to be traceable [19]. Recently, it is stated in [27] that Feng et al. proved several sufficient conditions, in terms of two distance-based graph parameters, the Wiener index or Harary index, for a graph to be ℓ -Hamiltonian and ℓ -edge-Hamiltonian. After one year, An et al. continued this program to explore new conditions in terms of the first Zagreb index for ℓ -Hamiltonian graphs, see [25] for more details. In what follows, we are interested in finding sufficient conditions in terms of a meaningful degree-based invariant, the forgotten topological index, for a graph to be Hamiltonian.

We begin by presenting a very elementary sufficient condition.

Lemma 3 (Chvátal [32]). Let $\pi = (d_1 \leq d_2 \leq \dots \leq d_n)$ be a degree sequence with $0 \leq \ell \leq n - 3$. If

$$d_i \leq i + \ell \Rightarrow d_{n-i-\ell} \geq n - i \text{ for } 1 \leq i < \frac{n - \ell}{2},$$

then π enforces ℓ -Hamiltonian.

The main result is the following:

Theorem 3. Let G be a connected graph of order $n \geq 6$ and $0 \leq \ell \leq n - 3$. If

$$F(G) > \begin{cases} (1 + \ell)^3 + (n - \ell - 2)(n - 2)^3 + (1 + \ell)(n - 1)^3 \doteq Q_3(n, \ell) \text{ if } \ell \neq n - 5, \\ (n - 3)(3(n - 3)^2 + (n - 1)^3) \text{ if } \ell = n - 5, \end{cases}$$

then G is ℓ -Hamiltonian.

Proof. For this, we assume that G is not ℓ -Hamiltonian. Then, it follows from Lemma 3 that there must exist an integer, i , such that $d_i \leq i + \ell$ and $d_{n-i-\ell} \leq n - i - 1$. Thus,

$$\begin{aligned} F(G) &\leq i(i + \ell)^3 + (n - 2i - \ell)(n - i - 1)^3 + (i + \ell)(n - 1)^3 \\ &= 3i^4 - (7n - 4\ell - 6)i^3 + (3\ell^2 + 6(n - 1)^2 + 3(n - \ell)(n - 1))i^2 \\ &\quad + (\ell^3 - (n - 1)^3 - 3(n - \ell)(n - 1)^2)i + n(n - 1)^3. \end{aligned}$$

For simplicity, we define the following function on $x \in [1, \frac{n-\ell-1}{2}]$:

$$\begin{aligned} \mathcal{H}_3(x) &= 3x^4 - (7n - 4\ell - 6)x^3 + (3\ell^2 + 6(n - 1)^2 + 3(n - \ell)(n - 1))x^2 \\ &\quad + (\ell^3 - (n - 1)^3 - 3(n - \ell)(n - 1)^2)x. \end{aligned}$$

Then the first and second derivatives of $\mathcal{H}_3(x)$, respectively, are presented as

$$\begin{aligned} \mathcal{H}'_3(x) &= 12x^3 - (21n - 12\ell - 18)x^2 + (6\ell^2 + 12(n - 1)^2 + 6(n - \ell)(n - 1))x \\ &\quad + \ell^3 - (n - 1)^3 - 3(n - \ell)(n - 1)^2 \end{aligned}$$

and

$$\mathcal{H}''_3(x) = 36x^2 - (42n - 24\ell - 36)x + 18n^2 + 6\ell^2 + 6\ell + 12 - 30n - 6\ell n.$$

For convenience, we distinguishing the following two cases.

Case 1. $n - \ell - 1$ is odd.

It is routine to check that $1 \leq x \leq \frac{n-\ell-2}{2}$ and $0 \leq \ell \leq n - 4$. The discriminant of $\mathcal{H}''_3(x) = 0$ is $\Delta_3 = -36(23n^2 + 32\ell n - 36n + 8\ell^2 - 24\ell + 12) < 0$. It follows that $\mathcal{H}''_3(x) > 0$, which implies that $\mathcal{H}_3(x)$ is a convex function in $[1, \frac{n-\ell-2}{2}]$. Hence, $\mathcal{H}_3(x) \leq \max \left\{ \mathcal{H}_3(1), \mathcal{H}_3\left(\frac{n-\ell-2}{2}\right) \right\}$.

Direct calculations yields that

$$\begin{aligned} \mathcal{H}_3(1) - \mathcal{H}_3\left(\frac{n-\ell-2}{2}\right) &= \frac{1}{16} \left(\ell^4 + (2n+8)\ell^3 - (12n-48)\ell^2 \right) \\ &\quad - \frac{1}{16} \left((10n^3 - 72n^2 + 192n - 184)\ell \right) \\ &\quad + \frac{1}{16} \left(7n^4 - 68n^3 + 240n^2 - 376n + 224 \right). \end{aligned}$$

Define a function on ℓ in $[0, n-4]$:

$$\begin{aligned} \mathcal{D}(\ell) &= \ell^4 + (2n+8)\ell^3 - (12n-48)\ell^2 - (10n^3 - 72n^2 + 192n - 184)\ell \\ &\quad + 7n^4 - 68n^3 + 240n^2 - 376n + 224. \end{aligned}$$

Then the first and second derivatives of $\mathcal{D}(\ell)$, respectively, are

$$\mathcal{D}'(\ell) = 4\ell^3 + (6n+24)\ell^2 - (24n-96)\ell - 10n^3 + 72n^2 - 192n + 184$$

and

$$\mathcal{D}''(\ell) = 12\ell^2 + 12n\ell + 48\ell - 24n + 96.$$

Subcase 1.1. $\ell = 0$.

In this case, it is routine to check that $\mathcal{D}(0) = 7n^4 - 68n^3 + 240n^2 - 376n + 224 > 0$, implying that $\mathcal{H}_3\left(\frac{n-\ell-2}{2}\right) - \mathcal{H}_3(1) < 0$. Consequently, $\mathcal{H}_3(x) \leq \mathcal{H}_3(1)$. Hence, we have $F(G) \leq (1+\ell)^3 + (n-\ell-2)(n-2)^3 + (1+\ell)(n-1)^3$, which contradicts to our initial hypothesis. Hence, G is ℓ -Hamiltonian.

Subcase 1.2. $\ell = 1$.

It is easily to calculate that $\mathcal{D}(1) = 7n^4 - 78n^3 + 312n^2 - 578n + 465 > 0$, which implies that $\mathcal{H}_3\left(\frac{n-\ell-2}{2}\right) - \mathcal{H}_3(1) < 0$. It immediately yields that $\mathcal{H}_3(x) \leq \mathcal{H}_3(1)$. Hence, $F(G) \leq (1+\ell)^3 + (n-\ell-2)(n-2)^3 + (1+\ell)(n-1)^3$, again a contradiction. Hence, G is ℓ -Hamiltonian.

Subcase 1.3. $\ell \geq 2$.

In this subcase, $\mathcal{D}''(\ell) > 0$. Hence, $\mathcal{D}'(\ell)$ is increasing in the interval $[2, n-4]$. Consequently, $\mathcal{D}'(\ell) \leq \mathcal{D}'(n-4) = -24n^2 + 96n - 72 < 0$. Therefore, $\mathcal{D}(\ell)$ is decreasing on $[2, n-4]$ and $\mathcal{D}(\ell) \geq \mathcal{D}(n-4) = 0$. This confirms that $\mathcal{H}_3\left(\frac{n-\ell-2}{2}\right) - \mathcal{H}_3(1) \leq 0$. Hence, $F(G) \leq (1+\ell)^3 + (n-\ell-2)(n-2)^3 + (1+\ell)(n-1)^3$, again a contradiction. Hence, G is ℓ -Hamiltonian.

Case 2. $n - \ell - 1$ is even.

In this case, it is not difficult to find that $1 \leq x \leq \frac{n-\ell-1}{2}$ and $\ell \in [0, n-3]$. For simplicity, we distinguishing the following cases:

Subcase 2.1. $\ell = n-3$.

It is routine to check that $n - \ell - 1 = 2$ and $x = 1$. Hence, $F(G) \leq (n-2)[2(n-2)^2 + (n-1)^3] = Q_4(n, n-3)$, which contradicts to the given condition. This confirms that G is ℓ -Hamiltonian.

Subcase 2.2. $\ell \in [0, n-5]$.

By similar discussion as in Case 1, we obtain that $\mathcal{H}_3(x)$ is a convex function in the interval $[1, \frac{n-\ell-1}{2}]$. Hence, $\mathcal{H}_3(x) \leq \max\{\mathcal{H}_3(1), \mathcal{H}_3(\frac{n-\ell-1}{2})\}$. Direct computations show that

$$\begin{aligned} \mathcal{H}_3(1) - \mathcal{H}_3\left(\frac{n-\ell-1}{2}\right) &= \frac{1}{16} \left(\ell^4 + (2n+12)\ell^3 - (6n-54)\ell^2 \right) \\ &\quad - \frac{1}{16} \left((10n^3 - 72n^2 + 162n - 164)\ell \right) \\ &\quad + \frac{1}{16} \left(7n^4 - 78n^3 + 288n^2 - 434n + 249 \right). \end{aligned}$$

Define a function on ℓ in $[0, n-5]$:

$$\begin{aligned} \mathcal{E}(\ell) &= \ell^4 + (2n+12)\ell^3 - (6n-54)\ell^2 - (10n^3 - 72n^2 + 162n - 164)\ell \\ &\quad + 7n^4 - 78n^3 + 288n^2 - 434n + 249. \end{aligned}$$

Then the first and second derivative of $\mathcal{E}(\ell)$, respectively, are

$$\mathcal{E}'(\ell) = 4\ell^3 + (6n+36)\ell^2 - (12n-108)\ell - 10n^3 + 72n^2 - 162n + 164$$

and

$$\begin{aligned} \mathcal{E}''(\ell) &= 12\ell^2 + (12n+72)\ell - (12n-108) \\ &= 12n(\ell-1) + 12\ell^2 + 72\ell + 108. \end{aligned}$$

■ $\ell = 0$.

It is routine to check that $\mathcal{E}(0) = 7n^4 - 78n^3 + 288n^2 - 434n + 249 > 0$. This implies that $\mathcal{H}_3\left(\frac{n-\ell-1}{2}\right) - \mathcal{H}_3(1) < 0$, and consequently $F(G) \leq (1+\ell)^3 + (n-\ell-2)(n-2)^3 + (1+\ell)(n-1)^3$, again a contradiction. Hence, G is ℓ -Hamiltonian.

■ $\ell \in [1, n-5]$.

Note that $1 \leq \ell \leq n-5$. Then, we have $\mathcal{E}''(\ell) = 12n(\ell-1) + 12\ell^2 + 72\ell + 108 > 0$. Thus, $\mathcal{E}'(\ell)$ is an increasing function in the interval $[1, n-5]$, implying that $\mathcal{E}'(\ell) \leq \mathcal{E}'(n-5) = -24n^2 + 96n + 24 < 0$ for $n \geq 6$. Hence, one can find that $\mathcal{E}(\ell)$ is a decreasing function for $\ell \in [1, n-5]$.

If $\ell = n-5$, then $\mathcal{E}(n-5) = -96 < 0$, implying that $\mathcal{H}_3\left(\frac{n-\ell-1}{2}\right) - \mathcal{H}_3(1) = \mathcal{H}_3(2) - \mathcal{H}_3(1) > 0$. Hence, $F(G) \leq \mathcal{H}_3(2) = (n-3)(3(n-3)^2 + (n-1)^3)$, which contradicts to the given condition. Hence, G is ℓ -Hamiltonian.

If $\ell \in [1, n-7]$, then it follows that $\mathcal{E}(\ell) \geq \mathcal{E}(n-7) = 96n^2 - 480n + 32 > 0$, implying that $\mathcal{H}_3\left(\frac{n-\ell-1}{2}\right) - \mathcal{H}_3(1) < 0$. Hence, $F(G) \leq (1+\ell)^3 + (n-\ell-2)(n-2)^3 + (1+\ell)(n-1)^3$, again a contradiction. Hence, G is ℓ -Hamiltonian. □

Remark 3. If $G \cong K_{\ell+1} + (K_1 \cup K_{n-\ell-2})$, then one can easily see that $F(G) = Q_3(n, \ell)$. Conversely, let $F(G) = Q_3(n, \ell)$. It immediately follows from the above that $F(G) = (1+\ell)^3 + (n-\ell-2)(n-2)^3 + (1+\ell)(n-1)^3$, implying that $d_1 = \ell + 1, d_2 = d_3 = \dots = d_{n-\ell-2} = n-2$ and $d_{n-\ell-1} = d_{n-\ell} = \dots = d_n = n-1$. Hence, $G \cong K_{\ell+1} + (K_1 \cup K_{n-\ell-2})$. This implies that the condition in Theorem 3 cannot be dropped.

By taking $\ell = 0$ in Theorem 3, we have the following consequence.

Corollary 1. Let G be a connected graph of order $n \geq 6$. If

$$F(G) > (n-2)^4 + (n-1)^3 + 1,$$

then G is Hamiltonian.

5. Sufficient Conditions for ℓ^- -Independent Graphs

Let S be a vertex subset of G , which is called an independent set of G if the induced subgraph $G[S]$ is a graph such that any pairs of vertices in S are not adjacent. The independence number, denoted by $\alpha(G)$, of G is the number of vertices in the largest independent set of G . For a fixed integer ℓ , we call a graph is ℓ^- -independent if its independence number does not exceed to ℓ . It is reported in [25] that An et al. characterized the ℓ^- -independent graphs in terms of a kind of degree-based graph invariants. In this section, we continue this program to explore such sufficient conditions.

Lemma 4 (Bauer et al. [33]). *Let $\pi = (d_1 \leq d_2 \leq \dots \leq d_n)$ be a degree sequence and $\ell \geq 1$. If $d_{\ell+1} \geq n - \ell$, then π enforces ℓ^- -independence.*

The complement of a graph G , denoted by \overline{G} , is the graph with vertex set $V(G)$ and two distinct vertices u and v are adjacent in \overline{G} if, and only if, they are not adjacent in the original graph G .

We conclude this paper with the following structural result.

Theorem 4. *Let G be a connected graph of order n . If*

$$F(G) > (\ell + 1)(n - \ell - 1)^3 + (n - \ell - 1)(n - 1)^3 \doteq Q_4(n, \ell),$$

then G is ℓ^- -independent.

Proof. This result we prove by contradiction. For this we assume that G is not ℓ^- -independent. Then, by Lemma 4, we obtain that $d_{\ell+1} \leq n - \ell - 1$. Consequently, from the definition of the forgotten topological index, we have $F(G) \leq (\ell + 1)(n - \ell - 1)^3 + (n - \ell - 1)(n - 1)^3$, which contradicts to the given condition. Hence, the result follows. \square

Remark 4. *If $G \cong \overline{K_{\ell+1}} + K_{n-\ell-1}$, then one can easily see that $F(G) = Q_4(n, \ell)$. Conversely, let $F(G) = Q_4(n, \ell)$. It immediately follows from the above that $F(G) = n^4 - 9n^3 - (3\ell^2 - 3\ell - 33)n^2 + (3\ell^3 + 9\ell^2 - 18\ell - 51)n - \ell^4 - 4\ell^3 - 6\ell^2 + 23\ell + 26$, implying that $d_1 = d_2 = \dots = d_{\ell+1} = n - \ell - 1$ and $d_{\ell+2} = d_{\ell+3} = \dots = d_n = n - 1$. Hence, $G \cong \overline{K_{\ell+1}} + K_{n-\ell-1}$. Therefore, the condition in Theorem 4 cannot be dropped.*

6. Conclusions

The problem of determining whether a graph keeps a certain reasonable property is interesting and meaningful in graph theory. In this paper, sufficient conditions, in terms of the forgotten topological index, for a graph to be ℓ -connected, ℓ -deficient, ℓ -Hamiltonian and ℓ^- -independent are presented. The conditions obtained cannot be dropped. The Randić-type invariants of a simple connected graph $G = (V(G), E(G))$ can be expressed in terms of the quantities $R_f(G) = \sum_{uv \in E(G)} f(d_G(u), d_G(v))$ for various choices of the function $f(x)$. In the future, we will explore sufficient conditions for a graph to be Hamiltonian, Hamilton-connected and traceable, or some other similar properties in terms of the Randić-type invariants of G or its complement.

Author Contributions: Conceptualization, G.S., S.W., J.D., M.G., K.C.D. and Y.S.; investigation, G.S., S.W., J.D., M.G., K.C.D. and Y.S.; writing—original draft preparation, G.S., S.W., J.D., M.G., K.C.D. and Y.S.; writing—review and editing, G.S., S.W., J.D., M.G., K.C.D. and Y.S. All authors have read and agreed to the published version of the manuscript.

Funding: G.S. is supported by the National Key Research and Development Project (2019YFB2006602) and Beijing Natural Science Foundation of China (No.1222012). K.C.D. is supported by National Research Foundation funded by the Korean government (Grant No. 2021R1F1A1050646).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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