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A SWITCHING FEEDBACK CONTROL APPROACH FOR PERSISTENCE OF MANAGED RESOURCES

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Abstract. An adaptive switching feedback control scheme is proposed for classes of discrete-time, positive difference equations, or systems of equations. In overview, the objective is to choose a control strategy which ensures persistence of the state, consequently avoiding zero which corresponds to absence or extinction. A robust feedback control solution is proposed as the effects of different management actions are assumed to be uncertain. Our motivating application is to the conservation of dynamic resources, such as populations, which are naturally positive quantities and where discrete and distinct courses of management actions, or control strategies, are available. The theory is illustrated with examples from population ecology.

1. Introduction.

We present a theoretical robust feedback control solution to the problem of conserving temporally-varying, but uncertain, quantities of interest, such as managed populations, through the choice of discrete control strategies. The problem of making decisions which lead to desirable outcomes arises in almost all scientific and engineering disciplines, including natural resource management and conservation. The academic literature is consequently vast, with monographs including [8, 10]. The motivation for our study is to establish theoretical results related to the management of poorly understood or poorly modelled, but important dynamic resources. Our starting point is that the quantity of interest, denoted $x(t)$, varies temporally with fixed discrete time-step t. Here $x(t)$ may be scalar- or vector-valued, the latter permitting the modelling of structured quantities. The variable $x(t)$ is naturally nonnegative, as its components denote necessarily nonnegative quantities, such as concentrations, densities or abundances.

To affect a change in the dynamics for x, we posit that q distinct control strategies (also termed courses of management action) are available, and that the choice of which control action is applied over time is determined by the user and may change. Accommodating the above considerations and the dependence of the dynamics on the control strategy naturally leads to a model for x comprising a so-called switched system of positive difference equations of the form

$$x(t+1) = F(h, x(t)), \quad x(0) = x_0, \quad t \in \mathbb{Z}_+ := \{0, 1, 2, \dots\}, \tag{1.1}$$

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where x_0 is the initial condition. Here the first variable h of the function F in (1.1) determines which of the q control strategies is being applied. For fixed first argument, the function $F(h, \cdot)$ describes the dynamics of x. The assumed property that $F(h, 0) = 0$ for all h means that zero is a constant (equilibrium) solution of (1.1), corresponding to absence of x.

Given the above setup, the problem is essentially to choose a control strategy which ensures persistence of x, that is, which avoids $x(t) \to 0$ as $t \to \infty$. Persistence is now a well-established concept, and captures the extent to which non-zero solutions are bounded away from zero; see, for instance [15, 30]. There are many possible solutions to the problem described so far. If the functions $F(h, \cdot)$ are known, then the particular goal is evidently achieved by choosing the appropriate h which gives the desired dynamic behaviour. However, in many real-world situations, the *effect* of the distinct control strategies is not known, meaning that the $F(h, \cdot)$ are *not* known exactly. Another approach is to seek to identify $F(h, \cdot)$, so that the above solution may be applied. For identifiability references in an ecological context, we refer to [20, 28]. Here we do not pursue this approach, one reason being that in ecological models, unlike many engineered systems, it is often not practicable to excite the system with specific known inputs to generate input-output data, see [19].

The novel solution we propose is a feedback control approach. We design an algorithm for switching between strategies which identifies (or learns) a suitable strategy that ensures persistence. To give an outline of our approach, we highlight our previous work [18] which addressed the problem of eradication of pests using a so-called adaptive feedback control scheme, where the feedback switches through a number of distinct control strategies. Adaptive control is a broad term, with no one single agreed definition, and traces its roots back to the control of aircraft in the 1950s. The early history is discussed in the review [2], and [3] is a more recent review. We note that in natural resource management the word "adaptive" generally means a feedback, see [36]. Under the assumption that at least one of these strategies is stabilizing, and by carefully exploiting the rules by which switching is determined, in [18] we were able to demonstrate convergence of the scheme with switching terminating at a strategy that was itself stabilizing. In developing this approach, much use was made of the underlying positive systems structure, that is, dynamical systems whose evolution map leaves a positive cone invariant; see, for instance [4, 5].

The current problem is, in some sense, the opposite problem to that in [18]. So rather than stabilization corresponding to the eradication of a resource, we instead seek persistence of that resource. Key to the present study is further exploitation of the underlying positive systems structure. In fact, in some sense this structure is far more crucial in a context of persistence than it is in a context of stabilization. Roughly, this is because, under reasonable conditions, the trajectory $x(t)$ of a system of positive difference equations can be bounded from above and this proves crucial in deriving the switching rules. Where positive systems differ from general systems is that we can also bound trajectories from below or, in fact, bound $1/\|x(t)\|$ from above. This simple observation then means we can develop switching mechanisms for persistence built around the behaviour of $1/\|x(t)\|$ in a way similar to the how the switching mechanisms for stabilization were built in [18] around $\|x(t)\|$.

Thus, here we present theoretical results relating to the dynamic behaviour of our so-called adaptive switching feedback control scheme under different scenarios for the dynamics of x, that is, the functions $F(h, \cdot)$ in (1.1). Our main results are Theorems 2.1 and 2.4 which, broadly, provide sufficient conditions on the functions $F(h, \cdot)$ in (1.1) under which the switching sequence asymptotically identifies and converges to a desirable strategy. We consider both linear and classes of nonlinear systems of positive difference equations, the latter including as a special case classes of scalar difference equations, sometimes called (nonlinear) maps in the difference equations literature.

The paper is organised as follows. We first gather some preliminaries. Section 2 is the technical heart of the manuscript and worked examples relating to the conservation of managed populations are presented in Section 3. A summary is contained in Section 4. Proofs of our results appear in the appendices. The present work shall contribute to the doctoral thesis of the third author, and shall appear in an expanded form in her forthcoming thesis [32].

1.1. **Preliminaries.** We collect notation and terminology used throughout our work. Let

$$\mathbb{Z}_+ := \{m \in \mathbb{Z} : m \geq 0\} \quad \text{and} \quad \mathbb{R}_+ := \{h \in \mathbb{R} : h \geq 0\}.$$

For $n \in \mathbb{N}$, we let $\mathbb{R}^n$ and $\mathbb{R}^{n \times n}$ denote the real n-dimensional Euclidean space and the set of $n \times n$ matrices with real entries, respectively. As usual, we denote the identity matrix by I.

We set $\underline{q} := \{1, 2, \ldots, q\}$ for $q \in \mathbb{N}$, to avoid repeatedly writing the more cumbersome $\{1, 2, \ldots, q\}$.

Given a matrix $A \in \mathbb{R}^{n\times n}$, we let $r(A)$ denote the spectral radius of A. For A, $B \in \mathbb{R}^{n\times n}$ with entries a_{ij} and b_{ij}, respectively, we write

$$\begin{aligned} A &\geq B \quad \text{if } a_{ij} \geq b_{ij} \quad \forall\, i \text{ and } j\,, \\ A &> B \quad \text{if } A \geq B \text{ and } A \neq B\,, \\ A &\gg B \quad \text{if } a_{ij} > b_{ij} \quad \forall\, i \text{ and } j\,. \end{aligned}$$

We let $\mathbb{R}^n_+$ denote the nonnegative orthant in $\mathbb{R}^n$ and let $\mathbb{R}^{n\times n}_+$ denote the set of nonnegative matrices, that is, $A \in \mathbb{R}^{n\times n}_+$ if $A \geq 0$. The matrix A is said to be *positive* or *strictly positive* if $A > 0$ or $A \gg 0$, respectively, with the corresponding conventions for vectors $v \in \mathbb{R}^n_+$. A nonnegative square matrix A is *irreducible* if, for every i and j, there exists nonnegative integer k such that $(A^k)_{ij} > 0$. We recall that the Perron-Frobenius theorem ensures that if A is irreducible then $r(A)$ is a positive eigenvalue of A, with corresponding left and right eigenvectors which can be chosen to be strictly positive. A nonnegative square matrix A is *primitive* if there exists a nonnegative integer k such that $A^k \gg 0$.

Throughout we equip Euclidean space $\mathbb{R}^n$ with the one-norm $\|\cdot\| := \|\cdot\|_1$. We also use the symbol $\|\cdot\|$ to denote the corresponding induced matrix norm. We comment that our results hold for any monotonic norm on $\mathbb{R}^n$.

2. An adaptive switching feedback control scheme.

We present our algorithm for switching between strategies. Recall the context that x is assumed to be governed by (1.1), where strategy $h \in \underline{q}$ is to be determined. To apply feedback control requires some per time-step measurements of the quantity x. We assume that the whole state $x(t)$ is not necessarily known. Indeed, in an ecological setting, there may be stage-classes which are expensive, laborious or ineffective to measure, such as pelagic or subterranean stage-classes. Thus, we assume that

$$y = Cx\,, \tag{2.1}$$

that is, $y(t)$ contains the information about $x(t)$ which is assumed available to the modeller at time-step t for feedback purposes. The matrix C is order $p \times n$, where n is the dimension of the state vector, and p denotes the number of per time-step measurements taken. Of course, the case $C = I$ corresponds to the situation where complete knowledge of $x(t)$ is available. Further, C is assumed throughout to have no zero rows as these correspond to trivial (zero) measurements of x, and are as such inappropriate.

We introduce a sequence τ satisfying

(T) τ is a positive, strictly increasing and unbounded (scalar) sequence with $\tau(0) = 0$ and such that

$$\frac{\tau(j+1)}{\tau(j)} \to \infty \quad \text{as} \quad j \to \infty\,.$$

Intuitively, **(T)** means that asymptotically τ grows faster than exponentially, for any exponent.

Given such a τ, we define $\mathcal{K} : \mathbb{R}_+ \to \{1, 2, \ldots, q\}$ by

$$\mathcal{K}(z) := \begin{cases} 1, & z = 0\,, \\ (j \mod q) + 1, & z \in (\tau(j-1), \tau(j)], \quad j \in \mathbb{N}\,. \end{cases}$$

Assumption **(T)** implies that $\mathcal{K}(z)$ is well-defined for all $z \geq 0$. Moreover, for given $z \geq 0$, the evaluation $\mathcal{K}(z)$ returns an integer in $\underline{q}$ which shall index the strategy to be applied.

We consider the following switched system

$$x(t+1) = F(\mathcal{K}(s(t)), x(t)), \quad x(0) = x_0, \quad t \in \mathbb{Z}_+\,, \tag{2.2}$$

where the sequence s is called the switching sequence and is to-be-determined as a function of the measured variable y.

We propose the following update law for the switching sequence

$$s(t+1) = s(t) + \begin{cases} 0, & M \leq \|y(t)\|,\ \|y(t)\| = 0\,, \\ \dfrac{1}{\|y(t)\|}, & \|y(t)\| < M\,, \end{cases} \qquad s(0) = s_0\,, \tag{2.3}$$

where $M > 0$ and s_0 are design parameters, and y is given by (2.1).

The feedback interconnection of (2.2) and (2.3) gives rise to the system of difference equations

$$\left.\begin{aligned} x(t+1) &= F\big(\mathcal{K}(s(t)), x(t)\big), \quad x(0) = x_0\,, \\ s(t+1) &= s(t) + \begin{cases} 0, & M \le \|y(t)\|,\ \|y(t)\| = 0\,, \\ \dfrac{1}{\|y(t)\|}, & \|y(t)\| < M\,, \end{cases} \qquad s(0) = s_0\,, \end{aligned}\right\} \quad t \in \mathbb{Z}_+\,, \tag{2.4}$$

which we call an adaptive switching feedback control scheme. It is clear that, for each fixed $(x_0, s_0) \in \mathbb{R}^n_+ \times \mathbb{R}_+$, and sequence τ satisfying **(T)**, there is a unique solution of (2.4) which we denote by (x, s). When $x_0 = 0$, this solution is the trivial solution $(0, s_0)$ which we shall avoid by assuming that $x_0 > 0$.

The proceeding two subsections investigate the asymptotic behaviour of (2.4) under different assumptions for the terms $F(h, \cdot)$ in (2.4).

2.1. **The linear case.** Here we shall assume that $F : \underline{q} \times \mathbb{R}^n_+ \to \mathbb{R}^n_+$ in (2.4) is given by:

$$F(h, z) := A_h z \quad \forall\,(h, z) \in \underline{q} \times \mathbb{R}^n_+\,, \tag{2.5}$$

for $A_1, \dots, A_q \in \mathbb{R}^{n\times n}_+$. Thus, associated with (2.4) are

$$x(t+1) = F(h, x(t)) = A_h x(t)\,, \quad x(0) = x_0\,, \quad t \in \mathbb{Z}_+,\ h \in \underline{q}\,, \tag{2.6}$$

which are linear systems of positive difference equations.

We formulate the following assumption.

(L1) For each $h \in \underline{q}$, the matrix $A_h \in \mathbb{R}^n_+$ is irreducible.

Here **L** stands for linear and **(L1)** ensures that solutions of the difference equation (2.6) remain nonnegative when x_0 is nonnegative, for any sequence of switches. The irreducibility assumption in **(L1)** is natural in many applied settings, for instance in ecological models, see [33].

As is well-known, for each $h \in \underline{q}$, the asymptotic dynamics of (2.6) are determined by $r(A_h)$. We formulate the following assumption for $h \in \underline{q}$:

(L2) One of the following holds:
(a) $r(A_h) < 1$ (b) $r(A_h) > 1$.

Clearly, for each fixed $h \in \underline{q}$ such that **(L1)** and **(L2)**(a) holds, there exist $N_h > 0$ and $\lambda_h \in (0, 1)$ such that the solution x of (2.6) satisfies

$$\|x(t+\theta)\| \le N_h \lambda_h^t \|x(\theta)\| \quad \forall\, t, \theta \in \mathbb{Z}_+\,. \tag{2.7}$$

In other words, under these strategies, the solution $x(t)$ decays to zero exponentially over time, which is the situation we wish to avoid, and consequently we term these strategies *undesirable*.

Similarly, for each fixed $h \in \underline{q}$, assumptions **(L1)** and **(L2)**(b) entail that the solution x of the difference equation (2.6) diverges in norm as $t \to \infty$, for all nonzero x_0. In other words, under these strategies the growth of x is unbounded, and consequently we term these *desirable* strategies. Unbounded exponential growth is not realistic in applied settings, and is a deficiency of linear models. These shortcomings are addressed in Section 2.2 where nonlinear models are considered. However, linear models are ubiquitous in applied sciences, a linear model serves to illustrate the key ideas, and may be valid for the initial growth of small quantities (such as populations, which are likely to be the subjects of conservation management).

An essential ingredient for the adaptive switching feedback control scheme (2.4) is a coupling condition between the dynamics generated by $F(h, \cdot)$ in (2.5), determined in this case by a common lower bound A_- for the A_h, and the measurements $y = Cx$. We propose the following.

(L3) There exists irreducible $A_- \in \mathbb{R}^{n\times n}_+$ such that $A_h \ge A_-$ for all $h \in \underline{q}$. Further, there exist $k \in \mathbb{Z}_+$ and $w \in \mathbb{R}^p_+$ such that $w^T C A_-^k \gg 0$.

Recalling that C is always assumed to have no zero rows, assumption **(L3)** is satisfied, for instance, if

- there exists primitive $A_- \in \mathbb{R}^{n\times n}_+$ such that $A_h \ge A_-$ for all $h \in \underline{q}$;
- $C \gg 0$, that is, C is strictly positive.

Briefly, a consequence of **(L3)** is that, for some constants $c_1, c_2 > 0$

$$c_1\|x(t)\| \le \|y(t)\| \le c_2\|x(t)\| \quad \forall\, t \in \mathbb{Z}_+,\ t \ge k\,,$$

so that, after k time-steps, the norm of the (known) measured variable $y(t)$ is equivalent in the above sense to that of (the unknown) $x(t)$.

Our main result of this section is the following.

Theorem 2.1. *Consider* (2.4) *where F is as in* (2.5) *with $q \geq 2$. Assume that τ satisfies* **(T)**, *that* **(L1)**–**(L3)** *hold, and that* **(L2)**(b) *holds for at least one $h \in \underline{q}$. Then, for each $(x_0, s_0) \in \mathbb{R}^n_+ \times \mathbb{R}_+$, with $x_0 \neq 0$, the following statements hold*

(i) *s is bounded, and hence (as non-decreasing) convergent;*
(ii) *$\mathcal{K}(s(t)) \to h$ as $t \to \infty$ where h is such that* **(L2)**(b) *holds;*
(iii) *x is divergent.*

We provide some commentary on the above theorem.

Remark 2.2. (a) In words, Theorem 2.1 states that the adaptive switching feedback control system (2.4) identifies (or learns) a desirable strategy, assuming that there is one to be found. This is without knowing the underlying model for the dynamics of x exactly or the effects of the strategies, rather, the qualitative assumptions **(L1)**–**(L3)** are imposed.

(b) For simplicity, we have excluded the case that there are strategies for which $r(A_h) = 1$, which corresponds to asymptotic stasis of the solution of (2.6). More discussion of this case shall appear in [32].

(c) We comment on the choice M. Whilst the conclusions of Theorem 2.1 hold for any $M > 0$, the choice of M can control the speed with which s and $\mathcal{K}(s)$ converge. Roughly speaking, if M is picked to be small, then s will grow slower as $\|y(t)\| > M$ leads to $s(t+1) = s(t)$. This may lead to an intolerably small $\|y(t)\|$ before a desirable strategy is chosen. Conversely, if M is large, then s is "more likely" to grow faster, which on the one hand may lead to a desirable strategy being chosen faster, but on the other may lead to inadvertently switching away from a desirable strategy. $\diamond$

As a corollary we consider the situation wherein $C = I$. In this special case we are able to drop the coupling condition **(L3)**.

Corollary 2.3. *Consider* (2.4) *where F is as in* (2.5) *with $q \geq 2$ and assume that $C = I$. Assume that τ satisfies* **(T)**, *that* **(L1)** *and* **(L2)** *hold, and that* **(L2)**(b) *holds for at least one $h \in \underline{q}$. Then the conclusions of Theorem 2.1 hold.*

2.2. **A nonlinear case.** We next consider $F : \underline{q} \times \mathbb{R}^n_+ \to \mathbb{R}^n_+$ in (2.4) with the following nonlinear structure

$$F(h, z) := A_h z + b_h g_h(f_h^T z) \quad \forall\, (h, z) \in \underline{q} \times \mathbb{R}^n_+ \,. \tag{2.8}$$

Here, for each $h \in \underline{q}$, we have $A_h \in \mathbb{R}^{n\times n}_+$, $b_h, f_h \in \mathbb{R}^n_+$, and further, $g_h : \mathbb{R}_+ \to \mathbb{R}_+$ are (nonlinear) functions. For each fixed $h \in \underline{q}$, the model

$$x(t+1) = F(h, x(t)) = A_h x(t) + b_h g_h(f_h^T x(t)), \quad x(0) = x_0, \quad t \in \mathbb{Z}_+ \,, \tag{2.9}$$

contains a linear component $A_h x(t)$, and a structured (rank-one) nonlinear component $b_h g_h(f_h^T x(t))$.

We formulate the following assumptions.

(NL1) There exist $A_\pm \in \mathbb{R}^{n\times n}_+$, $b_\pm, f_\pm \in \mathbb{R}^n_+$ with $b_-, f_- \neq 0$ such that

$$A_- \leq A_h \leq A_+, \quad b_- \leq b_h \leq b_+, \quad \text{and} \quad f_- \leq f_h \leq f_+ \quad \forall\, h \in \underline{q}\,.$$

Furthermore, $r(A_+) < 1$ and $A_- + b_- f_-^T$ is irreducible.

(NL2) The $g_h : \mathbb{R}_+ \to \mathbb{R}_+$ are locally Lipschitz, positive definite functions with $g_h(0) = 0$, for every $h \in \underline{q}$. Further, there exist $\chi > 0$ and $\eta \in (0, p_+)$ such that

$$g_h(z) \leq \eta z + \chi \quad \forall\, z \geq 0\,,$$

where $p_+ := 1/f_+^T(I - A_+)^{-1} b_+ \in (0, \infty)$.

Here **NL** stands for nonlinear. Assumptions **(NL1)** and **(NL2)** together entail that solutions of the system of nonlinear difference equations (2.9) for initial condition $x_0 \in \mathbb{R}^n_+$ are nonnegative for each $h \in \underline{q}$. We note that if g_h is bounded for every $h \in \underline{q}$, then the affine linear bound in **(NL2)** is satisfied, and the conjunction of **(NL1)** and **(NL2)** entails that solutions of (2.9) are bounded by [14, Theorem 4.4, statement (a)]. The assumption that $g_h(0) = 0$ implies that $(x, s) = (0, s_0)$ is a constant solution of (2.4), for any $s_0 > 0$.

For each $h \in \underline{q}$, the asymptotic dynamics of (2.9) are determined by the interplay of the linear data, namely A_h, b_h and f_h, captured through the quantity

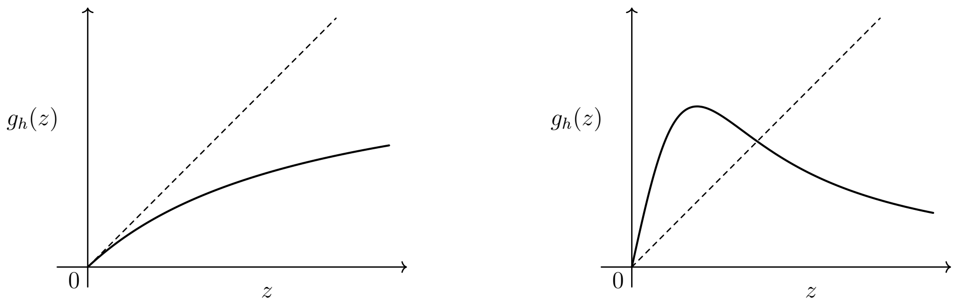
$$p_h := 1/(f_h^T (I - A_h)^{-1} b_h)\,,$$

and the nonlinear term g_h. Assumption **(NL1)** guarantees that p_h is positive and finite. We record the following qualitative properties of the functions g_h.

(NL3) One of the following holds:

$$\text{(a)}\ \liminf_{z\searrow 0}\frac{g_h(z)}{z}>0 \quad\text{and}\quad \sup_{z>0}\frac{g_h(z)}{z}<p_h \qquad \text{(b)}\ \liminf_{z\searrow 0}\frac{g_h(z)}{z}>p_h\,.$$

Figure 2.1 contains a typical illustration of the conditions **(NL3)**(a) and (b).



(A) (B)

FIGURE 2.1. Illustration of the conditions **(NL3)**(a) and **(NL3)**(b) in panels (A) and (B), respectively. The dashed straight lines have gradient $p_h > 0$.

Under assumptions **(NL1)**, **(NL2)**, and for $h \in \underline{q}$ such that **(NL3)**(a) holds, it follows from [14, Theorem 2.3] that there exist $N_h > 0$ and $\lambda_h \in (0, 1)$ such that solution x of the difference equation (2.9) satisfies (2.7). Consequently, we term strategies for which **(NL3)**(a) hold undesirable.

However, assumptions **(NL1)**, **(NL2)**, and **(NL3)**(b) together imply that there exists $K_h > 0$ such that, for all nonzero $x_0 \in \mathbb{R}^n_+$, there exists $t_* = t_*(x_0) \in \mathbb{Z}_+$ such that

$$\|x(t+t_*)\| \geq K_h \quad \forall\, t \in \mathbb{Z}_+\,.$$

In other words, under these strategies, the difference equation (2.9) is strongly $\|\cdot\|$-persistent in the terminology of [30, Definition 3.1]. We call such strategies desirable.

Finally, to parallel **(L3)**, a coupling condition between the dynamics generated by $F(h,\cdot)$ and the measurements $y = Cx$ is required. We propose the following.

(NL4) There exist $k \in \mathbb{Z}_+$ and $w \in \mathbb{R}^p_+$ such that $w^T C(A_- + b_- f_-^T)^k \gg 0$.

Recalling that C is always assumed to have no zero rows and $A_- + b_- f_-^T$ is assumed irreducible in **(NL1)**, it is routine to verify that assumption **(NL4)** is satisfied, for instance, if

- $A_- + b_- f_-$ is primitive;
- $C \gg 0$, that is, C is strictly positive.

Our main result of this section is the following.

Theorem 2.4. *Consider* (2.4) *where* F *is as in* (2.8) *with* $q \geq 2$. *Assume that* τ *satisfies* **(T)**, *that* **(NL1)**–**(NL4)** *hold, and that* **(NL3)**(b) *holds for at least one* $h \in \underline{q}$. *There exist* $M > 0$ *and* $K > 0$ *such that, for all* $(x_0, s_0) \in \mathbb{R}^n_+ \times \mathbb{R}_+$ *with* $x_0 \neq 0$, *the following statements hold*

(i) *s is bounded, and hence (as non-decreasing) convergent;*
(ii) *$\mathcal{K}(s(t)) \to h$ as $t \to \infty$ where h is such that* **(NL3)**(b) *holds;*
(iii) $\liminf_{t\to\infty} \|x(t)\| > K$.

We provide some commentary on the above theorem.

Remark 2.5. Although Theorem 2.4 does guarantee that a switching threshold M exists for the adaptive switching feedback control system (2.4) which ensures (asymptotic) selection of a desirable strategy, a drawback is that a suitable threshold M is not explicitly constructed. As outlined above the statement of theorem, a key argument in the proof of Theorem 2.4 is to exploit persistency-type results. Roughly, M must be chosen below a persistency threshold for $x(t)$ in order for that persistent strategy to be deemed desirable. Thus, in applications, the choice of M may need to be supported by other considerations. ◇

We conclude this section by noting that the material considered here encompasses certain classes of scalar difference equations. As is well-known, difference equations have been proposed as a suitable model for species with non-overlapping generations; see, for instance [21]. In particular, by taking $n = 1$, $A_h = 0$, $b_h = f_h = 1$ for all $h \in \underline{q}$, the model (2.9) reduces to the (switched) difference equation

$$x(t+1) = F(h, x(t)) = g_h(x(t)), \quad x(0) = x_0, \quad t \in \mathbb{Z}_+ . \tag{2.10}$$

Here the measured output y is assumed just to equal x.

Assumption **(NL1)** holds with $A_\pm = 0$, $b_\pm = f_\pm = 1$. Now $p_h = p_+ = 1$ for all $h \in \underline{q}$ and assumption **(NL2)** holds if $g_h : \mathbb{R}_+ \to \mathbb{R}_+$ is locally Lipschitz and positive definite for all $h \in \underline{q}$, and there exist $\gamma \in (0, 1)$, $\Gamma > 0$ such that

$$g_h(z) \le \gamma z + \Gamma \quad \forall\, (h, z) \in \underline{q} \times \mathbb{R}_+ . \tag{2.11}$$

Furthermore, assumption **(NL3)** becomes

(NL3)$'$ One of the following holds:

$$\text{(a)}\ \liminf_{z\searrow 0} \frac{g_h(z)}{z} > 0 \quad \text{and} \quad \sup_{z>0} \frac{g_h(z)}{z} < 1 \qquad\qquad \text{(b)}\ \liminf_{z\searrow 0} \frac{g_h(z)}{z} > 1 .$$

Finally, in this special case, assumption **(NL4)** is always satisfied. Therefore, the conclusions of Theorem 2.4 apply to (2.4) with F as in (2.10) provided that (2.11) and **(NL3)**$'$ hold, and that **(NL3)**$'$(b) is satisfied for at least one $h \in \underline{q}$.

3. **Examples.**

Here we apply the theory developed in the previous sections to several examples from population ecology. Our main results are Theorems 2.1 and 2.4 and, roughly, both state that the adaptive switching feedback control scheme (2.4) finds or selects a strategy under which x persists in some form, assuming that there is such a strategy to be found. This persistence could be: that x exhibits unbounded growth; that x exhibits persistent fluctuations, or; that x converges to a nonzero equilibrium. Moreover, the non-decreasing switching sequence s which determines the choice of strategy via $\mathcal{K}(s(t))$ converges.

The section is organised as follows. Example 3.1 illustrates the theory from Section 2.1, and Examples 3.2 and 3.3 illustrate the theory from Section 2.2. Some discussion of performance is considered in Section 3.1. All numerical simulations were performed in MATLAB R2018a, and random numbers are actually pseudorandomly generated. We note that in order to numerically simulate models, the models must be specified. By specifying a model, it can clearly *a fortiori* be seen which strategies are desirable, and which are undesirable. However, recall our standing assumption that the effect of the control strategies is not known in practice.

Example 3.1. We consider an example which fits the framework of Section 2.1. In an ecological setting the discrete-time linear system of difference equations (2.6) is called a matrix population project model (PPM); see, for instance [7]. The state $x(t)$ describes the discrete stage structure of the population at time-step $t \in \mathbb{Z}_+$. Discrete stage-classes may be structured according to age or developmental stages, such as insect instars. We illustrate our results from Section 2.1 by considering a matrix PPM for North Atlantic right whales (*Eubalaena glacialis*) [16] — which becomes a model of the form (2.5) under the inclusion of control by application of a discrete management strategy. In this model, time-steps correspond to years and units correspond to 100 whales. We use the female population model with four stage classes, where stage classes 1–4 represent: calves; immature females; mature females; and, mature females with newborn calves (mothers), respectively. Calves are defined to be individuals that are sighted along with their mother. Similarly, mothers are females that are sighted with a newborn offspring. Immature females are those that are known to be less than nine years old, whilst mature females are those that are known to be at least nine years old or have previously been spotted with a calf.

The North Atlantic right whale has a declining population and has been categorised as endangered by the IUCN Red List of Threatened Species [9], thus they are of conservation interest. The species is a partial migrant that is known to use feeding grounds in and around the Gulf of Maine during spring through to autumn and calving or overwintering grounds off the southeastern United States (SEUS) during the winter [17]. The SEUS can be used by all demographic groups as an overwintering ground, but there is much variation in the number of non-breeders carrying out the migration across the years [17]. However, the SEUS is established as a calving ground. Hence, mothers are more likely to be observed than non-breeders. The probability that mothers are captured (observed) at least once during a given winter is close to one [16, 17]. To account for this, C takes the form

$$C := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ,$$

meaning that mothers (stage 4) and their calves (stage 1) are observed per time step.

We assume that two management strategies are available. We assume that the population projection matrix, for both strategies, is of the form

$$A_h = \begin{pmatrix} 0 & f_{1,2} & f_{1,3} & 0 \\ s_{2,1} & s_{2,2} & 0 & 0 \\ 0 & s_{3,2} & s_{3,3} & s_{3,4} \\ 0 & s_{4,2} & s_{4,3} & 0 \end{pmatrix} \quad h \in \{1,2\}\,. \tag{3.1}$$

Here $s_{j,i}$ represents the transition probability from stage i to stage j (not to be mistaken with the switching sequence s), and $f_{1,i}$ represents the probability that a female in stage i gives birth to a female calf and that the calf survives long enough to be catalogued. It is assumed that calves are catalogued on average midway through their first year, and that the mother must also survive this long for the calf to survive. It is assumed that all probabilities are positive, and depend on the strategy indexed by $h \in \{1,2\}$. Thus, the matrix A_h is clearly nonnegative and is irreducible. Hence, assumption **(L1)** is satisfied. The vital rates used in (3.1) for each strategy are given in Table 3.1.

Strategy (h)	Vital rates								
	$s_{2,1}$	$s_{2,2}$	$s_{3,2}$	$s_{3,3}$	$s_{3,4}$	$s_{4,2}$	$s_{4,3}$	$f_{1,2}$	$f_{1,3}$
1	0.85	0.85	0.08	0.8	0.64	0.02	0.19	0.0080	0.0760
2	0.92	0.86	0.08	0.8	0.83	0.02	0.19	0.0091	0.0865

TABLE 3.1. Vital rates used in the population projection matrices A_h in (3.1).

Strategy 2 corresponds to the average vital rates from 1980–1995 in [7]. Whereas strategy 1 corresponds to the vital rates of 1995 in [7], where the authors note that the mortality has increased, especially in mother whales. Studies cited by Fujiwara and Caswell in [7], as well as more recent studies, attribute the increased mortality of mothers to: collisions with ships; entanglement with fishing gear; and, changes in prey availability caused by climate-associated fluctuations in prey availability [6, 24, 25]. The vital rates for strategy 1 lead to $\rho_1 := r(A_1) = 0.9762$. Thus, **(L2)**(a) is satisfied, in other words strategy 1 is undesirable in the present context. Whereas, $\rho_2 := r(A_2) = 1.0098$, hence **(L2)**(b) is satisfied and strategy 2, of the two strategies, is deemed desirable.

It is clear from Table 3.1 that $A_2 \geq A_1$ and a routine calculation shows that $CA_1^2 \gg 0$. Consequently, the coupling condition **(L3)** holds with $A_- := A_1$, $k = 2$ and for any $w \gg 0$.

In the simulations, we set $s_0 := 0.2$, $M := 1.2$, that is 120 whales, and define the sequence τ via

$$\tau(j+1) = 0.35 + (j+1)\tau(j), \quad \tau(0) = 0, \quad j \in \mathbb{Z}_+\,, \tag{3.2}$$

which evidently satisfies the growth assumption **(T)**.

We perform three simulations, each with a different initial condition x_0^i, given in (A.1) in Appendix A.1. The initial conditions are random perturbations of the so-called stable stage structure of either strategy 1 or 2 (randomly chosen), that is, perturbations of a strictly positive $w_h \in \mathbb{R}^n_+$ such that

$$A_h w_h = r(A_h) w_h \quad h \in \{1,2\}\,,$$

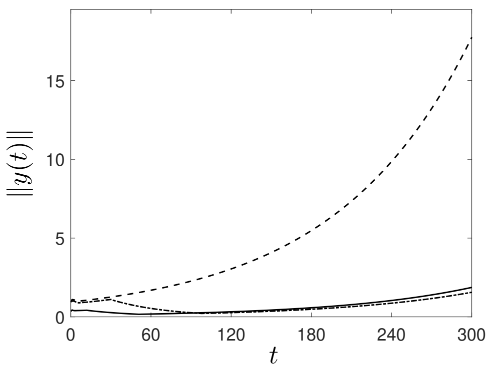
which are uniquely determined up to a multiplicative constant. We take x_0 such that

$$0.5 \times 4.58 \leq \|x_0\| \leq 1.5 \times 4.58\,.$$

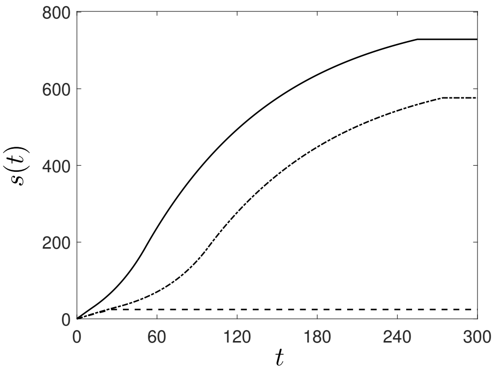
The figure 4.58 is a recent estimate of the population size of North Atlantic right whales from [26].

Numerical simulation results are plotted in Figure 3.1. Each panel contains three simulations, corresponding to the three initial conditions. Figure 3.1(a) plots the observed population size, $\|y(t)\|$, against time t. We see that for each of the initial conditions there is eventually unbounded exponential growth of y, and hence x. Figures 3.1(b) plots the switching sequence, $s(t)$, against time t. The switching sequences are bounded and eventually constant. The North Atlantic right whale has a generation length of 24 years [9], thus our model has been run for 12.5 generations.

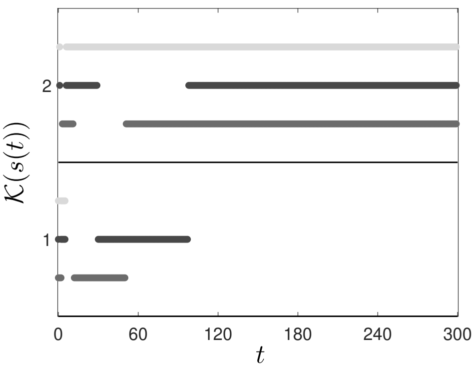
Figure 3.1(c) shows the time over which each strategy is applied, that is, $\mathcal{K}(s(t))$ is plotted against t. We see that $\mathcal{K}(s(t)) \to 2$ as $t \to \infty$, that is the switching sequence eventually settles on the second strategy, which recall is the desirable strategy in this example. Figure 3.1(d) illustrates the early switches in more detail, and shows how switches can skip strategies leading in this example to no change of strategy. For example, strategy 1 is applied at time $t = 0$ for each initial condition x_0; then, at $t = 1$, the first initial condition switches, but skips a whole τ interval, and so strategy 1 is still applied. The second and third initial conditions, however, switch to strategy 2 at $t = 1$. It is also interesting to note that, for small t, initial conditions 2 and 3 exhibit similar growth of s, however, from initial condition 3, we see that $\mathcal{K}(s(t))$ converges to a desirable strategy much faster than either of other initial conditions. $\diamond$



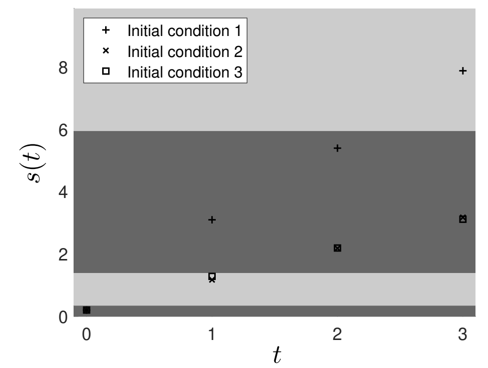
(A) Trajectories of the observed population size $\|y(t)\|$ at time t. The first, second and third initial conditions are represented by: a solid line; a dash-dot line; and a dashed line, respectively.



(B) Graph of the switching sequence $s(t)$ at time t. The first, second and third initial conditions are represented by: a solid line; a dash-dot line; and a dashed line, respectively.



(C) Graph of the strategy applied $\mathcal{K}(s(t))$ at time t. The first, second and third initial conditions are represented by: medium; dark; and light grey, respectively.



(D) Representation of the switching sequence $s(t)$ at time t, for the first three time steps. The dark and light grey shaded regions correspond to strategies 1 and 2 being applied, respectively.

FIGURE 3.1. Numerical simulations of the adaptive switching feedback control scheme (2.4) for the North Atlantic right whale model described in Example 3.1.

Example 3.2. We consider an example which fits the framework of Section 2.2. Before which, we give some further motivation and background in an ecological context for models of the form (2.9), that is,

$$x(t+1) = F(h, x(t)) = A_h x(t) + b_h g_h(f_h^T x(t)), \quad x(0) = x_0, \quad t \in \mathbb{Z}_+ . \tag{2.9}$$

As with the structured linear models in Section 2.1, here the state variable $x(t)$ describes the discrete stage structure of the population at time-step $t \in \mathbb{Z}_+$. In contrast to (2.6), the model (2.9) contains a structured, nonlinear component, and so (2.9) can model both so-called density-independent and density-dependent biological processes. As already stated in Section 2.2, the conjunction of **(NL1)** and **(NL2)** entails that solutions of both (2.4) and (2.9) are bounded.

Omitting the subscripts from (2.9) for clarity, typically, the matrix A in (2.9) captures survival and movement between stage-classes, whilst the term $bg(f^T x(t))$ models transitions which are limited by density, such as recruitment. In this case, the vector term b usually models the distribution into population structure of new recruits, $f^T x(t)$ is the density of possible recruits at time-step t. Then $g(f^T x(t))$ gives the establishment probability of a possible recruit, given $f^T x(t)$ possible recruits. Another interpretation is that f^T is a vector containing the per time-step fecundity of each stage class, leading to $f^T x(t)$ new individuals per time-step. The function $z \mapsto g(z)/z$ denotes the density-dependent per-capita survival probability of a new recruit, leading to $g(f^T x(t))$ new recruits per time-step. We refer the reader to [11] for further biological interpretation of models of the form (2.9), and note that there are now numerous papers which consider such models in an ecological setting, including [12, 13, 27, 31, 35]. Models of the form (2.9) are reasonably well-understood and amenable to mathematical analysis, yet also display a rich variety of realistic dynamical behaviour.

To illustrate our results we consider a density-dependent population projection matrix model for the trout cod (*Maccullochella macquariensis*) [34] — which becomes a model of the form (2.8) under the inclusion of control by application of a discrete management strategy. In this model units correspond to 10^3 fish. We use the female population with an annual time step and seven stage classes, where stage classes 1–4 represent juveniles, that is 1, 2, 3 and 4–year old individuals, respectively. Stage classes 5–7 represent adults, that is sexually mature female fish aged 5, 6 and 7+ years, respectively.

The trout cod has been categorised as vulnerable by the IUCN Red List of Threatened Species [23]. There is only one natural self-sustaining population [22, 34], located in a 200km stretch of the Murray River [34]. Thus, the trout cod is of conservation interest and has been the subject of reintroduction programs [22, 34]. In our simulations we have assumed that there are only two available strategies for management of the species, and that they only affect the nonlinear term g_h in (2.8). In particular, $A_1 = A_2 = A$, and similarly for b and f. We assume that the linear data are given by

$$A := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.3759 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.6014 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.7023 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.7591 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.7954 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.8203 & 0.8931 \end{pmatrix}, \quad b := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad f := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.9711 \\ 0.9711 \\ 2.5512 \end{pmatrix}. \tag{3.3}$$

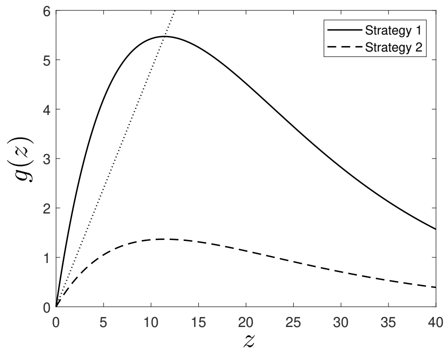
The spectral radius of A is $r(A) = 0.8931 < 1$. We set $A_\pm = A$, $b_\pm = b$ and $f_\pm = f$. In this case, $A_- + b_- f_-^T$ is primitive (and hence irreducible), and so the assumption **(NL1)** on the linear data holds.

We assume that for both strategies the functions $g_h : \mathbb{R}_+ \to \mathbb{R}_+$ are Ricker functions, that is,

$$g_h(z) = \sigma_h z e^{-z/RCC_h} \quad \forall\, z \geq 0,\ \forall\, h \in \{1, 2\}, \tag{3.4}$$

where σ_h and RCC_h are positive parameters given in Figure 3.2b. Specifically, RCC_h is the carrying capacity for larval recruits. The functions g_h evidently satisfy **(NL2)**, noting that the affine linear bound clearly holds as the functions g_h are bounded. Since the linear data are the same for both strategies considered, we have $p_h = p = 0.4792$. The functions g_h are plotted in Figure 3.2a, with parameters as in Figure 3.2b.

Figure 3.2a illustrates that strategy 1 satisfies **(NL3)**(b) and is, therefore, desirable. From the same figure we see that strategy 2 satisfies **(NL3)**(a) and is, therefore, deemed undesirable.



(A) Graphs of g_1 (solid) and g_2 (dashed) from (3.4). The dotted line has slope p.

Strategy (h)	Parameters	
	σ	RCC
1	1.3026	11.417
2	0.3257	11.417

(B) Parameters for g_h in (3.4).

FIGURE 3.2. Functions g_h, panel (a), with parameters, panel (b), from Example 3.2.

For our simulations, we assume that all adult fish can be observed, that is stage classes 5–7. Thus, C takes the form

$$C := \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.5}$$

Since $A_- + b_- f_-^T = A + bf^T$ is primitive, it follows that the coupling condition **(NL4)** holds. The sequence τ is defined by (3.2). Therefore, the hypotheses of Theorem 2.4 are satisfied.

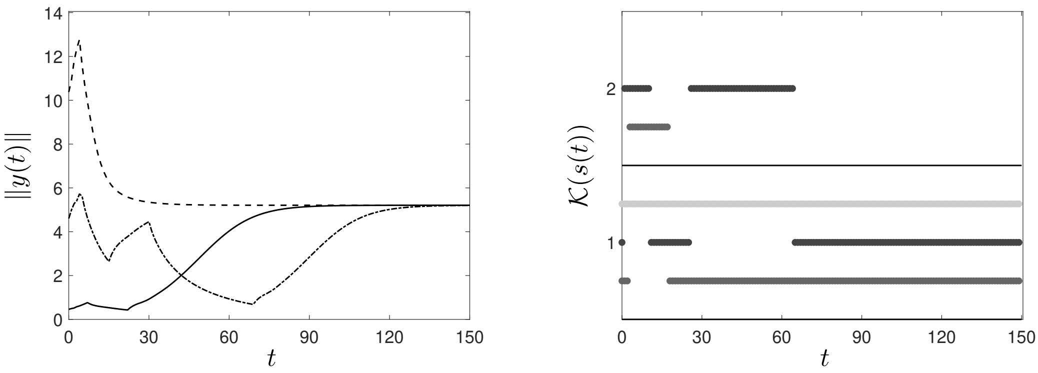
For the following numerical simulations, we set $s_0 := 0.2$, $M := 5$, that is, 5000 fish.

As in the linear case, we perform three simulations, each with a different initial condition x_0^i, given in (A.2) in Appendix A.1. The initial conditions are random perturbations of the equilibrium x^* of (2.9) associated with strategy 1, meaning

$$x^* := (I - A)^{-1} bz^* \quad \text{where } z^* > 0 \text{ solves} \quad g_1(z^*) = pz^* .$$

Numerical simulation results are plotted in Figure 3.3. Each panel contains three simulations, corresponding to the initial conditions in (A.2). The panels mirror the first and third panels of Figure 3.1. Figure 3.3a plots the observed population size, $\|y(t)\|$, against time t. We see that for each of the initial conditions, $\|y(t)\|$ eventually converges to a stable equilibrium, and importantly, persists at a level greater than M. This indicates that M has been chosen sufficiently small in this example. Figure 3.3b shows the time over which each strategy is applied, that is, $\mathcal{K}(s(t))$ is plotted against time t. We see that, for each initial condition, $\mathcal{K}(s(t)) \to 1$ as $t \to \infty$, which recall in this example corresponds to the desirable strategy where **(NL3)**(b) holds.

To illustrate the robustness of the adaptive feedback switching control model (2.4) with respect to uncertainty in initial conditions, we simulate (2.4) for the trout cod model considered here with 100 random initial conditions x_0. The results are plotted in Figure 3.4. In Figure 3.4a we see that, for all initial conditions, x converges to the equilibrium x^* as $t \to \infty$ and, hence persists, whilst Figure 3.4b shows the convergence of $s(t)$ as $t \to \infty$. ◇



(A) Trajectories of the observed population size $\|y(t)\|$ at time t. The first, second and third initial conditions are represented by: a solid line; a dash-dotted line; and a dashed line, respectively.

(B) Graph of the strategy applied $\mathcal{K}(s(t))$ at time t. The first, second and third initial conditions are represented by: medium; dark; and light grey, respectively.

FIGURE 3.3. Numerical simulations of the adaptive switching feedback control scheme (2.4) for the trout cod model from Example 3.2.

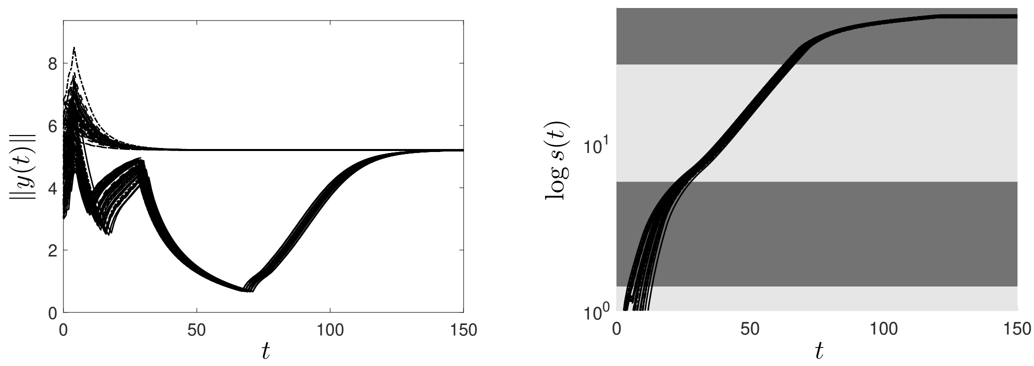
Example 3.3. We consider a scalar example which fits the framework of the switched difference equation (2.10) from Section 2.2. Specifically, we consider the Ricker model, see [29], namely

$$x(t+1) = g(x(t)) = x(t)e^{-(\mu+\eta)} + \alpha x(t) e^{-\beta x(t)} \quad \forall\, t \in \mathbb{Z}_+ , \tag{3.6}$$

for the Gold-spotted grenadier anchovy (*Coilia dussumieri*), where the state $x(t)$ describes the biomass of mature individuals in a population at time-step $t \in \mathbb{Z}_+$. The function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is given by

$$g(z) = e^{-(\mu+\eta)} z + \alpha z e^{-\beta z} \quad \forall\, z \geq 0 . \tag{3.7}$$

Here μ and η are nonnegative parameters denoting the natural mortality and fishing mortality, respectively. The positive parameter $\alpha > 0$ is the maximum per-capita reproduction rate and $\beta > 0$ affects the density-dependent mortality near equilibrium abundance [29, Supporting Information]. Recall that the model (3.6) is a special case of (2.9) with $n = 1$, $A_h = 0$ and $b_h = f_h = 1$ for all $h \in \underline{q}$.



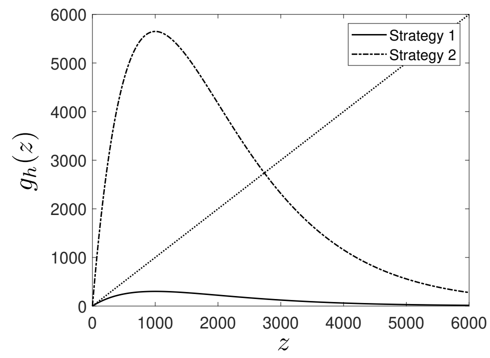
(A) Trajectories of the observed population size $\|y(t)\|$ at time t.

(B) Semilog plot of switching sequence $s(t)$ at time t. The dark and light grey shaded regions denote strategies 1 and 2, respectively.

FIGURE 3.4. Numerical simulations of the adaptive switching feedback control scheme (2.4) for the trout cod model from Example 3.2 with 100 random initial conditions x_0.

In the model (3.6), time-steps correspond to years and units correspond to biomass in kg. This anchovy is of economic importance and has a gradually increasing demand [1], which motivates appropriate management. The difference equation (3.6) becomes a model of the form (2.8) under the inclusion of control by application of a discrete management strategy, here meaning that $\mu = \mu_h$, $\eta = \eta_h$, $\alpha = \alpha_h$ and $\beta = \beta_h$, for strategies indexed by h, with corresponding function g_h of the form (3.7). In light of (3.7), it is clear that the functions g_h satisfy (2.11), provided that $\mu_h + \eta_h > 0$.

We assume that there are two management strategies available with associated parameter values recorded in Figure 3.5b. The functions g_h and associated parameter values are plotted in Figure 3.5. Figure 3.5a shows that **(NL3)**$'$(a) and (b) are satisfied by strategies 1 and 2, respectively. Thus, in this example, strategy 2 is the desirable strategy. We note that the linear component $e^{-(\mu_h+\eta_h)}z$ in the functions g_h yield that the g_h are unbounded. However, since $e^{-(\mu_1+\eta_1)} \approx 10^{-3}$, the contribution to $g_h(z)$ from the linear terms $e^{-(\mu_h+\eta_h)}z$ is very small relative to that from the nonlinear terms $\alpha_h z e^{-\beta_h z}$, certainly when $z \in [0, 0.5 \times 10^3]$, as seen in Figure 3.5a.



(A) Graph of g_h, see (3.7) for strategies $h \in \{1, 2\}$, with dotted line with unity slope.

Strategy (h)	Vital rates			
	μ	η	α	β
1	2.46	5.20	0.8187	0.001
2	1.68	3.10	15.3329	0.001

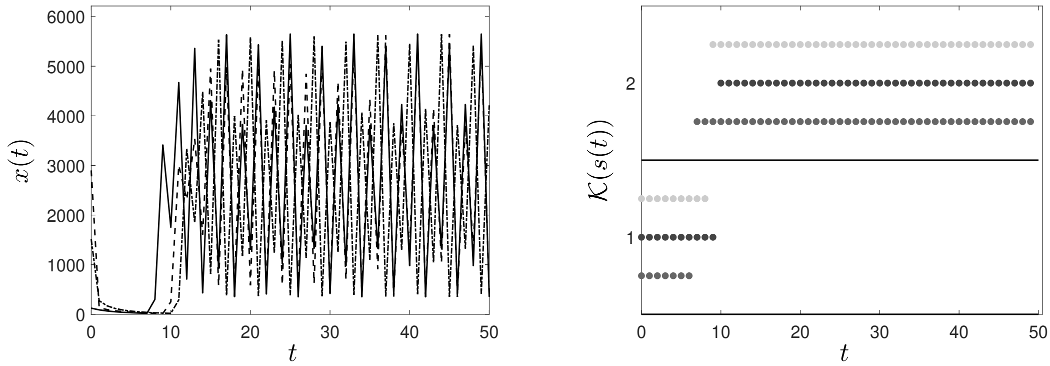
(B) Parameters for g_h.

FIGURE 3.5. Functions g_h, panel (a), with parameters, panel (b), from Example 3.3.

To simulate (2.4) in the current setting, we define the switching sequence τ via (3.2). With these choices, the hypothesis of Theorem 2.4 are satisfied. For the following simulations, we set $s_0 := 0.2$ and $M := 200$. As before, we perform three simulations, with the following randomly generated initial condition x_0^i,

$$x_0^1 := 123.70\,, \quad x_0^2 := 1515.1\,, \quad x_0^3 := 2899.2\,. \tag{3.8}$$

Numerical simulations are plotted in Figure 3.6. Each panel contains three simulations corresponding to the initial conditions in (3.8). The panels mirror those in Figure 3.3. Figure 3.6a plots the (scalar) population size $x(t)$ against time t. We see that for each of the initial conditions, there are eventually persistent fluctuations. Figure 3.6b plots $\mathcal{K}(s(t))$ against time t, that is, the strategy applied at time-step t. We see that in each case the desirable strategy is found. $\diamond$



(A) Trajectories of the observed population size $x(t)$ at time t. The first, second and third initial conditions are represented by: a solid line; a dash-dot line; and a dashed line, respectively.

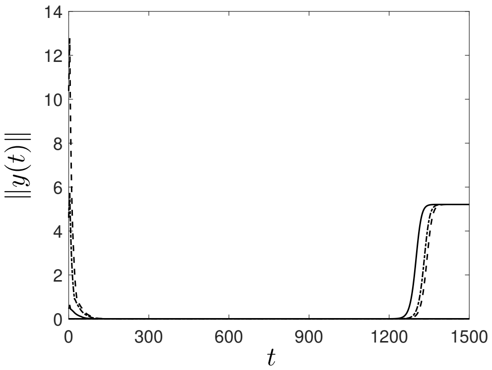
(B) Graph of the strategy applied $\mathcal{K}(s(t))$ at time t. The first, second and third initial conditions are represented by: medium; dark; and light grey, respectively.

FIGURE 3.6. Numerical simulations of the adaptive switching feedback control scheme (2.4) for the Gold-spotted grenadier anchovy model from Example 3.3.

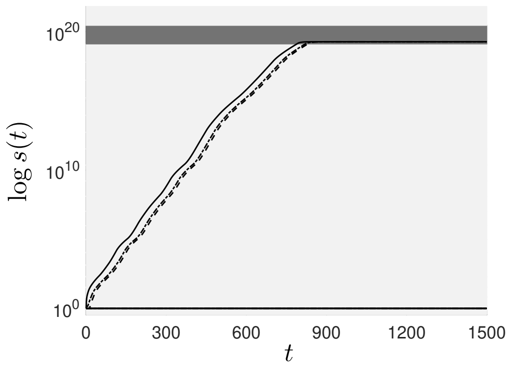
3.1. **Performance of the adaptive feedback switching control scheme.** We conclude this section by discussing some aspects of the performance of the adaptive feedback switching control scheme (2.4). First, our proof does not show this but, in the context of Theorem 2.4, it appears numerically that some persistent strategies may be ruled out by choosing M too large. In this sense, it appears numerically that the choice of M can filter between persistent strategies, so that some are deemed undesirable, and others desirable. This allows the situation, for instance, where *every* strategy is persistent, and M is used to asymptotically select a strategy which persists above a desired threshold.

Second, and as commented in Remark 2.2, our main results are asymptotic in nature. Of course, in the potential real-world applications we have in mind such as conservation, time is often of the essence, and it is imperative that control actions, or management strategies, perform well over short time periods. The power of our results is that they place relatively few constraints on required knowledge of the to-be-controlled models. This is advantageous when seeking to control highly uncertain or poorly understood systems. They are also (at least theoretically) very simple to implement. There is also some considerable freedom in certain design parameters, such as the switching threshold M, the initial state s_0 of the switching sequence, and the underlying sequence τ which determines the rate of switching via the defining property that $\mathcal{K}(z) = (k \mod q) + 1$ for all $z \in (\tau(k-1), \tau(k)]$ for given $k \in \mathbb{N}$ selects strategy $(k \mod q) + 1$. A tradeoff with the choice of τ is that if the τ intervals are too "small", then the strategy may change too often, and not give desirable strategies sufficient time to establish $\|y(t)\| \geq M$. If the τ intervals are too "large", then the dynamics may spend unnecessarily long under an undesirable strategy before switching again. We note that the sequence τ only needs to grow "faster than exponentially" asymptotically, and can be chosen to increase linearly or quadratically at first, for instance. The purpose of the present paper is to establish a theoretical underpinning of the novel adaptive feedback switching control scheme (2.4), and in our numerical simulations we have not tried to optimise or realistically tune any of these quantities. Other considerations may provide insight into how to choose these parameters in any given bespoke context.

We have observed that performance may be poor when there are many more undesirable strategies than desirable strategies, meaning informally that the system (2.4) spends considerable time applying undesirable strategies before trialling a desirable strategy. Although these situations satisfy the hypotheses of our main results, and a desirable strategy is eventually found, the time taken for $\mathcal{K}(s(t))$ to converge can become very large. As an illustration, we simulated the nonlinear model from Example 3.2 but introduced many more undesirable strategies. Recall that in this example the linear data A, b, f are fixed, and the nonlinear terms g_h depend on the strategy $h \in \underline{q}$. We retained the single desirable strategy from Example 3.2, but included 19 other undesirable strategies by randomly generating the σ_h parameter in (3.4) so that $\sigma_h \in (0, p)$. Numerical simulation results are plotted in Figure 3.7 from ten randomly generated initial conditions. In each case $x(t)$ persists asymptotically, see Figure 3.7a; and $s(t)$ does eventually converge, as seen in Figure 3.7b. However, the response time is very slow, as $\mathcal{K}(s(t))$ cycles through every undesirable strategy consecutively, during which the intervals $(\tau(k-1), \tau(k)]$ become very large, meaning that it takes even longer to switch strategy again. This situation can be mitigated against by having fewer strategies in total, or a higher ratio of desirable to undesirable strategies.



(A) Trajectories of the observed population size $\|y(t)\|$ at time t.



(B) Semilog plot of the switching sequence $s(t)$ at time t. The dark and light grey shaded regions correspond to desirable and undesirable strategies, respectively.

FIGURE 3.7. Numerical simulations of the adaptive switching feedback control scheme (2.4) for the trout cod model discussed in Section 3.1.

4. **Summary.**

A novel theoretical robust feedback control solution has been proposed for the problem of preservation of dynamic nonnegative quantities managed by choice of discrete control strategy. A motivating application is to the conservation of managed populations. We have proposed the so-called adaptive switching feedback control scheme (2.4) which uses a measured variable to inform the choice of control strategy. Our main results are Theorems 2.1 and 2.4 which provide sufficient conditions under which (2.4) identifies (or learns), and converges to, a strategy which results in persistence, under different assumptions on the class of underlying dynamic models $F(h, \cdot)$ in (1.1) for x. We prove our results by critically exploiting both the positivity and exponential rates of change of the underlying models, in conjunction with the faster-than-exponential growth of the sequence τ.

The assumptions we place on $F(h, \cdot)$ are structural, and are satisfied in reasonable physically-motivated scenarios. Our scheme does not require knowledge of the $F(h, \cdot)$ to be implemented and, as mentioned in the Introduction, our scheme is intended for use in the situation wherein the $F(h, \cdot)$ are unknown, as other solutions to the main problem considered are available otherwise. Some discussion of the performance of the models is provided in Section 3.1. Our work is in the spirit of robust control and, consequently, our results are not expected to be optimal in any sense. Arguably, optimality has been traded off against ensuring strong robustness properties. However, our results may have utility when models are so poor that optimal controls may not function or perform as intended. This comment also naturally raises a future research direction, which we hope to address, which is to combine elements of the theoretical foundation laid here with methods for improving performance in bespoke situations.

Appendix A. .

A.1. **Additional material for the examples.**

The initial conditions used in the simulations in Example 3.1 are

$$x_0^1 := \begin{pmatrix} 0.1224 \\ 0.6196 \\ 1.5709 \\ 0.2231 \end{pmatrix}, \quad x_0^2 := \begin{pmatrix} 0.2398 \\ 1.5278 \\ 3.5706 \\ 0.7903 \end{pmatrix}, \quad x_0^3 := \begin{pmatrix} 0.3014 \\ 2.1859 \\ 3.9035 \\ 0.6244 \end{pmatrix}, \tag{A.1}$$

where we recall the units of 100 whales.

The initial conditions used in the simulations in Example 3.2 are

$$x_0^1 := \begin{pmatrix} 0.9624 \\ 0.4000 \\ 0.1807 \\ 0.1256 \\ 0.1070 \\ 0.0647 \\ 0.2899 \end{pmatrix}, \quad x_0^2 := \begin{pmatrix} 8.5315 \\ 2.6647 \\ 1.8178 \\ 1.3227 \\ 0.8446 \\ 0.5622 \\ 3.1954 \end{pmatrix}, \quad x_0^3 := \begin{pmatrix} 17.2480 \\ 6.7629 \\ 4.3956 \\ 2.5322 \\ 1.8469 \\ 1.2930 \\ 7.2362 \end{pmatrix}, \tag{A.2}$$

where we recall the units of 1000 fish.

A.2. **Proofs of results.** We provide outline proofs of our results. For full details we refer the reader to [32]. The proofs are somewhat long, but intuitive and use elementary (if not careful) arguments.

Proofs for Section 2.1

We let $\underline{q}_{\mathrm{e}}, \underline{q}_{\mathrm{p}} \subseteq \underline{q}$ index the strategies for which **(L2)**(a) and **(L2)**(b) hold, respectively. By definition and assumption $\underline{q}_{\mathrm{e}}$ and $\underline{q}_{\mathrm{p}}$ partition $\underline{q}$, and $\underline{q}_{\mathrm{p}}$ is non empty.

A key estimate which is a routine consequence of **(L3)** is that there exist $k \in \mathbb{Z}_+$ and $d > 0$ such that

$$\|CA_-^k x\| \geq d\|x\| \quad \forall\, x \in \mathbb{R}_+^n\,. \tag{A.3}$$

Proof of Theorem 2.1. The proofs of statements (i)–(iii) are linked and the statements are, more or less, proven simultaneously. We proceed in steps.

STEP 1: s CANNOT ALWAYS AVOID DESIRABLE STRATEGIES. A consequence of the lower bounds $A_h \geq A_-$ for all $h \in \underline{q}$ and monotonicity of the one-norm is that

$$Cx(t+k) \geq CA_-^k x(t) \geq 0 \quad \text{and so} \quad \|Cx(t+k)\| \geq \|CA_-^k x(t)\| \quad \forall\, t \in \mathbb{Z}_+\,.$$

Therefore, invoking (A.3), there exist $\rho_- > 0$ and $\delta_- > 0$ such that

$$\|Cx(t+k)\| \geq \|CA_-^k x(t)\| \geq d\|x(t)\| \geq d\delta_-\rho_-^t\|x_0\| \quad \forall\, t \in \mathbb{Z}_+\,. \tag{A.4}$$

An application of (A.4) and a telescoping series argument gives the following upper bound for s,

$$s(t+k) \leq s(k) + \sum_{j=0}^{t-1} \frac{1}{\|Cx(j+k)\|} \leq s(k) + \frac{1}{d\delta_-\|x_0\|} \sum_{j=0}^{t-1} \left(\rho_-^{-1}\right)^j \quad \forall\, t \in \mathbb{N}\,. \tag{A.5}$$

We see that s grows at fastest exponentially. The faster-than-exponential growth assumption **(T)**, however, ensures that s cannot only switch between strategies indexed by $h \in \underline{q}_{\mathrm{e}}$.

STEP 2: s CANNOT BECOME BOUNDED UNDER AN UNDESIRABLE STRATEGY. Let $h \in \underline{q}_{\mathrm{e}}$, and let $m_1 \in \mathbb{Z}_+$ denote a time when the h-th strategy is entered. As a linear system of difference equations there exist $\delta_h > 0$ and $\rho_h \in (0,1)$ such that

$$\|Cx(\theta+t)\| \leq \delta_h\rho_h^t\|C\|\|x(\theta)\| \quad \forall\, t, \theta \in \mathbb{Z}_+ \quad \text{with} \quad \theta \geq m_1\,, \tag{A.6}$$

(strictly, at least until another switch happens). Since $\rho_h \in (0,1)$, it follows that $\|Cx(t)\| \to 0$ as $t \to \infty$, and so there exists $m_2 \in \mathbb{N}$, $m_2 \geq m_1$, such that $\|Cx(t+m_2)\| < M$ for all $t \in \mathbb{Z}_+$.

Therefore, invoking (A.6), we estimate that

$$s(t+m_2) = s(m_2) + \sum_{j=m_2}^{t+m_2-1} \frac{1}{\|Cx(j)\|} \geq s(m_2) + \frac{1}{\|C\|\delta_h\|x(m_2)\|} \sum_{j=0}^{t-1} (\rho_h^{-1})^j \quad \forall\, t \in \mathbb{N}\,.$$

We see that s grows at least exponentially, and thus diverges. Hence, at some future time a switch of strategy will occur.

To summarise the above two steps, for large times every strategy must be cycled through consecutively. Thus, at some (possibly large) time a desirable strategy is applied where **(L2)**(b) holds. Hence, statements (i)–(iii) are proven once we establish that the switching sequence is eventually bounded (constant, in fact) in a desirable strategy.

STEP 3: s CONVERGES UNDER A DESIRABLE STRATEGY. Let $h \in \underline{q}_{\mathrm{p}}$ and let θ be the first time that the h-th strategy is (re)applied. An application of (A.3) and routine estimates give that

$$\|y(t+\theta+k)\| = \|Cx(t+\theta+k)\| \geq d\|x(t+\theta)\| \geq c_3\rho_h^t\|x_0\| \quad \forall\, t \in \mathbb{Z}_+\,, \tag{A.7}$$

for some constant $c_3 = c_3(\theta)$, whilst the h-th strategy is applied. Here $\rho_h > 1$. Therefore, as y, and hence x, diverges in norm under this strategy, there exists $\psi = \psi(x_0) \in \mathbb{Z}_+$ such that $\|y(t+\theta)\| \geq M$ for all $t \in \mathbb{Z}_+$ with $t \geq \psi$. Therefore, $s(t+\psi+\theta) = s(\psi+\theta)$ for all $t \in \mathbb{Z}_+$, whilst still in this strategy.

Thus, all that remains to prove is that the strategy has not switched again. However, this essentially follows as, in light of the exponentially growing lower bound (A.7) for $\|y\|$, the switching sequence s admits the upper bound (A.5) but with ρ_-^{-1} replaced by $\rho_h^{-1} < 1$ (and a relabelling of constants), which is summable. Hence, s is convergent, and although its limit may be large, the faster-than-exponential growth **(T)** ensures that, at least for θ large enough, no further switching occurs. □

Proof of Corollary 2.3. The proof is very similar to that of Theorem 2.1, only differing in that the assumptions **(L1)** and **(L2)** together are sufficient for the estimates (A.4) and (A.7) to hold. □

Proofs for Section 2.2

The ideas behind the proofs for this section are very similar to those in the linear case, but the estimates become more technical.

Proof of Theorem 2.4. Steps 1 and 2 in the proof of Theorem 2.1 apply here and the proofs use similar estimates (adapted for the nonlinear setting) — the upshot being that the switching sequence cannot become bounded in an undesirable strategy, and grows at fatest exponentially, so cannot always avoid desirable strategies. Our assumptions imply that whilst $\|x\|$ is small, the solution of $x^+ = F(h, x)$ admits a linear lower bound which is exponentially growing (cf. [14, Theorem 4.4, statement (b)]). In particular, persistence of $\|x(t)\|$ follows. If M is chosen sufficiently small so that the linear lower bound for x applies, then a similar argument to that in Step 3 of the proof of Theorem 2.1 now shows that s is bounded, and hence convergent, under a desirable strategy. □

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