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STABILITY OF NONNEGATIVE LUR'E SYSTEMS*

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Abstract. A stability/instability trichotomy for a class of nonnegative continuous-time Lur'e systems is derived. Asymptotic, exponential, and input-to-state stability concepts are considered. The presented trichotomy rests on Perron–Frobenius theory, absolute stability theory, and recent input-to-state stability results for Lur'e systems. Applications of the results derived arise in various fields, including density-dependent population dynamics, and two examples are discussed in detail.

Key words. absolute stability, input-to-state-stability, Lur'e system, multiple equilibria, population modeling, positive systems

AMS subject classifications. 15B48, 92D25, 93C15, 93D05, 93D09, 93D10, 93D20

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1. Introduction.

In mathematical control theory, much attention has been devoted to a class of nonlinear systems referred to as Lur'e or Lurie systems [9, 18, 24, 29, 37, 47, 48]. These systems are comprised of two components: a linear system, with state x, input u, and output y, given by

$$\dot{x} = Ax + bu\,, \quad x(0) = \xi, \quad y = c^T x, \tag{1.1}$$

and a nonlinear feedback $u = f(y)$. The resulting nonlinear feedback system is given by

$$\dot{x} = Ax + bf(c^T x), \quad x(0) = \xi\,. \tag{1.2}$$

Lur'e systems arise in various contexts in circuit, control, and systems theory. Under the common assumption that the nonlinearity f satisfies $f(0) = 0$, it follows that 0 is an equilibrium of (1.2). The study of stability properties of the zero equilibrium of Lur'e systems is termed *absolute stability* and generally refers to the situation where the linear system (1.1) is known and the nonlinearity f is unknown but usually sector bounded. Absolute stability is a well-studied and active area of research, and we refer the reader to [9, 18, 24, 37, 47, 48] and the references therein. A typical absolute stability result provides conditions on the linear component (either in time or frequency domain) which ensure that zero is globally asymptotically stable (GAS) for a class of sector bounded nonlinearities. Crucially, stability of the Lur'e system is determined by the sector bounds and not by the individual nonlinearity f itself. Such inherent robustness makes absolute stability results especially powerful. Furthermore, if the Lur'e system (1.2) is subject to an external additive time-dependent disturbance d, that is, if (1.2) is replaced by

$$\dot{x} = Ax + bf(c^T x) + d, \quad x(0) = \xi, \tag{1.3}$$

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then recent research [18, 37] shows that the conditions of a well-known classical absolute stability result, the so-called circle criterion, guarantee input-to-state stability (ISS) of the forced system (1.3), thereby adding to the inherent robustness properties of stable Lur'e systems. Without going into details here, we mention that ISS means that the map $\mathbb{R}^n \times L^\infty(\mathbb{R}_+, \mathbb{R}^n) \to \mathbb{R}^n$, $(\xi, d) \mapsto x(t)$ has "nice" boundedness and asymptotic properties (see sections 2 and 5 for more details on ISS). For an overview of ISS theory, we refer the reader to [5, 41].

Systems of type (1.2) or (1.3) also arise naturally in biology, ecology, and chemistry, for example, in T-cell receptor signal transduction [31, 42]; enzyme synthesis [33, section 7.2], [38, Chapter 4.2], and [45]; and population dynamics [8]. Lur'e type models for economic fluctuations have also been suggested; see [7]. In a population model, the function f captures density-dependence, for example, a carrying capacity. In a chemical reaction model, f may describe a nonlinear reaction rate between certain components. In these applied contexts, a common key feature is that the components of the state x of the model, which may represent population abundances, chemical concentrations, or economical quantities (such as prices), are, necessarily, nonnegative. In this case, the matrix A is Metzler, whilst b and c are nonnegative, and f maps the interval $[0, \infty)$ into itself.

In the context of biological, ecological, and chemical models the focus is often on the existence and stability of nonzero equilibria which then correspond to the co-existence of populations or chemical compounds. Townley and collaborators have used Lur'e system ideas and small-gain techniques to develop stability/instability trichotomy results for classes of both finite-dimensional [44] and infinite-dimensional [35] discrete-time population models. For the situations considered in [35, 44], only one of three outcomes is possible: either zero is GAS, or there is a stable nonzero equilibrium which attracts all nonzero solutions, or else all nonzero solutions diverge componentwise. Further trichotomies of stability for various classes of monotone discrete-time dynamical systems have been established in [21, Chapter 6] and [22] for finite-dimensional systems and in [16, 39] for infinite-dimensional systems. The paper [22] also contains a limit set trichotomy for a class of periodic continuous-time systems satisfying certain monotonicity conditions. Here we develop stability/instability trichotomy results for continuous-time Lur'e systems using ideas from absolute stability theory (for instance, [18]). Our results cover asymptotic and exponential stability as well as ISS. We emphasize that the Lur'e systems considered in this paper are in general not monotone and therefore results from the theory of monotone dynamical systems [38] do not apply.

This paper is organized as follows. Sections 2 and 3, respectively, collect material on absolute stability and Metzler matrices that we shall require. Sections 4 and 5 contain our main results, namely stability/instability trichotomies for unforced and forced Lur'e systems, respectively. The paper concludes with section 6, in which detailed discussions of two examples are provided.

Notation and terminology. The set of positive integers is denoted by $\mathbb{N}$, whilst $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively. For $n \in \mathbb{N}$, $\mathbb{R}^n$ and $\mathbb{C}^n$ denote the familiar real and complex n-dimensional vector spaces, respectively, both equipped with a norm $\|\cdot\|$ (the 2-norm, unless said otherwise). For $m \in \mathbb{N}$, let $\mathbb{R}^{n\times m}$ and $\mathbb{C}^{n\times m}$ denote the normed linear spaces of $n \times m$ matrices with real and complex entries, respectively, both equipped with the operator norm, also denoted $\|\cdot\|$, induced by the norm on $\mathbb{R}^n$ or $\mathbb{C}^n$. As usual, let $\|\cdot\|_1$ denote the 1-norm on $\mathbb{R}^n$. We set $\mathbb{R}_+ := \{r \in \mathbb{R} : r \geq 0\}$, and the nonnegative orthant in $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$.

For $z \in \mathbb{C}$ and $r > 0$, set

$$\mathbb{D}(z,r) := \{s \in \mathbb{C} : |s - z| < r\} \subseteq \mathbb{C},$$

the open disk in the complex plane, centered at z and of radius r. Similarly, for $x \in \mathbb{R}^n$, $\mathbb{B}(x,r)$ denotes the open ball in $\mathbb{R}^n$ centered at x and with radius r.

For $M \in \mathbb{C}^{n\times n}$, $\sigma(M)$ and $\rho(M)$ denote the *spectrum* and the *spectral radius* of M, respectively. The *spectral abscissa* $\alpha(M)$ of M is defined by

$$\alpha(M) = \max\{\operatorname{Re}\lambda : \lambda \in \sigma(M)\},$$

and M is said to be *Hurwitz* if $\alpha(M) < 0$ (that is, every eigenvalue of M has negative real part).

Let $M = (m_{ij}) \in \mathbb{R}^{n\times m}$ and $N = (n_{ij}) \in \mathbb{R}^{n\times m}$. We write

$$\begin{aligned} M &\geq N \quad \text{if } m_{ij} \geq n_{ij}\ \forall i \text{ and } j,\\ M &> N \quad \text{if } M \geq N \text{ and } M \neq N,\\ M &\gg N \quad \text{if } m_{ij} > n_{ij}\ \forall i \text{ and } j. \end{aligned}$$

We say that M is *nonnegative* if $M \geq 0$. The matrix M is called *positive* if $M \gg 0$. A nonnegative square matrix M is said to be *primitive* if there exists $k \in \mathbb{N}$ such that M^k is positive. A square matrix $M = (m_{ij}) \in \mathbb{R}^{n\times n}$ is said to be *Metzler* (or *essentially nonnegative* or *quasi-positive*) if all its off-diagonal entries of M are nonnegative, that is, $m_{ij} \geq 0$ for all $1 \leq i, j \leq n$ with $i \neq j$.

As usual, the space of bounded holomorphic functions defined on the open right half of the complex plane is denoted by H^∞. For a function $h \in H^\infty$, its norm is defined by

$$\|h\|_{H^\infty} = \sup_{\operatorname{Re} s>0} |h(s)|.$$

Finally, we recall the definitions of certain classes of comparison functions. Let $\mathcal{K}$ denote the set of all continuous functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi(0) = 0$ and φ is strictly increasing. Moreover,

$$\mathcal{K}_\infty := \{\varphi \in \mathcal{K} : \varphi(s) \to \infty \text{ as } s \to \infty\}.$$

We denote by $\mathcal{KL}$ the set of functions $\psi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ with the following properties: $\psi(\cdot\,, t) \in \mathcal{K}$ for every $t \geq 0$, and $\psi(s, \cdot\,)$ is nonincreasing with $\lim_{t\to\infty} \psi(s,t) = 0$ for every $s \geq 0$. For more details on comparison functions, we refer the reader to [19].

2. Stability of Lur'e systems. Consider the Lur'e system

$$\dot{x} = Ax + bf(c^T x), \quad x(0) = \xi \in \mathbb{R}^n, \tag{2.1}$$

where $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}^n$, and $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz with $f(0) = 0$. The superscript T denotes both matrix and vector transposition. Let $x(\cdot\,;\xi)$ denote the continuously differentiable unique maximally defined forward solution of the initial-value problem (2.1) (the existence of which is guaranteed by, for example, [26, Theorem 4.22] or [43, Theorem 54]). If there exists an affine linear bound for the nonlinearity f, then $x(t;\xi)$ is defined for all $t \geq 0$ (see [26, Proposition 4.12]).

Application of linear output feedback of the form $u = \kappa y$ to (1.1) leads to $\dot{x} = (A + \kappa bc^T)x$, where κ is a constant (sometimes referred to as a feedback gain). Define

$$\mathbb{S}(A, b, c^T) := \{\kappa \in \mathbb{C} : A + \kappa bc^T \text{ is Hurwitz}\},$$

the set of *complex* stabilizing output feedback gains for the linear system (A, b, c^T).

The following stability result, developed and proved in [18], will play an important role in this paper.

THEOREM 2.1. *Let $A \in \mathbb{R}^{n\times n}$, $b, c \in \mathbb{R}^n$, and let $f : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz. Assume that*

$$\mathbb{D}(k, r) \subseteq \mathbb{S}(A, b, c^T), \tag{2.2}$$

where $k \in \mathbb{R}$ and $r > 0$.

(1) *If*

$$\frac{f(y)}{y} \in [k - r, k + r] \quad \forall\ y \in \mathbb{R}\backslash\{0\}, \tag{2.3}$$

then there exists $g \geq 1$ such that

$$\|x(t; \xi)\| \leq g\|\xi\| \quad \forall\ t \geq 0,\ \forall\ \xi \in \mathbb{R}^n.$$

In particular, the equilibrium 0 *of* (2.1) *is stable in the large.*

(2) *If*

$$\frac{f(y)}{y} \in (k - r, k + r) \quad \forall\ y \in \mathbb{R}\backslash\{0\}, \tag{2.4}$$

then the equilibrium 0 *of* (2.1) *is globally asymptotically stable.*

(3) *If there exists $r_1 \in (0, r)$ such that*

$$\frac{f(y)}{y} \in (k - r_1, k + r_1) \quad \forall\ y \in \mathbb{R}\backslash\{0\}, \tag{2.5}$$

then the equilibrium 0 *of* (2.1) *is globally exponentially stable, that is, there exists $\gamma > 0$ and $g \geq 1$ such that*

$$\|x(t; \xi)\| \leq g e^{-\gamma t}\|\xi\| \quad \forall\ t \geq 0,\ \forall\ \xi \in \mathbb{R}^n.$$

The well-known control theoretic circle criterion (see, for example, [47]) can be derived as a corollary of Theorem 2.1 (see [18]). Statement (2) of Theorem 2.1 says, roughly speaking, that linear stability (namely, $\mathbb{D}(k, r) \subseteq \mathbb{S}(A, b, c^T)$) implies global asymptotic stability for all nonlinearities $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(y)/y \in (k-r, k+r)$ for all nonzero $y \in \mathbb{R}$. In this sense, Theorem 2.1 is reminiscent of the Aizerman conjecture (see [14, section 5.6.3]) and [18, 23, 47]). We emphasise though that stability of the linear feedback system $\dot{x} = (A+\kappa bc^T)x$ has to hold for all *complex* κ satisfying $|\kappa-k| < r$. It is easy to see that the conclusions in Theorem 2.1 remain true for *complex* nonlinearities $f : \mathbb{C} \to \mathbb{C}$, provided that, in statements (1)–(3), conditions (2.3)–(2.5) are replaced by

$$\frac{f(y)}{y} \in \overline{\mathbb{D}(k, r)}, \quad \frac{f(y)}{y} \in \mathbb{D}(k, r), \quad \text{and} \quad \frac{f(y)}{y} \in \mathbb{D}(k, r_1), \tag{2.6}$$

respectively, where the conditions (2.6) hold for all complex nonzero y and $\overline{\mathbb{D}(k, r)}$ denotes the *closed* complex ball, centered at k, with radius r.

We will now present a special case wherein the complex condition $\mathbb{D}(k, r) \subseteq \mathbb{S}(A, b, c^T)$ can be replaced by its real counterpart $(k - r, k + r) \subseteq \mathbb{S}(A, b, c^T)$.

COROLLARY 2.2. *Let $A \in \mathbb{R}^{n\times n}$, $b, c \in \mathbb{R}^n$, let $f : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz, and let $k \in \mathbb{R}$ and $r > 0$. Assume that b and c are nonnegative, $A + kbc^T$ is Metzler, and*

$$(k - r, k + r) \subseteq \mathbb{S}(A, b, c^T). \tag{2.7}$$

Under these conditions, statements (1)–(3) *of* Theorem 2.1 *hold.*

Proof. Set $A_k := A + kbc^T$ and define

$$r_{\mathbb{F}}(A_k; b, c^T) := \inf\{|\kappa| : \kappa \in \mathbb{F},\ A_k + \kappa bc^T \text{ is not Hurwitz}\} \quad (\text{where } \mathbb{F} = \mathbb{R} \text{ or } \mathbb{F} = \mathbb{C}),$$

the stability radius of A_k with respect to the perturbation structure given by b and c^T. Invoking (2.7), we see that $r \le r_{\mathbb{R}}(A_k; b, c^T)$. By a stability radius result for nonnegative systems proved in [15], $r_{\mathbb{R}}(A_k; b, c^T) = r_{\mathbb{C}}(A_k; b, c^T)$, and consequently $\mathbb{D}(0, r) \subseteq \mathbb{S}(A_k, b, c^T)$, or equivalently $\mathbb{D}(k, r) \subseteq \mathbb{S}(A, b, c^T)$. The claim now follows from Theorem 2.1. ∎

In the following, let G denote the transfer function of the linear system (1.1), i.e., $G(s) = c^T(sI - A)^{-1}b$. We will also refer to G as the transfer function of (A, b, c^T). The next result considers a scenario wherein the Lur'e system (2.1) has an equilibrium $x^* \neq 0$ in addition to the zero equilibrium.

THEOREM 2.3. *Consider the unforced Lur'e system* (2.1) *and assume that A is Hurwitz, $\|G\|_{H^\infty} = |G(0)| > 0$, and there exists $y^* \neq 0$ such that $y^* = G(0)f(y^*)$. Then $x^* = -A^{-1}bf(y^*) \neq 0$ is an equilibrium of* (2.1) *and the following statements hold:*

(1) *If*

$$\left|\frac{f(y) - f(y^*)}{y - y^*}\right| \le \frac{1}{|G(0)|} \quad \forall\ y \in \mathbb{R},\ y \neq y^*, \tag{2.8}$$

then there exists $g \ge 1$ such that $\|x(t; \xi) - x^*\| \le g\|\xi - x^*\|$ for all $\xi \in \mathbb{R}^n$ and $t \ge 0$.

(2) *If*

$$\left|\frac{f(y) - f(y^*)}{y - y^*}\right| < \frac{1}{|G(0)|} \quad \forall\ y \in \mathbb{R},\ y \neq 0, y^*, \tag{2.9}$$

then, for every $\xi \in \mathbb{R}^n$, we have that $x(t; \xi) \to x^*$ or $x(t; \xi) \to 0$ as $t \to \infty$.

Note that (2.8) is a "sector" condition in the sense that the graph of f is "sandwiched" between the lines $l_1(y) = py$ and $l_2(y) = -py + 2py^*$, where $p = 1/G(0)$ (see Figure 2.1 for an illustration). In the case of the "strict" sector condition (2.9), the graph of f "touches" these lines only at the points $(0, 0)$ and $(y^*, f(y^*))$.

Before proving Theorem 2.3, we provide examples of linear systems satisfying the condition $\|G\|_{H^\infty} = |G(0)|$.

Example 2.4. (1) Let (A, b, c^T) be a symmetric system, that is, $A = A^T$ and $c = b$. If A is Hurwitz, then by [25, part (2) of Theorem 4.1]

$$\|G\|_{H^\infty} = G(0).$$

The assumptions on A imply that $-A^{-1}$ is positive definite and thus, if $b \neq 0$, then $G(0) = -b^T A^{-1} b > 0$.

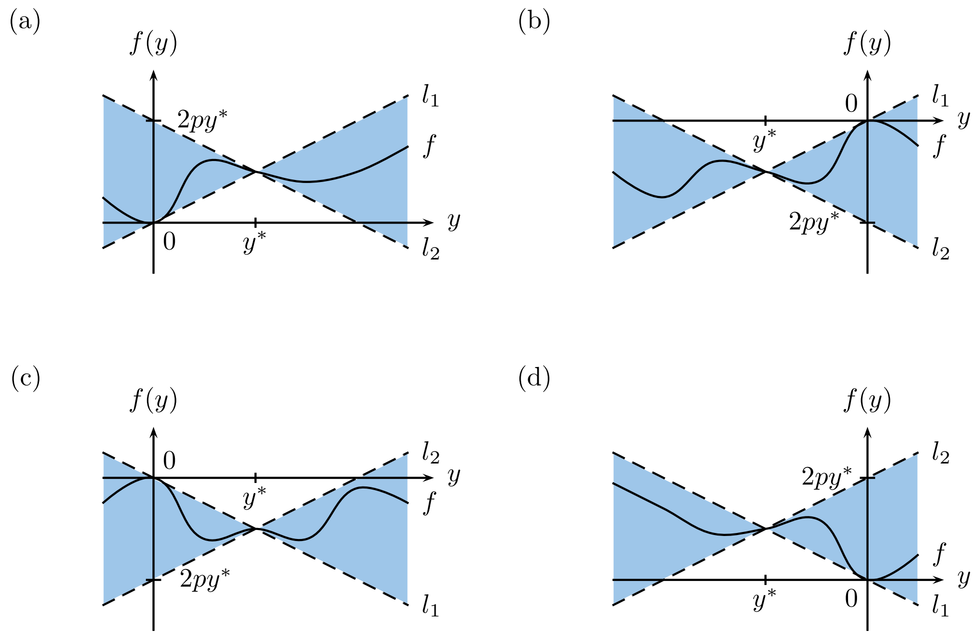


FIG. 2.1. *Graph of f "sandwiched" between the lines $l_1(y) = py$ and $l_2(y) = 2py^* - py$, where (a) $p > 0$ and $y^* > 0$, (b) $p > 0$ and $y^* < 0$, (c) $p < 0$ and $y^* > 0$, and (d) $p < 0$ and $y^* < 0$.*

(2) Let $\tilde{A} \in \mathbb{R}^{n\times n}$, $\tilde{b} \in \mathbb{R}^n$, $\tilde{c} \in \mathbb{R}^n$, and set $\tilde{G}(s) = \tilde{c}^T(sI - \tilde{A})^{-1}\tilde{b}$. Assume that $\tilde{A}$ is Hurwitz and $\tilde{G}(0) \neq 0$. Let k be a real parameter and consider the integral control system

$$\dot{x} = \tilde{A}x + \tilde{b}u, \quad y = \tilde{c}^T x, \quad \dot{u} = v - ky,$$

where v is an input (forcing) function. The above feedback system (with input v and output y) is described by the triple (A_k, b, c^T), where

$$A_k := \begin{pmatrix} \tilde{A} & \tilde{b} \\ -k\tilde{c}^T & 0 \end{pmatrix}, \quad b := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad c := \begin{pmatrix} \tilde{c} \\ 0 \end{pmatrix}.$$

A routine calculation shows that

$$\begin{aligned}\text{top right-hand block of } (sI - A_k)^{-1} &= \big(s + k\tilde{c}^T(sI - \tilde{A})^{-1}\tilde{b}\big)^{-1}(sI - \tilde{A})^{-1}\tilde{b} \\ &= \big(s + k\tilde{G}(s)\big)^{-1}(sI - \tilde{A})^{-1}\tilde{b},\end{aligned}$$

and so, the transfer function $G_k(s) = c^T(sI - A_k)^{-1}b$ of the system (A_k, b, c^T) satisfies

$$G_k(s) = \tilde{G}(s)\big(s + k\tilde{G}(s)\big)^{-1} = \frac{\tilde{G}(s)}{s}\left(1 + k\frac{\tilde{G}(s)}{s}\right)^{-1}.$$

It follows from [27] that there exists $k^* > 0$ such that, for all k with $k\tilde{G}(0) > 0$ and $0 < |k| < k^*$,

$$\|G_k\|_{H^\infty} = \frac{1}{|k|} = |G_k(0)|.$$

(3) Let (A, b, c^T) be a nonnegative continuous-time system, that is, A is Metzler and $b, c \in \mathbb{R}^n_+$. If A is Hurwitz, then $\|G\|_{H^\infty} = G(0) \geq 0$ (see Lemma 4.2).

Proof of Theorem 2.3. Since $G(0) \neq 0$ and $y^* \neq 0$, we have $f(y^*) \neq 0$ and $x^* \neq 0$. Noting that $c^T x^* = y^*$, we conclude that

$$Ax^* + bf(c^T x^*) = Ax^* + bf(y^*) = 0,$$

showing that x^* is an equilibrium of (2.1).

Let $\xi \in \mathbb{R}^n$ and set $\tilde{x}(t) = x(t;\xi) - x^*$ for all $t \geq 0$. Furthermore, defining $\tilde{f} : \mathbb{R} \to \mathbb{R}$ by $\tilde{f}(y) = f(y + y^*) - f(y^*)$ for all $y \in \mathbb{R}$, it follows that

$$\dot{\tilde{x}} = A\tilde{x} + b\tilde{f}(c^T \tilde{x}), \quad \tilde{x}(0) = \xi - x^*. \tag{2.10}$$

Setting $p := 1/|G(0)|$, it follows by hypothesis that $p = 1/\|G\|_{H^\infty}$ and thus, by elementary stability radius theory (see [13] or [14, section 5.3]),

$$\inf\left\{|\kappa| : \kappa \in \mathbb{C},\ A + \kappa bc^T \text{is not Hurwitz}\right\} = p.$$

Consequently,

$$\mathbb{D}(0, p) \subseteq \mathbb{S}(A, b, c^T). \tag{2.11}$$

To prove statement (1), note that

$$\left|\frac{\tilde{f}(y)}{y}\right| \leq p \quad \forall\ y \in \mathbb{R},\ y \neq 0. \tag{2.12}$$

Combining (2.10)–(2.12) with statement (1) of Theorem 2.1 yields the existence of a constant $g \geq 1$ such that, for every $\xi \in \mathbb{R}^n$,

$$\|x(t;\xi) - x^*\| \leq g\|\xi - x^*\| \quad \forall\ t \geq 0.$$

We proceed to prove statement (2) of Theorem 2.1. To this end, observe that

$$\left|\frac{\tilde{f}(y)}{y}\right| < p \quad \forall\ y \in \mathbb{R},\ y \neq 0, -y^*. \tag{2.13}$$

By [14, proof of Theorem 5.6.22] there exists a positive semidefinite $P = P^T \in \mathbb{R}^{n\times n}$ such that the quadratic form $V(z) = \langle Pz, z\rangle$ satisfies

$$V_{\mathrm{d}}(z) := \langle(\nabla V)(z), Az + b\tilde{f}(c^T z)\rangle \leq \tilde{f}^2(c^T z) - p^2(c^T z)^2 \leq 0 \quad \forall\ z \in \mathbb{R}^n, \tag{2.14}$$

where the last inequality follows from (2.13). By statement (1), $\tilde{x}$ is bounded and so its ω-limit set Ω is nonempty, compact, connected, and invariant. Furthermore, Ω is the smallest closed set with the property

$$\lim_{t\to\infty} \operatorname{dist}(\tilde{x}(t), \Omega) = 0;$$

see, for example, [26]. As a consequence of LaSalle's invariance principle, $\Omega \subseteq V_{\mathrm{d}}^{-1}(0)$. By (2.13) and (2.14),

$$V_{\mathrm{d}}^{-1}(0) \subseteq \{z \in \mathbb{R}^n : c^T z = 0 \text{ or } c^T z = -y^*\}.$$

Hence

$$\Omega \subseteq \ker c^T \cup (\ker c^T - x^*).$$

The sets $\ker c^T$ and $\ker c^T - x^*$ are closed and disjoint (as $c^T x^* = y^* \neq 0$). Now Ω is connected and, therefore,

$$\Omega \subseteq \ker c^T \quad \text{or} \quad \Omega \subseteq \ker c^T - x^*.$$

Consequently,

$$\lim_{t\to\infty} c^T \tilde{x}(t) = 0 \quad \text{or} \quad \lim_{t\to\infty} c^T \tilde{x}(t) = -y^*.$$

Hence by (2.10) and the Hurwitz property of A,

$$\lim_{t\to\infty} \tilde{x}(t) = 0 \quad \text{or} \quad \lim_{t\to\infty} \tilde{x}(t) = -A^{-1} b \tilde{f}(-y^*) = -x^*.$$

Thus,

$$\lim_{t\to\infty} x(t;\xi) = x^* \quad \text{or} \quad \lim_{t\to\infty} x(t;\xi) = 0,$$

completing the proof. ∎

Let us now consider forced Lur'e systems of the form

$$\dot{x} = Ax + bf(c^T x) + d, \quad x(0) = \xi \in \mathbb{R}^n, \tag{2.15}$$

where $A \in \mathbb{R}^{n\times n}$, $b, c \in \mathbb{R}^n$, $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz with $f(0) = 0$, and $d \in L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n)$ is an external disturbance (forcing, input). Let $x(\cdot\,;\xi, d)$ denote the unique absolutely continuous maximally defined forward solution of the initial-value problem (2.15) (see, for example, [43, Theorem 54]). In most applied contexts, the function d will be piecewise continuous, in which case $x(\cdot\,;\xi, d)$ is piecewise continuously differentiable. Furthermore, if the nonlinearity f satisfies an affine linear bound, then $x(t;\xi, d)$ is defined for all $t \geq 0$ (see [26, Proposition 4.12]). The following result is proved in [37].

THEOREM 2.5. *Let $A \in \mathbb{R}^{n\times n}$, $b, c \in \mathbb{R}^n$, and let $f : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz and $f(0) = 0$. Assume that*

$$\mathbb{D}(k, r) \subseteq \mathbb{S}(A, b, c^T), \tag{2.16}$$

where $k \in \mathbb{R}$ and $r > 0$. If there exists $\beta \in \mathcal{K}_\infty$ such that

$$|f(y) - ky| \leq r|y| - \beta(|y|) \quad \forall\ y \in \mathbb{R}, \tag{2.17}$$

then there exist $\psi \in \mathcal{KL}$ and $\varphi \in \mathcal{K}$ such that, for all $\xi \in \mathbb{R}^n$ and all $d \in L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n)$,

$$\|x(t;\xi, d)\| \leq \psi(\|\xi\|, t) + \varphi(\|d\|_{L^\infty(0,t)}) \quad \forall\ t \geq 0. \tag{2.18}$$

Obviously, if $d = 0$ in (2.15), then 0 is an equilibrium of (2.15). If there exist $\psi \in \mathcal{KL}$ and $\varphi \in \mathcal{K}$ such that (2.18) holds for all $\xi \in \mathbb{R}^n$ and all $d \in L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n)$, then 0 is said to be ISS. For more details on ISS theory, we refer the reader to [5, 41]. Note that there exists $\beta \in \mathcal{K}_\infty$ such that (2.17) holds if and only if $|f(y) - ky| < r|y|$ for all $y \neq 0$, and the difference $r|y| - |f(y) - ky|$ tends to ∞ as $|y| \to \infty$.

Trivially, condition (2.4) in Theorem 2.1 can be rewritten in the form

$$|f(y) - ky| < r|y| \quad \forall\ y \in \mathbb{R}\backslash\{0\}, \tag{2.19}$$

and thus we see that Theorem 2.5 is structurally very similar to Theorem 2.1. In particular, Theorem 2.5 says that linear stability (namely, $\mathbb{D}(k,r) \subseteq \mathbb{S}(A,b,c^T)$) implies ISS for all nonlinearities $f : \mathbb{R} \to \mathbb{R}$ satisfying (2.17). It is not difficult to construct counterexamples which show that Theorem 2.5 does not remain valid if the condition $\beta \in \mathcal{K}_\infty$ is replaced by $\beta \in \mathcal{K}$ (see [37]). In particular, the disc condition (2.16) together with (2.19) is not sufficient for ISS. Finally, we mention that if there exists $r_1 \in (0,r)$ such that (2.5) holds, then (2.17) is satisfied with $\beta(s) = r_0 s$, where r_0 is an arbitrary constant satisfying $0 < r_0 < r - r_1$. In this case, assuming that the linear condition (2.16) holds, it can be shown that the ISS estimate (2.18) holds with ψ and φ given by $\psi(s,t) = c_1 e^{-c_2 t} s$ and $\varphi(s) = c_3 s$, where c_1, c_2, and c_3 are suitable positive constants.

3. Metzler matrices. In this section we gather from the literature a number of results on Metzler matrices that shall play a key role in sections 4 and 5. Frequently, Metzler matrices are also called essentially nonnegative [1, p. 146] or quasi-positive [38, p. 60]. In a dynamical systems context, they are the continuous-time analogue of nonnegative matrices which arise naturally in discrete-time nonnegative dynamical systems (see, for example, [6] or [11]). We refer the reader to [1, 28, 46] for further background on Metzler matrices.

Let $M = (m_{ij}) \in \mathbb{R}^{n\times n}$. Recall that M is *Metzler* if all off-diagonal entries of M are nonnegative, that is, $m_{ij} \geq 0$ for all $1 \leq i,j \leq n$ with $i \neq j$. The matrix M is said to be *reducible* if there exist nonempty disjoint subsets $J_1, J_2 \subseteq \{1,\ldots,n\}$ such that $J_1 \cup J_2 = \{1,\ldots,n\}$ and $m_{ij} = 0$ for all $(i,j) \in J_1 \times J_2$. The matrix M is *irreducible* if it is not reducible. A primitive matrix is irreducible.

The following well-known result demonstrates that the Metzler property characterises linear flows which leave the nonnegative orthant invariant.

Lemma 3.1. *A matrix $M \in \mathbb{R}^{n\times n}$ is Metzler if and only if $\mathrm{e}^{Mt} > 0$ for all $t \geq 0$. A matrix $M \in \mathbb{R}^{n\times n}$ is Metzler and irreducible if and only if $\mathrm{e}^{Mt} \gg 0$ for all $t > 0$.*

Proof. The first claim is proved in, for example, [38, section 3.1] or [40, Theorem 3]. The second claim may be found in [46, Theorem 8.2] (see also [40, Proposition 1] for a more general version). $\square$

Lemma 3.2. *A matrix $M \in \mathbb{R}^{n\times n}$ is Metzler and Hurwitz if and only if $-M^{-1} > 0$.*

Proof. See [2, characterization N_{38} in section 6.2] or [34, characterization F_{15}], noting that the matrix M is Hurwitz and Metzler if and only if $-M$ is a nonsingular M-matrix (recall that $N \in \mathbb{R}^{n\times n}$ is a M-matrix if N is of the form $N = \nu I - P$, where P is nonnegative and $\nu \geq \varrho(P)$). $\square$

Classical Perron–Frobenius theory [2, 30] pertains to nonnegative matrices. Whilst a Metzler matrix M is, in general, not nonnegative, by defining

$$\delta(M) := -\min_{1\leq i\leq n} \big(m_{ii}, 0\big) \geq 0\,,$$

the matrix $\mu I + M$ is nonnegative for all $\mu \geq \delta(M)$. This observation, combined with the following lemma, enables applications of Perron–Frobenius theory to Metzler matrices; see Theorem 3.4.

Lemma 3.3. *Let $M \in \mathbb{R}^{n\times n}$ be an irreducible Metzler matrix. If $\mu > \delta(M)$, then $\mu I + M$ is a primitive matrix.*

Proof. It is clear that $\mu I + M$ is a nonnegative irreducible matrix with positive trace. It follows now from [2, Corollary 2.2.28] that $\mu I + M$ is primitive. ∎

THEOREM 3.4. *Let $M \in \mathbb{R}^{n\times n}$ be a Metzler matrix, set $a := \alpha(M)$, and let $\mu > \delta(M)$. Then*

(1) *$a \in \sigma(M)$ and $a = \rho(\mu I + M) - \mu$.*

(2) *If $\lambda \in \sigma(M)$ and $\lambda \neq a$, then $\operatorname{Re}\lambda < a$.*

Furthermore, under the additional assumption that M is irreducible, the following statements hold:

(3) *a is simple.*

(4) *There exist unique vectors $v, w \in \mathbb{R}^n$ satisfying*

$$v^T M = a v^T, \quad Mw = aw, \quad v, w \gg 0, \quad \text{and} \quad \|v\|_1 = \|w\|_1 = 1. \tag{3.1}$$

(5) *The following convergence result holds:*

$$\lim_{t\to\infty} e^{(M-aI)t} = \frac{1}{v^T w} w v^T \gg 0,$$

where v and w are the vectors satisfying (3.1).

Proof. Statement (1) is taken from [15, part (i) of Proposition 1 and equation (8)] and statement (2) from [15, part (ii) of Proposition 1]. Statements (3) and (4) follow from, for example, [6, Theorems 11 and 17]. Although statement (5) is undoubtedly known, we could not find a proof in the literature. Therefore, we prove it here. To this end, we invoke $a \in \sigma(M)$ and statement (3) to see that there exists an invertible matrix S such that

$$M = S^{-1}\begin{pmatrix} a & 0 \\ 0 & J \end{pmatrix} S, \quad \text{where } J \text{ is a Jordan matrix with } \sigma(J) = \sigma(M)\backslash\{a\}.$$

By Lemma 3.3, the matrix $\mu I + M$ is primitive and thus, appealing to statement (1) and Perron–Frobenius theory, we obtain

$$\mu + a = \rho(\mu I + M) > |\mu + \lambda| \quad \forall\ \lambda \in \sigma(J).$$

Consequently, $\rho((\mu + a)^{-1}(\mu I + J)) < 1$, and so

$$(\mu{+}a)^{-j}(\mu I{+}M)^j = (\mu{+}a)^{-j} S^{-1}\begin{pmatrix} \mu + a & 0 \\ 0 & \mu I + J \end{pmatrix}^j S \to S^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} S \quad \text{as } j \to \infty.$$

On the other hand, Perron–Frobenius theory applied to $\mu I + M$ guarantees that

$$(\mu + a)^{-j}(\mu I + M)^j \to \frac{1}{v^T w} w v^T \quad \text{as } j \to \infty,$$

where v and w are the unique vectors satisfying (3.1). Hence,

$$S^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} S = \frac{1}{v^T w} w v^T,$$

and furthermore,

$$e^{(M-aI)t} = e^{-at}e^{Mt} = S^{-1}\begin{pmatrix} 1 & 0 \\ 0 & e^{-at}e^{Jt} \end{pmatrix} S \to \frac{1}{v^T w} w v^T \quad \text{as } t \to \infty,$$

where we have used that, by statement (2), $a > \operatorname{Re}\lambda$ for all $\lambda \in \sigma(J)$. We may now conclude that $e^{-at}e^{Jt} \to 0$ as $t \to \infty$. ∎

We shall also make use of the following monotonicity property of the spectral abscissas of irreducible Metzler matrices a proof may be found in [38, Corollary 4.3.2].

LEMMA 3.5. *Let* $M, N \in \mathbb{R}^{n\times n}$ *denote Metzler matrices. If* $M > N$ *and* N *is irreducible, then* $\alpha(M) > \alpha(N)$.

4. Stability of unforced nonnegative Lur'e systems. Consider the Lur'e system (2.1). We introduce the following assumptions.

(A1) A is Metzler and $b, c \in \mathbb{R}^n_+ \backslash \{0\}$.

(A2) A is Hurwitz.

(A3) $A + bc^T$ is irreducible.

(A4) $f : \mathbb{R}_+ \to \mathbb{R}_+$ is locally Lipschitz and $f(0) = 0$.

If (A1) and (A4) hold, then for every $\xi \in \mathbb{R}^n_+$, the unique maximally defined forward solution $x(\cdot\,;\xi)$ of (2.1) satisfies $x(t;\xi) \in \mathbb{R}^n_+$ for all $t \geq 0$ for which the solution exists. The following remark is a straightforward consequence of the definition of irreducibility and Lemma 3.3.

Remark 4.1. (1) If (A1) and (A3) are satisfied, then $A + kbc^T$ is irreducible for every $k > 0$.

(2) If (A1) and (A3) hold, then for every $\mu > \delta(A)$ and every $k > 0$, $\mu I + A + kbc^T$ is primitive.

(3) If there exist $\mu, k \geq 0$ such that $\mu I + A + kbc^T$ is primitive, then (A3) holds.

The following result shows that under assumptions (A1)–(A3), the steady-state gain $G(0)$ of the linear system (A, b, c^T) is positive and equal to the H^∞-norm.

LEMMA 4.2. *Assume that* (A1) *and* (A2) *hold; then* $\|G\|_{H^\infty} = |G(0)| = G(0) \geq 0$. *If additionally* (A3) *holds, then* $G(0) > 0$.

Proof. The first claim is a consequence of [15, Theorem 5]. For the convenience of the reader, we provide a short direct proof. To this end, assume that (A1) and (A2) are satisfied. Then, by Lemma 3.1, $c^T e^{At} b \geq 0$ for all $t \geq 0$, and hence, for all $s \in \mathbb{C}$ with $\operatorname{Re} s \geq 0$,

$$|G(s)| = \left| \int_0^\infty c^T e^{At} b e^{-st}\, dt \right| \leq \int_0^\infty |c^T e^{At} b| |e^{-st}|\, dt \leq \int_0^\infty c^T e^{At} b\, dt = G(0),$$

showing that $\|G\|_{H^\infty} = G(0) \geq 0$.

Now assume that (A1)–(A3) hold. We prove that $G(0) > 0$ by invoking a contradiction argument. To this end, suppose that $G(0) = 0$. Since $\|G\|_\infty = G(0) = 0$, it follows that $G(s) \equiv 0$ and hence $c^T A^k b = 0$ for all $k \in \mathbb{N}_0$. Therefore, for $\mu > \delta(A)$,

$$\sum_{k=0}^\infty c^T \frac{(\mu I + A + bc^T)^k}{k!} b = 0\,. \tag{4.1}$$

On the other hand, we know, by part (2) of Remark 4.1, that $\mu I + A + bc^T$ is primitive, implying that the series in (4.1) is positive. This provides the desired contradiction and therefore $G(0) > 0$. ∎

LEMMA 4.3. *Assume that* (A1)–(A3) *hold, set*

$$p := 1/G(0)\,, \tag{4.2}$$

and let $r > p$. *Then* $0 = \alpha(A + pbc^T) < \alpha(A + rbc^T)$.

Proof. By Lemma 4.2, $p = 1/\|G\|_{H^\infty}$, and from stability radius theory for nonnegative linear systems [15] we know that the real and complex stability radii of A with respect to the weightings b and c^T coincide and are equal to p. Moreover, p is a minimal destabilizing perturbation, implying in particular that $\alpha(A + pbc^T) = 0$. Moreover, if $r > p$, then the Metzler matrices $A + pbc^T$ and $A + rbc^T$ satisfy $A + rbc^T > A + pbc^T$. By (A3) and part (1) of Remark 4.1, $A + pbc^T$ is irreducible, and thus, invoking Lemma 3.5, $\alpha(A + rbc^T) > \alpha(A + pbc^T)$. ∎

Assume that (A1)–(A4) hold. This means in particular that p given by (4.2) is well defined and positive. As for the nonlinearity f, we will consider the following three cases:

Case 1. $f(y)/y \le p$ for all $y > 0$.

Case 2. $\inf_{y>0} f(y)/y > p$.

Case 3. There exists $y^* > 0$ such that $f(y^*) = py^*$ and

$$\left| \frac{f(y) - f(y^*)}{y - y^*} \right| \le p \quad \forall\ y \ge 0, y \ne y^*. \tag{4.3}$$

See Figure 4.1 for an illustration of condition (4.3).

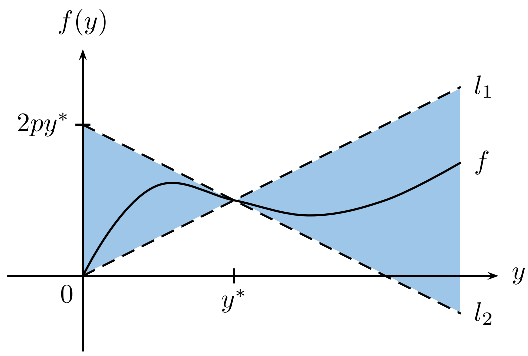


FIG. 4.1. *The graph of f is "sandwiched" between the lines $l_1(y) = py$ and $l_2(y) = 2py^* - py$.*

Roughly speaking, the three cases respectively correspond to the stability/instability trichotomy we will be establishing: the zero equilibrium of (2.1) being stable, all nonzero solutions of (2.1) diverging to infinity, and the existence of a "quasi-globally" stable nonzero equilibrium of (2.1).

Case 1. The following theorem shows that in this case the zero equilibrium of (2.1) has "nice" stability properties.

THEOREM 4.4. *Consider the unforced Lur'e system* (2.1) *and assume that* (A1)–(A4) *hold.*

(1) *If $f(y)/y \le p$ for all $y > 0$, then the equilibrium* 0 *is stable in the large in the sense that there exists $g \ge 1$ such that, for every $\xi \in \mathbb{R}^n_+$,*

$$\|x(t;\xi)\| \le g\|\xi\| \quad \forall\ t \ge 0.$$

(2) *If $f(y)/y < p$ for all $y > 0$, then the equilibrium* 0 *is globally asymptotically stable in the sense that* 0 *is stable in the large and, for every $\xi \in \mathbb{R}^n_+$, $x(t;\xi) \to 0$ as $t \to \infty$.*

(3) *If* $\sup_{y>0} f(y)/y < p$*, then the equilibrium* 0 *is globally exponentially stable, that is, there exists* $\gamma > 0$ *and* $g \geq 1$ *such that, for every* $\xi \in \mathbb{R}^n_+$*,*

$$\|x(t;\xi)\| \leq g e^{-\gamma t}\|\xi\| \quad \forall\ t \geq 0\,.$$

Proof. By Lemma 4.2, $p = 1/\|G\|_{H^\infty}$ and therefore $\mathbb{D}(0,p) \subseteq \mathbb{S}(A,b,c^T)$. To apply Theorem 2.1, we extend f to the whole real line by defining an extension $\tilde{f}:\mathbb{R}\to\mathbb{R}$ as follows:

$$\tilde{f}(y) = \begin{cases} f(y) & \text{for } y > 0, \\ 0 & \text{for } y \leq 0. \end{cases} \tag{4.4}$$

Note that by linear boundedness of f and assumptions (A1) and (A4), we have that for every $\xi \in \mathbb{R}^n_+$, $x(\cdot\,;\xi)$ is defined on $\mathbb{R}_+$ and $x(t;\xi) \in \mathbb{R}^n_+$ for all $t \geq 0$. Therefore, for every $\xi \in \mathbb{R}^n_+$, $x(\cdot;\xi)$ is also the unique maximally defined forward solution of

$$\dot{x} = Ax + b\tilde{f}(c^T x), \quad x(0) = \xi. \tag{4.5}$$

To prove statement (1), assume that $f(y)/y \leq p$ for all $y > 0$. Then, trivially,

$$\frac{\tilde{f}(y)}{y} \in [0,p] \quad \forall\ y \in \mathbb{R},\ y \neq 0\,,$$

and the claim follows from statement (1) of Theorem 2.1 applied to (4.5). Statements (2) and (3) are derived from Theorem 2.1 in a similar manner. For the sake of brevity, we omit the details. ∎

Remark 4.5. An alternative approach to establishing global asymptotic stability of the zero equilibrium of nonnegative Lur'e systems is based on dissipativity theory for nonnegative dynamical systems [11, Chapter 5]. The assumptions and conclusions of [11, Theorem 5.6] (see also [10, Theorem 7.2]) are similar to those in statement (2) of Theorem 4.4. The proof presented in [10, 11] invokes a linear Lyapunov function arising from a dissipativity assumption. We emphasise that the main topic of the current paper (see case 3), namely stability properties of a nonzero equilibrium in the presence of another equilibrium at the origin, is not considered in [10, 11].

Case 2. In this case, the zero equilibrium of (2.1) is "strongly" unstable, as the following result shows.

THEOREM 4.6. *Consider the unforced Lur'e system* (2.1)*. Assume that* (A1)–(A4) *hold and that* f *satisfies*

$$\inf_{y>0} \frac{f(y)}{y} > p\,.$$

If $\xi \in \mathbb{R}^n_+$*,* $\xi \neq 0$*, is such that the solution* $x(t;\xi)$ *exists for every* $t \geq 0$*, then*

$$\lim_{t\to\infty} x_i(t;\xi) = \infty \quad \forall\ i \in \{1,\ldots,n\},$$

where $x_i(t;\xi)$ *denotes the* i*th component of* $x(t;\xi)$*.*

Proof. Let $\xi \in \mathbb{R}^n_+$, $\xi \neq 0$, be such that the solution $x(t;\xi)$ exists for every $t \geq 0$. Write $x(t) := x(t;\xi)$ for all $t \geq 0$. By the hypothesis on f, there exists $q > p$ such that

$$f(y) \geq qy \quad \forall\ y \in \mathbb{R}_+.$$

By (A1), (A3), and part (1) of Remark 4.1, $A+qbc^T$ is an irreducible Metzler matrix. Invoking statement (5) of Theorem 3.4 shows that there exists a positive matrix L such that

$$\lim_{t\to\infty} e^{(A+qbc^T-aI)t} = L \gg 0, \tag{4.6}$$

where $a := \alpha(A+qbc^T)$. We note that $a>0$, as follows from Lemma 4.3. Moreover,

$$\dot{x} = (A+qbc^T)x + b(f(c^Tx) - qc^Tx)$$

and thus, by the variation of parameters formula,

$$x(t) = e^{(A+qbc^T)t}\xi + \int_0^t e^{(A+qbc^T)(t-s)}b\big(f(c^Tx(s)) - qc^Tx(s)\big)ds \geq e^{(A+qbc^T)t}\xi \quad \forall\, t\geq 0.$$

Since $a>0$, it follows from (4.6) that every component of $e^{(A+qbc^T)t}\xi$ diverges to ∞ as $t\to\infty$, completing the proof. ∎

Case 3. Assume that (A1)–(A4) hold, which implies that $G(0)>0$ and $p = 1/G(0)$ is well-defined. Since in the current case we are assuming that there exists $y^*>0$ such that $f(y^*) = py^*$, it follows from Theorem 2.3 that (2.1) has an additional equilibrium $x^* = -A^{-1}bpy^* \neq 0$. We shall provide sufficient conditions under which x^* has "nice" stability properties. To provide a detailed treatment, it is convenient to introduce the following assumptions on the nonlinearity f.

(A5) There exists $y^*>0$ such that $f(y^*) = py^*$.

(A6) f satisfies

$$\left|\frac{f(y)-f(y^*)}{y-y^*}\right| \leq p \quad \forall\; y\geq 0,\, y\neq y^*.$$

(A7) f satisfies

$$\left|\frac{f(y)-f(y^*)}{y-y^*}\right| < p \quad \forall\; y > 0,\, y\neq y^*.$$

Assumptions (A6) and (A7) are sector conditions in the sense that they are equivalent to the graph of f being "sandwiched" between the straight lines $l_1(y) = py$ and $l_2(y) = 2py^* - py$; see Figure 4.1. The following proposition provides a sufficient condition for (A5) and (A7) to hold.

PROPOSITION 4.7. *Assume that* $f:\mathbb{R}_+\to\mathbb{R}_+$ *is twice continuously differentiable. If* $f(0)=0$, $f'(0)>p$, $f'(y)\geq 0$, *and* $f''(y)\leq 0$ *for all* $y\in\mathbb{R}_+$, *and if* $\lim_{y\to\infty} f'(y) < p$, *then* (A5) *and* (A7) *hold.*

Note that the assumptions $f'\geq 0$ and $f''\leq 0$ together guarantee the existence of the limit of $f'(y)$ as $y\to\infty$. The proof of Proposition 4.7 is elementary and is therefore omitted.

Example 4.8. (1) Consider the nonlinearity $f:\mathbb{R}_+\to\mathbb{R}_+$ given by

$$f(y) = \frac{f_1 y}{f_2 + y}, \quad \text{where } f_1 \text{ and } f_2 \text{ are positive constants.} \tag{4.7}$$

In population dynamics, the nonlinearity (4.7) is sometimes said to be of Beverton–Holt type [3] or of Holling type II [17]. The function f satisfies $f(0)=0$, $f'(0) = f_1/f_2$,

$f'(y) > 0$, and $f''(y) < 0$ for all $y \in \mathbb{R}_+$ and $\lim_{y\to\infty} f'(y) = 0$. Therefore, by Proposition 4.7, (A5) and (A7) hold if $f_1/f_2 > p$. Moreover, if $f_1/f_2 \le p$, then (A5) does not hold.

(2) Consider the nonlinearity $f : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$f(y) = ye^{-ay}, \quad \text{where } a \text{ is a positive constant.} \tag{4.8}$$

In population dynamics, this function is referred to as Ricker nonlinearity [36]. By using elementary calculus, it can be shown that (A5) and (A7) hold if and only if $p \in [e^{-2}, 1)$ (see [44]) independently of $a > 0$.

LEMMA 4.9. *Assume that* (A1)–(A5) *are satisfied. Then* 0 *and* $x^* := -A^{-1}bpy^* > 0$ *are equilibria of system* (2.1). *If additionally* (A7) *holds, then there are no other equilibria in* $\mathbb{R}^n_+$.

Proof. Clearly, since $f(0) = 0$ by (A4), 0 is an equilibrium of (2.1). By Lemma 4.2, $\|G\|_{H^\infty} = G(0) > 0$, and so, appealing to Theorem 2.3 and Corollary 3.2, we see that x^* is an equilibrium of (2.1) satisfying $x^* > 0$.

Now assume that (A7) also holds and that $x_0 \in \mathbb{R}^n_+$ is an equilibrium of (2.1), that is, $Ax_0 + bf(c^T x_0) = 0$. We have to show that $x_0 = 0$ or $x_0 = x^*$. Since A is Hurwitz, A is invertible, and so,

$$x_0 = -A^{-1}bf(c^T x_0). \tag{4.9}$$

If $c^T x_0 = 0$, then $x_0 = 0$. Assume that $c^T x_0 > 0$. Since

$$c^T x_0 = -c^T A^{-1} bf(c^T x_0) = G(0)f(c^T x_0) = \frac{1}{p} f(c^T x_0),$$

it follows that

$$f(c^T x_0) - f(y^*) = p(c^T x_0 - y^*).$$

Invoking assumption (A7), we conclude that $c^T x_0 = y^*$, which together with (4.9) implies that $x_0 = x^*$. ∎

The following result shows in particular that under suitable assumptions, the equilibrium $x^* = -A^{-1}bpy^*$ is stable in the large and attracts every nonzero initial vector $\xi \ge 0$.

THEOREM 4.10. *Consider the unforced Lur'e system* (2.1) *and assume that* (A1)–(A5) *hold.*

(1) *Under the additional assumption that* (A6) *is satisfied, there exists* $g \ge 1$ *such that* $x^* = -A^{-1}bpy^*$ *is stable in the large in the sense that, for every* $\xi \in \mathbb{R}^n_+$,

$$\|x(t;\xi) - x^*\| \le g\|\xi - x^*\| \quad \forall\ t \ge 0.$$

(2) *Under the additional assumption that* (A7) *is satisfied, the equilibrium* $x^* = -A^{-1}bpy^*$ *is "globally" asymptotically stable in the sense that it is stable in the large and, for every* $\xi \in \mathbb{R}^n_+$, $\xi \ne 0$, $x(t;\xi) \to x^*$ *as* $t \to \infty$.

Proof. Let $\tilde{f}$ be defined by (4.4), thereby extending f to the whole real line in an obvious way. Note that by linear boundedness of f and assumptions (A1) and (A4), we have that, for every $\xi \in \mathbb{R}^n_+$, $x(\cdot\,;\xi)$ is defined on $\mathbb{R}_+$ and $x(t;\xi) \in \mathbb{R}^n_+$ for all $t \ge 0$. Therefore, for every $\xi \in \mathbb{R}^n_+$, $x(\cdot\,;\xi)$ is also the unique maximally defined forward solution of (4.5). Appealing to Lemma 4.2, we see that $\|G\|_{H^\infty} = G(0) > 0$.

Thus, via an application of statement (1) of Theorem 2.3 to the Lur'e system (4.5), it follows that statement (1) holds. Furthermore, if (A7) is satisfied, then by statement (2) of Theorem 2.3,

$$\lim_{t\to\infty} x(t;\xi) = x^* \quad \text{or} \quad \lim_{t\to\infty} x(t;\xi) = 0.$$

Fix $\xi \in \mathbb{R}^n_+$, $\xi \neq 0$, and write $x(t) := x(t;\xi)$ for all $t \geq 0$. Seeking a contradiction, suppose that $x(t) \to 0$ as $t \to \infty$. Then there exists $\tau \geq 0$ such that $c^T x(t) \leq y^*$ for all $t \geq \tau$. Thus, since

$$x(t+\tau) = e^{(A+pbc^T)t}x(\tau) + \int_\tau^{t+\tau} e^{(A+pbc^T)(t+\tau-s)} b(f(c^T x(s)) - pc^T x(s))ds \quad \forall\ t \geq 0$$

and

$$f(y) - py \geq 0 \quad \forall\ y \in [0, y^*],$$

we have

$$x(t+\tau) \geq e^{(A+pbc^T)t}x(\tau) \quad \forall\ t \geq 0. \tag{4.10}$$

By Lemma 4.3, $\alpha(A+pbc^T) = 0$, and so it follows from Theorem 3.4 that there exists $v \gg 0$ such that $v^T(A+pbc^T) = 0$. Consequently,

$$v^T e^{(A+pbc^T)t} = v^T \quad \forall\ t \geq 0.$$

By (4.10),

$$v^T x(t+\tau) \geq v^T x(\tau) \quad \forall\ t \geq 0.$$

Since $v \gg 0$ and $x(\tau) \in \mathbb{R}^n_+$, $x(\tau) \neq 0$, it is clear that $v^T x(\tau) > 0$, and so

$$v^T x(t) \geq v^T x(\tau) > 0 \quad \forall\ t \geq \tau,$$

contradicting the supposition that $\lim_{t\to\infty} x(t) = 0$. Thus, $x(t) \to x^* = -A^{-1}bpy^*$ as $t \to \infty$, completing the proof of statement (2). ∎

We now discuss the issue of exponential stability. To recap, assumptions (A1)–(A5) imply that the unforced Lur'e system (2.1) has at least two equilibria, including 0 and x^*. When (A6) and (A7) hold as well, then Theorem 4.10 shows that x^* is respectively stable in the large and asymptotically stable with domain of attraction equal to $\mathbb{R}^n_+ \setminus \{0\}$. Assumption (A7) also ensures that 0 and x^* are the only equilibria of (2.1) (see Lemma 4.9). We remark that "global" exponential stability of x^*, in the sense that there exist $g \geq 1$ and $\gamma > 0$ such that for every *nonzero* $\xi \in \mathbb{R}^n_+$,

$$\|x(t;\xi) - x^*\| \leq ge^{-\gamma t}\|\xi - x^*\| \quad \forall\ t \geq 0, \tag{4.11}$$

is not possible. This is a straightforward consequence of the continuity of the flow map $(t,\xi) \mapsto x(t;\xi)$ together with the facts that 0 is an equilibrium of (2.1) and $x^* \neq 0$. Indeed, $x(t;0) = 0$ for each $t \geq 0$, and so, if (t_n) is a sequence in $\mathbb{R}_+$ with $t_n \to \infty$, then there exists a sequence (ξ_n) in $\mathbb{R}^n_+ \setminus \{0\}$ such that

$$\xi_n \to 0 \quad \text{and} \quad x(t_n;\xi_n) \to 0 \qquad \text{as } n \to \infty,$$

and thus,

$$\|x(t_n;\xi_n) - x^*\| \to \|x^*\| > 0 \quad \text{as } n \to \infty.$$

Hence, there do not exist constants g and γ such the estimate (4.11) holds for all nonzero ξ in $\mathbb{R}^n_+$.[1]

We will now show that under suitable conditions, the equilibrium x^* is exponentially stable in a "quasi-global" sense (see Theorem 4.11 below). To this end, we introduce two further assumptions:

(A8) $\limsup_{y\to y^*} \left|\dfrac{f(y)-f(y^*)}{y-y^*}\right| < p.$

(A9) $\limsup_{y\to \infty} \left|\dfrac{f(y)-f(y^*)}{y-y^*}\right| < p.$

If f is differentiable at y^*, then (A8) is equivalent to the condition $|f'(y^*)| < p$. Assumption (A9) says that for all sufficiently large y, the points $(y, f(y))$ lie in a sector defined by two straight lines of slope r and $-r$, where $0 < r < p$. Note that neither (A8) nor (A9) is implied by assumption (A7).

Theorem 4.11. *Assume that* (A1)–(A5) *and* (A7)–(A9) *hold. Then the equilibrium* $x^* = -A^{-1}bpy^*$ *of the unforced Lur'e system* (2.1) *is quasi-globally exponentially stable in the sense that for every* $\varepsilon > 0$*, there exist constants* $\gamma > 0$ *and* $g \geq 1$ *such that* (4.11) *holds for every* $\xi \in \mathbb{R}^n_+$ *with* $\|\xi\| \geq \varepsilon$.

Note that, collectively, (A7)–(A9) are equivalent to the condition:

$$\sup_{y\geq y_0,\, y\neq y^*} \left|\frac{f(y)-f(y^*)}{y-y^*}\right| < p \quad \forall\, y_0 > 0.$$

Obviously, this condition can be seen as a "uniformized" version of (A7). Also note that the above inequality does not hold for $y_0 = 0$ (since $f(0) = 0$ and $f(y^*) = py^*$). The proof of Theorem 4.11 is facilitated by the following proposition, which is formulated in the context of the forced Lur'e system (2.15), because it will not only be useful in this section, but also in the next section on ISS for forced Lur'e systems.

Proposition 4.12. *Assume that* (A1)–(A5) *and* (A7) *hold and let* $\varepsilon > 0$*. Then there exist* $\eta > 0$ *and* $\theta \geq 0$ *such that for all* $\xi \in \mathbb{R}^n_+$ *with* $\|\xi\|_1 \geq \varepsilon$ *and all nonnegative disturbances* $d \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n_+)$*, the solution* $x(\cdot\,;\xi,d)$ *of the forced Lur'e system* (2.15) *satisfies*

$$c^T x(t;\xi,d) \geq \eta \quad \forall\ t \geq \theta\,. \tag{4.12}$$

Informally, the proposition says that under the stated assumptions, the output $c^T x(t;\xi,d)$ is uniformly ultimately bounded away from 0. In order to avoid breaking the flow of the presentation, we relegate the lengthy proof of Proposition 4.12 to the end of the section.

Proof of Theorem 4.11. Let $\varepsilon > 0$. By Proposition 4.12, there exist $\eta > 0$ and $\theta \geq 0$ such that $c^T x(t;\xi) \geq \eta$ for all $t \geq \theta$, all $\xi \in \mathbb{R}^n_+$ with $\|\xi\|_1 \geq \varepsilon$, and all nonnegative disturbances $d \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n_+)$. Invoking assumptions (A7)–(A9), it follows that

$$r := \sup\left\{|f(y+y^*) - f(y^*)|/|y| : -y^* + \eta \leq y < \infty,\ y \neq 0\right\} < p\,. \tag{4.13}$$

[1] We mention that this also follows from a more general result on domains of attractions; see [12, Theorem 33.2].

Consider a fixed, but arbitrary, $\xi \in \mathbb{R}^n_+$ with $\|\xi\|_1 \geq \varepsilon$, and write $\tilde{x}(t) := x(t;\xi) - x^*$ for $t \geq 0$. Choose a locally Lipschitz function $\tilde{f} : \mathbb{R} \to \mathbb{R}$ such that

$$\tilde{f}(y) = f(y + y^*) - f(y^*) \quad \forall\ y \in [-y^* + \eta, \infty) \quad \text{and} \quad |\tilde{f}(y)/y| \leq r \quad \forall\ y \in \mathbb{R}\backslash\{0\}\,. \tag{4.14}$$

Since $c^T\tilde{x}(t) \geq -y * +\eta$ for all $t \geq \theta$ and using (4.14), it is straightforward to show that

$$\dot{\tilde{x}} = A\tilde{x} + b\tilde{f}(c^T\tilde{x}) \quad \forall\ t \geq \theta\,.$$

By (4.13), $r < p$, and thus it follows from statement (3) of Theorem 2.1 that there exist $\gamma > 0$ and $h \geq 1$ (not depending on ξ) such that

$$\|\tilde{x}(t)\| \leq he^{-\gamma(t-\theta)}\|\tilde{x}(\theta)\| \quad \forall\ t \geq \theta\,.$$

Combined with the stability in the large of x^* (statement (1) of Theorem 4.10), this shows that there exists $g > h$ (not depending on ξ) such that

$$\|\tilde{x}(t)\| \leq ge^{-\gamma t}\|\xi - x^*\| \quad \forall\ t \geq 0\,,$$

completing the proof. ∎

Example 4.13. We revisit Example 4.8 and consider nonlinearities of Beverton–Holt and Ricker type.

(1) The Beverton–Holt nonlinearity f given by (4.7) has the following properties: $f'(0) = f_1/f_2$, $f'(y) < f_1/f_2$ for all $y > 0$, and $f(y)/y \to 0$ as $y \to \infty$. Combining this with part (1) of Example 4.8, we conclude that f satisfies (A5) and (A7)–(A9) if and only if $f_1/f_2 > p$. Consequently, since f trivially satisfies (A4), Theorem 4.11 applies to the Lur'e system (2.1) provided that the linear system (A, b, c^T) satisfies (A1)–(A3) and $f_1/f_2 > p$.

(2) The Ricker nonlinearity f given by (4.8) satisfies $f'(0) = 1$ and, since $e^{-ay^*} = p$,

$$f'(y^*) = e^{-ay^*}(1 - ay^*) = p(1 + \log p)\,.$$

In conjunction with part (2) of Example 4.8, this shows that f satisfies (A5) and (A7)–(A9) if and only if $p \in (e^{-2}, 1)$. Now it is obvious that f satisfies (A4), and so, Theorem 4.11 applies to the Lur'e system (2.1), provided that the linear system (A, b, c^T) satisfies (A1)–(A3) and $p \in (e^{-2}, 1)$.

The next result shows that if (A9) does not hold, but all other assumptions of Theorem 4.11 are satisfied, then x^* is still "semiglobally" exponentially stable.

THEOREM 4.14. *Assume that* (A1)–(A5), (A7), *and* (A8) *hold. Then the equilibrium* $x^* = -A^{-1}bpy^*$ *of the unforced Lur'e system* (2.1) *is semiglobally exponentially stable in the sense that for every compact set* $\Gamma \subseteq \mathbb{R}^n_+$ *with* $0 \notin \Gamma$, *there exist constants* $\gamma > 0$ *and* $g \geq 1$ *such that* (4.11) *holds for every* $\xi \in \Gamma$.

Proof. Let $\Gamma \subseteq \mathbb{R}^n_+$ be compact with $0 \notin \Gamma$. Then there exists $\varepsilon > 0$ such that $\|\xi\|_1 \geq \varepsilon$ for all $\xi \in \Gamma$. Consequently, invoking Proposition 4.12, there exist $\eta > 0$ and $\theta \geq 0$ such that

$$c^T x(t;\xi) \geq \eta \quad \forall\, t \geq \theta,\ \forall\, \xi \in \Gamma.$$

Furthermore, by statement (1) of Theorem 4.10, the equilibrium x^* is stable in the large, and thus, there exists a constant $h > 0$ such that

$$c^T x(t;\xi) \leq h \quad \forall\, t \geq 0,\ \forall\, \xi \in \Gamma.$$

Replacing the definitions of r and $\tilde{f}:\mathbb{R}\to\mathbb{R}$ in (4.13) and (4.14) by

$$r := \sup\big\{|f(y+y^*)-f(y^*)|/|y| : -y^*+\eta \le y \le h\},\, y\neq 0\big\}$$

and

$$\tilde{f}(y) = f(y+y^*)-f(y^*) \quad \forall\, y\in[-y^*+\eta,\infty) \quad \text{and} \quad |\tilde{f}(y)/y|\le r \quad \forall\, y\in\mathbb{R}\backslash\{0\}\,,$$

respectively, and noting that $r<p$ by (A7) and (A8), we can argue as in the proof of Theorem 4.11 to establish the claim. ∎

We mention that there is an alternative method of proving the semiglobal exponential stability property guaranteed by Theorem 4.14 that does not make use of Proposition 4.12. This proof rests on a combination of local exponential stability of x^* (which is not difficult to establish), statement (2) of Theorem 4.10, and a well-known uniformity property enjoyed by compact subsets of the region of attraction of an asymptotically stable equilibrium (see [26, Proposition 5.20]). We emphasize that this approach cannot be used to establish the quasi-global exponential stability property (Theorem 4.11), which pertains to initial vectors of arbitrarily large norm.

It remains to prove Proposition 4.12. To this end, it is convenient to introduce some notation. Assuming that (A1)–(A3) hold, let $q\ge p$, set $a_q := \alpha(A+qbc^T)$, and let $v_q,\, w_q\in\mathbb{R}^n_+$ denote the unique positive vectors such that

$$v_q^T(A+qbc^T) = a_q v_q^T, \quad (A+qbc^T)w_q = a_q w_q, \quad \|v_q\|_1 = \|w_q\|_1 = 1.$$

The existence of such vectors follows from an application of statement (4) of Theorem 3.4 to the irreducible Metzler matrix $A+qbc^T$. By Lemma 4.3,

$$0 = a_p < a_q \quad \forall\, q>p.$$

Invoking part (5) of Theorem 3.4, there exist $\tau_q>0$ such that

$$e^{-a_q t}e^{(A+qbc^T)t} \ge \frac{1}{2v_q^T w_q} w_q v_q^T =: L_q \gg 0 \quad \forall\, t\ge\tau_q\,. \tag{4.15}$$

We define the constants

$$\mu := \delta(A+pbc^T)\ge 0, \quad \lambda_q := \text{smallest component of } c^T L_q. \tag{4.16}$$

Note that $\lambda_q>0$ by the positivity of L_q. Furthermore, for every $l>0$, we set

$$\omega(l) := \inf\{\|z\|_1 : c^T z\ge l\} > 0. \tag{4.17}$$

The following technical lemma will be useful for the proof of Proposition 4.12.

LEMMA 4.15. *Assume that* (A1)–(A5) *and* (A7) *hold, fix* $y^\dagger\in(0,y^*)$, *and let* $d\in L^\infty_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R}^n_+)$.

(1) *If for* $\xi\in\mathbb{R}^n_+\backslash\{0\}$, *there exists* $t^\dagger\ge 0$ *such that* $c^T x(t^\dagger;\xi,d) = y^\dagger$, *then*

$$c^T x(t;\xi,d) \ge \min\big\{\lambda_p\omega(y^\dagger), e^{-\mu\tau_p}y^\dagger\big\} > 0 \quad \forall\; t\ge t^\dagger\,.$$

(2) *For each* $\varepsilon>0$, *there exists* $t_\varepsilon\in(0,\infty)$ *such that for all* $\xi\in\mathbb{R}^n_+$ *with* $\|\xi\|_1\ge\varepsilon$ *and* $c^T\xi<y^\dagger$, *there exists* $t^\dagger\in(0,t_\varepsilon]$ *such that* $c^T x(t^\dagger;\xi,d) = y^\dagger$.

Proof. Let $\xi \in \mathbb{R}^n_+$, $\xi \neq 0$, $d \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n_+)$, and set $x(t) := x(t; \xi, d)$ for all $t \geq 0$.

To prove statement (1), let $t^\dagger \geq 0$ be such that $c^T x(t^\dagger) = y^\dagger$. If $c^T x(t) \geq y^\dagger$ for all $t \geq t^\dagger$, then there is nothing to show (because $y^\dagger \geq \min\{\lambda_p \omega(y^\dagger), e^{-\mu\tau_p} y^\dagger\}$). Therefore, let us assume that there exists $t_1 > t^\dagger$ such that $c^T x(t_1) < y^\dagger$. It is sufficient to show that

$$c^T x(t_1) \geq \min\left\{\lambda_p \omega(y^\dagger), e^{-\mu\tau_p} y^\dagger\right\}.$$

To this end, as $t \mapsto c^T x(t)$ is continuous, note that there exists $t_0 \in [t^\dagger, t_1)$ such that $c^T x(t_0) = y^\dagger$ and

$$c^T x(t) \leq y^\dagger \quad \forall\ t \in [t_0, t_1].$$

Invoking the sector condition (A7), we obtain

$$f(c^T x(t)) \geq p c^T x(t) \quad \forall\ t \in [t_0, t_1]. \tag{4.18}$$

Now, for $t \geq 0$, it follows from the variation of parameters formula that

$$x(t+t_0) = e^{(A+pbc^T)t} x(t_0) + \int_{t_0}^{t+t_0} e^{(A+pbc^T)(t+t_0-s)} \big[b\big(f(c^T x(s)) - pc^T x(s)\big) + d(s)\big] ds. \tag{4.19}$$

By the hypotheses and (4.18), the integrand on the right-hand side of (4.19) is nonnegative for all $s \in [t_0, t_1]$, and so

$$x(t+t_0) \geq e^{(A+pbc^T)t} x(t_0) \quad \forall\ t \in [0, t_1 - t_0]. \tag{4.20}$$

By definition of μ in (4.16), it follows that $\mu I + A + pbc^T$ is nonnegative, and thus

$$e^{(A+pbc^T)t} \geq e^{-\mu t} I \geq e^{-\mu\tau_p} I \quad \forall\ t \in [0, \tau_p], \tag{4.21}$$

with τ_p defined in (4.15). Combining (4.20) and (4.21), we see that

$$c^T x(t+t_0) \geq e^{-\mu\tau_p} c^T x(t_0) = e^{-\mu\tau_p} y^\dagger \quad \text{for all } t \text{ such that } 0 \leq t \leq \min\left\{\tau_p, t_1 - t_0\right\}.$$

Hence, if $t_1 - t_0 \leq \tau_p$, then

$$c^T x(t_1) = c^T x(t_1 - t_0 + t_0) \geq e^{-\mu\tau_p} y^\dagger. \tag{4.22}$$

Furthermore, if $t_1 - t_0 > \tau_p$, then, by (4.15), (4.16), and (4.20),

$$c^T x(t+t_0) \geq c^T L_p x(t_0) \geq \lambda_p \|x(t_0)\|_1 \quad \forall\ t \in (\tau_p, t_1 - t_0].$$

Since $c^T x(t_0) = y^\dagger$, we have $\|x(t_0)\|_1 \geq \omega(y^\dagger)$, and so

$$c^T x(t_1) = c^T x(t_1 - t_0 + t_0) \geq \lambda_p \omega(y^\dagger). \tag{4.23}$$

Combining (4.22) and (4.23) yields that

$$c^T x(t_1) \geq \min\left\{\lambda_p \omega(y^\dagger), e^{-\mu\tau_p} y^\dagger\right\},$$

which completes the proof of statement (1).

We proceed to prove statement (2). To this end, given $\varepsilon > 0$, let $\xi \in \mathbb{R}^n_+$ be such that $c^T \xi < y^\dagger$ and $\|\xi\|_1 \geq \varepsilon$. We consider two exhaustive cases.

Case 1. There does not exist $t \in (0, \tau_p]$ such that $c^T x(t) = y^\dagger$, in which case

$$c^T x(t) < y^\dagger \quad \forall\ t \in [0, \tau_p]\,. \tag{4.24}$$

Set

$$\mathcal{T} := \bigl\{t \geq 0 : c^T x(s) \leq y^\dagger \ \ \forall\, s \in [\tau_p, \tau_p{+}t]\bigr\} \text{ and } r := \inf\bigl\{f(y)/y : \lambda_p\varepsilon \leq y \leq y^\dagger\bigr\} > p.$$

Note that assumption (A7) guarantees that $r > p$. It is clear that $\mathcal{T} \neq \emptyset$ and $t^* := \sup \mathcal{T}$ satisfies $0 < t^* \leq \infty$ (by (4.24) and the definition of t^*).

Noting that $c^T x(t) \leq y^\dagger$ for all $t \in [0, \tau_p + t^*)$, we can argue as in the derivation of statement (1) (see, especially, (4.18)–(4.20)) to obtain

$$c^T x(t) \geq c^T e^{(A+pbc^T)t}\xi \quad \forall\ t \in [0, \tau_p + t^*)\,.$$

Now $c^T e^{(A+pbc^T)t}\xi \geq c^T L_p \xi \geq \lambda_p \varepsilon$ for all $t \geq \tau_p$, and so

$$c^T x(t) \geq \lambda_p \varepsilon \quad \forall\ t \in [\tau_p, \tau_p + t^*)\,. \tag{4.25}$$

Consequently,

$$\lambda_p \varepsilon \leq c^T x(t) \leq y^\dagger \quad \forall\ t \in [\tau_p, \tau_p + t^*), \tag{4.26}$$

and thus, by the definition of r,

$$f(c^T x(t)) \geq r c^T x(t) \quad \forall\ t \in [\tau_p, \tau_p + t^*).$$

The variation of parameters formula then yields

$$\begin{aligned} x(t) &= e^{(A+rbc^T)(t-\tau_p)} x(\tau_p) + \int_{\tau_p}^{t} e^{(A+rbc^T)(t-s)}\bigl[b\bigl(f(c^T x(s)) - r c^T x(s)\bigr) + d(s)\bigr] ds \\ &\geq e^{(A+rbc^T)(t-\tau_p)} x(\tau_p) \quad \forall\, t \in [\tau_p, \tau_p + t^*). \end{aligned} \tag{4.27}$$

Since

$$e^{(A+rbc^T)(t-\tau_p)} x(\tau_p) \geq e^{a_r(t-\tau_p)} L_r x(\tau_p) \quad \forall\, t \geq \tau_p + \tau_r, \tag{4.28}$$

we use the positivity of $a_r > 0$ to conclude from (4.26) and (4.27) that $t^* < \infty$. Setting $t^\dagger := \tau_p + t^*$, it is clear that $c^T x(t^\dagger) = y^\dagger$. If $t^\dagger > \tau_p + \tau_r$ (equivalently, $t^* > \tau_r$), then, by (4.27) and (4.28),

$$y^\dagger = c^T x(t^\dagger) \geq c^T e^{(A+rbc^T)(t^\dagger - \tau_p)} x(\tau_p) \geq e^{a_r(t^\dagger - \tau_p)} c^T L_r x(\tau_p),$$

and so, invoking (4.25),

$$y^\dagger \geq e^{a_r(t^\dagger - \tau_p)} \lambda_r \omega(\lambda_p \varepsilon),$$

which in turn leads to

$$t^\dagger \leq \tau_p + \frac{1}{a_r} \ln \frac{y^\dagger}{\lambda_r \omega(\lambda_p \varepsilon)} =: s_\varepsilon.$$

Consequently, we have that

$$t^\dagger \leq \max\{s_\varepsilon, \tau_p + \tau_r\} =: t_\varepsilon. \tag{4.29}$$

Case 2. There exists $t \in (0, \tau_p]$ such that $c^T x(t) = y^\dagger$. In which case, setting $t^\dagger := t$, (4.29) is trivially satisfied. ∎

We are now in position to prove Proposition 4.12.

Proof of Proposition 4.12. Fix $y^\dagger \in (0, y^*)$ and $\varepsilon > 0$. For $\xi \in \mathbb{R}^n_+$ with $\|\xi\|_1 \geq \varepsilon$ and $d \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n_+)$, set $x(t) := x(t; \xi, d)$. Furthermore, define

$$\eta := \min\left\{\lambda_p \omega(y^\dagger), e^{-\mu\tau_p} y^\dagger\right\}$$

and $\theta := t_\varepsilon > 0$, where t_ε is the number guaranteed to exist by statement (2) of Lemma 4.15. We will show that

$$c^T x(t) \geq \eta \quad \forall\, t \geq \theta \tag{4.30}$$

by considering three exhaustive cases.

Case 1. If $c^T x(0) = c^T \xi < y^\dagger$, appealing to statement (2) of Lemma 4.15, we see that there exists $t^\dagger \in (0, \theta]$ such that $c^T x(t^\dagger) = y^\dagger$. An application of statement (1) of Lemma 4.15 yields that $c^T x(t) \geq \eta$ for all $t \geq t^\dagger$, and so (4.30) is satisfied.

Case 2. $c^T x(0) = c^T \xi = y^\dagger$. In this case, by statement (1) of Lemma 4.15, $c^T x(t) \geq \eta$ for all $t \geq 0$, and hence (4.30) holds.

Case 3. $c^T x(0) = c^T \xi > y^\dagger$. If $c^T x(t) > y^\dagger$ for all $t \geq 0$, then (4.30) is satisfied since $y^\dagger \geq \eta$. Alternatively, there exists $t^\dagger > 0$ such that

$$c^T x(t^\dagger) = y^\dagger \quad \text{and} \quad c^T x(t) > y^\dagger \;\; \forall\, t \in [0, t^\dagger).$$

By statement (1) of Lemma 4.15, $c^T x(t) \geq \eta$ for all $t \geq t^\dagger$. It now follows that (4.30) holds, since $c^T x(t) > y^\dagger \geq \eta$ for all $t \in [0, t^\dagger)$. ∎

5. Stability of forced nonnegative Lur'e systems. Consider a forced nonnegative Lur'e system of the form (2.15) where, as usual, nonnegative means that (A1) and (A4) hold. The unique maximally defined forward solution of (2.15) is denoted by $x(\cdot\,; \xi, d)$. If f is affine-linearly bounded and the disturbance (forcing, input) $d \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$ is nonnegative, then the maximally defined forward solution exists for all times $t \geq 0$ (that is, there is no finite escape time from the nonnegative orthant). Obviously, if $d \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$ is not nonnegative, then the interval of existence of the maximally defined forward solution may be bounded (finite escape time from the nonnegative orthant).

We introduce a further assumption (recalling that $p := 1/G(0) > 0$).

(A10) $py - f(y) \to \infty$ as $y \to \infty$.

The following result provides the counterpart of statement (2) of Theorem 4.4 for Lur'e systems with disturbances.

THEOREM 5.1. *Assume that* (A1)–(A4) *and* (A10) *hold, and $f(y)/y < p$ for all $y > 0$. Then, for the forced Lur'e system* (2.15), 0 *is ISS in the sense that there exist $\psi \in \mathcal{KL}$ and $\varphi \in \mathcal{K}$ such that, for all $\xi \in \mathbb{R}^n_+$ and all nonnegative disturbances $d \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n_+)$,*

$$\|x(t; \xi, d)\| \leq \psi(\|\xi\|, t) + \varphi(\|d\|_{L^\infty(0,t)}) \quad \forall\; t \geq 0. \tag{5.1}$$

Proof. By Lemma 4.2, $p = 1/\|G\|_{H^\infty} > 0$, and therefore $\mathbb{D}(0, p) \subsetneq \mathbb{S}(A, b, c^T)$. To apply Theorem 2.5 (with $r = p$ and $k = 0$), consider the function $\tilde{f} : \mathbb{R} \to \mathbb{R}$ given by (4.4), which extends f to the whole real line. Furthermore, by the hypotheses on f, we have that

$$p|y| - |\tilde{f}(y)| > 0 \quad \forall\; y \neq 0 \qquad \text{and} \qquad p|y| - |\tilde{f}(y)| \to \infty \quad \text{as } |y| \to \infty.$$

Hence, there exists $\beta \in \mathcal{K}_\infty$ such that

$$|\tilde{f}(y)| \leq p|y| - \beta(|y|) \quad \forall\, y \in \mathbb{R}.$$

Note that by linear boundedness of f and assumptions (A1) and (A4), we have that, for every $\xi \in \mathbb{R}^n_+$ and every $d \in L^\infty(\mathbb{R}_+, \mathbb{R}^n_+)$, $x(\cdot\,;\xi,d)$ is defined on $\mathbb{R}_+$ and $x(t;\xi,d) \in \mathbb{R}^n_+$ for all $t \geq 0$. Therefore, for every $\xi \in \mathbb{R}^n_+$ and every $d \in L^\infty(\mathbb{R}_+, \mathbb{R}^n_+)$, $x(\cdot;\xi,d)$ is also the unique maximally defined forward solution of

$$\dot{x} = Ax + b\tilde{f}(c^T x) + d, \quad x(0) = \xi. \tag{5.2}$$

An application of Theorem 2.5 to (5.2) shows that there exist $\psi \in \mathcal{KL}$ and $\varphi \in \mathcal{K}$ such that, for all $\xi \in \mathbb{R}^n_+$ and all $d \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n_+)$,

$$\|x(t;\xi,d)\| \leq \psi(\|\xi\|, t) + \varphi(\|d\|_{L^\infty(0,t)}) \quad \forall\, t \geq 0,$$

completing the proof. ∎

The next result shows that Theorem 4.6 extends to disturbed Lur'e systems.

THEOREM 5.2. *Assume that* (A1)–(A4) *hold and* f *satisfies*

$$\inf_{y>0} \frac{f(y)}{y} > p.$$

If $\xi \in \mathbb{R}^n_+$, $\xi \neq 0$, *and* $d \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n_+)$ *are such that the solution* $x(t;\xi,d)$ *of the forced Lur'e system* (2.15) *exists for every* $t \geq 0$, *then*

$$\lim_{t\to\infty} x_i(t;\xi,d) = \infty \quad \forall\, i \in \{1,\ldots,n\},$$

where $x_i(t;\xi,d)$ *denotes the* ith *component of* $x(t;\xi,d)$.

We omit the proof of Theorem 5.2 because the proof of Theorem 4.6, mutatis mutandis, carries over to Lur'e systems with disturbances.

We now state and prove the main result of this section. It can be viewed as a counterpart of statement (2) of Theorem 4.10 for Lur'e systems with disturbances.

THEOREM 5.3. *Assume that* (A1)–(A5), (A7), *and* (A10) *hold. Then, for the forced Lur'e system* (2.15), $x^* = -A^{-1}bpy^*$ *is "quasi-globally" ISS in the sense that, for all* $\varepsilon > 0$, *there exist* $\psi \in \mathcal{KL}$ *and* $\varphi \in \mathcal{K}$ *such that, for all* $\xi \in \mathbb{R}^n_+$ *with* $\|\xi\| \geq \varepsilon$ *and all nonnegative disturbances* $d \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n_+)$,

$$\|x(t;\xi,d) - x^*\| \leq \psi(\|\xi - x^*\|, t) + \varphi(\|d\|_{L^\infty(0,t)}) \quad \forall\, t \geq 0\,. \tag{5.3}$$

Proof. Let $\varepsilon > 0$. By Proposition 4.12, there exist $\eta > 0$ and $\theta \geq 0$ such that for all $\xi \in \mathbb{R}^n_+$ with $\|\xi\| \geq \varepsilon$ and all $d \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n_+)$,

$$c^T x(t;\xi,d) \geq \eta \quad \forall\, t \geq \theta\,. \tag{5.4}$$

Define $\tilde{f} : \mathbb{R} \to \mathbb{R}$ by

$$\tilde{f}(y) = \begin{cases} f(y+y^*) - f(y^*) & \text{for } y \geq -y^* + \eta, \\ f(\eta) - f(y^*) & \text{for } y < -y^* + \eta; \end{cases} \tag{5.5}$$

see Figure 5.1 for illustrations of f and $\tilde{f}$. Then

$$p|y| - |\tilde{f}(y)| > 0 \quad \forall\, y \neq 0 \qquad \text{and} \qquad p|y| - |\tilde{f}(y)| \to \infty \quad \text{as } |y| \to \infty.$$

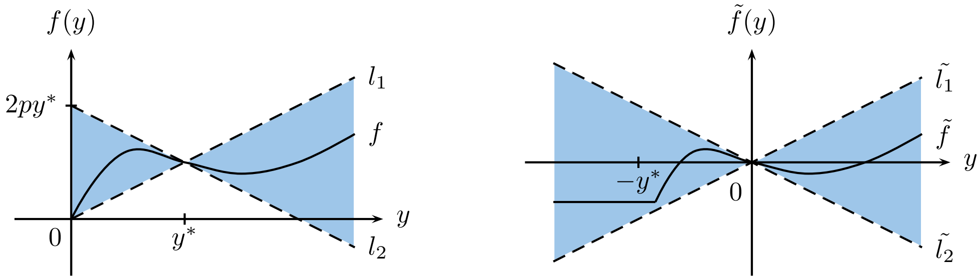


FIG. 5.1. *The left figure shows the "original" nonlinearity f bounded by the lines $l_1(y) = py$ and $l_2(y) = 2py^* - py$. The right figure shows the graph of the shifted and extended nonlinearity $\tilde{f}$ bounded by the lines $\tilde{l_1}(y) = py$ and $\tilde{l_2}(y) = -py$.*

Hence, there exists $\beta \in \mathcal{K}_\infty$ such that

$$|\tilde{f}(y)| \le p|y| - \beta(|y|) \quad \forall\ y \in \mathbb{R}\,.$$

Combining this with the fact that $\mathbb{D}(0,p) \subseteq \mathbb{S}(A,b,c^T)$, it follows from Theorem 2.5 that the system

$$\dot{z} = Az + b\tilde{f}(c^T z) + \tilde{d}, \quad z(0) = \zeta\,, \tag{5.6}$$

is ISS in the sense that there exist $\tilde{\psi} \in \mathcal{KL}$ and $\tilde{\varphi} \in \mathcal{K}$ such that, for every $\zeta \in \mathbb{R}^n$ and every $\tilde{d} \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$,

$$\|z(t;\zeta,\tilde{d})\| \le \tilde{\psi}(\|\zeta\|, t) + \tilde{\varphi}(\|\tilde{d}\|_{L^\infty(0,t)}) \quad \forall\ t \ge 0\,, \tag{5.7}$$

where $z(\cdot\,;\zeta,\tilde{d})$ denotes the unique forward solution of (5.6).

Let $\xi \in \mathbb{R}^n_+$ with $\|\xi\| \ge \varepsilon$, and let $d \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n_+)$ be a nonnegative disturbance. Define $\tilde{x}(t) := x(t;\xi,d) - x^*$ for all $t \ge 0$, and set

$$\tilde{x}_\theta(t) := \tilde{x}(t+\theta) \quad \text{and} \quad d_\theta(t) := d(t+\theta) \qquad \forall\ t \ge 0\,.$$

By (5.4),

$$c^T \tilde{x}_\theta(t) \ge -y^* + \eta \quad \forall\ t \ge 0\,,$$

and it is easy to see that $\tilde{x}_\theta$ solves (5.6) with $\zeta = \tilde{x}_\theta(0) = x(\theta;\xi,d) - x^*$ and $\tilde{d} = d_\theta$. Hence, by (5.7) we have that

$$\|\tilde{x}_\theta(t)\| \le \tilde{\psi}(\|\tilde{x}_\theta(0)\|, t) + \tilde{\varphi}(\|d_\theta\|_{L^\infty(0,t)}) \quad \forall\ t \ge 0\,. \tag{5.8}$$

Moreover, on the interval $[0,\theta]$, $\tilde{x}$ satisfies

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + bf^*(c^T\tilde{x}(t)) + d(t) \quad \forall\ t \in [0,\theta]\,,$$

where the function $f^* : [-y^*, \infty) \to [-py^*, \infty)$ is defined by

$$f^*(y) = f(y+y^*) - f(y^*) = f(y+y^*) - py^* \quad \forall\ y \ge -y^*.$$

It is clear that $|f^*(y)| \leq p|y|$ for all $y \geq -y^*$, and using the variation of parameters formula, it follows that there exist constants $k_1 > 0$ and $k_2 > 0$ (not depending on ξ and d) such that

$$\|\tilde{x}(t)\| \leq k_1\big(\|\xi - x^*\| + \|d\|_{L^\infty(0,\theta)}\big) + k_2 \int_0^t \|\tilde{x}(s)\| ds \quad \forall\ t \in [0,\theta]. \tag{5.9}$$

Applying Gronwall's lemma to the estimate (5.9) yields

$$\|\tilde{x}(t)\| \leq k_1 e^{k_2\theta}\big(\|\xi - x^*\| + \|d\|_{L^\infty(0,\theta)}\big) = k\big(\|\xi - x^*\| + \|d\|_{L^\infty(0,\theta)}\big) \quad \forall\, t \in [0,\theta]\,, \tag{5.10}$$

where $k := k_1 e^{k_2\theta}$. Defining $\psi_1 \in \mathcal{KL}$ and $\varphi_1 \in \mathcal{K}$ by

$$\psi_1(s,t) := k e^{\theta} e^{-t} s \quad \forall\ s,t \geq 0 \qquad \text{and} \qquad \varphi_1(s) := ks \quad \forall\ s \geq 0,$$

respectively, and noting that $ks \leq ke^{\theta}e^{-t}s = \psi_1(s,t)$ for all $t \in [0,\theta]$ and $s \geq 0$, it follows from (5.10) that

$$\|\tilde{x}(t)\| \leq \psi_1(\|\xi - x^*\|, t) + \varphi_1(\|d\|_{L^\infty(0,t)}) \quad \forall\ t \in [0,\theta]\,. \tag{5.11}$$

Note that we have made use here of the causality of the underlying Lur'e system (on the right-hand side of (5.11), the L^∞-norm is taken over the interval $[0,t]$ and not over $[0,\theta]$ as in (5.10)). Furthermore, evaluating (5.10) at $t = \theta$, we see that

$$\|\tilde{x}_\theta(0)\| = \|\tilde{x}(\theta)\| \leq k\big(\|\xi - x^*\| + \|d\|_{L^\infty(0,\theta)}\big)\,. \tag{5.12}$$

Inserting (5.12) into (5.8) and invoking the inequality

$$\tilde{\psi}(s_1 + s_2, t) \leq \tilde{\psi}(2s_1, t) + \tilde{\psi}(2s_2, t) \leq \tilde{\psi}(2s_1, t) + \tilde{\psi}(2s_2, 0) \quad \forall\, s_1, s_2, t \geq 0,$$

we obtain

$$\|\tilde{x}(t+\theta)\| \leq \tilde{\psi}(2k\|\xi - x^*\|, t) + \tilde{\psi}(2k\|d\|_{L^\infty(0,\theta)}, 0) + \tilde{\varphi}(\|d\|_{L^\infty(0,\,t+\theta)}) \quad \forall\, t \geq 0\,. \tag{5.13}$$

Defining $\psi_2 \in \mathcal{KL}$ and $\varphi_2 \in \mathcal{K}$ by

$$\varphi_2(s) := \tilde{\varphi}(s) + \tilde{\psi}(2ks, 0) \quad \forall\ s \geq 0$$

and

$$\psi_2(s,t) := \begin{cases} \tilde{\psi}(2ks, 0), & (s,t) \in \mathbb{R}_+ \times [0,\theta], \\ \tilde{\psi}(2ks, t - \theta), & (s,t) \in \mathbb{R}_+ \times (\theta, \infty), \end{cases}$$

respectively, the estimate (5.13) can be written as

$$\|\tilde{x}(t+\theta)\| \leq \psi_2(\|\xi - x^*\|, t + \theta) + \varphi_2(\|d\|_{L^\infty(0,t+\theta)}) \quad \forall\ t \geq 0\,.$$

Finally, setting

$$\psi := \max(\psi_1, \psi_2) \in \mathcal{KL} \qquad \text{and} \qquad \varphi := \max(\varphi_1, \varphi_2) \in \mathcal{K},$$

it is clear that ψ and φ do not depend on ξ and d. Invoking (5.11), we obtain

$$\|\tilde{x}(t)\| \leq \psi(\|\xi - x^*\|, t) + \varphi(\|d\|_{L^\infty(0,t)}) \quad \forall\ t \geq 0,$$

and hence (5.3), completing the proof. ∎

Example 5.4. Once again we consider nonlinearities of Beverton–Holt and Ricker type; see Examples 4.8 and 4.13.

(1) Let f be the Beverton–Holt nonlinearity given by (4.7). Invoking Examples 4.8 and 4.13, we see that Theorem 5.3 applies to the Lur'e system (2.1), provided that the linear system (A, b, c^T) satisfies (A1)–(A3) and $f_1/f_2 > p$.

(2) Consider the Lur'e system (2.1), where f is the Ricker nonlinearity f given by (4.8), and assume that the linear system (A, b, c^T) satisfies (A1)–(A3). Then, if $p \in [e^{-2}, 1)$, the conclusion of Theorem 5.3 holds (cf. Examples 4.8 and 4.13).

Finally, we comment on forced Lur'e systems with arbitrary, not necessarily nonnegative, disturbances. As has already been mentioned, if the disturbance d is not nonnegative, then the solution may not exist on the whole interval $\mathbb{R}_+$, that is, the solution may approach the boundary of the nonnegative orthant in finite time. The following result shows that if $x^* \gg 0$, then, under the assumptions of Theorem 5.3, the forward solution exists on $\mathbb{R}_+$ (with values in the interior of the nonnegative orthant) for all initial conditions $\xi \in \mathbb{R}^n_+$ and all (not necessarily nonnegative) disturbances $d \in L^\infty(\mathbb{R}_+, \mathbb{R}^n)$ with $\|\xi - x^*\| + \|d\|_{L^\infty(\mathbb{R}_+)}$ sufficiently small.

PROPOSITION 5.5. *Assume that* (A1)–(A5), (A7), *and* (A10) *hold, and that* $x^* = -A^{-1}bpy^* \gg 0$. *Then there exists* $\varepsilon > 0$ *such that, for all* $\xi \in \mathbb{R}^n_+$ *and all disturbances* $d \in L(\mathbb{R}_+, \mathbb{R}^n)$ *with* $\|\xi - x^*\| + \|d\|_{L^\infty(\mathbb{R}_+)} < \varepsilon$, *the maximally defined forward solution* $x(\cdot\,; \xi, d)$ *exists on* $\mathbb{R}_+$ *with values in the interior of* $\mathbb{R}^n_+$.

The proof is based on an ISS argument combined with ideas similar to those used in the proof of Theorem 5.3 and can be found in Appendix A.

6. Examples.

Nonnegative Lur'e systems occur naturally in numerous biological contexts, and here we present a detailed discussion of two examples. The first arises in both population modeling [8] and reaction kinetics; see, for example, [33, section 7.2] or [45]. The second example comes from a spatial discretization of a nonlinear integro-partial differential equation that also arises in population dynamics.

Example 6.1. Consider the n coupled differential equations

$$\begin{aligned} \dot{x}_1 &= -a_1 x_1 + f(x_n) + d_1\,, \quad x_1(0) = \xi_1\,, \\ \dot{x}_k &= a_{2(k-1)} x_{k-1} - a_{2k-1} x_k + d_k\,, \quad x_k(0) = \xi_k \quad \text{for } k \in \{2, \dots, n\}\,, \end{aligned} \tag{6.1}$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$, $a_i > 0$ for all $i \in \{1, \dots, 2n-1\}$, $d_i \in L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R})$ is nonnegative for all $i \in \{1, \dots, n\}$, and $\xi_i \geq 0$ for all $i \in \{1, \dots, n\}$.

Systems of the form (6.1) describe the dynamics of a population over time, partitioned into n discrete age- or stage-classes; see, for example [8]. Here the a_{2k-1} denote mortality rates, the a_{2k} denote growth rates into the next stage class, and f models nonlinear recruitment. The functions d_i represent nonnegative disturbances.

Introducing $x := (x_1, \dots, x_n)^T$ and $d := (d_1, \dots, d_n)^T$, system (6.1) may be rewritten in the form (2.15) with

$$A := \begin{pmatrix} -a_1 & 0 & \cdots & 0 \\ a_2 & -a_3 & \ddots & \vdots \\ & \ddots & \ddots & 0 \\ 0 & & a_{2n-2} & -a_{2n-1} \end{pmatrix}, \quad b := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad c := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Obviously, (A1) holds. Since

$$\sigma(A) = \{-a_1, -a_3, \dots, -a_{2n-1}\},$$

and $a_i > 0$ for all $i \in \{1, \dots, 2n-1\}$, assumption (A2) is also satisfied. Moreover, it is readily verified that (A3) holds. A straightforward calculation yields that

$$p = \frac{1}{G(0)} = -\frac{1}{c^T A^{-1} b} = \frac{\prod_{i=1}^{n} a_{2i-1}}{\prod_{i=1}^{n-1} a_{2i}}.$$

Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a Beverton–Holt type nonlinearity given by

$$f(y) = \frac{f_1 y}{f_2 + y}, \quad \text{where } f_1 \text{ and } f_2 \text{ are positive constants;} \tag{6.2}$$

cf. Examples 4.8, 4.13, and 5.4. In the unforced case ($d = 0$), it follows from part (1) of Example 4.13 that Theorem 4.11 applies to (6.1), provided that $f_1/f_2 > p$. Similarly, in the context of the forced system, if $f_1/f_2 > p$, then, by part (1) of Example 5.4, Theorem 5.3 applies to (6.1).

For the following numerical simulation, we take

$$n = 3, \quad a_1 = 1, \quad a_2 = 0.8, \quad a_3 = 0.9, \quad a_4 = 0.6, \quad a_5 = 0.8, \quad f_1 = 3, \quad f_2 = 1. \tag{6.3}$$

For these choices $p = 3/2$, the unique positive y^* satisfying $f(y^*) = py^*$ is given by $y^* = 1$ and, furthermore,

$$x^* := -A^{-1}bpy^* = \begin{pmatrix} 3/2 & 4/3 & 1 \end{pmatrix}^T$$

is the unique nonzero equilibrium of (6.1) with $d_i = 0$ for all $i \in \{1, 2, 3\}$.

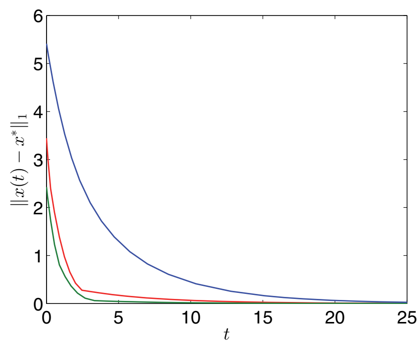
As $f_1/f_2 = 3 > 3/2 = p$, we may conclude that

- in the unforced case ($d_1 = d_2 = d_3 = 0$), the equilibrium x^* is quasi-globally exponentially stable in the sense of Theorem 4.11;
- in the forced case, x^* is ISS in the sense of Theorem 5.3.

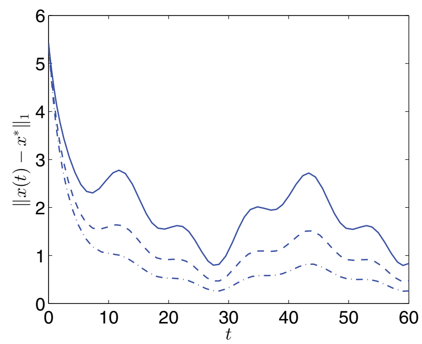
Figure 6.1(a) shows simulations of the system given by (6.1)–(6.3) with $d = (d_1, d_2, d_3)^T = 0$ for three initial conditions $\xi^i \in \mathbb{R}^3_+$, $i \in \{1, 2, 3\}$. To illustrate the ISS property of the system given by (6.1)–(6.3), Figure 6.1(b) contains simulations of (6.1)–(6.3) with initial condition ξ^3, and where the dynamics (6.1) are subject to the disturbances

$$d^i(t) = k_i \begin{pmatrix} 0.1\big(1 + \sin(0.2t)\big) \\ 0.2\big(1 + \cos(0.6t)\big) \\ 0.15\big(1 + \sin(0.2t)\big) \end{pmatrix} \quad \forall\, t \geq 0, \quad i = 1, 2, 3, \tag{6.4}$$

with $k_1 = 1$, $k_2 = 0.5$, and $k_3 = 0.25$.



(a)



(b)

FIG. 6.1. *Numerical simulations of the system given by* (6.1)–(6.3). (a) *Unforced case: evolution of* $\|x(t) - x^*\|$ *from three different initial conditions.* (b) *Forced case: evolution of* $\|x(t) - x^*\|$ *with three different disturbances* d^1, d^2, *and* d^3 *given by* (6.4), *plotted in solid, dashed, and dashed-dotted, respectively.*

Remark 6.2. The asymptotic dynamics of the unforced model (6.1) have been considered elsewhere in the literature, and here we highlight some existing results. In [45], the Popov criterion is used to prove, for a specific class of nonlinearities, global asymptotic stability results in the case that the unforced system (6.1) has a unique nonzero equilibrium (a scenario not considered in this paper). For the situation wherein the unforced model (6.1) has multiple equilibria, a global asymptotic result [38, Proposition 2.1] holds provided that f is strictly increasing. The latter result has some overlap with part (2) of Theorem 2.3. However, we emphasize that these two results are difficult to compare; for example, neither monotonicity of f nor the Metzler property of A is assumed in Theorem 2.3.

Example 6.3. Consider the following partial differential equation which models a population $q = q(z,t)$ at the (continuous) age-class $z \in [0,1]$ and time $t \geq 0$:

$$q_t(z,t) = -\delta q(z,t) - q_z(z,t), \quad t \geq 0\,,\ z \in [0,1], \tag{6.5a}$$

where subscripts denote partial derivatives. The initial condition is given by

$$q(z,0) = \xi(z) \quad \forall\, z \in [0,1] \tag{6.5b}$$

for continuous $\xi : [0,1] \to \mathbb{R}_+$, and we impose the nonlinear boundary condition

$$q(0,t) = f\left(\int_0^1 \beta(z) q(z,t)\, dz\right) \quad \forall\, t \geq 0\,. \tag{6.5c}$$

The nonnegative constant δ and the continuous nonnegative $\beta : [0,1] \to \mathbb{R}_+$ denote mortality and birth weighting, respectively, where we assume that $\beta(z) \not\equiv 0$. The function $f : \mathbb{R}_+ \to \mathbb{R}_+$ models density-dependence affecting recruitment into the population, and is assumed locally Lipschitz with $f(0) = 0$.

If f is the identity function (that is, $f(y) = y$ for all $y \geq 0$), then (6.5) is known as the McKendrick equation (also called the McKendrick–von Foerster equation); see [4, Chapter 2] and [20, 32].

We note that system (6.5) can be obtained from the linear controlled and observed PDE

$$q_t(z,t) = -\delta q(z,t) - q_z(z,t), \quad q(0,t) = u(t), \quad y(t) = \int_0^1 \beta(z) q(z,t)\, dz\,, \tag{6.6}$$

via application of the feedback $u = f(y)$. The transfer function G of (6.6) is given by

$$G(s) = \int_0^1 \beta(z) e^{-(\delta+s)z} dz. \tag{6.7}$$

For complex s with $\operatorname{Re} s \geq 0$, we have

$$G(0) \geq \int_0^1 |\beta(z) e^{-(\delta+s)z}| dz \geq |G(s)|,$$

showing that $G \in H^\infty$ and $\|G\|_{H^\infty} = G(0) > 0$, where the inequality follows from the assumption that $\beta(z) \not\equiv 0$.

A simple calculation shows that any equilibrium (steady-state solution) $q^* : [0,1] \to \mathbb{R}_+$ of (6.5) is of the form

$$q^*(z) = e^{-\delta z} r^* \quad \forall\ z \in [0,1]\,, \tag{6.8}$$

where $r^* \geq 0$ satisfies

$$r^* = f(G(0) r^*). \tag{6.9}$$

As $f(0) = 0$, (6.9) admits the solution $r^* = 0$, corresponding to the zero equilibrium of (6.5). Obviously, there may be other equilibria, depending on f and $G(0)$. Defining $p := 1/G(0)$ and setting $y^* := r^*/p$, (6.9) can be written in the familiar form $f(y^*) = py^*$.

Obviously, the theory in section 4 was developed in the context of Lur'e systems defined by ordinary differential equations and does not apply to the Lur'e system given by the PDE (6.5). Instead, we use the finite-dimensional results in section 4 to analyze the (finite-dimensional) Lur'e system

$$\dot{x}_N = A_N x_N + b_N f(c_N^T x_N), \quad x_N(0) = \xi_N, \tag{6.10a}$$

obtained by applying feedback with characteristic f to a finite-difference approximation of the PDE (6.6). In (6.10a), $N \in \mathbb{N}$, $(A_N, b_N, c_N) \in \mathbb{R}^{N \times N} \times \mathbb{R}^N \times \mathbb{R}^N$ are given by

(6.10b)

$$A_N = \begin{pmatrix} -\delta - N & 0 & \dots & & 0 \\ N & -\delta - N & 0 & \dots & \vdots \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & & & \\ 0 & \dots & 0 & N & -\delta - N \end{pmatrix}, \quad b_N = \begin{pmatrix} N \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}, \quad c_N = \frac{1}{N} \begin{pmatrix} \beta(z_1) \\ \beta(z_2) \\ \vdots \\ \\ \beta(z_N) \end{pmatrix},$$

and

$$\xi_N := (\xi(z_1), \dots, \xi(z_N))^T, \qquad \text{with } z_i := i/N,\ i \in \{1, 2, \dots, N\}.$$

To arrive at the approximation (6.10) of (6.5), set $x_N(t) := \big(q(z_1,t),\ldots,q(z_N,t)\big)^T$ for $t \geq 0$, and note that by approximating the spatial derivative $q_z(z_i,t)$ by a difference quotient, it follows from the PDE (6.5a) that

$$\begin{aligned} q_t(z_i,t) &\approx -\delta q(z_i,t) - \frac{q(z_i,t)-q(z_{i-1},t)}{z_i - z_{i-1}} \\ &= (-\delta - N)q(z_i,t) + Nq(z_{i-1},t) \quad \text{for } i \in \{2,\ldots,N\}. \end{aligned} \tag{6.11}$$

Moreover, for $i = 1$, we use a difference quotient approximation and the boundary condition (6.5c) to obtain

$$q_t(z_1,t) \approx -\delta q(z_1,t) - \frac{q(z_1,t)-q(0,t)}{z_1} = (-\delta - N)q(z_1,t) + Nq(0,t),$$

whence

$$\begin{aligned} q_t(z_1,t) &\approx (-\delta - N)q(z_1,t) + Nf\left(\int_0^1 \beta(z)q(z,t)\,dz\right) \\ &\approx (-\delta - N)q(z_1,t) + Nf\left(\sum_{j=1}^N \frac{\beta(z_j)}{N} q(z_j,t)\right), \end{aligned} \tag{6.12}$$

where we have approximated the integral by a Riemann sum. The equations (6.10) now follow from (6.11) and (6.12).

The triple (A_N, b_N, c_N) satisfies (A1) and (A2), and it is easily verified that (A3) holds if and only if $\beta(z_N) = \beta(1) > 0$, which we shall always impose. Denoting the transfer function of (6.10) by G_N, that is, $G_N(s) = c_N^T(sI - A_N)^{-1}b_N$, and setting $p_N := 1/G_N(0)$, a straightforward calculation shows that

$$G_N(0) = -c_N A_N^{-1} b_N = \frac{1}{N}\sum_{j=1}^N \frac{\beta(z_j)}{(1+\delta/N)^j} = \frac{1}{p_N} > 0. \tag{6.13}$$

We now assume for the remainder of the example that $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a Ricker type function given by (4.8):

$$f(y) = ye^{-ay}, \quad \text{where } a \text{ is a positive constant.}$$

We consider three cases.

Case a: $p_N < e^{-2}$. In this case, the results of section 4 do not apply to (6.10).

Case b: $p_N \in (e^{-2}, 1)$. Let y_N^* be the unique positive number such that $p_N y_N^* = f(y_N^*)$. Then

$$x_N^* := -A_N^{-1} b_N p_N y_N^*$$

is an equilibrium of (6.10), and by part (2) of Example 4.13, x_N^* is quasi-globally exponentially stable in the sense of Theorem 4.11.

Case c: $p_N > 1$. In this case, since $f'(y) \leq 1$ for all $y \geq 0$, it follows from part (3) of Theorem 4.4 that the equilibrium 0 of (6.10) is globally exponentially stable.

It can be shown (see Appendix A) that

$$\lim_{N\to\infty} G_N(0) = G(0), \tag{6.14}$$

and hence $\|G_N\|_{H^\infty} \to \|G\|_{H^\infty}$ as $N \to \infty$. As an immediate consequence of (6.14) we have

$$y_N^* = \frac{1}{a}\ln G_N(0) \to \frac{1}{a}\ln G(0) = y^* \quad \text{as } N \to \infty.$$

Moreover, a straightforward calculation shows that

$$(x_N^*)^{(j)} = \left(\frac{N}{N+\delta}\right)^j p_N y_N^* \quad \forall\, j \in \{1, 2, \ldots, N\},$$

where $(x_N^*)^{(j)}$ denotes the jth component of x_N^*. Consequently, for rational $z = n_z/d_z \in (0, 1]$ (n_z and d_z co-prime integers) and with $N_k := kd_z$, $k \in \mathbb{N}$, it follows, as $k \to \infty$,

$$\begin{aligned}(x_{N_k}^*)^{(zN_k)} &= \left(\frac{1}{1+\delta/N_k}\right)^{zN_k} p_{N_k} y_{N_k}^* \\ &= \left((1+\delta/N_k)^{N_k}\right)^{-z} \frac{y_{N_k}^*}{G_{N_k}(0)} \to e^{-\delta z}\frac{y^*}{G(0)} = q^*(z).\end{aligned}$$

In the following, we shall focus on cases b and c. For given N, it depends on δ which of these two cases is relevant. To make this statement more precise, assume that β is such that

$$\|\beta\|_{L^1} = \int_0^1 \beta(z)dz \in (1, e^2). \tag{6.15}$$

Then, writing $p_N = p_N(\delta)$, we have

$$1 < \int_0^1 \beta(z)dz = \lim_{N\to\infty}\left(\frac{1}{N}\sum_{j=1}^N \beta(z_j)\right) = \lim_{N\to\infty}\frac{1}{p_N(0)} < e^2.$$

Note that the function $\delta \mapsto 1/p_N(\delta)$ is strictly decreasing, converging to 0 as $\delta \to \infty$. Hence, choosing N such that $1 < 1/p_N(0) < e^2$, it follows that we move from case b to case c as δ increases from 0 to ∞.

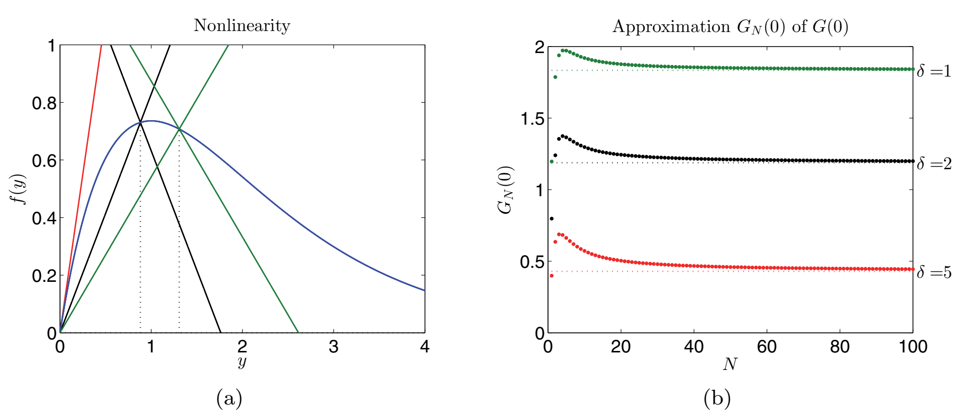


FIG. 6.2. (a) *The Ricker function f from* (4.8) *with $a = 2$ (blue) and lines with slopes $\pm p_{60}(1)$ (green), $\pm p_{60}(2)$ (black), and $p_{60}(5)$ (red).* (b) *Sequences of $G_N(0)$ from* (6.13) *for increasing N labeled with corresponding δ and limiting $G(0)$ (dotted line). (Figure in color on-line.)*

For the following numerical simulations, we assume a log-normal type birth weighting β defined by

$$\beta(0) = 0, \quad \beta(z) = \frac{6}{\sqrt{2\pi}z}e^{-\ln^2(z)/2} \quad \forall\, z \in (0, 1], \tag{6.16}$$

which satisfies (6.15) since $\|\beta\|_{L^1} = 3$. As initial population distribution, we take

$$\xi(z) = e^{8(z-1)} \quad \forall\, z \in [0, 1].$$

We choose $N = 60$, $a = 2$ (Ricker parameter), and we consider three numerical values for δ, namely 1, 2, and 5.

Figure 6.2(a) shows the graph of f and the relevant straight lines, and Figure 6.2(b) plots the sequence of approximations $G_N(0)$ and $G(0)$ for the three δ values considered. The simulations are plotted in Figures 6.3(a), 6.3(b), and 6.3(c), respectively. Panels (a) and (b) of Figure 6.3 illustrate quasi-global exponential stability of nonzero equilibria (in the sense of Theorem 4.11) and panel (c) of Figure 6.3 shows zero global exponential stability of the zero equilibria.

As for the original PDE model (6.5), the considerations made in this example seem to indicate that the nonzero equilibrium distribution q^* given by (6.8) and (6.9) is quasi-globally exponentially stable for $\delta = 1, 2$, and that the zero distribution is globally exponentially stable for $\delta = 5$.

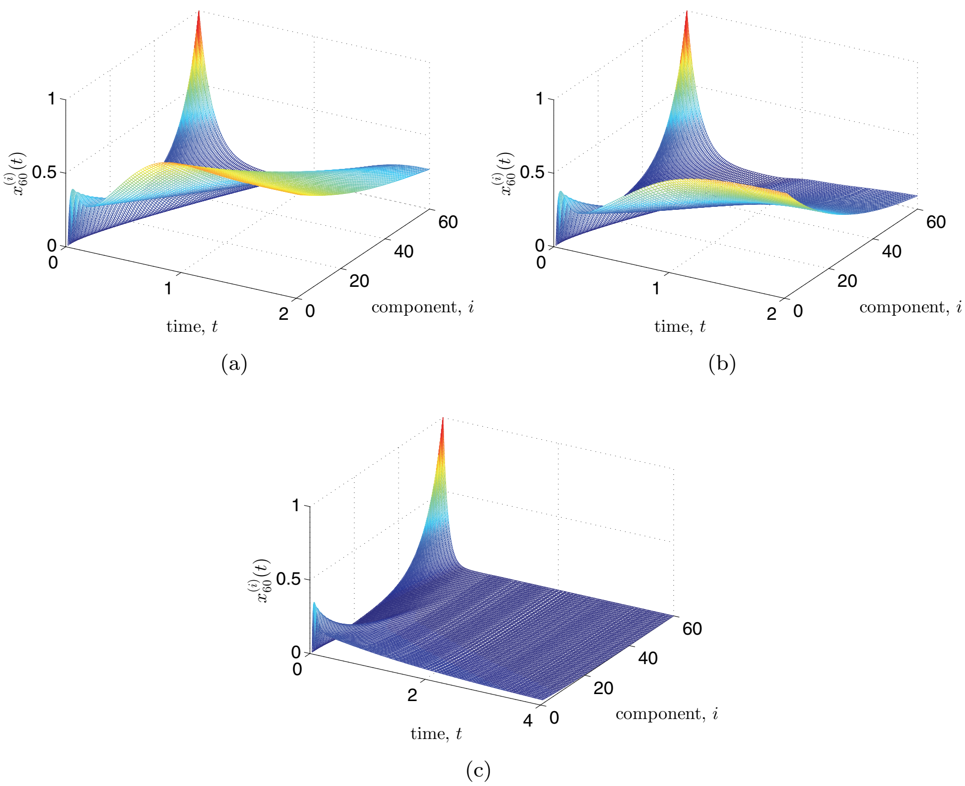


FIG. 6.3. *Numerical simulation of the discretization* (6.10) *from Example* 6.3*, corresponding to* (a) $\delta = 1$*,* (b) $\delta = 2$*, and* (c) $\delta = 5$.

Appendix A. Proofs of results. The proofs of some results were not included in the relevant sections in order to maintain the flow of the development of the key points of the work. These proofs are given below.

Proof of Proposition 5.5. Since $x^* \gg 0$ (by hypothesis) and $c^T x^* = y^*$, there exists $\varepsilon_0 > 0$ such that

$$\mathbb{B}(x^*, \varepsilon_0) \subseteq \operatorname{int} \mathbb{R}^n_+ \tag{A.1}$$

and

$$c^T z \geq \frac{y^*}{2} \quad \forall\, z \in \mathbb{B}(x^*, \varepsilon_0)\,. \tag{A.2}$$

Defining the nonlinearity $\tilde{f} : \mathbb{R} \to \mathbb{R}$ by (5.5) with $\eta = y^*/2$, there exists $\beta \in \mathcal{K}_\infty$ such that $|\tilde{f}(y)| \leq p|y| - \beta(|y|)$ for all $y \in \mathbb{R}$, and thus, in the context of the system,

$$\dot{w} = Aw + b\tilde{f}(c^T w) + d, \quad \tilde{w}(0) = \zeta, \tag{A.3}$$

the origin is ISS, as follows from Theorem 2.5. Consequently, there exist $\psi \in \mathcal{KL}$ and $\varphi \in \mathcal{K}$ such that

$$\|w(t;\zeta,d)\| \leq \psi(\|\zeta\|, t) + \varphi(\|d\|_{L^\infty}) \quad \forall\, t \geq 0,\ \forall\, \zeta \in \mathbb{R}^n_+,\ d \in L^\infty(\mathbb{R}_+, \mathbb{R}^n), \tag{A.4}$$

where $w(\cdot\,;\zeta,d)$ denotes the unique solution of (A.3). Obviously, $w(t;\zeta,d)$ is defined for all $t \geq 0$. Note that in (A.4), all disturbances $d \in L^\infty(\mathbb{R}_+, \mathbb{R}^n)$ (not necessarily nonnegative) are considered. Now choose $\varepsilon > 0$ such that

$$\psi(\|\zeta\|, 0) + \varphi(\|d\|_{L^\infty}) < \varepsilon_0 \quad \forall (\zeta, d) \in \mathbb{R}^n \times L^\infty(\mathbb{R}_+, \mathbb{R}^n) \text{ such that } \|\zeta\| + \|d\|_{L^\infty} < \varepsilon.$$

With this choice of ε, it follows from (A.4) that

$$\|w(t;\zeta,d)\| < \varepsilon_0 \ \ \forall\, t \geq 0 \text{ and } \forall\, (\zeta, d) \in \mathbb{R}^n \times L^\infty(\mathbb{R}_+, \mathbb{R}^n) \text{ such that } \|\zeta\| + \|d\|_{L^\infty} < \varepsilon. \tag{A.5}$$

Finally, let $(\xi, d) \in \mathbb{R}^n \times L^\infty(\mathbb{R}_+, \mathbb{R}^n)$ such that $\|\xi - x^*\| + \|d\|_{L^\infty} < \varepsilon$, and set $z(t) := w(t; \xi - x^*, d) + x^*$. By (A.1), (A.2), and (A.5),

$$z(t) \in \mathbb{B}(x^*, \varepsilon_0) \subseteq \operatorname{int} \mathbb{R}^n_+ \quad \forall\ t \geq 0 \tag{A.6}$$

and

$$c^T z(t) \geq \frac{y^*}{2} \quad \forall\ t \geq 0\,.$$

Consequently, by the latter,

$$c^T w(t; \xi - x^*, d) = c^T z(t) - y^* \geq -\frac{y^*}{2} \quad \forall\ t \geq 0\,,$$

from which it follows that

$$\tilde{f}(c^T w(t; \xi - x^*, d)) = f(c^T w(t; \xi - x^*, d) + y^*) - f(y^*) = f(c^T z(t)) - py^* \quad \forall\, t \geq 0\,.$$

An immediate consequence of this identity is that $\dot{z} = Az + bf(c^T z) + d$. Now $z(0) = \xi$, and thus, by uniqueness of solutions, $z(t) = x(t; \xi, d)$ for all $t \geq 0$. The claim now follows from (A.6). ∎

Proof of (6.14) *in Example* 6.3. Since β is continuous, the Riemann sum

$$\sum_{j=1}^{N} \beta(z_j) e^{-\delta z_j} (z_j - z_{j-1}), \quad \text{where } z_0 := 0,$$

converges to $\int_0^1 \beta(z)e^{-\delta z}dz$ as $N \to \infty$. Consequently,

$$G(0) = \int_0^1 \beta(z)e^{-\delta z}dz = \lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} \beta(z_j)e^{-\delta j/N}.$$

To prove that $G_N(0) \to G(0)$ as $N \to \infty$, it is sufficient to show that

$$\left| \frac{1}{N}\sum_{j=1}^{N} \beta(z_j)e^{-\delta j/N} - G_N(0) \right| \to 0 \quad \text{as } N \to \infty.$$

Setting $\|\beta\|_\infty := \sup_{z\in[0,1]} |\beta(z)|$, we estimate

$$\begin{aligned}\left| \frac{1}{N}\sum_{j=1}^{N} \beta(z_j)e^{-\delta j/N} - G_N(0) \right| &\le \frac{\|\beta\|_\infty}{N} \sum_{j=1}^{N} \left| \left(e^{-\delta/N}\right)^j - \left(\frac{1}{1+\delta/N}\right)^j \right| \\ &= \frac{\|\beta\|_\infty}{N} \left| e^{-\delta/N} - \frac{1}{1+\delta/N} \right| \sum_{j=1}^{N}\sum_{k=0}^{j-1} \left(e^{-\delta/N}\right)^{j-1-k} \left(\frac{1}{1+\delta/N}\right)^k.\end{aligned}$$

Obviously,

$$\left(e^{-\delta/N}\right)^{j-1-k} \left(\frac{1}{1+\delta/N}\right)^k \le 1 \quad \text{for all } k = 0, \dots, j-1,$$

and thus

$$\begin{aligned}\left| \frac{1}{N}\sum_{j=1}^{N} \beta(z_j)e^{-\delta j/N} - G_N(0) \right| &\le \frac{\|\beta\|_\infty}{N} \left| e^{-\delta/N} - \frac{1}{1+\delta/N} \right| \sum_{j=1}^{N} j \\ &= \|\beta\|_\infty \left(\frac{N}{N+\delta} - e^{-\delta/N} \right) \frac{N+1}{2}.\end{aligned} \tag{A.7}$$

The expression (A.7) converges to zero, which can be seen by noting that

$$\begin{aligned} N\left(\frac{N}{N+\delta} - e^{-\delta/N}\right) &= N\left(\frac{\delta^2}{N(N+\delta)} - \sum_{j=2}^{\infty} \frac{(-\delta/N)^j}{j!}\right) \\ &= \frac{\delta^2}{N+\delta} - \frac{\delta^2}{N}\sum_{j=0}^{\infty} \frac{(-\delta/N)^j}{(j+2)!} \to 0 \qquad \text{as } N \to \infty,\end{aligned}$$

completing the proof. ∎

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