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THE GENERALISED SINGULAR PERTURBATION APPROXIMATION FOR BOUNDED REAL AND POSITIVE REAL CONTROL SYSTEMS

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Abstract. The generalised singular perturbation approximation (GSPA) is considered as a model reduction scheme for bounded real and positive real linear control systems. The GSPA is a state-space approach to truncation with the defining property that the transfer function of the approximation interpolates the original transfer function at a prescribed point in the closed right half complex plane. Both familiar balanced truncation and singular perturbation approximation are known to be special cases of the GSPA, interpolating at infinity and at zero, respectively. Suitably modified, we show that the GSPA preserves classical dissipativity properties of the truncations, and existing *a priori* error bounds for these balanced truncation schemes are satisfied as well.

1. **Introduction.** Model reduction of finite-dimensional, continuous-time, linear control systems of the form

$$\left.\begin{aligned} \dot{x} &= Ax + Bu, \quad x(0) = x^0\,, \\ y &= Cx + Du\,, \end{aligned}\right\} \tag{1.1}$$

by the generalised singular perturbation approximation (GSPA) is considered. Here, as usual, u, x and y denote the input, state and output, respectively, and A, B, C and D are appropriately sized matrices. Model reduction in this context refers to approximating the input-output relationship $u \mapsto y$ in (1.1) by a simpler one, which is ideally both qualitatively and quantitatively close to the original. Model reduction is important for both simulation and controller design [39]. There are a multitude of different approaches to model reduction in the literature, see [13] and in particular [13, Fig. 2.1], including, for example, state-space methods, polynomial and rational interpolation and error minimisation methods to name but a few. The GSPA is in the spirit of the classic control theoretic model reduction scheme called (Lyapunov) balanced truncation, proposed in [31], and its close relation, the singular perturbation approximation, first considered in the context of model reduction of linear control systems in [11, 12].

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Lyapunov balanced realisations of stable systems are computed by finding a state-space similarity transform under which the solutions P and Q of the controller and observer Lyapunov equations, respectively,

$$AP + PA^* + BB^* = 0 \quad \text{and} \quad A^*Q + QA + C^*C = 0\,,$$

are equal. States in (1.1) are omitted in a reduced order model, the so-called balanced truncation, according to the relative size of the square roots of the eigenvalues of the product PQ (which are similarity invariants), which are in fact equal to the singular values of the Hankel operator associated with (1.1). Lyapunov balanced truncations retain stability and minimality of the original model — properties established in [40] — and another appealing property is the *a priori* error bound

$$\|\mathbf{G} - \mathbf{G}_r\|_{H^\infty} \leq 2 \sum_{j=r+1}^{n} \sigma_j\,, \tag{1.2}$$

between the transfer function $\mathbf{G}$ and its reduced order approximation $\mathbf{G}_r$. Here σ_j denote the distinct Hankel singular values, and the summation on the right hand side of (1.2) contains the singular values *omitted* from the reduced-order system. The error bound (1.2) was derived independently in [10] and [15]. The upper bound (1.2) is known to be achieved (that is, equality holds in (1.2)) for certain single-input single-output (SISO) systems, see [29], and a lower bound in the multi-input multi-output (MIMO) case has recently been derived in [38]. For more information on balanced truncation, the reader is referred to the survey paper [18] or the textbooks [3, 13, 17, 36]. The popularity of balanced realisations and balanced truncation has led to numerous further developments, some of which we discuss further below, as well as, for example, to infinite-dimensional systems: [8, 14, 16, 24, 35, 49].

In the frequency domain, balanced truncation for rational functions is a model reduction scheme which yields a rational approximation with the property that it interpolates the original function at infinity. Roughly, by applying the same method to a rational function now with argument $1/s$ instead of s, another reduced order rational transfer function is obtained, which now interpolates the original at zero. Interpolating at zero is a frequency domain property of the so-called singular perturbation approximation (SPA), in particular meaning that the steady-state gains are equal. From a dynamical systems perspective, singular perturbation approximation decomposes the state variables into those with "fast" and "slow" dynamics, and assumes that the "fast" variables are at equilibrium, meaning that differential equations simplify to algebraic equations. For linear systems these algebraic equations are easily solvable, which leads to a model with fewer differential equations, and hence fewer states. The mapping s to $1/s$ mentioned above is called the reciprocal transformation and provides a relationship between SPA and balanced truncation. This relationship was exploited in [30] to show that the singular perturbation approximation of a balanced, minimal, linear system admits the same H^∞ error bound (1.2), as well as retaining minimality and stability of the original. To the best of our knowledge, the provenance of the reciprocal transformation in systems and control theory is unclear, and it now forms part of the subjects' "folklore". It appears in numerous areas, for instance, when working with the technical difficulties which arise in infinite dimensional systems see, for example, [44, Section 12.4] and [6, 7].

The generalised singular perturbation approximation (GSPA) is a generalisation of both balanced truncation and singular perturbation approximation as it is a state-space truncation scheme with the property that the approximate transfer function interpolates the original at a prescribed point in the closed right half complex plane. The GSPA was proposed in a control theoretic context in [11], and was the subject of a number of papers around that time, see [1, 30, 27, 32]. Both balanced truncation and the SPA are special cases of the GSPA.

Here we demonstrate that when suitably adapted, the GSPA provides a dissipativity preserving model reduction scheme with error bounds and the additional interpolation property. To motivate our study we note that a disadvantage of balanced truncation or SPA is that any dissipativity property of the original system need not be retained in the truncation. Dissipativity (or passivity) theory as commonly used in systems and control theory dates back to the seminal work of [47, 48], where the notions of supply rate and storage function were introduced and which capture (and generalise) the notion of a system storing and dissipating energy over time. Dissipative systems are central to control, in part owing to a plethora of natural and important examples such as RLC circuits and mass-spring-damper systems. Much attention has been devoted to the situation when the supply rate is *quadratic*, as multiple notions of energy are quadratic in state variables, such as kinetic energy. Two classical notions of quadratic dissipative systems which first arose in circuit theory go by the names of impedance passive and scattering passive, also known as passive and contractive, or bounded real and positive real, respectively, the latter term being introduced in [4]. Two famous results, sometimes called the Bounded Real Lemma and Positive Real Lemma, provide a complete state-space characterisation of these two notions of dissipativity, respectively, see, for example, [2]. The latter is also known as the Kalman-Yakubovich-Popov (KYP) Lemma in recognition of its original contributors. We refer the reader to [25] or [41] and the references therein for more background on the KYP Lemma.

In response, balanced truncation has been extended to bounded real and positive real systems in [9] and [37], respectively, and to the infinite-dimensional case in [22]. Here the truncations *do* retain the respective dissipativity property and error bounds have also been established see, for example, [18] and [23]. We note that there is a false bound in [5], see [21]. By using the reciprocal transformation, it was shown in [33] that when the SPA is defined in terms of a dissipative balanced realisation, then the reduced order system inherits dissipativity from the original system, and satisfies corresponding error bounds. There have been other variations in dissipativity preserving model reduction schemes, including to descriptor systems [42], and certain classes of finite-dimensional behavioral systems [23]. To summarise, the bounded real and positive real GSPA generalises the results of [9], [37] and [33] and provides a truncation scheme which retains the relevant dissipativity property, error bounds, and interpolation at a prescribed point.

The manuscript is organised as follows. After recording notation and terminology, Section 2 recalls model reduction by generalised singular perturbation approximation. Our main results are contained in Sections 3 and 4, namely, bounded real and positive real preserving generalised singular perturbation approximation. Examples are contained in Section 5. In an attempt to streamline the presentation, the proofs of our main results appear in Section 6 and the Appendix.

Notation: Most mathematical notation we use is standard, or defined when introduced. The set of positive integers is denoted by $\mathbb{N}$, whilst $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively. For $k \in \mathbb{N}$, $\underline{k} := \{1, 2, \ldots, k\}$ and for $\xi \in \mathbb{C}$, $\mathrm{Re}\,(\xi)$, $\mathrm{Im}\,(\xi)$, $\overline{\xi}$ and $|\xi|$ denote its real part, imaginary part, complex conjugate and modulus, respectively. We let $\mathbb{C}_0$ denote the set of all complex numbers with positive real part. For $n \in \mathbb{N}$, $\mathbb{R}^n$ and $\mathbb{C}^n$ denote the familiar real and complex n-dimensional Hilbert spaces, respectively, both equipped with the inner product $\langle \cdot, \cdot \rangle$ which induces the usual 2-norm $\| \cdot \|_2$. For $m \in \mathbb{N}$, let $\mathbb{R}^{n\times m}$ and $\mathbb{C}^{n\times m}$ denote the normed linear spaces of $n \times m$ matrices with real and complex entries, respectively, both equipped with the operator norm, also denoted $\| \cdot \|_2$, induced by the $\| \cdot \|_2$ norm on $\mathbb{R}^n$ or $\mathbb{C}^n$. The superscript $*$ denotes the complex-conjugate transpose (and, importantly, the adjoint with respect to the above inner product).

For $M, N \in \mathbb{C}^{n\times n}$, $\sigma(M)$ denotes the spectrum of M and we write $M \geq N$ or $N \leq M$ if $M - N$ is positive semi-definite, and $M > N$ or $N < M$ if the difference $M - N$ is positive definite. It is well-known that, as $\mathbb{C}^n$ is a complex Hilbert space, if $M \succeq 0$, then $M = M^*$.

For $m, p \in \mathbb{N}$, the space of analytic functions $\mathbb{C}_0 \to \mathbb{C}^{p\times m}$ is denoted by $H(\mathbb{C}_0, \mathbb{C}^{p\times m})$. The subset of functions which are additionally bounded with respect to the norm

$$\|\mathbf{G}\|_{H^\infty} = \sup_{s\in\mathbb{C}_0} \|\mathbf{G}(s)\|_2 \,,$$

is denoted by $H^\infty(\mathbb{C}_0, \mathbb{C}^{p\times m})$.

2. The generalised singular perturbation approximation.

We gather elementary and notational preliminaries before recalling the generalised singular perturbation approximation and describing some properties.

We consider the linear control system (1.1) where, as usual, u, x and y denote the input, state and output and

$$(A, B, C, D) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times m} \times \mathbb{C}^{p\times n} \times \mathbb{C}^{p\times m} \,,$$

for some $m, n, p \in \mathbb{N}$[1]. In practice, the quadruple (A, B, C, D) is real-valued and in many situations, the matrix D does not play a role. As such, we use the triple (A, B, C) when the choice of D, which need not be zero, is unimportant.

The triple (A, B, C) is said to be stable if A is Hurwitz, that is, every eigenvalue of A has negative real part. The dimension of the triple (A, B, C) is equal to the dimension of its A term, and the triple is minimal if the pair (A, B) is controllable and the pair (C, A) is observable, see [43, Theorem 27, p.286].

Naturally, associated to the quadruple (A, B, C, D) is the linear system (1.1). The transfer function of the linear system (1.1) or quadruple (A, B, C, D) is the rational function

$$s \mapsto \mathbf{G}(s) := D + C(sI - A)^{-1}B \,, \tag{2.1}$$

which is certainly defined for all complex s with $\mathrm{Re}\, s > \alpha(A)$, the spectral abscissa of A. Conversely, given a proper rational function $\mathbf{G}$ defined on a right-half complex plane, a quadruple (A, B, C, D) is called a realisation of $\mathbf{G}$ if (2.1) holds on that half-plane. Realisations are never unique. The McMillan degree of a proper rational

[1]The material which follows holds if we assume that $A : \mathcal{X} \to \mathcal{X}$, $B : \mathcal{U} \to \mathcal{X}$, $C : \mathcal{X} \to \mathcal{Y}$ and $D : \mathcal{U} \to \mathcal{Y}$ are bounded linear operators between finite-dimensional complex Hilbert spaces $\mathcal{U}$, $\mathcal{X}$ and $\mathcal{Y}$ which, of course, is equivalent to our formulation once bases are chosen for $\mathcal{U}$, $\mathcal{X}$ and $\mathcal{Y}$.

transfer function is the dimension of a minimal state-space realisation, see [43, Remark 6.7.4, p.299].

Recall that the stable triple (A, B, C) is called (internally or Lyapunov) balanced if there exists a Σ such that

$$A\Sigma + \Sigma A^* + BB^* = 0 \quad \text{and} \quad A^*\Sigma + \Sigma A + C^*C = 0\,. \tag{2.2}$$

If Σ satisfies (2.2), then necessarily Σ equals both the controllability and observability Gramians of the linear system specified by (A, B, C), that is,

$$\Sigma = \int_{\mathbb{R}_+} \mathrm{e}^{At} BB^* \mathrm{e}^{A^*t}\, dt = \int_{\mathbb{R}_+} \mathrm{e}^{A^*t} C^*C \mathrm{e}^{At}\, dt\,,$$

(hence the terminology balanced) and is consequently self-adjoint and positive semi-definite. It is well-known that it is always possible to construct a balanced realisation from a given one via a state-space similarity transformation [3, Lemma 7.3, p.210]. The triple (A, B, C) is minimal if, and only if, Σ is positive definite. The eigenvalues of Σ are precisely the singular values of the Hankel operator corresponding to the triple (A, B, C). We shall let $(\sigma_j)_{j=1}^n$ denote the n *distinct* Hankel singular values of (A, B, C), which we shall assume throughout the paper are simple (that is, each has algebraic and geometric multiplicity equal to one). As singular values, the σ_j are ordered so that

$$\sigma_1 > \sigma_2 > \cdots \geq 0\,. \tag{2.3}$$

In practical applications, a basis of the state-space is chosen so that Σ is a diagonal matrix, with the terms σ_j on the diagonal.

Singular perturbation approximations are defined in terms of conformal partitions of (A, B, C), denoted by

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad C = \begin{pmatrix} C_1 & C_2 \end{pmatrix}\,, \tag{2.4}$$

where $A_{11} \in \mathbb{R}^{r\times r}$, $B_1 \in \mathbb{R}^{r\times m}$, $C_1 \in \mathbb{R}^{p\times r}$ and so on, for some $r \in \underline{n-1}$. Of course, the partitions in (2.4) depend on both the realisation and r, which are degrees of freedom.

Definition 2.1. Given the quadruple (A, B, C, D), partitioned according to (2.4) for some $r \in \underline{n-1}$ and $\xi \in \mathbb{C}$, $\mathrm{Re}(\xi) \geq 0$ assume that $\xi \notin \sigma(A_{22})$. The quadruple $(A_\xi, B_\xi, C_\xi, D_\xi)$ given by

$$\left.\begin{aligned} A_\xi &:= A_{11} + A_{12}(\xi I - A_{22})^{-1}A_{21}\,, \quad & B_\xi &:= B_1 + A_{12}(\xi I - A_{22})^{-1}B_2\,, \\ C_\xi &:= C_1 + C_2(\xi I - A_{22})^{-1}A_{21}\,, \quad & D_\xi &:= D + C_2(\xi I - A_{22})^{-1}B_2\,, \end{aligned}\right\} \tag{2.5}$$

is called the generalised singular perturbation approximation of (1.1).

Remark 2.2. Throughout this remark, we assume that $\xi \in \mathbb{C}$, $\mathrm{Re}(\xi) \geq 0$.

(a) The generalised singular perturbation approximation may be defined for any realisation (A, B, C) and choice of partition in (2.4). In this section we shall assume that (A, B, C) is stable and balanced and a partition in (2.4) is chosen with respect to two unions of eigenspaces of Σ corresponding to *distinct* eigenvalues. With respect to such a partition, Σ has the block form

$$\Sigma := \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix}, \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad C = \begin{pmatrix} C_1 & C_2 \end{pmatrix}\,. \tag{2.6}$$

In light of the ordering (2.3), Σ_1 and Σ_2 contain the larger and smaller eigenvalues of Σ, respectively.

(b) Given the stable, minimal, balanced quadruple (A, B, C, D) with transfer function $\mathbf{G}$, let $\mathbf{G}_r^\xi$ denote the transfer function of the generalised singular perturbation approximation. The motivation and defining property of the generalised singular perturbation approximation is that

$$\mathbf{G}_r^\xi(\xi) = \mathbf{G}(\xi)\,, \tag{2.7}$$

that is, the transfer function interpolates the original at ξ, see [30, Lemma 2.4]. Of course, a downside of the GSPA for practical applications is that $(A_\xi, B_\xi, C_\xi, D_\xi)$ will in general be complex when $\operatorname{Im}\xi \neq 0$, even if (A, B, C, D) is real.

(c) If the realisation (A, B, C) is stable, minimal, balanced and decomposed as in (2.6), then by [40, Theorem 3.2] both A_{11} and A_{22} are Hurwitz. Consequently, the generalised singular perturbation approximation is well-defined. Furthermore, in the limit $\xi \to \infty$, we obtain from (2.5)

$$A_\infty := A_{11}\,, \quad B_\infty := B_1\,, \quad C_\infty := C_1\,, \quad D_\infty := D\,,$$

and the linear system specified by the quadruple $(A_\infty, B_\infty, C_\infty, D_\infty)$ is called the balanced truncation of (1.1). The case $\xi = 0$ in (2.5) leads to

$$\begin{aligned} A_0 &:= A_{11} - A_{12}A_{22}^{-1}A_{21}\,, \quad & B_0 &:= B_1 - A_{12}A_{22}^{-1}B_2\,, \\ C_0 &:= C_1 - C_2A_{22}^{-1}A_{21}\,, \quad & D_0 &:= D - C_2A_{22}^{-1}B_2\,, \end{aligned}$$

and the linear system specified by the quadruple (A_0, B_0, C_0, D_0) is called the singular perturbation approximation of (1.1). We see that the balanced truncation and singular perturbation approximation are special cases of the generalised singular perturbation approximation, hence the terminology. In state-space terms, the GSPA assumes that x in (1.1) is partitioned into x_1 and x_2 and

$$\dot{x}_2(t) = \xi x_2(t). \tag{2.8}$$

By substituting (2.8) into (1.1) and eliminating x_2, the linear system specified by $(A_\xi, B_\xi, C_\xi, D_\xi)$ is obtained (with state x_1). The assumption (2.8) highlights the input-output motivation of the GSPA, at least for stable systems. Indeed, $\dot{x}_2(t) = \xi x_2(t)$ and $\operatorname{Re}(\xi) \geq 0$ implies that $\|x_2(t)\|$ does not decrease as $t \to \infty$. Under the assumption that A is Hurwitz, we would of course expect $\|x_2(t)\| \to 0$ as $t \to \infty$ in the absence of control, that is, when $u = 0$. ♢

We recall two results which shall play a key role in constructing the dissipativity preserving GSPA in Sections 3 and 4.

Theorem 2.3. *Given $\xi \in \mathbb{C}$ with $\operatorname{Re}(\xi) \geq 0$ and stable, minimal, balanced quadruple (A, B, C, D), assume that the Hankel singular values are simple. Then $(A_\xi, B_\xi, C_\xi, D_\xi)$, the generalised singular perturbation approximation of order $r \in \underline{n-1}$, is well-defined and the following statements hold.*

(i) *A_ξ is Hurwitz and (A_ξ, B_ξ, C_ξ) is minimal.*
(ii) *If $\xi \in i\mathbb{R}$, then (A_ξ, B_ξ, C_ξ) is balanced.*

Statement (i) of Theorem 2.3 appears in the special case that $\xi \in \mathbb{R}$, $\xi > 0$ in [27, Theorem 5.4], but does not appear in [30]. It is claimed in [27, Remark 5.5] that statement (i) extends to all $\xi \in \mathbb{C}_0$, but no proof is given there. For completeness, we have provided a proof in the Appendix. Statement (ii) is novel.

Theorem 2.4. *Let* $\mathbf{G} \in H^\infty(\mathbb{C}_0, \mathbb{C}^{p\times m})$ *be rational with simple Hankel singular vales* $(\sigma_j)_{j=1}^n$*, ordered as in* (2.3)*, let* $r \in \underline{n-1}$ *and* $\xi \in \mathbb{C}$ *with* $\mathrm{Re}(\xi) \geq 0$*. Then there exists a rational* $\mathbf{G}_r^\xi \in H^\infty(\mathbb{C}_0, \mathbb{C}^{p\times m})$ *of McMillan degree* r *such that the interpolation property* (2.7) *holds and*

$$\|\mathbf{G} - \mathbf{G}_r^\xi\|_{H^\infty} \leq 2 \sum_{j=r+1}^{n} \sigma_j \,. \tag{2.9}$$

The proof of Theorem 2.4 is constructive — a transfer function $\mathbf{G}_r^\xi$ which satisfies (2.7) and (2.9) is realised by the generalised singular perturbation approximation of a stable, minimal, balanced realisation of $\mathbf{G}$. The error bound (2.9) has been established when $\xi = 0$ or $\xi = \infty$ as these correspond to the singular perturbation approximation and balanced truncation, respectively, as well as when $\xi \in i\mathbb{R}$ (see [30, Theorem 3.4]) and when $\xi \in \mathbb{R}$, $\xi > 0$ (see [27, Theorem 5.4]). Again, it is claimed in [27, Remark 5.5] that the error bound (2.9) holds for all $\mathrm{Re}(\xi) \geq 0$, but no proof is given. Again for completeness, a proof is provided in the Appendix.

3. **Bounded real generalised singular perturbation approximation.**

In this section we define the bounded real GSPA of a quadruple with bounded real transfer function, and show that it gives rise to a bounded real reduced order system, with properties including the point interpolation (2.7) and error bounds. Recall that $\mathbf{G} \in H^\infty(\mathbb{C}_0, \mathbb{C}^{p\times m})$ is said to be bounded real if $\|\mathbf{G}\|_{H^\infty} \leq 1$, and strictly bounded real if $\|\mathbf{G}\|_{H^\infty} < 1$. Bounded realness is the frequency domain name of the property called scattering passive or contractive in the time-domain. From many possible references the reader is referred to, for example, [45, 46]. The term 'real' in bounded real refers to the sometimes-made assumption that $\mathbf{G}$ is real on the real axis. It is true that many physically motivated systems enjoy such a property, but we do not enforce it because there is no mathematical need to. Although we acknowledge that the terminology 'bounded' or 'contractive' would suffice, in keeping with existing literature we persevere with the term 'bounded real'.

Bounded real balanced truncation, proposed in [37], and bounded real singular perturbation approximation, proposed in [33], are morally similar to the (Lyapunov) balanced versions. However, instead of balancing the solutions of two Lyapunov equations, for the bounded real model reduction schemes certain solutions of the so-called primal and dual Bounded Real Lur'e (or Algebraic Riccati) equations are balanced. The existence of these solutions is ensured by the Bounded Real Lemma. There are numerous treatments of bounded real balanced truncation in the literature, examples in addition to [37] and [33] include [3, 18, 19, 22, 23]. For brevity, here we describe only the aspects required to define the bounded real GSPA.

For which purpose, recall that if the stable, minimal quadruple (A, B, C, D) is bounded real, then there exist P_m and P_M, positive definite solutions of the Bounded Real Lur'e equations

$$\left.\begin{aligned} A^*Z + ZA + C^*C &= -K^*K\,, \\ ZB + C^*D &= -K^*W\,, \\ I - D^*D &= W^*W\,, \end{aligned}\right\} \tag{3.1}$$

(with variable Z), for some $K \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{m\times m}$, which are extremal in the sense that any other positive semi-definite solution P of (3.1) satisfies $P_m \leq P \leq$

P_M. It is straightforward to show that P_M^{-1} is also equal to the minimal solution (in the previous sense) of the dual Bounded Real Lur'e equations

$$\left.\begin{aligned} AZ + ZA^* + BB^* &= -LL^*\,, \\ ZC^* + BD^* &= -LX^*\,, \\ I - DD^* &= XX^*\,, \end{aligned}\right\} \tag{3.2}$$

(also with variable Z) for some $L \in \mathbb{C}^{n\times p}$ and $X \in \mathbb{C}^{p\times p}$. We say that the realisation (A, B, C, D) is bounded real balanced if

$$P_m = P_M^{-1} =: \Sigma\,.$$

In particular, when (A, B, C, D) is bounded real balanced, then Σ is a solution of both (3.1) and (3.2). The bounded real singular values, denoted $(\sigma_k)_{k=1}^n$, are the nonnegative square roots of the eigenvalues of $P_m P_M^{-1}$, and so the eigenvalues of Σ in a bounded real balanced realisation. We note that they are called characteristic values by some authors, such as in [42]; see [23, Remark 3.6].

Definition 3.1. The bounded real generalised singular perturbation of stable, minimal quadruple (A, B, C, D), for $\xi \in \mathbb{C}$ with $\mathrm{Re}(\xi) \geq 0$, is given by (2.5) when (A, B, C, D) is bounded real balanced, provided that it is well-defined.

Our two main results of this section are stated and proven next. They parallel the results in Section 2: the first contains state-space properties of the bounded real GSPA and the second contains a frequency domain error bound.

Theorem 3.2. *Given $\xi \in \mathbb{C}$ with $\mathrm{Re}(\xi) \geq 0$ and stable, minimal, and bounded real balanced quadruple (A, B, C, D), assume that the bounded real singular values are simple. Then $(A_\xi, B_\xi, C_\xi, D_\xi)$, the bounded real generalised singular perturbation approximation of order $r \in \underline{n-1}$, is well-defined and the following statements hold.*

(i) *$(A_\xi, B_\xi, C_\xi, D_\xi)$ is bounded real, and is bounded real balanced if $\xi \in i\mathbb{R}$.*

(ii) *A_ξ is Hurwitz.*

(iii) *If (A, B, C, D) is strictly bounded real, then $(A_\xi, B_\xi, C_\xi, D_\xi)$ is minimal and strictly bounded real.*

Special cases of the above theorem appear in [37, Theorem 2] and [33, Theorem 2 (a)], corresponding to the cases $\xi = \infty$ (the bounded real balanced truncation) and $\xi = 0$ (the bounded real singular perturbation approximation), respectively. Even in these special cases, the claim in statement (iii) above that strict bounded realness is preserved in the respective truncations does not appear in [37] or [33].

Theorem 3.3. *Let $\mathbf{G} \in H^\infty(\mathbb{C}_0, \mathbb{C}^{p\times m})$ be rational and bounded real with simple bounded real singular vales $(\sigma_j)_{j=1}^n$, ordered as in (2.3), let $r \in \underline{n-1}$ and $\xi \in \mathbb{C}$ with $\mathrm{Re}(\xi) \geq 0$. Then there exists a rational, bounded real $\mathbf{G}_r^\xi \in H^\infty(\mathbb{C}_0, \mathbb{C}^{p\times m})$ which has a state-space realisation of dimension r, such that (2.7) holds and*

$$\|\mathbf{G} - \mathbf{G}_r^\xi\|_{H^\infty} \leq 2 \sum_{j=r+1}^{n} \sigma_j\,. \tag{3.3}$$

If $\|\mathbf{G}\|_{H^\infty} < 1$, then $\mathbf{G}_r^\xi$ may be chosen with the above properties and, additionally, to have McMillan degree r and $\|\mathbf{G}_r^\xi\|_{H^\infty} < 1$.

The next result pertains to existence and approximation of so-called spectral factors, and spectral "sub"-factors, particularly of reduced order transfer functions obtained by bounded real GSPA. Here $\mathbf{H}^*$ denotes $s \mapsto (\mathbf{H}(s))^*$ for matrix-valued rational functions $\mathbf{H}$ of a complex variable.

Proposition 3.4. *Imposing the notation and assumptions of Theorem 3.3, the following statements hold.*

(i) *There exist rational* $\mathbf{R} \in H^\infty(\mathbb{C}_0, \mathbb{C}^{m\times m})$, $\mathbf{S} \in H^\infty(\mathbb{C}_0, \mathbb{C}^{p\times p})$ *such that*

$$I - \mathbf{G}^*\mathbf{G} = \mathbf{R}^*\mathbf{R} \quad \text{and} \quad I - \mathbf{G}\mathbf{G}^* = \mathbf{S}\mathbf{S}^* \quad \text{on } i\mathbb{R}.$$

(ii) *If* $\xi \in i\mathbb{R}$, *then there exist rational* $\mathbf{R}_r^\xi \in H^\infty(\mathbb{C}_0, \mathbb{C}^{m\times m})$, $\mathbf{S}_r^\xi \in H^\infty(\mathbb{C}_0, \mathbb{C}^{p\times p})$ *such that*

$$I - (\mathbf{G}_r^\xi)^*\mathbf{G}_r^\xi = (\mathbf{R}_r^\xi)^*\mathbf{R}_r^\xi \quad \text{and} \quad I - \mathbf{G}_r^\xi(\mathbf{G}_r^\xi)^* = \mathbf{S}_r^\xi(\mathbf{S}_r^\xi)^* \quad \text{on } i\mathbb{R}, \tag{3.4}$$

and

$$\max\left\{ \left\| \begin{pmatrix} \mathbf{G} - \mathbf{G}_r^\xi \\ \mathbf{R} - \mathbf{R}_r^\xi \end{pmatrix} \right\|_{H^\infty}, \left\| \begin{pmatrix} \mathbf{G} - \mathbf{G}_r^\xi & \mathbf{S} - \mathbf{S}_r^\xi \end{pmatrix} \right\|_{H^\infty} \right\} \leq 2 \sum_{j=r+1}^{n} \sigma_j \,, \tag{3.5}$$

so that in particular

$$\|\mathbf{R} - \mathbf{R}_r^\xi\|_{H^\infty}, \|\mathbf{S} - \mathbf{S}_r^\xi\|_{H^\infty} \leq 2 \sum_{j=r+1}^{n} \sigma_j \,. \tag{3.6}$$

The spectral factors $\mathbf{R}_r^\xi$ *and* $\mathbf{S}_r^\xi$ *have state-space realisations with the same dimension as those for* $\mathbf{G}_r^\xi$ *and may be chosen with the interpolation property*

$$\mathbf{R}(\xi) = \mathbf{R}_r^\xi(\xi) \quad \text{and} \quad \mathbf{S}(\xi) = \mathbf{S}_r^\xi(\xi)\,. \tag{3.7}$$

(iii) *If* $\xi \in \mathbb{C}_0$, *then there exist rational* $\mathbf{R}_r^\xi \in H^\infty(\mathbb{C}_0, \mathbb{C}^{m\times m})$, $\mathbf{S}_r^\xi \in H^\infty(\mathbb{C}_0, \mathbb{C}^{p\times p})$, *such that properties* (3.5)–(3.7) *from statement* (ii) *hold, and* (3.4) *is replaced by*

$$I - (\mathbf{G}_r^\xi)^*\mathbf{G}_r^\xi \geq (\mathbf{R}_r^\xi)^*\mathbf{R}_r^\xi \quad \text{and} \quad I - \mathbf{G}_r^\xi(\mathbf{G}_r^\xi)^* \geq \mathbf{S}_r^\xi(\mathbf{S}_r^\xi)^* \quad \text{on } i\mathbb{R}. \tag{3.8}$$

4. Positive real generalised SPA.

In this section we define the positive real GSPA of a quadruple with positive real transfer function, and show that it gives rise to a positive real reduced order system, with properties including the point interpolation (2.7) and error bounds. Recall that positive realness is a property of "square" systems, meaning the input and output spaces have the same dimension, $m = p$, and that a rational, $\mathbb{C}^{m\times m}$-valued function $\mathbf{G}$ is said to be positive real if

$$\operatorname{Re}\mathbf{G}(s) = \mathbf{G}(s) + [\mathbf{G}(s)]^* \geq 0, \quad \forall\, s \in \mathbb{C}_0 \setminus \Delta\,, \tag{4.1}$$

where Δ is the set of poles of $\mathbf{G}$. The assumption that $\mathbf{G}$ is rational implies that $\mathbf{G}$ is analytic on $\mathbb{C}_0 \setminus \Delta$, and it is well-known (see [20, Proposition 3.3]) that analyticity and the positive realness condition (4.1) together imply that $\mathbf{G}$ in fact has no poles in $\mathbb{C}_0$, and hence $\mathbf{G} \in H(\mathbb{C}_0, \mathbb{C}^{m\times m})$. Rational positive real functions may have simple imaginary axis poles, such as $s \mapsto 1/s$, and need not be proper, such as $s \mapsto s$.

Positive realness is the frequency domain term for systems which are called impedance passive, or sometimes just passive, in the time domain. For scalar, rational functions, the terms positive and positive real were introduced in [4], with the former used for functions which satisfy (4.1), and the latter for functions which satisfy (4.1) and are also real on the real axis. As with bounded realness, although many physically motivated transfer functions are real on the real axis, we do not impose this assumption simply because it is not required. However, we adopt the convention of calling such functions positive real, which agrees with much existing literature and as it captures that the real part of the function under consideration is positive (non-negative, to be precise). Positive realness and bounded realness are related via the mapping which goes by the name of the diagonal transformation, (external) Cayley transform or Möbius transform, see [19, Ch. 7], [34, Ch. 5] or [46], which we exploit in the present section to make use of the material established previously.

Positive real balanced truncation, proposed in [9] and further developed in [26], and positive real singular perturbation approximation, proposed in [33], are defined in the same spirit as their bounded real counterparts, where now extremal solutions of the primal and dual Positive Real Lur'e equations (or Riccati equations) are balanced. The theoretical result underpinning the process is the Positive Real Lemma. We note the potential confusion between the original nomenclature 'balanced stochastic truncation' and the more recent 'positive real balanced truncation', see [21, Remark 1]. As with the bounded real case, there are a myriad of references to these model reduction approaches for positive real systems, including those cited above and [3, 18, 19, 23, 22]. For brevity, here we describe only the key aspects which we shall require to define the positive real GSPA and establish its properties.

To that end, recall that if the stable, minimal quadruple (A, B, C, D) is positive real, then there exist P_m and P_M, positive definite solutions of the Positive Real Lur'e equations

$$\left.\begin{aligned} A^*Z + ZA &= -K^*K\,, \\ ZB - C^* &= -K^*W\,, \\ D + D^* &= W^*W\,, \end{aligned}\right\} \tag{4.2}$$

(with variable Z), for some $K \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{m\times m}$, which are extremal in the sense that any other positive semi-definite solution P of (4.2) satisfies $P_m \leq P \leq P_M$. It is straightforward to show that P_M^{-1} is also equal to the minimal solution (in the previous sense) of the dual Positive Real Lur'e equations

$$\left.\begin{aligned} AZ + ZA^* &= -LL^*\,, \\ ZC^* - B^* &= -LX^*\,, \\ D^* + D &= XX^*\,, \end{aligned}\right\} \tag{4.3}$$

(also with variable Z) for some $L \in \mathbb{C}^{n\times m}$ and $X \in \mathbb{C}^{m\times m}$. We say that (A, B, C, D) is positive real balanced if

$$P_m = P_M^{-1} = \Sigma\,.$$

In particular, when (A, B, C, D) is positive real balanced, then Σ is a solution of both (4.2) and (4.3). The positive real singular values, denoted $(\sigma_k)_{k=1}^n$, are the nonnegative square roots of the eigenvalues of $P_m P_M^{-1}$, although like bounded real singular values, they are called characteristic values by some authors, see [42].

Definition 4.1. The positive real generalised singular perturbation of a stable, minimal quadruple (A, B, C, D), for $\xi \in \mathbb{C}$ with $\mathrm{Re}(\xi) \geq 0$, is given by (2.5) when (A, B, C, D) is positive real balanced, provided that it is well-defined.

Our two main results of this section are stated and proven next. They parallel the results in Section 3: the first contains state-space properties of the positive real GSPA and the second contains frequency domain properties and error bounds. Adopting the nomenclature convention used in [20], we say that the rational, $\mathbb{C}^{m\times m}$-valued function $\mathbf{G}$ is strongly positive real if

$$\operatorname{Re}\mathbf{G}(s) = \mathbf{G}(s) + [\mathbf{G}(s)]^* \geq \delta I, \quad \forall\, s \in \mathbb{C}_0 \setminus \Delta\,,$$

for some $\delta > 0$, and where Δ denotes the set of poles of $\mathbf{G}$. Strongly positive real functions are clearly positive real.

Theorem 4.2. *Given $\xi \in \mathbb{C}$ with $\operatorname{Re}(\xi) \geq 0$ and stable, minimal, and positive real balanced quadruple (A, B, C, D), assume that the positive real singular values are simple. Then $(A_\xi, B_\xi, C_\xi, D_\xi)$, the positive real generalised singular perturbation approximation of order $r \in \underline{n-1}$, is well-defined and the following statements hold.*

(i) *$(A_\xi, B_\xi, C_\xi, D_\xi)$ is positive real, and is positive real balanced if $\xi \in i\mathbb{R}$.*
(ii) *A_ξ is Hurwitz.*
(iii) *If (A, B, C, D) is strongly positive real, then $(A_\xi, B_\xi, C_\xi, D_\xi)$ is minimal and strongly positive real.*

Theorem 4.3. *Let $\mathbf{G} \in H(\mathbb{C}_0, \mathbb{C}^{m\times m})$ be proper, rational, and positive real with simple positive real singular vales $(\sigma_j)_{j=1}^n$, ordered as in (2.3), let $r \in \underline{n-1}$ and $\xi \in \mathbb{C}$ with $\operatorname{Re}(\xi) \geq 0$ which is not a pole of $\mathbf{G}$. Then there exists proper, rational, and positive real $\mathbf{G}_r^\xi \in H(\mathbb{C}_0, \mathbb{C}^{m\times m})$ which has a state-space realisation of dimension r, such that (2.7) holds and*

$$\hat{\delta}(\mathbf{G}, \mathbf{G}_r^\xi) \leq 2 \sum_{j=r+1}^{n} \sigma_j\,, \tag{4.4}$$

where $\hat{\delta}$ denotes the gap metric [28, p.197, p.201]. If $\mathbf{G} \in H^\infty(\mathbb{C}_0, \mathbb{C}^{m\times m})$, then $\mathbf{G}_r^\xi$ with the previous properties may be chosen to be in $H^\infty(\mathbb{C}_0, \mathbb{C}^{m\times m})$ as well, and

$$\begin{aligned}\|\mathbf{G} - \mathbf{G}_r^\xi\|_{H^\infty} \leq 2\min\big\{&(1 + \|\mathbf{G}\|_{H^\infty}^2)(1 + \|\mathbf{G}_r^\xi\|_{H^\infty}),\\ &(1 + \|\mathbf{G}\|_{H^\infty})(1 + \|\mathbf{G}_r^\xi\|_{H^\infty}^2)\big\} \sum_{j=r+1}^{n} \sigma_j\,,\end{aligned} \tag{4.5}$$

holds. Finally, if $\mathbf{G}$ is strongly positive real, then $\mathbf{G}_r^\xi$ as above may be chosen to have McMillan degree r and be strongly positive real as well.

In certain cases, the error bound (4.5) may be used to derive a more conservative (that is, worse), but *a priori*, bound. The reader is referred to [19, Remark 3.6.11] for more details.

Our final result pertains to existence of so-called spectral factors, now in the positive real case, and is the positive real analogue of Proposition 3.4. Although our approach is to use the Cayley transform and Proposition 3.4, 'natural' error bounds in the gap metric for the distance between spectral factors and their approximations in the positive real case sadly do not seemingly follow from those in the bounded real case. For completeness, we do provide an H^∞ error bound in the special case that $\mathbf{G} \in H^\infty$ which, in keeping with the GSPA, does depend linearly on the sum of omitted singular values. The constant which appears in the bound may be somewhat conservative, however.

Proposition 4.4. *Imposing the notation and assumptions of Theorem 4.3, let Δ denote the set of poles of $\mathbf{G}$ on $i\mathbb{R}$. The following statements hold.*

(i) *There exists a proper, rational, $\mathbb{C}^{m\times m}$-valued function* $\mathbf{R}$ *such that*

$$\mathbf{G}+\mathbf{G}^* = \mathbf{R}^*\mathbf{R} \quad \text{on } i\mathbb{R}\backslash\Delta.$$

(ii) *If $\xi \in i\mathbb{R}$, then there exists a proper, rational $\mathbb{C}^{m\times m}$-valued function $\mathbf{R}_r^\xi$ such that*

$$\mathbf{G}_r^\xi + (\mathbf{G}_r^\xi)^* = (\mathbf{R}_r^\xi)^*\mathbf{R}_r^\xi \quad \text{on } i\mathbb{R}\backslash\Delta.$$

The functions $\mathbf{R}$ and $\mathbf{R}_r^\xi$ may be chosen with the property that $\mathbf{R}(\xi)=\mathbf{R}_r^\xi(\xi)$ and, further, $\mathbf{R}_r^\xi$ and $\mathbf{G}_r^\xi$ have state-space realisations with the same dimension.

If $\mathbf{G}\in H^\infty$, then $\mathbf{R}$ and $\mathbf{R}_r^\xi$ may be chosen to belong to H^∞ as well. In this case it follows that

$$\big\|\mathbf{R}-\mathbf{R}_r^\xi\big\|_{H^\infty} \le \min\big\{2a\big\|\mathbf{R}(I+\mathbf{G})^{-1}\big\|_{H^\infty}+\sqrt{2}\big\|I+\mathbf{G}_r^\xi\big\|_{H^\infty},\\ 2a\big\|\mathbf{R}_r^\xi(I+\mathbf{G}_r^\xi)^{-1}\big\|_{H^\infty}+\sqrt{2}\big\|I+\mathbf{G}\big\|_{H^\infty}\big\}\sum_{j=r+1}^{n}\sigma_j\,,$$

where

$$a := \min\big\{(1+\big\|\mathbf{G}\big\|^2_{H^\infty})(1+\big\|\mathbf{G}_r^\xi\big\|_{H^\infty}),(1+\big\|\mathbf{G}\big\|_{H^\infty})(1+\big\|\mathbf{G}_r^\xi\big\|^2_{H^\infty})\big\}\,.$$

5. **Examples.**

Example 5.1. Let $\mathbf{G}$ denote the strictly bounded real transfer function

$$s\mapsto \mathbf{G}(s) = \frac{(s+1)(s+2)}{(s+3)(s+4)(s+5)}\,,$$

considered in [37, Section V] and then [33, Example 1]. A minimal realisation of $\mathbf{G}$ is

$$A=\begin{pmatrix}-12 & -5.875 & -3.75\\ 8 & 0 & 0\\ 0 & 2 & 0\end{pmatrix},\quad B=\begin{pmatrix}1\\0\\0\end{pmatrix},\quad C=\begin{pmatrix}1 & 0.375 & 0.125\end{pmatrix},\quad D=0\,,$$

and the bounded real singular values are

$$\sigma_1 = 5.21\times10^{-2},\quad \sigma_2=3.61\times10^{-2},\quad \sigma_3=6.35\times10^{-4}\,.$$

Figures 5.1 and 5.2 plot the combined error

$$\left\|\begin{pmatrix}\mathbf{G}(s)-\mathbf{G}_r^{\xi_j}(s)\\ \mathbf{R}(s)-\mathbf{R}_r^{\xi_j}(s)\end{pmatrix}\right\|_2\,,$$

against real $s>0$ for several $\xi_j>0$, for the cases $r=1$ and $r=2$, respectively. Here $\mathbf{R}$ is a spectral factor for $I-\mathbf{G}^*\mathbf{G}$ and $\mathbf{R}_r^\xi$ is a sub-spectral factor for $I-(\mathbf{G}_r^{\xi_j})^*\mathbf{G}_r^{\xi_j}$, in the sense of statement (iii) of Proposition 3.4. We see in the plots the interpolation properties (2.7) and (3.7) holding. As expected from inspection of the bounded real singular values — the first two are of the same order — the errors are much smaller when $r=2$, compare the y-axes of Figures 5.1 and 5.2. Figure 5.3 plots the error $|\mathbf{G}(\omega i)-\mathbf{G}_r^{\xi_j}(\omega i)|$ on an interval of the imaginary axis. Recall that the infinity norm error $\|\mathbf{G}-\mathbf{G}_r^{\xi_j}\|_{H^\infty}$ will be achieved at some such ω. Observe that the choice of point of interpolation ξ_j seemingly leads to a trade-off between the error of the approximations at $\omega=0$ (the steady state gain) and $\omega=\infty$ (the feedthrough). □

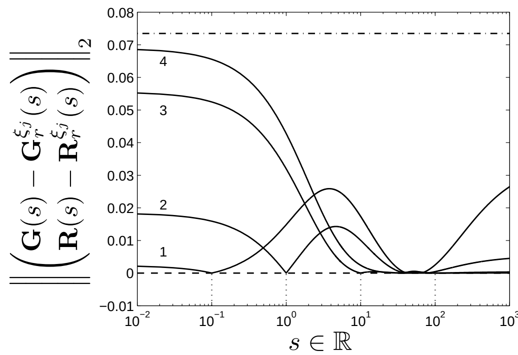


FIGURE 5.1. Semi-log plot of combined errors on the real axis for the bounded real GSPA from Example 5.1, with $r = 1$. The lines numbered 1–4 correspond to $\xi_1 = 0.1$, $\xi_2 = 1$, $\xi_3 = 10$ and $\xi_4 = 100$, respectively. Note the interpolation properties (2.7) and (3.7) hold and are highlighted with vertical dotted lines. The dashed dotted line is the bound (3.3).

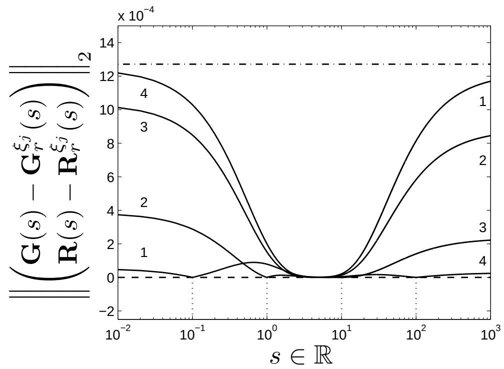
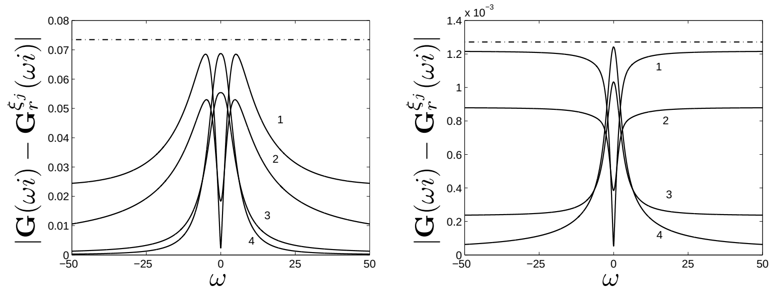


FIGURE 5.2. Semi-log plot of combined errors on the real axis for the bounded real GSPA from Example 5.1, with $r = 2$. The lines numbered 1–4 correspond to $\xi_1 = 0.1$, $\xi_2 = 1$, $\xi_3 = 10$ and $\xi_4 = 100$, respectively. Note the interpolation properties (2.7) and (3.7) hold and are highlighted with vertical dotted lines. The dashed dotted line is the error bound (3.3).

Example 5.2. The paper [38, Section V] considers model reduction of RC ladder circuit arrangements. The first circuit in that paper, which we consider here, has two current sources which gives rise to MIMO control system with the state-space realisation

$$\left.\begin{aligned} &A = \begin{pmatrix} -\frac{3}{2\mathcal{R}\mathcal{C}} & \frac{1}{2\mathcal{R}\mathcal{C}} & 0 & 0 \\ \frac{1}{\mathcal{R}\mathcal{C}} & -\frac{2}{\mathcal{R}\mathcal{C}} & \frac{1}{\mathcal{R}\mathcal{C}} & 0 \\ 0 & \frac{1}{\mathcal{R}\mathcal{C}} & -\frac{2}{\mathcal{R}\mathcal{C}} & \frac{1}{\mathcal{R}\mathcal{C}} \\ 0 & 0 & \frac{1}{\mathcal{R}\mathcal{C}} & -\frac{3}{2\mathcal{R}\mathcal{C}} \end{pmatrix}, \quad B = \begin{pmatrix} -\frac{1}{\mathcal{C}} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -\frac{1}{\mathcal{C}} \end{pmatrix}, \\ &C = B^T, \quad D = \begin{pmatrix} \frac{\mathcal{R}}{2} & 0 \\ 0 & \frac{\mathcal{R}}{2} \end{pmatrix}. \end{aligned}\right\} \tag{5.1}$$



(a) (b)

FIGURE 5.3. Plots of errors on the imaginary axis for the bounded real GSPA from Example 5.1, with $r = 1$ and $r = 2$ in panels (a) and (b), respectively. The lines numbered 1–4 correspond to $\xi_1 = 0.1$, $\xi_2 = 1$, $\xi_3 = 10$ and $\xi_4 = 100$, respectively, and are symmetric around $\omega = 0$. The dashed dotted lines are the bounds (3.3).

Here the terms $\mathcal{R}$ and $\mathcal{C}$ are positive parameters (resistances and capacitances, respectively). The inputs are currents at the sources, the outputs are voltages at the sources, and the state variables are voltages at the capacitors. We refer the reader to [38, Section V] for more details. The quadruple in (5.1) is strongly positive real, as $A + A^* \leq 0$, $B = C^*$ and $D + D^* > 0$. With $\mathcal{R} = \mathcal{C} = 1$, the positive real singular values are (to three significant figures)

$$\sigma_1 = 0.153, \quad \sigma_2 = 0.0870 \quad \sigma_3 = 0.0190 \quad \sigma_4 = 0.00190\,,$$

which, note, are different to the *Hankel* singular values of (5.1) computed in [38]. Figure 5.4 plots the error $\left\|\mathbf{G}(s) - \mathbf{G}_r^{\xi}(s)\right\|_2$, where $\mathbf{G}_r^{\xi}$ now denotes the positive real GSPA, against real $s > 0$ for fixed $\xi = 10$, for $r \in \{1, 2, 3\}$.

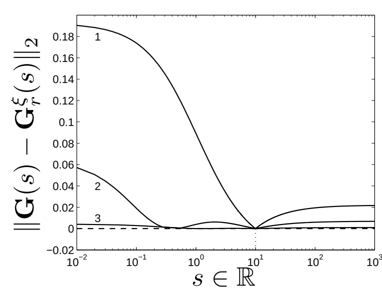


FIGURE 5.4. Semi-log plot of combined errors on the real axis for the positive real GSPA from Example 5.2, with $\xi = 10$. The lines numbered 1–3 correspond to $r \in \{1, 2, 3\}$ respectively. Note the interpolation property (2.7) holds.

The circuit in [38, Section V] may be easily be extended by adding identical "rungs" of the ladder, with each capacitor adding another state variable. As an illustrative example, we chose $N = 15$ capacitors, giving 15 states, with the same inputs and outputs as before. It is readily established from Kirchoff's laws and elementary circuit theory that the resulting matrix A has the same tri-banded structure as that in (5.1). The new B matrix has the same first and last row as that in (5.1), but with more rows of zeros in the middle. Further, $C = B^T$ still holds and D is unchanged. Fixing $\xi = 10$, we computed the error in the gap metric between $\mathbf{G}$ and $\mathbf{G}_r^\xi$ for $r \in \{1, 2, \ldots, 13\}$, as well as the error bounds from (4.4). The results are plotted on a semi-log axis in Figure 5.5. Although the errors are larger than the bound for $r \geq 10$, we expect that this is a consequence of the Matlab's function `gapmetric` maximal error tolerance of 1×10^{-5}. $\square$

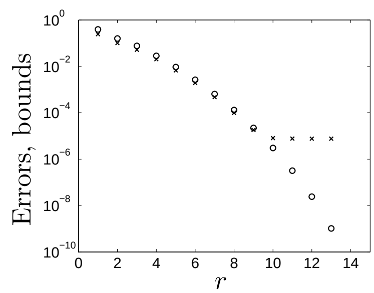


FIGURE 5.5. Semi-log plot of gap metric error $\hat{\delta}(\mathbf{G}, \mathbf{G}_r^\xi)$ (crosses) and error bounds (4.4) (circles) for extended circuit model from Example 5.2. Here $\xi = 10$.

6. **Proofs of results in Sections 3 and 4.** We divide the section into two subsections, considering the bounded real and positive real cases separately.

6.1. **The bounded real generalised singular perturbation approximation.** In order to prove Theorems 3.2 and 3.3, we draw on the material presented in Section 2, and also require three technical lemmas, stated and proven first.

Lemma 6.1. *If stable (A, B, C, D) with transfer function $\mathbf{G}$ and $\Sigma = \Sigma^* \geq 0$ are such that*

$$\left.\begin{aligned} A^*\Sigma + \Sigma A + C^*C &= -K^*K - P^*P \\ \Sigma B + C^*D &= -K^*W - P^*Q \\ I - D^*D &= W^*W + Q^*Q \end{aligned}\right\}, \tag{6.1}$$

hold for some $K, P \in \mathbb{C}^{m \times n}$ and $Q, W \in \mathbb{C}^{m \times m}$, then

(i) *(A, B, C, D) is bounded real.*

(ii) *$\mathbf{R} \in H^\infty(\mathbb{C}_0, \mathbb{C}^{2m \times m})$ with realisation $(A, B, \left[\begin{smallmatrix} K \\ P \end{smallmatrix}\right], \left[\begin{smallmatrix} W \\ Q \end{smallmatrix}\right])$ is a spectral factor for $I - \mathbf{G}^*\mathbf{G}$ in the sense that*

$$I - (\mathbf{G}(s))^*\mathbf{G}(s) = (\mathbf{R}(s))^*\mathbf{R}(s) \quad \forall\, s \in i\mathbb{R}\,.$$

Further, if the dual equations

$$\left.\begin{aligned} A\Sigma + \Sigma A^* + BB^* &= -LL^* - RR^* \\ \Sigma C^* + BD^* &= -LX^* - RS^* \\ I - DD^* &= XX^* + SS^* \end{aligned}\right\}, \tag{6.2}$$

hold for some $L, R \in \mathbb{C}^{n \times p}$ and $X, S \in \mathbb{C}^{p \times p}$, then

(iii) $\mathbf{S} \in H^\infty(\mathbb{C}_0, \mathbb{C}^{p\times 2p})$ *with realisation* $(A, [\,B\ R\,], C, [\,X\ S\,])$ *is a spectral factor for* $I - \mathbf{G}\mathbf{G}^*$ *in the sense that*

$$I - \mathbf{G}(s)(\mathbf{G}(s))^* = \mathbf{S}(s)(\mathbf{S}(s))^* \quad \forall\, s \in i\mathbb{R}\,.$$

Observe that in the above lemma, if $P = 0$ and $Q = 0$, then (A, B, K, W) is a realisation of a spectral factor $\mathbf{R}$. Similarly, if $R = 0$ and $S = 0$, then (A, L, C, X) is a realisation of a spectral factor $\mathbf{S}$.

Proof of Lemma 6.1: To prove statement (i), let $x^0 \in \mathbb{C}^n$, u be a continuous control and $x = x(\cdot; u, x^0)$ the corresponding differentiable state. From (6.1) we have that for all $\tau \geq 0$

$$\begin{aligned}
&\frac{d}{d\tau}\langle x(\tau), \Sigma x(\tau)\rangle + \|y(\tau)\|^2 - \|u(\tau)\|^2 \\
&\qquad = \left\langle \begin{pmatrix} A^*\Sigma + \Sigma A + C^*C & \Sigma B + C^*D \\ B^*\Sigma + D^*C & D^*D - I \end{pmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}, \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \right\rangle \\
&\qquad \leq - \left\| \begin{pmatrix} K & W \end{pmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \right\|^2 - \left\| \begin{pmatrix} P & Q \end{pmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \right\|^2 \\
&\qquad \leq 0\,.
\end{aligned} \tag{6.3}$$

Integrating both sides of (6.3) between 0 and $t \geq 0$ gives

$$\int_0^t \frac{d}{d\tau}\langle x(\tau), \Sigma x(\tau)\rangle + \|y(\tau)\|^2 - \|u(\tau)\|^2 \, d\tau \leq 0 \quad \forall\, t \geq 0\,,$$

whence

$$\int_0^t \|y(\tau)\|^2 - \|u(\tau)\|^2 \, d\tau \leq \langle x^0, \Sigma x^0\rangle - \langle x(t), \Sigma x(t)\rangle \quad \forall\, t \geq 0\,. \tag{6.4}$$

By a continuity and density argument, the inequality (6.4) holds for all $u \in L^2$ with corresponding continuous state x. With zero initial state $x^0 = 0$, it follows that the input u and output y satisfy $\|y\|_{L^2} \leq \|u\|_{L^2}$, and hence (A, B, C, D) is bounded real.

Statement (ii) follows from an elementary calculation using the equalities in (6.1). Indeed, let $s \in i\mathbb{R}$ and consider

$$\begin{aligned}
I - (\mathbf{G}(s))^*\mathbf{G}(s) &= I - (D + C(sI - A)^{-1}B)^*(D + C(sI - A)^{-1}B) \\
&= I - D^*D - D^*C(sI - A)^{-1}B - B^*(sI - A)^{-*}C^*D \\
&\qquad - B^*(sI - A)^{-*}C^*C(sI - A)^{-1}B \\
&= W^*W^* + Q^*Q + \big(B^*\Sigma + W^*K + Q^*P\big)(sI - A)^{-1}B \\
&\qquad + B^*(sI - A)^{-*}\big(\Sigma B + K^*W + P^*Q\big) \\
&\qquad + B^*(sI - A)^{-*}\big(A^*\Sigma + \Sigma A + K^*K + P^*P\big)(sI - A)^{-1}B \\
&= W^*W^* + Q^*Q + \big(W^*K + Q^*P\big)(sI - A)^{-1}B \\
&\qquad + B^*(sI - A)^{-*}\big(K^*W + P^*Q\big) \\
&\qquad + B^*(sI - A)^{-*}\big(K^*K + P^*P\big)(sI - A)^{-1}B \\
&= \left(\begin{pmatrix} W \\ Q \end{pmatrix} + \begin{pmatrix} K \\ P \end{pmatrix}(sI - A)^{-1}B\right)^* \left(\begin{pmatrix} W \\ Q \end{pmatrix} + \begin{pmatrix} K \\ P \end{pmatrix}(sI - A)^{-1}B\right) \\
&= (\mathbf{R}(s))^*\mathbf{R}(s)\,.
\end{aligned}$$

Statement (iii) is proven similarly, only instead using the equalities in (6.2). The details are omitted. ☐

For $\xi \in \mathbb{C}$ with $\mathrm{Re}(\xi) \geq 0$ and stable (A, B, C, D), set

$$\left.\begin{aligned} \mathcal{A} &:= (A - \xi I)^{-1}, & \mathcal{B} &:= (A - \xi I)^{-1} B\,, \\ \mathcal{C} &:= C(A - \xi I)^{-1}, & \mathcal{D} &:= D - C(A - \xi I)^{-1} B\,, \end{aligned}\right\} \tag{6.5}$$

which are well-defined and based on the reciprocal transformation. For given $r \in \underline{n-1}$, let the decomposition $(\mathcal{A}_{11}, \mathcal{B}_1, \mathcal{C}_1)$ be analogous to those in (2.4). The next lemma describes properties of $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ and relationships with (A_ξ, B_ξ, C_ξ).

Lemma 6.2. *For $\xi \in \mathbb{C}$ with $\mathrm{Re}(\xi) \geq 0$ and stable (A, B, C), assume that (A_ξ, B_ξ, C_ξ) given by (2.5) is well-defined and let $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ be given by (6.5). The following statements hold.*

(1) *If (A, B, C) is controllable or observable, then $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ has the same respective property.*

(2) *$\xi \notin \sigma(A_\xi)$ so that $A_\xi - \xi I$ is invertible and*

$$\mathcal{A}_{11} = (A_\xi - \xi I)^{-1}, \quad \mathcal{B}_1 = (A_\xi - \xi I)^{-1} B_\xi, \quad \mathcal{C}_1 = C_\xi (A_\xi - \xi I)^{-1}\,. \tag{6.6}$$

(3) *If $M \in \mathbb{C}^{n \times n}$ is Hurwitz and $\xi \in \mathbb{C}_0$, then $\sigma((M - \xi I)^{-1}) \subseteq \mathbb{E}_\xi$, where*

$$\mathbb{E}_\xi := \big\{ s \in \mathbb{C} \,:\, |s + 1/(2\mathrm{Re}(\xi))| < 1/(2\mathrm{Re}(\xi)) \big\}\,. \tag{6.7}$$

If $\xi \in i\mathbb{R}$, then $(M - \xi I)^{-1}$ is Hurwitz.

(4) *$\mathcal{A}$ in (6.5) is Hurwitz.*

Proof. (1): We use the Hautus criterion for observability. Assume that $v \in \mathbb{C}^n$ is such that $\mathcal{A}v = \lambda v$ and $\mathcal{C}v = 0$. Since $\mathcal{A}$ is invertible, if $\lambda = 0$, then $v = 0$ and there is nothing to prove. If $\lambda \neq 0$, then rearranging gives $Av = (\xi + 1/\lambda)v$ and

$$0 = \mathcal{C}v = C(A - \xi I)^{-1} v = \lambda C v\,,$$

so that $Cv = 0$. As the pair (C, A) is observable, it follows that $v = 0$, and thus the pair $(\mathcal{C}, \mathcal{A})$ is also observable. The proof of the controllability claim is similar, and so the details are omitted.

(2): We prove that $\xi \notin \sigma(A_\xi)$ by contraposition. If $v \neq 0$ and $\xi \in \mathbb{C}$ are such that $A_\xi v = \xi v$, then

$$\begin{aligned} A \begin{pmatrix} v \\ (\xi I - A_{22})^{-1} A_{21} v \end{pmatrix} &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} v \\ (\xi I - A_{22})^{-1} A_{21} v \end{pmatrix} \\ &= \begin{pmatrix} A_\xi v \\ \xi v \end{pmatrix} = \xi \begin{pmatrix} v \\ (\xi I - A_{22})^{-1} A_{21} v \end{pmatrix}\,, \end{aligned}$$

and we conclude that $\xi \in \sigma(A)$. The claim now follows as A is assumed Hurwitz, but $\mathrm{Re}(\xi) \geq 0$.

The equalities in (6.6) follow from block-wise matrix inversion and the definitions in (2.5) and (6.5).

(3): Let $\lambda \in \sigma((M-\xi I)^{-1})$ (so that necessarily $\lambda \neq 0$). Then $\xi + 1/\lambda \in \sigma(M)$, so that

$$\begin{aligned} \mathrm{Re}(\xi + 1/\lambda) < 0 \quad &\Rightarrow \quad \mathrm{Re}(\xi) < -\mathrm{Re}(1/\lambda) = \frac{-\mathrm{Re}(\lambda)}{|\lambda|^2} \\ &\Rightarrow \quad \mathrm{Re}(\lambda) < -\mathrm{Re}(\xi)|\lambda|^2 \,. \end{aligned} \tag{6.8}$$

If $\xi \in \mathbb{C}_0$, then (6.8) gives that $\lambda \in \mathbb{E}_\xi$, as required. If $\mathrm{Re}(\xi) = 0$, then (6.8) now yields that $\mathrm{Re}(\lambda) < 0$.

(4): Follows from (3), upon noticing that $\mathbb{E}_\xi \subset \mathbb{C}_0$. ☐

In the sequel we shall require the simple observation that for $\xi \in \mathbb{C}_0$

$$\lambda \in \partial\mathbb{E}_\xi \quad \Longleftrightarrow \quad \mathrm{Re}\,(\lambda) = -\mathrm{Re}(\xi)|\lambda|^2 \,, \tag{6.9}$$

where $\partial\mathbb{E}_\xi$ denotes the boundary of $\mathbb{E}_\xi$ — the circle in the complex plane with radius $1/(2\mathrm{Re}(\xi))$ and centre $-1/(2\mathrm{Re}(\xi))$.

Lemma 6.3. *Given $\xi \in \mathbb{C}$ with $\mathrm{Re}(\xi) \geq 0$, suppose that stable (A, B, C, D) has transfer function $\mathbf{G}$. Define $\mathbf{G}_r^\xi$, $\mathbf{H}$ and $\mathbf{H}_r$ as the transfer functions with realisations $(A_\xi, B_\xi, C_\xi, D_\xi)$, $(\mathcal{A}, \mathcal{B}, -\mathcal{C}, \mathcal{D})$ and $(\mathcal{A}_{11}, \mathcal{B}_1, -\mathcal{C}_1, \mathcal{D})$, respectively. Assume that $\sigma(\mathcal{A}_{11}) \subseteq \mathbb{E}_\xi$ if $\xi \in \mathbb{C}_0$, or $\mathcal{A}_{11}$ is Hurwitz if $\xi \in i\mathbb{R}$. Then*

$$\mathbf{G}(z) = \mathbf{H}\left(\frac{1}{z-\xi}\right) \quad \forall\, z \in \mathbb{C}_0,\ \mathrm{Re}(z) \geq 0, \quad z \neq \xi \,, \tag{6.10}$$

and

$$\mathbf{G}_r^\xi(z) = \mathbf{H}_r\left(\frac{1}{z-\xi}\right) \quad \forall\, z \in \mathbb{C}_0,\ \mathrm{Re}(z) \geq 0, \quad z \neq \xi \,. \tag{6.11}$$

Proof. Invoking Lemma 6.2, as A is Hurwitz, either $\sigma(\mathcal{A}) \subseteq \mathbb{E}_\xi$ or $\mathcal{A}$ is Hurwitz, depending on whether $\xi \in \mathbb{C}_0$ or $\xi \in i\mathbb{R}$, respectively. For $z \in \mathbb{C}$, $\mathrm{Re}(z) \geq 0$ and $z \neq 0$, we compute that

$$\begin{aligned} \mathbf{G}(\xi + 1/z) &= D + C\left((\xi + 1/z)I - A\right)^{-1} B = D + C\left(1/zI - (A - \xi I)\right)^{-1} B \\ &= D - Cz\left(zI - (A-\xi I)^{-1}\right)^{-1}(A-\xi I)^{-1}B \\ &= D - C(A-\xi I)^{-1}B - C(A-\xi I)^{-1}\left(zI - (A-\xi I)^{-1}\right)^{-1}(A-\xi I)^{-1}B \\ &= \mathcal{D} - \mathcal{C}(zI - \mathcal{A})^{-1}\mathcal{B} \\ &= \mathbf{H}(z)\,. \end{aligned} \tag{6.12}$$

Similarly, using the relationships (6.6), we have that

$$\begin{aligned} \mathbf{H}_r(z) &= \mathcal{D} - \mathcal{C}_1(zI - \mathcal{A}_{11})^{-1}\mathcal{B}_1 \\ &= \mathcal{D} - C_\xi(A_\xi - \xi I)^{-1}\left(zI - (A_\xi - \xi I)^{-1}\right)^{-1}(A_\xi - \xi I)^{-1}B_\xi \\ &= \mathcal{D} + C_\xi(A_\xi - \xi I)^{-1}B_\xi + C_\xi((\xi + 1/z)I - A_\xi)^{-1}B_\xi \\ &= \mathbf{G}_r^\xi(\xi + 1/z)\,, \end{aligned} \tag{6.13}$$

where we have used (2.7) to infer that

$$\begin{aligned} \mathcal{D} + C_\xi(A_\xi - \xi I)^{-1}B_\xi &= D - C(A-\xi I)^{-1}B + C_\xi(A_\xi - \xi I)^{-1}B_\xi \\ &= \mathbf{G}(\xi) - (\mathbf{G}_r^\xi(\xi) - D_\xi) \\ &= D_\xi\,. \end{aligned}$$

Therefore, combining (6.12) and (6.13) with a change of variables yields (6.10) and (6.11), respectively. □

Proof of Theorem 3.2. Let $\xi \in \mathbb{C}$ with $\mathrm{Re}(\xi) \geq 0$. An application of [40, Theorem 3.2] to the first equations in (3.1) and (3.2), both with $Z = \Sigma$, shows that A_{22} is Hurwitz, so that $(A_\xi, B_\xi, C_\xi, D_\xi)$ is well-defined. Elementary calculations using the definitions of $(A_\xi, B_\xi, C_\xi, D_\xi)$ in (2.5) and the equalities (3.1) and (3.2) considered block wise show that

$$\left.\begin{aligned} A_\xi^*\Sigma_1 + \Sigma_1 A_\xi + C_\xi^* C_\xi &= -K_\xi^* K_\xi - 2\mathrm{Re}(\xi) A_{21}^* \phi^* \Sigma_2 \phi A_{21}\,, \\ \Sigma_1 B_\xi + C_\xi^* D_\xi &= -K_\xi^* W_\xi - 2\mathrm{Re}(\xi) A_{21}^* \phi^* \Sigma_2 \phi B_2\,, \\ I - D_\xi^* D_\xi &= W_\xi^* W_\xi + 2\mathrm{Re}(\xi) B_2^* \phi^* \Sigma_2 \phi B_2\,, \end{aligned}\right\} \tag{6.14}$$

and

$$\left.\begin{aligned} A_\xi\Sigma_1 + \Sigma_1 A_\xi^* + B_\xi B_\xi^* &= -L_\xi L_\xi^* - 2\mathrm{Re}(\xi) A_{12} \phi \Sigma_2 \phi^* A_{12}^*\,, \\ \Sigma_1 C_\xi^* + B_\xi D_\xi^* &= -L_\xi X_\xi^* - 2\mathrm{Re}(\xi) A_{12} \phi \Sigma_2 \phi^* C_2^*\,, \\ I - D_\xi D_\xi^* &= X_\xi X_\xi^* + 2\mathrm{Re}(\xi) C_2 \phi \Sigma_2 \phi^* C_2^*\,, \end{aligned}\right\} \tag{6.15}$$

where $\phi := (\xi I - A_{22})^{-1}$ and

$$\left.\begin{aligned} K_\xi &:= K_1 + K_2\phi A_{21}, \quad W_\xi := W + K_2 \phi B_2\,, \\ L_\xi &:= L_1 + A_{12}\phi L_2, \quad X_\xi := X + C_2\phi L_2\,. \end{aligned}\right\} \tag{6.16}$$

In light of Lemma 6.1 and (6.14), it follows that $(A_\xi, B_\xi, C_\xi, D_\xi)$ is bounded real. Evidently, if $\xi \in i\mathbb{R}$, then the resulting simplification of (6.14) and (6.15) implies that $(A_\xi, B_\xi, C_\xi, D_\xi)$ is bounded real balanced, completing the proof of statement (i).

We proceed to prove statements (ii) and (iii), treating the cases $\xi \in \mathbb{C}_0$ and $\xi \in i\mathbb{R}$ separately. Assume that $\xi \in \mathbb{C}_0$. The first equation in (6.14) implies that every eigenvalue of A_ξ has non-positive real part. Suppose that $A_\xi v = \eta i\, v$ for some $\eta \in \mathbb{R}$ and $v \in \mathbb{C}^r$. Forming the inner product

$$\langle (A_\xi^*\Sigma_1 + \Sigma_1 A_\xi + C_\xi^* C_\xi) v, v\rangle\,,$$

and using (6.14), it follows that

$$0 \leq \|C_\xi v\|^2 = -\|K_\xi v\|^2 - 2\mathrm{Re}(\xi)\langle \Sigma_2 (\xi I - A_{22})^{-1} A_{21} v, (\xi I - A_{22})^{-1} A_{21} v\rangle \leq 0\,,$$

whence

$$\langle \Sigma_2 (\xi I - A_{22})^{-1} A_{21} v, (\xi I - A_{22})^{-1} A_{21} v\rangle = 0\,,$$

as $\mathrm{Re}\,(\xi) > 0$. Since $\Sigma_2 > 0$, we infer that

$$(\xi I - A_{22})^{-1} A_{21} v = 0\,.$$

Consequently

$$A\begin{pmatrix} v \\ 0 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} v \\ (\xi I - A_{22})^{-1} A_{21} v \end{pmatrix} = \begin{pmatrix} A_\xi v \\ \xi(\xi I - A_{22})^{-1} A_{21} v \end{pmatrix} = \eta i \begin{pmatrix} v \\ 0 \end{pmatrix}\,,$$

and, as A is Hurwitz, we deduce that $v = 0$. Recalling our supposition that $A_\xi v = \eta i\, v$, we conclude that A_ξ is Hurwitz as well.

For $\xi \in \mathbb{C}_0$, and for statement (iii), we shall require $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ defined in (6.5), which is stable by statement (3) of Lemma 6.2. Calculations starting from (3.1) and (3.2) respectively show that

$$\left.\begin{aligned} \mathcal{A}^*\Sigma + \Sigma\mathcal{A} + \mathcal{C}^*\mathcal{C} &= -\mathcal{K}^*\mathcal{K} - 2\mathrm{Re}(\xi)\mathcal{A}^*\Sigma\mathcal{A} \\ \Sigma\mathcal{B} - \mathcal{C}^*\mathcal{D} &= \mathcal{K}^*\mathcal{W} - 2\mathrm{Re}(\xi)\mathcal{A}^*\Sigma\mathcal{B} \\ I - \mathcal{D}^*\mathcal{D} &= \mathcal{W}^*\mathcal{W} + 2\mathrm{Re}(\xi)\mathcal{B}^*\Sigma\mathcal{B} \end{aligned}\right\}, \tag{6.17}$$

and

$$\left.\begin{aligned} \mathcal{A}\Sigma + \Sigma\mathcal{A}^* + \mathcal{B}\mathcal{B}^* &= -\mathcal{L}\mathcal{L}^* - 2\mathrm{Re}(\xi)\mathcal{A}\Sigma\mathcal{A}^* \\ \Sigma(-\mathcal{C}^*) + \mathcal{B}\mathcal{D}^* &= -\mathcal{L}\mathcal{X}^* - 2\mathrm{Re}(\xi)\mathcal{A}\Sigma(-\mathcal{C})^* \\ I - \mathcal{D}\mathcal{D}^* &= \mathcal{X}\mathcal{X}^* + 2\mathrm{Re}(\xi)\mathcal{C}\Sigma\mathcal{C}^* \end{aligned}\right\}, \tag{6.18}$$

where

$$\mathcal{K} := K\mathcal{A}, \quad \mathcal{W} := W - K\mathcal{B}, \quad \mathcal{L} := \mathcal{A}L, \quad \text{and} \quad \mathcal{X} := X - \mathcal{C}L\,. \tag{6.19}$$

The first equations in (6.17) and (6.18) may respectively be rewritten as

$$\mathcal{A}^*\Sigma + \Sigma\mathcal{A} + \begin{pmatrix}\mathcal{C}^* & \mathcal{K}^*\end{pmatrix}\begin{pmatrix}\mathcal{C}\\ \mathcal{K}\end{pmatrix} = -2\mathrm{Re}(\xi)\mathcal{A}^*\Sigma\mathcal{A}\,, \tag{6.20}$$

and

$$\mathcal{A}\Sigma + \Sigma\mathcal{A}^* + \begin{pmatrix}\mathcal{B} & \mathcal{L}\end{pmatrix}\begin{pmatrix}\mathcal{B}^*\\ \mathcal{L}^*\end{pmatrix} = -2\mathrm{Re}(\xi)\mathcal{A}\Sigma\mathcal{A}^*\,. \tag{6.21}$$

If $\xi \in i\mathbb{R}$, then a consequence of the simplification of (6.21) and (6.20) is that

$$\left(\mathcal{A}, \begin{pmatrix}\mathcal{B} & \mathcal{L}\end{pmatrix}, \begin{pmatrix}\mathcal{C}\\ \mathcal{K}\end{pmatrix}\right),$$

is Lyapunov balanced. An application of [40, Theorem 3.2] yields that $\mathcal{A}_{11}$ is Hurwitz, again invoking the assumption that the singular values are simple implies that the spectra of Σ_1 and Σ_2 are disjoint. Statement (2) of Lemma 6.2 implies that $\xi \notin \sigma(A_\xi)$ and that (6.6) holds, from which it is routine to verify that A_ξ is Hurwitz, since $\mathcal{A}_{11}$ is, and $\xi \in i\mathbb{R}$. The proof of statement (ii) is complete.

To prove statement (iii), we additionally assume that (A, B, C, D) is strictly bounded real. Suppose first that $\xi \in \mathbb{C}_0$. To establish minimality, let $\lambda \in \mathbb{C}$ and $v \in \mathbb{C}^n$ be such that $A_\xi v = \lambda v$ and $C_\xi v = 0$. We compute that

$$A\begin{pmatrix} v \\ (\xi I - A_{22})^{-1}A_{21}v\end{pmatrix} = \begin{pmatrix} A_\xi v \\ \xi(\xi I - A_{22})^{-1}A_{21}v\end{pmatrix} = \begin{pmatrix}\lambda & 0\\ 0 & \xi\end{pmatrix}\begin{pmatrix} v \\ (\xi I - A_{22})^{-1}A_{21}v\end{pmatrix},$$

so that

$$Az = Ez\,,$$

where

$$E := \begin{pmatrix}\lambda & 0\\ 0 & \xi\end{pmatrix} \quad \text{and} \quad z := \begin{pmatrix} v \\ (\xi I - A_{22})^{-1}A_{21}v\end{pmatrix}.$$

An application of [40, Theorem 3.1] to the (Lyapunov) balanced realisation

$$\left(A, \begin{pmatrix}B & L\end{pmatrix}, \begin{pmatrix}C\\ K\end{pmatrix}\right),$$

implies that

$$\|\mathrm{e}^{At}z\|^2 = \|\mathrm{e}^{Et}z\|^2 = \left\|\begin{pmatrix}\mathrm{e}^{\lambda t} & 0\\ 0 & \mathrm{e}^{\xi t}\end{pmatrix}\begin{pmatrix}z_1\\ z_2\end{pmatrix}\right\|^2 < \|z\|^2 \quad \forall\, t > 0\,,$$

whence

$$\mathrm{e}^{2\mathrm{Re}(\xi)t}\|z_2\|^2 \le \mathrm{e}^{2\mathrm{Re}(\lambda)t}\|z_1\|^2 + \mathrm{e}^{2\mathrm{Re}(\xi)t}\|z_2\|^2 < \|z_1\|^2 + \|z_2\|^2 \quad \forall\, t > 0\,. \tag{6.22}$$

Since $\xi \in \mathbb{C}_0$, the inequality (6.22) yields that

$$z_2 = (\xi I - A_{22})^{-1} A_{21} v = 0\,,$$

from which

$$\lambda v = A_\xi v = A_{11} v + A_{12}(\xi I - A_{22})^{-1} A_{21} v = A_{11} v\,,$$

and

$$0 = C_\xi v = C_1 v + C_2(\xi I - A_{22})^{-1} A_{21} v = C_1 v\,.$$

As (A, B, C, D) is bounded real balanced and strictly bounded real, the pair (C_1, A_{11}) is observable by [37, Theorem 2], and so we deduce that $v = 0$, proving that (C_ξ, A_ξ) is observable. The proof that the pair (A_ξ, B_ξ) is controllable is similar, and thus is omitted.

Let $\mathbf{G}$ and $\mathbf{H}$ be realised by (A, B, C, D) and $(\mathcal{A}, \mathcal{B}, -\mathcal{C}, \mathcal{D})$, respectively. If $\xi \in i\mathbb{R}$, then the equality (6.12) gives

$$\|\mathbf{H}\|_{H^\infty} = \sup_{z \in \mathbb{C}_0} \|\mathbf{H}(z)\|_2 = \sup_{z \in \mathbb{C}_0} \|\mathbf{G}(\xi + 1/z)\|_2 = \|\mathbf{G}\|_{H^\infty} < 1\,, \tag{6.23}$$

so that $\mathbf{H}$ is strictly bounded real. It follows from the equalities in (6.17) and (6.18) that $(\mathcal{A}, \mathcal{B}, -\mathcal{C}, \mathcal{D})$ is bounded real balanced, and so $(\mathcal{A}_{11}, \mathcal{B}_1, -\mathcal{C}_1, \mathcal{D})$ is the bounded real balanced truncation. Invoking [37, Theorem 2] yields that $(\mathcal{A}_{11}, \mathcal{B}_1, -\mathcal{C}_1)$ is minimal, and hence so is (A_ξ, B_ξ, C_ξ) via the relationships in (6.6), establishing minimality.

To establish the strict bounded realness of $(A_\xi, B_\xi, C_\xi, D_\xi)$, again we consider $\xi \in \mathbb{C}_0$ and $\xi \in i\mathbb{R}$ separately. In both cases, let the realisation $(\mathcal{A}_{11}, \mathcal{B}_1, -\mathcal{C}_1, \mathcal{D})$ have transfer function denoted $\mathbf{H}_r^\xi$. For $\xi \in \mathbb{C}_0$ we use proof by contraposition; suppose that $\omega_0 \in \mathbb{R}$ and $u_0 \in \mathbb{C}^m$ with $\|u_0\|_2 = 1$ are such that

$$\left\|\mathbf{G}_r^\xi(i\omega_0)\right\|_2 = \left\|\mathbf{G}_r^\xi(i\omega_0) u_0\right\|_2 = 1\,.$$

It follows from Lemma 6.3, notably (6.11), that

$$\|\mathbf{H}_r^\xi(p_0) u_0\|_2 = \left\|\mathbf{H}_r^\xi\left(\frac{1}{i\omega_0 - \xi}\right) u_0\right\|_2 = \|\mathbf{G}_r^\xi(i\omega_0) u_0\|_2 = 1\,,$$

where $p_0 := 1/(i\omega_0 - \xi) \in \partial\mathbb{E}_\xi$.

An elementary sequence of calculations using (6.9) and (6.17), which are relegated to Appendix B, shows that

$$\begin{aligned} I - &[\mathbf{H}_r^\xi(p_0)]^* \mathbf{H}_r^\xi(p_0) \\ &= q^2(\mathcal{B}_2 + \mathcal{A}_{21}(p_0 I - \mathcal{A}_{11})^{-1}\mathcal{B}_1)^* \Sigma_2 (\mathcal{B}_2 + \mathcal{A}_{21}(p_0 I - \mathcal{A}_{11})^{-1}\mathcal{B}_1) \\ &\quad + (\mathcal{W} - \mathcal{K}_1(p_0 I - \mathcal{A}_{11})^{-1}\mathcal{B}_1)^* (\mathcal{W} - \mathcal{K}_1(p_0 I - \mathcal{A}_{11})^{-1}\mathcal{B}_1)\,, \end{aligned} \tag{6.24}$$

where $q := \sqrt{2\mathrm{Re}(\xi)} > 0$. Since $\Sigma_2 > 0$, in light of (6.24), it follows that

$$\big(\mathcal{B}_2 + \mathcal{A}_{21}(p_0 I - \mathcal{A}_{11})^{-1}\mathcal{B}_1\big) u_0 = 0\,, \tag{6.25}$$

and

$$\big(\mathcal{W} - \mathcal{K}_1(p_0 I - \mathcal{A}_{11})^{-1}\mathcal{B}_1\big) u_0 = 0\,. \tag{6.26}$$

Setting

$$z_0 := \begin{pmatrix} (p_0 I - \mathcal{A}_{11})^{-1}\mathcal{B}_1 u_0 \\ 0 \end{pmatrix}\,,$$

and appealing to (6.25), we have that

$$\begin{aligned}\mathcal{A}z_0 + \mathcal{B}u_0 &= \begin{pmatrix}\mathcal{A}_{11} & \mathcal{A}_{12}\\ \mathcal{A}_{21} & \mathcal{A}_{22}\end{pmatrix}\begin{pmatrix}(p_0 I - \mathcal{A}_{11})^{-1}\mathcal{B}_1 u_0\\ 0\end{pmatrix} + \begin{pmatrix}\mathcal{B}_1\\ \mathcal{B}_2\end{pmatrix}u_0\\ &= \begin{pmatrix}p_0(p_0 I - \mathcal{A}_{11})^{-1}\mathcal{B}_1 u_0\\ 0\end{pmatrix}\\ &= p_0 z_0\,. \end{aligned} \tag{6.27}$$

Since $\sigma(\mathcal{A}) \subseteq \mathbb{E}_\omega$, $p_0 \notin \sigma(\mathcal{A})$, and so rearranging (6.27) yields

$$z_0 = (p_0 I - \mathcal{A})^{-1}\mathcal{B}u_0\,.$$

We conclude that

$$\begin{aligned}(\mathcal{W} - \mathcal{K}(pI - \mathcal{A})^{-1}\mathcal{B})u_0 = \mathcal{W}u_0 - \mathcal{K}z_0 &= \begin{pmatrix}\mathcal{K}_1 & \mathcal{K}_2\end{pmatrix}\begin{pmatrix}(p_0 I - \mathcal{A}_{11})^{-1}\mathcal{B}_1 u_0\\ 0\end{pmatrix}\\ &= \mathcal{W}u_0 - \mathcal{K}_1(p_0 I - \mathcal{A}_{11})^{-1}\mathcal{B}_1 u_0\\ &= 0\,, \end{aligned} \tag{6.28}$$

by (6.26). Another elementary series of calculations using (6.9) and (6.17), relegated to Appendix C, shows that

$$I - [\mathbf{H}(p_0)]^*\mathbf{H}(p_0) = (\mathcal{W} - \mathcal{K}(p_0 I - \mathcal{A})^{-1}\mathcal{B})^*(\mathcal{W} - \mathcal{K}(p_0 I - \mathcal{A})^{-1}\mathcal{B})\,, \tag{6.29}$$

which, in conjunction with (6.28), implies that

$$\|\mathbf{H}(p_0)u_0\|_2 = 1\,.$$

Invoking (6.10), we now see that

$$\|\mathbf{G}(i\omega_0)u\|_2 = \|\mathbf{H}(p_0)u_0\|_2 = 1\,,$$

implying that $\mathbf{G}$ is not strictly bounded real. The above proof is easily altered by taking $p_0 = 0$ in the case that

$$\lim_{\substack{\omega\in\mathbb{R}\\ \omega\to\infty}} \|\mathbf{G}_r^\xi(i\omega)\|_2 = 1\,,$$

as $\mathbf{G}_r^\xi$ is continuous at infinity.

It remains to consider $\xi \in i\mathbb{R}$. We first establish that $(\mathcal{A}_{11}, \mathcal{B}_1, -\mathcal{C}_1, \mathcal{D})$ is strictly bounded real. For which purpose, the inequality (6.23) implies that $\|\mathcal{D}\|_2 < 1$, and hence $I - \mathcal{D}^*\mathcal{D}$ is invertible. Since $(\mathcal{A}_{11}, \mathcal{B}_1, -\mathcal{C}_1, \mathcal{D})$ is bounded real balanced, it follows from the Bounded Real Lemma and by construction that Σ_1 and Σ_1^{-1} are solutions of the bounded real algebraic Riccati equation

$$\mathcal{A}_{11}^* Z + Z\mathcal{A}_{11} + \mathcal{C}_1^*\mathcal{C}_1 + (Z\mathcal{B}_1 - \mathcal{C}_1^*\mathcal{D})(I - \mathcal{D}^*\mathcal{D})^{-1}(Z\mathcal{B}_1 - \mathcal{C}_1^*\mathcal{D})^* = 0\,, \tag{6.30}$$

with the property that $\Sigma_1^{-1} > I > \Sigma_1$. For notational convenience, define

$$\mathcal{R} := I - \mathcal{D}^*\mathcal{D} = \mathcal{R}^* > 0, \quad \mathcal{S} = I - \mathcal{D}\mathcal{D}^* = \mathcal{S}^* > 0\,,$$

and

$$\mathcal{A}_E := \mathcal{A}_{11} + \mathcal{B}_1 R^{-1}(\mathcal{B}_1^*\Sigma_1 - \mathcal{D}^*\mathcal{C}_1)\,.$$

In light of [50, Theorem 13.19], it suffices to prove that $\mathcal{A}_E$ is Hurwitz, that is, that Σ_1 is a stabilizing solution of (6.30). Elementary manipulation of (6.30) for both $Z = \Sigma_1$ and $Z = \Sigma_1^{-1}$ shows that

$$\mathcal{A}_E^*\Sigma_1 + \Sigma_1\mathcal{A}_E + \mathcal{C}_1^*\mathcal{S}^{-1}\mathcal{C}_1 - \Sigma_1\mathcal{B}_1\mathcal{R}^{-1}\mathcal{B}_1^*\Sigma_1 = 0\,, \tag{6.31}$$

and

$$\mathcal{A}_E^*\Sigma_1^{-1} + \Sigma_1^{-1}\mathcal{A}_E + \mathcal{C}_1^*\mathcal{S}^{-1}\mathcal{C}_1 + \Pi\mathcal{B}_1\mathcal{R}^{-1}\mathcal{B}_1^*\Sigma_1\Pi - \Sigma_1\mathcal{B}_1\mathcal{R}^{-1}\mathcal{B}_1^*\Sigma_1 = 0\,, \tag{6.32}$$

hold, where $\Pi = \Sigma_1^{-1} - \Sigma_1 = \Pi^* > 0$. Subtracting (6.31) from (6.32) gives

$$\mathcal{A}_E^*\Pi + \Pi\mathcal{A}_E + \Pi\mathcal{B}_1\mathcal{R}^{-1}\mathcal{B}_1^*\Pi = 0\,,$$

from which we see that every eigenvalue of $\mathcal{A}_E$ has non-positive real part. Now suppose that $v \in \mathbb{C}^r$ and $\omega \in \mathbb{R}$ are such that $\mathcal{A}_E v = i\omega v$. Forming the inner product

$$\big\langle \big[A_E^*\Pi + \Pi A_E + \Pi\mathcal{B}_1\mathcal{R}^{-1}\mathcal{B}_1^*\Pi\big]v, v\big\rangle = 0\,,$$

it follows that

$$\mathcal{B}_1^*\Pi v = 0\,. \tag{6.33}$$

Since

$$\big\langle \big[\mathcal{A}_E^*\Pi + \Pi\mathcal{A}_E + \Pi\mathcal{B}_1\mathcal{R}^{-1}\mathcal{B}_1^*\Pi\big]x, v\big\rangle = 0 \quad \forall\, x \in \mathbb{C}^r\,,$$

we see that

$$\begin{aligned} \big\langle \big[\mathcal{A}_E^*\Pi + \Pi\mathcal{A}_E\big]x, v\big\rangle = 0 \quad \forall\, x \in \mathbb{C}^r \quad &\Rightarrow \quad \langle x, [\mathcal{A}_E^* + i\omega I]\Pi v\rangle = 0 \quad \forall\, x \in \mathbb{C}^r \\ &\Rightarrow \quad \mathcal{A}_E^*\Pi v = -i\omega\Pi v\,. \end{aligned} \tag{6.34}$$

Finally, noting that $(\mathcal{A}_E, \mathcal{B}_1)$ is controllable, as $(\mathcal{A}_{11}, \mathcal{B}_1)$ is, we conclude from (6.33) and (6.34) that $\Pi v = 0$, and so $v = 0$. Hence, $\mathcal{A}_E$ is Hurwitz and so $(\mathcal{A}_{11}, \mathcal{B}_1, -\mathcal{C}_1, \mathcal{D})$ is strictly bounded real. Finally, invoking (6.11) and that $\xi \in i\mathbb{R}$, we estimate that

$$\|\mathbf{G}_r^\xi\|_{H^\infty} = \sup_{z\in\mathbb{C}_0} \|\mathbf{G}_r^\xi(z)\|_2 = \sup_{z\in\mathbb{C}_0} \|\mathbf{H}_r^\xi(1/(z-\xi))\|_2 = \|\mathbf{H}_r^\xi\|_{H^\infty} < 1\,,$$

whence $(A_\xi, B_\xi, C_\xi, D_\xi)$ is strictly bounded real. □

Proof of Theorem 3.3. Let (A, B, C, D) denote a minimal, bounded real balanced, and stable, realisation of $\mathbf{G}$. For K, W, L, X as in (3.1) and (3.2), it follows that the realisation

$$\left(A, \begin{pmatrix} B & L \end{pmatrix}, \begin{pmatrix} C \\ K \end{pmatrix}, \begin{pmatrix} D & X \\ W & 0 \end{pmatrix}\right), \tag{6.35}$$

with transfer function $\mathbf{J}$, is Lyapunov balanced. Let $(A_\xi, B_\xi, C_\xi, D_\xi)$, with transfer function $\mathbf{G}_r^\xi$, denote the bounded real GSPA of (A, B, C, D), which is well-defined for all $\xi \in \mathbb{C}_0 \cup i\mathbb{R}$ by Theorem 3.2. By construction, the realisation

$$\left(A_\xi, \begin{pmatrix} B_\xi & L_\xi \end{pmatrix}, \begin{pmatrix} C_\xi \\ K_\xi \end{pmatrix}, \begin{pmatrix} D_\xi & X_\xi \\ W_\xi & 0 \end{pmatrix}\right)\,, \tag{6.36}$$

is the GSPA of that in (6.35), where K_ξ, L_ξ, W_ξ and X_ξ are given by (6.16).

Letting $\mathbf{J}_r^\xi$ denote the transfer function of (6.36) and invoking Theorem 2.4 yields

$$\|\mathbf{J} - \mathbf{J}_r^\xi\|_{H^\infty} \le 2\sum_{j=r+1}^{n} \sigma_j\,, \tag{6.37}$$

where $(\sigma_j)_{j=1}^n$ are the Hankel singular values of $\mathbf{J}$, which are equal to the bounded real singular values of $\mathbf{G}$. Combining (6.37) with the easily established estimate

$$\|\mathbf{G} - \mathbf{G}_r^\xi\|_{H^\infty} \le \|\mathbf{J} - \mathbf{J}_r^\xi\|_{H^\infty}\,,$$

gives (3.3), as required. The function $\mathbf{G}_r^\xi$ has the properties claimed.

The final claim follows from statement (iii) of Theorem 3.2. □

Proof of Proposition 3.4: The proof builds on that of Theorem 3.3.

For statement (i), define $\mathbf{R} \in H^\infty(\mathbb{C}_0, \mathbb{C}^{m\times m})$ and $\mathbf{S} \in H^\infty(\mathbb{C}_0, \mathbb{C}^{p\times p})$ by the realisations

$$(A, B, K, W) \quad \text{and} \quad (A, L, C, X)\,,$$

respectively. In light of (3.1) and (3.2), it follows from statements (ii) and (iii) of Lemma 6.1 that $\mathbf{R}$ and $\mathbf{S}$ are spectral factors of $I - \mathbf{G}^*\mathbf{G}$ and $I - \mathbf{G}\mathbf{G}^*$, respectively, as required.

For statement (ii), let $\xi \in i\mathbb{R}$, and let $\mathbf{R}_r^\xi \in H^\infty(\mathbb{C}_0, \mathbb{C}^{m\times m})$ and $\mathbf{S}_r^\xi \in H^\infty(\mathbb{C}_0, \mathbb{C}^{p\times p})$ be defined by the realisations

$$(A_\xi, B_\xi, K_\xi, W_\xi) \quad \text{and} \quad (A_\xi, L_\xi, C_\xi, X_\xi)\,, \tag{6.38}$$

respectively, where K_ξ, L_ξ, W_ξ and X_ξ are given by (6.16). Appealing to (6.14), (6.15), and invoking statements (ii) and (iii) of Lemma 6.1, it follows that $\mathbf{R}_r^\xi$ and $\mathbf{S}_r^\xi$ are spectral factors of $\mathbf{G}_r^\xi$ in the sense of (3.4), as required. By their definitions in.

The error bound (3.5) follows by combining (6.37) with the identity

$$\begin{pmatrix} \mathbf{G} - \mathbf{G}_r^\xi & \mathbf{S} - \mathbf{S}_r^\xi \\ \mathbf{R} - \mathbf{R}_r^\xi & \sharp \end{pmatrix} = \mathbf{J} - \mathbf{J}_r^\xi\,,$$

(which follows by construction) where $\sharp$ denotes an entry we are not concerned with. The error bounds (3.6) are a straightforward consequence of (3.5).

The interpolation equalities (3.7) hold owing to the definition (6.16) of the realisation (6.38) (compare with (2.5)).

For statement (iii), we define $\mathbf{R}_r^\xi \in H^\infty(\mathbb{C}_0, \mathbb{C}^{m\times m})$ and $\mathbf{S}_r^\xi \in H^\infty(\mathbb{C}_0, \mathbb{C}^{p\times p})$ as above, which, as with the proof of statement (ii), satisfy properties (3.5)–(3.7). Appealing to (6.14), an application of statement (ii) of Lemma 6.1, the function $\mathbf{U}_r^\xi \in H^\infty(\mathbb{C}_0, \mathbb{C}^{2m\times m})$ with realisation

$$\left(A_\xi, B_\xi, \begin{pmatrix} K_\xi \\ q\sqrt{\Sigma_2}\phi A_{21} \end{pmatrix}, \begin{pmatrix} W_\xi \\ q\sqrt{\Sigma_2}\phi B_2 \end{pmatrix}\right)\,,$$

where $q := \sqrt{2\mathrm{Re}(\xi)} > 0$ and $\phi = (\xi I - A_{22})^{-1}$, is a spectral factor of $I - (\mathbf{G}_r^\xi)^*\mathbf{G}_r^\xi$. A straightforward calculation shows that

$$(\mathbf{U}_r^\xi)^*\mathbf{U}_r^\xi \geq (\mathbf{R}_r^\xi)^*\mathbf{R}_r^\xi \quad \text{on } i\mathbb{R},$$

establishing the first inequality in (3.8). The dual case is proven similarly, using (6.15), and invoking statement (iii) of Lemma 6.1 with $\mathbf{V}_r^\xi \in H^\infty(\mathbb{C}_0, \mathbb{C}^{p\times 2p})$ defined by the realisation

$$\left(A_\xi, \begin{pmatrix} L_\xi & qA_{12}\phi\sqrt{\Sigma_2} \end{pmatrix}, C_\xi, \begin{pmatrix} X_\xi & qC_2\phi\sqrt{\Sigma_2} \end{pmatrix}\right)\,.$$

□

6.2. **The positive real generalised singular perturbation approximation.** The proof of the next lemma is very similar to that of Lemma 6.1, and is thus omitted. We have also omitted the corresponding statements pertaining to the dual positive real equations as, although they do hold, we shall not require them.

Lemma 6.4. *If (A,B,C,D) with transfer function $\mathbf{G}$ and $\Sigma \geq 0$ are such that*

$$\begin{aligned} A^*\Sigma + \Sigma A &= -K^*K - P^*P\,, \\ \Sigma B - C^* &= -K^*W - P^*Q\,, \\ D^* + D &= W^*W + Q^*Q\,, \end{aligned}$$

for some appropriately sized K, P, Q and W, then the following statements hold.

(i) *(A,B,C,D) is positive real.*

(ii) *$\mathbf{R}$ with realisation $(A,B,\left[\begin{smallmatrix}K\\P\end{smallmatrix}\right],\left[\begin{smallmatrix}W\\Q\end{smallmatrix}\right])$ is a spectral factor for $\mathbf{G}^*+\mathbf{G}$ in the sense that*

$$(\mathbf{G}(s))^* + \mathbf{G}(s) = (\mathbf{R}(s))^*\mathbf{R}(s) \quad \forall\, s \in i\mathbb{R}\setminus\Delta\,,$$

where Δ denotes the set of poles of $\mathbf{G}$.

We shall employ the so-called Cayley Transform $\mathcal{S} : H(\mathbb{C}_0,\mathbb{C}^{m\times m}) \supseteq D(\mathcal{S}) \to H(\mathbb{C}_0,\mathbb{C}^{m\times m})$, which is given by

$$\mathcal{S}(\mathbf{G})(s) = (I - \mathbf{G}(s))(I + \mathbf{G}(s))^{-1} \quad s \in \mathbb{C}_0\,.$$

Here $D(\mathcal{S})$ contains all $\mathbf{G} \in H(\mathbb{C}_0,\mathbb{C}^{m\times m})$ where the above formula makes sense (at least) for all $s \in \mathbb{C}_0$. Further, it is well-known (see, instance, [19, Lemma 7.1.8]) that if $\mathbf{G}$ is positive real, then $\mathbf{G} \in D(\mathcal{S})$ and $\mathcal{S}(\mathbf{G})$ is bounded real, and so in particular, belongs to $H^\infty(\mathbb{C}_0,\mathbb{C}^{m\times m})$. It is evident that the Cayley transform maps rational functions to rational functions.

If (A,B,C,D) is a minimal realisation of $\mathbf{G} \in D(\mathcal{S})$, then $(\tilde{A},\tilde{B},\tilde{C},\tilde{D})$ given by

$$\left.\begin{aligned} \tilde{A} &:= A - B(I+D)^{-1}C & \tilde{B} &:= \sqrt{2}B(I+D)^{-1} \\ \tilde{C} &:= -\sqrt{2}(I+D)^{-1}C & \tilde{D} &:= (I-D)(I+D)^{-1} \end{aligned}\right\}\,, \tag{6.39}$$

is well-defined and a minimal realisation of $\mathcal{S}(\mathbf{G})$. Since $\mathcal{S} : D(\mathcal{S}) \to D(\mathcal{S})$ and $\mathcal{S}^2 = \mathrm{id}$, the identity function, meaning that $\mathcal{S}$ is self-inverse, it follows that

$$(\tilde{\tilde{A}},\tilde{\tilde{B}},\tilde{\tilde{C}},\tilde{\tilde{D}}) \quad \text{is well-defined and} \quad (\tilde{\tilde{A}},\tilde{\tilde{B}},\tilde{\tilde{C}},\tilde{\tilde{D}}) = (A,B,C,D)\,.$$

The next lemma shows that the following diagram

$$\left.\begin{tikzcd} (A,B,C,D) \arrow[r, "\text{GSPA}"] \arrow[d, leftrightarrow, "\text{Cayley}"] & (A_\xi,B_\xi,C_\xi,D_\xi) \arrow[d, leftrightarrow, "\text{Cayley}"] \\ (\tilde{A},\tilde{B},\tilde{C},\tilde{D}) \arrow[r, "\text{GSPA}"] & ((\tilde{A})_\xi,(\tilde{B})_\xi,(\tilde{C})_\xi,(\tilde{D})_\xi) \end{tikzcd}\right\} \tag{6.40}$$

commutes. The proof is a tedious series of elementary calculations, and is relegated to Appendix D.

Lemma 6.5. *Given $\xi \in \mathbb{C}$ with $\mathrm{Re}(\xi) \geq 0$ and (A,B,C,D), assume that each of the quadruples in* (6.40) *are well-defined. Then*

$$(\widetilde{(A_\xi)},\widetilde{(B_\xi)},\widetilde{(C_\xi)},\widetilde{(D_\xi)}) = ((\tilde{A})_\xi,(\tilde{B})_\xi,(\tilde{C})_\xi,(\tilde{D})_\xi)\,,$$

and so the diagram (6.40) *commutes.*

Proof of Theorem 4.2. Let $\xi \in \mathbb{C}$ with $\mathrm{Re}(\xi) \geq 0$. An application of [40, Theorem 3.2] to the first two equations in (4.2) and (4.3) shows that A_{22} is Hurwitz, so that

$(A_\xi, B_\xi, C_\xi, D_\xi)$ is well-defined. Elementary calculations using the definitions of $(A_\xi, B_\xi, C_\xi, D_\xi)$ in (2.5) and the equalities (4.2) considered block wise show that

$$\left.\begin{aligned} A_\xi^*\Sigma_1 + \Sigma_1 A_\xi &= -K_\xi^* K_\xi - 2\mathrm{Re}(\xi) A_{21}^*\phi^*\Sigma_2\phi A_{21} \\ \Sigma_1 B_\xi - C_\xi^* &= -K_\xi^* W_\xi - 2\mathrm{Re}(\xi) A_{21}^*\phi^*\Sigma_2\phi B_2 \\ D_\xi^* + D_\xi &= W_\xi^* W_\xi + 2\mathrm{Re}(\xi) B_2^*\phi^*\Sigma_2\phi B_2 \end{aligned}\right\}, \tag{6.41}$$

and

$$\left.\begin{aligned} A_\xi\Sigma_1 + \Sigma_1 A_\xi^* &= -L_\xi L_\xi^* - 2\mathrm{Re}(\xi) A_{12}\phi\Sigma_2\phi^* A_{12}^* \\ \Sigma_1 C_\xi^* - B_\xi &= -L_\xi X_\xi^* - 2\mathrm{Re}(\xi) A_{12}\phi\Sigma_2\phi^* C_2^* \\ D_\xi + D_\xi^* &= X_\xi X_\xi^* + 2\mathrm{Re}(\xi) C_2\phi\Sigma_2\phi^* C_2^* \end{aligned}\right\}, \tag{6.42}$$

where $\phi = (\xi I - A_{22})^{-1}$ and K_ξ, W_ξ, L_ξ, X_ξ are given by (6.16).

In light of (6.41), an application of statement (i) of Lemma 6.4 yields that $(A_\xi, B_\xi, C_\xi, D_\xi)$ is positive real. Evidently, if $\xi \in i\mathbb{R}$, then the resulting simplification of (6.41) and (6.42) implies that $(A_\xi, B_\xi, C_\xi, D_\xi)$ is positive real balanced, completing the proof of statement (i).

The proof that A_ξ is Hurwitz when $\xi \in \mathbb{C}_0$ is the same as that in the proof of Theorem 3.2, only using the first equation in (6.41), instead of (6.14). The details are therefore omitted.

Next, define $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ as in (6.5) and note that $\mathcal{A} = (A - \xi I)^{-1}$ is Hurwitz by statement (3) of Lemma 6.2. Calculations starting from (4.2) and (4.3) respectively show that

$$\left.\begin{aligned} \mathcal{A}^*\Sigma + \Sigma\mathcal{A} &= -\mathcal{K}^*\mathcal{K} - 2\mathrm{Re}(\xi)\mathcal{A}^*\Sigma\mathcal{A} \\ \Sigma\mathcal{B} - (-\mathcal{C})^* &= \mathcal{K}^*\mathcal{W} - 2\mathrm{Re}(\xi)\mathcal{A}^*\Sigma\mathcal{B} \\ \mathcal{D}^* + \mathcal{D} &= \mathcal{W}^*\mathcal{W} + 2\mathrm{Re}(\xi)\mathcal{B}^*\Sigma\mathcal{B} \end{aligned}\right\}, \tag{6.43}$$

and

$$\left.\begin{aligned} \mathcal{A}\Sigma + \Sigma\mathcal{A}^* &= -\mathcal{L}\mathcal{L}^* - 2\mathrm{Re}(\xi)\mathcal{A}\Sigma\mathcal{A}^* \\ \Sigma(-\mathcal{C})^* - \mathcal{B} &= -\mathcal{L}\mathcal{X}^* - 2\mathrm{Re}(\xi)\mathcal{A}\Sigma(-\mathcal{C})^* \\ \mathcal{D} + \mathcal{D}^* &= \mathcal{X}\mathcal{X}^* + 2\mathrm{Re}(\xi)\mathcal{C}\Sigma\mathcal{C}^* \end{aligned}\right\}, \tag{6.44}$$

where $\mathcal{K}$, $\mathcal{W}$, $\mathcal{L}$ and $\mathcal{X}$ are given by (6.19).

When $\xi \in i\mathbb{R}$, then a consequence of the first equations in (6.43) and (6.44) is that the realisation $(\mathcal{A}, \mathcal{L}, \mathcal{K})$ is Lyapunov balanced. Thus $\mathcal{A}_{11}$ is Hurwitz by [40, Theorem 3.2], again invoking the assumption that the singular values are simple implies that the spectra of Σ_1 and Σ_2 are disjoint. Statement (2) of Lemma 6.2 yields that $\xi \notin \sigma(A_\xi)$. Consequently, $A_\xi - \xi I$ is invertible, and thus from (6.6) we see that $\mathcal{A}_{11} = (A_\xi - \xi I)^{-1}$. It is now routine to verify that A_ξ is Hurwitz, since $\mathcal{A}_{11}$ is, and $\xi \in i\mathbb{R}$. We have proven statement (ii).

To prove statement (iii), assume that (A, B, C, D) is strongly positive real, so that $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is well-defined and strictly bounded real. Further, $\tilde{A}$ is Hurwitz, since the realisation $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is minimal, and the transfer function is strictly bounded real (and hence belongs to H^∞).

As (A, B, C, D) is assumed positive real balanced, it follows that $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is bounded real balanced (by [37, Lemma 5]). Invoking statement (iii) of Theorem 3.2, it follows that

$$((\tilde{A})_\xi, (\tilde{B})_\xi, (\tilde{C})_\xi, (\tilde{D})_\xi)\,,$$

is minimal and strictly bounded real, and so is

$$\big(\widetilde{(A_\xi)}, \widetilde{(B_\xi)}, \widetilde{(C_\xi)}, \widetilde{(D_\xi)}\big)\,,$$

by Lemma 6.5. Since the Cayley transform is self-inverse, preserves minimality and maps strictly bounded real systems to strongly positive real systems [19, Lemma 7.1.8, p.159], it follows that $(A_\xi, B_\xi, C_\xi, D_\xi)$ is minimal and strongly positive real, proving statement (iii). □

Proof of Theorem 4.3. Let (A, B, C, D) denote a minimal, positive real balanced realisation of $\mathbf{G}$ and $\xi \in \mathbb{C}$ with $\mathrm{Re}(\xi) \geq 0$ which is not a pole of $\mathbf{G}$. Therefore, ξ is not an eigenvalue of A, as (A, B, C) is minimal. Arguing as in the proof of [40, Theorem 3.2] from the first equations in (4.2) and (4.3) shows that $\xi \notin \sigma(A_{22})$, and so $(A_\xi, B_\xi, C_\xi, D_\xi)$ is well defined.

Let $\mathbf{G}_r^\xi$ and $\mathbf{H}$ be defined by the realisations

$$(A_\xi, B_\xi, C_\xi, D_\xi) \quad \text{and} \quad (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})\,,$$

respectively. In light of (6.41), an application of statement (i) of Lemma 6.4 yields that $\mathbf{G}_r^\xi$ is positive real. Therefore, $\mathbf{G}_r^\xi \in D(\mathcal{S})$, in particular meaning that $\big(\widetilde{(A_\xi)}, \widetilde{(B_\xi)}, \widetilde{(C_\xi)}, \widetilde{(D_\xi)}\big)$ is well-defined. Next, note that $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is minimal, stable, bounded real, and bounded real balanced, whence $\tilde{A}_{22}$ is Hurwitz and so $((\tilde{A})_\xi, (\tilde{B})_\xi, (\tilde{C})_\xi, (\tilde{D})_\xi)$ is well-defined; we denote its transfer function by $\mathbf{H}_r^\xi$.

A consequence of Lemma 6.5 is that $\mathcal{S}(\mathbf{G}_r^\xi) = \mathbf{H}_r^\xi$. An application of Theorem 3.3 shows that

$$\|\mathbf{H} - \mathbf{H}_r^\xi\|_{H^\infty} \leq 2 \sum_{j=r+1}^{n} \sigma_j\,,$$

since the positive real singular values of $\mathbf{G}$ are precisely the bounded real singular values of $\mathbf{H}$, see [23, Corollary 9.6]. The remainder of the proof of (4.4) follows using the arguments given in [19, Theorem 7.2.12] or [22, Theorem 1.2]. The bound (4.5) follows from (4.4) and the equivalence of the gap metric restricted to bounded, linear operators and the operator norm, see [19, Corollary 3.6.9].

If $\mathbf{G} \in H^\infty(\mathbb{C}_0, \mathbb{C}^{m\times m})$, then, in addition to its other properties, the realisation (A, B, C, D) may be chosen to be stable. It follows from statement (ii) of Theorem 4.2 that A_ξ is Hurwitz and so $\mathbf{G}_r^\xi \in H^\infty(\mathbb{C}_0, \mathbb{C}^{m\times m})$ as well. If $\mathbf{G}$ is strongly positive real, then, by construction of $\mathbf{G}_r^\xi$, statement (iii) of Theorem 4.2 implies that $\mathbf{G}_r^\xi$ is strongly positive real as well. □

Proof of Proposition 4.4. (i) Since $\mathbf{G}$ is positive real, $\mathbf{G} \in D(\mathcal{S})$ and $\mathbf{H} := \mathcal{S}(\mathbf{G})$ is bounded real. Applying statement (i) of Proposition 3.4 to $\mathbf{H} \in H^\infty$ yields $\mathbf{T} \in H^\infty$ such that

$$I - \mathbf{H}^*\mathbf{H} = \mathbf{T}^*\mathbf{T} \quad \text{on } i\mathbb{R}. \tag{6.45}$$

Since $\mathbf{H} \in D(\mathcal{S})$ and $\mathcal{S}$ is self-inverse, we have that $\mathbf{G} = \mathcal{S}(\mathbf{H})$ and a straightforward calculation invoking (6.45) shows that

$$\begin{aligned}
\mathbf{G} + \mathbf{G}^* &= \mathcal{S}(\mathbf{H}) + [\mathcal{S}(\mathbf{H})]^* = (I - \mathbf{H})(I + \mathbf{H})^{-1} + \big[(I - \mathbf{H})(I + \mathbf{H})^{-1}\big]^* \\
&= 2(I + \mathbf{H})^{-*}\big[I - \mathbf{H}^*\mathbf{H}\big](I + \mathbf{H})^{-1} \\
&= (\mathbf{R})^*\mathbf{R} \quad \forall\, s \in i\mathbb{R} \setminus \Delta\,,
\end{aligned}$$

where $\mathbf{R} := \sqrt{2}\mathbf{T}(I+\mathbf{H})^{-1}$, which is evidently rational. Moreover, upon calculating

$$(I + \mathbf{H})^{-1} = (I + \mathcal{S}(\mathbf{G}))^{-1} = \frac{1}{2}(I + \mathbf{G}) \,,$$

it follows that $\mathbf{R}$ is proper.

(ii) The proof mimics that of statement (i), only replacing $\mathbf{G}$ by $\mathbf{G}_r^\xi$ from Theorem 4.3 and $\mathbf{H}_r^\xi := \mathcal{S}(\mathbf{G}_r^\xi)$. Then (6.45) becomes

$$I - (\mathbf{H}_r^\xi)^*\mathbf{H}_r^\xi = (\mathbf{T}_r^\xi)^*\mathbf{T}_r^\xi \quad \text{on } i\mathbb{R}, \tag{6.46}$$

for some $\mathbf{T}_r^\xi \in H^\infty$. The desired proper, rational spectral factor $\mathbf{R}_r^\xi$ is given by $\mathbf{R}_r^\xi := \sqrt{2}\mathbf{T}_r^\xi(I + \mathbf{H}_r^\xi)^{-1} = (\sqrt{2}/2)\mathbf{T}(I + \mathbf{G}_r^\xi)$. Note that since $\mathbf{G}(\xi) = \mathbf{G}_r^\xi$, we have that

$$\mathbf{H}(\xi) = (I - \mathbf{G}(\xi))(I + \mathbf{G}(\xi))^{-1} = (I - \mathbf{G}_r^\xi(\xi))(I + \mathbf{G}_r^\xi(\xi))^{-1} = \mathbf{H}_r^\xi(\xi) \,.$$

Therefore, we verify that

$$\mathbf{R}(\xi) = \sqrt{2}\mathbf{T}(\xi)(I + \mathbf{H}(\xi))^{-1} = \sqrt{2}\mathbf{T}_r^\xi(\xi)(I + \mathbf{H}_r^\xi(\xi))^{-1} = \mathbf{R}_r^\xi(\xi) \,,$$

where we have used $\mathbf{T}(\xi) = \mathbf{T}_r^\xi(\xi)$, which follows from (3.7).

By Theorem 4.3, if $\mathbf{G} \in H^\infty$, then $\mathbf{G}_r^\xi \in H^\infty$ as well, whence so are $\mathbf{R}, \mathbf{R}_r^\xi$.

Finally, using the definitions of $\mathbf{R}$ and $\mathbf{R}_r^\xi$, we estimate

$$\begin{aligned} \frac{1}{\sqrt{2}}\big\|\mathbf{R} - \mathbf{R}_r^\xi\big\|_{H^\infty} &= \big\|\mathbf{T}(I+\mathbf{H})^{-1} - \mathbf{T}_r^\xi(I + \mathbf{H}_r^\xi)^{-1}\big\|_{H^\infty} \\ &\leq \big\|\mathbf{T}\big((I+\mathbf{H})^{-1} - (I + \mathbf{H}_r^\xi)^{-1}\big)\big\|_{H^\infty} + \big\|(\mathbf{T} - \mathbf{T}_r^\xi)(I + \mathbf{H}_r^\xi)^{-1}\big\|_{H^\infty} \\ &\leq \frac{1}{2}\big\|\mathbf{T}\big\|_{H^\infty}\big\|\mathbf{G} - \mathbf{G}_r^\xi\big\|_{H^\infty} + \big\|\mathbf{T} - \mathbf{T}_r^\xi\big\|_{H^\infty}\big\|(I + \mathbf{H}_r^\xi)^{-1}\big\|_{H^\infty} \\ &\leq \big(a\big\|\mathbf{T}\big\|_{H^\infty} + 2\big\|(I + \mathbf{H}_r^\xi)^{-1}\big\|_{H^\infty}\big) \sum_{j=r+1}^{n} \sigma_j \,, \end{aligned} \tag{6.47}$$

where we have invoked (4.5) and (3.6) in the final inequality above. Using expressions for $\mathbf{T}$ and $(I + \mathbf{H}_r^\xi)^{-1}$ yields that

$$\big\|\mathbf{R} - \mathbf{R}_r^\xi\big\|_{H^\infty} \leq \Big(2a\big\|\mathbf{R}(I+\mathbf{G})^{-1}\big\|_{H^\infty} + \sqrt{2}\big\|I + \mathbf{G}_r^\xi\big\|_{H^\infty}\Big) \sum_{j=r+1}^{n} \sigma_j \,. \tag{6.48}$$

If in (6.47) we add and subtract $\mathbf{T}_r^\xi(I+\mathbf{H})^{-1}$ (instead of $\mathbf{T}(I+\mathbf{H}_r^\xi)^{-1}$) and perform the analogous steps, *mutatis mutandis*, we arrive at the bound

$$\big\|\mathbf{R} - \mathbf{R}_r^\xi\big\|_{H^\infty} \leq \Big(2a\big\|\mathbf{R}_r^\xi(I+\mathbf{G}_r^\xi)^{-1}\big\|_{H^\infty} + \sqrt{2}\big\|I + \mathbf{G}\big\|_{H^\infty}\Big) \sum_{j=r+1}^{n} \sigma_j \,. \tag{6.49}$$

Combining (6.48) and (6.49) gives the required bound. □

Appendix A. Proofs of Theorems 2.3 and 2.4. We need the following lemma.

Lemma A.1. *Given $\xi \in \mathbb{C}_0$, suppose that $(A, B, -C, D)$ with transfer function $\mathbf{H}$ satisfies*

$$A\Sigma + \Sigma A^* + BB^* \leq -2\mathrm{Re}(\xi)A\Sigma A^* \,, \tag{A.1}$$

and

$$A^*\Sigma + \Sigma A + C^*C \leq -2\mathrm{Re}(\xi)A^*\Sigma A \,. \tag{A.2}$$

Further assume that $\Sigma = \Sigma^* > 0$ *has simple eigenvalues* $(\sigma_j)_{j=1}^n$*, ordered according to* (2.3)*, and that for each* $k \in \{r, \ldots, n\}$ *the truncation* $A_{11}^{(k)} \in \mathbb{C}^{k\times k}$ *satisfies*

$$\sigma(A_{11}^{(k)}) \subseteq \mathbb{E}_\xi\,, \tag{A.3}$$

where $A_{11}^{(r)} = A_{11}$ *and* $A_{11}^{(n)} = A$*. Let* $\mathbf{H}_r$ *have realisation* $(A_{11}, B_1, -C_1, D)$*. Then*

$$\|\mathbf{H}(s) - \mathbf{H}_r(s)\|_2 \le 2 \sum_{j=r+1}^{n} \sigma_j \quad \forall\, s \in \partial\mathbb{E}_\xi\,. \tag{A.4}$$

If $\xi \in i\mathbb{R}$, (A.1) *and* (A.2) *hold, and* (A.3) *is replaced by*

$$A_{11}^{(k)} \text{ is Hurwitz for all } k \in \{r, \ldots, n\}\,,$$

then

$$\|\mathbf{H}(s) - \mathbf{H}_r(s)\|_2 \le 2 \sum_{j=r+1}^{n} \sigma_j \quad \forall\, s \in i\mathbb{R}\,. \tag{A.5}$$

Proof. First let $\xi \in \mathbb{C}_0$. For $s \in \partial\mathbb{E}_\xi$, let

$$\begin{aligned} A_s &:= A_{22} + A_{21}(sI - A_{11})^{-1}A_{12} \\ B_s &:= B_2 + A_{21}(sI - A_{11})^{-1}B_1 \\ C_s &:= C_2 + C_1(sI - A_{11})^{-1}A_{12} \end{aligned}\quad ,$$

which are well-defined by assumption (A.3).

Block wise inspection of the two inequalities (A.1) and (A.2) yields the relationships:

$$\begin{aligned} A_{11}\Sigma_1 + \Sigma_1 A_{11}^* + B_1B_1^* &\le -2\mathrm{Re}(\xi)\left(A_{11}\Sigma_1A_{11}^* + A_{12}\Sigma_2A_{12}^*\right)\,, \\ A_{12}\Sigma_2 + \Sigma_1 A_{21}^* + B_1B_2^* &\le -2\mathrm{Re}(\xi)\left(A_{11}\Sigma_1A_{21}^* + A_{12}\Sigma_2A_{22}^*\right)\,, \\ A_{22}\Sigma_2 + \Sigma_2 A_{22}^* + B_2B_2^* &\le -2\mathrm{Re}(\xi)\left(A_{21}\Sigma_1A_{21}^* + A_{22}\Sigma_2A_{22}^*\right)\,, \end{aligned} \tag{A.6}$$

and

$$\begin{aligned} A_{11}^*\Sigma_1 + \Sigma_1 A_{11} + C_1^*C_1 &\le -2\mathrm{Re}(\xi)\left(A_{11}^*\Sigma_1A_{11} + A_{21}^*\Sigma_2A_{21}\right)\,, \\ A_{21}^*\Sigma_2 + \Sigma_1 A_{12} + C_1^*C_2 &\le -2\mathrm{Re}(\xi)\left(A_{11}^*\Sigma_1A_{12} + A_{21}^*\Sigma_2A_{22}\right)\,, \\ A_{22}^*\Sigma_2 + \Sigma_2 A_{22} + C_2^*C_2 &\le -2\mathrm{Re}(\xi)\left(A_{12}^*\Sigma_1A_{12} + A_{22}^*\Sigma_2A_{22}\right)\,. \end{aligned} \tag{A.7}$$

An elementary sequence of calculations, using the definitions of A_s, B_s and C_s and the above inequalities, gives

$$A_s\Sigma_2 + \Sigma_2A_s^* + B_sB_s^* \le -2\mathrm{Re}(\xi)A_s\Sigma_2A_s^*\,, \tag{A.8}$$

and

$$A_s^*\Sigma_2 + \Sigma_2A_s + C_s^*C_s \le -2\mathrm{Re}(\xi)A_s^*\Sigma_2A_s\,. \tag{A.9}$$

We claim that for all $s \in \partial\mathbb{E}_\xi$, $s \notin \sigma(A_s)$ so that $sI - A_s$ is invertible. To establish the claim, if $v \in \mathbb{C}^{n-r}$ is such that $A_sv = sv$, then

$$\begin{aligned} Az = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} (sI - A_{11})^{-1}A_{12}v \\ v \end{pmatrix} &= \begin{pmatrix} A_{11}(sI - A_{11})^{-1}A_{12}v + A_{12}v \\ A_sv \end{pmatrix} \\ = s\begin{pmatrix} (sI - A_{11})^{-1}A_{12}v \\ v \end{pmatrix} &= sz\,. \end{aligned} \tag{A.10}$$

Since $s \notin \sigma(A)$ (indeed, $\sigma(A) \subseteq \mathbb{E}_\xi$), it follows from (A.10) that $z = 0$ and thus $v = 0$, proving that $s \notin \sigma(A_s)$.

Moreover, since $\|C_s v\|^2 \geq 0$ for all $v \in \mathbb{C}^{n-r}$, by considering any eigenvalue λ of A_s with corresponding eigenvector v and the inequality

$$\langle (A_s^* \Sigma_2 + \Sigma_2 A_s + C_s^* C_s)v, v\rangle \leq -2\mathrm{Re}(\xi)\langle A_s^* \Sigma_2 A_s v, v\rangle \,,$$

it follows that

$$2\mathrm{Re}(\lambda)\langle \Sigma_2 v, v\rangle \leq 2\mathrm{Re}(\lambda)\langle \Sigma_2 v, v\rangle + \|C_s v\|^2 \leq -2\mathrm{Re}(\xi)|\lambda|^2\langle \Sigma_2 v, v\rangle \,.$$

Hence,

$$\sigma(A_s) \subseteq \mathbb{E}_\xi \cup \partial\mathbb{E}_\xi \,, \tag{A.11}$$

see (6.9). The arguments which follow are, in part, in the spirit of those used in [10] — deriving the H^∞ error bound for Lyapunov balanced truncation. Setting $\Delta = \Delta(s) := sI - A_s$, straightforward calculations show that

$$\mathbf{H}(s) - \mathbf{H}_r(s) = C_s \Delta^{-1} B_s \quad \forall\, s \in \partial\mathbb{E}_\xi \,,$$

where we have used that $s \notin \sigma(A_s)$, and so

$$\begin{aligned}\|\mathbf{H}(s) - \mathbf{H}_r(s)\|_2^2 &= \lambda_{\mathrm{m}}\big(C_s\Delta^{-1}B_s(C_s\Delta^{-1}B_s)^*\big) = \lambda_{\mathrm{m}}\big(C_s\Delta^{-1}B_sB_s^*\Delta^{-*}C_s^*\big)\\ &= \lambda_{\mathrm{m}}\big(\Delta^{-1}B_sB_s^*\Delta^{-*}C_s^*C_s\big) \quad \forall\, s \in \partial\mathbb{E}_\xi \,.\end{aligned} \tag{A.12}$$

Here we have used that for square matrices M, N and $\lambda \neq 0$, $\lambda \in \sigma(MN)$ if, and only if, $\lambda \in \sigma(NM)$, and

$$\|M\|_2^2 = \lambda_{\mathrm{m}}(M^*M) =: \max\big\{\lambda \,:\, \lambda \in \sigma(M^*M)\big\} \,,$$

that is, the 2-norm of M is equal to the non-negative squareroot of the largest eigenvalue of M^*M.

For notational convenience in the following arguments set $\zeta = \mathrm{Re}(\xi) > 0$. Rearranging (A.8) yields that

$$B_sB_s^* \leq -\big(2\zeta A_s\Sigma_2A_s^* + A_s\Sigma_2 + \Sigma_2A_s^*\big) \,,$$

whence

$$\begin{aligned}\Delta^{-1}B_sB_s^*\Delta^{-*} &\leq -(sI - A_s)^{-1}\left[2\zeta A_s\Sigma_2A_s^* + A_s\Sigma_2 + \Sigma_2A_s^*\right](sI - A_s)^{-*} \,,\\ &= -2\zeta\big((sI - A_s) - sI\big)\Sigma_2\big((sI - A_s) - sI\big)^*\\ &\qquad + (sI - A_s)\Sigma_2 + \Sigma_2(sI - A_s)^* - 2\mathrm{Re}(s)\Sigma_2 \,,\\ &= -2\zeta\Sigma_2 + p\Delta^{-1}\Sigma_2 + \overline{p}\Sigma_2\Delta^{-*} \,,\end{aligned} \tag{A.13}$$

where $p := 1 + 2\zeta s$ and we have used (6.9). Similarly, from (A.9), we see that

$$\begin{aligned}C_s^*C_s &\leq -2\big(\zeta A_s^*\Sigma_2A_s + A_s^*\Sigma_2 + \Sigma_2A_s\big)\\ &= -2\zeta\big((sI - A_s) - sI\big)^*\Sigma_2\big((sI - A_s) - sI\big) + (sI - A_s)^*\Sigma_2 + \Sigma_2(sI - A_s)\\ &\qquad - 2\mathrm{Re}(s)\Sigma_2 \,,\\ &= -2\zeta\Delta^*\Sigma_2\Delta + \overline{p}\Sigma_2\Delta + p\Delta^*\Sigma_2 \,,\end{aligned} \tag{A.14}$$

where again we have used (6.9). Combining (A.13) and (A.14) gives

$$\begin{aligned}&\lambda_{\mathrm{m}}(\Delta^{-1}B_sB_s^*\Delta^{-*}C_s^*C_s)\\ &\leq \lambda_{\mathrm{m}}\big((-2\zeta\Sigma_2 + p\Delta^{-1}\Sigma_2 + \overline{p}\Sigma_2\Delta^{-*})(-2\zeta\Delta^*\Sigma_2\Delta + \overline{p}\Sigma_2\Delta + p\Delta^*\Sigma_2)\big)\\ &= \lambda_{\mathrm{m}}\big((-2\zeta\Delta\Sigma_2\Delta^* + p\Sigma_2\Delta^* + \overline{p}\Delta\Sigma_2)(-2\zeta\Sigma_2 + \overline{p}\Delta^{-*}\Sigma_2 + p\Sigma_2\Delta^{-1})\big).\end{aligned}$$

Now assume that just one singular value is omitted in the reduced order system, so that $\Sigma_2 = \sigma_n I$. Invoking the assumption that the singular values are simple, it follows that the reduced order system has a scalar state. Then

$$\begin{aligned}
&\lambda_{\mathrm{m}}(\Delta^{-1}B_sB_s^*\Delta^{-*}C_s^*C_s) \\
&\le \sigma_n^2(-2\zeta\Delta\Delta^* + p\Delta^* + \overline{p}\Delta)(-2\zeta + \overline{p}\Delta^{-*} + p\Delta^{-1}) \\
&= \sigma_n^2\big(4\zeta^2\Delta\Delta^* - 4\zeta p\Delta^* - 4\zeta\overline{p}\Delta + |p|^2 + \overline{p}^2\Delta\Delta^{-*} + p^2\Delta^*\Delta^{-1} + |\overline{p}|^2\big) \\
&= \sigma_n^2\big((1+\overline{p}^2\Delta\Delta^{-*})(1+p^2\Delta^*\Delta^{-1}) + 4[(\zeta\Delta^* - \overline{p})(\zeta\Delta - p) - 1]\big)\,, \qquad \text{(A.15)}
\end{aligned}$$

where we have used that $|p| = |\overline{p}| = 1$ and that Δ and $\Delta^* = \overline{\Delta}$ are scalar quantities.

We investigate the second term in (A.15) and estimate that

$$\begin{aligned}
(\zeta\Delta^* - \overline{p})(\zeta\Delta - p) &= |\zeta\Delta - p|^2 = |\zeta(sI - A_s) - (1+2\zeta s)|^2 \\
&= |(-1-\zeta s) - \zeta A_s|^2 \le 1\,,
\end{aligned}$$

by geometric considerations and in light of (A.11). Thus the second term in (A.15) is non-positive, and so

$$\lambda_{\mathrm{m}}(\Delta^{-1}B_sB_s^*\Delta^{-*}C_s^*C_s) \le \sigma_n^2(1+\overline{p}^2\Delta\Delta^{-*})(1+p^2\Delta^*\Delta^{-1}) \quad \forall\, s \in \partial\mathbb{E}_\xi\,.$$

Writing $f(s) = \overline{p}^2\Delta(s)\Delta^{-*}(s)$, it follows that

$$|f(s)| = \left|\overline{p}^2\Delta(s)/\overline{\Delta(s)}\right| = 1 \quad \forall\, s \in \partial\mathbb{E}_\xi\,,$$

therefore

$$\lambda_{\mathrm{m}}(\Delta^{-1}B_sB_s^*\Delta^{-*}C_s^*C_s) \le \sigma_n^2|1+f(s)|^2 \le \sigma_n^2(1+|f(s)|)^2 = 4\sigma_n^2\,,$$

which, when combined with (A.12), proves the one-step bound

$$\|\mathbf{H}_n(s) - \mathbf{H}_{n-1}(s)\|_2 \le 2\sigma_n \quad \forall\, s \in \partial\mathbb{E}_\xi\,,$$

where $\mathbf{H}_k$ for $k \in \{1,2,\ldots,n\}$ denotes the reduced order system with k singular values retained so that, in particular, $\mathbf{H}_n = \mathbf{H}$. To establish the intermediate one-step bounds

$$\|\mathbf{H}_j(s) - \mathbf{H}_{j-1}(s)\|_2 \le 2\sigma_n \quad \forall\, s \in \partial\mathbb{E}_\xi \quad \forall\, j \in \{r+1,\ldots,n-1\}\,,$$

we repeat the above arguments with $(A, B, -C)$ and $(A_{11}, B_1, -C_1)$ replaced by

$$(A_{11}, B_1, -C_1) \quad \text{and} \quad ((A_{11})_{11}, (B_1)_1, (-C_1)_1)\,,$$

respectively. As such, we see $\mathbf{H}_{j-1}$ as the one-step truncation of $\mathbf{H}_j$. Note that by (A.6) and (A.7), $(A_{11}, B_1, -C_1)$ satisfy the inequalities

$$A_{11}\Sigma_1 + \Sigma_1A_{11}^* + B_1B_1^* \le -2\mathrm{Re}(\xi)A_{11}\Sigma_1A_{11}^*\,,$$

and

$$A_{11}^*\Sigma_1 + \Sigma_1A_{11} + C_1^*C_1 \le -2\mathrm{Re}(\xi)A_{11}^*\Sigma_1A_{11}\,,$$

which are of the form (A.1) and (A.2), respectively.

We now use a telescoping series and the triangle inequality to show that

$$\begin{aligned}
\|\mathbf{H}(s) - \mathbf{H}_r(s)\|_2 &= \left\|\sum_{j=r+1}^{n}\big[\mathbf{H}_j(s) - \mathbf{H}_{j-1}(s)\big]\right\|_2 \le \sum_{j=r+1}^{n}\|\mathbf{H}_j(s) - \mathbf{H}_{j-1}(s)\|_2 \\
&\le 2\sum_{j=r+1}^{n}\sigma_j \quad \forall\, s \in \partial\mathbb{E}_\xi\,,
\end{aligned}$$

which is (A.4), as required.

The proof of (A.5) in the case that $\xi \in i\mathbb{R}$ follows via the same argument used in [10], the only difference being that the Lyapunov equations (2.2) are replaced by Lyapunov inequalities (A.1) and (A.2). $\square$

Proof of Theorem 2.3. Since (A, B, C, D) is a minimal, balanced and stable, it follows from [40, Theorem 3.2] that A_{22} is Hurwitz, yielding that $(A_\xi, B_\xi, C_\xi, D_\xi)$ is well-defined for all $\xi \in \mathbb{C}_0 \cup i\mathbb{R}$. Suppose first that $\xi \in \mathbb{C}_0$. Straightforward algebraic manipulation using the definition of $(A_\xi, B_\xi, C_\xi, D_\xi)$ in (2.5), the decomposition (2.6) and the equations (2.2) shows that the following Lyapunov inequalities

$$A_\xi \Sigma_1 + \Sigma_1 A_\xi^* + B_\xi B_\xi^* = -2\mathrm{Re}(\xi) A_{12}(\xi I - A_{22})^{-1}\Sigma_2(\xi I - A_{22})^{-*}A_{12}^* \leq 0\,, \quad \text{(A.16)}$$

and

$$A_\xi^* \Sigma_1 + \Sigma_1 A_\xi + C_\xi^* C_\xi = -2\mathrm{Re}(\xi) A_{21}^*(\xi I - A_{22})^{-*}\Sigma_2(\xi I - A_{22})^{-1}A_{21} \leq 0\,. \quad \text{(A.17)}$$

hold. If $\xi \in i\mathbb{R}$, then it follows immediately from inspection of (A.16) and (A.17) that (A_ξ, B_ξ, C_ξ) is balanced, proving statement (ii).

We prove statement (i) first assuming that $\xi \in \mathbb{C}_0$. Inequality (A.17) implies that every eigenvalue of A_ξ has non-positive real part. Suppose that $A_\xi v = \eta i\, v$ for some $\eta \in \mathbb{R}$ and $v \in \mathbb{C}^r$. Forming the inner product

$$\langle (A_\xi^* \Sigma_1 + \Sigma_1 A_\xi + C_\xi^* C_\xi) v, v\rangle\,,$$

and using (A.17), it follows that

$$0 \leq \|C_\xi v\|^2 = -2\mathrm{Re}(\xi)\langle \Sigma_2(\xi I - A_{22})^{-1}A_{21}v, (\xi I - A_{22})^{-1}A_{21}v\rangle \leq 0\,,$$

whence

$$\langle \Sigma_2(\xi I - A_{22})^{-1}A_{21}v, (\xi I - A_{22})^{-1}A_{21}v\rangle = 0\,,$$

as $\mathrm{Re}\,(\xi) > 0$. Since $\Sigma_2 > 0$, we infer that

$$(\xi I - A_{22})^{-1}A_{21}v = 0\,.$$

Consequently

$$A\begin{pmatrix} v \\ 0 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} v \\ (\xi I - A_{22})^{-1}A_{21}v \end{pmatrix} = \begin{pmatrix} A_\xi v \\ \xi(\xi I - A_{22})^{-1}A_{21}v \end{pmatrix} = \eta i \begin{pmatrix} v \\ 0 \end{pmatrix}\,,$$

and, as A is Hurwitz, we deduce that $v = 0$. Recalling our supposition that $A_\xi v = \eta i\, v$, we conclude that A_ξ is Hurwitz as well.

For observability, let $\lambda \in \mathbb{C}$ and $v \in \mathbb{C}^n$ be such that $A_\xi v = \lambda v$ and $C_\xi v = 0$. Note that

$$\begin{aligned} A\begin{pmatrix} v \\ (\xi I - A_{22})^{-1}A_{21}v \end{pmatrix} &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} v \\ (\xi I - A_{22})^{-1}A_{21}v \end{pmatrix} \\ &= \begin{pmatrix} A_\xi v \\ \xi(\xi I - A_{22})^{-1}A_{21}v \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \xi \end{pmatrix}\begin{pmatrix} v \\ (\xi I - A_{22})^{-1}A_{21}v \end{pmatrix}\,, \end{aligned}$$

so that

$$Az = Ez\,,$$

where

$$E := \begin{pmatrix} \lambda & 0 \\ 0 & \xi \end{pmatrix} \quad \text{and} \quad z := \begin{pmatrix} v \\ (\xi I - A_{22})^{-1}A_{21}v \end{pmatrix}\,.$$

We conclude that

$$\|e^{At}z\|^2 = \|e^{Et}z\|^2 = \left\| \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\xi t} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right\|^2 < \|z\|^2 \quad \forall\, t > 0\,,$$

by [40, Theorem 3.1] applied to the balanced realisation (A, B, C), so that

$$e^{2\mathrm{Re}(\lambda)t}\|z_1\|^2 + e^{2\mathrm{Re}(\xi)t}\|z_2\|^2 < \|z_1\|^2 + \|z_2\|^2 \quad \forall\, t > 0\,.$$

Since $\xi \in \mathbb{C}_0$, it follows that

$$z_2 = (\xi I - A_{22})^{-1}A_{21}v = 0\,,$$

from which

$$\lambda v = A_\xi v = A_{11}v + A_{12}(\xi I - A_{22})^{-1}A_{21}v = A_{11}v$$

and

$$0 = C_\xi v = C_1 v + C_2(\xi I - A_{22})^{-1}A_{21}v = C_1 v\,.$$

The pair (C_1, A_{11}) is observable, and so we deduce that $v = 0$, proving that (C_ξ, A_ξ) is observable. The proof that (A_ξ, B_ξ) is controllable is similar, using instead that (A_{11}, B_1) is controllable, and so is omitted.

We now consider the situation wherein $\xi \in i\mathbb{R}$. Statement (1) of Lemma 6.2 yields that $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is minimal and it is easily shown that $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ satisfies the Lyapunov inequalities

$$\mathcal{A}\Sigma + \Sigma\mathcal{A}^* + \mathcal{B}\mathcal{B}^* = -2\mathrm{Re}(\xi)\mathcal{A}\Sigma\mathcal{A}^* \le 0\,, \tag{A.18}$$

and

$$\mathcal{A}^*\Sigma + \Sigma\mathcal{A} + \mathcal{C}^*\mathcal{C} = -2\mathrm{Re}(\xi)\mathcal{A}^*\Sigma\mathcal{A} \le 0\,. \tag{A.19}$$

Since $\mathrm{Re}(\xi) = 0$, these simplify to the Lyapunov equations

$$\mathcal{A}\Sigma + \Sigma\mathcal{A}^* + \mathcal{B}\mathcal{B}^* = 0 \quad \text{and} \quad \mathcal{A}^*\Sigma + \Sigma\mathcal{A} + \mathcal{C}^*\mathcal{C} = 0\,. \tag{A.20}$$

Note that (A.20) implies that $\mathcal{A}$ is Hurwitz and $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is balanced. From usual balanced truncation theory [40, Theorem 3.2, Corollary 2], we see that $\mathcal{A}_{11}$ is Hurwitz and $(\mathcal{A}_{11}, \mathcal{B}_1, \mathcal{C}_1)$ is minimal. In particular, it is here where we have used that the singular values are simple, implying that the spectra of Σ_1 and Σ_2 are disjoint.

Next, by statement (2) of Lemma 6.2, $\xi \notin \sigma(A_\xi)$, as A is Hurwitz and the equalities in (6.6) hold. From these and the minimality of $(\mathcal{A}_{11}, \mathcal{B}_1, \mathcal{C}_1)$ it follows that (A_ξ, B_ξ, C_ξ) is minimal. The Lyapunov equation (A.17) now shows that A_ξ is Hurwitz. $\square$

Proof of Theorem 2.4: Let (A, B, C, D) denote a minimal, balanced, stable, realisation of $\mathbf{G}$ which, by Theorem 2.3, implies that $(A_\xi, B_\xi, C_\xi, D_\xi)$ is well-defined for all $\xi \in \mathbb{C}_0 \cup i\mathbb{R}$. Further, A_ξ is Hurwitz. Let $\mathbf{G}_r^\xi$, $\mathbf{H}$ and $\mathbf{H}_r$ be defined as in Lemma 6.3. With these choices, we first assume that $\xi \in \mathbb{C}_0$.

Invoking statement (3) of Lemma 6.2 to $\mathcal{A}$ and the first equality in (6.6) implies that

$$\sigma(\mathcal{A}), \sigma(\mathcal{A}_{11}) \subseteq \mathbb{E}_\xi\,. \tag{A.21}$$

The error bound (2.9) now follows from subtracting (6.11) from (6.10) in Lemma 6.3 and an application of Lemma A.1. In the former result we are using that the map

$$i\,\mathbb{R} \cup \{\infty\} \ni z \mapsto \frac{1}{z - w}\,,$$

a bijection onto $\partial\mathbb{E}_\xi$, where $\mathbb{E}_\xi$ is given by (6.7) and we see from (A.21) that $\mathbf{H}$ and $\mathbf{H}_r$ are well-defined on $\partial\mathbb{E}_\xi$, respectively. In the latter result we take (A, B, C, D)

equal to $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$. Note that the equalities in (A.18) and (A.19) imply that the inequalities (A.1) and (A.2) respectively hold. That assumption (A.3) holds follows from (6.6), as every partition in (2.6) gives rise to a Hurwitz A_ξ, by Theorem 2.3.

If $\xi \in i\mathbb{R}$, then the result follows from the error bound (A.5), also in Lemma A.1. Here we have applied statement (3) of Lemma 6.2 to the first equality in (6.6) to infer that $\mathcal{A}_{11}$ is Hurwitz. $\square$

Appendix B. **Derivation of** (6.24)**.** Considering (6.17) block wise, we have that

$$\mathcal{A}_{11}^*\Sigma_1 + \Sigma_1\mathcal{A}_{11} + \mathcal{C}_1^*\mathcal{C}_1 = -\mathcal{K}_1^*\mathcal{K}_1 - q^2(\mathcal{A}_{11}^*\Sigma_1\mathcal{A}_{11} + \mathcal{A}_{21}^*\Sigma_2\mathcal{A}_{21}), \tag{B.1}$$

and

$$\Sigma_1\mathcal{B}_1 - \mathcal{C}_1^*\mathcal{D} = \mathcal{K}_1^*\mathcal{W} - q^2(\mathcal{A}_{11}^*\Sigma_1\mathcal{B}_1 + \mathcal{A}_{21}^*\Sigma_2\mathcal{B}_2) \tag{B.2}$$

Given $p \in \partial\mathbb{E}_\xi$, for notational convenience set $\Gamma := (pI - \mathcal{A}_{11})$ and let

$$\mathcal{I}_1 = \mathcal{A}_{11}^*\Sigma_1\mathcal{A}_{11} + \mathcal{A}_{21}^*\Sigma_2\mathcal{A}_{21}, \quad \mathcal{I}_2 := \mathcal{A}_{11}^*\Sigma_1\mathcal{B}_1 + \mathcal{A}_{21}^*\Sigma_2\mathcal{B}_2\,.$$

Using (B.1) and (B.2), we compute that

$$\begin{aligned}
I - [\mathbf{H}_r^\xi(p)]^*\mathbf{H}_r^\xi(p) &= I - (\mathcal{D} - \mathcal{C}_1(pI - \mathcal{A}_{11})^{-1}\mathcal{B}_1)^*(\mathcal{D} - \mathcal{C}_1(pI - \mathcal{A}_{11})^{-1}\mathcal{B}_1) \\
&= I - (\mathcal{D} - \mathcal{C}_1\Gamma^{-1}\mathcal{B}_1)^*(\mathcal{D} - \mathcal{C}_1\Gamma^{-1}\mathcal{B}_1) \\
&= I - \mathcal{D}^*\mathcal{D} + \mathcal{B}_1^*\Gamma^{-*}\mathcal{C}_1^*\mathcal{D} + \mathcal{D}^*\mathcal{C}_1\Gamma^{-1}\mathcal{B}_1 - \mathcal{B}_1^*\Gamma^{-*}\mathcal{C}_1^*\mathcal{C}_1\Gamma^{-1}\mathcal{B}_1 \\
&= \mathcal{W}^*\mathcal{W} + q^2\mathcal{B}_1^*\Sigma_1\mathcal{B}_1 + q^2\mathcal{B}_2^*\Sigma_2\mathcal{B}_2 \\
&\quad + \mathcal{B}_1^*\Gamma^{-*}(\Sigma_1\mathcal{B}_1 - \mathcal{K}_1^*\mathcal{W} + q^2\mathcal{I}_2) \\
&\quad + (\mathcal{B}_1^*\Sigma_1 - \mathcal{W}^*\mathcal{K}_1 + q^2\mathcal{I}_2^*)\Gamma^{-1}\mathcal{B}_1 \\
&\quad + \mathcal{B}_1^*\Gamma^{-*}(\mathcal{A}_{11}^*\Sigma_1 + \Sigma_1\mathcal{A}_{11} + \mathcal{K}_1^*\mathcal{K}_1 + q^2\mathcal{I}_1)\Gamma^{-1}\mathcal{B}_1 \\
&= (\mathcal{W} - \mathcal{K}_1\Gamma^{-1}\mathcal{B}_1)^*(\mathcal{W} - \mathcal{K}_1\Gamma^{-1}\mathcal{B}_1) + \mathcal{R}\,,
\end{aligned} \tag{B.3}$$

where

$$\begin{aligned}
\mathcal{R} &:= q^2\mathcal{B}_2^*\Sigma_2\mathcal{B}_2 + q^2\mathcal{B}_1^*\Gamma^{-*}\mathcal{I}_2 + q^2\mathcal{I}_2^*\Gamma^{-1}\mathcal{B}_1 \\
&\quad + \mathcal{B}_1^*\Gamma^{-*}(q^2\Gamma^*\Sigma_1\Gamma + \Sigma_1\Gamma + \Gamma^*\Sigma_1 + \mathcal{A}_{11}^*\Sigma_1 + \Sigma_1\mathcal{A}_{11} + q^2\mathcal{I}_1)\Gamma^{-1}\mathcal{B}_1 \\
&= q^2\big[\mathcal{B}_2^*\Sigma_2\mathcal{B}_2 + \mathcal{B}_1^*\Gamma^{-*}(\mathcal{A}_{11}^*\Sigma_1\mathcal{B}_1 + \mathcal{A}_{21}^*\Sigma_2\mathcal{B}_2) \\
&\quad + (\mathcal{B}_1^*\Sigma_1\mathcal{A}_{11} + \mathcal{B}_2^*\Sigma_2\mathcal{A}_{21})\Gamma^{-1}\mathcal{B}_1\big] \\
&\quad + \mathcal{B}_1^*\Gamma^{-*}(q^2\Gamma^*\Sigma_1\Gamma + 2\mathrm{Re}(p)\Sigma_1 + q^2(\mathcal{A}_{11}^*\Sigma_1\mathcal{A}_{11} + \mathcal{A}_{21}^*\Sigma_2\mathcal{A}_{21}))\Gamma^{-1}\mathcal{B}_1 \\
&= q^2(\mathcal{B}_2 + \mathcal{A}_{21}\Gamma^{-1}\mathcal{B}_1)^*\Sigma_2(\mathcal{B}_2 + \mathcal{A}_{21}\Gamma\mathcal{B}_1) \\
&\quad + \mathcal{B}_1^*\Gamma^{-*}(q^2(\Gamma^*\Sigma_1\Gamma + \mathcal{A}_{11}^*\Sigma_1\mathcal{A}_{11} + \Gamma^*\Sigma_1\mathcal{A}_{11} + \mathcal{A}_{11}^*\Sigma_1\Gamma) + 2\mathrm{Re}(p)\Sigma_1)\Gamma^{-1}\mathcal{B}_1 \\
&= q^2(\mathcal{B}_2 + \mathcal{A}_{21}\Gamma^{-1}\mathcal{B}_1)^*\Sigma_2(\mathcal{B}_2 + \mathcal{A}_{21}\Gamma\mathcal{B}_1) \\
&\quad + 2\mathcal{B}_1^*\Gamma^{-*}(\mathrm{Re}(p) + \mathrm{Re}(\xi)|p|^2)\Sigma_1\Gamma^{-1}\mathcal{B}_1 \\
&= q^2(\mathcal{B}_2 + \mathcal{A}_{21}\Gamma^{-1}\mathcal{B}_1)^*\Sigma_2(\mathcal{B}_2 + \mathcal{A}_{21}\Gamma\mathcal{B}_1)\,.
\end{aligned} \tag{B.4}$$

In the final equality above we have used that $p \in \partial\mathbb{E}_\xi$ and (6.9). Combining (B.3) and (B.4) gives (6.24), as required.

Appendix C. **Derivation of** (6.29)**.** The arguments are identical in spirit to those used in Appendix B. Given $p \in \partial\mathbb{E}_\xi$, for notational convenience set $\Theta := (pI - \mathcal{A})$. Using (6.17), we compute that

$$\begin{aligned}
I - [\mathbf{H}(p)]^*\mathbf{H}(p) &= I - (\mathcal{D} - \mathcal{C}(pI - \mathcal{A})^{-1}\mathcal{B})^*(\mathcal{D} - \mathcal{C}(pI - \mathcal{A})^{-1}\mathcal{B}) \\
&= I - (\mathcal{D} - \mathcal{C}\Theta^{-1}\mathcal{B})^*(\mathcal{D} - \mathcal{C}\Theta^{-1}\mathcal{B}) \\
&= I - \mathcal{D}^*\mathcal{D} + \mathcal{B}^*\Theta^{-*}\mathcal{C}^*\mathcal{D} + \mathcal{D}^*\mathcal{C}\Theta^{-1}\mathcal{B} - \mathcal{B}^*\Theta^{-*}\mathcal{C}^*\mathcal{C}\Theta^{-1}\mathcal{B} \\
&= \mathcal{W}^*\mathcal{W} + q^2\mathcal{B}^*\Sigma\mathcal{B} + \mathcal{B}^*\Theta^{-*}(\Sigma\mathcal{B} - \mathcal{K}^*\mathcal{W} + q^2\mathcal{A}^*\Sigma\mathcal{B}) \\
&\quad + (\mathcal{B}^*\Sigma - \mathcal{W}^*\mathcal{K} + q^2\mathcal{B}^*\Sigma\mathcal{A})\Theta^{-1}\mathcal{B} \\
&\quad + \mathcal{B}^*\Theta^{-*}(\mathcal{A}^*\Sigma + \Sigma\mathcal{A} + \mathcal{K}^*\mathcal{K} + q^2\mathcal{A}^*\Sigma\mathcal{A})\Theta^{-1}\mathcal{B} \\
&= (\mathcal{W} - \mathcal{K}\Theta^{-1}\mathcal{B})^*(\mathcal{W} - \mathcal{K}\Theta^{-1}\mathcal{B}) + \mathcal{S}\,.
\end{aligned} \tag{C.1}$$

Here

$$\begin{aligned}
\mathcal{S} :&= \mathcal{B}^*\Theta^{-*}(q^2(\Theta^*\Sigma\Theta + \mathcal{A}^*\Sigma\mathcal{A} + \mathcal{A}^*\Sigma\Theta + \Theta^*\Sigma\mathcal{A}) + 2\mathrm{Re}(p)\Sigma)\Theta^{-1}\mathcal{B} \\
&= 2\mathcal{B}^*\Theta^{-*}(\mathrm{Re}(p) + \mathrm{Re}(\xi)|p|^2)\Sigma\Theta^{-1}\mathcal{B} \\
&= 0\,.
\end{aligned} \tag{C.2}$$

In the final equality above we have used that $p \in \partial\mathbb{E}_\xi$ and (6.9). Combining (C.1) and (C.2) gives (6.29), as required.

Appendix D. **Proof of Lemma 6.5.** The proof is by direct calculation. For notation convenience, set $\Psi := (\xi I - A_{22})^{-1}$, $\Phi := (I + D)^{-1}$ and

$$X_B := B_2\Phi, \quad X_C := C_2\Psi, \quad N := (I + X_C X_B)^{-1}, \quad M := (I + X_B X_C)^{-1}\,. \tag{D.1}$$

Note that M and N are well-defined by our assumption that all the terms which appear in the commuting diagram are. Straightforward calculations show that

$$N = I - X_C X_B N, \quad X_B N = M X_B, \quad \text{and} \quad X_C M = N X_C\,. \tag{D.2}$$

Using the definitions in (2.5), (6.39) and (D.1) and the properties (D.2), we have that

$$\begin{aligned}
\widetilde{(A_\xi)} &= A_\xi - B_\xi(I + D_\xi)^{-1}C_\xi \\
&= A_\xi - (B_1 + A_{12}\Psi B_2)(I + D + C_2\Psi B_2)^{-1}(C_1 + C_2\Psi A_{21}) \\
&= A_\xi - (B_1\Phi + A_{12}\Psi B_2\Phi)(I + C_2\Psi B_2\Phi)^{-1}(C_1 + C_2\Psi A_{21}) \\
&= A_\xi - (B_1\Phi + A_{12}\Psi X_B)N(C_1 + X_C A_{21}) \\
&= A_\xi - (B_1\Phi + A_{12}\Psi X_B)(I - X_C X_B N)(C_1 + X_C A_{21})\,.
\end{aligned} \tag{D.3}$$

Similarly

$$\begin{aligned}
(\tilde{A})_\xi &= (\tilde{A})_{11} + (\tilde{A})_{12}(\xi I - (\tilde{A})_{22})(\tilde{A})_{21} \\
&= (A - B\Phi C)_{11} + (A - B\Phi C)_{12}(\xi I - (A - B\Phi C)_{22})^{-1}(A - B\Phi C)_{21} \\
&= A_{11} - B_1\Phi C_1 + (A_{12} - B_1\Phi C_2)(\xi I - A_{22} + B_2\Phi C_2)^{-1}(A_{21} - B_2\Phi C_1) \\
&= A_{11} - B_1\Phi C_1 + (A_{12}\Psi - B_1\Phi C_2\Psi)(I + B_2\Phi C_2\Psi)^{-1}(A_{21} - B_2\Phi C_1) \\
&= A_{11} - B_1\Phi C_1 + (A_{12}\Psi - B_1\Phi X_C)M(A_{21} - X_B C_1)\,.
\end{aligned} \tag{D.4}$$

Inspection of (D.3) and (D.4) reveals that they are equal. Next, we compute that

$$\begin{aligned}
\frac{1}{\sqrt{2}}\widetilde{(B_\xi)} &= B_\xi(I+D_\xi)^{-1} = (B_1 + A_{12}\Psi B_2)(I + D + C_2\Psi B_2)^{-1} \\
&= (B_1\Phi + A_{12}\Psi B_2\Phi)(I + C_2\Psi B_2\Phi)^{-1} = (B_1\Phi + A_{12}\Psi X_B)N \\
&= B_1\Phi + (A_{12}\Psi - B_1\Phi X_C)MX_B \\
&= B_1\Phi + (A_{12}\Psi - B_1\Phi C_2\Psi)(I + B_2\Phi C_2\Psi)^{-1}X_B \\
&= B_1\Phi + (A_{12} - B_1\Phi C_2)(\xi I - A_{22} + B_2\Phi C_2)^{-1}B_2\Phi \\
&= \frac{1}{\sqrt{2}}\big((\tilde{B})_1 + (\tilde{A})_{12}(\xi I - (\tilde{A})_{22})^{-1}(\tilde{B})_2\big) = \frac{1}{\sqrt{2}}(\tilde{B})_\xi\,.
\end{aligned}$$

Further,

$$\begin{aligned}
-\frac{1}{\sqrt{2}}\widetilde{(C_\xi)} &= (I+D_\xi)^{-1}C_\xi = (I + D + C_2\Psi B_2)^{-1}(C_1 + C_2\Psi A_{21}) \\
&= \Phi(I + C_2\Psi B_2\Phi)^{-1}(C_1 + C_2\Psi A_{21}) = \Phi N(C_1 + X_C A_{21}) \\
&= \Phi C_1 + \Phi X_C M(A_{21} - X_B C_1) \\
&= \Phi C_1 + \Phi C_2\Psi(I + B_2\Phi C_2\Psi)^{-1}(A_{21} - B_2\Phi C_1) \\
&= \Phi C_1 + \Phi C_2(\xi I - A_{22} + B_2\Phi C_2)^{-1}(A_{21} - B_2\Phi C_1) \\
&= -\frac{1}{\sqrt{2}}\big((\tilde{C})_1 + (\tilde{C})_2(\xi I - (\tilde{A})_{22})^{-1}(\tilde{A})_{21}\big) = -\frac{1}{\sqrt{2}}(\tilde{C})_\xi\,.
\end{aligned}$$

Finally,

$$\begin{aligned}
\widetilde{(D_\xi)} &= (I - D_\xi)(I + D_\xi)^{-1} = (I - D - C_2\Psi B_2)(I + D + C_2\Psi B_2)^{-1} \\
&= ((I - D)\Phi - C_2\Psi B_2\Phi)(I + C_2\Psi B_2\Phi)^{-1} = (\tilde{D} - X_C X_B)N \\
&= \tilde{D} - 2\Phi X_C M X_B \qquad \text{(D.5)} \\
&= \tilde{D} - 2\Phi C_2\Psi(I + B_2\Phi C_2\Psi)^{-1}B_2\Phi \\
&= \tilde{D} - 2\Phi C_2(\xi I - A_{22} + B_2\Phi C_2)^{-1}B_2\Phi = \tilde{D} + (\tilde{C})_2(\xi I - (\tilde{A})_{22})^{-1}(\tilde{B})_2 \\
&= (\tilde{D})_\xi\,.
\end{aligned}$$

To establish (D.5) we used that

$$\tilde{D} - \tilde{D}N + X_C X_B N - 2\Phi X_C M X_B = 0\,.$$

The proof is complete. □

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