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On the Procedure for the Series Solution of Second-Order Homogeneous Linear Differential Equation via the Complex Integration Method

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Abstract

The theory of series solutions for second-order linear homogeneous ordinary differential equation is developed *ab initio*, using an elementary complex integral expression (based on Herrera' work [3]) derived and applied in previous papers [8, 9]. As well as reproducing the usual expression for the recurrence relations for second-order equations, the general solution method is straight-forward to apply as an algorithm on its own, with the integral algorithm replacing the manipulation of power series by reducing the task of finding a series solution for second-order equations to the solution, instead, of a system of uncoupled simple equations in a single unknown. The integral algorithm also simplifies the construction of 'logarithmic solutions' to second-order Fuchs, equations. Examples, from the general science and mathematics literature, are presented throughout.

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1. Introduction

In this paper, we continue the project [8, 9] of developing power series and Frobenius series solutions of ordinary differential equations (ODE) using a particular complex integration procedure. As before we find that the technique reduces the solution of the original ODE, through a complex integral transformation, to a system of simple equations for the indices of the series coefficients that define the series recurrence relation. In fact, as well as presenting further examples of the technique, we apply the solution methodology to the general or abstract second-order linear homogeneous ODE (all suffixes below in brackets represent differentiation with respect to z)

$$f^{(2)}(z)+P(z)f^{(1)}(z)+Q(z)f^{(0)}(z)=0 \tag{1.1}$$

In the case of (1.1) being a Fuchs' equation, we assume that [10]

$$p(z)=(z-z_0)P(z) \quad \text{and} \quad q(z)=(z-z_0)^2Q(z) \tag{1.2}$$

are *analytic* functions of the independent variable z, with z_0 being a regular singular point of (1.1). In other words, we assume that

$$P(z)=\sum_{i=0}^{\infty}p_i(z-z_0)^{i-1} \quad \text{and} \quad Q(z)=\sum_{i=0}^{\infty}q_i(z-z_0)^{i-2} \tag{1.3}$$

for given constants $\{p_i\}_{i=0}^{\infty}$ and $\{q_i\}_{i=0}^{\infty}$. In this case, when (1.1) is a Fuchs' equation, we seek a Frobenius series solution of (1.1) about z_0, that is [10]

$$f(z)=\sum_{m=0}^{\infty}a_m(z-z_0)^{m+r} \tag{1.4}$$

when the series coefficients, $\{a_m\}_{m=0}^{\infty}$, are determined via the contour integral [9]

$$a_m=\frac{1}{[m+r]_k 2\pi\hat{i}}\oint_C \frac{f^{(k)}(z)}{(z-z_0)^{m+r-k+1}}dz \tag{1.5}$$

where $\hat{i}^2=-1$ and, following the notation of Ince [6], for positive integers k

$$[m+r]_k=(m+r)(m+r-1)(m+r-2)\cdots(m+r-k+1)=\frac{\Gamma(m+r+1)}{\Gamma(m+r-k+1)} \tag{1.6}$$

while $[m+r]_0=1$.

If z_0 is an ordinary point of (1.1), then $P(z)$ and $Q(z)$ are analytic functions at z_0 and we seek a power series solution of an ODE about z_0, that is [10]

$$f(z) = \sum_{m=0}^{\infty} a_m (z - z_0)^m \tag{1.7}$$

In this case the series coefficients, $\{a_m\}_{m=0}^{\infty}$, are, from (1.5) and (1.6) with $r = 0$, determined via [8]

$$a_m = \frac{(m-k)!}{m!\,2\pi\hat{i}} \oint_C \frac{f^{(k)}(z)}{(z - z_0)^{m-k+1}} dz \tag{1.8}$$

It is to be noted that in both (1.5) and (1.8) the contour C is a closed contour, taken in the positive or anti-clockwise direction, encircling the point z_0, but avoiding any (other) singularities. Further, regardless of the type of ODE (1.1) represents, we will consider (1.1) solved once the recurrence relation for the series coefficients, $\{a_m\}_{m=0}^{\infty}$, is obtained. Finally, it is to be emphasised that the general methodology presented below, in sections 2,3 and 4, whereby (1.1) is solved, yields not only the general series solutions to (1.1), but also 'expresses' itself as an algorithm for the solution of arbitrary ODE of the form of (1.1). This is made clearer through the examples.

The paper is organized as follows. In section 2 we solve (1.1) for the case of z_0 being an ordinary point and present the general series solution for $P(z)$ and $Q(z)$ arbitrary analytic functions; the algorithm is then exemplified using two problems requiring the solution of the Schrödinger equation [1, 11]. In section 3 we solve (1.1) for the case of z_0 being a regular singular point and present the general Frobenius series solution; the algorithm is then exemplified, again, via a problem requiring the solution of the Schrödinger equation [5]. Next, in section 4, we examine the case where the second solution of (1.1), when z_0 is a regular singular point, does *not* yield a second Frobenius series; this time the algorithm is applied to the solution of the extended confluent hypergeometric equation [2].

Note that the examples in sections 2, 3 and 4 are solved using the complex integral method as an algorithm; the general formulae ((2.5) and (3.5)/(3.6)) may also be used, but here are (implicitly) applied to provide a check on the results of using the complex integral algorithm directly.

2. Series Solutions about an Ordinary Point

We consider, first, in this section the case where $P(z)$ and $Q(z)$ are analytic at the point z_0 so that we may write

$$P(z)=\sum_{i=0}^{\infty}p_i(z-z_0)^i \quad \text{and} \quad Q(z)=\sum_{i=0}^{\infty}q_i(z-z_0)^i \tag{2.1}$$

for given constants $\{p_i\}_{i=0}^{\infty}$ and $\{q_i\}_{i=0}^{\infty}$. In this case, we look for a power series solution of (1.1) of the form (1.7), with the coefficients given by (1.8). The procedure for obtaining the recurrence relation for the coefficients (1.8) of the series solution (1.7) is as follows. First, we substitute $P(z)$ and $Q(z)$ from (2.1) into (1.1) and then divide through by $(z-z_0)^{n+1}$ to get

$$\frac{f^{(2)}(z)}{(z-z_0)^{n-1}}+\sum_{i=0}^{\infty}p_i\frac{f^{(1)}(z)}{(z-z_0)^{n-i-1}}+\sum_{i=0}^{\infty}q_i\frac{f^{(0)}(z)}{(z-z_0)^{n-i-1}}=0 \tag{2.2}$$

We now integrate through (2.2), with the integral being a contour integral round a closed-path, C, taken anti-clockwise and containing z_0 while avoiding any singularities of $f^{(0)}(z)$; this procedure leads to

$$\oint_C\frac{f^{(2)}(z)}{(z-z_0)^{n+1}}dz+\sum_{i=0}^{\infty}p_i\oint_C\frac{f^{(1)}(z)}{(z-z_0)^{n-i+1}}dz+\sum_{i=0}^{\infty}q_i\oint_C\frac{f^{(0)}(z)}{(z-z_0)^{n-i+1}}dz=0 \tag{2.3}$$

Next, we compare the denominators in (2.3) with that of (1.8), term by term, to get three equations for the dummy index m in terms of the dummy index n, one for each of the values of k (the order of the derivative of $f(z)$) in each integral, (two, one and zero, respectively). So, we find that (2.3) yields the three equations

$$m-k+1=m-2+1=n+1\Leftrightarrow m=n+2 \tag{2.4a}$$
$$m-k+1=m-1+1=n-i+1\Leftrightarrow m=n-i+1 \tag{2.4b}$$
$$m-k+1=m-0+1=n-i+1\Leftrightarrow m=n-i \tag{2.4c}$$

Utilizing (1.7) and (1.8) again, with the results of (2.4) in hand, we see that equation (2.3) transforms, term by term, into

$$(n+2)(n+1)a_{n+2}+\sum_{i=0}^{n}[(n-i+1)p_ia_{n-i+1}+q_ia_{n-i}]=0 \tag{2.5a}$$

after cancellation and recalling that $a_m=0,\, m<0$.

For comparative purposes, we change the dummy variable i in the sum in (2.5a) to $n-k$, when we get

$$(n+2)(n+1)a_{n+2}+\sum_{k=0}^{n}[(k+1)p_{n-k}a_{k+1}+q_{n-k}a_k]=0 \tag{2.5b}$$

which is just equation (19) of chapter 5, section 27, of Simmons [10]. The recurrence relation (2.5) is the ‘in principle’ series solution to (1.1).

Before presenting a couple of examples involving the Schrodinger equation, we pause to point out that the index of the divisor, $(z-z_0)^{n+1}$ above, sets thevalues of the subscripts in the recurrence relation for the series coefficients. To vary the value of the subscripts, we simply vary the value of the index of the divisor. So, for example, if we had divided through (1.1) by $(z-z_0)^{n-1}$ instead of $(z-z_0)^{n+1}$, then the overall effect would have been to subtract two from n wherever it occurred in (2.5). Alternatively, we could just subtract two from n wherever it occurred in (2.5) directly, which amounts to the same thing. The choice of $(z-z_0)^{n+1}$ was made so that the recurrence relation (2.5) would be exactly that of Simmons’ [10] and a direct comparison made possible without further manipulation (see below).

We look, now, at a couple of example of the series solution of second-order linear homogeneous ODE about an ordinary point. Instead of fitting the given equations into the straight-jacket of the formulae given above, we use the basic method used in the derivation of (2.5) as an (integral) algorithm. This is, of course, the usual way that series solutions of second-order linear homogeneous ODE are normally discovered. As a first example, we consider the following Schrödinger equation (Alhendi and Lashin [1])

$$f^{(2)}(z)+[E-(\mu^2z^2+gz^{2l})]f^{(0)}(z)=0 \tag{2.6}$$

with E,μ,g and l independent of z. We seek a basic power series solution of (2.6) about the origin ($z_0=0$), so we divide through (2.6) by z^{n+1} and then integrate through the resultant (in the same manner as before) to get

$$\oint_C \frac{f^{(2)}(z)}{z^{n+1}}dz+E\oint_C \frac{f^{(0)}(z)}{z^{n+1}}dz-\mu^2\oint_C \frac{f^{(0)}(z)}{z^{n-1}}dz-g\oint_C \frac{f^{(0)}(z)}{z^{n-2l+1}}dz=0 \tag{2.7}$$

Next, we compare the denominators in (2.7) with that of (1.8), term by term, to get four equations for the dummy index m in terms of the dummy index n, one for each of the values of k (the order of the derivative of $f(z)$) in each integral, (two, zero, zero and zero, respectively). So, we find that (2.7) yields the four equations

$$m-k+1=m-2+1=n+1 \Leftrightarrow m=n+2 \tag{2.8a}$$
$$m-k+1=m-0+1=n+1 \Leftrightarrow m=n \tag{2.8b}$$
$$m-k+1=m-0+1=n-1 \Leftrightarrow m=n-2 \tag{2.8c}$$

$$m-k+1=m-0+1=n-2l+1 \Leftrightarrow m=n-2l \tag{2.8d}$$

Utilizing (1.7) and (1.8) again, with the results of (2.8) in hand, we see that equation (2.7) transforms, term by term, into (after cancellation)

$$(n+2)(n+1)a_{n+2}+Ea_n-\mu^2 a_{n-2}-ga_{n-2l}=0 \tag{2.9a}$$

which, with a switch of subscript ($n+2 \to n$, etc.) reduces to Alhendi's [1] recurrence formula, that is

$$n(n-1)a_n+Ea_{n-2}-\mu^2 a_{n-4}-ga_{n-2l-2}=0 \tag{2.9b}$$

As a second example, we take the Schrödinger equation solved by Taşeli [11]

$$\Phi^{(2)}(z)+\frac{2\ell+1}{z}\Phi^{(1)}+[E-z^2]\Phi^{(0)}(z)=0 \tag{2.10}$$

with E and ℓ independent of z. For comparative purposes, we follow Taşeli [11] and make the standard transformation of (2.10), that is we set

$$\Phi^{(0)}(z)=e^{-\frac{z^2}{2}}f^{(0)}(z) \tag{2.11}$$

and (2.10) is transformed into the following equation for $f^{(0)}(z)$, that is

$$zf^{(2)}(z)+(2\ell+1)f^{(1)}(z)-2z^2 f^{(1)}(z)+[E-2(\ell+1)]zf^{(0)}(z)=0 \tag{2.12}$$

As before, we seek a basic power series solution of (2.12) about the origin, so we divide through (2.12) by z^{n+1} and then integrate through the resultant, to get

$$\oint_C \frac{f^{(2)}(z)}{z^n}dz+(2\ell+1)\oint_C \frac{f^{(1)}(z)}{z^{n+1}}dz-2\oint_C \frac{f^{(1)}(z)}{z^{n-1}}dz$$
$$+[E-2(\ell+1)]\oint_C \frac{f^{(0)}(z)}{z^n}dz=0 \tag{2.13}$$

Following our previous procedure, we get from (2.13) four equations (again) for m, one for each value of k (two, one, one and zero, respectively)

$$m-k+1=m-2+1=n \Leftrightarrow m=n+1 \tag{2.14a}$$
$$m-k+1=m-1+1=n+1 \Leftrightarrow m=n+1 \tag{2.14b}$$
$$m-k+1=m-1+1=n-1 \Leftrightarrow m=n-1 \tag{2.14c}$$
$$m-k+1=m-0+1=n \Leftrightarrow m=n-1 \tag{2.14d}$$

Utilizing (1.7) and (1.8) again, with the results of (2.14) in hand, we see that equation (2.13) transforms, term by term, into (after cancellation and re-arrangement)

$$(n+1)(n+2\ell+1)a_{n+1}+[E-2(n+\ell)]a_{n-1}=0 \tag{2.15}$$

Finally, setting $n=2s+1,$ (2.15) may be rewritten as

$$a_{2s+2}=-\frac{E-2(2s+\ell+1)}{4(s+1)(s+\ell+1)}a_{2s} \tag{2.16}$$

It is apparent that the series determined by the recurrence relation (2.16) will terminate whenever the eigenvalues are determined by [11]

$$E=E_s=2(2s+\ell+1) \tag{2.17}$$

3. Series Solutions About a Regular Singular Point

In this section we consider the case where (1.1) represents the general second-order linear homogeneous Fuchs equation. On seeking a Frobenius series solution (1.4) of (1.1), we substitute $P(z)$ and $Q(z)$ from (1.3) into (1.1) and then divide through by $(z-z_0)^{n+r-1}$ to get

$$\frac{f^{(2)}(z)}{(z-z_0)^{n+r-1}}+\sum_{i=0}^{\infty}p_i\frac{f^{(1)}(z)}{(z-z_0)^{n+r-i}}+\sum_{i=0}^{\infty}q_i\frac{f^{(0)}(z)}{(z-z_0)^{n+r-i+1}}=0 \tag{3.1}$$

We now integrate through (3.1), with the integral being a contour integral round a closed-path, $C,$ taken anti-clockwise and containing z_0 while avoiding any other singularities of $f^{(0)}(z)$; this procedure leads to

$$\oint_C\frac{f^{(2)}(z)}{(z-z_0)^{n+r-1}}dz+\sum_{i=0}^{\infty}p_i\oint_C\frac{f^{(1)}(z)}{(z-z_0)^{n+r-i}}dz+\sum_{i=0}^{\infty}q_i\oint_C\frac{f^{(0)}(z)}{(z-z_0)^{n+r-i+1}}dz=0 \tag{3.2}$$

Next, we compare the denominators in (3.2) with that of (1.5), term by term, to get three equations for the dummy index m in terms of the dummy index $n,$ one for each of the values of k (the order of the derivative of $f(z)$) in each integral, (two, one and zero, respectively). So, we find that (2.4) yields the three equations

$$m+r-k+1=m+r-2+1=n+r-1\Leftrightarrow m=n \tag{3.3a}$$

$$m + r - k + 1 = m + r - 1 + 1 = n + r - i \Leftrightarrow m = n - i \tag{3.3b}$$
$$m + r - k + 1 = m + r - 0 + 1 = n + r - i + 1 \Leftrightarrow m = n - i \tag{3.3c}$$

Utilizing (1.4) and (1.5) again, with the results of (3.3) in hand, we see that equation (3.2) transforms, term by term, into

$$(n + r)(n + r - 1)a_n + \sum_{i=0}^{n} [(n + r - i)p_i + q_i]a_{n-i} = 0 \tag{3.4a}$$

after cancellation and recalling that $a_m = 0, m < 0$.

At this point, as in section 2, we make the substitution $i \to n - k$ for comparative purposes and (3.4a) transforms to

$$(n + r)(n + r - 1)a_n + \sum_{k=0}^{n} [(k + r)p_{n-k} + q_{n-k}]a_k = 0 \tag{3.4b}$$

Finally, collecting all terms involving a_n together, we see that (3.4b) becomes

$$[(n + r)(n + r - 1) + (n + r)p_0 + q_0]a_n + \sum_{k=0}^{n-1} [(k + r)p_{n-k} + q_{n-k}]a_k = 0 \tag{3.5}$$

which is just equation (4) of chapter 5, section 29, of Simmons [10].

Setting $n = 0$ in (3.5), we get the *indicial equation* for (1.1), that is

$$r(r - 1) + rp_0 + q_0 = 0 \tag{3.6}$$

as $a_0 \neq 0$, by *hypothesis* [10]. (3.5) in combination with (3.6) is the 'in principle' Frobenius solution to (1.1), when (1.1) represents the general second-order linear homogeneous Fuchs equation.

As with the series solution of (1.1) about an ordinary point the divisor $(z - z_0)^{n+r-1}$, used to derive the Frobenius series solution of (1.1) about a regular singular point, was chosen for comparison purposes (with the solution quoted in Simmons [10]). If we vary the index in $(z - z_0)^{n+r-1}$, then we simply shift the value of n in the recurrence relation. Indeed, for comparison purposes, in our next example, involving a series about the origin, we use z^{n+r+1} as our divisor.

So, as our only example in this section, we consider the Schrödinger equation solved by Kościk and Okopińska [5]

$$z^2 f^{(2)}(z) - \ell(\ell + 1) f^{(0)}(z) - 2\left(\sum_{i=0}^{\infty} d_{i-2} z^i\right) f^{(0)}(z) + 2\lambda z f^{(0)}(z) = 0 \tag{3.7}$$

Again, we seek a basic Frobenius series solution of (3.7) about the origin, so we divide through (3.7) by z^{n+r+1} and then integrate through the resultant (in the same manner as before), to get

$$\oint_C \frac{f^{(2)}(z)}{z^{n+r-1}}dz - \ell(\ell+1)\oint_C \frac{f^{(0)}(z)}{z^{n+r+1}}dz - 2\sum_{i=0}^{\infty} d_{i-2}\oint_C \frac{f^{(0)}(z)}{z^{n+r-i+1}}dz + 2\lambda\oint_C \frac{f^{(0)}(z)}{z^{n+r-1}}dz = 0 \qquad (3.8)$$

Following the usual comparison procedure, but this time with equation (1.5), we get from (3.8) four equations (again!) for m, one for each value of k (two, zero, zero and zero, respectively)

$$m + r - k + 1 = m + r - 2 + 1 = n + r - 1 \Leftrightarrow m = n \qquad (3.9a)$$

$$m + r - k + 1 = m + r - 0 + 1 = n + r + 1 \Leftrightarrow m = n \qquad (3.9b)$$

$$m + r - k + 1 = m + r - 0 + 1 = n + r - i + 1 \Leftrightarrow m = n - i \qquad (3.9c)$$

$$m + r - k + 1 = m + r - 0 + 1 = n + r - 1 \Leftrightarrow m = n - 2 \qquad (3.9d)$$

Utilizing (1.4) and (1.5) again, with the results of (3.9) in hand, we see that (after cancellation and rearrangement) equation (3.8) transforms, term by term, into

$$[(n+r)(n+r-1) - \ell(\ell+1)]a_n - 2\sum_{i=0}^{n} d_{i-2}a_{n-i} + 2\lambda a_{n-2} = 0 \qquad (3.10)$$

(where we take $a_m = 0, m < 0$). Setting $n = 0$ in (3.10), we get the indicial equation for our Frobenius solution of (3.7), that is

$$r(r-1) - \ell(\ell+1) - 2d_{-2} = 0 \qquad (3.11)$$

as $a_0 \neq 0$ by hypothesis. The recurrence relation (3.10) is identical to equation (8) of Kościk and Okopińska [5], the superficial difference being due the sum in

(3.10) being performed in 'the opposite direction' to that of Kościk and Okopińska [5] (set $i \to n - k$).

4. Frobenius Series: The Second Solution

As well as delivering the basic Frobenius solution, the current methodology can also be used to find a logarithmic solution, when such is required [7]. All that is necessary is for us to re-express the formalism of the previous section in such a manner that we bring it into line with the standard formalism presented in text-books, in particular [7]. It will then be possible simply to *quote* the required results and apply the formalism of section 3 to an appropriate example. First, we define, as usual [7], the linear operator L via

$$L[f(z)] = f^{(2)}(z) + P(z) f^{(1)}(z) + Q(z) f^{(0)}(z) \tag{4.1}$$

Next, we note that the Frobenius solution (1.4) to (1.1) has recurrence relation (3.5), which is just the coefficient of $(z-z_0)^{n+r}$ in the infinite series expansion of $L[f(z)] = 0$. That is, on extracting the $n = 0$ term, we may write

$$L[f(z)] = [r(r-1) + p_0 r + q_0]a_0(z-z_0)^r + \sum_{n=1}^{\infty}\left([(n+r)(n+r-1) + (n+r)p_0 + q_0]a_n + \sum_{k=0}^{n-1}[(k+r)p_{n-k} + q_{n-k}]a_k\right)(z-z_0)^{n+r} \tag{4.2}$$

and $L[f(z)] = 0$ whenever r is a solution of (3.6). Now, in the standard discussion of the second solution [7], we *choose* (in (4.2)), as before, for *any* r and $n \geq 1$

$$[(n+r)(n+r-1) + (n+r)p_0 + q_0]a_n + \sum_{k=0}^{n-1}[(k+r)p_{n-k} + q_{n-k}]a_k = 0 \tag{4.3}$$

which is the recurrence relation (3.5) for $n \geq 1$, so that (4.2) with (4.3) in mind becomes, for arbitrary r

$$L[f(z)] = [r(r-1) + p_0 r + q_0]a_0(z-z_0)^r \tag{4.4}$$

The relations (4.3) and (4.4) form the foundations of the standard discussion of the 'second solution' to (1.1) in the Fuchs case [7] (for real r at this point) and, as stated above, we will now simply quote the results.

If we denote the Frobenius solution to (1.1), for the larger or equal root, as $f_1(z)$, then the second solution to (1.1), for the smaller or equal root (shown, again, as r), may be taken as [7, 10]

$$f_2(z) = c\ln(z-z_0) f_1(z) + g(z) \tag{4.5a}$$

with

$$g(z) = \sum_{m=0}^{\infty} b_m (z-z_0)^{m+r} \tag{4.5b}$$

with the coefficients c and $\{b_m\}_{m=0}^{\infty}$ to be determined. In fact, if we consider the general second-order Fuchs equation (1.1), then substitution of (4.5a) into (1.1) produces the following ODE for the unknown $g(z)$ (following [7])

$$z^2 g^{(2)}(z) + z^2 P(z) g^{(1)}(z) + z^2 Q(z) g^{(0)}(z) \\ = c[1 - zP(z)] f_1^{(0)}(z) - 2czf_1^{(1)}(z) \qquad (4.6)$$

The point of the coefficient c is that, in some cases [10], $c = 0$ and the second solution does not contain a logarithmic term; otherwise we take $c = 1$. To determine whether or not $c = 0$ a careful examination of (4.3) is required, so we leave it at that and continue-on to an example of a logarithmic second solution.

As an example of the logarithmic second solution, we consider the extended confluent hypergeometric equation (following Campos [2])

$$zf^{(2)}(z) + (\gamma - z) f^{(1)}(z) - (\alpha + \sum_{i=1}^{M} A_i z^i)] f^{(0)}(z) = 0 \qquad (4.7)$$

with α, γ and the $\{A_i\}_{i=1}^{M}$ independent of z. We seek a Frobenius power series solution of (4.7) about the origin, so, as before, we divide through (4.7) by z^{n+r+1} and then integrate through the resultant, to get

$$\oint_C \frac{f^{(2)}(z)}{z^{n+r}} dz + \gamma \oint_C \frac{f^{(1)}(z)}{z^{n+r+1}} dz - \oint_C \frac{f^{(1)}(z)}{z^{n+r}} dz - \alpha \oint_C \frac{f^{(0)}(z)}{z^{n+r+1}} dz \\ - \sum_{i=1}^{M} A_i \oint_C \frac{f^{(0)}(z)}{z^{n+r-i+1}} dz = 0 \qquad (4.8)$$

Following the usual comparison procedure, again with equation (1.4), we get five equations for m, one for each value of k (two, one, one, zero and zero, respectively)

$$m + r - k + 1 = m + r - 2 + 1 = n + r \Leftrightarrow m = n + 1 \qquad (4.9a)$$
$$m + r - k + 1 = m + r - 1 + 1 = n + r + 1 \Leftrightarrow m = n + 1 \qquad (4.9b)$$
$$m + r - k + 1 = m + r - 1 + 1 = n + r \Leftrightarrow m = n \qquad (4.9c)$$
$$m + r - k + 1 = m + r - 0 + 1 = n + r + 1 \Leftrightarrow m = n \qquad (4.9d)$$
$$m + r - k + 1 = m + r - 0 + 1 = n + r - i - 1 \Leftrightarrow m = n - i \qquad (4.9e)$$

Utilizing (1.4) and (1.5) again, with the results of (4.9) in hand, we see that (after cancellation and rearrangement) equation (4.8) transforms, term by term, into

$$[(n+r)(n+r+\gamma-1)a_n-(n+r+\alpha-1)a_{n-1}-\sum_{i=1}^{M}A_i a_{n-i-1}=0 \qquad (4.10)$$

where we have changed $n \to n-1$.

Setting $n=0$ in (4.10), we get the indicial equation for our Frobenius solution, that is, as $a_0 \neq 0$ by hypothesis

$$r(r+\gamma-1)=0 \qquad (4.11)$$

The recurrence relation (4.11) is identical to that of Campos [2], from whom we have taken this example. Unless $M=0$, the recurrence relation (4.10) is of an order higher than two and it proves necessary to encode the procedure.

On examining the indicial equation, (4.11), Campos [2] shows that in some cases a ‘logarithmic solution’ of the form (4.5) (with $z_0=0$) is required, as the second root fails to deliver a basic series solution. Specifically, from the original ODE (4.7), we get, on substituting (4.5a) in (4.7), the following equation for $g(z)$

$$z^2 g^{(2)}(z)+z(\gamma-z)g^{(1)}(z)-z(\alpha+\sum_{i=1}^{M}A_i z^i)]g^{(0)}(z)$$
$$=c(z-\gamma+1)f_1^{(0)}(z)-2czf_1^{(1)}(z) \qquad (4.12)$$

The solution procedure for (4.12) is just the same as before: we divide through (4.12) by z^{n+r+1} and then integrate through (in the usual manner) to get

$$\oint_C \frac{g^{(2)}(z)}{z^{n+r-1}}dz+\gamma\oint_C \frac{g^{(1)}(z)}{z^{n+r}}dz-\oint_C \frac{g^{(0)}(z)}{z^{n+r-1}}dz-\alpha\oint_C \frac{g^{(0)}(z)}{z^{n+r}}dz-\sum_{i=1}^{M}A_i\oint_C \frac{g^{(0)}(z)}{z^{n+r-i}}dz$$
$$=c\oint_C \frac{f_1^{(0)}(z)}{z^{n+r}}dz+(1-\gamma)c\oint_C \frac{f_1^{(0)}(z)}{z^{n+r+1}}dz-2c\oint_C \frac{f_1^{(1)}(z)}{z^{n+r}}dz \qquad (4.13)$$

Following the usual comparison procedure, again with equation (1.5), we get from (4.13) eight (five plus three) equations for m, one for each value of k (two, one, one, zero and zero, respectively, on the left of (4.13) – for the $b's$ – and zero, zero and one, respectively on the right of (4.13) – for the $a's$)

$$m+r-k+1=m+r-2+1=n+r-1 \Leftrightarrow m=n \qquad (4.14a)$$
$$m+r-k+1=m+r-1+1=n+r \Leftrightarrow m=n \qquad (4.14b)$$
$$m+r-k+1=m+r-1+1=n+r-1 \Leftrightarrow m=n-1 \qquad (4.14c)$$
$$m+r-k+1=m+r-0+1=n+r \Leftrightarrow m=n-1 \qquad (4.14d)$$
$$m+r-k+1=m+r-0+1=n+r-i \Leftrightarrow m=n-i-1 \qquad (4.14e)$$

and

$$m+r-k+1=m+r-0+1=n+r \Leftrightarrow m=n-1 \qquad (4.14f)$$

$$m+r-k+1=m+r-0+1=n+r+1 \Leftrightarrow m=n \tag{4.14g}$$
$$m+r-k+1=m+r-1+1=n+r \Leftrightarrow m=n \tag{4.14h}$$

Utilizing (1.4)/(4.5b) and (1.5) again, with the results of (4.14) in hand, we see that (after cancellation and rearrangement) equation (4.13) transforms, term by term, into

$$[(n+r)(n+r+\gamma-1)b_n-(n+r+\alpha-1)b_{n-1}-\sum_{i=1}^{M}A_i b_{n-i-1}$$

$$=-c(2n+2r+\gamma-1)a_n+ca_{n-1} \qquad (4.15)$$

in agreement with equation (20a) ($n \to n+1$) of Campos [2], in case $c=1$.

Naturally, we may apply the usual procedure to the more general case of (4.6).

5. Discussion and Conclusions

There are a couple of obvious points that need mentioning, before we go on to some more general points of discussion: the existence of complex conjugate roots and the possibility of tackling inhomogeneous ODE with the present method.

Naturally, being a quadratic equation, it is possible that the indicial equation may give rise to complex conjugate roots. We make the existence of this real possibility the subject of our next example [6]. So, following Neuringer [6], we consider the second-order ODE

$$z^2 f^{(2)}(z)-zf^{(1)}(z)+(2+z)f^{(0)}(z)=0 \tag{5.1}$$

Applying the complex integration method about the origin (divide through by z^{n+r+1} and integrate round an appropriate closed contour) we find that the recurrence relation for our equation (5.1) is

$$(n+r)(n+r-1)a_n-(n+r)a_n+2a_n+a_{n-1}=0 \tag{5.2}$$

Setting $n=0$ in (5.2), we get the indicial equation as

$$r(r-1)-r+2=r^2-2r+2=0 \tag{5.3}$$

with complex conjugate solutions $r=1\pm\hat{i}$, with $\hat{i}^2=-1$, and the recurrence equation (5.2) reduces to

$$a_n=-\frac{a_{n-1}}{n(n\pm 2\hat{i})}, \quad n=1,2,3,\ldots \tag{5.4}$$

Apparently, then, the case of complex conjugate roots of the indicial equation leads to complex coefficients in the Frobenius series. For further discussion of this complex conjugate roots case, see Neuringer [6].

As to the second point, at first sight it does appears that we use can the present methodology to tackle inhomogeneous ODE. Indeed, when seeking a logarithmic second solution that is exactly what, *formally*, we have done. However, although the contour integration method will work in finding particular solutions to inhomogeneous ODE, the mechanics of the calculation become more involved.
For example, consider the case, where the right-hand-side of the inhomogeneous ODE is an analytic function, that is (with the coefficients $\{c_i\}_{i=0}^{\infty}$ given)

$$f_p^{(2)}(z) + P(z) f_p^{(1)}(z) + Q(z) f_p^{(0)}(z) = \sum_{n=0}^{\infty} c_n z^n \tag{5.5}$$

where $f_p(z)$ is the required particular integral. Applying the contour integration method of section 2 to (5.5) yields the recurrence relation

$$(n+2)(n+1)a_{n+2} + \sum_{i=0}^{n} [(n-i+1) p_i a_{n-i+1} + q_i a_{n-i}] = c_n \tag{5.6}$$

with, as usual, a_0 and a_1 arbitrary constants. However, to solve the recurrence equation (5.6) for $f_p(z)$, we must have *particular* values for a_0 and a_1, as $f_p(z)$, being a *particular* integral, can have no arbitrariness about it.

As an example, consider the second-order inhomogeneous ODE

$$f_p^{(2)}(z) + f_p^{(0)}(z) = 2e^z \tag{5.7}$$

with particular integral (by sight) $f_p(z) = e^z$. Applying the contour integration method of section 2 to (5.7) yields the recurrence relation

$$a_{n+2} = \frac{2}{(n+2)!} - \frac{a_n}{(n+2)(n+1)}, \quad \text{n} = 0,1,2,3,\ldots \tag{5.8}$$

with, of course, a_0 and a_1 arbitrary constants. But, how to choose a_0 and a_1?
In this case, if we set $a_1 = a_0$, then the recurrence relation (5.8) becomes (after close inspection)

$$a_{n+2} = \frac{a_{n+1}}{(n+2)}, \quad \text{n} = 0,1,2,3,\ldots \tag{5.9}$$

when we have $f_p(z) = a_0 \sum_{n=0}^{\infty} \frac{z^n}{n!} = a_0 e^z$ and we can determine a_0 by substitution in the original equation (5.7) (the usual 'try') and get $a_0 = 1$. The problem with this is that the choice of a_0 and a_1 in the above example was 'goal oriented', there being no obvious logical reason for setting $a_1 = a_0$ and continuing as we did; the choice was based on knowing the answer already. However, as a_0 and a_1 are both *arbitrary*, we are at liberty to *search* for a solution as best we can….

Moving on, on a more general note, the question of solving higher-order ODEs arises. In principle, the complex integral method should still be useful when solving higher-order ODEs, there being simply 'more of the same' involved. Naturally, though, the calculations become more involved and the type of solution, especially in the Frobenius series case, more involved also [4]. We leave this matter at this point, but it is to be noted that, on a case-by-case basis, any particular higher-order linear homogeneous ODE will yield an appropriate series solution on applying the complex integral methodology, in the same manner as in sections 2 and 3 above. (in fact, so will first-order linear homogeneous ODE [8]).

There are certain topics that we have not discussed. here. As our procedure has been essentially formal, we have not discussed convergence of the solutions. This is covered, for real roots of the indicial equation, in standard textbooks [7, 10]; for the complex roots case, see Neuringer [6]. Also, we have not touched on the concept of 'the point at infinity'; again, the reader is referred to a standard textbook [10] for a discussion of this topic. Finally, we have stopped our discussion, in general, at the point where the recurrence relations has been obtained. The solution of recurrence relations is a major independent problem, but in the case of the examples presented here the solutions of the recurrence relations and the actual series solutions can be obtained from the original papers from which the present examples were sourced [1, 2, 5, 6, and 11]. (See, also [7].)

In conclusion, we have presented a complex integration approach to the problem of finding series solution of the *general form* of the second-order linear homogeneous ODE, along with the solution of some typical examples from mathematical physics, especially examples involving the Schrödinger equation. The complex integral method reduces the solution process of the ODE to the solution of simultaneous, but independent, simple equations for the subscripts of the recurrence relation defining the infinite series solution's coefficients. The usual recurrence relations [10] for both the case of the ordinary point and the regular singular point have been determined, in a much simpler and direct manner than is generally supposed possible. In the case of Frobenius solutions to Fuchs' class of second-order ODE, if a logarithmic solution exists, then the method extends, in an elementary manner, to this type of solution also.

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