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On Compact Uniform Analytical Approximations to the Blasius Velocity Profile

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Abstract

It is shown that approximate analytical representations of the Blasius function may be developed by using the error function, $erf(ax)$, for positive constant a, as a basic *compact* approximate form for the *derivative* of the Blasius function, that is, the dimensionless velocity profile for the Blasius problem. This compact approximate analytical representation of the Blasius velocity function is then refined by the addition, following Savaş [13], of another parameter, to obtain further approximate analytical representations of the Blasius function.

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1. Introduction

In boundary layer theory, the Blasius function, $F(x)$, with x a dimensionless distance, is the solution of the nonlinear ordinary differential equation (ODE) [3]

$$F'''(x) + \frac{1}{2}F(x)F''(x) = 0 \tag{1.1}$$

with the boundary/initial conditions

$$F(0) = F'(0) = 0; \; F'(\infty) = 1 \tag{1.2}$$

Recently, there has been a considerable interest (see, for example, [2, 6, 8–17]) in approximate analytical representations of $F(x)$ and, hence, approximate solutions of the initial/boundary-value problem represented by (1.1) and (1.2). In this paper, we reinvestigate the approximate analytical representation of $F(x)$ and present new uniform (valid for $x \in [0,\infty)$) analytical approximations of $F(x)$ based on the error function [4] and the ideas of Savaş [13].

Rather than deal with (1.1)/(1.2) directly, it is technically easier to deal with the dimensionless velocity [3] $f(x) = F'(x)$, that is, the nonlinear ODE

$$f'' + \frac{1}{2}Ff' = 0 \tag{1.3}$$

along with the boundary/initial conditions

$$f(0) = 0, \quad f(\infty) = 1 \tag{1.4}$$

Then, *given* $f(x) = F'(x)$, we have [1]

$$F(x) = \int_0^x f(u)du \tag{1.5}$$

and an approximation to $f(x) = F'(x)$ leads to an approximation to $F(x)$ via (1.5).

The 'reduced' problem (1.3)/(1.4) proves more tractable than (1.1)/(1.2) as the velocity profile $f(x) = F'(x)$ is well known [1, 3] and of a 'convenient' shape for the development of families of analytical approximations to $f(x) = F'(x)$ using the *trial function* (or, more generally, trial functions) approach [1], the choice of trial function(s) being dictated by a knowledge of the general shape of the solution curve to the problem in question [1]. The trial function approach has the further advantage [1] of being as *elementary* or compact as possible: in this work all the trial functions for $f(x)$ are based on a single function the *error function* [4], although the 'fitting' of the trial function(s) to the details of (1.3)/(1.4) requires *free parameters* to be present in the trial function(s) (which will be introduced as required below).

To implement the trial function approach, it is handy to have certain basic data available which has been taken from references [3, 9] and is presented in Table 1.

Symbol	Definition	Numerical Value
κ	$F''(0)$	0.33205733621519630
B	$\lim_{x\to\infty}(F(x) - x)$	1.7207876575205038

Table 1. Basic Properties of the Blasius Function [3, 9].

The body of the paper is arranged as follows. As mentioned already, the initial/boundary-value problem for $f(x)=F'(x)$ 'dictates' the basic approximate trial function and so, in section 2, we concentrate our attention on various initial standard possibilities involving consideration of the error function, $erf(ax)$, for positive constant a, as an approximate analytical solution of (1.3)/(1.4); hence, through (1.5), we obtain various approximate analytical solutions of (1.1)/(1.2) also. Next, in section 3, we adapt some ideas of Savaş [13] to our particular starting format, $erf(ax)$. Specifically [13], we introducing another free parameter into the functional form, $erf(ax)$, of the analytical approximation to $f(x)=F'(x)$ and obtain various improved approximate fits to $f(x)$ and, via (1.5) again, $F(x)$. The paper concludes, in section 4, with a brief discussion of the current results along with a comparison with the approaches and results of other similar attempts at producing approximate analytical representations of the Blasius function [13]. Note that we quote our results to four decimal places (at most); further, we concentrate on quoting the results for $F(x)$.

2. The Basic Error Function Approximation Scheme

In this section we consider the simplest form of our trial function, that is, for positive constant a, we consider as a trial function for $f(x)=F'(x)$

$$f_t(x)=erf(ax) \tag{2.1}$$

On examination, it is apparent that $f_t(x)=erf(ax)$ satisfies both of the conditions (1.4), while, following (1.5), integration shows that [4]

$$F_t(x)=\int_0^x f_t(u)du=\frac{e^{-a^2x^2}}{\sqrt{\pi}a}+xerf(ax)-\frac{1}{\sqrt{\pi}a} \tag{2.2}$$

which satisfies the additional initial condition (from (1.1)) $F_t(0)=0$ and has, also, the correct *asymptotic* functional form (see Table 1, row two).

The nub of the problem, then, is to find a satisfactory means of determining the parameter a. In regards to this, we note that, from (2.1)

$$f_t'(0)=F_t''(0)=\frac{2a}{\sqrt{\pi}} \tag{2.3}$$

which means that we may essentially *assume* the value of a, via (2.3) and the given value of $F''(0)$ in Table 1. Otherwise, we must provide a 'not-unreasonable' method for *determining* a independent of the known value of $F''(0)$.

To begin with, as it is the simplest way to manufacture an approximation to the Blasius function, we assume we have (from row one, Table 1) for our first trial function $f_1(x) = F_1'(x)$

$$f_1'(0) = F_1''(0) \approx 0.332057 \tag{2.4}$$

So, from (2.3) and (2.4), we have $a \approx \dfrac{332057\sqrt{\pi}}{2000000} \approx 0.294278$ and our first trial function becomes

$$f_1(x) = erf\left(\tfrac{332057\sqrt{\pi}}{2000000}x\right) \tag{2.5}$$

when, from (1.5) or (2.2)

$$F_1(x) = \int_0^x erf\left(\tfrac{332057\sqrt{\pi}}{2000000}u\right)du \tag{2.6}$$

The results of evaluating the approximation (2.6) for a fixed set of values of x is compared with the 'exact' (figure of speech) numerical results [7] in Table 2.

The other approaches to determining a depend on the manipulation of the *residual* of (1.3), that is [1]

$$R_t(x,a) = f_t'' + \frac{1}{2}F_t f_t' \tag{2.7}$$

So, our second possibility is to collocate (2.7), following reference [1], by requiring $R_t(x,a) = R_2(x,a) = 0$ when $x = \ln 2/a$. In other words, we must determine a in $f_2(x) = erf(ax)$ such that

$$R_2(x,a) = f_2'' + \frac{1}{2}F_2 f_2' = erf''(ax) + \frac{1}{2}\left(\int_0^x erf(au)du\right)erf'(ax) = 0 \tag{2.8}$$

when $x = \ln 2/a$. The numerical solution to this second problem is $a \approx 0.3010484$ and so

$$f_2(x) = erf(0.3010484x) \tag{2.9}$$

when, from (1.5) or (2.2)

$$F_2(x) = \int_0^x erf(0.3010484u)du \tag{2.10}$$

The approximate analytical expression (2.10) is compared with the approximation (2.6) and the 'exact' numerical results [7] in Table 2.

Moving on, our next approach is to determine (numerically) a value of a in $f_3(x) = erf(ax)$ *uniformly*, in other words determine a such that ([5], *unit* weight)

$$\int_0^\infty R_3(x,a)dx = \int_0^\infty (f_3'' + \frac{1}{2}F_3 f_3')dx = 0 \qquad (2.11)$$

The numerical solution to this third problem is $a \approx 0.3217971$, so that

$$f_3(x) = erf(0.3217971x) \qquad (2.12)$$

when, from (1.5) or (2.2)

$$F_3(x) = \int_0^x erf(0.3217971u)du \qquad (2.13)$$

The approximate expression (2.13) is compared with the approximations (2.6) and (2.10), along with the ‘exact’ numerical results [7], in Table 2.

The fourth and final scheme we consider to determine a and $f_4(x) = erf(ax)$ is the method of *least squares*, where we find (numerically) a value of a in $f_4(x) = erf(ax)$ such that (see, for example, [10])

$$\frac{\partial}{\partial a}\int_0^\infty R_4^2(x,a)dx = 2\int_0^\infty R_4(x,a)\frac{\partial}{\partial a}[R_4(x,a)]dx = 0 \qquad (2.14)$$

with $R_4(x,a) = f_4'' + \frac{1}{2}F_4 f_4'$. Solving (2.14) numerically gives $a \approx 0.3174155$ and

$$f_4(x) = erf(0.3174155x) \qquad (2.15)$$

is our fourth approximate velocity profile, so that, from (1.5) or (2.2)

$$F_4(x) = \int_0^x erf(0.3174155u)du \qquad (2.16)$$

x	$F_1(x)$	$F_2(x)$	$F_3(x)$	$F_4(x)$	'Exact' [7]
0	0	0	0	0	0
0.4	0.0265	0.0271	0.0290	0.0286	0.0266
0.8	0.1053	0.1077	0.1149	0.1134	0.1061
1.2	0.2342	0.2394	0.2551	0.2518	0.2379
1.6	0.4100	0.4187	0.4453	0.4397	0.4203
2.0	0.6283	0.6411	0.6799	0.6718	0.6500
2.4	0.8841	0.9013	0.9530	0.9422	0.9223
2.8	1.1722	1.1938	1.2580	1.2447	1.2310
3.2	1.4872	1.5129	1.5889	1.5732	1.5691
3.6	1.8242	1.8537	1.9400	1.9223	1.9295
4.0	2.1785	2.2112	2.3063	2.2869	2.3057
4.4	2.5462	2.5816	2.6838	2.6630	2.6924
4.6	2.7340	2.7705	2.8757	2.8543	2.8882
4.8	2.9239	2.9614	3.0692	3.0473	3.0853
5.0	3.1156	3.1540	3.2640	3.2417	3.2833
5.2	3.3088	3.3480	3.4600	3.4373	3.4819
5.4	3.5033	3.5432	3.6568	3.6338	3.6809
5.6	3.6989	3.7393	3.8543	3.8311	3.8803
5.8	3.8953	3.9363	4.0524	4.0290	4.0799
6.0	4.0925	4.1339	4.2510	4.2274	4.2796
6.4	4.4885	4.5305	4.6490	4.6252	4.6794
6.8	4.8861	4.9285	5.0480	5.0240	5.0793
7.0	5.0853	5.1278	5.2476	5.2236	5.2792
7.4	5.4842	5.5270	5.6472	5.6231	5.6792
8.0	6.0833	6.1263	6.2469	6.2227	6.2792
10	8.0828	8.1259	8.2468	8.2226	8.2792
20	18.0830	18.1260	18.2470	18.2230	18.2792
100	98.0830	98.1260	98.2470	98.2230	98.2792

Table 2. Uniform Approximations to $F(x)$ based-on $f(x) = erf(ax)$.

The approximate expression (2.16) is compared with the approximations (2.6), (2.10) and (2.13), along with the 'exact' numerical results [7], in Table 2 again.

A careful look through the results presented in Table 2 shows that, while 'out' somewhat at the start of the interval, the approximations (2.13) and (2.16) soon 'pick-up' and appear to present an overall better 'fit' to the problem. A brief discussion of the *form* of the functions leading to Table 2 is presented in Section 4.

3. Improving on the Basic Error Function Approximation

In this section we first adopt an idea of Savaş [13] and interpolate a second parameter into our error function trial function. Specifically, we introduce a second parameter, *n*, through the new Blasius velocity trial solution

$$f_{nt}(x) = [erf([ax]^n)]^{\frac{1}{n}} \tag{3.1}$$

The trial function (3.1) satisfies the conditions (1.4). To see that (3.1) can be made to satisfy the condition on the first derivative ($f'_{nt}(0) = F''_{nt}(0) = \kappa \approx 0.332057$) also, we determine the values of the parameters a and n by following the lead of Savaş [13] by expanding (3.1) for 'small' values of x to get

$$f_{nt}(x) \approx \left(\frac{2}{\sqrt{\pi}}\right)^{\frac{1}{n}} \left(ax - \frac{(ax)^{2n+1}}{3n}\right) \tag{3.2}$$

which we compare with the first couple of terms of the known Maclaurin expansion of the Blasius velocity function [13] (κ is given in Table 1)

$$f(x) \approx \kappa x - \frac{1}{24}\kappa^2 x^4 \tag{3.3}$$

Still following Savaş [13], we compare (3.2) with (3.3) and we choose (with *error*) $n = 3/2$, so that $a \approx 0.3063674$, which gives us our fifth trial function, from (3.1), as

$$f_5(x) = [erf([0.3063675x]^{\frac{3}{2}})]^{\frac{2}{3}} \tag{3.4}$$

so that, from (1.5) or (2.2)

$$F_5(x) = \int_0^x [erf([0.3063675u]^{\frac{3}{2}})]^{\frac{2}{3}} du \tag{3.5}$$

x	$F_5(x)$	$F_6(x)$	$F_7(x)$	$F_S(x)$	$F_{SI}(x)$
0	0	0	0	0	0
0.4	0.0266	0.0266	0.0266	0.0266	0.0266
0.8	0.1061	0.1061	0.1064	0.1061	0.1063
1.2	0.2380	0.2377	0.2384	0.2378	0.2385
1.6	0.4207	0.4195	0.4208	0.4195	0.4217
2.0	0.6510	0.6483	0.6502	0.6478	0.6527
2.4	0.9245	0.9193	0.9220	0.9174	0.9265
2.8	1.2351	1.2267	1.2300	1.2220	1.2366
3.2	1.5758	1.5636	1.5676	1.5546	1.5755
3.6	1.9393	1.9233	1.9277	1.9086	1.9359
4.0	2.3186	2.2991	2.3040	2.2783	2.3114
4.4	2.7080	2.6858	2.6909	2.6588	2.6968
4.6	2.9050	2.8817	2.8870	2.8520	2.8921
4.8	3.1031	3.0790	3.0843	3.0466	3.0885
5.0	3.3018	3.2770	3.2824	3.2425	3.2859
5.2	3.5010	3.4758	3.4812	3.4392	3.4840
5.4	3.7006	3.6750	3.6804	3.6367	3.6826
5.6	3.9003	3.8744	3.8799	3.8348	3.8815
5.8	4.1002	4.0741	4.0796	4.0334	4.0808
6.0	4.3001	4.2739	4.2794	4.2323	4.2803
6.4	4.7000	4.6738	4.6793	4.6308	4.6797
6.8	5.1000	5.0737	5.0792	5.0300	5.0794
7.0	5.3000	5.2737	5.2792	5.2297	5.2793
7.4	5.7000	5.6737	5.6792	5.6294	5.6792
8.0	6.3000	6.2737	6.2792	6.2292	6.2792
10	8.3000	8.2737	8.2792	8.2290	8.2791
20	18.3000	18.2737	18.2792	18.2290	18.2791
100	98.3000	98.2737	98.2792	98.2290	98.2791

Table 3. Uniform Approximations to $F(x)$ based-on (3.1) and (3.8).

The results for the approximate Blasius function (3.5) are presented in Table 3.

We see from Table 3 that, while $F_5(x)$ is an advance on the previous four approximations, there is still room for further improvement and if we continue to follow Savaş [13], we find that this is indeed feasible. First, we remind ourselves that Savaş's approximate Blasius velocity function [13]

$$f_S(x) = [\tanh([0.33206x]^{\frac{3}{2}})]^{\frac{2}{3}} \tag{3.6}$$

leading to the approximate Blasius function

$$F_S(x) = \int_0^x [\tanh([0.33206u]^{\frac{3}{2}})]^{\frac{2}{3}} du \tag{3.7}$$

was obtained [13] using the same argument leading to (3.4), but starting with the trial function

$$f_{St}(x) = [\tanh([ax]^n)]^{\frac{1}{n}} \tag{3.8}$$

instead of (3.1). Using a somewhat heuristic argument, rather than one based on general principles [13], Savaş's improved his approximation (3.6) to the Blasius velocity function and obtained

$$f_{SI}(x) = [\tanh([0.33245x]^{\frac{5}{3}})]^{\frac{3}{5}} \tag{3.9}$$

leading to

$$F_{SI}(x) = \int_0^x [\tanh([0.33245u]^{\frac{5}{3}})]^{\frac{3}{5}} du \tag{3.10}$$

The approximation (3.10) is indeed, a significant improvement over the original, equation (3.7) [13]. Values of $F_S(x)$ and $F_{SI}(x)$ are presented in Table 3.

In a similar manner to Savaş [13], we may also search for another simple rational exponent n (other than $3/2$) for (3.1) and hope to determine a value of a to go along with it. We find, after a bit of experimentation, that setting $n = 7/5$ in (3.1), while leaving $a \approx 0.3063674$ isn't too bad an approximation. However, if we examine (3.2) we see that

$$f'_{nt}(0) = F''_{nt}(0) \approx \left(\frac{2}{\sqrt{\pi}}\right)^{\frac{1}{n}} a \tag{3.11}$$

and we investigate leaving $n = 7/5$ in (3.1), while solving for a using (3.11) with $n = 7/5$ and the value of $f'(0) = F''(0) = \kappa$ from row one of Table 1. Following this route, we find that $a \approx 0.304610$, so that our next (our *sixth*) approximate velocity function is

$$f_6(x) = [erf([0.30461x]^{\frac{7}{5}})]^{\frac{5}{7}} \tag{3.12}$$

with the corresponding approximate Blasius function given by

$$F_6(x) = \int_0^x [erf([0.30461u]^{\frac{7}{5}})]^{\frac{5}{7}} du \tag{3.13}$$

In fact, if we investigate further, we find (numerically) that (3.12) and (3.13) can be further ‘improved’ (uniformly) to

$$f_7(x) = [erf([0.305584x]^{\frac{7}{5}})]^{\frac{5}{7}} \tag{3.14}$$

and

$$F_7(x) = \int_0^x [erf([0.305584u]^{\frac{7}{5}})]^{\frac{5}{7}} du \tag{3.15}$$

The results corresponding to (3.13) and (3.15) are also given in Table 3.

4. Discussion and Conclusions

The compact (based on a single function) analytical approximations to the Blasius velocity function, and hence the Blasius function itself, have obvious strengths and weaknesses. To make this clearer, we first note that the general expression for the type of compact analytical approximate velocity presented here and in [13] is

$$[\varphi([ax]^n)]^{\frac{1}{n}} \tag{4.1}$$

with $\varphi(x) = erf(x)$ used in this paper and $\varphi(x) = \tanh(x)$ used by Savaş [13]. The basic strength of expression (4.1) lies in its simplicity, with only two parameters (n and a) to be determined once an appropriate choice of $\varphi(x)$ has been made. On the other hand, it is just this simplicity that limits the accuracy of the compact analytical approximation: there is limited room for manoeuvre. Having said that, it is still surprising that such close fits to the actual solution to the Blasius problem can be obtained. Furthermore, the compact analytical approximations to the Blasius problem presented here have the added merit of being *uniform* approximations, valid along the *entire* half-line $x \geq 0$; this is no mean feat. Indeed, the error in $F_7(x)$ lies within 0.3% of the ‘exact’ solution presented in Table 2, while that of $F_{SI}(x)$ lies within 0.5% of the same, along the *entire* half-line.

Other types of compact analytical approximations to the Blasius problem are possible, but based, instead, on rational functions (see, for example [2, 11]); as these are of a different character to equation (3.1), we gloss over them and turn back the consideration of equation (3.1) itself.

Finally, following Savaş [13] yet again, with a little patience and some further numerical investigation, we can further refine our format and determine that

$$f_8(x) = [erf([0.3054x]^{1.405})]^{\frac{1}{1.405}} \tag{4.2}$$

and

$$F_8(x) = \int_0^x [erf([0.3054u]^{1.405})]^{\frac{1}{1.405}} du \tag{4.3}$$

provide even better *uniform* analytical approximations to the Blasius problem, with the *uniform* error in (4.3) being better than 0.13%. The main difference between (3.15) and (4.3) occurs at the start of the interval, $0 \le x \le 2$, where (4.3) is a much better fit than (3.15); after this (3.15) is slightly the better fit. (Savaş [13] encountered a similar phenomenon in his more extensive analysis also.) The observation that (4.3) is a better fit than (3.15) over the interval $0 \le x \le 2$ is reinforced by the fact that (from (3.11)) $F_8''(0) \approx 0.3328$ gives the closest *estimate* (overall) to $F''(0) \approx 0.3321$ (from Table 1), a difference of less than 0.22%.

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