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Josef Janyška, Marco Modugno

Smooth and F–smooth systems

with applications to Covariant Quantum Mechanics

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Smooth and F–smooth systems

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We review the geometric theory of *smooth systems of smooth maps, of* smooth systems of smooth sections of a smooth double fibred manifold and of *smooth systems of smooth connections of a smooth fibred manifold.*

Moreover, after reviewing the concept of F–smooth space due to A. Frölicher, we discuss the *F–smooth systems of smooth maps, the* F–smooth systems of fibrewisely smooth sections of a smooth double fibred manifold, the *F–smooth systems of fibrewisely smooth connections of a smooth fibred manifold and of* F–smooth connections of an F–smooth system of fibrewisely smooth sections.

Further, we discuss three applications of the general geometric theory, which are taken in the framework of Covariant Quantum Mechanics. Namely, we discuss the *smooth upper quantum connection, the* F–smooth sectional quantum bundle and the Schrödinger operator regarded as an *F–smooth connection of the F–smooth sectional quantum bundle.*

Key words: systems of maps, systems of sections, systems of connections, universal connection, F–smooth spaces, F–smooth space of maps, F–smooth space of sections, F–smooth space of connections, F–smooth connections, Covariant Quantum Mechanics, quantum bundle, upper quantum connection, system of observed quantum connections, sectional quantum bundle, Schrödinger operator.

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Contents

Introduction

The present report is aimed at combining two independent geometric constructions: *smooth systems of smooth sections of a smooth double fibred manifold* and *F–smooth spaces.*

The notion of *smooth systems of connections* goes back to the paper [7], where P.L. García studies the *systems of principal connections* of a principal bundle and the associated "*universal connection*". Indeed, this is a fruitful geometric idea which exploits the properties of the Lie algebra associated with the principal bundle and deserves several applications.

Later, it has been shown that the above geometric construction can be extended to any fibred manifold, at a more basic level, regardless of a possible symmetry group, so detaching the notion of "system" from principal bundles and their structure group (see, for instance, [2, 19, 22]). In a few words, given a double fibred manifold $\boldsymbol{G} \to \boldsymbol{F} \to \boldsymbol{B}$, a "system of sections" is defined to be a 3–plet $(\boldsymbol{S}, \zeta, \epsilon)$, where $\zeta : \boldsymbol{S} \to \boldsymbol{B}$ is a fibred manifold and $\epsilon : \boldsymbol{S} \times_{\boldsymbol{B}} \boldsymbol{F} \to \boldsymbol{G}$ a fibred morphism over $\boldsymbol{F}$. Thus, the fibred space $\boldsymbol{S}$ behaves as a space of "parameters" and the "evaluation map" ϵ maps sections $s : \boldsymbol{B} \to \boldsymbol{S}$ of the fibred manifold $\boldsymbol{S} \to \boldsymbol{B}$ to sections $\breve{s} : \boldsymbol{F} \to \boldsymbol{G}$ of the fibred manifold $\boldsymbol{G} \to \boldsymbol{F}$. Therefore, the choice of such a system of sections turns out to be just a smooth selection of a distinguished family of sections of the fibred manifold $\boldsymbol{G} \to \boldsymbol{F}$.

In particular, the above notion can be easily used to define a "system of connections" of a fibred manifold $\boldsymbol{F} \to \boldsymbol{B}$, by setting $\boldsymbol{G} := T^*\boldsymbol{B} \otimes T\boldsymbol{F}$. Actually, in this basic framework, we can recover the "universal connection" of the system, along with its properties, without explicit reference to the structure group and its Lie algebra. In practice, the choice of the "fibred manifold of parameters" $\boldsymbol{S}$ and of the "evaluation map" ϵ play the selective role that is played by the equivariance with respect to the structure group in the language of systems of principal connections.

The notion of *F–smooth space* goes back to the paper [6], where A. Frölicher introduces a notion of "smoothness" which is alternative with respect to standard notion. Actually, in the present report, we use a definition of F–smoothness with a mild simplification with respect to the original one (see [25]).

In a few words, the notion of *standard smoothness* for a topological manifold $\boldsymbol{M}$ is based on the choice of a family $\mathcal{A}$ of local charts $(x^i) : \boldsymbol{M} \to \mathbb{R}^n$, which fulfills a standard smoothness compatibility condition. However, the *F–smoothness* for any set $\boldsymbol{S}$ is based on the choice of a family $\mathcal{C}$ of curves $c_{\boldsymbol{I}} : \boldsymbol{I} \to \boldsymbol{S}$, which fulfills a condition of smooth re–parametrisation, but no mutual compatibility condition.

At a first insight, it might appear that the two notions of smoothness above be mutually dual and rather equivalent. But, it is not so. Indeed, the F–smoothness turns out to be a rather weak condition: it is even more feeble than continuity! Nevertheless, one can prove that in the framework of F–smooth spaces it is possible to achieve several advanced geometric constructions (see, for instance, [3, 4, 14, 16]). It is also worth mentioning further deep investigation in this framework (see, for instance, [17, 21]).

In the present report, we are mainly concerned with F–smooth spaces consisting of smooth maps between standard smooth manifolds. In our opinion, this is a quite fruitful application of the general notion of F–smoothness.

Accordingly, we generalise the concept of system of sections of a double fibred manifold by replacing the smooth fibred manifold of parameters $\zeta : \boldsymbol{S} \to \boldsymbol{B}$ with an F–smooth fibred set $\zeta : \boldsymbol{S} \to \boldsymbol{B}$. In this way, we can achieve systems of sections, which, in a sense, are infinite dimensional, by skipping the hard methods of infinite dimensional manifolds.

At a first insight, some constructions of the present report might appear very cumbersome. But, their core idea is quite simple and intuitive. Unfortunately, a detailed account requires odd subtleties; but the reader can grasp the basic simple ideas at a first reading and go throughout details in a second reading, when necessary.

As far as applications to mathematical physics are concerned, we have used, in the framework of Covariant Quantum Mechanics, the smooth systems of connections and their universal connection in order to define the "upper quantum connection", the F–smooth systems of sections for the definition of "sectional quantum bundle" over time and the "F–smooth connections" in order to regard the Schrödinger operator as a connection of the sectional quantum bundle (see, for instance, [9, 11]).

We use the following symbols:

- given two smooth manifolds $\boldsymbol{M}$ and $\boldsymbol{N}$, we denote the set of *global* *smooth maps* between the two manifolds by $\mathrm{Map}(\boldsymbol{M}, \boldsymbol{N})$,

- given a smooth fibred manifold $\boldsymbol{F} \to \boldsymbol{B}$, we denote the set of *global smooth sections* $s : \boldsymbol{B} \to \boldsymbol{F}$ by $\mathrm{Sec}(\boldsymbol{B}, \boldsymbol{F})$.

- given a smooth fibred manifold $\boldsymbol{F} \to \boldsymbol{B}$, we denote the sheaf of *local smooth sections* $s : \boldsymbol{B} \to \boldsymbol{F}$ by $\mathrm{sec}(\boldsymbol{B}, \boldsymbol{F})$.

Chapter 1

Smooth manifolds and F–smooth spaces

We briefly recall the standard definition of *smooth manifold* and discuss the notion of *F–smooth space*. Moreover, we compare these concepts.

The reader can be interested to go back to the original literature concerning F–smooth spaces (see [6] and [3, 4, 14, 16, 17, 25]).

1.1 Smooth manifolds

We briefly recall the standard definition of *smooth manifold* just in view of a comparison with the forthcoming definition of F–smooth space (see next Section 1.2.1).

We observe that the basic original concept of derivative and the consequent concept of smoothness can be achieved in the framework of affine spaces (see, for instance, [24]). However, for practical reasons, one usually replaces a generic modelling affine space with $\mathbb{R}^n$.

Accordingly, one can introduce the standard notion of smooth manifold as follows.

Definition 1.1.1. A *smooth manifold* of dimension m is defined to be a pair $(\boldsymbol{M}, \mathcal{A})$, where $\boldsymbol{M}$ is a topological manifold of dimension n and $\mathcal{A}$ is a topological atlas whose local charts $(x^i) : \boldsymbol{M} \to \mathbb{R}^m$, fulfill "smooth transition rules" (in the sense of smoothness of affine spaces), in the intersections of their domains, and a "maximality condition".

Moreover, a map $f : \boldsymbol{M} \to \boldsymbol{N}$ between two smooth manifolds of dimension m and n, respectively, is said to be *smooth* if it yields locally "local smooth maps" (in the sense of smoothness of affine spaces) $\mathbb{R}^m \to \mathbb{R}^n$. □

Thus, the standard smoothness of a manifold $\boldsymbol{M}$ involves a background structure of topological manifold and a smoothness compatibility condition of local topological charts. Hence, the smoothness of manifolds turns out to be a global property achieved via a local condition of compatibility between local topological charts.

1.2 F–smooth spaces

We discuss the notion of *F–smooth space* $(\boldsymbol{S}, \mathcal{C})$ and of *F–smooth maps* $f : \boldsymbol{S} \to \acute{\boldsymbol{S}}$ between F–smooth spaces.

Moreover, we compare the notions of smoothness and F–smoothness.

1.2.1 F–smooth spaces

We introduce the notion of "*F–smooth space*", with minor changes, with respect to the original definition due to A. Frölicher [6] (see also [4, 15, 16, 17, 25]).

Roughly speaking, an "*F–smooth space*" is defined to be a set $\boldsymbol{S}$ equipped with a family $\mathcal{C}$ of curves $c : \boldsymbol{I}_c \to \boldsymbol{S}$ which fulfills a certain feeble requirement of reparametrisation, without compatibility conditions.

This apparently simple difference of the two concepts of smooth manifold and F–smooth space yields great difference in their consequences.

Actually, the notion of "F–smooth space" is weaker than that of "smooth manifold". Nevertheless, it allows us to achieve several geometric constructions.

Indeed, any smooth manifold turns out to be an F–smooth space in a natural way (see the forthcoming Section 1.3).

We can define the notion of "*F–smooth subspace*" of an F–smooth space in a natural way. Indeed, the F–smooth spaces fulfill a remarkable property concerning F–subspaces, which has no analogue for topological or smooth manifolds.

The notion of "F–smooth space" provides the geometric context suitable for the further discussions on "F–smooth systems of smooth maps" and "F–smooth systems of smooth sections" (see the forthcoming Section 2.2.1 and Section 3.2.1).

Definition 1.2.1. An *F–smooth space* is defined to be a pair $(\boldsymbol{S}, \mathcal{C})$, where $\boldsymbol{S}$ is a non empty set and $\mathcal{C}$ is a family of curves, called *basic curves*,

$$\mathcal{C} \equiv \{c : \boldsymbol{I}_c \to \boldsymbol{S}\},$$

where $\boldsymbol{I}_c \subset \mathbb{R}$ is an open subset, such that:

1) for each $s \in \boldsymbol{S}$, there is at least one curve $c : \boldsymbol{I}_c \to \boldsymbol{S}$ belonging to $\mathcal{C}$ and a $\lambda \in \boldsymbol{I}_c$, such that

$$c(\lambda) = s,$$

2) if $c \in \mathcal{C}$ and $\gamma : \boldsymbol{I}_\gamma \to \boldsymbol{I}_c$ is a smooth map defined on an open subset $\boldsymbol{I}_\gamma \subset \mathbb{R}$, then

$$c \circ \gamma \in \mathcal{C}. \square$$

We have immediate consequences of the above Definition.

Proposition 1.2.1. Let $(\boldsymbol{S}, \mathcal{C})$ be an F–smooth space.

1) All constant curves belong to $\mathcal{C}$.

2) If the curve $c : \boldsymbol{I}_c \to \boldsymbol{S}$ belongs to $\mathcal{C}$, then its restriction c' to any open subset $\boldsymbol{I}_{c'} \subset \boldsymbol{I}_c$ belongs to $\mathcal{C}$.

PROOF. 1) Let $s \in \boldsymbol{S}$. Then, in virtue of condition 1) in Definition 1.2.1, there exists a curve $c : \boldsymbol{I}_c \to \boldsymbol{S}$ which belongs to $\mathcal{C}$ and such that, for a certain $\lambda \in \boldsymbol{I}_c$, we have $c(\lambda) = s$.

Moreover, let us consider any open subset $\boldsymbol{I}_\gamma \subset \mathbb{R}$, the above $\lambda \in \boldsymbol{I}_c$ and the constant map $\gamma : \boldsymbol{I}_\gamma \to \boldsymbol{I}_c : \mu \mapsto \lambda$. Then, according to condition 2) in Definition 1.2.1, the constant curve $c \circ \gamma : \boldsymbol{I}_\gamma \to \boldsymbol{S}$ belongs to $\mathcal{C}$.

2) The curve $c' : \boldsymbol{I}_{c'} \to \boldsymbol{S}$ can be regarded as the composition $c' = c \circ \gamma$, where $\gamma : \boldsymbol{I}_{c'} \hookrightarrow \boldsymbol{I}_c$ is the smooth inclusion map. QED

The families of basic curves of F–smooth spaces fulfill the following remarkable property, which has no analogue concerning the charts of smooth manifolds.

Proposition 1.2.2. Let us consider a set $\boldsymbol{S}$ and two families $\mathcal{C}$ and $\acute{\mathcal{C}}$ of $\boldsymbol{S}$, which fulfill the requirements of Definition 1.2.1.

Then, the union $\mathcal{C} \cup \acute{\mathcal{C}}$ of the two families of curves fulfills the requirements of Definition 1.2.1. $\square$

Next, we compare different F–smooth structures of a set $\boldsymbol{S}$.

Note 1.2.1. Any non empty set $\boldsymbol{S}$ admits at least one F–smooth structure. Even more, if $\boldsymbol{S}$ has infinitely many elements, then it admits infinitely many F–smooth structures. $\square$

Definition 1.2.2. Given two F–smooth structures $\mathcal{C}$ and $\acute{\mathcal{C}}$ of a set $\boldsymbol{S}$, such that $\mathcal{C} \subset \acute{\mathcal{C}}$, then we say that $\acute{\mathcal{C}}$ is *finer* than $\mathcal{C}$. □

It is worth comparing the notions of F–smooth space and smooth manifold.

Remark 1.2.1. In a sense,

a) the assignment of the set of curves $\mathcal{C}$ for an F–smooth space plays a role analogous to the assignment of a maximal smooth atlas $\mathcal{A}$ for a smooth manifold,

b) the assignment of a set $\overline{\mathcal{C}}$ as in the fothcoming Example 1.2.1 plays a role analogous to the assignment of a smooth atlas $\bar{\mathcal{A}}$ for a smooth manifold,

c) the conditions 1) and 2) in Definition 1.2.1 for an F–smooth space play a role analogous to the conditions of smooth transitions for the charts of a smooth atlas. □

Eventually, we consider a few trivial or exotic examples of F–smooth spaces, just to account for the range of the notion of F–smooth space.

Example 1.2.1. Let us consider any set $\boldsymbol{S}$.

Moreover, let us choose any set $\overline{\mathcal{C}}$ consisting of curves $c : \boldsymbol{I}_c \to \boldsymbol{S}$ fulfilling condition 1) in Definition 1.2.1.

Furthermore, let us define the set $\mathcal{C}$ consisting of all curves of the type

$$c \circ \gamma : \boldsymbol{I}_\gamma \to \boldsymbol{S}\,, \qquad \text{where} \qquad c \in \overline{\mathcal{C}}\,,$$

and where $\gamma : \boldsymbol{I}_\gamma \to \boldsymbol{I}_c$ is any smooth map defined in a open subset $\boldsymbol{I}_\gamma \subset \mathbb{R}$.

Then, the pair $(\boldsymbol{S}, \mathcal{C})$ turns out to be an F–smooth space. □

Example 1.2.2. Let us consider any set $\boldsymbol{S}$ and the set $\mathcal{C}$ consisting just of all constant curves $c : \boldsymbol{I}_c \to \boldsymbol{S}$. Then, the pair $(\boldsymbol{S}, \mathcal{C})$ turns out to be an F–smooth space.

In this case, all other possible F–smooth structures are finer than the above structure. □

Example 1.2.3. Let us consider the set $\boldsymbol{S} := \mathbb{R}^2$, along with the natural smooth chart (x, y), and the set $\mathcal{C}$ consisting of all smooth curves c whose coordinate expression is of the type

$$c^x(\lambda) = a\,, \qquad c^y(\lambda) = \gamma(\lambda)\,,$$

where $a \in \mathbb{R}$ and $\gamma : \boldsymbol{I}_\gamma \to \mathbb{R}$ is any smooth curve.

Then, the pair $(\boldsymbol{S}, \mathcal{C})$ turns out to be an F–smooth space. □

Example 1.2.4. Let us consider the set $\boldsymbol{S} := \mathbb{R}$ and the set $\mathcal{C}$ consisting of all smooth curves $c : \boldsymbol{I}_c \to \boldsymbol{S}$, such that

$$0 \le |c(\lambda_2) - c(\lambda_1)| \le 1\,, \qquad \text{for each} \qquad \lambda_1, \lambda_2 \in \boldsymbol{I}_c\,.$$

Then, the pair $(\boldsymbol{S}, \mathcal{C})$ turns out to be an F–smooth space. □

1.2.2 F–smooth subspaces

Further, we discuss the F–smooth subspaces of an F–smooth space and emphasise an interesting property, which has no analogue for topological or smooth manifolds.

Definition 1.2.3. Let $(\boldsymbol{S}, \mathcal{C})$ be an F-smooth space. Then, an *F–smooth subspace* is defined to be a pair

$$(\boldsymbol{S}', \mathcal{C}') \subseteq (\boldsymbol{S}, \mathcal{C})\,,$$

where $\boldsymbol{S}' \subseteq \boldsymbol{S}$ is a non empty subset and $\mathcal{C}' \subseteq \mathcal{C}$ is the subset consisting of all curves belonging to $\mathcal{C}$ and with values in $\boldsymbol{S}'$.

For short, we say also that $\boldsymbol{S}' \subseteq \boldsymbol{S}$ is an *F–smooth subspace.* □

Proposition 1.2.3. Let us consider an F–smooth space $(\boldsymbol{S}, \mathcal{C})$ and *any* non empty subset $\boldsymbol{S}' \subseteq \boldsymbol{S}$. Then, $\boldsymbol{S}'$ turns out to be an F–smooth space in a natural way.

In fact, let $\mathcal{C}' \subseteq \mathcal{C}$ be the subset consisting of all curves $c \in \mathcal{C}$, whose image is contained in $\boldsymbol{S}'$. Then, the pair $(\boldsymbol{S}', \mathcal{C}')$ turns out to be an F–smooth space.

PROOF. Let us check the two conditions of Definiton 1.2.1.

1) The 1st condition is fulfilled because all constant curves of $\boldsymbol{S}'$ belong to $\mathcal{C}'$.

2) The 2nd condition is also fulfilled. In fact, let $c \in \mathcal{C}'$ and let $\gamma : \boldsymbol{I}_\gamma \to \boldsymbol{I}_c$ be a smooth map. Then, the composition $c \circ \gamma$ belongs to $\mathcal{C}$ and has values in $\boldsymbol{S}'$. QED

Remark 1.2.2. Let us consider an F–smooth space $(\boldsymbol{S}, \mathcal{C})$ and any non empty subset $\boldsymbol{S}' \subseteq \boldsymbol{S}$.

Then, besides the above structure of F–smooth subspace (see Definition 1.2.3), the subset $\boldsymbol{S}'$ might be equipped with other F-smooth structures given by a further subset $\mathcal{C}'' \subseteq \mathcal{C}' \subseteq \mathcal{C}$.

Thus, the above F–smooth structure $\mathcal{C}'$ is the finest F–smooth structure among those which are comparable with the F–smooth structure $\mathcal{C}$ of the environmental space $\boldsymbol{S}$. □

We exhibit an exotic example of F–smooth subspace.

Example 1.2.5. Let us consider the F–smooth space $(\boldsymbol{S}, \mathcal{C})$, where $\boldsymbol{S} := \mathbb{R}$ and $\mathcal{C}$ is the set of all smooth curves of $\mathbb{R}$.

Moreover, consider the subset $\boldsymbol{S}' \subset \boldsymbol{S}$ consisting of all irrational numbers.

Then, the curves belonging to $\mathcal{C}$ and with values in $\boldsymbol{S}'$ are just the constant curves with irrational image. Let $\mathcal{C}'$ be the set of these curves.

Indeed, the pair $(\boldsymbol{S}', \mathcal{C}')$ turns out to be an F–smooth subspace of $(\boldsymbol{S}, \mathcal{C})$. □

1.2.3 F–smooth maps

We introduce the notion of "*F–smooth map*" between F–smooth spaces in a natural way. According to this definition, F–smooth spaces along with global F–smooth maps constitute a category.

Indeed, the restriction property of F–smooth maps is analogous to a property holding for smooth maps; however a gluing property of F–smooth maps does not hold for F–smooth spaces.

Definition 1.2.4. Let us consider two F–smooth spaces $(\boldsymbol{S}, \mathcal{C})$ and $(\boldsymbol{S}', \mathcal{C}')$.

Then, a (local) map

$$f : \boldsymbol{S} \to \boldsymbol{S}'$$

(defined on a subset $\boldsymbol{U} \subset \boldsymbol{S}$) is said to be *F–smooth* if, for each $c \in \mathcal{C}$ (with values in $\boldsymbol{U}$) we have

$$c' \equiv f \circ c \in \mathcal{C}' . \square$$

Proposition 1.2.4. If $\boldsymbol{S}$ is an F-smooth space, then the global map

$$\mathrm{id}_{\boldsymbol{S}} : \boldsymbol{S} \to \boldsymbol{S}$$

is F–smooth.

Mooreover, if $\boldsymbol{S}$, $\boldsymbol{S}'$, $\boldsymbol{S}''$ are F–smooth spaces and

$$f : \boldsymbol{S} \to \boldsymbol{S}' \qquad \text{and} \qquad f' : \boldsymbol{S}' \to \boldsymbol{S}''$$

are global F–smooth maps, then $f' \circ f : \boldsymbol{S} \to \boldsymbol{S}''$ turns out to be a global F–smooth map. □

Remark 1.2.3. Let us consider a set $\boldsymbol{S}$ equipped with two F–smooth structures $\mathcal{C}$ and $\mathcal{C}'$, and suppose that $\mathcal{C}' \subset \mathcal{C}$.

Then,

$$\mathrm{id}_{\boldsymbol{S}} : (\boldsymbol{S}, \mathcal{C}') \to (\boldsymbol{S}, \mathcal{C})$$

turns out to be F–smooth, but

$$\mathrm{id}_{\boldsymbol{S}} : (\boldsymbol{S}, \mathcal{C}) \to (\boldsymbol{S}, \mathcal{C}')$$

is not F–smooth. □

We have a restriction property of F–smooth maps analogous to a property of smooth maps.

Proposition 1.2.5. Let us consider two F–smooth spaces $(\boldsymbol{S}, \mathcal{C})$ and $(\boldsymbol{S}', \mathcal{C}')$.

If $f : \boldsymbol{S} \to \boldsymbol{S}'$ is an F–smooth map, defined on a subset $\boldsymbol{U} \subset \boldsymbol{S}$, then the restriction $\bar{f}$ of f to a further subset $\bar{\boldsymbol{U}} \subset \boldsymbol{U}$ turns out to be F–smooth.

Proof. If $c' \equiv f \circ c \in \mathcal{C}'$, for each $c \in \mathcal{C}$ with values in $\boldsymbol{U}$, then $c' \equiv f \circ \bar{c} \in \mathcal{C}'$, for each $\bar{c} \in \mathcal{C}$ with values in $\bar{\boldsymbol{U}}$. QED

The F–smooth maps do not have a gluing property analogous to the gluing property of smooth maps.

The reason for this difference of behaviour between smooth and F–smooth maps is due to the fact that F–smoothness of maps is a *global property*, while smoothness of maps is a *local property*.

Remark 1.2.4. Let us consider two F–smooth spaces $(\boldsymbol{S}, \mathcal{C})$ and $(\boldsymbol{S}', \mathcal{C}')$ and two maps

$$f_1 : \boldsymbol{S} \to \boldsymbol{S}' \qquad \text{and} \qquad f_2 : \boldsymbol{S} \to \boldsymbol{S}',$$

defined respectively on the subsets $\boldsymbol{U}_1 \subset \boldsymbol{S}$ and $\boldsymbol{U}_2 \subset \boldsymbol{S}$.

Clearly, if $f_1 = f_2$ on $\boldsymbol{U}_1 \cap \boldsymbol{U}_2$, then f_1 and f_2 yield a "glued map" $f : \boldsymbol{S} \to \boldsymbol{S}'$, defined on $\boldsymbol{U}_1 \cup \boldsymbol{U}_2$.

However, the hypothesis that f_1 and f_2 be F–smooth does not imply that f be F–smooth.

In fact, let us provide an exotic countre–example.

Let us consider the F–smooth spaces $(\boldsymbol{S}, \mathcal{C})$ and $(\boldsymbol{S}', \mathcal{C}')$, where $\boldsymbol{S} = \boldsymbol{S}' = \mathbb{R}$.

Suppose that $\mathcal{C}$ consists of all smooth curves of $\boldsymbol{S}$ and that $\mathcal{C}'$ consists of all smooth curves of $\boldsymbol{S}'$ considered in Example 1.2.4.

Next consider the maps

$$f_1 = \mathrm{id} : \boldsymbol{U}_1 = (0,1) \to \boldsymbol{S}' \qquad \text{and} \qquad f_2 = \mathrm{id} : \boldsymbol{U}_2 = (1/2, 3/2) \to \boldsymbol{S}' \,.$$

Clearly, these maps are F–smooth and coincide in the intersection of their domains.

However, the glued map $f : \boldsymbol{S} \to \boldsymbol{S}'$, defined in $(0, 3/2)$ is not F–smooth. □

Next, we discuss the behaviour of F–smooth maps with respect different smooth structures.

Proposition 1.2.6. Let $\boldsymbol{S}$ be a set equipped with two different F–smooth structures $(\boldsymbol{S}, \mathcal{C})$ and $(\boldsymbol{S}, \overline{\mathcal{C}})$, with $\mathcal{C} \subseteq \overline{\mathcal{C}}$ and let $(\boldsymbol{S}', \mathcal{C}')$ be another F–smooth space.

Let

$$\mathrm{Map}_{\mathcal{C}\mathcal{C}'}(\boldsymbol{S},\, \boldsymbol{S}')\,, \quad \mathrm{Map}_{\overline{\mathcal{C}}\mathcal{C}'}(\boldsymbol{S},\, \boldsymbol{S}') \quad \text{and} \quad \mathrm{Map}_{\mathcal{C}'\mathcal{C}}(\boldsymbol{S}',\, \boldsymbol{S})\,, \quad \mathrm{Map}_{\mathcal{C}'\overline{\mathcal{C}}}(\boldsymbol{S}',\, \boldsymbol{S})$$

be, respectively, the sets of maps $\boldsymbol{S} \to \boldsymbol{S}'$ and $\boldsymbol{S}' \to \boldsymbol{S}$, which are F–smooth with respect to the two different F–smooth structures of $\boldsymbol{S}$.

Then, we have

$$\mathrm{Map}_{\overline{\mathcal{C}}\mathcal{C}'}(\boldsymbol{S},\, \boldsymbol{S}') \subseteq \mathrm{Map}_{\mathcal{C}\mathcal{C}'}(\boldsymbol{S},\, \boldsymbol{S}') \quad \text{and} \quad \mathrm{Map}_{\mathcal{C}'\mathcal{C}}(\boldsymbol{S}',\, \boldsymbol{S}) \subseteq \mathrm{Map}_{\mathcal{C}'\overline{\mathcal{C}}}(\boldsymbol{S}',\, \boldsymbol{S})\,. \square$$

Example 1.2.6. Let $(\boldsymbol{S}, \mathcal{C})$ be an F–smooth space and let $\mathcal{C}$ consist just of constant curves of $\boldsymbol{S}$ (see Example 1.2.2).

Moreover, let $(\boldsymbol{S}', \mathcal{C}')$ be another F–smooth space and let $\mathcal{C}'$ contain also non constant curves of $\boldsymbol{S}'$ passing through each point of $\boldsymbol{S}'$.

Then, a map $f : \boldsymbol{S}' \to \boldsymbol{S}$ is F–smooth if and only if it is constant.

Moreover, all maps $f : \boldsymbol{S} \to \boldsymbol{S}'$ are F–smooth. □

Next, we consider an exotic example of F–smooth maps, just to account for the range of the notion of F–smooth map.

Example 1.2.7. Let us consider the set $\boldsymbol{S} := \mathbb{R}$ and equip it with two different smooth structures induced respectively by the two charts (see also Remark 2.1.1)

$$x : \boldsymbol{S} \to \mathbb{R} : s \mapsto s \qquad \text{and} \qquad \acute{x} : \boldsymbol{S} \to \mathbb{R} : s \mapsto s^3 \,.$$

Clearly, the 1st smooth structure is the natural one, while the 2nd one is an exotic smooth structure. Indeed, these smooth structures are different because the transition rule

$$x = (\acute{x})^{1/3}$$

is not differentiable in $s = 0 \in \boldsymbol{S}$.

Next, let us define two F–smooth structures $(\boldsymbol{S}, \mathcal{C})$ and $(\boldsymbol{S}, \mathcal{C}')$ on $\boldsymbol{S}$, where, respectively, $\mathcal{C}$ and $\acute{\mathcal{C}}$ consist of the curves $c : \boldsymbol{I}_c \to \boldsymbol{S}$ and $\acute{c} : \boldsymbol{I}_{\acute{c}} \to \boldsymbol{S}$ whose coordinate expressions

$$x \circ c : \boldsymbol{I}_c \to \mathbb{R} \qquad \text{and} \qquad \acute{x} \circ \acute{c} : \boldsymbol{I}_c \to \mathbb{R}\,,$$

are smooth in the standard sense, with respect to the above smooth structures of $\boldsymbol{S}$, respectively.

Indeed, the above F–smooth structures are different. To prove this fact, the curves with coordinate expressions

$$x \circ c : \boldsymbol{I}_c \to \mathbb{R} : \lambda \mapsto \lambda \qquad \text{and} \qquad \acute{x} \circ \acute{c} : \boldsymbol{I}_{\acute{c}} \to \mathbb{R} : \lambda \mapsto \lambda$$

belong, respectively, to $\mathcal{C}$ and $\acute{\mathcal{C}}$.

Actually, we have $\acute{c} \notin \mathcal{C}$, because the coordinate expression

$$x \circ \acute{c} : \boldsymbol{I}_{\acute{c}} \to \mathbb{R} : \lambda \mapsto \lambda^{1/3}$$

is not differentiable in $s = 0 \in \boldsymbol{S}$.

However, we have $\mathcal{C} \subset \acute{\mathcal{C}}$, because, for each $c \in \mathcal{C}$, the coordinate expression

$$\acute{x} \circ c : \boldsymbol{I}_c \to \mathbb{R} : \lambda \mapsto c(\lambda)^3$$

is smooth in the standard sense. □

1.2.4 Cartesian product of F–smooth spaces

Eventually, we analyse the F–smooth cartesian product of F–smooth spaces.

Proposition 1.2.7. Let $(\boldsymbol{S}', \mathcal{C}')$ and $(\boldsymbol{S}'', \mathcal{C}'')$ be F-smooth spaces and set

$$\mathcal{C} := \Big\{ (c', c'') \in \mathcal{C}' \times \mathcal{C}'' \mid \boldsymbol{I}_{c'} = \boldsymbol{I}_{c''} \Big\}\,.$$

Then, the pair $(\boldsymbol{S}' \times \boldsymbol{S}'', \mathcal{C})$ turns out to be an F–smooth space.

Moreover, the natural projections

$$\mathrm{pro}' : \boldsymbol{S}' \times \boldsymbol{S}'' \to \boldsymbol{S}' \qquad \text{and} \qquad \mathrm{pro}'' : \boldsymbol{S}' \times \boldsymbol{S}'' \to \boldsymbol{S}''$$

turn out to be F-smooth.

PROOF. 1) The pair $(\boldsymbol{S}' \times \boldsymbol{S}'', \mathcal{C})$ is an F–smooth space.

In fact, we can easily see that the set $\mathcal{C}$ fulfills the properties of Definition 1.2.1.

2) The maps

$$\mathrm{pro}' : \boldsymbol{S}' \times \boldsymbol{S}'' \to \boldsymbol{S}' \qquad \text{and} \qquad \mathrm{pro}'' : \boldsymbol{S}' \times \boldsymbol{S}'' \to \boldsymbol{S}''$$

are F–smooth.

In fact, for each F–smooth curve $c := (c', c'') : \boldsymbol{I}_c \to \boldsymbol{S}' \times \boldsymbol{S}''$, the curves

$$c' = \mathrm{pro}' \circ c \qquad \text{and} \qquad c'' = \mathrm{pro}'' \circ c$$

are F–smooth by hypothesis. QED

1.3 Smooth manifolds as F–smooth spaces

Each smooth manifold turns out to be an F–smooth space in a natural way.

However, each set $\boldsymbol{S}$ might be equipped with many smooth structures; hence, these smooth manifolds turn out to be F–smooth spaces in many ways.

Indeed, we can achieve a "universal" F–smooth structure of a set $\boldsymbol{S}$ by taking the union of all families of basic curves of all F–smooth structures.

Moreover, an F-smooth space $\boldsymbol{S}$ might be a smooth manifold in many ways.

Here, we do not address the above relation between smooth structures and F–smooth structures in detail and full generality. But, we just clarify this question by an example.

Theorem 1.3.1. *Each smooth manifold $\boldsymbol{M}$, along with the set $\mathcal{C}$ consisting of all smooth curves $c : \boldsymbol{I}_c \to \boldsymbol{M}$, turns out to be an F–smooth space.*

Moreover, a map between smooth manifolds turns out to be F–smooth (in the sense of the above natural F–smooth structures) if and only if it is smooth.

PROOF. The non trivial proof of this result is due to A. Frölicher [6] and is based on the Boman's theorem [1].QED

Definition 1.3.1. According to the above Theorem 1.3.1, for each smooth manifold $\boldsymbol{M}$, we define its *natural F–smooth structure* to be provided by the set $\mathcal{C}$ consisting of all smooth curves $c : \boldsymbol{I}_c \to \boldsymbol{M}$. □

Indeed, from now on, for each smooth manifold $\boldsymbol{M}$, we shall refer to its natural F–smooth structure.

Corollary 1.3.1. Let $\boldsymbol{M}$ be a smooth manifold. Then, a function $f : \boldsymbol{M} \to \mathbb{R}$ is F–smooth if and only if the map $f \circ c : \boldsymbol{I}_c \to \mathbb{R}$ is smooth for each smooth curve $c : \boldsymbol{I}_c \to \boldsymbol{M}$. □

Corollary 1.3.2. Let $(\boldsymbol{S}, \mathcal{C})$ be an F–smooth space and $c : \boldsymbol{I}_c \to \boldsymbol{S}$ any curve.

Then, c is F–smooth if and only if $c \in \mathcal{C}$.

PROOF. Let us consider $\boldsymbol{I}_c$ as a smooth manifold equipped with its natural F–smooth structure.

1) If $c \in \mathcal{C}$, then, in virtue of Definition 1.2.1, for each smooth curve $\gamma : \boldsymbol{I}_\gamma \to \boldsymbol{I}_c$, we have $c \circ \gamma \in \mathcal{C}$.

Hence, in virtue of Definition 1.2.4, c is F–smooth.

2) If c is F–smooth, then, in virtue of Definition 1.2.4, for each smooth curve $\gamma : \boldsymbol{I}_\gamma \to \boldsymbol{I}_c$, we have $c \circ \gamma \in \mathcal{C}$.

Hence, in particular, we have $c = c \circ \mathrm{id}_{\boldsymbol{I}_c} \in \mathcal{C}$. QED

Eventually, we provide an example suitable to compare different smooth and F-smooth structures.

Example 1.3.1. Let us consider the set $\boldsymbol{S} := \mathbb{R}$.

On this set $\boldsymbol{S}$ we have a "natural" smooth structure provided by the natural global chart $x := \mathrm{id} : \boldsymbol{S} \to \mathbb{R}$.

Moreover, we can equip the set $\boldsymbol{S}$ with the further "exotic" smooth structure provided by the global chart $\acute{x} := x^3$.

If we regard $\boldsymbol{S}$ as an F–smooth space according to its natural smooth structure x, then the family of F–smooth curves $\mathcal{C}_x$ consists of all curves $c : \boldsymbol{I}_c \to \boldsymbol{S}$, such that the function $x \circ c : \boldsymbol{I}_c \to \mathbb{R}$ are smooth.

If we regard $\boldsymbol{S}$ as an F–smooth space according to its exotic smooth structure $\acute{x}$, then the family of F–smooth curves $\mathcal{C}_{\acute{x}}$ consists of all curves $c : \boldsymbol{I}_c \to \boldsymbol{S}$, such that the function $\acute{x} \circ c : \boldsymbol{I}_c \to \mathbb{R}$ are smooth.

Clearly, we have

$$\mathcal{C}_x \subset \mathcal{C}_{\acute{x}} .$$

In an analogous way, we can equip $\boldsymbol{S}$ with other "exotic" smooth structures in infinitely many ways and regard $\boldsymbol{S}$ as an F–smooth space in infinitely many ways according to each smooth structure. Indeed, the union of all families of F–smooth curves obtained in this way equips $\boldsymbol{S}$ with a "universal" F–smooth structure.

Thus, conversely, the above construction provides an example of an F–smooth space, which can be regarded as a smooth manifold in infinitely many ways. □

Chapter 2

Systems of maps

First, we discuss the *smooth systems* $(\boldsymbol{S}, \epsilon)$ of *smooth maps* $f : \boldsymbol{M} \to \boldsymbol{N}$. Here, the "space of parameters" $\boldsymbol{S}$ is a *smooth manifold* and the "evaluation map" $\epsilon : \boldsymbol{S} \times \boldsymbol{M} \to \boldsymbol{N}$ a *smooth map*.

Then, we discuss the *F–smooth systems* $(\boldsymbol{S}, \epsilon)$ of *smooth maps* $f : \boldsymbol{M} \to \boldsymbol{N}$. Here, the "space of parameters" $\boldsymbol{S}$ is an *F–smooth space* and the "evaluation map" $\epsilon : \boldsymbol{S} \times \boldsymbol{M} \to \boldsymbol{N}$ an *F–smooth map*.

The above notions are intended as an introduction to the more sophisticated notions of systems of sections discussed in the next Chapter §3.

The reader can find further discussions concerning the present subject in [19].

2.1 Smooth systems of smooth maps

We discuss the notion of *smooth system of smooth maps* and the tangent prolongations of a smooth system of smooth maps as an introduction to the subsequent notions of smooth systems of smooth sections and smooth systems of smooth connections.

2.1.1 Smooth systems of smooth maps

We start by defining the concept of *smooth system of smooth maps* between two smooth manifolds.

In simple words, a "smooth system of smooth maps" is defined to be a family $\{f\}$ of global smooth maps $f : \boldsymbol{M} \to \boldsymbol{N}$ between two smooth manifolds $\boldsymbol{M}$ and $\boldsymbol{N}$, which is smoothly parametrised by the points $s \in \boldsymbol{S}$ of a smooth manifold $\boldsymbol{S}$.

This simple concept will serve as an introduction to the further notions of "smooth system of smooth sections" of a smooth double fibred manifold and "smooth system of smooth connections" of a smooth fibred manifold (see, later, Definition 3.1.3 and Definition 4.1.1).

Later, this concept will serve also as an introduction to the further more sophisticated notions of "F–smooth system of smooth maps" between smooth manifolds, of "F–smooth system of fibrewisely smooth sections" of a smooth double fibred manifold and of "F–smooth system of fibrewisely F–smooth connections" of a smooth double fibred manifold (see, later, Definition 2.2.1 and Definition 3.2.2).

Let us consider two smooth manifolds $\boldsymbol{M}$ and $\boldsymbol{N}$, and denote the set of global smooth maps between the two manifolds by $\mathrm{Map}(\boldsymbol{M},\boldsymbol{N}) := \big\{ f : \boldsymbol{M} \to \boldsymbol{N} \big\}$.

Definition 2.1.1. We define a *smooth system of smooth maps*, between the smooth manifolds $\boldsymbol{M}$ and $\boldsymbol{N}$, to be a pair $(\boldsymbol{S}, \epsilon)$, where

1) $\boldsymbol{S}$ is a smooth manifold,
2) ϵ is a global smooth map, called *evaluation map*,

$$\epsilon : \boldsymbol{S} \times \boldsymbol{M} \to \boldsymbol{N} .$$

Thus, the evaluation map ϵ yields the map

$$\epsilon_{\boldsymbol{S}} : \boldsymbol{S} \to \mathrm{Map}(\boldsymbol{M},\boldsymbol{N}) : s \mapsto \breve{s} ,$$

where, for each $s \in \boldsymbol{S}$, the global smooth map $\breve{s}$ is defined by

$$\breve{s} : \boldsymbol{M} \to \boldsymbol{N} : m \mapsto \epsilon(s, m) .$$

Therefore, the map $\epsilon_{\boldsymbol{S}} : \boldsymbol{S} \to \mathrm{Map}(\boldsymbol{M},\boldsymbol{N})$ provides a *selection* of the global smooth maps $\boldsymbol{M} \to \boldsymbol{N}$, given by the subset

$$\mathrm{Map}_{\boldsymbol{S}}(\boldsymbol{M},\boldsymbol{N}) := \epsilon_{\boldsymbol{S}}(\boldsymbol{S}) \subset \mathrm{Map}(\boldsymbol{M},\boldsymbol{N}) .$$

The smooth system of smooth maps $(\boldsymbol{S}, \epsilon)$ is said to be *injective* if the map

$$\epsilon_{\boldsymbol{S}} : \boldsymbol{S} \to \mathrm{Map}(\boldsymbol{M},\boldsymbol{N})$$

is injective, i.e. if, for each $s, \acute{s} \in \boldsymbol{S}$,

$$\epsilon_{\boldsymbol{S}}(s) = \epsilon_{\boldsymbol{S}}(\acute{s}) \qquad \Rightarrow \qquad s = s' .$$

If the system is injective, then we obtain the bijection

$$\epsilon_{\boldsymbol{S}} : \boldsymbol{S} \to \mathrm{Map}_{\boldsymbol{S}}(\boldsymbol{M}, \boldsymbol{N}) : s \mapsto \breve{s} \,,$$

whose inverse is denoted by

$$(\epsilon_{\boldsymbol{S}})^{-1} : \mathrm{Map}_{\boldsymbol{S}}(\boldsymbol{M}, \boldsymbol{N}) \to \boldsymbol{S} : f \mapsto \widehat{f} \,. \square$$

Indeed, we are essentially interested in injective F–smooth systems of smooth maps.

We denote the local smooth charts of $\boldsymbol{M}$, $\boldsymbol{N}$, $\boldsymbol{S}$ respectively, by

$$(y^i) : \boldsymbol{M} \to \mathbb{R}^{d_{\boldsymbol{M}}} \,, \qquad (z^a) : \boldsymbol{N} \to \mathbb{R}^{d_{\boldsymbol{N}}} \,, \qquad (w^A) : \boldsymbol{S} \to \mathbb{R}^{d_{\boldsymbol{S}}} \,.$$

We have the following elementary examples of smooth systems of smooth maps.

Example 2.1.1. If $\boldsymbol{M}$ and $\boldsymbol{N}$ are vector spaces, then we obtain the injective smooth system of linear maps by setting

$$\boldsymbol{S} := \mathrm{lin}(\boldsymbol{M}, \boldsymbol{N}) \,.$$

In this case, the smooth manifold $\boldsymbol{S}$ turns out to be a vector space of dimension

$$d_{\boldsymbol{S}} = d_{\boldsymbol{M}} \cdot d_{\boldsymbol{N}} \,.$$

The choice of a basis of $\boldsymbol{M}$ and a basis of $\boldsymbol{N}$ yields the distinguished linear (global) charts

$$(y^i) : \boldsymbol{M} \to \mathbb{R}^{d_{\boldsymbol{M}}} \,, \qquad (z^a) : \boldsymbol{N} \to \mathbb{R}^{d_{\boldsymbol{N}}} \,, \qquad (w^a_i) : \boldsymbol{S} \to \mathbb{R}^{d_{\boldsymbol{M}} \cdot d_{\boldsymbol{M}}} \,.$$

Then, the coordinate expression of ϵ becomes

$$\epsilon^a = w^a_i \, y^i \,. \square$$

Example 2.1.2. If $\boldsymbol{M}$ and $\boldsymbol{N}$ are affine spaces, then we obtain the injective smooth system of affine maps by setting

$$\boldsymbol{S} := \mathrm{aff}(\boldsymbol{M}, \boldsymbol{N}) \,.$$

In this case, the smooth manifold $\boldsymbol{S}$ turns out to be an affine space, of dimension

$$d_{\boldsymbol{S}} = d_{\boldsymbol{M}} \cdot d_{\boldsymbol{N}} + d_{\boldsymbol{N}} \,,$$

which is associated with the vector space $\operatorname{aff}(\boldsymbol{M}, \bar{\boldsymbol{N}})$, where $\bar{\boldsymbol{N}}$ is the vector space associated with the affine space $\boldsymbol{N}$.

The choice of a basis and an origin of $\boldsymbol{M}$ and a basis and an origin of $\boldsymbol{N}$ yields the distinguished affine (global) charts

$$(y^i) : \boldsymbol{M} \to \mathbb{R}^{d_{\boldsymbol{M}}}, \qquad (z^a) : \boldsymbol{N} \to \mathbb{R}^{d_{\boldsymbol{N}}}, \qquad (w^a_i, w^a) : \boldsymbol{S} \to \mathbb{R}^{d_{\boldsymbol{M}} \cdot d_{\boldsymbol{M}}} \times \mathbb{R}^{d_{\boldsymbol{N}}}.$$

Accordingly, the coordinate expression of ϵ becomes

$$\epsilon^a = w^a_i \, y^i + w^a . \square$$

Example 2.1.3. If $\boldsymbol{M}$ and $\boldsymbol{N}$ are affine spaces, then, analogously to the above Example 2.1.2, we can define

- the injective smooth system of polynomial maps of a given degree r, with $0 \leq r$,

- the injective smooth system of polynomial maps of any degree r, with $0 \leq r \leq k$, where k is a given positive integer. $\square$

All examples above deal with finite dimensional smooth systems of maps, as it is implicitly requested in Definition 2.1.1.

However, we can easily extend the concept of smooth system of smooth maps between two smooth manifolds, by considering an infinite dimensional system, which is the direct limit of finite dimensional systems, according to the following Example 2.1.4. In a sense, this example is intermediate between the finite dimensional case and the infinite dimensional case.

Example 2.1.4. If $\boldsymbol{M}$ and $\boldsymbol{N}$ are affine spaces, then we obtain the injective (infinite dimensional) smooth system of all polynomial maps by considering the set

$$\boldsymbol{S} := \operatorname{pol}(\boldsymbol{M}, \boldsymbol{N}),$$

consisting of all polynomials $\boldsymbol{M} \to \boldsymbol{N}$ of any degree r, with $0 \leq r < \infty$.

Later, we shall see that such a smooth system has a natural F–smooth structure (see, later, Definition 1.2.1). $\square$

Eventually we compare three simple examples in order to emphasise further features of smooth systems of global smooth maps.

Example 2.1.5. Let us consider the smooth manifolds

$$\boldsymbol{M} := \mathbb{R}, \qquad \boldsymbol{N} := \mathbb{R}, \qquad \boldsymbol{S} := \mathbb{R},$$

equipped with their standard charts

$$(y) : \boldsymbol{M} \to \mathbb{R}\,, \qquad (z) : \boldsymbol{N} \to \mathbb{R}\,, \qquad (w) : \boldsymbol{S} \to \mathbb{R}\,.$$

Moreover, let us consider three smooth systems of smooth maps defined, respectively, by the following evaluation maps

$$\epsilon_1 : \boldsymbol{S} \times \boldsymbol{M} \to \boldsymbol{N}\,, \qquad \epsilon_2 : \boldsymbol{S} \times \boldsymbol{M} \to \boldsymbol{N}\,, \qquad \epsilon_3 : \boldsymbol{S} \times \boldsymbol{M} \to \boldsymbol{N}\,,$$

with coordinate expressions

$$z \circ \epsilon_1 = w\, y\,, \qquad z \circ \epsilon_2 = w^3\, y\,, \qquad z \circ \epsilon_3 = w^2\, y\,.$$

Then,

- $(\boldsymbol{S}, \epsilon_1)$ is the system of linear maps $f : \boldsymbol{M} \to \boldsymbol{N}$,
- $(\boldsymbol{S}, \epsilon_2)$ is the system of linear maps $f : \boldsymbol{M} \to \boldsymbol{N}$,
- $(\boldsymbol{S}, \epsilon_3)$ is the system of linear maps $f : \boldsymbol{M} \to \boldsymbol{N}$ with coefficients $w^2 \geq 0$.

Thus, the systems $(\boldsymbol{S}, \epsilon_1)$ and $(\boldsymbol{S}, \epsilon_2)$ select the same smooth maps, but with different smooth parametrisations. In these cases we have assumed the same smooth structure of $\boldsymbol{S}$, but the transition between the two parametrisations is non smooth in one way.

Moreover, the systems $(\boldsymbol{S}, \epsilon_1)$ and $(\boldsymbol{S}, \epsilon_2)$ are injective, while the system $(\boldsymbol{S}, \epsilon_3)$ is non injective. □

It is useful to compare the above systems $(\boldsymbol{S}, \epsilon_1)$ and $(\boldsymbol{S}, \epsilon_2)$ with the system $(\boldsymbol{S}, \epsilon)$ discussed in the forthcoming Example 2.1.7.

Eventually, we exhibit a further weird example of F–smooth systems of smooth maps.

Example 2.1.6. Let us consider two vector spaces $\boldsymbol{M}$ and $\boldsymbol{N}$, along with the smooth manifold $\boldsymbol{S} := \mathrm{lin}(\boldsymbol{M}, \boldsymbol{N})$ and the smooth map

$$\epsilon : \boldsymbol{S} \times \boldsymbol{M} \to \boldsymbol{N} : (s, m) \mapsto n + s(X)\,,$$

where $n \in \boldsymbol{N}$ and $X \in \boldsymbol{M}$ is a non vanishing given element.

Then, the pair $(\boldsymbol{S}, \epsilon)$ turns out to be an injective smooth system of smooth maps. □

2.1.2 Smooth structure of $(\boldsymbol{S}, \epsilon)$

In our definition of "smooth system of smooth maps" between two smooth manifolds (see Definition 2.1.1) we have required a priori that the set of parameters $\boldsymbol{S}$ be finite dimensional and smooth and that the evaluation map ϵ be smooth as well.

On the ontrary, we might ask whether the fact that the manifolds $\boldsymbol{M}$ and $\boldsymbol{N}$ are smooth and that the maps $f : \boldsymbol{M} \to \boldsymbol{N}$ selected by the system are smooth allows us to recover uniquely the smooth structure of $\boldsymbol{S}$.

The answer to the above question is negative. Here, we do not fully address this question. But, we present a simple example (see Example 2.1.7), where we show that, if $\boldsymbol{S}$ admits a finite dimensional smooth structure compatible with ϵ, then this structure needs not to be unique.

Moreover, we observe that, if we do not assume a priori a finite dimensional smooth structure on $\boldsymbol{S}$, then it might be that no finite dimensional smooth structure at all could be recovered on $\boldsymbol{S}$. To prove this, just consider the system $(\boldsymbol{S}, \epsilon)$, where $\boldsymbol{S}$ is the set of *all* smooth maps $f : \boldsymbol{M} \to \boldsymbol{N}$ (see, later, §2.2.1).

The present question might arise also in comparison with a result which will be achieved later, in the next Section, in the context of "F–smooth systems of smooth maps", where we do not assume a priori any smooth structure of $\boldsymbol{S}$, but we recover uniquely its F–smooth structure (see Definition 2.2.1 and Theorem 2.2.1).

Example 2.1.7. Let us consider the following smooth system $(\boldsymbol{S}, \epsilon)$ of smooth maps between smooth manifolds.

Let us consider the manifolds

$$\boldsymbol{M} := \mathbb{R}, \qquad \boldsymbol{N} := \mathbb{R}, \qquad \boldsymbol{S} := \mathbb{R}.$$

We consider the "natural" smooth structures of $\boldsymbol{M}$ and $\boldsymbol{N}$ induced by their "natural" charts

$$y : \boldsymbol{M} \to \mathbb{R} \qquad \text{and} \qquad z : \boldsymbol{N} \to \mathbb{R}.$$

Further, we consider the "natural" smooth structure and the "exotic" smooth structure of $\boldsymbol{S}$ given respectively by the natural and the exotic charts

$$w : \boldsymbol{S} \to \mathbb{R} \qquad \text{and} \qquad \acute{w} := w^3 : \boldsymbol{S} \to \mathbb{R}.$$

Moreover, let us consider the evaluation map $\epsilon : \boldsymbol{S} \times \boldsymbol{M} \to \boldsymbol{N}$, whose coordinate expression in the above charts reads, respectively, as

$$z \circ \epsilon = w^3 y \qquad \text{and} \qquad z \circ \epsilon = \acute{w}\, y\,.$$

Indeed, the evaluation map ϵ turns out to be the same and smooth with respect to the above different smooth structures of $\boldsymbol{S}$. □

It is useful to compare the previous systems $(\boldsymbol{S}, \epsilon_1)$ and $(\boldsymbol{S}, \epsilon_2)$, discuss in Example 2.1.5, with the above system $(\boldsymbol{S}, \epsilon)$.

Indeed, in the first two systems above we deal with the same smooth structure of $\boldsymbol{S}$ and with different parametrisations of the selected global smooth maps.

Conversely, in the third system we deal with a different smooth structure of $\boldsymbol{S}$.

However, these three systems select the same global smooth maps in different ways.

2.1.3 Smooth tangent prolongation of $(\boldsymbol{S}, \epsilon)$

Given a smooth system of smooth maps $(\boldsymbol{S}, \epsilon)$ between two smooth manifolds $\boldsymbol{M}$ and $\boldsymbol{N}$, we discuss

- the smooth system $(T\boldsymbol{S}, T\epsilon)$ of smooth maps between the smooth manifolds $T\boldsymbol{M}$ and $T\boldsymbol{N}$, which is achieved via the tangent prolongation $T\epsilon$ of ϵ with respect to the both factors $\boldsymbol{S}$ and $\boldsymbol{M}$,

- the smooth system $(T\boldsymbol{S}, T_1\epsilon)$ of smooth maps between the smooth manifolds $\boldsymbol{M}$ and $T\boldsymbol{N}$, which is achieved via the tangent prolongation $T_1\epsilon$ of ϵ with respect to the 1st factor $\boldsymbol{S}$,

- the smooth system $(\boldsymbol{S}, T_2\epsilon)$ of smooth maps between the smooth manifolds $T\boldsymbol{M}$ and $T\boldsymbol{N}$, which is achieved via the tangent prolongation $T_2\epsilon$ of ϵ with respect to the 2nd factor $\boldsymbol{M}$.

We stress that, if the system $(\boldsymbol{S}, \epsilon)$ is injective, then its tangent prolongations $(T\boldsymbol{S}, T\epsilon)$ and $(T\boldsymbol{S}, T_1\epsilon)$ need not to be injective.

Later (see §2.2.1), we shall be involved with a generalised concept of system, where $\boldsymbol{S}$ is no longer a finite dimensional smooth manifold, hence we cannot avail of the standard approach to define its tangent space $T\boldsymbol{S}$. Actually, we will achieve the tangent space of $\boldsymbol{S}$ by an indirect procedure, via smooth maps between smooth manifolds. Indeed, the tangent prolongations of the system $(\boldsymbol{S}, \epsilon)$ discussed in the present Section will be a hint for the more sophisticated cases discussed in the next Chapter (see Section 2.2.2 and Section 3.2.3).

Let us consider two smooth manifolds $\boldsymbol{M}$ and $\boldsymbol{N}$, and a smooth system of smooth maps $(\boldsymbol{S}, \epsilon)$ between these manifolds (see Definition 2.1.1).

We denote the charts of $\boldsymbol{M}$, $\boldsymbol{N}$, $\boldsymbol{S}$, $T\boldsymbol{M}$, $T\boldsymbol{N}$, $T\boldsymbol{S}$ respectively, by

$$\begin{aligned}
(y^i) &: \boldsymbol{M} \to \mathbb{R}^{d_{\boldsymbol{M}}}\,, \qquad & (y^i, \dot{y}^i) &: T\boldsymbol{M} \to \mathbb{R}^{2d_{\boldsymbol{M}}}\,, \\
(z^a) &: \boldsymbol{N} \to \mathbb{R}^{d_{\boldsymbol{N}}}\,, \qquad & (z^a, \dot{z}^a) &: T\boldsymbol{N} \to \mathbb{R}^{2d_{\boldsymbol{N}}}\,, \\
(w^A) &: \boldsymbol{S} \to \mathbb{R}^{d_{\boldsymbol{S}}}\,, \qquad & (w^A, \dot{w}^A) &: T\boldsymbol{S} \to \mathbb{R}^{2d_{\boldsymbol{S}}}\,.
\end{aligned}$$

Proposition 2.1.1. We consider the following *tangent prolongations* of the system.

1) We have the *total tangent prolongation* of ϵ, which yields the smooth map

$$T\epsilon : T\boldsymbol{S} \times T\boldsymbol{M} \to T\boldsymbol{N}\,,$$

with coordinate expression

$$(z^a \circ T\epsilon) = \epsilon^a \qquad \text{and} \qquad (\dot{z}^a \circ T\epsilon) = \partial_A \epsilon^a \, \dot{w}^A + \partial_i \epsilon^a \, \dot{y}^i\,.$$

2) We have the *partial tangent prolongation* of ϵ, *with respect to the 1st factor* which yield the smooth map

$$T_1\epsilon : T\boldsymbol{S} \times \boldsymbol{M} \to T\boldsymbol{N}\,,$$

with coordinate expression

$$(z^a \circ T_1\epsilon) = \epsilon^a \qquad \text{and} \qquad (\dot{z}^a \circ T_1\epsilon) = \partial_A \epsilon^a \, \dot{w}^A\,.$$

3) We have the *partial tangent prolongation* of ϵ, *with respect to the 2nd factor*, which yields the smooth map

$$T_2\epsilon : \boldsymbol{S} \times T\boldsymbol{M} \to T\boldsymbol{N}\,,$$

with coordinate expression

$$(z^a \circ T_2\epsilon) = \epsilon^a \qquad \text{and} \qquad (\dot{z}^a \circ T_2\epsilon) = \partial_i \epsilon^a \, \dot{y}^i\,.$$

Indeed, the following diagrams commute

$$\begin{array}{ccc}
T\boldsymbol{S} \times T\boldsymbol{M} & \xrightarrow{T\epsilon} & T\boldsymbol{N} \\
\downarrow & & \downarrow \\
\boldsymbol{S} \times \boldsymbol{M} & \xrightarrow{\epsilon} & \boldsymbol{N}
\end{array}
\qquad
\begin{array}{ccc}
T\boldsymbol{S} \times \boldsymbol{M} & \xrightarrow{T_1\epsilon} & T\boldsymbol{N} \\
\downarrow & & \downarrow \\
\boldsymbol{S} \times \boldsymbol{M} & \xrightarrow{\epsilon} & \boldsymbol{N}
\end{array}
\qquad
\begin{array}{ccc}
\boldsymbol{S} \times T\boldsymbol{M} & \xrightarrow{T_2\epsilon} & T\boldsymbol{N} \\
\downarrow & & \downarrow \\
\boldsymbol{S} \times \boldsymbol{M} & \xrightarrow{\epsilon} & \boldsymbol{N}
\end{array}\,.$$

Thus, according to the above commutative diagrams,

- the pair $(T\boldsymbol{S}, T\epsilon)$ turns out to be a smooth system of smooth maps between the smooth manifolds $T\boldsymbol{M}$ and $T\boldsymbol{N}$,

- the pair $(T\boldsymbol{S}, T_1\epsilon)$ turns out to be a smooth system of smooth maps between the smooth manifolds $\boldsymbol{M}$ and $T\boldsymbol{N}$,

- the pair $(\boldsymbol{S}, T_2\epsilon)$ turns out to be a smooth system of smooth maps between the smooth manifolds $T\boldsymbol{M}$ and $T\boldsymbol{N}$.

All above prolonged smooth systems project over the smooth system $(\boldsymbol{S}, \epsilon)$.

Indeed, the smooth system $(T\boldsymbol{S}, T\epsilon)$ is characterised by the smooth system $(T\boldsymbol{S}, T_2\epsilon)$ and, conversely, the smooth system $(T\boldsymbol{S}, T_2\epsilon)$ is characterised by the smooth system $(T\boldsymbol{S}, T\epsilon)$ in virtue of the equalities

$$T\epsilon = T_1\epsilon + T_2\epsilon \qquad \text{and} \qquad T_2\epsilon = T\epsilon - T_1\epsilon \,. \square$$

Corollary 2.1.1. If the smooth system $(\boldsymbol{S}, \epsilon)$ is injective, then also the smooth system $(\boldsymbol{S}, T_2\epsilon)$ turns out to be injective, in virtue of the projectability on $(\boldsymbol{S}, \epsilon)$. $\square$

Proposition 2.1.2. The following implications hold:

- if the smooth system $(T\boldsymbol{S}, T\epsilon)$ is injective then the smooth system $(\boldsymbol{S}, \epsilon)$ is injective as well,

- if the smooth system $(T\boldsymbol{S}, T_1\epsilon)$ is injective then the smooth system $(\boldsymbol{S}, \epsilon)$ is injective as well.

Proof. In fact, the smooth system $(\boldsymbol{S}, \epsilon)$ is obtained by projection of its tangent prolongations, according to the commutative diagrams in Proposition 2.1.1. QED

Remark 2.1.1. If the smooth system $(\boldsymbol{S}, \epsilon)$ is injective, then its tangent prolongations $(T\boldsymbol{S}, T\epsilon)$ and $(T\boldsymbol{S}, T_1\epsilon)$ need not to be injective. $\square$

Now, let us examine the tangent prolongations of two elementary examples of smooth systems of smooth maps.

Example 2.1.8. Let us refer to the smooth system $(\boldsymbol{S}, \epsilon)$ of linear maps discussed in Example 2.1.1.

Then, the coordinate expressions of the smooth systems

$$T\epsilon : T\boldsymbol{S} \times T\boldsymbol{M} \to T\boldsymbol{N} \qquad \text{and} \qquad T_1\epsilon : T\boldsymbol{S} \times \boldsymbol{M} \to T\boldsymbol{N}$$

are

$$z^a \circ T\epsilon = w^a_i \, y^i \qquad \text{and} \qquad \dot{z}^a \circ T\epsilon \; = \dot{w}^a_i \, y^i + w^a_i \, \dot{y}^i \, ,$$
$$z^a \circ T_1\epsilon = w^a_i \, y^i \qquad \text{and} \qquad \dot{z}^a \circ T_1\epsilon = \dot{w}^a_i \, y^i \, .$$

Indeed, the above coordinate expressions show that the two tangent prolongations are injective. □

Example 2.1.9. Let us refer to the smooth system $(\boldsymbol{S}, \epsilon)$ of affine maps discussed in Example 2.1.2.

Then, the coordinate expressions of the smooth systems

$$T\epsilon : T\boldsymbol{S} \times T\boldsymbol{M} \to T\boldsymbol{N} \qquad \text{and} \qquad T_1\epsilon : T\boldsymbol{S} \times \boldsymbol{M} \to T\boldsymbol{N}$$

are

$$z^a \circ T\epsilon = w^a_i \, y^i + w^a \qquad \text{and} \qquad \dot{z}^a \circ T\epsilon \; = \dot{w}^a_i \, y^i + w^a_i \, \dot{y}^i + \dot{w}^a \, ,$$
$$z^a \circ T_1\epsilon = w^a_i \, y^i + w^a \qquad \text{and} \qquad \dot{z}^a \circ T_1\epsilon = \dot{w}^a_i \, y^i + \dot{w}^a \, .$$

Indeed, the above coordinate expressions show that the two tangent prolongations are injective. □

In a similar way we can define (see Example 2.1.3)

- the tangent prolongations of the smooth system of polynomial maps of a given degree r, with $0 \le r$,

- the tangent prolongations of the system of polynomial maps of any degree r, with $0 \le r \le k$,

- the tangent prolongations of the smooth system of polynomial maps of any degree r, with $0 \le r \le \infty$.

2.2 F–smooth systems of smooth maps

We discuss the *F–smooth systems* $(\boldsymbol{S}, \epsilon)$ of smooth maps $f : \boldsymbol{M} \to \boldsymbol{N}$ between smooth manifolds and its F–smooth tangent prolongation $(\mathsf{T}\boldsymbol{S}, \mathsf{T}\epsilon)$.

Moreover, we compare the F–smooth systems of smooth maps with the smooth systems of smooth maps.

2.2.1 F–smooth systems of smooth maps

In the previous Section 2.1, we have discussed the concept of a "smooth system of smooth maps" $(\boldsymbol{S}, \epsilon)$ between two smooth manifolds $\boldsymbol{M}$ and $\boldsymbol{N}$ (see Section 2).

Here, we introduce the generalised notion of "*F–smooth system of smooth maps*" $(\boldsymbol{S}, \epsilon)$ between two smooth manifolds $\boldsymbol{M}$ and $\boldsymbol{N}$, by releasing the hypotheses of smoothness and finite dimension of $\boldsymbol{S}$.

The general concept of F–smooth space, which has been introduced in the above Chapter (see Section 1.2) can be applied to a large spectrum of contexts.

Indeed, for our purposes, the most interesting examples of F–smooth spaces are the F–smooth spaces of smooth maps between two smooth manifolds. Even more, this kind of examples provide the true reason of our interest on F–smooth spaces in the present report.

To be more precise, our main interest deals with the particular case of F–smooth systems of fibrewisely smooth sections, which will be introduced in Section 3.2.1. Thus, here the concept of "F–smooth system of smooth maps" is intended as an introduction to the more sophisticated concept of "F–smooth system of fibrewisely smooth sections".

Indeed, a geometric approach to the space of smooth maps between smooth manifolds, in terms of the standard differential geometry, would possibly involve subtle and hard problems concerning infinite dimensional smooth manifolds. Conversely, the spaces of smooth maps between two smooth manifolds, regarded as F–smooth spaces, allow us to achieve several geometric results, even if the structure of F–smooth space is weaker than that of smooth manifold (see, for instance [3, 4, 16]).

In the next Section 2.2.2, as a starting example of such geometric constructions, we sketch the tangent space of F–smooth spaces of global smooth maps.

Let us consider two smooth manifolds $\boldsymbol{M}$ and $\boldsymbol{N}$.

The following Definition provides a generalisation of the concept of smooth system of smooth maps (see Definition 2.1.1), as here we do not require that $\boldsymbol{S}$ be a finite dimensional smooth manifold (hence, that the map ϵ be smooth).

Even more, we do not assume a priori any kind of smoothness on the set $\boldsymbol{S}$, but later we will uniquely recover its F–smooth structure by Theorem 2.2.1. Indeed, the specification "*F–smooth*" system, which is anticipated in the following Definition 2.2.1, will be justified later by this Theorem.

Definition 2.2.1. We define an *F–smooth system of smooth maps* between the smooth manifolds $\boldsymbol{M}$ and $\boldsymbol{N}$ to be a pair $(\boldsymbol{S}, \epsilon)$, where

1) $\boldsymbol{S}$ is a set,

2) $\epsilon : \boldsymbol{S} \times \boldsymbol{M} \to \boldsymbol{N}$ is a map, called *evaluation map*, such that, for each $s \in \boldsymbol{S}$, the induced map

$$\epsilon_s : \boldsymbol{M} \to \boldsymbol{N} : m \mapsto \epsilon(s, m)$$

is *globally defined and smooth.*

Thus, the evaluation map ϵ yields the map

$$\epsilon_{\boldsymbol{S}} : \boldsymbol{S} \to \mathrm{Map}(\boldsymbol{M}, \boldsymbol{N}) : s \mapsto \breve{s}\,,$$

where, for each $s \in \boldsymbol{S}$, the global smooth map $\breve{s}$ is defined by

$$\breve{s} : \boldsymbol{M} \to \boldsymbol{N} : m \mapsto \epsilon(s, m)\,.$$

Therefore, the map

$$\epsilon_{\boldsymbol{S}} : \boldsymbol{S} \to \mathrm{Map}(\boldsymbol{M}, \boldsymbol{N})$$

provides a *selection* of the global smooth maps $\boldsymbol{M} \to \boldsymbol{N}$, given by the subset

$$\mathrm{Map}_{\boldsymbol{S}}(\boldsymbol{M}, \boldsymbol{N}) := \epsilon_{\boldsymbol{S}}(\boldsymbol{S}) \subset \mathrm{Map}(\boldsymbol{M}, \boldsymbol{N})\,.$$

The F–smooth system of smooth maps $(\boldsymbol{S}, \epsilon)$ is said to be *injective* if the map

$$\epsilon_{\boldsymbol{S}} : \boldsymbol{S} \to \mathrm{Map}(\boldsymbol{M}, \boldsymbol{N})$$

is injective, i.e. if, for each $s\,, \acute{s} \in \boldsymbol{S}$,

$$\epsilon_{\boldsymbol{S}}(s) = \epsilon_{\boldsymbol{S}}(\acute{s}) \qquad \Rightarrow \qquad s = s'\,.$$

If the system is injective, then we obtain the bijection

$$\epsilon_{\boldsymbol{S}} : \boldsymbol{S} \to \mathrm{Map}_{\boldsymbol{S}}(\boldsymbol{M}, \boldsymbol{N}) : s \mapsto \breve{s}\,,$$

whose inverse is denoted by

$$(\epsilon_{\boldsymbol{S}})^{-1} : \mathrm{Map}_{\boldsymbol{S}}(\boldsymbol{M}, \boldsymbol{N}) \to \boldsymbol{S} : f \mapsto \widehat{f}\,. \square$$

Indeed, we are essentially interested in injective F–smooth systems of smooth maps.

Next, we prove that any system of smooth maps has a natural F–smooth structure. Actually, this property justifies the fact that we have anticipated the F–smoothness in Definition 2.2.1.

Theorem 2.2.1. *Let us consider an F–smooth system* $(\boldsymbol{S}, \epsilon)$ *of global smooth maps and let*

$$\mathcal{C} := \{c : \boldsymbol{I}_c \to \boldsymbol{S}\},$$

be the set consisting of all *curves, such that the induced map*

$$c^*(\epsilon) : \boldsymbol{I}_c \times \boldsymbol{M} \to \boldsymbol{N} : (\lambda, m) \mapsto \epsilon\big(c(\lambda), m\big)$$

be smooth.

Then, the pair $(\boldsymbol{S}, \mathcal{C})$ *turns out to be an F–smooth space.*

Proof. Let us check that $\mathcal{C}$ fulfills the two requirements of Definition 1.2.1.

1) For each $s \in \boldsymbol{S}$, the constant curve $c : \mathbb{R} \to \boldsymbol{S} : \lambda \mapsto s$ yields the map

$$c^*(\epsilon) : \boldsymbol{I}_c \times \boldsymbol{M} \to \boldsymbol{N} : (\lambda, m) \mapsto \epsilon\big(c(\lambda), m\big) = \epsilon(s, m),$$

which is smooth in virtue of condition 2) in Definition 2.2.1. Hence, c turns out to be an element of $\mathcal{C}$ passing through s.

2) Let us consider a curve $c \in \mathcal{C}$ and a smooth curve $\gamma : \boldsymbol{I}_\gamma \to \boldsymbol{I}_c$.

Then, the induced map

$$(c \circ \gamma)^*(\epsilon) : \boldsymbol{I}_\gamma \times \boldsymbol{M} \to \boldsymbol{N} : (\lambda, m) \mapsto \epsilon\Big(c\big(\gamma(\lambda)\big), m\Big)$$

is the composition of two smooth maps

$$\boldsymbol{I}_\gamma \times \boldsymbol{M} \xrightarrow{(\gamma, \mathrm{id}_{\boldsymbol{M}})} \boldsymbol{I}_c \times \boldsymbol{M} \xrightarrow{c^*(\epsilon)} \boldsymbol{N},$$

hence it turns out to be smooth. QED

Corollary 2.2.1. Let us consider an F–smooth system $(\boldsymbol{S}, \epsilon)$ of smooth maps.

Then, the map $\epsilon : \boldsymbol{S} \times \boldsymbol{M} \to \boldsymbol{N}$ turns out to be F-smooth, with reference to the F-smooth structure of $\boldsymbol{S}$ and the natural F-smooth structures of $\boldsymbol{M}$ and $\boldsymbol{N}$ (see the above Theorem 2.2.1, Theorem 1.3.1, Definition 1.3.1 and Proposition 1.2.7).

Proof. By definition (see Definition 1.2.4), the map $\epsilon : \boldsymbol{S} \times \boldsymbol{M} \to \boldsymbol{N}$ is F–smooth if, for each F–smooth curve $c : \boldsymbol{I}_c \to \boldsymbol{S} \times \boldsymbol{M}$, the curve $\epsilon \circ c : \boldsymbol{I}_c \to \boldsymbol{N}$, given by the commutative diagram

$$\begin{array}{ccc} \boldsymbol{S} \times \boldsymbol{M} & \xrightarrow{\quad \epsilon \quad} & \boldsymbol{N} \\ {\scriptstyle c}\uparrow & & \downarrow{\scriptstyle \mathrm{id}_{\boldsymbol{N}}} \\ \boldsymbol{I}_c & \xrightarrow{\quad \epsilon \circ c \quad} & \boldsymbol{N} \end{array},$$

is smooth.

Thus, let us consider any F–smooth curve $c : \boldsymbol{I}_c \to \boldsymbol{S} \times \boldsymbol{M}$.

In virtue of Proposition 1.2.7 and Theorem 1.3.1, we can write

$$c \equiv (c_{\boldsymbol{S}}, c_{\boldsymbol{M}}) : \boldsymbol{I}_c \to \boldsymbol{S} \times \boldsymbol{M},$$

where $c_{\boldsymbol{S}} : \boldsymbol{I}_c \to \boldsymbol{S}$ is an F–smooth curve and $c_{\boldsymbol{M}} : \boldsymbol{I}_c \to \boldsymbol{M}$ is a smooth curve.

Let us define the smooth map

$$c^*_{\boldsymbol{S}}(\epsilon) : \boldsymbol{I}_c \times \boldsymbol{M} \to \boldsymbol{N} : (\lambda, m) \mapsto \epsilon\big(c_{\boldsymbol{S}}(\lambda), m\big).$$

Then, the curve

$$\epsilon \circ c = \epsilon \circ (c_{\boldsymbol{S}}, c_{\boldsymbol{M}}) : \boldsymbol{I}_c \to \boldsymbol{N} : \lambda \mapsto \epsilon\big(c_{\boldsymbol{S}}(\lambda),\, c_{\boldsymbol{M}}(\lambda)\big) = c^*_{\boldsymbol{S}}(\epsilon)(\lambda, m),$$

which is a composition of smooth maps according to the following digram

$$\begin{array}{ccc}
\boldsymbol{I}_c & \xrightarrow{(\mathrm{id},\, c_{\boldsymbol{M}})} & \boldsymbol{I}_c \times \boldsymbol{M} \\
{\scriptstyle \mathrm{id}}\big\uparrow & & \big\downarrow {\scriptstyle c^*_{\boldsymbol{S}}(\epsilon)} \\
\boldsymbol{I}_c & \xrightarrow{\epsilon \circ c} & \boldsymbol{N}
\end{array} \quad ,$$

turns out to be smooth hence F–smooth (see Theorem 2.2.1). QED

Now, we provide examples of F–smooth systems of smooth maps, by starting with "infinite dimensional" examples of systems of smooth maps.

Example 2.2.1. Let us consider two smooth manifolds $\boldsymbol{M}$ and $\boldsymbol{N}$.

Then, the set

$$\boldsymbol{S} := \mathrm{Map}(\boldsymbol{M}, \boldsymbol{N}) := \big\{ f : \boldsymbol{M} \to \boldsymbol{N} \big\}$$

consisting of *all global smooth maps* $f : \boldsymbol{M} \to \boldsymbol{N}$ yields an F–smooth system of smooth maps. □

Example 2.2.2. Let us consider a smooth manifold $\boldsymbol{M}$ and a vector space $\boldsymbol{N}$.

Then, the subset

$$\boldsymbol{S} := \mathrm{Map}_{\mathrm{cpt}}(\boldsymbol{M}, \boldsymbol{N}) \subset \mathrm{Map}(\boldsymbol{M}, \boldsymbol{N}) := \big\{ f : \boldsymbol{M} \to \boldsymbol{N} \big\}$$

consisting of *all global smooth maps* $f : \boldsymbol{M} \to \boldsymbol{N}$ *with compact support* yields an F–smooth system of smooth maps.

Indeed, this system is a subsystem of the system of all global smooth maps $\boldsymbol{M} \to \boldsymbol{N}$ considered in the above Example 2.2.2. □

We can reconsider, in the present context of F–smooth spaces, "finite dimensional" examples of systems of global smooth maps already discussed in Section 2.1.1.

Example 2.2.3. According to Example 2.1.1, if $\boldsymbol{M}$ and $\boldsymbol{N}$ are vector spaces, then we obtain the F–smooth system of linear maps by setting

$$\boldsymbol{S} := \text{lin}(\boldsymbol{M}, \boldsymbol{N}) .$$

Indeed, the F–smooth structure of $\boldsymbol{S}$ turns out to be just the natural F–smooth structure underlying the smooth structure of $\boldsymbol{S}$, according to Theorem 2.2.1 and Example 2.1.1. □

Example 2.2.4. According to Example 2.1.2, if $\boldsymbol{M}$ and $\boldsymbol{N}$ are affine spaces, then we obtain the F–smooth system of affine maps by setting

$$\boldsymbol{S} := \text{aff}(\boldsymbol{M}, \boldsymbol{N}) .$$

Indeed, the F–smooth structure of $\boldsymbol{S}$ turns out to be just the natural F–smooth structure underlying the smooth structure of $\boldsymbol{S}$, according to Theorem 2.2.1 and Example 2.1.2. □

Example 2.2.5. According to Example 2.1.3, if $\boldsymbol{M}$ and $\boldsymbol{N}$ are affine spaces, then we obtain the F–smooth systems $\boldsymbol{S}$ of polynomial maps of a given degree r and the F–smooth system $\boldsymbol{S}$ of polynomial maps of all degrees less than a given integer k.

Indeed, the F–smooth structures of the above systems of smooth maps turn out to be just the natural F–smooth structures underlying the smooth structures of $\boldsymbol{S}$, according to Theorem 2.2.1 and Example 2.1.3. □

Eventually, we revisit in the present context of F–smooth spaces the "infinite dimensional" example of polynomial maps of any degree.

Example 2.2.6. According to Example 2.1.3, if $\boldsymbol{M}$ and $\boldsymbol{N}$ are affine spaces, then we obtain the F–smooth system $\boldsymbol{S}$ of polynomial maps of any degree. □

Further, we consider two examples which play an intermediate role between the F–smooth systems of smooth maps and the F–smooth systems of fibrewisely smooth sections, which will be discussed later (see §3.2.1).

Example 2.2.7. Let us suppose that the smooth manifold $\boldsymbol{N}$ be a smooth bundle $q : \boldsymbol{N} \to \boldsymbol{M}$ over the smooth manifold $\boldsymbol{M}$.

Then, the subset

$$\boldsymbol{S} := \mathrm{Sec}(\boldsymbol{M}, \boldsymbol{N}) \subset \mathrm{Map}(\boldsymbol{M}, \boldsymbol{N})$$

consisting of *all global smooth sections* $\phi : \boldsymbol{M} \to \boldsymbol{N}$ is an F–smooth system of smooth maps.

Indeed, this system is a subsystem of the system of all smooth maps $\boldsymbol{M} \to \boldsymbol{N}$ considered in the Example 2.2.1. □

Example 2.2.8. Let us suppose that the smooth manifold $\boldsymbol{N}$ be a vector space and a smooth bundle $q : \boldsymbol{N} \to \boldsymbol{M}$ over the smooth manifold $\boldsymbol{M}$.

Then, the subset

$$\boldsymbol{S} \subset \mathrm{Sec}(\boldsymbol{M}, \boldsymbol{N}) \subset \mathrm{Map}(\boldsymbol{M}, \boldsymbol{N})$$

consisting of *all global smooth sections with compact support* $\phi : \boldsymbol{M} \to \boldsymbol{N}$ is an F–smooth system of smooth maps.

Indeed, this system is a subsystem of the above system of all global smooth sections considered in the above Example 2.2.7. □

2.2.2 F–smooth tangent prolongation of $(\boldsymbol{S}, \epsilon)$

Now, we consider an F–smooth system $(\boldsymbol{S}, \epsilon)$ of smooth maps between two smooth manifolds $\boldsymbol{M}$ and $\boldsymbol{N}$ (see Definition 2.2.1) and introduce the concept of "*F–smooth tangent space*" $\mathsf{T}\boldsymbol{S}$ of the F–smooth space of parameters $\boldsymbol{S}$.

Our formal construction of the tangent space $\mathsf{T}\boldsymbol{S}$ reflects the intuitive idea, by which, for each $s \in \boldsymbol{S}$, a tangent vector $\mathsf{X}_s \in \mathsf{T}_s\boldsymbol{S}$ is to be an "infinitesimal variation" of the global smooth map $\epsilon_s : \boldsymbol{M} \to \boldsymbol{N}$. It is remarkable the fact that this construction involves only global smooth maps between smooth manifolds, by exploiting the smooth structures of the manifolds $\boldsymbol{M}$ and $\boldsymbol{N}$ and the smoothness of the selected maps $f : \boldsymbol{M} \to \boldsymbol{N}$ of the system.

Actually, we define a tangent vector X_s of $\boldsymbol{S}$, at $s \in \boldsymbol{S}$, to be an equivalence class of F–smooth curves $\widehat{c} : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{S}$, such that the induced smooth maps between smooth manifolds $\widehat{c}^*(\epsilon) : \boldsymbol{I}_c \times \boldsymbol{M} \to \boldsymbol{N}$ have a 1st order contact in s .

The definition of 1st order contact shows that every $\mathsf{X}_s \in \mathsf{T}_s\boldsymbol{S}$ turns out to be represented by suitable smooth map $\Xi_s : \boldsymbol{M} \to T\boldsymbol{N}$, or that it can, equivalently, be represented by a suitable smooth map $\overline{\Xi}_s : T\boldsymbol{M} \to T\boldsymbol{N}$. Both Ξ_s and $\overline{\Xi}_s$ project over the smooth map $\epsilon_s : \boldsymbol{M} \to \boldsymbol{N}$.

Moreover, we show that the tangent space $\mathsf{T}\boldsymbol{S}$ is equipped with a natural F–smooth structure and that the natural maps

$$\tau_{\boldsymbol{S}} : \mathsf{T}\boldsymbol{S} \to \boldsymbol{S} \qquad \text{and} \qquad \mathsf{T}_1\epsilon : \mathsf{T}\boldsymbol{S} \times \boldsymbol{M} \to T\boldsymbol{N}$$

turn out to be F–smooth.

Indeed, the fibres of $\tau_{\boldsymbol{S}} : \mathsf{T}\boldsymbol{S} \to \boldsymbol{S}$ are naturally equipped with a vector structure.

This discussion on the F–smooth tangent space $\mathsf{T}\boldsymbol{S}$ of an F–smooth system of smooth maps between smooth manifolds is intended as an introduction to the more sophisticated case of the tangent space $\mathsf{T}\boldsymbol{S}$ of an F–smooth system of fibrewisely smooth sections (see Section 3.2.3).

Thus, let us consider two smooth manifolds $\boldsymbol{M}$ and $\boldsymbol{N}$ and denote the smooth charts of $\boldsymbol{M}$, $T\boldsymbol{N}$, $\boldsymbol{N}$, $T\boldsymbol{N}$, respectively, by

$$(y^i) : \boldsymbol{M} \to \mathbb{R}^{d_{\boldsymbol{M}}}, \qquad (y^i, \dot{y}^i) : T\boldsymbol{M} \to \mathbb{R}^{2d_{\boldsymbol{M}}},$$
$$(z^a) : \boldsymbol{N} \to \mathbb{R}^{d_{\boldsymbol{N}}}, \qquad (z^a, \dot{z}^a) : T\boldsymbol{N} \to \mathbb{R}^{2d_{\boldsymbol{N}}}.$$

Moreover, let us consider an F–smooth system $(\boldsymbol{S}, \epsilon)$ of smooth maps between the smooth manifolds $\boldsymbol{M}$ and $\boldsymbol{N}$ (see Definition 2.2.1), along with the set of F–smooth curves $\mathcal{C}$ of $\boldsymbol{S}$ defined in Theorem 2.2.1.

Then, we define the F–smooth tangent space $\mathsf{T}\boldsymbol{S}$ of the F–smooth space of parameters $\boldsymbol{S}$, via equivalence classes of suitable smooth maps between smooth manifolds, in the following way.

Definition 2.2.2. We say that two F–smooth curves

$$\widehat{c}_1 : \boldsymbol{I}_1 \to \boldsymbol{S} \qquad \text{and} \qquad \widehat{c}_2 : \boldsymbol{I}_2 \to \boldsymbol{S}$$

have a *1st order contact* in $(\lambda_1, \lambda_2) \in \boldsymbol{I}_1 \times \boldsymbol{I}_2$ if the induced smooth maps (see Theorem 2.2.1)

$$\widehat{c}^*_1(\epsilon) : \boldsymbol{I}_1 \times \boldsymbol{M} \to \boldsymbol{N} \qquad \text{and} \qquad \widehat{c}^*_2(\epsilon) : \boldsymbol{I}_2 \times \boldsymbol{M} \to \boldsymbol{N}$$

fulfill the following condition

$$*) \qquad \big(T_1\widehat{c}^*_1(\epsilon)\big)|_{(\lambda_1,1)} = \big(T_1\widehat{c}^*_2(\epsilon)\big)|_{(\lambda_2,1)} : \boldsymbol{M} \to T\boldsymbol{N},$$

i.e. if, in coordinates,

$$\widehat{c}^*_1(\epsilon)^a|_{\lambda_1} = \widehat{c}^*_2(\epsilon)^a|_{\lambda_2},$$
$$\partial_0\widehat{c}^*_1(\epsilon)^a|_{\lambda_1} = \partial_0\widehat{c}^*_2(\epsilon)^a|_{\lambda_2},$$

where ∂_0 denotes the partial derivative with respect to the parameter.

Clearly, the above 1st order contact yields an equivalence relation $\sim$ in the set of pairs

$$\big(\widehat{c} : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{S}, \quad \lambda \in \boldsymbol{I}_{\widehat{c}}\big) \,,$$

where $\widehat{c} : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{S}$ are F–smooth curves of $\boldsymbol{S}$.□

Definition 2.2.3. We define a *tangent vector* X_s of $\boldsymbol{S}$, at $s \in \boldsymbol{S}$, to be an equivalence class (see the above Definition 2.2.2)

$$\mathsf{X}_s := \big[(\widehat{c}_{\,s}, \lambda)\big]_\sim$$

where $\widehat{c}_{\,s} : I_{\widehat{c}_s} \to \boldsymbol{S}$ are F–smooth curves, such that $\widehat{c}_{\,s}(\lambda) = s$.

Then, we define:

1) the *F–smooth tangent space* of $\boldsymbol{S}$, at $s \in \boldsymbol{S}$, to be the set of tangent vectors of $\boldsymbol{S}$, at s ,

$$\mathsf{T}_s\boldsymbol{S} := \big\{\mathsf{X}_s\big\}$$

2) the *F–smooth tangent space* of $\boldsymbol{S}$ to be the disjoint union

$$\mathsf{T}\boldsymbol{S} := \bigsqcup_{s \in \boldsymbol{S}} \mathsf{T}_s\boldsymbol{S} \,.\square$$

Thus, in virtue of the above Definition 2.2.2 and Definition 2.2.3, the tangent vectors $\mathsf{X}_s \in \mathsf{T}_s\boldsymbol{S}$ can be represented through suitable smooth maps $\Xi : \boldsymbol{M} \to T\boldsymbol{N}$, as follows.

Theorem 2.2.2. *Let $(\boldsymbol{S}, \epsilon)$ be an F–smooth system of smooth maps.*

Then, in virtue of Definition 2.2.2 and Definition 2.2.3, every tangent vector

$$\mathsf{X}_s := [(\widehat{c}, \lambda)]_\sim \in \mathsf{T}_s\boldsymbol{S} \,,$$

can be regarded as the smooth map

$$\Xi_s := \big(T_1(\widehat{c}^{\,*}(\epsilon))\big)_{|(\lambda,1)} : \boldsymbol{M} \to T\boldsymbol{N} \,,$$

which projects on ϵ_s , according to the following commutative diagram

$$\begin{CD} \boldsymbol{M} @>{\Xi_s}>> T\boldsymbol{N} \\ @V{\mathrm{id}_{\boldsymbol{M}}}VV @VV{\tau_{\boldsymbol{N}}}V \\ \boldsymbol{M} @>{\epsilon_s}>> \boldsymbol{N} \end{CD} \quad .$$

We have the coordinate expression

$$(z^a, \dot z^a) \circ \Xi_s = (\epsilon^a_s,\ \Xi^a_s)\,, \qquad \text{with} \qquad \Xi^a_s = \partial_0\big(c^*(\epsilon)^a\big)_{|(\lambda,1)} \in \mathrm{map}(\boldsymbol{M},\,\mathbb{R})\,.$$

Thus, in this way, we obtain a natural injective map

$$\mathsf{r}_s \;:\; \mathsf{X}_s \in \mathsf{T}_s\boldsymbol{S} \mapsto \Xi_s : \boldsymbol{M} \to T\boldsymbol{N}\,,$$

which yields a representation of the tangent vectors $\mathsf{X}_s \in \mathsf{T}_s\boldsymbol{S}$, *through suitable smooth maps* $\Xi_s : \boldsymbol{M} \to T\boldsymbol{N}$, *which project on* $\epsilon_s : \boldsymbol{M} \to \boldsymbol{N}$. □

The F–smooth tangent space $\mathsf{T}\boldsymbol{S}$ can be also represented in another equivalent way through smooth maps between smooth manifolds, as follows.

Corollary 2.2.2. The tangent vectors

$$\Xi_s \,\simeq\, \mathsf{X}_s := \big[(\widehat{c}, \lambda)\big]_\sim \in \mathsf{T}_s\boldsymbol{S}$$

can be regarded as affine fibred morphisms over the smooth map $\epsilon_s : \boldsymbol{M} \to \boldsymbol{N}$

$$\overline{\Xi}_s : T\boldsymbol{M} \to T\boldsymbol{N}\,,$$

whose fibre derivative is equal to the smooth tangent prolongation of $\epsilon_s : \boldsymbol{M} \to \boldsymbol{N}$

$$D\overline{\Xi}_s = T(\epsilon_s) : T\boldsymbol{M} \to T\boldsymbol{N}\,,$$

where we have considered the natural identifications

$$VT\boldsymbol{M} \,\simeq\, T\boldsymbol{M} \underset{\boldsymbol{M}}{\times} T\boldsymbol{M} \qquad \text{and} \qquad VT\boldsymbol{N} \,\simeq\, T\boldsymbol{N} \underset{\boldsymbol{M}}{\times} T\boldsymbol{N}\,.$$

Their coordinate expressions are of the type

$$(z^a, \dot z^{\mathrm{a}}) \circ \overline{\Xi}_s = (\epsilon^a_s,\ \Xi^a_s + \partial_i\epsilon^a_s\,\dot y^i)\,, \qquad \text{with} \qquad \Xi^a_s \in \mathrm{map}(\boldsymbol{M},\,\mathbb{R})\,.$$

Proof. Each smooth map $\Xi_s : \boldsymbol{M} \to T\boldsymbol{N}$, which projects over the smooth map $\epsilon_s : \boldsymbol{M} \to \boldsymbol{N}$, yields the affine fibred morphism over $\epsilon_s : \boldsymbol{M} \to \boldsymbol{N}$

$$\overline{\Xi}_s := \Xi_s \circ \tau_{\boldsymbol{M}} + T(\epsilon_s)\,,$$

with coordinate expression

$$(z^a, \dot z^a) \circ \overline{\Xi}_s = (\epsilon^a_s,\ \Xi^a_s + \partial_i\epsilon^a_s\,\dot y^i)\,, \qquad \text{where} \qquad \Xi^a_s \in \mathrm{map}(\boldsymbol{M},\,\mathbb{R})\,.$$

Conversely, each smooth affine fibred morphism over $\epsilon_s : \boldsymbol{M} \to \boldsymbol{N}$,

$$\overline{\Xi}_s : T\boldsymbol{M} \to T\boldsymbol{N}\,,$$

whose fibre derivative is $D\overline{\Xi}_s = T(\epsilon_s)$, yields the smooth map

$$\Xi_s := \overline{\Xi}_s \circ 0_{\boldsymbol{M}}\,, \qquad \text{where} \qquad 0_{\boldsymbol{M}} : \boldsymbol{M} \to T\boldsymbol{M}\,,$$

which projects over $\epsilon_s : \boldsymbol{M} \to \boldsymbol{N}$.

Indeed, the above correspondence

$$\Xi_s \mapsto \overline{\Xi}_s \mapsto \Xi_s$$

is a bijection. QED

Remark 2.2.1. In the above Corollary 2.2.2, we have exploited the fact that, given $s \in \boldsymbol{S}$, we know the smooth map $\epsilon_s : \boldsymbol{M} \to \boldsymbol{N}$, along with its smooth tangent prolongation $T(\epsilon_s) : T\boldsymbol{M} \to T\boldsymbol{N}$. □

Note 2.2.1. By the way, by means of a reasoning analogous to that of the above Corollary 2.2.2, a representation of $\mathsf{T}\boldsymbol{S}$ can be obtained by considering smooth maps of the type

$$\overline{\Xi}^k_s : T^k\boldsymbol{M} \to T^k\boldsymbol{N}\,,$$

for any order k of tangent prolongation. □

In particular, the above results can be applied to the F–smooth system of compact support global smooth maps (see Example 2.2.2).

Next, we introduce two natural maps associated with $\mathsf{T}\boldsymbol{S}$ and show the natural vector structure of the fibres of $\mathsf{T}\boldsymbol{S}$.

Lemma 2.2.1. We have the *natural projection*

$$\tau_{\boldsymbol{S}} : \mathsf{T}\boldsymbol{S} \to \boldsymbol{S} : \Xi_s \mapsto s$$

and the *natural evaluation map*

$$\mathsf{T}_1\epsilon : \mathsf{T}\boldsymbol{S} \times \boldsymbol{M} \to T\boldsymbol{N} : (\Xi_s, m) \mapsto \Xi_s(m)\,. \square$$

Proposition 2.2.1. The fibres of $\tau_{\boldsymbol{S}} : \mathsf{T}\boldsymbol{S} \to \boldsymbol{S}$ are naturally equipped with a vector structure induced by the maps

$$(\Xi_s + \acute{\Xi}_s) : \boldsymbol{M} \to T\boldsymbol{N} : m \mapsto \Xi_s(m) + \acute{\Xi}_s(m) \in T_{\epsilon(s,m)}\boldsymbol{N}\,,$$
$$(k\,\Xi_s) : \boldsymbol{M} \to T\boldsymbol{N} : m \mapsto k\,\Xi_s(m) \in T_{\epsilon(s,m)}\boldsymbol{N}\,. \square$$

We have not assumed a priori any kind of smoothness on the set $\mathsf{T}\boldsymbol{S}$, but we can uniquely recover its natural F–smooth structure by the following Theorem 2.2.3.

Indeed, the specification "*F–smooth*" tangent space, that we have anticipated in Definition 2.2.3, is justified by this Theorem.

Theorem 2.2.3. *Let us define the set*

$$\mathsf{T}\mathcal{C} := \{\widehat{\mathsf{c}} : \boldsymbol{I}_{\widehat{\mathsf{c}}} \to \mathsf{T}\boldsymbol{S}\}$$

consisting of all curves of $\mathsf{T}\boldsymbol{S}$ *such that*

1) the base curve

$$\widehat{c}_{\boldsymbol{S}} := \tau_{\boldsymbol{S}} \circ \widehat{\mathsf{c}} : \boldsymbol{I}_{\widehat{\mathsf{c}}} \to \boldsymbol{S}$$

be F–smooth,

2) the induced map between smooth manifolds

$$\Xi_{\widehat{\mathsf{c}}} : \boldsymbol{I}_{\widehat{\mathsf{c}}} \times \boldsymbol{M} \to T\boldsymbol{N} : (\lambda, m) \mapsto \Xi_{\widehat{c}_{\boldsymbol{S}}(\lambda)}(m)$$

be smooth.

Then, the pair $(\mathsf{T}\boldsymbol{S}, \mathsf{T}\mathcal{C})$ *turns out to be an F–smooth space and the maps*

$$\tau_{\boldsymbol{S}} : \mathsf{T}\boldsymbol{S} \to \boldsymbol{S} \qquad \text{and} \qquad T_1\epsilon : \mathsf{T}\boldsymbol{S} \times \boldsymbol{M} \to T\boldsymbol{N}$$

turn out to be F–smooth.

PROOF. a) First of all, let us check that the set $\mathsf{T}\mathcal{C}$ fulfills the two requirements of Definition 1.2.1.

1) For each $\mathsf{X} \in \mathsf{T}\boldsymbol{S}$, the constant curve $\widehat{\mathsf{c}} : \mathbb{R} \to \mathsf{T}\boldsymbol{S} : \lambda \mapsto \mathsf{X}$ turns out to be an element of $\mathsf{T}\mathcal{C}$ passing through X.

2) Let us consider a curve $\widehat{\mathsf{c}} \in \mathsf{T}\mathcal{C}$ and a smooth curve $\gamma : \boldsymbol{I}_\gamma \to \boldsymbol{I}_{\widehat{\mathsf{c}}}$.

Then, the induced map

$$\Xi_{\widehat{\mathsf{c}}} \circ \gamma_{\boldsymbol{M}} : \boldsymbol{I}_\gamma \times \boldsymbol{M} \to T\boldsymbol{N} : (\lambda, m) \mapsto \epsilon_{(\widehat{\mathsf{c}} \circ \gamma)(\lambda)}(m)$$

is the composition of two smooth maps

$$\boldsymbol{I}_\gamma \times \boldsymbol{M} \xrightarrow{\gamma_{\boldsymbol{M}} := (\gamma, \mathrm{id}_{\boldsymbol{M}})} \boldsymbol{I}_{\widehat{\mathsf{c}}} \times \boldsymbol{M} \xrightarrow{\Xi_{\widehat{\mathsf{c}}}} T\boldsymbol{N} \quad ,$$

hence it turns out to be smooth.

b) In order to prove that $\tau_{\boldsymbol{S}} : \mathsf{T}\boldsymbol{S} \to \boldsymbol{S}$ be F–smooth, we have to show that, for each F–smooth curve $\widehat{\mathsf{c}} : \boldsymbol{I}_{\widehat{\mathsf{c}}} \to \mathsf{T}\boldsymbol{S}$, the induced curve $\widehat{c}_{\boldsymbol{S}} := \tau_{\boldsymbol{S}} \circ \widehat{\mathsf{c}} : \boldsymbol{I}_{\widehat{\mathsf{c}}} \to \boldsymbol{S}$ be F–smooth.

In fact, let us consider any F–smooth curve and the associated smooth map

$$\widehat{\mathsf{c}} : \boldsymbol{I}_{\widehat{\mathsf{c}}} \to \mathsf{T}\boldsymbol{S} \qquad \text{and} \qquad \Xi_{\widehat{\mathsf{c}}} : \boldsymbol{I}_{\widehat{\mathsf{c}}} \times \boldsymbol{M} \to T\boldsymbol{N}\,.$$

Indeed, the induced curve

$$\widehat{c}_{\boldsymbol{S}} := \tau_{\boldsymbol{S}} \circ \widehat{\mathsf{c}} : \boldsymbol{I}_{\widehat{\mathsf{c}}} \to \boldsymbol{S}$$

is characterised by the composition of smooth maps

$$\tau_{\boldsymbol{N}} \circ \Xi_{\widehat{\mathsf{c}}} : \boldsymbol{I}_{\widehat{\mathsf{c}}} \times \boldsymbol{M} \to \boldsymbol{N}\,,$$

hence it turns out to be smooth.

c) In order to prove that $\mathsf{T}_1\epsilon : \mathsf{T}\boldsymbol{S} \times \boldsymbol{M} \to T\boldsymbol{N}$ be F–smooth, we have to show that, for each F–smooth curve $\widehat{\mathsf{c}} = (\widehat{\mathsf{c}}_{\mathsf{T}\boldsymbol{S}}, c_{\boldsymbol{M}}) : \boldsymbol{I}_{\widehat{\mathsf{c}}} \to \mathsf{T}\boldsymbol{S} \times \boldsymbol{M}$, the induced curve $\mathsf{T}_1\epsilon \circ \widehat{\mathsf{c}} : \boldsymbol{I}_{\widehat{\mathsf{c}}} \to T\boldsymbol{N}$ be F–smooth.

In fact, let us consider any F–smooth curve and the associated smooth map

$$\widehat{\mathsf{c}}_{\mathsf{T}\boldsymbol{S}} : \boldsymbol{I}_{\widehat{\mathsf{c}}} \to \mathsf{T}\boldsymbol{S} \qquad \text{and} \qquad \Xi_{c} : \boldsymbol{I}_{\widehat{\mathsf{c}}} \times \boldsymbol{M} \to T\boldsymbol{N}\,.$$

Then, the curve

$$\mathsf{T}_1\epsilon \circ (\widehat{\mathsf{c}}_{T\boldsymbol{S}}, c_{\boldsymbol{M}}) : \boldsymbol{I}_{\widehat{\mathsf{c}}} \to T\boldsymbol{N} : \lambda \mapsto \mathsf{T}_1\epsilon\big(\widehat{\mathsf{c}}_{\mathsf{T}\boldsymbol{S}}(\lambda),\, c_{\boldsymbol{M}}(\lambda)\big)$$

which is a composition of smooth maps according to the following diagram

$$\begin{array}{ccc}
\boldsymbol{I}_{\widehat{\mathsf{c}}} & \xrightarrow{(\mathrm{id}, c_{\boldsymbol{M}})} & \boldsymbol{I}_{\widehat{\mathsf{c}}} \times \boldsymbol{M} \\
{\scriptstyle \mathrm{id}_{I_{\widehat{\mathsf{c}}}}}\uparrow & & \downarrow{\scriptstyle \Xi_{\widehat{\mathsf{c}}}} \\
\boldsymbol{I}_{\widehat{\mathsf{c}}} & \xrightarrow{\mathsf{T}_1\epsilon \circ (\widehat{\mathsf{c}}_{\mathsf{T}\boldsymbol{S}}, c_{\boldsymbol{M}})} & T\boldsymbol{N}
\end{array} \quad ,$$

turns out to be smooth hence F–smooth (see Theorem 2.2.1). QED

We say that $\tau_{\boldsymbol{S}} : \mathsf{T}\boldsymbol{S} \to \boldsymbol{S}$ is an *F–smooth fibred set.*

Eventually, we discuss the tangent prolongation of F–smooth curves of $\boldsymbol{S}$.

Definition 2.2.4. Let $\widehat{c} : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{S}$ be an F–smooth curve. Then, we define its *tangent prolongation* to be the curve

$$d\,\widehat{c} : \boldsymbol{I}_{\widehat{c}} \to \mathsf{T}\boldsymbol{S} : \lambda \mapsto d\widehat{c}\,(\lambda) := [(\widehat{c}, \lambda)]_{\sim}$$

defined in Definition 2.2.3. □

Proposition 2.2.2. If $\widehat{c} : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{S}$ is an F–smooth curve, then its tangent prolongation $d\,\widehat{c} : \boldsymbol{I}_{\widehat{c}} \to \mathsf{T}\boldsymbol{S}$ turns out to be F–smooth.

PROOF. Let $\gamma : \boldsymbol{I}_{\gamma} \to \boldsymbol{I}_{\widehat{c}}$ be a smooth curve. Then, the induced map

$$\big(d\,(\widehat{c} \circ \gamma)\big)^{*}(T_1\epsilon) : \boldsymbol{I}_{\gamma} \times \boldsymbol{M} \to T\boldsymbol{N}$$

turns out to be the composition of smooth maps. Hence, $d\,\widehat{c}$ is F–smooth. QED

2.3 F–smooth vs smooth systems of maps

In the above Sections 2.1.3 and 2.2.2 we have studied the smooth tangent space $T\boldsymbol{S}$ and the F–smooth tangent space $\mathsf{T}\boldsymbol{S}$ with reference to a smooth system and to an F–smooth system $(\boldsymbol{S}, \epsilon)$ of smooth maps between two smooth manifolds, respectively.

Clearly, a smooth system is a particular case of an F–smooth system, because a smooth system can be regarded as an F–smooth system along with the additional assumptions on the smoothness of $\boldsymbol{S}$ and ϵ.

Then, the need of a comparison between the smooth approach to $T\boldsymbol{S}$ and the F–smooth approach to $\mathsf{T}\boldsymbol{S}$ arises naturally, having in mind Theorem 1.3.1.

Actually, by regarding a smooth system $(\boldsymbol{S}, \epsilon)$ of smooth maps as a particular F–smooth system of smooth maps, we show a natural bijection

$$\imath : T\boldsymbol{S} \to \mathsf{T}\boldsymbol{S} : X \mapsto \mathsf{X}\,,$$

which, in terms of the representation of X via the smooth map

$$\Xi := \mathsf{r}(\mathsf{X}) : \boldsymbol{M} \to T\boldsymbol{N}$$

reads as

$$\Xi = (T_1\epsilon)_X\,.$$

Note 2.3.1. Summing up previous results, let us compare two types of systems of smooth maps between smooth manifolds.

1) In the case of an F–smooth system $(\boldsymbol{S}, \epsilon)$ of smooth maps $f : \boldsymbol{M} \to \boldsymbol{N}$, we make no assumptions on any kind of smoothness of the set $\boldsymbol{S}$ (see Definition 2.2.1).

Hence, we cannot avail of a smooth structure of $\boldsymbol{S}$ and, in order to achieve the F–smooth tangent space $\mathsf{T}\boldsymbol{S}$, we need to follow an indirect abstract procedure.

In fact, we have defined the tangent vectors $\mathsf{X}_s \in \mathsf{T}_s\boldsymbol{S}$, with $s \in \boldsymbol{S}$, as equivalence classes $\mathsf{X}_s := [(\widehat{c}, \lambda)]_\sim$ of pairs $(\widehat{c}, \lambda)$, where $\widehat{c} : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{S}$ are F–smooth curves and $\lambda \in \boldsymbol{I}_{\widehat{c}}$, which have a 1st order contact in s (see Definition 2.2.3).

Here, the 1st order contact is defined through the smooth map (see Definition 2.2.2)

$$\big(T_1\widehat{c}^*(\epsilon)\big)|_{(\lambda,1)} : \boldsymbol{M} \to T\boldsymbol{N}\,.$$

Then, we obtain the representation of tangent vectors $\mathsf{X}_s := [(\widehat{c}, \lambda)]_\sim \in \mathsf{T}_s\boldsymbol{S}$, through the smooth maps

$$\Xi_s := \big(T_1(\widehat{c}^*_s(\epsilon))\big)|_{(\lambda,1)} : \boldsymbol{M} \to T\boldsymbol{N}\,,$$

which project on $\epsilon_s : \boldsymbol{M} \to \boldsymbol{N}$ (see Theorem 2.2.2). $\square$

2) In the case of a finite dimensional smooth system $(\boldsymbol{S}, \epsilon)$ of smooth maps $f : \boldsymbol{M} \to \boldsymbol{N}$, we assume that $\boldsymbol{S}$ is a smooth manifold and ϵ a smooth map.

Let us denote the typical smooth chart of $\boldsymbol{S}$ by (w^A).

Hence, we can avail of the assumed smooth structure of $\boldsymbol{S}$ to achieve a direct approach of the tangent space $T\boldsymbol{S}$.

In fact, according to a standard definition in Differential Geometry, we define the tangent vectors $X_s \in T_s\boldsymbol{S}$, with $s \in \boldsymbol{S}$, as equivalence classes $X_s := [(\widehat{c}, \lambda)]_\sim$ of pairs $(\widehat{c}, \lambda)$, where $\widehat{c} : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{S}$ are smooth curves and $\lambda \in \boldsymbol{I}_{\widehat{c}}$, which have a 1st order contact in s.

Here, the 1st order contact is defined directly through the smooth structure of $\boldsymbol{S}$ (without the need to consider the smooth map $\widehat{c}^*(\epsilon) : \boldsymbol{I}_{\widehat{c}} \times \boldsymbol{M} \to \boldsymbol{N}$).

Then, the smooth tangent map (see Section 2.2.2)

$$T_1\epsilon : T\boldsymbol{S} \times \boldsymbol{M} \to T\boldsymbol{N},$$

with coordinate expression

$$z^a \circ T_1\epsilon = \epsilon^a \qquad \text{and} \qquad \dot{z}^a \circ T_1\epsilon = \partial_A \epsilon^a \, \dot{w}^A,$$

provides, for each $X_s \in T_s\boldsymbol{S}$, the smooth map

$$(T_1\epsilon)_{X_s} : \boldsymbol{M} \to T\boldsymbol{N}. \square$$

Proposition 2.3.1. Let us consider a smooth system $(\boldsymbol{S}, \epsilon)$ of smooth maps $f : \boldsymbol{M} \to \boldsymbol{N}$ and regard it as a particular F–smooth system.

Then, in virtue of the standard definition of the smooth tangent space $T\boldsymbol{S}$ and of the definition of the F–smooth tangent space $\mathsf{T}\boldsymbol{S}$ (see Definition 2.2.3), we obtain a natural F–smooth map

$$\imath : T_s\boldsymbol{S} \to \mathsf{T}_s\boldsymbol{S} : X_s \mapsto \mathsf{X}_s, \qquad \text{for each} \qquad s \in \boldsymbol{S}.$$

Indeed, in terms of the smooth representation of $\mathsf{X}_s \in \mathsf{T}_s\boldsymbol{S}$, the above map turns out to be given by the smooth map (see Theorem 2.2.2)

$$\Xi_s := \mathsf{r}(\mathsf{X}_s) = (T_1\epsilon)_{|X_s} : \boldsymbol{M} \to T\boldsymbol{N}, \qquad \text{for each} \qquad s \in \boldsymbol{S},$$

i.e., in coordinates,

$$z^a \circ \Xi_s = \epsilon^a_s \qquad \text{and} \qquad \dot{z}^a \circ \Xi_s = (\partial_A \epsilon^a)_s \, X^A_s.$$

PROOF. Each smooth curve $\widehat{c} : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{S}$ makes the map

$$\widehat{c}^*(\epsilon) : \boldsymbol{I}_{\widehat{c}} \times \boldsymbol{M} \to \boldsymbol{N}$$

smooth, hence $\widehat{c} : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{S}$ turns out to be also F–smooth (see Theorem 2.2.1).

Moreover, we can easily see that, if two smooth pairs $(\widehat{c}_1, \lambda_1)$ and $(\widehat{c}_2, \lambda_2)$ have a 1st order contact in $s \in \boldsymbol{S}$, in the sense of smooth manifolds, then they have also a 1st order contact in the sense of F–smooth spaces (see Definition 2.2.2). QED

In the above Proposition 2.3.1, we have assumed a given smooth structure of $\boldsymbol{S}$. Now, we discuss what happens if we change this smooth structure.

Remark 2.3.1. We have already observed (see Example 1.3.1) that, given a smooth system $(\boldsymbol{S}, \epsilon)$ of smooth maps, there might exist infinitely many possible smooth structures of $\boldsymbol{S}$ which make the map $\epsilon : \boldsymbol{S} \times \boldsymbol{M} \to \boldsymbol{N}$ smooth, hence, which yield the same family of selected smooth maps $f : \boldsymbol{M} \to \boldsymbol{N}$.

Indeed, different smooth structures of $\boldsymbol{S}$ yield different F–smooth fibred morphisms

$$\imath : T\boldsymbol{S} \to \mathsf{T}\boldsymbol{S} : X_s \mapsto \mathsf{X}_s \,.$$

Therefore, the F–smooth tangent space $\mathsf{T}_s\boldsymbol{S}$ contains the images $\imath(X_s)$ of the tangent vectors $X_s \in T\boldsymbol{S}$, for all possible smooth structures of $\boldsymbol{S}$. □

Remark 2.3.2. In general, the map $\imath : T\boldsymbol{S} \to \mathsf{T}\boldsymbol{S}$ is not injective. We prove this fact by means of a counter–example.

Let us consider the smooth manifolds $\boldsymbol{S} := \mathbb{R}$, $\boldsymbol{M} := \mathbb{R}$ and $\boldsymbol{N} := \mathbb{R}$, along with their natural smooth charts $w : \boldsymbol{S} \to \mathbb{R}$, $y : \boldsymbol{M} \to \mathbb{R}$ and $z : \boldsymbol{N} \to \mathbb{R}$.

Moreover, let us consider the non–injective smooth system of smooth maps given by the evaluation map $\epsilon : \boldsymbol{S} \times \boldsymbol{M} \to \boldsymbol{N}$, with coordinate expression

$$z \circ \epsilon = w^2 \, y \,.$$

Let us consider the element $s = 1 \in \boldsymbol{S}$ and the two smooth curves

$$\widehat{c}_1 : \mathbb{R} \to \boldsymbol{S} : \lambda \mapsto \lambda \qquad \text{and} \qquad \widehat{c}_2 : \mathbb{R} \to \boldsymbol{S} : \lambda \mapsto -\lambda \,.$$

Then, we obtain

$$\widehat{c}_1(1) = 1 = \widehat{c}_2(-1)$$

and two different tangent vectors

$$X_1 = T\widehat{c}_1(1, 0) = (1, 1) \in T_1\boldsymbol{S} \quad \text{and} \quad X_2 = T\widehat{c}_2(-1, 0) = (1, -1) \in T_1\boldsymbol{S} \,.$$

However, the pairs $(\widehat{c}_1, 1)$ and $(\widehat{c}_2, -1)$ are equivalent in the sense of Definition 2.2.2.

In fact, we have

$$w \circ \big(\widehat{c}_1^*(\epsilon)\big)(1) = y = w \circ \big(\widehat{c}_2^*(\epsilon)\big)(-1)$$

and

$$\big(\partial_0(\widehat{c}_1^*(\epsilon))\big)|_1 = 2\,y = \big(\partial_0(\widehat{c}_2^*(\epsilon))\big)|_{-1}\,.$$

Then, we have

$$\iota(X_1) = \iota(X_2)\,.\,\square$$

Chapter 3

Systems of sections

First, we discuss the *smooth systems* $(\boldsymbol{S}, \zeta, \epsilon)$ of *smooth sections* $\phi : \boldsymbol{F} \to \boldsymbol{G}$ of a smooth double fibred manifold $\boldsymbol{G} \to \boldsymbol{F} \to \boldsymbol{B}$. Here, the "fibred space of parameters" $\zeta : \boldsymbol{S} \to \boldsymbol{B}$ is a *smooth fibred manifold* and the "evaluation map" $\epsilon : \boldsymbol{S} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \to \boldsymbol{G}$ a *smooth fibred morphism* over $\boldsymbol{B}$.

Then, we discuss the *F–smooth systems* $(\boldsymbol{S}, \zeta, \epsilon)$ of *fibrewisely smooth sections* $\phi : \boldsymbol{F} \to \boldsymbol{G}$ of a smooth double fibred manifold $\boldsymbol{G} \to \boldsymbol{F} \to \boldsymbol{B}$. Here, we have a weaker smoothness requirement, as the "fibred space of parameters" $\zeta : \boldsymbol{S} \to \boldsymbol{B}$ turns out to be an *F–smooth fibred* space and the "evaluation map" $\epsilon : \boldsymbol{S} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \to \boldsymbol{G}$ an *F–smooth fibred morphism* over $\boldsymbol{B}$.

The above notions are intended as an introduction to the particular cases of systems of connections discussed in the next Chapter §4.

The reader can find further discussions concerning the present subject in [2, 3, 4, 5, 7, 9, 14, 15, 16, 19, 20, 23].

3.1 Smooth systems of smooth sections

We discuss the notion of *smooth systems of smooth sections* of a smooth double fibred manifold.

3.1.1 Smooth systems of smooth sections

First of all, given a smooth double fibred manifold $\boldsymbol{G} \xrightarrow{q} \boldsymbol{F} \xrightarrow{p} \boldsymbol{B}$, we define the *"tubelike" smooth sections* $\phi : \boldsymbol{F} \to \boldsymbol{G}$.

Next, we discuss the notion of *"smooth system of smooth sections"* of the smooth double fibred manifold.

In simple words, a "smooth system of smooth sections" is defined to be a selected family $\{\phi\}$ of smooth sections $\phi : \boldsymbol{F} \to \boldsymbol{G}$ of the smooth fibred manifold $q : \boldsymbol{G} \to \boldsymbol{F}$, which is parametrised by the smooth sections $\sigma : \boldsymbol{B} \to \boldsymbol{S}$ of a smooth fibred manifold $\zeta : \boldsymbol{S} \to \boldsymbol{B}$.

The notion of smooth system of smooth sections will be used as an introduction to the concept of "smooth system of smooth connections", which is developed in the forthcoming Section 4.1.1.

Later, in the next Section 3.2.1, we shall revisit the notion of system of sections in a larger context, detached from the hypothesis of smoothness and finite dimension and approached by means of the concept of F–smoothness (see Section 1.2.1).

Let us consider a smooth double fibred manifold $\boldsymbol{G} \xrightarrow{q} \boldsymbol{F} \xrightarrow{p} \boldsymbol{B}$.

We start by introducing the preliminary notions of "tubelike subset" and "tubelike section".

Definition 3.1.1. We define the *tubelike* subsets of $\boldsymbol{F}$ and of $\boldsymbol{G}$ to be the open subsets of the type

$$p^{-1}(\boldsymbol{U}) \subset \boldsymbol{F} \qquad \text{and} \qquad (p \circ q)^{-1}(\boldsymbol{U}) \subset \boldsymbol{G},$$

where $\boldsymbol{U} \subset \boldsymbol{B}$ is an open subset, according to the following commutative diagrams

$$\begin{array}{ccc} p^{-1}(\boldsymbol{U}) & \xrightarrow{\subset} & \boldsymbol{F} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \boldsymbol{U} & \xrightarrow{\subset} & \boldsymbol{B} \end{array} \qquad\qquad \begin{array}{ccc} (p \circ q)^{-1}(\boldsymbol{U}) & \xrightarrow{\subset} & \boldsymbol{G} \\ {\scriptstyle p \circ q}\downarrow & & \downarrow{\scriptstyle p \circ q} \\ \boldsymbol{U} & \xrightarrow{\subset} & \boldsymbol{B} \end{array} \;. \square$$

Note 3.1.1. The tubelike open subsets $p^{-1}(\boldsymbol{U}) \subset \boldsymbol{F}$ and $(p \circ q)^{-1}(\boldsymbol{U}) \subset \boldsymbol{G}$ yield a topology on $\boldsymbol{F}$ and $\boldsymbol{G}$.

In the following, we shall usually refer to this topology. □

Definition 3.1.2. A smooth section $\phi \in \sec(\boldsymbol{F}, \boldsymbol{G})$ is said to be *tubelike* if it is globally defined on a tubelike open subset $p^{-1}(\boldsymbol{U}) \subset \boldsymbol{F}$.

We denote the subsheaf (with respect to the tubelike topology) of *tubelike smooth sections* $\phi \in \sec(\boldsymbol{F}, \boldsymbol{G})$ by

$$\operatorname{tub}(\boldsymbol{F}, \boldsymbol{G}) \subset \sec(\boldsymbol{F}, \boldsymbol{G}) \,. \square$$

Then, we introduce the notion of "smooth system of smooth sections".

Definition 3.1.3. We define a *smooth system of smooth sections* of the smooth fibred manifold $q : \boldsymbol{G} \to \boldsymbol{F}$ to be a 3–plet $(\boldsymbol{S}, \zeta, \epsilon)$, where

1) $\zeta : \boldsymbol{S} \to \boldsymbol{B}$ is a smooth fibred manifold,

2) $\epsilon : \boldsymbol{S} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \to \boldsymbol{G}$ is a smooth fibred map over $\boldsymbol{F}$, according to the following commutative diagram

$$\begin{array}{ccc} \boldsymbol{S} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} & \xrightarrow{\ \epsilon\ } & \boldsymbol{G} \\ {\scriptstyle \mathrm{pro}_2}\downarrow & & \downarrow{\scriptstyle q} \\ \boldsymbol{F} & \xrightarrow{\ \mathrm{id}_{\boldsymbol{F}}\ } & \boldsymbol{F} \end{array} \quad .$$

We call ϵ the *evaluation map* of the system.

Thus, the evaluation map ϵ yields the sheaf morphism

$$\epsilon_{\boldsymbol{S}} : \sec(\boldsymbol{B}, \boldsymbol{S}) \to \mathrm{tub}(\boldsymbol{F}, \boldsymbol{G}) : \sigma \mapsto \breve{\sigma} \,,$$

where, for each $\sigma \in \sec(\boldsymbol{B}, \boldsymbol{S})$, the tubelike smooth section $\breve{\sigma}$ is defined by

$$\breve{\sigma} : \boldsymbol{F} \to \boldsymbol{G} : f_b \mapsto \epsilon(\sigma(b), f_b) \,, \qquad \text{for each} \qquad b \in \boldsymbol{B} \,.$$

Therefore, the map $\epsilon_{\boldsymbol{S}} : \sec(\boldsymbol{B}, \boldsymbol{S}) \to \mathrm{tub}(\boldsymbol{F}, \boldsymbol{G})$ provides a *selection* of the tubelike smooth sections $\phi : \boldsymbol{F} \to \boldsymbol{G}$, given by the subset

$$\mathrm{tub}_{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{G}) := \epsilon_{\boldsymbol{S}}\big(\sec(\boldsymbol{B}, \boldsymbol{S})\big) \subset \mathrm{tub}(\boldsymbol{F}, \boldsymbol{G}) \,.$$

The smooth system of smooth sections $(\boldsymbol{S}, \zeta, \epsilon)$ is said to be *injective* if the map $\epsilon_{\boldsymbol{S}} : \sec(\boldsymbol{B}, \boldsymbol{S}) \to \mathrm{tub}(\boldsymbol{F}, \boldsymbol{G})$ is injective, i.e. if, for each $\sigma, \acute{\sigma} \in \sec(\boldsymbol{B}, \boldsymbol{S})$,

$$\breve{\sigma} \equiv \epsilon_{\boldsymbol{S}}(\sigma) = \breve{\acute{\sigma}} \equiv \epsilon_{\boldsymbol{S}}(\acute{\sigma}) \qquad \Rightarrow \qquad \sigma = \acute{\sigma} \,.$$

If the system is injective, then we obtain the bijection

$$\epsilon_{\boldsymbol{S}} : \sec(\boldsymbol{B}, \boldsymbol{S}) \to \mathrm{tub}_{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{G}) : \sigma \mapsto \breve{\sigma} \,,$$

whose inverse is denoted by

$$(\epsilon_{\boldsymbol{S}})^{-1} : \mathrm{tub}_{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{G}) \to \sec(\boldsymbol{B}, \boldsymbol{S}) : \phi \mapsto \widehat{\phi} \,. \square$$

Indeed, we are essentially interested in injective smooth systems of smooth sections.

We shall denote the local fibred smooth charts of $\boldsymbol{B}$, $\boldsymbol{S}$, $\boldsymbol{F}$, $\boldsymbol{G}$, respectively, by

$$(x^\lambda) : \boldsymbol{B} \to \mathbb{R}^{d_{\boldsymbol{B}}}, \qquad (x^\lambda, w^A) : \boldsymbol{S} \to \mathbb{R}^{d_{\boldsymbol{S}}},$$
$$(x^\lambda, y^i) : \boldsymbol{F} \to \mathbb{R}^{d_{\boldsymbol{F}}}, \qquad (x^\lambda, y^i, z^a) : \boldsymbol{G} \to \mathbb{R}^{d_{\boldsymbol{G}}}.$$

We discuss the following elementary examples of smooth systems of smooth sections.

Example 3.1.1. If $p : \boldsymbol{F} \to \boldsymbol{B}$ and $p \circ q : \boldsymbol{G} \to \boldsymbol{B}$ are vector bundles, then the family of tubelike sections $\breve{\sigma} : \boldsymbol{F} \to \boldsymbol{G}$, which are linear fibred morphisms over $\boldsymbol{B}$, yields the injective smooth system of smooth sections

$$\boldsymbol{S} := \mathrm{lin}_{\boldsymbol{B}}(\boldsymbol{F}, \boldsymbol{G}).$$

With reference to a linear fibred chart of the smooth double fibred manifold, the coordinate expression of the tubelike sections $\breve{\sigma} : \boldsymbol{F} \to \boldsymbol{G}$ of this system is of the type

$$x^\lambda \circ \breve{\sigma} = x^\lambda, \quad y^i \circ \breve{\sigma} = y^i, \quad z^a \circ \breve{\sigma} = K^a_i\, y^i, \quad \text{with} \quad K^a_i \in \mathrm{map}(\boldsymbol{B}, \mathbb{R}).$$

Hence, the above fibred chart of the smooth double fibred manifold yields a distinguished fibred chart (x^λ, w^a_i) of $\boldsymbol{S}$ and the coordinate expression of ϵ becomes

$$\epsilon^a = w^a_j\, y^j. \,\square$$

Example 3.1.2. If $p : \boldsymbol{F} \to \boldsymbol{B}$ and $p \circ q : \boldsymbol{G} \to \boldsymbol{B}$ are affine bundles, then the family of tubelike sections $\breve{\sigma} : \boldsymbol{F} \to \boldsymbol{G}$, which are affine fibred morphisms over $\boldsymbol{B}$, yields the injective smooth system of smooth sections

$$\boldsymbol{S} := \mathrm{aff}_{\boldsymbol{B}}(\boldsymbol{F}, \boldsymbol{G}).$$

With reference to an affine fibred chart of the smooth double fibred manifold, the coordinate expression of the tubelike sections $\breve{\sigma} : \boldsymbol{F} \to \boldsymbol{G}$ of this system is of the type

$$x^\lambda \circ \breve{\sigma} = x^\lambda, \quad y^i \circ \breve{\sigma} = y^i, \quad z^a \circ \breve{\sigma} = K^a_i\, y^i + K^a,$$

with K^a_i, $K^a \in \mathrm{map}(\boldsymbol{B}, \mathbb{R})$.

Hence, the above fibred chart of the smooth double fibred manifold yields the distinguished fibred chart (x^λ, w^a_i, w^a) of $\boldsymbol{S}$ and the coordinate expression of ϵ becomes

$$\epsilon^a = w^a_j\, y^j + w^a. \,\square$$

Example 3.1.3. If $p : \boldsymbol{F} \to \boldsymbol{B}$ and $p \circ q : \boldsymbol{G} \to \boldsymbol{B}$ are affine bundles, then analogously to the above Example 3.1.2, we can define

- the injective smooth system of polynomial sections of degree r, with $0 \leq r$,

- the injective smooth system of polynomial sections of any degree $0 \leq r \leq k$, where k is a given positive integer. □

All examples above deal with finite dimensional smooth systems of smooth sections, as it is implicitly requested by Definition 3.1.3.

However, we can easily extend the concept of smooth system of smooth sections, by considering an infinite dimensional smooth system, which is the direct limit of finite dimensional smooth systems, according to the following Example 3.1.4.

Example 3.1.4. If $p : \boldsymbol{F} \to \boldsymbol{B}$ and $p \circ q : \boldsymbol{G} \to \boldsymbol{B}$ are affine bundles, then we obtain the smooth system of polynomial sections by considering the family of polynomial sections $s : \boldsymbol{F} \to \boldsymbol{G}$ of any degree r, with $0 \leq r \leq \infty$.

Later, we shall see that such a smooth system has a natural F–smooth structure (see, later, Definition 1.2.1) □

3.1.2 Smooth structure of $(\boldsymbol{S}, \zeta, \epsilon)$

In our definition of "smooth system of smooth sections" (see Definition 3.1.3) we have required a priori that the fibred set of parameters $\zeta : \boldsymbol{S} \to \boldsymbol{B}$ be smooth and that the evaluation map ϵ be smooth as well.

On the contrary, we might ask whether the fact that the manifold $\boldsymbol{B}$, the fibred manifolds $p : \boldsymbol{F} \to \boldsymbol{B}$ and $q : \boldsymbol{G} \to \boldsymbol{F}$ are smooth and that the sections $\phi : \boldsymbol{F} \to \boldsymbol{G}$ selected by the system are smooth allows us to recover uniquely the smooth structure of $\zeta : \boldsymbol{S} \to \boldsymbol{B}$.

The answer to the above question is negative. Here we do not fully address this problem, which is too far from the true scope of the present report. But, we present a simple example (see Example 3.1.5), where we show that, if $\boldsymbol{S}$ admits a finite dimensional smooth structure compatible with ϵ, then this structure needs not to be unique.

Moreover, we observe that if we do not assume a priori a finite dimensional smooth structure on $\zeta : \boldsymbol{S} \to \boldsymbol{B}$, then it might be that no finite dimensional smooth structure at all could be recovered on $\zeta : \boldsymbol{S} \to \boldsymbol{B}$. To prove this, just consider the system $(\boldsymbol{S}, \zeta, \epsilon)$ of *all* smooth sections $\phi : \boldsymbol{F} \to \boldsymbol{G}$ (see, later, Section 3.2.1).

The above question might arise also in comparison with a result which will be achieved later, in the next Section, in the context of "F–smooth systems of smooth sections", where we do not assume a priori any smooth structure of the fibred set $\zeta : \boldsymbol{S} \to \boldsymbol{B}$, but we recover uniquely its F–smooth structure (see Definition 3.2.2 and Theorem 3.2.1).

Analogously to Example 2.1.7, we can exhibit a system of smooth sections $(\boldsymbol{S}, \zeta, \epsilon)$, where $\boldsymbol{S}$ is equipped with different smooth structures.

Example 3.1.5. Let us consider the following system $(\boldsymbol{S}, \zeta, \epsilon)$ of smooth sections:

1) we consider the manifolds

$$\boldsymbol{B} := \mathbb{R}, \qquad \boldsymbol{F} := \mathbb{R} \times \mathbb{R}, \qquad \boldsymbol{G} := \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \\ \boldsymbol{S} := \mathbb{R} \times \mathbb{R},$$

equipped with their "natural" smooth structures, and denote their "natural" charts by

$$x : \boldsymbol{B} \to \mathbb{R}, \qquad (x, y) : \boldsymbol{F} \to \mathbb{R} \times \mathbb{R}, \qquad (x, y, z) : \boldsymbol{G} \to \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \\ (x, w) : \boldsymbol{S} \to \mathbb{R} \times \mathbb{R}.$$

Moreover, let us consider the smooth evaluation map

$$\epsilon : \boldsymbol{S} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \to \boldsymbol{G},$$

defined by the coordinate expression

$$1) \qquad (x, y;\ z) \circ \epsilon = (x, y;\ w\, y).$$

Besides the above "natural" smooth structure of $\boldsymbol{S}$, we can consider several further smooth structures of $\boldsymbol{S}$, which make the *same* evaluation map ϵ smooth, hence which essentially yield the "same" system of sections.

For instance, let us consider the "exotic" smooth structure of $\boldsymbol{S}$ induced by the fibred bijection

$$(x, \acute{w}) := (x, w^3) : \boldsymbol{S} \to \mathbb{R} \times \mathbb{R}.$$

Then, the evaluation map ϵ reads, in the above exotic fibred chart as

$$2) \qquad (x, y;\ z) \circ \epsilon = (x, y;\ w^3\, y).$$

Clearly, the equalities 1) and 2) define the same evaluation map ϵ. Moreover, ϵ turns out to be smooth with respect to both smooth structures of $\boldsymbol{S}$.

By the way, the system $(\boldsymbol{S}, \epsilon)$ turns out to be injective in both cases. □

We might even ask whether we can characterise the "natural" smooth structure of $\boldsymbol{S}$ via a suitable feature of the system; this question might arise later if we compare "smooth systems of smooth maps" and "F–smooth systems of smooth maps" (see, later, in the next Section 2.2.1).

Here, we skip a general answer to this question, which is too far from the true scope of the present report.

However, we observe, as a hint, that the set of curves of $\boldsymbol{S}$ which are smooth with respect to its natural smooth structure is smaller than the set of curves of $\boldsymbol{S}$ which are smooth with respect to its exotic smooth structure. The converse occurs for the sets of smooth functions.

3.1.3 Smooth lifted fibred manifold

Given a smooth double fibred manifold $\boldsymbol{G} \xrightarrow{q} \boldsymbol{F} \xrightarrow{p} \boldsymbol{B}$ and a smooth system of smooth sections $(\boldsymbol{S}, \zeta, \epsilon)$, it is useful to define the "*lifted smooth fibred manifold*"

$$p^{\uparrow} : \boldsymbol{F}^{\uparrow} := \boldsymbol{S} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \to \boldsymbol{S}$$

of the fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$.

Thus, let us consider a smooth double fibred manifold $\boldsymbol{G} \xrightarrow{q} \boldsymbol{F} \xrightarrow{p} \boldsymbol{B}$ and a smooth system $(\boldsymbol{S}, \zeta, \epsilon)$ of smooth sections, where $\epsilon : \boldsymbol{S} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \to \boldsymbol{G}$ (see Definition 3.1.3).

Definition 3.1.4. We define the *lifted smooth fibred manifold* of the smooth fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ to be the fibred product over $\boldsymbol{B}$

$$\boldsymbol{F}^{\uparrow} := \left\{(s_b, f_b) \in \boldsymbol{S} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \mid s_b \in \boldsymbol{S}_b,\ f_b \in \boldsymbol{F}_b,\ b \in \boldsymbol{B}\right\} = \boldsymbol{S} \underset{\boldsymbol{B}}{\times} \boldsymbol{F},$$

which can be regarded as the pullback of the smooth fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$, with respect to the smooth projection $\zeta : \boldsymbol{S} \to \boldsymbol{B}$.

The natural map

$$p^{\uparrow} : \boldsymbol{F}^{\uparrow} \to \boldsymbol{S} : (s_b, f_b) \to s_b$$

makes $\boldsymbol{F}^{\uparrow}$ a smooth fibred manifold over $\boldsymbol{S}$.

In simple words, the smooth fibred manifold $p^{\uparrow} : \boldsymbol{F}^{\uparrow} \to \boldsymbol{S}$ can be regarded as the "extension" of the smooth fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ obtained by "extending" the base space $\boldsymbol{B}$ to $\boldsymbol{S}$.

Then, $\boldsymbol{F}^{\uparrow}$ turns out to be also a smooth double fibred manifold

$$\boldsymbol{F}^{\uparrow} \xrightarrow{\mathrm{pro}_2} \boldsymbol{F} \xrightarrow{p} \boldsymbol{B} \ ,$$

according to the following commutative diagram

$$\begin{array}{ccc} \boldsymbol{S} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} & \xrightarrow{\mathrm{pro}_2} & \boldsymbol{F} \\ {\scriptstyle \mathrm{pro}_1}\downarrow & & \downarrow{\scriptstyle p} \\ \boldsymbol{S} & \xrightarrow{\zeta} & \boldsymbol{B} \end{array} \ .$$

Hence, we can regard the smooth evaluation map $\epsilon : \boldsymbol{S} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \to \boldsymbol{G}$ as a smooth fibred morphism over $\boldsymbol{F}$

$$\epsilon : \boldsymbol{F}^{\uparrow} \to \boldsymbol{G} \,.$$

The induced fibred chart of $p^{\uparrow} : \boldsymbol{F}^{\uparrow} \to \boldsymbol{S}$ is $(x^{\lambda}, w^{A}, y^{i})$. □

3.2 F–smooth systems of smooth sections

We discuss the notions of *F–smooth systems of fibrewisely smooth sections* $(\boldsymbol{S}, \zeta, \epsilon)$ and of its *F–smooth tangent prolongation* $(\mathsf{T}\boldsymbol{S}, \mathsf{T}\zeta, \mathsf{T}\epsilon)$.

Moreover, we compare the F–smooth and smooth structure of $\boldsymbol{S}$ for the particular case of a smooth system of smooth sections.

Eventually, we discuss the F–smooth differential operators.

3.2.1 F–smooth systems of smooth sections

In the previous Section 3.1.1, we have studied the smooth systems $(\boldsymbol{S}, \zeta, \epsilon)$ of smooth sections of a smooth double fibred manifold $\boldsymbol{G} \to \boldsymbol{F} \to \boldsymbol{B}$.

Now, we analyse the generalised notion of *F–smooth system* $(\boldsymbol{S}, \zeta, \epsilon)$ of *fibrewisely smooth sections* of a smooth double fibred manifold $\boldsymbol{G} \to \boldsymbol{F} \to \boldsymbol{B}$, by releasing the hypotheses of smoothness and finite dimension of $\boldsymbol{S}$.

Let us consider a smooth double fibred manifold $\boldsymbol{G} \overset{q}{\to} \boldsymbol{F} \overset{p}{\to} \boldsymbol{B}$.

Definition 3.2.1. We denote by (see Definition 3.1.2)

$$\underline{\mathrm{tub}}(\boldsymbol{F}, \boldsymbol{G}) \subset \{s : \boldsymbol{F} \to \boldsymbol{G}\}$$

the subsheaf consisting of tubelike sections $s : \boldsymbol{F} \to \boldsymbol{G}$, which are *fibrewiselyly smooth*, i.e. which fulfill the following condition, *without any further local smoothness requirement*,

- $s_b : \boldsymbol{F}_b \to \boldsymbol{G}_b$ is global and smooth, for each $b \in \boldsymbol{B}$.

Thus, the sheaf of tubelike smooth sections turns out to be a subsheaf (see Definition 3.1.2)

$$\mathrm{tub}(\boldsymbol{F}, \boldsymbol{G}) \subset \underline{\mathrm{tub}}(\boldsymbol{F}, \boldsymbol{G}) \,.\square$$

The following Definition is a generalisation of Definition 3.1.3, as here we do not require that $\boldsymbol{S}$ be a finite dimensional smooth manifold (hence, that the maps ζ and ϵ be smooth).

Definition 3.2.2. We define an *F–smooth system of fibrewisely smooth sections* of the smooth double fibred manifold $\boldsymbol{G} \to \boldsymbol{F} \to \boldsymbol{B}$ to be a 3–plet $(\boldsymbol{S}, \zeta, \epsilon)$, where

1) $\boldsymbol{S}$ is a set,

2) $\zeta : \boldsymbol{S} \to \boldsymbol{B}$ is a fibred set (i.e. $\boldsymbol{S}$ is a set and ζ a surjective map, without any smoothness requirements),

3) $\epsilon : \boldsymbol{S} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \to \boldsymbol{G}$ is a fibred map over $\boldsymbol{F}$, according to the following commutative diagram

$$\begin{array}{ccc}
\boldsymbol{S} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} & \xrightarrow{\ \epsilon\ } & \boldsymbol{G} \\
{\scriptstyle \mathrm{pro}_2}\downarrow & & \downarrow{\scriptstyle q} \\
\boldsymbol{F} & \xrightarrow{\ \mathrm{id}_{\boldsymbol{F}}\ } & \boldsymbol{F}
\end{array} \quad ,$$

which fulfills the following condition:

*) for each $s \in \boldsymbol{S}_b$, with $b \in \boldsymbol{B}$, the induced section

$$\epsilon_s : \boldsymbol{F}_b \to \boldsymbol{G}_b$$

of the restricted smooth fibred manifold $q_b : \boldsymbol{G}_b \to \boldsymbol{F}_b$ is *smooth and globally defined on* $\boldsymbol{F}_b$.

The map $\epsilon : \boldsymbol{S} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \to \boldsymbol{G}$ is called the *evaluation map* of the system.

We denote by

$$\underline{\mathrm{sec}}(\boldsymbol{B}, \boldsymbol{S}) \subset \{s : \boldsymbol{B} \to \boldsymbol{S}\}$$

the subsheaf consisting of *local* sections $s : \boldsymbol{B} \to \boldsymbol{S}$, *without any smoothness requirement.*

Thus, the evaluation map ϵ yields the sheaf morphism

$$\epsilon_{\boldsymbol{S}} : \underline{\mathrm{sec}}(\boldsymbol{B}, \boldsymbol{S}) \to \underline{\mathrm{tub}}(\boldsymbol{F}, \boldsymbol{G}) : \sigma \mapsto \breve{\sigma} ,$$

where, for each $\sigma \in \underline{\mathrm{sec}}(\boldsymbol{B}, \boldsymbol{S})$, the tubelike fibrewisely smooth section $\breve{\sigma}$ is defined by

$$\breve{\sigma} : \boldsymbol{F}_b \to \boldsymbol{G}_b : f_b \mapsto \epsilon\big(\sigma(b), f_b\big) , \qquad \text{for each} \qquad b \in \boldsymbol{S} .$$

Therefore, the map $\epsilon_{\boldsymbol{S}} : \underline{\mathrm{sec}}(\boldsymbol{B}, \boldsymbol{S}) \to \underline{\mathrm{tub}}(\boldsymbol{F}, \boldsymbol{G})$ provides a *selection* of the tubelike fibrewisely smooth sections $\phi : \boldsymbol{F} \to \boldsymbol{G}$, given by the subset

$$\underline{\mathrm{tub}}_{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{G}) := \epsilon_{\boldsymbol{S}}\big(\underline{\mathrm{sec}}(\boldsymbol{B}, \boldsymbol{S})\big) \subset \underline{\mathrm{tub}}(\boldsymbol{F}, \boldsymbol{G}) .$$

The smooth system of fibrewisely smooth sections $(\boldsymbol{S}, \zeta, \epsilon)$ is said to be *injective* if the map $\epsilon_{\boldsymbol{S}} : \underline{\mathrm{sec}}(\boldsymbol{B}, \boldsymbol{S}) \to \underline{\mathrm{tub}}(\boldsymbol{F}, \boldsymbol{G})$ is injective, i.e. if, for each $\sigma , \acute{\sigma} \in \underline{\mathrm{sec}}(\boldsymbol{B}, \boldsymbol{S})$,

$$\breve{\sigma} \equiv \epsilon_{\boldsymbol{S}}(\sigma) = \breve{\acute{\sigma}} \equiv \epsilon_{\boldsymbol{S}}(\acute{\sigma}) \qquad \Rightarrow \qquad \sigma = \acute{\sigma} .$$

If the system is injective, then we obtain the bijection

$$\epsilon_{\boldsymbol{S}} : \underline{\mathrm{sec}}(\boldsymbol{B}, \boldsymbol{S}) \to \underline{\mathrm{tub}}_{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{G}) : \sigma \mapsto \breve{\sigma} ,$$

whose inverse is denoted by

$$(\epsilon_{\boldsymbol{S}})^{-1} : \underline{\mathrm{tub}}_{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{G}) \to \underline{\mathrm{sec}}(\boldsymbol{B}, \boldsymbol{S}) : \phi \mapsto \widehat{\phi} . \square$$

Indeed, we are essentially interested in injective F–smooth systems of tubelike fibrewisely smooth sections.

Remark 3.2.1. We stress that we have not assumed a priori any smooth or F–smooth structure on $\boldsymbol{S}$. However, the specification "*F–smooth*" system in the above Definition 3.2.2 will be justified later by Theorem 3.2.1.

Moreover, we stress that the local sections $\sigma : \boldsymbol{B} \to \boldsymbol{S}$, without any F–smoothness requirement, yield tubelike sections $\breve{\sigma} : \boldsymbol{F} \to \boldsymbol{G}$, which need not to be smooth, even if their restrictions $\breve{\sigma}_b : \boldsymbol{F}_b \to \boldsymbol{G}_b$, are smooth, for each $b \in \boldsymbol{B}$, (according to condition *) in Definition 3.2.2).

Later, we shall see that the set $\boldsymbol{S}$ is an F–smooth space in a natural way (see Theorem 3.2.1) and that the selected fibrewisely smooth sections $\breve{\sigma} \in \underline{\mathrm{tub}}_{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{G})$ turn out to be smooth if and only if their source sections $\sigma \in \underline{\mathrm{sec}}(\boldsymbol{B}, \boldsymbol{S})$ are F–smooth (see Theorem 3.2.2).

This result further justifies the name "*F–smooth systems of fibrewisely smooth sections*" in Definition 3.2.2. □

Let us examine some examples of F–smooth systems of fibrewisely smooth sections.

We start by considering "infinite dimensional" examples, by following the analogous thread of smooth systems of global smooth maps (see §3).

Example 3.2.1. For each smooth double fibred manifold $\boldsymbol{G} \to \boldsymbol{F} \to \boldsymbol{B}$, the set (see Definition 3.2.1)

$$\underline{\text{tub}}_{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{G}) := \underline{\text{tub}}(\boldsymbol{F}, \boldsymbol{G})$$

consisting of *all* $\breve{\phi} \in \underline{\text{tub}}(\boldsymbol{F}, \boldsymbol{G})$ yields in a natural way an injective F–smooth system of fibrewisely smooth sections. □

Example 3.2.2. For each smooth double fibred manifold $\boldsymbol{G} \to \boldsymbol{F} \to \boldsymbol{B}$, where $\boldsymbol{G} \to \boldsymbol{F}$ is a *vector bundle*, the subset

$$\underline{\text{tub}}_{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{G}) \subset \underline{\text{tub}}(\boldsymbol{F}, \boldsymbol{G})$$

consisting of *all* $\breve{\phi} \in \underline{\text{tub}}(\boldsymbol{F}, \boldsymbol{G})$, whose fibrewisely restrictions $\breve{\sigma}_b : \boldsymbol{F}_b \to \boldsymbol{G}_b$ have *compact support* for each $b \in \boldsymbol{B}$, yields in a natural way an injective F–smooth system of tubelike fibrewisely smooth sections.

Indeed, this system is a subsystem of the system of sections $\phi : \boldsymbol{F} \to \boldsymbol{G}$ considered in the above Example 3.2.1. □

We can reconsider "finite dimensional" examples of smooth systems of tubelike fibrewisely smooth sections in the present context of F–smooth spaces (see Section 3.1.1).

Example 3.2.3. For each smooth double fibred manifold $\boldsymbol{G} \to \boldsymbol{F} \to \boldsymbol{B}$, where $\boldsymbol{F} \to \boldsymbol{B}$ and $\boldsymbol{G} \to \boldsymbol{B}$ are *vector bundles*, the subset

$$\underline{\text{tub}}_{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{G}) \subset \underline{\text{tub}}(\boldsymbol{F}, \boldsymbol{G})$$

consisting of *all* $\breve{\phi} \in \underline{\text{tub}}(\boldsymbol{F}, \boldsymbol{G})$, whose fibrewise restrictions $\breve{\sigma}_b : \boldsymbol{F}_b \to \boldsymbol{G}_b$ are *linear maps* for each $b \in \boldsymbol{B}$, yields in a natural way an injective F–smooth system of fibrewisely smooth sections.

Indeed, the F–smooth structure of $\boldsymbol{S}$ turns out to be just the natural F–smooth structure underlying the natural smooth structure of $\boldsymbol{S}$ (see, later, Theorem 3.2.1). □

Example 3.2.4. For each smooth double fibred manifold $\boldsymbol{G} \to \boldsymbol{F} \to \boldsymbol{B}$, where $\boldsymbol{F} \to \boldsymbol{B}$ and $\boldsymbol{G} \to \boldsymbol{B}$ are *affine bundles*, the subset

$$\underline{\text{tub}}_{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{G}) \subset \underline{\text{tub}}(\boldsymbol{F}, \boldsymbol{G})$$

consisting of *all* $\breve{\phi} \in \underline{\mathrm{tub}}(\boldsymbol{F}, \boldsymbol{G})$, whose fibrewise restrictions $\breve{s}_b : \boldsymbol{F}_b \to \boldsymbol{G}_b$, are *affine maps* for each $b \in \boldsymbol{B}$, yields in a natural way an injective F–smooth system of fibrewisely smooth sections.

Indeed, the F–smooth structure of $\boldsymbol{S}$ turns out to be just the natural F–smooth structure underlying the natural smooth structure of $\boldsymbol{S}$ (see, later, Theorem 3.2.1). □

Example 3.2.5. For each smooth double fibred manifold $\boldsymbol{G} \to \boldsymbol{F} \to \boldsymbol{B}$, where $\boldsymbol{F} \to \boldsymbol{B}$ and $\boldsymbol{G} \to \boldsymbol{B}$ are *affine bundles*, the subset

$$\underline{\mathrm{tub}}_{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{G}) \subset \underline{\mathrm{tub}}(\boldsymbol{F}, \boldsymbol{G})$$

consisting of *all* $\breve{\phi} \in \underline{\mathrm{tub}}(\boldsymbol{F}, \boldsymbol{G})$, whose fibrewise restrictions $\breve{\phi}_b : \boldsymbol{F}_b \to \boldsymbol{G}_b$ are *polynomial maps* of a *given degree* for each $b \in \boldsymbol{B}$, yields in a natural way an injective F–smooth system of fibrewisely smooth sections.

Indeed, the F–smooth structure of $\boldsymbol{S}$ turns out to be just the natural F–smooth structure underlying the natural smooth structure of $\boldsymbol{S}$ (see, later, Theorem 3.2.1). □

We can also reconsider the "infinite dimensional" example of F–smooth systems of polynomial sections of any degree in the present context of F–smooth spaces (see §3).

Example 3.2.6. For each smooth double fibred manifold $\boldsymbol{G} \to \boldsymbol{F} \to \boldsymbol{B}$, where $\boldsymbol{F} \to \boldsymbol{B}$ and $\boldsymbol{G} \to \boldsymbol{B}$ are *affine bundles*, the subset

$$\underline{\mathrm{tub}}_{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{G}) \subset \underline{\mathrm{tub}}(\boldsymbol{F}, \boldsymbol{G})$$

consisting of *all* $\breve{\sigma} \in \underline{\mathrm{tub}}(\boldsymbol{F}, \boldsymbol{G})$, whose fibrewise restrictions $\breve{\sigma}_b : \boldsymbol{F}_b \to \boldsymbol{G}_b$, are *polynomial maps* of *any degree* for each $b \in \boldsymbol{B}$, yields in a natural way an injective F–smooth system of fibrewisely smooth sections.

Indeed, the F–smooth structure of $\boldsymbol{S}$ turns out to be just the natural F–smooth structure underlying the natural infinite dimensional smooth structure of $\boldsymbol{S}$ (see, later, Theorem 3.2.1). □

3.2.2 F–smooth structure of $\boldsymbol{S}$

We show that an F–smooth system $(\boldsymbol{S}, \zeta, \epsilon)$ of fibrewisely smooth sections turns out to have a natural F–smooth structure.

Namely, the set $\boldsymbol{S}$ has a natural F–smooth structure and the maps

$$\zeta : \boldsymbol{S} \to \boldsymbol{B} \qquad \text{and} \qquad \epsilon : \boldsymbol{S} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \to \boldsymbol{G}$$

turn out to be F–smooth.

Furthermore, we exhibit a bijection between F–smooth sections $\sigma : \boldsymbol{B} \to \boldsymbol{S}$ and tubelike smooth sections $\breve{\sigma} : \boldsymbol{F} \to \boldsymbol{G}$.

Let us consider an F–smooth system of fibrewisely smooth sections $(\boldsymbol{S}, \zeta, \epsilon)$.

Let us start by exhibiting a natural F–smooth structure of the set $\boldsymbol{S}$. Preliminarily, we need a few technical Lemmas, which provide some pull-back objects.

Lemma 3.2.1. If $c : \boldsymbol{I}_c \to \boldsymbol{B}$ is a smooth curve, then we obtain the smooth submanifold

$$c^*(\boldsymbol{F}) := \{(\lambda, f) \in \boldsymbol{I}_c \times \boldsymbol{F} \mid c(\lambda) = p(f)\} \subset \boldsymbol{I}_c \times \boldsymbol{F},$$

along with the smooth projections

$$c^*(p) : c^*(\boldsymbol{F}) \to \boldsymbol{I}_c : (\lambda, f) \mapsto \lambda \qquad \text{and} \qquad c^*_{\boldsymbol{F}} : c^*(\boldsymbol{F}) \to \boldsymbol{F} : (\lambda, f) \mapsto f . \square$$

Lemma 3.2.2. If $c : \boldsymbol{I}_c \to \boldsymbol{B}$ and $\gamma : \boldsymbol{I}_\gamma \to \boldsymbol{I}_c$ are smooth maps, then we obtain the smooth map

$$\gamma^* : (c \circ \gamma)^*(\boldsymbol{F}) \to c^*(\boldsymbol{F}) : (\lambda, f) \mapsto \big(\gamma(\lambda), f\big),$$

which provides just a smooth reparametrisation of $c^*(\boldsymbol{F})$. $\square$

Lemma 3.2.3. If $\widehat{c} : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{S}$ is a curve, which projects on a smooth curve $c := \zeta \circ \widehat{c} : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{B}$, then we obtain the map

$$\widehat{c}^*(\epsilon) : c^*(\boldsymbol{F}) \to \boldsymbol{G} : (\lambda, f) \mapsto \epsilon\big(\widehat{c}(\lambda), f\big) . \square$$

Theorem 3.2.1. *Let us consider the set*

$$\mathcal{C} := \{\widehat{c} : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{S}\}$$

consisting of all curves $\widehat{c} : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{S}$, such that the following induced maps between smooth manifolds be smooth (see the above Lemma 3.2.1 and Lemma 3.2.3)

$$(a) \qquad c : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{B} : \lambda \mapsto \zeta\big(\widehat{c}(\lambda)\big),$$

$$(b) \qquad \widehat{c}^*(\epsilon) : c^*(\boldsymbol{F}) \to \boldsymbol{G} : (\lambda, f) \mapsto \epsilon\big(\widehat{c}(\lambda), f\big).$$

Then, the pair $(\boldsymbol{S}, \mathcal{C})$ turns out to be an F–smooth space.

PROOF. Let us prove that the pair $(\boldsymbol{S}, \mathcal{C})$ be an F–smooth space, by showing that it fulfills the two conditions 1) and 2) in Definition 1.2.1.

1) For every element $s \in \boldsymbol{S}$, let us consider the constant curve $\widehat{c} : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{S} : \lambda \mapsto s$, which clearly passes through s. Then, the following facts hold.

a) The induced curve $c := \zeta \circ \widehat{c} : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{B}$ turns out to be constant as well, hence smooth.

b) The map

$$\widehat{c}^*(\epsilon) : c^*(\boldsymbol{F}) \to \boldsymbol{G} : (\lambda, f) \mapsto \epsilon\big(\widehat{c}(\lambda), f\big)$$

can be regarded as the map

$$\epsilon_s : \boldsymbol{F}_{\zeta(s)} \to \boldsymbol{G}_{\zeta(s)} ,$$

hence, in virtue of condition *) in Definition 3.2.2, it turns out to be smooth.

Therefore, the constant curves $\widehat{c} : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{S}$ belong to $\mathcal{C}$.

2) If $\gamma : \boldsymbol{I}_\gamma \to \boldsymbol{I}_{\widehat{c}}$ is any smooth curve, then

a) the map

$$c \circ \gamma : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{B}$$

is a composition of smooth maps between smooth manifolds, hence it turns out to be smooth,

b) in virtue of the above Lemma 3.2.2, the map

$$(\widehat{c} \circ \gamma)^*(\epsilon) : (c \circ \gamma)^*(\boldsymbol{F}) \to \boldsymbol{G} : (\lambda, f) \mapsto \epsilon\Big(\widehat{c}\big((\gamma(\lambda)\big),\, f\Big)$$

is given by the composition of smooth maps between smooth manifolds

$$(c \circ \gamma)^*(\boldsymbol{F}) \xrightarrow{\ \gamma^*\ } c^*(\boldsymbol{F}) \xrightarrow{\ \widehat{c}^*(\epsilon)\ } \boldsymbol{G} \quad :$$

$$\Big(\lambda, f_{c(\gamma(\lambda))}\Big) \xrightarrow{\ \gamma^*\ } \Big(\gamma(\lambda), f_{c(\gamma(\lambda))}\Big) \xrightarrow{\ \widehat{c}^*(\epsilon)\ } \epsilon\Big(\widehat{c}\big(\gamma(\lambda)\big), f_{c(\gamma(\lambda))}\Big) \ ,$$

hence it turns out to be smooth.

Therefore, according to the above conditions (a) and (b), the curves $\widehat{c} \circ \gamma : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{S}$ belong to $\mathcal{C}$. QED

Then, we show that the maps ζ and ϵ are F–smooth. Preliminarily, we need a technical Lemma, which provides some pullback objects.

Lemma 3.2.4. Let us consider an F–smooth curve

$$(\widehat{c}, c_{\boldsymbol{F}}) : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{S} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} ,$$

where

$$\widehat{c} : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{S} \qquad \text{and} \qquad c_{\boldsymbol{F}} : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{F}$$

are, respectively, an F–smooth curve and a smooth curve, which project on the same smooth base curve $c : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{B}$.

Then, the F–smooth curve $(\widehat{c}, c_{\boldsymbol{F}}) : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{S} \times_{\boldsymbol{B}} \boldsymbol{F}$ factorises through a smooth curve (see Lemma 3.2.1)

$$(c, c_{\boldsymbol{F}}) : \boldsymbol{I}_{\widehat{c}} \to c^*(\boldsymbol{F}),$$

according to the following commutative diagram

$$\begin{array}{ccc} \boldsymbol{I}_{\widehat{c}} & \xrightarrow{(\widehat{c}, c_{\boldsymbol{F}})} & \boldsymbol{S} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \\ {\scriptstyle \mathrm{id}_{\boldsymbol{I}_{\widehat{c}}}} \downarrow & & \uparrow {\scriptstyle \widehat{c} \times \mathrm{id}_{\boldsymbol{F}}} \\ \boldsymbol{I}_{\widehat{c}} & \xrightarrow{(\mathrm{id}_{\boldsymbol{I}_{\widehat{c}}}, c_{\boldsymbol{F}})} & c^*(\boldsymbol{F}) \end{array} .$$

PROOF. In fact, for each $\lambda \in \boldsymbol{I}_{\widehat{c}}$, the following diagram commutes

$$\begin{array}{ccc} \lambda & \xrightarrow{(\widehat{c}, c_{\boldsymbol{F}})} & \big(\widehat{c}(\lambda), c_{\boldsymbol{F}}(\lambda)\big) \\ {\scriptstyle \mathrm{id}} \downarrow & & \uparrow {\scriptstyle \widehat{c} \times \mathrm{id}_{\boldsymbol{F}}} \\ \lambda & \longrightarrow & \big(\lambda, c_{\boldsymbol{F}}(\lambda)\big) \end{array} . \text{ QED}$$

Proposition 3.2.1. The maps

$$\zeta : \boldsymbol{S} \to \boldsymbol{B} \qquad \text{and} \qquad \epsilon : \boldsymbol{S} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \to \boldsymbol{G}$$

turn out to be F–smooth.

PROOF. According to Definition 1.2.4, we have to prove that ζ and ϵ map F–smooth curves of the source space into F–smooth curves of the target space.

1) For each F–smooth curve $\widehat{c} : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{S}$, the composed map $\zeta \circ \widehat{c} : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{B}$ is a smooth curve, by assumption, hence it is an F–smooth curve (see Definition 1.3.1).

2) Let $(\widehat{c}, c_{\boldsymbol{F}}) : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{S} \underset{\boldsymbol{B}}{\times} \boldsymbol{F}$ be an F–smooth curve, where

$$\widehat{c} : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{S} \qquad \text{and} \qquad c_{\boldsymbol{F}} : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{F}$$

are, respectively an F–smooth curve and a smooth curve, which project on the same smooth base curve $c : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{B}$.

Then, in virtue of the above Lemma 3.2.4, the curve

$$\epsilon \circ (\widehat{c}, c^*_{\boldsymbol{F}}) : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{G}$$

turns out to be the composition of smooth curves

$$\epsilon \circ (\widehat{c}, c^*_{\boldsymbol{F}}) = c^*(\epsilon) \circ (\mathrm{id}_{\boldsymbol{I}_{\widehat{c}}}, c_{\boldsymbol{F}}) \,,$$

according to the following commutative diagram

$$\begin{array}{ccccc} \boldsymbol{I}_{\widehat{c}} & \xrightarrow{(\widehat{c},\, c_{\boldsymbol{F}})} & \boldsymbol{S} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} & \xrightarrow{\epsilon} & \boldsymbol{G} \\ {\scriptstyle \mathrm{id}_{\boldsymbol{I}_{\widehat{c}}}} \big\downarrow & & \big\uparrow {\scriptstyle \widehat{c} \times \mathrm{id}_{\boldsymbol{F}}} & & \big\uparrow {\scriptstyle \mathrm{id}_{\boldsymbol{G}}} \\ \boldsymbol{I}_{\widehat{c}} & \xrightarrow{(\mathrm{id},\, c_{\boldsymbol{F}})} & c^*(\boldsymbol{F}) & \xrightarrow{c^*(\epsilon)} & \boldsymbol{G} \end{array} \quad ,$$

hence it is smooth. QED

Next, we show a natural bijection between local F–smooth sections $\sigma : \boldsymbol{B} \to \boldsymbol{S}$ and tubelike smooth sections $\breve{\sigma} : \boldsymbol{F} \to \boldsymbol{G}$. Preliminarily, we need a technical Lemma.

Lemma 3.2.5. Let us consider a section $\sigma \in \underline{\mathrm{sec}}(\boldsymbol{B}, \boldsymbol{S})$, the induced tubelike fibrewisely smooth section $\breve{\sigma} \in \underline{\mathrm{tub}}_{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{G})$ and a smooth curve $c : \boldsymbol{I}_c \to \boldsymbol{B}$ (see Definition 3.2.2).

Then, we obtain, by pullback, the fibred morphism over $\boldsymbol{F}$

$$c^*(\breve{\sigma}) : c^*(\boldsymbol{F}) \to \boldsymbol{G} \,,$$

given by the composition (see Lemma 3.2.1)

$$c^*(\boldsymbol{F}) \xrightarrow{(c^*(p),\, c^*_{\boldsymbol{F}})} \boldsymbol{I}_c \times \boldsymbol{F} \xrightarrow{c \times \mathrm{id}_{\boldsymbol{F}}} \boldsymbol{B} \times \boldsymbol{F} \xrightarrow{\sigma \times \mathrm{id}_{\boldsymbol{F}}} \boldsymbol{S} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \xrightarrow{\epsilon} \boldsymbol{G}. \ \square$$

We denote by

$$\text{F–sec}(\boldsymbol{B}, \boldsymbol{S}) \subset \underline{\mathrm{sec}}(\boldsymbol{B}, \boldsymbol{S})$$

the subsheaf consisting of *F–smooth* local sections of the fibred set $\zeta : \boldsymbol{S} \to \boldsymbol{B}$.

Moreover, we denote by

$$\mathrm{tub}_{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{G}) := \underline{\mathrm{tub}}_{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{G}) \cap \mathrm{tub}(\boldsymbol{F}, \boldsymbol{G})$$

the subsheaf consisting of tubelike *smooth* sections of the smooth fibred manifold $\boldsymbol{G} \to \boldsymbol{F}$, which are selected by the F–smooth system $(\boldsymbol{S}, \zeta, \epsilon)$.

Theorem 3.2.2. *Let $\sigma \in \underline{\mathrm{sec}}(\boldsymbol{B}, \boldsymbol{S})$ be a local section and*

$$\breve{\sigma} := \epsilon_{\boldsymbol{S}}(\sigma) \in \underline{\mathrm{tub}}_{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{G})$$

the induced tubelike section (see Definition 3.2.2).

Then, $\breve{\sigma}$ is smooth if and only if σ is F–smooth.

In other words, the sheaf morphism

$$\epsilon_{\boldsymbol{S}} : \underline{\mathrm{sec}}(\boldsymbol{B}, \boldsymbol{S}) \to \underline{\mathrm{tub}}_{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{G})$$

restricts to a sheaf isomorphism *(denoted by the same symbol)*

$$\epsilon_{\boldsymbol{S}} : \mathrm{F\text{–}sec}(\boldsymbol{B}, \boldsymbol{S}) \to \mathrm{tub}_{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{G}),$$

according to the following diagram commutes

$$\begin{array}{ccc} \underline{\mathrm{sec}}(\boldsymbol{B}, \boldsymbol{S}) & \xrightarrow{\epsilon_{\boldsymbol{S}}} & \underline{\mathrm{tub}}_{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{G}) \\ \cup \uparrow & & \uparrow \cup \\ \mathrm{F\text{–}sec}(\boldsymbol{B}, \boldsymbol{S}) & \xrightarrow{\epsilon_{\boldsymbol{S}}} & \mathrm{tub}_{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{G}) \end{array} \quad .$$

PROOF. 1) Let us prove that, if σ is F–smooth, then $\breve{\sigma}$ is smooth.

In fact, the map between smooth manifolds $\breve{\sigma} : \boldsymbol{F} \to \boldsymbol{G}$ is given by a composition of F–smooth maps, according to the following commutative diagram (see Definition 1.3.1 and Proposition 3.2.1)

$$\begin{array}{ccccccc} \boldsymbol{F} & \xrightarrow{(p, \mathrm{id}_{\boldsymbol{F}})} & \boldsymbol{B} \times \boldsymbol{F} & \xrightarrow{\sigma \times \mathrm{id}_{\boldsymbol{F}}} & \boldsymbol{S} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} & \xrightarrow{\epsilon} & \boldsymbol{G} \\ \mathrm{id}_{\boldsymbol{F}} \uparrow & & & & & & \downarrow \mathrm{id}_{\boldsymbol{G}} \\ \boldsymbol{F} & & & \xrightarrow{\breve{\sigma}} & & & \boldsymbol{G} \end{array} \quad .$$

Hence, according to Proposition 1.2.4, $\breve{\sigma} : \boldsymbol{F} \to \boldsymbol{G}$ is F-smooth. Indeed, in virtue of Definition 1.3.1 and Theorem 1.3.1, it means that the map $\breve{\sigma} : \boldsymbol{F} \to \boldsymbol{G}$ is smooth.

2) Let us prove that, if $\breve{\sigma}$ is smooth, then σ is F–smooth.

By definition of F–smooth map between F–smooth spaces (see Definition 1.2.4), we have to prove that, for smooth curve $c : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{B}$, the composed curve

$$\widehat{c}_{\sigma} := \sigma \circ c : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{S}$$

be F–smooth.

Hence, according to Theorem 3.2.1, we have to prove that the induced maps between smooth manifolds

$$c_{\sigma} := \zeta \circ \widehat{c}_{\sigma} : \boldsymbol{I}_{c_{\sigma}} \to \boldsymbol{B} \qquad \text{and} \qquad (\widehat{c}_{\sigma})^{*}(\epsilon) : (c_{\sigma})^{*}(\boldsymbol{F}) \to \boldsymbol{G},$$

be smooth.

a) The curve $c_\sigma := \zeta \circ \widehat{c}_\sigma : \boldsymbol{I}_{c_\sigma} \to \boldsymbol{B}$ is F–smooth because it is just the smooth curve $c_\sigma : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{B}$.

b) The map $(\widehat{c}_\sigma)^*(\epsilon) : (c_\sigma)^*(\boldsymbol{F}) \to \boldsymbol{G}$ turns out to be smooth, because it is the restriction to the smooth sub manifold over $c : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{B}$

$$(c_\sigma)^*(\boldsymbol{F}) \subset \boldsymbol{F}\,,$$

according to the following commutative diagram

$$\begin{array}{ccc} (c_\sigma)^*(\boldsymbol{F}) & \xrightarrow{(\widehat{c}_\sigma)^*(\epsilon)} & \boldsymbol{G} \\ {\scriptstyle (c_\sigma)^*_{\boldsymbol{F}}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\boldsymbol{G}}} \\ \boldsymbol{F} & \xrightarrow{\breve{\sigma}} & \boldsymbol{G} \end{array} \quad .\ \text{QED}$$

Remark 3.2.2. The above Theorem 3.2.2 clarifies the name "F–smooth system of fibrewisely *smooth* sections" in Definition 3.2.2.

Roughly speaking, we can interpret the above Theorem 3.2.2 in the following way.

Given a section $\sigma \in \underline{\mathrm{sec}}(\boldsymbol{B}, \boldsymbol{S})$, the induced section $\breve{\sigma} \in \underline{\mathrm{tub}}(\boldsymbol{F}, \boldsymbol{G})$ is smooth along the fibres of the fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ in virtue of condition *) in Definition 3.2.2.

Then, in order to check whether $\breve{\sigma} \in \underline{\mathrm{tub}}(\boldsymbol{F}, \boldsymbol{G})$ is fully smooth, we have to show that it is smooth "transversally", i.e. along smooth curves $c : \boldsymbol{I}_c \to \boldsymbol{B}$.

However, in virtue of the definition of F-smooth map (see Definition 1.2.4), the section $\sigma \in \underline{\mathrm{sec}}(\boldsymbol{B}, \boldsymbol{S})$ is F–smooth if and only if its composition with any smooth curve $c : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{B}$ is smooth.

Thus, to check the smoothness of both $\breve{\sigma} \in \underline{\mathrm{tub}}(\boldsymbol{F}, \boldsymbol{G})$ and the F–smoothness of $\sigma \in \underline{\mathrm{sec}}(\boldsymbol{B}, \boldsymbol{S})$ is subject to analysing the behaviour of both sections along smooth curves $c : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{B}$.

In the particular case when $\boldsymbol{B} = \mathbb{R}$, the above Theorem 3.2.2 reduces to a tautology.

In fact, a local section $\sigma : \boldsymbol{B} \to \boldsymbol{S}$ turns out to be a local curve $\widehat{c} := \sigma : \mathbb{R} \to \boldsymbol{S}$. Moreover, in this case we have locally $c^*(\boldsymbol{F}) = \boldsymbol{F}$.

So, checking that $\widehat{\sigma}$ maps F–smooth curves of $\boldsymbol{B}$ into F–smooth curves of $\boldsymbol{S}$ reduces to check that the fibrewisely smooth section $\sigma : \boldsymbol{F} \to \boldsymbol{G}$ be smooth. Indeed, no other independent check is required. In other words, in the particular case when $\boldsymbol{B} = \mathbb{R}$, the above Theorem reduces to say that $\mathrm{sec}(\boldsymbol{B}, \boldsymbol{S})$ is just, by definition, the subsheaf $\mathrm{sec}(\boldsymbol{B}, \boldsymbol{S}) \subset \underline{\mathrm{sec}}(\boldsymbol{B}, \boldsymbol{S})$ which yields $\mathrm{tub}(\boldsymbol{F}, \boldsymbol{G})$. □

Remark 3.2.3. In our definition of "smooth system of smooth sections" (see Definition 3.1.3), we have assumed a priori a finite dimensional smooth structure of $\boldsymbol{S}$, while, for our concept of "F–smooth system of fibrewisely smooth sections" (see Definition 3.2.2), we have not assumed a priori any smooth structure of $\boldsymbol{S}$, but have recovered an F–smooth structure in a unique way (see Theorem 3.2.1).

Now, the procedure we have used for F–smooth systems of fibrewisely smooth sections to recover the F–smooth structure of $\boldsymbol{S}$ can be applied to smooth systems of smooth sections as well and a natural question arises: "is the assumed smooth structure of $\boldsymbol{S}$ compatible with the recovered F–smooth structure?"

In general, the answer is negative. In fact, in Example 3.1.5, we have shown that we can assume several smooth structures on $\boldsymbol{S}$ providing the same system of smooth sections. □

Next, we show that, in the case when the smooth fibred manifold $q : \boldsymbol{G} \to \boldsymbol{F}$ is an affine (vector) bundle, the F–smooth fibred space $\zeta : \boldsymbol{S} \to \boldsymbol{B}$ of an injective system $(\boldsymbol{S}, \zeta, \epsilon)$ of smooth sections $\phi : \boldsymbol{F} \to \boldsymbol{G}$ inherits in a natural way an affine (vector) structure.

Proposition 3.2.2. Let us suppose that the fibred manifold $q : \boldsymbol{G} \to \boldsymbol{F}$ be a vector bundle and consider an *injective* F–smooth system $(\boldsymbol{S}, \zeta, \epsilon)$ of fibrewisely smooth sections $\phi : \underline{\mathrm{tub}}(\boldsymbol{F}, \boldsymbol{G})$ (see Example 3.2.1).

Then, the fibres of the F–smooth fibred space $\zeta : \boldsymbol{S} \to \boldsymbol{B}$ inherit in a natural way a vector structure given, for each $k \in \mathbb{R}$ and $s,\ \acute{s} \in \boldsymbol{S}_b$, with $b \in \boldsymbol{B}$, by

$$k\, s := \widehat{k\, \breve{s}} \qquad \text{and} \qquad s + \acute{s} := \widehat{\breve{s} + \breve{\acute{s}}}\ . \square$$

Proposition 3.2.3. Let us suppose that the fibred manifold $q : \boldsymbol{G} \to \boldsymbol{F}$ be an affine bundle associated with the vector bundle $\bar{q} : \bar{\boldsymbol{G}} \to \boldsymbol{F}$, and consider an *injective* F–smooth system $(\boldsymbol{S}, \zeta, \epsilon)$ of fibrewisely smooth sections $\phi : \underline{\mathrm{tub}}(\boldsymbol{F}, \boldsymbol{G})$ and the associated injective F–smooth system $(\bar{\boldsymbol{S}}, \bar{\zeta}, \bar{\epsilon})$ of fibrewisely smooth sections $\bar{\phi} : \underline{\mathrm{tub}}(\boldsymbol{F}, \bar{\boldsymbol{G}})$.

Then, the fibres of the F–smooth fibred space $\zeta : \boldsymbol{S} \to \boldsymbol{B}$ inherit in a natural way an affine structure given, for each $s \in \boldsymbol{S}_b$ and $\bar{s} \in \bar{\boldsymbol{S}}_b$ with $b \in \boldsymbol{B}$, by

$$s + \bar{s} := \widehat{\breve{s} + \breve{\bar{s}}}\ . \square$$

3.2.3 F–smooth tangent prolongation of $(\boldsymbol{S}, \zeta, \epsilon)$

Now, we consider an F–smooth system $(\boldsymbol{S}, \zeta, \epsilon)$ of fibrewisely smooth sections of a smooth double fibred manifold $\boldsymbol{G} \xrightarrow{q} \boldsymbol{F} \xrightarrow{p} \boldsymbol{B}$ (see Definition 3.2.2) and introduce the concept of "F–smooth tangent space" $\mathsf{T}\boldsymbol{S}$ of the F–smooth space of parameters $\boldsymbol{S}$, which is equipped with its family $\mathcal{C}$ of F–smooth curves $\widehat{c} : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{S}$ (see Theorem 3.2.1).

Our formal construction of $\mathsf{T}\boldsymbol{S}$ reflects the intuitive idea, by which, for each $s \in \boldsymbol{S}_b$, with $b \in \boldsymbol{B}$, a tangent vector $\mathsf{X}_s \in \mathsf{T}_s\boldsymbol{S}$ is to be an "infinitesimal variation" of the global smooth map $\epsilon_s : \boldsymbol{F}_b \to \boldsymbol{G}_b$. It is remarkable the fact that this construction involves only smooth maps between smooth manifolds, by taking into account the smooth structure of the smooth double fibred manifold $\boldsymbol{G} \xrightarrow{q} \boldsymbol{F} \xrightarrow{p} \boldsymbol{B}$.

Actually, we define a tangent vector X_s of $\boldsymbol{S}$, at $s \in \boldsymbol{S}$, as an equivalence class of F–smooth curves $\widehat{c} : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{S}$, such that the induced smooth maps between smooth manifolds $\widehat{c}^*(\epsilon) : c^*(\boldsymbol{F}) \to c^*(\boldsymbol{G})$ have a 1st order contact in s.

Then, we show that a tangent vector X_s can be naturally represented by a pair (u, Ξ_u), where $u \in T_b\boldsymbol{B}$ and $\Xi_u : (T\boldsymbol{F})_u \to (T\boldsymbol{G})_u$ is a suitable smooth section.

Moreover, we show that the F–smooth tangent space $\mathsf{T}\boldsymbol{S}$ is equipped with a natural F–smooth structure and exhibit the natural F–smooth maps

$$\tau_{\boldsymbol{S}} : \mathsf{T}\boldsymbol{S} \to \boldsymbol{S} \qquad \text{and} \qquad \mathsf{T}_1\epsilon : \mathsf{T}\boldsymbol{S} \times \boldsymbol{M} \to T\boldsymbol{N}\,.$$

Indeed, the fibres of $\tau_{\boldsymbol{S}} : \mathsf{T}\boldsymbol{S} \to \boldsymbol{S}$ are naturally equipped with a vector structure.

The procedure followed to achieve the above results is partially similar to that followed for the tangent space of the F–smooth systems of smooth maps between two smooth manifolds (see §2.2.2). However, the present context is more complex and requires an additional care. In the present case, the reason of the difficulty and of the consequent complication is due to the fact that the map $\epsilon : \boldsymbol{S} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \to \boldsymbol{G}$ acts on a fibred product $\boldsymbol{S} \underset{\boldsymbol{B}}{\times} \boldsymbol{F}$, not just on a product $\boldsymbol{S} \times \boldsymbol{F}$. Hence, a curve $\widehat{c} : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{S}$, which moves the base points of $\boldsymbol{S}$ in $\boldsymbol{B}$, moves at the same time also the base points of $\boldsymbol{F}$ (and of $\boldsymbol{G}$) in $\boldsymbol{B}$.

Let us consider a smooth double fibred manifold $\boldsymbol{G} \xrightarrow{q} \boldsymbol{F} \xrightarrow{p} \boldsymbol{B}$ and denote the smooth fibred charts of $\boldsymbol{G}$ by (x^μ, y^i, z^a).

Moreover, let us consider an F–smooth system $(\boldsymbol{S}, \zeta, \epsilon)$ of fibrewisely smooth sections $\phi \in \underline{\text{tub}}(\boldsymbol{F}, \boldsymbol{G})$, along with the set $\mathcal{C}$ of F–smooth curves $\widehat{c} : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{S}$ defined in Theorem 3.2.1.

Then, we define the tangent space of the F–smooth space $(\boldsymbol{S}, \mathcal{C})$, via equivalence classes of suitable smooth maps between smooth manifolds, in the following way.

Lemma 3.2.6. If $(c_1, c_2) : (\boldsymbol{I}_1, \boldsymbol{I}_2) \to \boldsymbol{B}$ is a pair of smooth curves and the pair $(\lambda_1, \lambda_2) \in \boldsymbol{I}_1 \times \boldsymbol{I}_2$ is an element, such that

$$dc_1(\lambda_1) = dc_2(\lambda_2) \in T\boldsymbol{B},$$

then we have

$$\big(c_1^*(\boldsymbol{F})\big)_{\lambda_1} = \big(c_2^*(\boldsymbol{F})\big)_{\lambda_2} \qquad \text{and} \qquad T\big(c_1^*(\boldsymbol{F})\big)_{\lambda_1} = T\big(c_2^*(\boldsymbol{F})\big)_{\lambda_2} . \square$$

Lemma 3.2.7. If $\widehat{c} : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{S}$ is an F–smooth curve which projects on a smooth curve $c : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{B}$, then we obtain the smooth map (see Theorem 3.2.1)

$$T\big(\widehat{c}^*(\epsilon)\big) : T\big(c^*(\boldsymbol{F})\big) \to T\boldsymbol{G} . \square$$

Then, we introduce the concept of 1st order contact for the F–smooth curves of the type $\widehat{c} : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{S}$.

Definition 3.2.3. We say that two F–smooth curves (see Theorem 3.2.1)

$$\widehat{c}_1 : \boldsymbol{I}_1 \to \boldsymbol{S} \qquad \text{and} \qquad \widehat{c}_2 : \boldsymbol{I}_2 \to \boldsymbol{S},$$

which project, respectively, on smooth curves

$$c_1 := \zeta \circ \widehat{c}_1 : \boldsymbol{I}_1 \to \boldsymbol{B} \qquad \text{and} \qquad c_2 := \zeta \circ \widehat{c}_1 : \boldsymbol{I}_2 \to \boldsymbol{B},$$

have a *1st order contact* in $(\lambda_1, \lambda_2) \in \boldsymbol{I}_1 \times \boldsymbol{I}_2$ if they fulfill the following conditions involving smooth manifolds and maps

1) $$dc_1(\lambda_1) = dc_2(\lambda_2),$$

2) $$T_{\lambda_1}\big(\widehat{c}_1^*(\epsilon)\big) = T_{\lambda_2}\big(\widehat{c}_2^*(\epsilon)\big),$$

i.e., in coordinates,

1'a) $$c_1^\mu(\lambda_1) = x^\mu(b) = c_2^\mu(\lambda_2),$$

1'b) $$(\partial_0 c_1^\mu)(\lambda_1) \equiv \Xi_0^\mu \equiv (\partial_0 c_2^\mu)(\lambda_2),$$

2'a) $$\partial_0\big(\widehat{c}_1^*(\epsilon)^a\big)_{|\lambda_1} \equiv \Xi^a_0 \equiv \partial_0\big(\widehat{c}_2^*(\epsilon)^a\big)_{|\lambda_2},$$

2'b) $$\partial_i\big(\widehat{c}_1^*(\epsilon)^a\big)_{|\lambda_1} \equiv \Xi^a_i \equiv \partial_i\big(\widehat{c}_2^*(\epsilon)^a\big)_{|\lambda_2}.$$

Clearly, the above 1st order contact yields equivalence relations $\sim$ in the sets of pairs

$$\big(c : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{B}, \;\; \lambda \in \boldsymbol{I}_{\widehat{c}}\big) \qquad \text{and} \qquad \big(\widehat{c} : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{S}, \;\; \lambda \in \boldsymbol{I}_{\widehat{c}}\big),$$

where $\widehat{c} : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{S}$ are F–smooth curves of $\boldsymbol{S}$ and $c := \zeta \circ \widehat{c} : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{B}$ are the associated smooth curves of $\boldsymbol{B}$. □

Then, we define the tangent vectors of $\boldsymbol{S}$ via 1st order equivalence classes of F–smooth curves of $\widehat{c} : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{S}$ (see Theorem 3.2.1).

Definition 3.2.4. We define a *tangent vector* of $\boldsymbol{S}$, at $s \in \boldsymbol{S}_b$, with $b \in \boldsymbol{B}$, to be an equivalence class (see Definition 3.2.3)

$$\mathsf{X}_s := \big[(\widehat{c}, \lambda)\big]_\sim$$

where $\widehat{c} : I_{\widehat{c}} \to \boldsymbol{S}$ are F–smooth curves, such that

$$\widehat{c}(\lambda) = s.$$

Then, we define:

1) the *tangent space* of $\boldsymbol{S}$, at $s \in \boldsymbol{S}_b$, with $b \in \boldsymbol{B}$, to be the set of tangent vectors of $\boldsymbol{S}$ at s

$$\mathsf{T}_s\boldsymbol{S} := \{\mathsf{X}_s\},$$

2) the *tangent space* of $\boldsymbol{S}$ to be the disjoint union

$$\mathsf{T}\boldsymbol{S} := \bigsqcup_{s \in \boldsymbol{S}} \mathsf{T}_s\boldsymbol{S}. \,\square$$

Remark 3.2.4. In order to mimic the tangent space of standard manifolds, we call the elements X_s "tangent vectors". However, so far, we do not know yet that these objects are really elements of a vector space. This fact will be proved later in Theorem 3.2.5. □

Thus, in virtue of the above Definition 3.2.3 and Definition 3.2.4, the tangent vectors $\mathsf{X}_s \in \mathsf{T}_s\boldsymbol{S}$ can be represented through suitable pairs (u, Ξ_u), consisting of a base vector $u \in T_b\boldsymbol{B}$ and a smooth map $\Xi_u : (T\boldsymbol{F})_u \to (T\boldsymbol{G})_u$, as follows.

Theorem 3.2.3. *Let $(\boldsymbol{S}, \zeta, \epsilon)$ be an F–smooth system of fibrewisely smooth sections.*

Then, in virtue of the above Definition 3.2.3 and Definition 3.2.4, every tangent vector

$$\mathsf{X}_s := [(\widehat{c}, \lambda)]_\sim \in \mathsf{T}_s \boldsymbol{S} \,,$$

can be regarded as the pair (u, Ξ_u), consisting of

a) the base vector

$$u := \big[(c, \lambda)\big]_\sim \in T_b \boldsymbol{B} \,,$$

b) the smooth map

$$\Xi_u : (T\boldsymbol{F})_u \to (T\boldsymbol{G})_u \,,$$

which is a global smooth section of the smooth fibred manifold $Tq : (T\boldsymbol{G})_u \to (T\boldsymbol{F})_u$ and an affine fibred morphisms over $s^*(\epsilon) : \boldsymbol{F}_b \to \boldsymbol{G}_b$, whose smooth fibre derivative

$$D\Xi_u : (V\boldsymbol{F})_b \to (V\boldsymbol{G})_b$$

fulfills the equality

$$D\Xi_u = T_b\big(s^*(\epsilon)\big) : T(\boldsymbol{F}_b) \to T(\boldsymbol{G}_b) \,.$$

Indeed, the following diagram commutes

$$\begin{array}{ccc}
(T\boldsymbol{F})_u & \xrightarrow{\mathrm{id}_{(T\boldsymbol{F})_u}} & (T\boldsymbol{F})_u \\
\uparrow{\scriptstyle \mathrm{id}_{T\boldsymbol{F}_u}} & & \uparrow{\scriptstyle Tq} \\
(T\boldsymbol{F})_u & \xrightarrow{\Xi_u} & (T\boldsymbol{G})_u \\
\downarrow{\scriptstyle \tau_{\boldsymbol{F}}} & & \downarrow{\scriptstyle \tau_{\boldsymbol{G}}} \\
\boldsymbol{F}_b & \xrightarrow{s^*(\epsilon)} & \boldsymbol{G}_b
\end{array} \quad ,$$

We have the coordinate expressions

$$\begin{aligned}
u &= u^\mu \, \partial_\mu \,, \\
(x^\mu, \, y^i, \, z^a) \circ \Xi_u &= \big(x^\mu(b), \, y^i, \, s^*(\epsilon)^a\big) \,, \\
(\dot{x}^\mu, \, \dot{y}^i_{|u}, \, \dot{z}^a_{|u}) \circ \Xi_u &= \big(u^\mu, \, \dot{y}^i_{|u}, \, \Xi^a\big) \,, \\
(x^\mu, \, y^i, \, z^a) \circ D\Xi_u &= \big(x^\mu(b), \, y^i, \, s^*(\epsilon)^a\big) \,, \\
(\dot{x}^\mu, \, \dot{y}^i, \, \dot{z}^a) \circ D\Xi_u &= \big(0, \, \dot{y}^i, \, (D\Xi)^a\big) \,,
\end{aligned}$$

where

$$u^\mu = (\partial_0 c^\mu)(\lambda)\,,$$
$$\Xi^a = \Xi^a_0 + \partial_i\big(s^*(\epsilon)^a\big)\,\dot y^i_{|u}\,, \qquad (D\Xi)^a = \partial_i\big(s^*(\epsilon)^a\big)\,\dot y^i_{|0}\,,$$

with

$$u^\mu \in \mathbb{R}\,, \qquad \Xi^a_0 \in \mathrm{map}(\boldsymbol{F}_b, \mathbb{R})\,.$$

Thus, in this way, we obtain a natural map

$$\mathsf{r}_s : \mathsf{X}_s \in \mathsf{T}_s\boldsymbol{S} \mapsto (u, \Xi_u)\,,$$

where $u \in T_b\boldsymbol{B}$ *and* $\Xi_u : (T\boldsymbol{F})_u \to (T\boldsymbol{G})_u$ *is a smooth map as above.*

For each $s \in \boldsymbol{S}$, *the map* r_s *turns out to be injective.* □

Remark 3.2.5. With reference to the above Theorem 3.2.3, we stress that a vector $\mathsf{X}_s := (u, \Xi_u) \in \mathsf{T}_s\boldsymbol{S}$ is characterised, in coordinates, by its components

$$u^\mu \in \mathbb{R} \qquad \text{and} \qquad \Xi^a_0 \in \mathrm{map}(\boldsymbol{F}_b, \mathbb{R})\,.$$

Moreover, the equality

$$D\Xi_u = T_b\big(s^*(\epsilon)\big) : T(\boldsymbol{F}_b) \to T(\boldsymbol{G}_b)\,.$$

shows that $D\Xi_u$ depends only on the element $s \in \boldsymbol{S}_b$. □

The set $\mathsf{T}\boldsymbol{S}$ turns out to be equipped with the natural maps

$$\tau_{\boldsymbol{S}} : \mathsf{T}\boldsymbol{S} \to \boldsymbol{S}\,, \qquad \mathsf{T}\zeta : \mathsf{T}\boldsymbol{S} \to T\boldsymbol{B}\,, \qquad \mathsf{T}\epsilon : \mathsf{T}\boldsymbol{S} \underset{T\boldsymbol{B}}{\times} T\boldsymbol{F} \to T\boldsymbol{G}\,.$$

Proposition 3.2.4. We obtain in a natural way the following maps:

1) the natural surjective map (see Theorem 3.2.3)

$$\tau_{\boldsymbol{S}} : \mathsf{T}\boldsymbol{S} \to \boldsymbol{S} : \mathsf{X}_s \mapsto s\,,$$

2) the natural surjective map

$$\mathsf{T}\zeta : \mathsf{T}\boldsymbol{S} \to T\boldsymbol{B}\,,$$

given, according to Theorem 3.2.3, by

$$\mathsf{T}\zeta : \mathsf{X}_s := (u, \Xi_u) \mapsto u\,, \qquad \text{for each} \qquad s \in \boldsymbol{S}_b\,, \quad \text{with} \quad b \in \boldsymbol{B}\,,$$

3) the natural fibred map

$$\mathsf{T}\epsilon : \mathsf{T}\boldsymbol{S} \underset{T\boldsymbol{B}}{\times} T\boldsymbol{F} \to T\boldsymbol{G}\,,$$

given, according to Theorem 3.2.3, by

$$\mathsf{T}\epsilon : (\mathsf{X}_s, Y) := \big((u, \Xi_u), Y\big) \mapsto \Xi_u(Y)\,,$$
$$\text{for each} \qquad s \in \boldsymbol{S}_b\,, \quad Y \in (T\boldsymbol{F})_u\,, \quad \text{with} \quad u \in T_b\boldsymbol{B}\,.$$

Indeed, the following natural diagrams commute

$$\begin{array}{ccc} \mathsf{T}\boldsymbol{S} & \xrightarrow{\ \mathsf{T}\zeta\ } & T\boldsymbol{B} \\ {\scriptstyle \tau_{\boldsymbol{S}}}\downarrow & & \downarrow{\scriptstyle \tau_{\boldsymbol{B}}} \\ \boldsymbol{S} & \xrightarrow{\ \zeta\ } & \boldsymbol{B} \end{array} \quad ,$$

$$\begin{array}{ccc} T\boldsymbol{F} & \xrightarrow{\ \mathrm{id}_{T\boldsymbol{F}}\ } & T\boldsymbol{F} \\ {\scriptstyle \mathrm{pro}_2}\uparrow & & \uparrow{\scriptstyle Tq} \\ \mathsf{T}\boldsymbol{S} \underset{T\boldsymbol{B}}{\times} T\boldsymbol{F} & \xrightarrow{\ \mathsf{T}\epsilon\ } & T\boldsymbol{G} \\ {\scriptstyle \tau_{\boldsymbol{S}} \times \tau_{\boldsymbol{F}}}\downarrow & & \downarrow{\scriptstyle \tau_{\boldsymbol{G}}} \\ \boldsymbol{S} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} & \xrightarrow{\ \epsilon\ } & \boldsymbol{G} \end{array} \quad . \square$$

Moreover, the set $\mathsf{T}\boldsymbol{S}$ turns out to be equipped with the natural subset

$$\mathsf{V}\boldsymbol{S} \subset \mathsf{T}\boldsymbol{S}\,.$$

Proposition 3.2.5. The elements of the subset

$$\mathsf{V}\boldsymbol{S} := (T\zeta)^{-1}(0) \subset \mathsf{T}\boldsymbol{S}$$

can be regarded as the pairs of the type

$$\underline{\mathsf{X}}_s = (0, \Xi_0) \in V_s\boldsymbol{S}\,, \qquad \text{for each} \qquad s \in \boldsymbol{S}_b\,,$$

where

$$\Xi_0 : (V\boldsymbol{F})_b \to (V\boldsymbol{G})_b\,,$$

a) is a global smooth section of the smooth fibred manifold $Tq : (V\boldsymbol{G})_b \to (V\boldsymbol{F})_b$ and an affine fibred morphisms over $s^*(\epsilon) : \boldsymbol{F}_b \to \boldsymbol{G}_b$, according to the following diagram commutative

$$\begin{array}{ccc}
(V\boldsymbol{F})_b & \xrightarrow{\mathrm{id}_{(V\boldsymbol{F})_b}} & (V\boldsymbol{F})_b \\
\Big\uparrow{\scriptstyle \mathrm{id}_{(V\boldsymbol{F})_b}} & & \Big\uparrow{\scriptstyle Tq} \\
(V\boldsymbol{F})_b & \xrightarrow{\Xi_0} & (V\boldsymbol{G})_b \\
\Big\downarrow{\scriptstyle \tau_{\boldsymbol{F}}} & & \Big\downarrow{\scriptstyle \tau_{\boldsymbol{G}}} \\
\boldsymbol{F}_b & \xrightarrow{s^*(\epsilon)} & \boldsymbol{G}_b
\end{array} \quad ,$$

b) whose smooth fibre derivative

$$D\Xi_u : (V\boldsymbol{F})_b \to (V\boldsymbol{G})_b$$

fulfills the equality

$$D\Xi_u = T_b\big(s^*(\epsilon)\big) : (V\boldsymbol{F})_b \to (V\boldsymbol{G})_b \,.$$

We have the coordinate expression

$$(x^\mu,\, y^i,\, z^a) \circ \Xi_0 = \big(x^\mu(b),\, y^i,\, s^*_b(\epsilon)^a\big) \quad \text{and} \quad (\dot{x}^\mu_{|0},\, \dot{y}^i_{|0},\, \dot{z}^a_{|0}) \circ \Xi_0 = \big(0,\, \dot{y}^i_{|0},\, \Xi^a\big),$$

where

$$\Xi^a = \Xi^a_0 + \partial_i\big(s^*(\epsilon)^a\big)\, \dot{y}^i_{|0}\,, \qquad \text{with} \qquad \Xi^a_0 \in \mathrm{map}(\boldsymbol{F}_b, \mathbb{R})\,.$$

PROOF. The Corollary follows easily from Theorem 3.2.3 and Proposition 3.2.4 item 1).

An alternative direct proof could be obtained by rephrasing the proof of Theorem 3.2.3, taking into account vertical F–smooth curves of $\boldsymbol{S}$. QED

We can achieve two important simplifications in the representation of vertical elements of $\mathsf{T}\boldsymbol{S}$.

In fact, we can represent such an element

1) as defined on $\boldsymbol{F}$, instead of $V\boldsymbol{F}$,

2) as valued in $V_{\boldsymbol{F}}\boldsymbol{G}$, instead of $V_{\boldsymbol{B}}\boldsymbol{G}$.

Corollary 3.2.1. We can equivalently regard the elements of the vertical subset

$$\mathsf{V}\boldsymbol{S} \subset \mathsf{T}\boldsymbol{S}$$

as the pairs of the type

$$\underline{\mathrm{X}}_s = (0, \underline{\Xi}_0) \in \mathrm{V}_s \boldsymbol{S}\,, \qquad \text{for each} \qquad s \in \boldsymbol{S}_b\,,$$

where

$$\underline{\Xi}_0 := \Xi_0 \circ 0_{\boldsymbol{F}} : \boldsymbol{F}_b \to (V_{\boldsymbol{F}}\boldsymbol{G})_b\,, \qquad \text{with} \qquad 0_{\boldsymbol{F}} : \boldsymbol{F}_b \to (V\boldsymbol{F})_b\,,$$

is a global section of the smooth fibred manifold $\tau_{\boldsymbol{F}} \circ Tq : (V\boldsymbol{G})_b \to \boldsymbol{F}_b$ according to the following diagram commutative

$$\begin{array}{ccc}
\boldsymbol{F}_b & \xleftarrow{\quad q \quad} & (\boldsymbol{G})_b \\
{\scriptstyle \mathrm{id}_{\boldsymbol{F}_b}} \uparrow & & \uparrow {\scriptstyle \tau_{\boldsymbol{G}}} \\
\boldsymbol{F}_b & \xrightarrow{\quad \underline{\Xi}_0 \quad} & (V_{\boldsymbol{F}}\boldsymbol{G})_b \\
{\scriptstyle 0} \downarrow & & \downarrow {\scriptstyle \cap} \\
(V\boldsymbol{F})_b & \xrightarrow{\quad \Xi_0 \quad} & (V\boldsymbol{G})_b\,.
\end{array}$$

We have the coordinate expression

$$(x^\mu, y^i, z^a) \circ \underline{\Xi}_0 = \big(x^\mu(b), y^i, s_b^*(\epsilon)^a\big) \quad \text{and} \quad (\dot{x}^\mu_{|0}, \dot{y}^i_{|0}, \dot{z}^a_{|0}) \circ \underline{\Xi}_0 = \big(0, 0, \underline{\Xi}^a\big),$$

where

$$\underline{\Xi}^a = \Xi^a_0\,, \qquad \text{with} \qquad \Xi^a_0 \in \mathrm{map}(\boldsymbol{F}_b, \mathbb{R})\,.$$

PROOF. It follows easily from the above Proposition 3.2.5. QED

Corollary 3.2.2. Let us suppose that $q : \boldsymbol{G} \to \boldsymbol{F}$ be a vector bundle and consider the natural fibred isomorphism over $\boldsymbol{F}$

$$V_{\boldsymbol{F}}\boldsymbol{G} \simeq \boldsymbol{G} \underset{\boldsymbol{F}}{\times} \boldsymbol{G}\,.$$

Then, the elemets

$$\underline{\mathrm{X}}_s = (0, \underline{\Xi}_0) \in \mathrm{V}_s \boldsymbol{S}\,, \qquad \text{for each} \qquad s \in \boldsymbol{S}_b\,,$$

are characterised by the global smooth sections

$$\underline{\Xi} : \boldsymbol{F}_b \to \boldsymbol{G}_b\,,$$

according to the following commutative diagram

$$\begin{array}{ccc} \boldsymbol{F}_b & \xrightarrow{\Xi_0} & (V_{\boldsymbol{F}}\boldsymbol{G})_b \\ {\scriptstyle \mathrm{id}_{\boldsymbol{F}_b}}\downarrow & & \downarrow{\scriptstyle \mathrm{pro}_2} \\ \boldsymbol{F}_b & \xrightarrow{\Xi} & \boldsymbol{G}_b \end{array} \quad .$$

Hence, if the F–smooth system $(\boldsymbol{S}, \zeta, \epsilon)$ is injective, then we obtain a natural fibred isomorphism over $\boldsymbol{B}$

$$\mathsf{V}\boldsymbol{S} \simeq \boldsymbol{S} \underset{\boldsymbol{B}}{\times} \boldsymbol{S} \,.$$

PROOF. The 1st claim follows directly from the above Corollary 3.2.1.

Further, the splitting of $V\boldsymbol{S}$ follows from the fact that the smooth sections $\Xi : \boldsymbol{F}_b \to \boldsymbol{G}_b$ are just the selected sections of the system $(\boldsymbol{S}, \zeta, \epsilon)$. QED

The set $\mathsf{T}\boldsymbol{S}$ and the associated maps

$$\tau_{\boldsymbol{S}} : \mathsf{T}\boldsymbol{S} \to \boldsymbol{S} \,, \qquad \mathsf{T}\zeta : \mathsf{T}\boldsymbol{S} \to T\boldsymbol{B} \qquad \mathsf{T}\epsilon : \mathsf{T}\boldsymbol{S} \underset{T\boldsymbol{B}}{\times} T\boldsymbol{F} \to T\boldsymbol{G}$$

turn out to be F–smooth in a natural way, according to the following Theorem.

Theorem 3.2.4. *Let us consider the set* $\mathsf{T}\mathcal{C}$ *consisting of all curves*

$$d\widehat{c} : \boldsymbol{I}_{\widehat{c}} \to \mathsf{T}\boldsymbol{S} \,,$$

given, for each $\widehat{c} \in \mathcal{C}$, *according to Theorem 3.2.3, by*

$$d\widehat{c} : \lambda \mapsto \mathsf{X}_{\widehat{c}(\lambda)} \,,$$

Indeed, the set $T\mathcal{C}$ *equips the set* $\mathsf{T}\boldsymbol{S}$ *with an F–smooth structure.*
Moreover, the maps

$$\tau_{\boldsymbol{S}} : \mathsf{T}\boldsymbol{S} \to \boldsymbol{S} \,, \qquad T\zeta : \mathsf{T}\boldsymbol{S} \to T\boldsymbol{B} \qquad T\epsilon : \mathsf{T}\boldsymbol{S} \underset{T\boldsymbol{B}}{\times} T\boldsymbol{F} \to T\boldsymbol{G}$$

turn out to be F–smooth.

Thus, the 3–plet $(\mathsf{T}\boldsymbol{S}, \mathsf{T}\zeta, \mathsf{T}\epsilon)$ *turns out to be an F–smooth system of fibrewisely sections of the smooth double fibred manifold*

$$T\boldsymbol{G} \xrightarrow{Tq} T\boldsymbol{F} \xrightarrow{Tp} T\boldsymbol{B} \quad . \square$$

The fibred set $\tau_{\boldsymbol{S}} : \mathsf{T}\boldsymbol{S} \to \boldsymbol{S}$ inherits a vector structure in a natural way.

Actually, the proof of this result is not straightforward because we have to achieve invariant linear algebraic operations on bundles which are not vector bundles.

Theorem 3.2.5. *The fibres of the F–smooth fibred set* $\tau_{\boldsymbol{S}} : \mathsf{T}\boldsymbol{S} \to \boldsymbol{S}$ *inherit a natural vector structure given, for each* $s \in \boldsymbol{S}_b$, *with* $b \in \boldsymbol{B}$, *by means of the equalities*

$$r\,(u, \Xi_u) = (r\,u, r \,\bar{\cdot}\, \Xi_u) \quad \text{and} \quad (u, \Xi_u) \,\bar{+}\, (\acute{u}, \acute{\Xi}_u) := \big((u + \acute{u}), (\Xi_u \,\bar{+}\, \acute{\Xi}_u)\big),$$

where the smooth maps

$$r \,\bar{\cdot}\, \Xi_u : (T\boldsymbol{F})_{r\,u} \to (T\boldsymbol{G})_{r\,u} \quad \text{and} \quad \Xi_u \,\bar{+}\, \acute{\Xi}_{\acute{u}} : (T\boldsymbol{F})_{u+\acute{u}} \to (T\boldsymbol{G})_{u+\acute{u}}$$

are defined through the equivariant coordinate equalities

$$(r \,\bar{\cdot}\, \Xi)^\mu := r\,\Xi^\mu\,, \quad (r \,\bar{\cdot}\, \Xi)^i := r\,\Xi^i\,, \quad (r \,\bar{\cdot}\, \Xi)^a_0 := r\,\Xi^a_0\,,$$
$$(\Xi \,\bar{+}\, \acute{\Xi})^\mu := \Xi^\mu + \acute{\Xi}^\mu\,, \quad (\Xi \,\bar{+}\, \acute{\Xi})^i := \Xi^i + \acute{\Xi}^i\,, \quad (\Xi \,\bar{+}\, \acute{\Xi})^a_0 := \Xi^a_0 + \acute{\Xi}^a_0\,,$$

with

$$(r \,\bar{\cdot}\, \Xi)^i := \dot{y}^i_{|ru} \quad \text{and} \quad (\Xi \,\bar{+}\, \acute{\Xi})^i := \dot{y}^i_{u+\acute{u}}\,.$$

PROOF. Let us consider the smooth maps

$$X : (T\boldsymbol{F})_u \to (T\boldsymbol{G})_u \qquad \text{and} \qquad Y : (T\boldsymbol{F})_u \to (T\boldsymbol{G})_u\,,$$

with coordinate expressions

$$X^\mu = u^\mu\,, \qquad X^i = \dot{y}^i_{|u}\,, \qquad X^a = X^a_0 + \phi^a_i\,\dot{y}^i_{|u}\,,$$
$$Y^\mu = v^\mu\,, \qquad Y^i = \dot{y}^i_{|v}\,, \qquad Y^a = Y^a_0 + \phi^a_i\,\dot{y}^i_{|v}\,,$$

and

$$X^a_0,\, Y^a_0,\, \phi^a_i \in \mathrm{map}(\boldsymbol{F}, \mathbb{R})\,.$$

Then, we consider their local smooth extensions to the total tangent space $T\boldsymbol{F}$, which are induced by the chosen smooth fibred chart (x^λ, y^i, z^a),

$$\widetilde{X} : T\boldsymbol{F} \to T\boldsymbol{G} \qquad \text{and} \qquad \widetilde{Y} : T\boldsymbol{F} \to T\boldsymbol{G}\,,$$

with coordinate expressions

$$\widetilde{X}^\mu = \dot{x}^\mu\,, \qquad \widetilde{X}^i = \dot{y}^i\,, \qquad \widetilde{X}^a = X^a_0 + \phi^a_i\,\dot{y}^i\,,$$
$$\widetilde{Y}^\mu = \dot{x}^\mu\,, \qquad \widetilde{Y}^i = \dot{y}^i\,, \qquad \widetilde{Y}^a = Y^a_0 + \phi^a_i\,\dot{y}^i\,.$$

Next, let us take into account the vector structure of the smooth bundle $T\boldsymbol{G} \to \boldsymbol{G}$, by which we can define the algebraic operations

$$r\,\widetilde{X} : T\boldsymbol{F} \to T\boldsymbol{G} \qquad \text{and} \qquad \widetilde{X} + \widetilde{Y} : T\boldsymbol{F} \to T\boldsymbol{G}\,,$$

with coordinate expressions

$$r\,\widetilde{X}^\mu = r\,\dot{x}^\mu\,, \quad r\,\widetilde{X}^i = r\,\dot{y}^i\,, \quad r\,\widetilde{X}^a = r\,X^a_0 + \phi^a_i\,r\,\dot{y}^i\,,$$
$$(\widetilde{X} + \widetilde{Y})^\mu = \dot{x}^\mu + \dot{x}^\mu\,, \quad (\widetilde{X} + \widetilde{Y})^i = \dot{y}^i + \dot{y}^i\,,$$
$$(\widetilde{X} + \widetilde{Y})^a = X^a_0 + Y^a_0 + \phi^a_i\,(\dot{y}^i + \dot{y}^i)\,.$$

Moreover, by restricting the above maps to the zero section $0 \subset T\boldsymbol{F}$, we obtain the smooth maps

$$(r\,\widetilde{X})(0) : \boldsymbol{F} \to T\boldsymbol{G} \qquad \text{and} \qquad (\widetilde{X} + \widetilde{Y})(0) : \boldsymbol{F} \to T\boldsymbol{G}\,,$$

with coordinate expressions

$$r\,\widetilde{X}^\mu(0) = 0\,, \qquad r\,\widetilde{X}^i(0) = 0\,, \qquad r\,\widetilde{X}^a(0) = r\,X^a_0\,,$$
$$(\widetilde{X} + \widetilde{Y})^\mu(0) = 0\,, \qquad (\widetilde{X} + \widetilde{Y})^i(0) = 0\,, \qquad (\widetilde{X} + \widetilde{Y})^a(0) = X^a_0 + Y^a_0\,.$$

Furthermore, by taking into account the affine structure of the smooth bundle $Tq : T\boldsymbol{G} \to T\boldsymbol{F}$, we observe that the maps $\widetilde{X}$ and $\widetilde{Y}$ are affine and their derivatives are the maps

$$D\widetilde{X} = D\widetilde{Y} = T\epsilon_s : T\boldsymbol{F} \to V_{\boldsymbol{F}}\boldsymbol{G}\,,$$

with coordinate expression

$$(D\widetilde{X})^\mu = (D\widetilde{Y})^\mu = \dot{x}^\mu\,, \quad (D\widetilde{X})^i = (D\widetilde{Y})^i = \dot{y}^i\,, \quad (D\widetilde{X})^a = (D\widetilde{Y})^a = \phi^a_i\,\dot{y}^i\,.$$

Then, the following smooth maps are well defined

$$r \mathbin{\bar{\cdot}} \widetilde{X} : T\boldsymbol{F} \to T\boldsymbol{G} : v \mapsto (r\,\widetilde{X})(0) + D\widetilde{X}(v)\,,$$
$$\widetilde{X} \mathbin{\bar{+}} \widetilde{Y} : T\boldsymbol{F} \to T\boldsymbol{G} : v \mapsto (\widetilde{X} + \widetilde{Y})(0) + D\widetilde{X}(v)$$

and have coordinate expressions

$$r \mathbin{\bar{\cdot}} \widetilde{X}^\mu = \dot{x}^\mu\,, \qquad r \mathbin{\bar{\cdot}} \widetilde{X}^i = \dot{y}^i\,, \qquad r \mathbin{\bar{\cdot}} \widetilde{X}^a = r\,X^a_0 + \phi^a_i\,\dot{y}^i\,,$$
$$(\widetilde{X} \mathbin{\bar{+}} \widetilde{Y})^\mu = \dot{x}^\mu\,, \qquad (\widetilde{X} \mathbin{\bar{+}} \widetilde{Y})^i = \dot{y}^i\,, \qquad (\widetilde{X} \mathbin{\bar{+}} \widetilde{Y})^a = X^a_0 + Y^a_0 + \phi^a_i\,\dot{y}^i\,.$$

Further, we consider the smooth restrictions of the above smooth maps

$$r \mathbin{\bar{\cdot}} \widetilde{X} : T\boldsymbol{F} \to T\boldsymbol{G} \qquad \text{and} \qquad \widetilde{X} \mathbin{\bar{+}} \widetilde{Y} : T\boldsymbol{F} \to T\boldsymbol{G}$$

to the smooth subbundles $(T\boldsymbol{F})_{ru}$ and $(T\boldsymbol{F})_{u+v}$, respectively,

$$(r \mathbin{\bar{\cdot}} \widetilde{X})_{ru} : (T\boldsymbol{F})_{ru} \to T\boldsymbol{G} \qquad \text{and} \qquad (\widetilde{X} \mathbin{\bar{+}} \widetilde{Y})_{u+v} : (T\boldsymbol{F})_{u+v} \to T\boldsymbol{G}\,.$$

We have the coordinate expressions

$$(r \,\bar{\cdot}\, \widetilde{X})^{\mu}_{ru} = r\, u^{\mu}\,, \quad (r \,\bar{\cdot}\, \widetilde{X})^{i}_{ru} = \dot{y}^{i}\,, \quad (r \,\bar{\cdot}\, \widetilde{X})^{a}_{ru} = r\, X^{a}_{0} + \phi^{a}_{i}\, \dot{y}^{i}\,,$$
$$(\widetilde{X} \,\bar{+}\, \widetilde{Y})^{\mu}_{u+v} = r\, u^{\mu}\,, \quad (\widetilde{X} \,\bar{+}\, \widetilde{Y})^{i}_{u+v} = \dot{y}^{i}\,, \quad (\widetilde{X} \,\bar{+}\, \widetilde{Y})^{a}_{u+v} = X^{a}_{0} + Y^{a}_{0} + \phi^{a}_{i}\, \dot{y}^{i}\,.$$

The above coordinate expressions show that the maps

$$(r \,\bar{\cdot}\, \widetilde{X})_{ru} : (T\boldsymbol{F})_{ru} \to T\boldsymbol{G} \quad \text{and} \quad (\widetilde{X} \,\bar{+}\, \widetilde{Y})_{u+v} : (T\boldsymbol{F})_{u+v} \to T\boldsymbol{G}$$

factorise through maps (denoted by the same symbols)

$$(r \,\bar{\cdot}\, \widetilde{X})_{ru} : (T\boldsymbol{F})_{ru} \to (T\boldsymbol{G})_{ru} \quad \text{and} \quad (\widetilde{X} \,\bar{+}\, \widetilde{Y})_{u+v} : (T\boldsymbol{F})_{u+v} \to (T\boldsymbol{G})_{u+v}\,,$$

according to the following commutative diagrams

$$\begin{array}{ccc} (T\boldsymbol{F})_{ru} & \xrightarrow{(r \,\bar{\cdot}\, \widetilde{X})_{ru}} & (T\boldsymbol{G})_{ru} \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \subset} \\ (T\boldsymbol{F})_{ru} & \xrightarrow{(r \,\bar{\cdot}\, \widetilde{X})_{ru}} & T\boldsymbol{G} \end{array} \qquad \begin{array}{ccc} (T\boldsymbol{F})_{u+v} & \xrightarrow{(\widetilde{X} \,\bar{+}\, \widetilde{Y})_{u+v}} & (T\boldsymbol{G})_{u+v} \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \subset} \\ (T\boldsymbol{F})_{u+v} & \xrightarrow{(\widetilde{X} \,\bar{+}\, \widetilde{Y})_{u+v}} & T\boldsymbol{G} \end{array}\,.$$

Moreover, the above coordinate expressions show that the maps

$$(r \,\bar{\cdot}\, \widetilde{X})_{ru} : (T\boldsymbol{F})_{ru} \to (T\boldsymbol{G})_{ru} \quad \text{and} \quad (\widetilde{X} \,\bar{+}\, \widetilde{Y})_{u+v} : (T\boldsymbol{F})_{u+v} \to (T\boldsymbol{G})_{u+v}$$

do not depend on the extensions $\widetilde{X}$ and $\widetilde{Y}$ induced by the chart, but depend only by the original maps X and Y.

For this reason, we can write

$$\begin{aligned} (r \,\bar{\cdot}\, X)_{ru} &:= (r \,\bar{\cdot}\, \widetilde{X})_{ru} &&: (T\boldsymbol{F})_{ru} \to (T\boldsymbol{G})_{ru}\,, \\ (X \,\bar{+}\, Y)_{u+v} &:= (\widetilde{X} \,\bar{+}\, \widetilde{Y})_{u+v} &&: (T\boldsymbol{F})_{u+v} \to (T\boldsymbol{G})_{u+v} \end{aligned}$$

Hence, the above definition of the maps $(r \,\bar{\cdot}\, X)_{ru}$ and $(X \,\bar{+}\, Y)_{u+v}$ is coordinate free. QED

Remark 3.2.6. The fact that the algebraic operations defined in the above Theorem 3.2.5 be coordinate free can be confirmed by the following explicit check.

Let $(\acute{x}^{\mu}, \acute{y}^{i}, \acute{z}^{a})$ be another fibred chart of $\boldsymbol{G}$.

Then, we have the following transition formulas

$$\acute{X}^{i} = \acute{\partial}_{\mu} y^{i}\, u^{\mu} + \acute{\partial}_{j} y^{i}\, X^{j}\,, \quad \acute{X}^{a} = \acute{\partial}_{\mu} z^{a}\, u^{\mu} + \acute{\partial}_{j} z^{a}\, X^{j} + \acute{\partial}_{b} z^{a}\, (X^{b}_{0} + \phi^{b}_{j}\, \dot{y}^{j}_{|u})\,,$$
$$\acute{Y}^{i} = \acute{\partial}_{\mu} y^{i}\, v^{\mu} + \acute{\partial}_{j} y^{i}\, Y^{j}\,, \quad \acute{Y}^{a} = \acute{\partial}_{\mu} z^{a}\, v^{\mu} + \acute{\partial}_{j} z^{a}\, Y^{j} + \acute{\partial}_{b} z^{a}\, (Y^{b}_{0} + \phi^{b}_{j}\, \dot{y}^{j}_{|v})\,,$$

which yield also the following equalities

$$\acute{X}^i = \acute{\partial}_j y^i \, X^j + \acute{\partial}_\mu y^i \, u^\mu \,, \qquad \acute{X}^a_0 = \acute{\partial}_b z^a \, X^b_0 + \acute{\partial}_\mu z^a \, u^\mu \,,$$
$$\acute{Y}^i = \acute{\partial}_j y^i \, Y^j + \acute{\partial}_\mu y^i \, v^\mu \,, \qquad \acute{Y}^a_0 = \acute{\partial}_b z^a \, Y^b_0 + \acute{\partial}_\mu z^a \, v^\mu \,.$$

Hence, we obtain

$$r \, \acute{X}^i = \acute{\partial}_\mu y^i \, (r \, u^\mu) + \acute{\partial}_j y^i \, (r \, X^j) \,,$$
$$r \, \acute{X}^a_0 = \acute{\partial}_b z^a \, (r \, X^b_0) + \acute{\partial}_\mu z^a \, (r \, u^\mu) \,,$$

and

$$\acute{X}^i + \acute{Y}^i = \acute{\partial}_j y^i \, (X^j + Y^j) + \acute{\partial}_\mu y^i \, (u^\mu + v^\mu) \,,$$
$$\acute{X}^a_0 + \acute{Y}^a_0 = \acute{\partial}_b z^a \, (X^b_0 + Y^b_0) + \acute{\partial}_\mu z^a \, (u^\mu + v^\mu) \,.$$

Moreover, we have the following transition formulas

$$\dot{\acute{y}}^i_{|u} = \acute{\partial}_\mu y^i \, u^\mu + \acute{\partial}_j \acute{y}^i \, \dot{y}^j_{|u} \quad \text{and} \quad \dot{\acute{y}}^i_{|v} = \acute{\partial}_\mu y^i \, v^\mu + \acute{\partial}_j \acute{y}^i \, \dot{y}^j_{|v}$$

and

$$\dot{\acute{y}}^i_{|ru} = \acute{\partial}_\mu y^i \, (r \, u^\mu) + \acute{\partial}_j \acute{y}^i \, \dot{y}^j_{|ru} \quad \text{and} \quad \dot{\acute{y}}^i_{|u+v} = \acute{\partial}_\mu y^i \, (u^\mu + v^\mu) + \acute{\partial}_j \acute{y}^i \, \dot{y}^j_{|u+v} \,.$$

Hence, if in one chart we have

$$(r \, X)^i := \dot{y}^i_{|ru} \quad \text{and} \quad (X + Y)^i := \dot{y}^i_{|u+v} \,,$$

then analogous formulas hold in the other chart. □

Corollary 3.2.3. The fibred subset over $\boldsymbol{S}$

$$\mathsf{V}\boldsymbol{S} \subset \mathsf{T}\boldsymbol{S}$$

turns out to be a vector fibred subset. □

Corollary 3.2.4. The fibres of the F–smooth fibred set $\mathsf{T}\zeta : \mathsf{T}\boldsymbol{S} \to T\boldsymbol{B}$ inherit a natural affine structure, whose associated vector spaces are the fibres of $V\boldsymbol{S}$. □

We can define the F–smooth tangent prolongation $T\sigma : T\boldsymbol{B} \to \mathsf{T}\boldsymbol{S}$ of F–smooth sections $\sigma \in$ F–sec$(\boldsymbol{B}, \boldsymbol{S})$ analogously to the case of smooth systems of sections (see §3).

Definition 3.2.5. We define the tangent prolongation of an F–smooth section

$$\sigma \in \text{F–sec}(\boldsymbol{B}, \boldsymbol{S})$$

to be the tubelike F–smooth section

$$\mathsf{T}\sigma : T\boldsymbol{B} \to \mathsf{T}\boldsymbol{S}\,,$$

given, according to Theorem 3.2.3, by

$$\mathsf{T}\sigma : u \in T_b\boldsymbol{B} \mapsto \mathsf{T}_u\sigma := \mathsf{X}_{\sigma(b)} := (u, \Xi_u) \in \mathsf{T}_u\boldsymbol{S}\,,$$

where

$$\Xi_u = T_u\big(\sigma^*(b)(\epsilon)\big) : (T\boldsymbol{F})_u \to (T\boldsymbol{G})_u\,. \square$$

Proposition 3.2.6. For each F–smooth section $\sigma \in \text{F–sec}(\boldsymbol{B}, \boldsymbol{S})$, the following diagram commutes

$$\begin{array}{ccc}
\boldsymbol{B} & \xrightarrow{\ \sigma\ } & \boldsymbol{S} \\
\uparrow{\scriptstyle \tau_{\boldsymbol{B}}} & & \uparrow{\scriptstyle \tau_{\boldsymbol{S}}} \\
T\boldsymbol{B} & \xrightarrow{\ \mathsf{T}\sigma\ } & \mathsf{T}\boldsymbol{S} \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle T\zeta} \\
T\boldsymbol{B} & \xrightarrow{\ \mathrm{id}_{\boldsymbol{B}}\ } & T\boldsymbol{B}
\end{array} \quad . \square$$

Note 3.2.1. By a certain mild abuse of language, we can write the equality

$$\mathsf{T}\sigma = (\mathsf{T}\sigma)^*(\mathsf{T}\epsilon)\,. \square$$

3.2.4 F–smooth differential operators

In view of a discussion on F–smooth connections of an F–smooth system of fibrewisely smooth sections (see the forthcoming Section 5.1), we analyse the smooth operators.

Given two injective F–smooth systems $(\boldsymbol{S}, \zeta, \epsilon)$ and $(\acute{\boldsymbol{S}}, \acute{\zeta}, \acute{\epsilon})$ of fibrewisely smooth sections of the smooth double fibred manifolds $\boldsymbol{G} \xrightarrow{q} \boldsymbol{F} \xrightarrow{p} \boldsymbol{B}$ and $\acute{\boldsymbol{G}} \xrightarrow{\acute{q}} \boldsymbol{F} \xrightarrow{p} \boldsymbol{B}$ a sheaf morphism

$$\mathcal{D} : \text{tub}(\boldsymbol{F}, \boldsymbol{G}) \to \text{tub}(\boldsymbol{F}, \acute{\boldsymbol{G}})\,,$$

which is compatible with the above F–smooth systems, yields in a natural way a sheaf morphism

$$\widehat{\mathcal{D}} : \text{F–sec}(\boldsymbol{B}, \boldsymbol{S}) \to \text{F–sec}(\boldsymbol{B}, \acute{\boldsymbol{S}})\,.$$

Thus, let us consider two smooth double fibred manifolds

$$\boldsymbol{G} \xrightarrow{q} \boldsymbol{F} \xrightarrow{p} \boldsymbol{B} \qquad \text{and} \qquad \boldsymbol{G}' \xrightarrow{\acute{q}} \boldsymbol{F} \xrightarrow{p} \boldsymbol{B}\,.$$

and denote the smooth fibred charts of $\boldsymbol{G}$ and $\acute{\boldsymbol{G}}$, respectively, by

$$(x^\mu, y^i, z^a) \qquad \text{and} \qquad (x^\mu, y^i, \acute{z}^a)\,.$$

Moreover, let us consider two *injective* F–smooth systems of fibrewisely smooth sections of the two smooth fibred manifolds above

$$(\boldsymbol{S}, \zeta, \epsilon) \qquad \text{and} \qquad (\acute{\boldsymbol{S}}, \acute{\zeta}, \acute{\epsilon})\,.$$

Definition 3.2.6. A sheaf morphism

$$\mathcal{D} : \mathrm{tub}(\boldsymbol{F}, \boldsymbol{G}) \to \mathrm{tub}(\boldsymbol{F}, \acute{\boldsymbol{G}})$$

is said to be *compatible* with the F–smooth systems of smooth sections $(\boldsymbol{S}, \zeta, \epsilon)$ and $(\acute{\boldsymbol{S}}, \acute{\zeta}, \acute{\epsilon})$ if it restricts to a sheaf morphism

$$\mathcal{D}_{\boldsymbol{S}} : \mathrm{tub}_{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{G}) \to \mathrm{tub}_{\acute{\boldsymbol{S}}}(\boldsymbol{F}, \acute{\boldsymbol{G}})\,,$$

according to the following commutative diagram

$$\begin{array}{ccc} \mathrm{tub}(\boldsymbol{F}, \boldsymbol{G}) & \xrightarrow{\quad \mathcal{D} \quad} & \mathrm{tub}(\boldsymbol{F}, \acute{\boldsymbol{G}}) \\ \uparrow & & \uparrow \\ \mathrm{tub}_{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{G}) & \xrightarrow{\quad \mathcal{D}_{\boldsymbol{S}} \quad} & \mathrm{tub}_{\acute{\boldsymbol{S}}}(\boldsymbol{F}, \acute{\boldsymbol{G}}) \end{array} \quad .\,\square$$

Proposition 3.2.7. Let us consider a sheaf morphism of compatible smooth operators

$$\mathcal{D} : \mathrm{tub}(\boldsymbol{F}, \boldsymbol{G}) \to \mathrm{tub}(\boldsymbol{F}, \acute{\boldsymbol{G}})\,.$$

Then, in virtue of Theorem 3.2.2, we obtain the sheaf morphism

$$\widehat{\mathcal{D}} : \text{F–sec}(\boldsymbol{B}, \boldsymbol{S}) \to \text{F–sec}(\boldsymbol{B}, \acute{\boldsymbol{S}}) : \widehat{\sigma} \mapsto \widehat{\mathcal{D}(\sigma)}\ .\,\square$$

3.3 F–smooth vs smooth systems of sections

In the above Section 3.2.3, we have discussed the F–smooth tangent space $\mathsf{T}\boldsymbol{S}$ of each F–smooth system $\boldsymbol{S}$ of fibrewisely smooth sections $\phi : \boldsymbol{F} \to \boldsymbol{G}$ of a smooth double fibred manifold $\boldsymbol{G} \xrightarrow{q} \boldsymbol{F} \xrightarrow{p} \boldsymbol{B}$ (see Definition 3.2.4) and studied its main properties (see Theorem 3.2.3 and Theorem 3.2.4).

However, if the set $\boldsymbol{S}$ is assumed a priory to be a finite dimensional smooth manifold and ϵ to be a smooth map, then we can achieve the tangent space $T\boldsymbol{S}$ directly in terms of the standard differential geometry of smooth manifolds.

Thus, the need of a comparison between the F–smooth approach to $\mathsf{T}\boldsymbol{S}$ and the smooth approach to $T\boldsymbol{S}$ arises naturally, having in mind Theorem 1.3.1.

Actually, by regarding a smooth system $(\boldsymbol{S}, \zeta, \epsilon)$ as a particular F–smooth system, we obtain a natural map

$$\imath : T\boldsymbol{S} \to \mathsf{T}\boldsymbol{S} : X \mapsto \mathsf{X}$$

and can prove that the representation of X in terms of the smooth map

$$\Xi_u := \mathsf{r}(\mathsf{X}_u) : (T\boldsymbol{F})_u \to (T\boldsymbol{G})_u$$

turns out to be given by the equality $\Xi_u = T_1 \epsilon_{X_u}$.

We leave to the reader the task to develop in detail the above considerations, by rephrasing to the present context of F–smooth systems of fibrewisely smooth sections the considerations that we have discussed in Section 2.3, which is devoted to such a comparison in the context of systems of maps.

Here, in order to clarify the “odd terms” appearing in the representation of F–smooth tangent space of an F–smooth system of fibrewisely smooth sections, we just discuss the above items in terms of standard smooth manifolds in the particular case of a finite dimensional smooth system of linear sections.

Example 3.3.1. Let us consider two vector bundles

$$p : \boldsymbol{F} \to \boldsymbol{B} \qquad \text{and} \qquad p \circ q : \boldsymbol{G} \to \boldsymbol{B}$$

and the system $(\boldsymbol{S}, \zeta, \epsilon)$ of linear sections $\breve{\sigma} \in \operatorname{tub}(\boldsymbol{F}, \boldsymbol{G})$, which has been discussed in Example 3.1.1, in the framework of smooth systems of smooth sections.

In this case, $\boldsymbol{S}$ has been assumed to be a priori a smooth manifold. Hence, we can avail of this smooth structure to introduce and analyse the tangent space $T\boldsymbol{S}$.

The smooth double fibred chart (x^λ, y^i, z^a) of $\boldsymbol{G}$ induces naturally the smooth fibred charts (x^λ, w^a_i) of $\boldsymbol{S}$ and $(x^\lambda, w^a_i; \dot{x}^\lambda, \dot{w}^a_i)$ of $T\boldsymbol{S}$.

Hence, we obtain the following coordinate expressions of the smooth maps ζ and ϵ

$$(x^\lambda) \circ \zeta = (x^\lambda) \qquad \text{and} \qquad (x^\lambda, y^i, z^a) \circ \epsilon = (x^\lambda, y^i, w^a_j\, y^j)\,.$$

The coordinate expressions of the smooth maps

$$T\zeta : T\boldsymbol{S} \to T\boldsymbol{B} \qquad \text{and} \qquad T\epsilon : T\boldsymbol{S} \underset{T\boldsymbol{B}}{\times} T\boldsymbol{F} \to T\boldsymbol{G}$$

are

$$(x^\lambda, \dot{x}^\lambda) \circ T\zeta = (x^\lambda, \dot{x}^\lambda)\,,$$
$$(x^\lambda, y^i, z^a; \dot{x}^\lambda, \dot{y}^i, \dot{z}^a) \circ T\epsilon = (x^\lambda, y^i, w^a_j\, y^j; \dot{x}^\lambda, \dot{y}^i, \dot{w}^a_j\, y^j + w^a_j\, \dot{y}^j)\,.$$

Thus, the smooth map $T\epsilon$ yields a natural representation of tangent vectors of $\boldsymbol{S}$ via the smooth map, which, for each $X_u \in (T_s\boldsymbol{S})_u$, where $s \in \boldsymbol{S}_b$, $u \in T_b\boldsymbol{B}$, $b \in \boldsymbol{B}$, yields the smooth section

$$\Xi_u := (T\epsilon)_{X_u} : (T\boldsymbol{F})_u \to (T\boldsymbol{G})_u\,,$$

with coordinate expression

$$\Xi^\lambda = X^\lambda = u^\lambda\,, \qquad \Xi^i = \dot{y}^i_{|u}\,, \qquad \Xi^a = X^a_j\, y^j_{|b} + w^a_j\, \dot{y}^j_{|u} = \Xi^a_0 + \partial_j \epsilon^a\, \dot{y}^j_{|u}\,,$$

where

$$X^\lambda \in \mathbb{R}\,, \qquad \Xi^a_0 := X^a_j\, y^j_{|b}\,.$$

Moreover, given $r \in \mathbb{R}$ and $X_u \in (T_s\boldsymbol{S})_u$, $\acute{X}_v \in (T_s\boldsymbol{S})_v$, the natural vector structure of the smooth vector bundle $\tau_{\boldsymbol{S}} : T\boldsymbol{S} \to \boldsymbol{S}$ yields the standard coordinate expressions

$$(r\, X_u)^\lambda = r\, u^\lambda\,, \qquad (r\, X_u)^a_i = r\, X^a_i\,,$$
$$(X_u + \acute{X}_v)^\lambda = u^\lambda + v^\lambda\,, \qquad (X_u + \acute{X}_v)^a_i = X^a_i + \acute{X}^a_i\,,$$

and the zero vector $X_0 := 0 \in T_s\boldsymbol{S}$ has coordinate expression

$$X^\lambda = 0\,, \qquad X^a_i = 0\,.$$

However, the representation of above formulas in terms of fibred morphisms turns out to read as follows

$$(r\,\Xi_u)^\lambda = r\,\Xi^\lambda\,, \qquad (r\,\Xi_u)^i = r\,\Xi^i\,,$$
$$(r\,\Xi_u)^a_i = r\,\Xi^a_j\,y^j_{|b} + w^a_j\,\dot{y}^j_{|ru}\,,$$
$$(\Xi_u + \acute{\Xi}_v)^\lambda = \Xi^\lambda + \acute{\Xi}^\lambda\,, \qquad (\Xi_u + \acute{\Xi}_v)^i = \Xi^i + \acute{\Xi}^i\,,$$
$$(\Xi_u + \acute{\Xi}_v)^a_i = (\Xi^a_j + \acute{\Xi}^a_j)\,y^j_{|b} + w^a_j\,\dot{y}^j_{|u+v}\,,$$

and the representation of the zero vector turns out to be

$$\Xi^\lambda = 0\,, \qquad \Xi^i = \dot{y}^i_{|0}\,, \qquad \Xi^a_0 = w^a_j\,\dot{y}^j_{|0}\,.$$

Thus, in agreement with the general results found for F–smooth systems of sections (see Theorem 3.2.3 and Theorem 3.2.5), the "odd" term of the type $w^a_j\,\dot{y}^j_{|u}$ appears in the representation of all vectors X_u and the zero vector is not represented by a vanishing map but by the map $w^a_j\,\dot{y}^j_{|0}$. □

Chapter 4

Systems of connections

We start by discussing the *smooth systems* $(\boldsymbol{C}, \zeta, \epsilon)$ of *smooth connections*

$$c : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F}$$

of a smooth fibred manifold $\boldsymbol{F} \to \boldsymbol{B}$.

Indeed, such smooth systems can be regarded as be a particular case of smooth systems of smooth sections of the smooth double fibred manifold $T^*\boldsymbol{B} \otimes T\boldsymbol{F} \to \boldsymbol{F} \to \boldsymbol{B}$.

Here, the "space of parameters" $\zeta : \boldsymbol{C} \to \boldsymbol{B}$ is a *smooth space* and the "evaluation map" $\epsilon : \boldsymbol{C} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F}$ a *smooth fibred morphism* over $\boldsymbol{B}$.

Moreover, we discuss the *smooth universal connection*

$$c^{\uparrow} : \boldsymbol{F}^{\uparrow} \to T^*\boldsymbol{C} \otimes T\boldsymbol{F}^{\uparrow}$$

of a smooth system of smooth connections.

Then, we discuss the *F–smooth systems* $(\boldsymbol{C}, \zeta, \epsilon)$ of *fibrewisely smooth connections*

$$c : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F}$$

of a smooth double fibred manifold $\boldsymbol{G} \to \boldsymbol{F} \to \boldsymbol{B}$.

Indeed, such F–smooth systems of connections can be regarded as be a particular case of F–smooth systems of fibrewisely smooth sections of the smooth double fibred manifold $T^*\boldsymbol{B} \otimes T\boldsymbol{F} \to \boldsymbol{F} \to \boldsymbol{B}$.

Here, the "space of parameters" $\zeta : \boldsymbol{C} \to \boldsymbol{B}$ is an *F–smooth space* and the "evaluation map" $\epsilon : \boldsymbol{C} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \to \boldsymbol{G}$ an *F–smooth fibred morphism* over $\boldsymbol{B}$.

The reader can find further discussions concerning the present subject in [2, 3, 4, 5, 7, 14, 15, 16, 17, 19, 20, 25].

4.1 Smooth systems of smooth connections

We discuss the *smooth systems of smooth connections*

$$c : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F}$$

of a smooth fibred manifold $\boldsymbol{F} \to \boldsymbol{B}$.

Moreover, we discuss the *smooth universal connection*

$$c^{\uparrow} : \boldsymbol{F}^{\uparrow} \to T^*\boldsymbol{C} \otimes T\boldsymbol{F}^{\uparrow}$$

of a smooth system of smooth connections and show that $c^{\uparrow}$ characterises the system.

4.1.1 Smooth systems of smooth connections

Given a smooth fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$, a "*smooth system of smooth connections*" is just a smooth system of smooth sections (see Definition 3.1.3) of the smooth double fibred manifold

$$\boldsymbol{G} := T^*\boldsymbol{B} \otimes T\boldsymbol{F} \to \boldsymbol{F} \to \boldsymbol{B}$$

consisting of a selected family of smooth connections

$$c : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F}$$

of the smooth fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$.

Therefore, all developments discussed in the previous Section 3.1.1 can be applied to the present Section.

Let us consider a smooth fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ and denote its fibred charts by (x^λ, y^i).

Then, we define the fibred manifold

$$q : T^*\boldsymbol{B} \otimes T\boldsymbol{F} \to \boldsymbol{F}$$

and obtain the smooth double fibred manifold

$$T^*\boldsymbol{B} \otimes T\boldsymbol{F} \xrightarrow{\ q\ } \boldsymbol{F} \xrightarrow{\ p\ } \boldsymbol{B}\,.$$

In the present context, it is convenient to deal with the definition of "smooth connection" of the smooth fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ as a smooth tangent valued form

$$c : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F}\,,$$

which makes the following diagram commutative

$$\begin{array}{ccc} \boldsymbol{F} & \xrightarrow{\;c\;} & T^*\boldsymbol{B} \otimes T\boldsymbol{F} \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}_{T^*\boldsymbol{B}} \otimes Tp} \\ \boldsymbol{B} & \xrightarrow{\;\mathbf{1}_{\boldsymbol{B}}\;} & T^*\boldsymbol{B} \otimes T\boldsymbol{B} \end{array} \quad .$$

Then, the coordinate expression of c is of the type

$$c = d^\lambda \otimes (\partial_\lambda + c_\lambda{}^i \, \partial_i) \,, \qquad \text{with} \qquad c_\lambda{}^i \in \mathrm{map}(\boldsymbol{F}, \mathbb{R}) \,.$$

We denote the subsheaf of smooth tubelike connections of the fibred manifold $\boldsymbol{F} \to \boldsymbol{B}$ by (see Definition 3.1.2)

$$\mathrm{cns}\,\mathrm{tub}(\boldsymbol{F}, T^*\boldsymbol{B} \otimes T\boldsymbol{F}) \subset \mathrm{tub}(\boldsymbol{F}, T^*\boldsymbol{B} \otimes T\boldsymbol{F}) \,.$$

Definition 4.1.1. A *smooth system of smooth connections* is defined to be a 3–plet $(\boldsymbol{C}, \zeta, \epsilon)$ where

1) $\zeta : \boldsymbol{C} \to \boldsymbol{B}$ is a smooth fibred manifold,

2) $\epsilon : \boldsymbol{C} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F}$ is a smooth fibred morphism over

$$\mathrm{id}_{\boldsymbol{F}} : \boldsymbol{F} \to \boldsymbol{F} \qquad \text{and} \qquad \mathbf{1}_{\boldsymbol{B}} : \boldsymbol{B} \to T^*\boldsymbol{B} \otimes T\boldsymbol{B} \,,$$

according to the following commutative diagrams

$$\begin{array}{ccc} \boldsymbol{C} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} & \xrightarrow{\;\epsilon\;} & T^*\boldsymbol{B} \otimes T\boldsymbol{F} \\ {\scriptstyle \mathrm{pro}_2}\big\downarrow & & \big\downarrow{\scriptstyle \tau_{\boldsymbol{F}}} \\ \boldsymbol{F} & \xrightarrow{\;\mathrm{id}_{\boldsymbol{F}}\;} & \boldsymbol{F} \end{array} \qquad \begin{array}{ccc} \boldsymbol{C} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} & \xrightarrow{\;\epsilon\;} & T^*\boldsymbol{B} \otimes T\boldsymbol{F} \\ \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}_{T^*\boldsymbol{B}} \otimes Tp} \\ \boldsymbol{B} & \xrightarrow{\;\mathbf{1}_{\boldsymbol{B}}\;} & T^*\boldsymbol{B} \otimes T\boldsymbol{B} \end{array} \quad .$$

We call ϵ the *evaluation map* of the system.

Thus, the evaluation map ϵ yields the sheaf morphism

$$\epsilon_{\boldsymbol{C}} : \sec(\boldsymbol{B}, \boldsymbol{C}) \to \mathrm{cns}\,\mathrm{tub}(\boldsymbol{F}, T^*\boldsymbol{B} \otimes T\boldsymbol{F}) : \gamma \mapsto c := \breve{\gamma} \,,$$

where, for each $\gamma \in \sec(\boldsymbol{B}, \boldsymbol{C})$, the tubelike smooth connection $c := \breve{\gamma}$ is defined by

$$\breve{\gamma} : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes \boldsymbol{F} : f_b \mapsto \epsilon\big(\gamma(b), f_b\big) \,, \qquad \text{for each} \qquad b \in \boldsymbol{B} \,.$$

Therefore, the map $\epsilon_{\boldsymbol{C}} : \mathrm{sec}(\boldsymbol{B}, \boldsymbol{C}) \to \mathrm{tub}(\boldsymbol{F}, T^*\boldsymbol{B} \otimes T\boldsymbol{F})$ provides a *selection* of the tubelike smooth connections $c : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F}$, given by the subset

$$\mathrm{tub}_{\boldsymbol{C}}(\boldsymbol{F}, T^*\boldsymbol{B} \otimes T\boldsymbol{F}) := \epsilon_{\boldsymbol{C}}\big(\mathrm{sec}(\boldsymbol{B}, \boldsymbol{C})\big) \subset \mathrm{cns\,tub}(\boldsymbol{F}, T^*\boldsymbol{B} \otimes T\boldsymbol{F}) \,.$$

We have the coordinate expressions

$$\epsilon = d^\lambda \otimes (\partial_\lambda + \epsilon_\lambda{}^i \, \partial_i) \,, \qquad \text{with} \qquad \epsilon_\lambda{}^i \in \mathrm{map}(\boldsymbol{F}^\uparrow, \mathbb{R}) \,,$$
$$\breve{\gamma} := \epsilon^*(\gamma) = d^\lambda \otimes \big(\partial_\lambda + (\epsilon_\lambda{}^i \circ \gamma_{\boldsymbol{F}}) \, \partial_i\big) \,,$$

where (see Definition 3.1.4)

$$\boldsymbol{F}^\uparrow := \boldsymbol{C} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \to \boldsymbol{F} \qquad \text{and} \qquad \gamma_{\boldsymbol{F}} : \boldsymbol{F} \to \boldsymbol{C} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} : f_b \mapsto \big(\gamma(b), f_b\big) \,.$$

The smooth system of smooth connections $(\boldsymbol{C}, \zeta, \epsilon)$ is said to be *injective* if the map $\epsilon_{\boldsymbol{C}} : \mathrm{sec}(\boldsymbol{B}, \boldsymbol{C}) \to \mathrm{cns\,tub}(\boldsymbol{F}, T^*\boldsymbol{B} \otimes T\boldsymbol{F})$ is injective, i.e. if, for each $\gamma \,, \acute{\gamma} \in \mathrm{sec}(\boldsymbol{B}, \boldsymbol{C})$,

$$\breve{\gamma} \, \equiv \, \epsilon^*(\gamma) = \breve{\acute{\gamma}} \, \equiv \, \epsilon^*(\acute{\gamma}) \qquad \Rightarrow \qquad \gamma = \acute{\gamma} \,.$$

If the system is injective, then we obtain the bijection

$$\epsilon_{\boldsymbol{C}} : \mathrm{sec}(\boldsymbol{B}, \boldsymbol{C}) \to \mathrm{cns\,tub}_{\boldsymbol{C}}(\boldsymbol{F}, T^*\boldsymbol{B} \otimes T\boldsymbol{F}) : \gamma \mapsto \breve{\gamma} \,,$$

whose inverse is denoted by

$$(\epsilon_{\boldsymbol{C}})^{-1} : \mathrm{cns\,tub}_{\boldsymbol{C}}(\boldsymbol{F}, T^*\boldsymbol{B} \otimes T\boldsymbol{F}) \to \mathrm{sec}(\boldsymbol{B}, \boldsymbol{C}) : c \mapsto \gamma \, \equiv \, \widehat{c} \,. \square$$

Let us examine a few distinguished examples of injective smooth systems of smooth connections.

Indeed, in the case of smooth systems of linear connections, affine connections and principal connections, our bundle $\zeta : \boldsymbol{C} \to \boldsymbol{B}$ is just the standard bundle of coefficients of such connections.

Example 4.1.1. If $p : \boldsymbol{F} \to \boldsymbol{B}$ is a vector bundle, then the linear connections constitute an injective smooth system $(\boldsymbol{C}, \zeta, \epsilon)$, where $\zeta : \boldsymbol{C} \to \boldsymbol{B}$ is an affine subbundle

$$\boldsymbol{C} \subset \mathrm{lin}_{\boldsymbol{B}}(\boldsymbol{F}, \, T^*\boldsymbol{B} \otimes T\boldsymbol{F}) \,,$$

which is associated with the vector bundle

$$\bar{\boldsymbol{C}} = \mathrm{lin}_{\boldsymbol{B}}(\boldsymbol{F}, \, T^*\boldsymbol{B} \otimes \boldsymbol{F}) \,.$$

The fibred charts induced on $\boldsymbol{C}$ are of the type $(x^\lambda, w_\lambda{}^i{}_j)$ and the coordinate expression of ϵ is

$$\epsilon = d^\lambda \otimes (\partial_\lambda + w_\lambda{}^i{}_j \, y^j \, \partial_i) \, . \square$$

Example 4.1.2. If $p : \boldsymbol{F} \to \boldsymbol{B}$ is an affine bundle, associated with the vector bundle $\bar{p} : \bar{\boldsymbol{F}} \to \boldsymbol{B}$, then the affine connections constitute an injective smooth system $(\boldsymbol{C}, \zeta, \epsilon)$, where $\zeta : \boldsymbol{C} \to \boldsymbol{B}$ is an affine subbundle

$$\boldsymbol{C} \subset \mathrm{aff}_{\boldsymbol{B}}(\boldsymbol{F}, \, T^*\boldsymbol{B} \otimes T\boldsymbol{F}) \, ,$$

which is associated with the vector bundle

$$\bar{\boldsymbol{C}} = \mathrm{aff}_{\boldsymbol{B}}(\boldsymbol{F}, \, T^*\boldsymbol{B} \otimes \bar{\boldsymbol{F}}) \, .$$

The fibred charts induced on $\boldsymbol{C}$ are of the type $(x^\lambda, w_\lambda{}^i{}_j, w_\lambda{}^i)$ and the coordinate expression of ϵ is

$$\epsilon = d^\lambda \otimes \big(\partial_\lambda + (w_\lambda{}^i{}_j \, y^j + w_\lambda{}^i) \, \partial_i\big) \, . \square$$

Example 4.1.3. If $p : \boldsymbol{F} \to \boldsymbol{B}$ is an affine bundle, then analogously to the above Example 4.1.2, we can define

- the injective smooth system of polynomial connections of degree r, with $1 \leq r$,

- the injective smooth system of polynomial connections of any degree r, with $1 \leq r \leq k$, where k is a given positive integer. $\square$

All examples above deal with finite dimensional smooth systems of smooth connections, as it is implicitly requested in Definition 4.1.1.

However, we can easily extend the concept of smooth system of smooth connections, by considering an infinite dimensional system, which is the direct limit of finite dimensional smooth systems, according to the following Example 4.1.4.

Example 4.1.4. If $p : \boldsymbol{F} \to \boldsymbol{B}$ is an affine bundle, then we obtain the smooth system of all polynomial connections by considering the family of all polynomial connections $c : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F}$ of any degree r, with $1 \leq r < \infty$.

However, we stress that such a system has a natural infinite dimensional smooth structure. $\square$

Example 4.1.5. If $p : \boldsymbol{F} \to \boldsymbol{B}$ is a smooth left principal bundle, with structure group G, then the smooth principal connections constitute an injective smooth system $(\boldsymbol{C}, \zeta, \epsilon)$, where $\zeta : \boldsymbol{C} \to \boldsymbol{B}$ is the quotient bundle with respect to the group of smooth fibred left actions over $\boldsymbol{B}$

$$\mathrm{id} \times TL_h : T^*\boldsymbol{B} \otimes T\boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F}\,, \qquad \text{where} \qquad h \in G\,.\square$$

Exercise 4.1.1. Let us consider a smooth manifold $\boldsymbol{M}$ and the trivial smooth principal bundle $\boldsymbol{F} := \boldsymbol{M} \times \mathbb{R} \to \boldsymbol{B} := \boldsymbol{M}$ whose structure group is the abelian group $\mathbb{R}$.

Show that the system of smooth principal connections of this bundle can be naturally identified with the family of 1–forms $\alpha : \boldsymbol{M} \to T^*\boldsymbol{M}\,.\square$

4.1.2 Smooth universal connection

Eventually, given a smooth fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$, we discuss the notions of *reducible smooth connection* and *universal smooth connection*. Moreover, we discuss the natural bijection between smooth systems of smooth connections and reducible smooth connections.

Namely, a smooth system of smooth connections $(\boldsymbol{C}, \zeta, \epsilon)$ yields a distinguished smooth connection

$$\epsilon^{\uparrow} : \boldsymbol{F}^{\uparrow} \underset{\boldsymbol{C}}{\times} T\boldsymbol{C} \to T\boldsymbol{F}^{\uparrow}\,,$$

called *universal*, on the pullback smooth fibred manifold

$$p^{\uparrow} : \boldsymbol{F}^{\uparrow} := \boldsymbol{C} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \to \boldsymbol{C}\,.$$

Indeed, this universal connection fulfills a universal property; in fact, all connections of the systems can be obtained from the universal connection by pullback.

This notion was originally introduced by P.L. Garcia [7] in the context of principal connections of a principal bundle. Later, this theory has been generalised to any fibred manifold, detached from any structure group of symmetries (see, for instance, [2, 3, 19]). Here, we follow this generalised approach.

Let us consider two smooth fibred manifolds

$$p : \boldsymbol{F} \to \boldsymbol{B} \qquad \text{and} \qquad \zeta : \boldsymbol{C} \to \boldsymbol{B}\,.$$

Then, we focus our attention on the fibred manifold (see Definition 3.1.4)

$$p^{\uparrow} : \boldsymbol{F}^{\uparrow} := \boldsymbol{C} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \to \boldsymbol{C}\,.$$

The fibred charts (x^λ, y^i) and (x^λ, w^A) of $\boldsymbol{F}$ and $\boldsymbol{C}$, respectively, yield the fibred chart $(x^\lambda, w^A; y^i)$ of $\boldsymbol{F}^\uparrow$.

We recall the equality

$$T\boldsymbol{F}^\uparrow = T\boldsymbol{C} \underset{T\boldsymbol{B}}{\times} T\boldsymbol{F}\,.$$

We recall that the smooth connections of the smooth fibred manifolds

$$p : \boldsymbol{F} \to \boldsymbol{B} \qquad \text{and} \qquad p^\uparrow : \boldsymbol{F}^\uparrow \to \boldsymbol{C}$$

can be regarded equivalently

1) as smooth fibred morphisms

$$c : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F} \qquad \text{and} \qquad c^\uparrow : \boldsymbol{F}^\uparrow \to T^*\boldsymbol{C} \otimes T\boldsymbol{F}^\uparrow$$

whose coordinate expressions are

$$c = d^\lambda \otimes (\partial_\lambda + c_\lambda{}^i\, \partial_i)\,,$$
$$c^\uparrow = d^\lambda \otimes (\partial_\lambda + c^\uparrow{}_\lambda{}^i\, \partial_i) + d^A \otimes (\partial_A + c^\uparrow{}_A{}^i\, \partial_i)\,,$$

2) as smooth fibred morphisms

$$c : \boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} \to T\boldsymbol{F} \qquad \text{and} \qquad c^\uparrow : \boldsymbol{F}^\uparrow \underset{\boldsymbol{C}}{\times} T\boldsymbol{C} \to T\boldsymbol{F}^\uparrow\,,$$

whose coordinate expressions are

$$(x^\lambda,\, y^i;\, \dot{x}^\lambda,\, \dot{y}^i) \circ c = (x^\lambda,\, y^i;\, \dot{x}^\lambda,\, c_\lambda{}^i\, \dot{x}^\lambda)\,,$$
$$(x^\lambda,\, w^A,\, y^i;\, \dot{x}^\lambda,\, \dot{w}^A,\, \dot{y}^i) \circ c^\uparrow = (x^\lambda,\, w^A,\, y^i;\, \dot{x}^\lambda,\, \dot{w}^A,\, c^\uparrow{}_A{}^i\, \dot{w}^A + c^\uparrow{}_\lambda{}^i\, \dot{x}^\lambda)\,,$$

where

$$c_\lambda{}^i \in \mathrm{map}(\boldsymbol{F}, \mathbb{R}) \qquad \text{and} \qquad c^\uparrow{}_A{}^i,\; c^\uparrow{}_\lambda{}^i \in \mathrm{map}(\boldsymbol{F}^\uparrow, \mathbb{R})\,.$$

Definition 4.1.2. A smooth connection

$$c^\uparrow : \boldsymbol{F}^\uparrow \underset{\boldsymbol{C}}{\times} T\boldsymbol{C} \to T\boldsymbol{F}^\uparrow$$

of the smooth fibred manifold $p^\uparrow : \boldsymbol{F}^\uparrow \to \boldsymbol{C}$ is said to be an *upper connection* of the smooth system of smooth connections.

Moreover, such a smooth connection $c^\uparrow$ is said to be *reducible* if it factorises through a smooth system $(\boldsymbol{C}, \zeta, \epsilon)$ of smooth connections of the smooth fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ according to the following commutative diagram (see Definition 4.1.1)

$$\begin{array}{ccc} \boldsymbol{F}^{\uparrow} \underset{\boldsymbol{C}}{\times} T\boldsymbol{C} & \xrightarrow{c^{\uparrow}} & T\boldsymbol{F}^{\uparrow} \\ \downarrow & & \downarrow \\ \boldsymbol{C} \underset{\boldsymbol{B}}{\times} (\boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B}) & \xrightarrow{\epsilon} & T\boldsymbol{F} \end{array} \quad . \square$$

The above intrinsic condition can be translated in coordinates as follows.

Proposition 4.1.1. A smooth connection $c^{\uparrow} : \boldsymbol{F}^{\uparrow} \underset{\boldsymbol{C}}{\times} T\boldsymbol{C} \to T\boldsymbol{F}^{\uparrow}$ is reducible if and only if, in coordinates,

$$c^{\uparrow}{}_A{}^i = 0 \,.$$

Thus, smooth a connection $c^{\uparrow} : \boldsymbol{F}^{\uparrow} \underset{\boldsymbol{C}}{\times} T\boldsymbol{C} \to T\boldsymbol{F}^{\uparrow}$ is reducible if and only if its coordinate expression is of the type

$$c^{\uparrow} = d^{\lambda} \otimes (\partial_{\lambda} + c^{\uparrow}{}_{\lambda}{}^i \, \partial_i) + d^A \otimes \partial_A \,.$$

PROOF. The coordinate expression of a connection $c^{\uparrow} : \boldsymbol{F}^{\uparrow} \underset{\boldsymbol{C}}{\times} T\boldsymbol{C} \to T\boldsymbol{F}^{\uparrow}$ is of the type

$$c^{\uparrow} = d^{\lambda} \otimes (\partial_{\lambda} + c^{\uparrow}{}_{\lambda}{}^i \, \partial_i) + d^A \otimes (\partial_A + c^{\uparrow}{}_A{}^i \, \partial_i) \,, \quad \text{where} \quad c^{\uparrow}{}_{\lambda}{}^i, \; c^{\uparrow}{}_A{}^i \in \operatorname{map}(\boldsymbol{F}^{\uparrow}, \mathbb{R}) \,.$$

Hence, the coordinate expression of $\mathrm{pro}_2 \circ c^{\uparrow} : \boldsymbol{F}^{\uparrow} \underset{\boldsymbol{C}}{\times} T\boldsymbol{C} \to T\boldsymbol{F}$ is

$$c^{\uparrow} = d^{\lambda} \otimes (\partial_{\lambda} + c^{\uparrow}{}_{\lambda}{}^i \, \partial_i) + d^A \otimes (c^{\uparrow}{}_A{}^i \, \partial_i) \,.$$

Therefore, the composition of smooth maps

$$\boldsymbol{F}^{\uparrow} \underset{\boldsymbol{C}}{\times} T\boldsymbol{C} \xrightarrow{c^{\uparrow}} T\boldsymbol{F}^{\uparrow} \xrightarrow{\mathrm{pro}_2} T\boldsymbol{F}$$

factorises through a smooth fibred morphism

$$\epsilon : \boldsymbol{C} \underset{\boldsymbol{B}}{\times} (\boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B}) \to T\boldsymbol{F}$$

over $\boldsymbol{F}$ if and only if

$$c^{\uparrow}{}_A{}^i = 0 \,.$$

Indeed, the smooth fibred morphism

$$\epsilon : \boldsymbol{C} \underset{\boldsymbol{B}}{\times} (\boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B}) \to T\boldsymbol{F} \,,$$

i.e. equivalently, the smooth fibred morphism

$$\epsilon : \boldsymbol{C} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F}\,,$$

turns out to be a smooth system of smooth connections. QED

Remark 4.1.1. Let us consider a generic smooth connection of a generic smooth fibred manifold; if some symbols of the connection vanish in a chart, they need not to vanish in another chart.

However, for each reducible connection $c^{\uparrow} : \boldsymbol{F}^{\uparrow} \underset{\boldsymbol{C}}{\times} T\boldsymbol{C} \to T\boldsymbol{F}^{\uparrow}$ of the fibred manifold $p^{\uparrow} : \boldsymbol{F}^{\uparrow} \to \boldsymbol{C}$, we have shown the following equality, in any fibred chart,

$${c^{\uparrow}}_A{}^i = 0\,.$$

Indeed, this unusual vanishing property of some symbols of the connection in any fibred chart is possible because $\boldsymbol{F}^{\uparrow} := \boldsymbol{C} \underset{\boldsymbol{B}}{\times} \boldsymbol{F}$ is a fibred product of manifolds.

Accordingly, if (x^{λ}, w^A, y^i) and $(\acute{x}^{\mu}, \acute{w}^B, \acute{y}^j)$ are two fibred charts of $\boldsymbol{F}^{\uparrow}$, then we have

$$\partial_A \acute{y}^j = 0 \qquad \text{and} \qquad \partial_i \acute{w}^B = 0\,.\square$$

We can exhibit a natural bijection between smooth systems of smooth connections $(\boldsymbol{C}, \zeta, \epsilon)$ of the smooth fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ and reducible smooth connections of the smooth fibred manifold $p^{\uparrow} : \boldsymbol{F}^{\uparrow} \to \boldsymbol{C}$, according to the following Proposition 4.1.2.

Even more, the reducible smooth connections fulfill a "universal property" with respect to the smooth connections of the associated smooth system of smooth connections, according to the following Theorem 4.1.1.

Proposition 4.1.2. We have a natural bijection between smooth systems of smooth connections of the smooth fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ and reducible smooth connections of the smooth fibred manifold $p^{\uparrow} : \boldsymbol{F}^{\uparrow} \to \boldsymbol{C}$ in the following way.

1) If $\epsilon : \boldsymbol{C} \underset{\boldsymbol{B}}{\times} (\boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B}) \to T\boldsymbol{F}$ is a smooth system of smooth connections of the smooth fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$, then the smooth map

$$\epsilon^{\uparrow} : \boldsymbol{F}^{\uparrow} \underset{\boldsymbol{C}}{\times} T\boldsymbol{C} \to T\boldsymbol{F}^{\uparrow} = T\boldsymbol{C} \times T\boldsymbol{F} : (f^{\uparrow}, X) \mapsto \Big(X\,,\, \big(\epsilon(f^{\uparrow})\big)\big((T\zeta)(X)\big)\Big)\,,$$

with coordinate expression

$$\epsilon^{\uparrow} = d^{\lambda} \otimes (\partial_{\lambda} + \epsilon_{\lambda}{}^i\, \partial_i) + d^A \otimes \partial_A\,,$$

can be regarded as a reducible smooth connection of the smooth fibred manifold $p^{\uparrow} : \boldsymbol{F}^{\uparrow} \to \boldsymbol{C}$.

2) If $\epsilon^{\uparrow} : \boldsymbol{F}^{\uparrow} \underset{\boldsymbol{C}}{\times} T\boldsymbol{C} \to T\boldsymbol{F}^{\uparrow}$ is a reducible smooth connection of the smooth fibred manifold $p^{\uparrow} : \boldsymbol{F}^{\uparrow} \to \boldsymbol{C}$, then the factor map (see Definition 4.1.2)

$$\epsilon : \boldsymbol{F}^{\uparrow} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} \to T\boldsymbol{F} ,$$

with coordinate expression

$$\epsilon = d^{\lambda} \otimes (\partial_{\lambda} + \epsilon_{\lambda}{}^{i} \, \partial_i) ,$$

turns out to be a smooth system of connections of the smooth fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$.

3) The above coordinate expressions exhibit a natural bijection

$$\epsilon \mapsto \epsilon^{\uparrow}$$

between the smooth systems of smooth connections of the smooth fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ and the reducible smooth connections of the smooth fibred manifold $p^{\uparrow} : \boldsymbol{F}^{\uparrow} \to \boldsymbol{C}$. □

Theorem 4.1.1. *Let* $\epsilon : \boldsymbol{C} \underset{\boldsymbol{B}}{\times} (\boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B}) \to T\boldsymbol{F}$ *be a smooth system of smooth connections of the smooth fibred manifold* $p : \boldsymbol{F} \to \boldsymbol{B}$.

Then, the following facts hold.

1) The smooth connection

$$\epsilon^{\uparrow} : \boldsymbol{F}^{\uparrow} \underset{\boldsymbol{C}}{\times} T\boldsymbol{C} \to T\boldsymbol{F}^{\uparrow}$$

of the smooth fibred manifold $p^{\uparrow} : \boldsymbol{F}^{\uparrow} \to \boldsymbol{C}$ *turns out to be the* "universal connection" *of the above system, i.e. it fulfills the following* "universal property"*:*

- all smooth connections $\breve{\gamma}$ *of the system can be obtained as pullback of* $\epsilon^{\uparrow}$*, through the equality*

$$\breve{\gamma} = \gamma^{*}(\epsilon^{\uparrow}) , \qquad \text{for each} \qquad \gamma \in \sec(\boldsymbol{B}, \boldsymbol{C}) ,$$

where

$$\gamma^{*}(\epsilon^{\uparrow}) \in \mathrm{fib}(\boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B}, T\boldsymbol{F})$$

is the smooth connection of the smooth fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ *defined by the following commutative diagram*

$$
\begin{array}{ccccc}
\boldsymbol{F}^{\uparrow} \underset{\boldsymbol{C}}{\times} T\boldsymbol{C} & & \xrightarrow{\quad \epsilon^{\uparrow} \quad} & & T\boldsymbol{F}^{\uparrow} \\
\downarrow & & & & \downarrow \\
\boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} & \xrightarrow{\gamma_{\boldsymbol{C}} \times \mathrm{id}} & \boldsymbol{F}^{\uparrow} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} & \xrightarrow{\quad \epsilon \quad} & T\boldsymbol{F} \\
\mathrm{id} \uparrow & & & & \uparrow \mathrm{id} \\
\boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} & & \xrightarrow{\quad \gamma^{*}(\epsilon)^{\uparrow} \quad} & & T\boldsymbol{F} \quad .
\end{array}
$$

2) The smooth curvature tensor (see, for instance, [15, 23])

$$R[\epsilon^{\uparrow}] : \boldsymbol{F}^{\uparrow} \underset{\boldsymbol{C}}{\times} \Lambda^2 T\boldsymbol{C} \to V_{\boldsymbol{C}} \boldsymbol{F}^{\uparrow}$$

fulfills the following "universal property"*:*

- the curvature ensors $R[\breve{\gamma}]$ *of all connections* $\breve{\gamma}$ *of the smooth system can be obtained as pullback of* $R[\epsilon^{\uparrow}]$ *through the equality*

$$R[\breve{\gamma}] = \gamma^{*}\big(R[\epsilon^{\uparrow}]\big)\,, \qquad \text{for each} \qquad \gamma \in \sec(\boldsymbol{B}, \boldsymbol{C})\,,$$

where

$$\gamma^{*}\big(R[\epsilon^{\uparrow}]\big) \in \mathrm{fib}(\boldsymbol{F} \underset{\boldsymbol{B}}{\times} \Lambda^2 T\boldsymbol{B}, V\boldsymbol{F})$$

is defined by the following commutative diagram

$$
\begin{array}{ccc}
\boldsymbol{F}^{\uparrow} \underset{\boldsymbol{C}}{\times} \Lambda^2 T\boldsymbol{C} & \xrightarrow{\quad R[\epsilon^{\uparrow}] \quad} & V_{\boldsymbol{C}} \boldsymbol{F}^{\uparrow} \\
\gamma_{\boldsymbol{C}} \times T\gamma \uparrow & & \downarrow \\
\boldsymbol{F} \underset{\boldsymbol{B}}{\times} \Lambda^2 T\boldsymbol{B} & \xrightarrow{\quad \gamma^{*}(R[\epsilon^{\uparrow}]) \quad} & V\boldsymbol{F} \quad .
\end{array}
$$

Proof. The coordinate expressions of $\epsilon^{\uparrow}$ and $\breve{\gamma}$ are

$$\epsilon^{\uparrow} = d^{\lambda} \otimes (\partial_{\lambda} + \epsilon_{\lambda}{}^{i}\, \partial_i) + d^{A} \otimes \partial_A\,,$$
$$\breve{\gamma} = d^{\lambda} \otimes \big(\partial_{\lambda} + (\epsilon_{\lambda}{}^{i} \circ \gamma_{\lambda}{}^{i})\, \partial_i\big)\,.$$

Hence, the universal property of $\epsilon^{\uparrow}$ follows from the equality

$$
\begin{aligned}
\gamma^{*}\epsilon^{\uparrow} &= \gamma^{*}\big(d^{\lambda} \otimes (\partial_{\lambda} + \epsilon_{\lambda}{}^{i}\, \partial_i) + d^{A} \otimes \partial_A\big) \\
&= d^{\lambda} \otimes \big(\partial_{\lambda} + (\epsilon_{\lambda}{}^{i} \circ \gamma)\, \partial_i\big) \\
&= \breve{\gamma}\,.
\end{aligned}
$$

The coordinate expressions of $R[\epsilon^{\uparrow}]$ and $R[\breve{\gamma}]$ are

$$R[\epsilon^{\uparrow}] = -2\Big((\partial_\lambda \epsilon_\mu{}^i + \epsilon_\lambda{}^j\, \partial_j \epsilon_\mu{}^i)\, d^\lambda \wedge d^\mu + \partial_A \epsilon_\mu{}^i\, d^A \wedge d^\mu\Big) \otimes \partial_i\,,$$
$$R[\breve{\gamma}] = -2\left(\partial_\lambda(\epsilon_\mu{}^i \circ \gamma) + (\epsilon_\lambda{}^j \circ \gamma)\, \partial_j(\epsilon_\mu{}^i \circ \gamma)\right) d^\lambda \wedge d^\mu \otimes \partial_i\,.$$

Hence, by taking into account the equalities

$$\partial_\lambda(\epsilon_\mu{}^i \circ \gamma) = (\partial_\lambda \epsilon_\mu{}^i) \circ \gamma + (\partial_A \epsilon_\mu{}^i) \circ \gamma\, \partial_\lambda \gamma^A \quad \text{and} \quad \partial_j(\epsilon_\mu{}^i \circ \gamma) = (\partial_j \epsilon_\mu{}^i) \circ \gamma\,,$$

the universal property of $R[\epsilon^{\uparrow}]$ follows from the equality

$$\begin{aligned}
\gamma^* R[\epsilon^{\uparrow}] &= -2\,\gamma^*\left((\partial_\lambda \epsilon_\mu{}^i + \epsilon_\lambda{}^j\, \partial_j \epsilon_\mu{}^i)\, d^\lambda \wedge d^\mu + \partial_A \epsilon_\mu{}^i\, d^A \wedge d^\mu\right) \otimes \partial_i \\
&= -2\left((\partial_\lambda \epsilon_\mu{}^i) \circ \gamma + (\epsilon_\lambda{}^j \circ \gamma)\,(\partial_j \epsilon_\mu{}^i) \circ \gamma + (\partial_A \epsilon_\mu{}^i) \circ \gamma\, \partial_\lambda \gamma^A\right) d^\lambda \wedge d^\mu \otimes \partial_i \\
&= -2\left(\partial_\lambda(\epsilon_\mu{}^i \circ \gamma) - (\partial_A \epsilon_\mu{}^i) \circ \gamma\, \partial_\lambda \gamma^A + (\epsilon_\lambda{}^j \circ \gamma)\, \partial_j(\epsilon_\mu{}^i \circ \gamma)\right. \\
&\qquad \left. + (\partial_A \epsilon_\mu{}^i) \circ \gamma\, \partial_\lambda \gamma^{\uparrow A}\right) d^\lambda \wedge d^\mu \otimes \partial_i \\
&= -2\left(\partial_\lambda(\epsilon_\mu{}^i \circ \gamma) + (\epsilon_\lambda{}^j \circ \gamma)\, \partial_j(\epsilon_\mu{}^i \circ \gamma)\right) d^\lambda \wedge d^\mu \otimes \partial_i \\
&= R[\breve{\gamma}]\,.\ \text{QED}
\end{aligned}$$

Let us examine a few distinguished examples of universal connections.

Example 4.1.6. Let us refer to the smooth system of linear connections of the vector bundle $p : \boldsymbol{F} \to \boldsymbol{B}$ (see Example 4.1.1).

Then, the associated universal connection of the system has coordinate expression

$$\epsilon^{\uparrow} = d^\lambda \otimes (\partial_\lambda + \epsilon_\lambda{}^i{}_j\, y^j\, \partial_i) + d_\lambda{}^i{}_j \otimes \partial^\lambda{}_i{}^j\,.\,\square$$

Example 4.1.7. Let us refer to the smooth system of affine connections of the affine bundle $p : \boldsymbol{F} \to \boldsymbol{B}$ (see Example 4.1.2).

Then, the associated universal connection of the system has coordinate expression

$$\epsilon^{\uparrow} = d^\lambda \otimes \left(\partial_\lambda + (\epsilon_\lambda{}^i{}_j\, y^j + \epsilon_\lambda{}^i)\, \partial_i\right) + d_\lambda{}^i{}_j \otimes \partial^\lambda{}_i{}^j + d_\lambda{}^i \otimes \partial^\lambda{}_i\,.\,\square$$

Eventually, we show that the natural Liouville form and symplectic form of a smooth manifold fulfill a well known property, that can be reinterpreted in terms of universal connection and curvature tensor of a smooth system of smooth connections.

Exercise 4.1.2. Let us refer to the smooth system of smooth principal connections of the trivial principal bundle $\boldsymbol{M} \times \mathbb{R} \to \boldsymbol{M}$ (see Exercise 4.1.1).

Then, show the following facts:

- the universal connection of the system can be naturally identified with the Liouville form $\lambda : T^*\boldsymbol{M} \to T^*T^*\boldsymbol{M}$, with coordinate expression

$$\lambda = \dot{x}_\mu \, d^\mu .$$

- the universal curvature of the system can be naturally identified with the symplectic form $\omega : T^*\boldsymbol{M} \to \Lambda^2 T^*T^*\boldsymbol{M}$, with coordinate expression

$$\omega := - \, d\lambda = d_\mu \wedge \dot{d}^\mu .$$

- the well known universal properties of the 1–form λ and of the 2–form $\omega := - \, d\lambda$ (see [8]) fit the universal properties of the universal connection and of its curvature. □

4.2 F–smooth systems of connections

We discuss the *F–smooth systems of "fibrewisely smooth connections"* of a smooth manifold (see Section 3.2.1).

The concept of *universal connection* that we have discussed for smooth systems of smooth connections can be easily extended to F–smooth systems of F–smooth connections. The reader who is interested in this subject can refer to [4].

Let us consider a smooth fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ and denote its fibred charts by (x^λ, y^i).

Then, let us consider the smooth double fibred manifold

$$T^*\boldsymbol{B} \otimes T\boldsymbol{F} \xrightarrow{\;q\;} \boldsymbol{F} \xrightarrow{\;p\;} \boldsymbol{B} \quad .$$

Definition 4.2.1. We denote by (see Definition 3.1.2)

$$\operatorname{cns} \underline{\operatorname{tub}}(\boldsymbol{F}, T^*\boldsymbol{B} \otimes T\boldsymbol{F}) \subset \big\{ c : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F} \big\}$$

the subsheaf consisting of tubelike sections $c : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F}$, which fulfill the following condition, *without any further local smoothness requirement,*

- $c_b : \boldsymbol{F}_b \to (T^*\boldsymbol{B} \otimes T\boldsymbol{F})_b$ is global and smooth, for each $b \in \boldsymbol{B}$;

- c_b projects over $\mathbf{1}_b$ according to the following commutative diagram

$$\begin{array}{ccc} \boldsymbol{F}_b & \xrightarrow{c_b} & (T^*\boldsymbol{B} \otimes T\boldsymbol{F})_b \\ \downarrow & & \downarrow \\ \{b\} & \xrightarrow{\mathbf{1}_b} & (T^*\boldsymbol{B} \otimes T\boldsymbol{B})_b \end{array} \quad . \square$$

Thus, let us consider a tubelike connection $c : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F}$.

We say that it is

- *fibrewisely smooth* if it is smooth along the fibres $\boldsymbol{F}_b \subset \boldsymbol{F}$, for each $b \in \boldsymbol{B}$,

- *smooth* if it is smooth in its full domain $p^{-1}(\boldsymbol{U}) \subset \boldsymbol{F}$, for each $\boldsymbol{U} \in \boldsymbol{B}$.

Therefore, the sheaf of smooth tubelike connections

$$c : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F}$$

of the smooth fibred manifold $\boldsymbol{F} \to \boldsymbol{B}$ turns out to be a subsheaf of

$$\operatorname{cns}\operatorname{tub}(\boldsymbol{F}, T^*\boldsymbol{B} \otimes T\boldsymbol{F}) \subset \operatorname{cns}\underline{\operatorname{tub}}(\boldsymbol{F}, T^*\boldsymbol{B} \otimes T\boldsymbol{F}) .$$

The following Definition is a generalisation of Definition 4.1.1, as here we do not require that $\boldsymbol{C}$ be a smooth finite dimensional manifold (hence, that the maps ζ and ϵ be smooth).

Definition 4.2.2. We define an *F–smooth system of fibrewisely smooth connections* of the smooth fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ to be a 3–plet $(\boldsymbol{C}, \zeta, \epsilon)$, where

1) $\boldsymbol{C}$ is a set,

2) $\zeta : \boldsymbol{C} \to \boldsymbol{B}$ is a surjective map,

3) $\epsilon : \boldsymbol{C} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F}$ is a fibred map over $\boldsymbol{F}$ and over $\mathbf{1}_{\boldsymbol{B}} : \boldsymbol{B} \to T^*\boldsymbol{B} \otimes T\boldsymbol{B}$, according to the following commutative diagrams

$$\begin{array}{ccc} \boldsymbol{C} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} & \xrightarrow{\epsilon} & T^*\boldsymbol{B} \otimes T\boldsymbol{F} \\ {\scriptstyle \mathrm{pro}_2}\downarrow & & \downarrow{\scriptstyle \tau_{\boldsymbol{F}}} \\ \boldsymbol{F} & \xrightarrow{\mathrm{id}_{\boldsymbol{F}}} & \boldsymbol{F} \end{array} \qquad \begin{array}{ccc} \boldsymbol{C} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} & \xrightarrow{\epsilon} & T^*\boldsymbol{B} \otimes T\boldsymbol{F} \\ \downarrow & & \downarrow{\scriptstyle \mathrm{id}_{T^*\boldsymbol{B}} \otimes Tp} \\ \boldsymbol{B} & \xrightarrow{\mathbf{1}_{\boldsymbol{B}}} & T^*\boldsymbol{B} \otimes T\boldsymbol{B} \end{array} \quad ,$$

which fulfills the following condition:

*) for each $c \in \boldsymbol{C}_b$, with $b \in \boldsymbol{B}$, the induced section

$$\epsilon_c : \boldsymbol{F}_b \to (T^*\boldsymbol{B} \otimes T\boldsymbol{F})_b$$

of the restricted smooth fibred manifold $(T^*\boldsymbol{B}\otimes T\boldsymbol{F})_b \to \boldsymbol{F}_b$ is *smooth and globally defined on* $\boldsymbol{F}_b$.

The map $\epsilon : \boldsymbol{C} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F}$ is called the *evaluation map* of the system.

We denote by

$$\underline{\mathrm{sec}}(\boldsymbol{B},\boldsymbol{C}) \subset \big\{\gamma : \boldsymbol{B} \to \boldsymbol{C}\big\}$$

the subsheaf consisting of *local* sections $\gamma : \boldsymbol{B} \to \boldsymbol{C}$, *without any smoothness requirement.* □

We leave to the reader the easy task to rephrase in the present context the notions and developments that have been previously established for F–smooth systems of smooth sections.

Chapter 5

F–smooth connections

Given an F–smooth system $(\boldsymbol{S}, \zeta, \epsilon)$ of fibrewisely smooth sections of a smooth double fibred manifold $\boldsymbol{G} \xrightarrow{q} \boldsymbol{F} \xrightarrow{p} \boldsymbol{B}$, we discuss the "*F–smooth connections*"

$$\mathsf{K} : \boldsymbol{S} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} \to \mathsf{T}\boldsymbol{S}$$

of the F–smooth fibred space $\zeta : \boldsymbol{S} \to \boldsymbol{B}$ (see, for instance, [3, 4, 16]).

We mention that the curvature of an F–smooth connection K as above can be defined via the generalised Frölicher–Nijenhuis bracket on F–smooth spaces in a way analogous to the curvature of a smooth connection on a smooth fibred manifold. The reader who is interested in this subject can refer, for instance, to [16, 23].

5.1 F–smooth connections

Given an F–smooth system $(\boldsymbol{S}, \zeta, \epsilon)$ of fibrewisely smooth sections of a smooth double fibred manifold $\boldsymbol{G} \xrightarrow{q} \boldsymbol{F} \xrightarrow{p} \boldsymbol{B}$, we define the "*F–smooth connections*"

$$\mathsf{K} : \boldsymbol{S} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} \to \mathsf{T}\boldsymbol{S}$$

of the F–smooth fibred space $\zeta : \boldsymbol{S} \to \boldsymbol{B}$ and show that such a K is characterised by a smooth section of a smooth bundle of the type

$$\Xi_{(s,u)} : (T\boldsymbol{F})_u \to (T\boldsymbol{G})_u \,, \qquad \text{for each} \qquad s \in \boldsymbol{S} \,, \quad u \in T_{\zeta(s)}\boldsymbol{B} \,.$$

Thus, let us consider a smooth double fibred manifold $\boldsymbol{G} \xrightarrow{q} \boldsymbol{F} \xrightarrow{p} \boldsymbol{B}$ and denote the typical smooth fibred chart of $\boldsymbol{G}$ by (x^μ, y^i, z^a).

Moreover, let us consider an F–smooth system $(\boldsymbol{S}, \zeta, \epsilon)$ of fibrewisely smooth sections of the above smooth double fibred manifold.

Definition 5.1.1. We define an *F–smooth connection* of the F–smooth fibred space $\zeta : \boldsymbol{S} \to \boldsymbol{B}$ to be an F–smooth fibred morphism over $\boldsymbol{S}$ and over $T\boldsymbol{B}$,

$$\mathsf{K} : \boldsymbol{S} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} \to \mathsf{T}\boldsymbol{S} ,$$

which is linear with respect to the 2nd factor $T\boldsymbol{B}$, according to the following commutative diagram

$$\begin{array}{ccc}
\boldsymbol{S} & \xrightarrow{\ \mathrm{id}_{\boldsymbol{S}}\ } & \boldsymbol{S} \\
{\scriptstyle \mathrm{pro}_1}\uparrow & & \uparrow{\scriptstyle \tau_{\boldsymbol{S}}} \\
\boldsymbol{S} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} & \xrightarrow{\ \mathsf{K}\ } & \mathsf{T}\boldsymbol{S} \\
{\scriptstyle \mathrm{pro}_2}\downarrow & & \downarrow{\scriptstyle T\zeta} \\
T\boldsymbol{B} & \xrightarrow{\ \mathrm{id}_{\boldsymbol{B}}\ } & T\boldsymbol{B}
\end{array} \quad ,$$

or, equivalently, to be an F–smooth tangent valued 1–form

$$\mathsf{K} : \boldsymbol{S} \to T^*\boldsymbol{B} \otimes \mathsf{T}\boldsymbol{S} ,$$

which projects on $\mathbf{1}_{\boldsymbol{B}} : \boldsymbol{B} \to T^*\boldsymbol{B} \otimes T\boldsymbol{B}$, according to the commutative diagram

$$\begin{array}{ccc}
\boldsymbol{S} & \xrightarrow{\ \mathsf{K}\ } & T^*\boldsymbol{B} \otimes \mathsf{T}\boldsymbol{S} \\
{\scriptstyle \zeta}\downarrow & & \downarrow{\scriptstyle \mathrm{id} \otimes \mathsf{T}\zeta} \\
\boldsymbol{B} & \xrightarrow{\ \mathbf{1}_{\boldsymbol{B}}\ } & T^*\boldsymbol{B} \otimes T\boldsymbol{B}
\end{array} \quad .$$

By recalling the representation of $\mathsf{T}\boldsymbol{S}$ provided by Theorem 3.2.3, the F–smooth connection K is characterised by a map of the type

$$\mathsf{K} : \boldsymbol{S} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} \to \mathsf{T}\boldsymbol{S} : (s, u) \mapsto \Xi_{(s,u)} ,$$

where

$$\Xi_{(s,u)} : (T\boldsymbol{F})_u \to (T\boldsymbol{G})_u$$

is a smooth section (see Theorem 3.2.5). □

Let us consider an F–smooth connection $\mathsf{K} : \boldsymbol{S} \to T^*\boldsymbol{B} \otimes \mathsf{T}\boldsymbol{S}$.

Definition 5.1.2. We define the *F–smooth covariant differential* of an F–smooth section $\sigma \in \text{F–sec}(\boldsymbol{B}, \boldsymbol{S})$, with respect to the F–smooth connection K, to be the F–smooth section

$$\nabla\sigma := \mathsf{T}\sigma - \mathsf{K} \circ \sigma : \boldsymbol{B} \to T^*\boldsymbol{B} \otimes \mathsf{V}\boldsymbol{S}\,,$$

according to the commutative diagram

$$\begin{array}{ccc} \boldsymbol{S} & \xrightarrow{(\mathsf{T}\sigma,\, \mathsf{K}\circ\sigma)} & (T^*\boldsymbol{B} \otimes \mathsf{T}\boldsymbol{S}) \underset{\boldsymbol{S}}{\times} (T^*\boldsymbol{B} \otimes \mathsf{T}\boldsymbol{S}) \\ {\scriptstyle\sigma}\big\uparrow & & \big\downarrow{\scriptstyle -_{\boldsymbol{S}}} \\ \boldsymbol{B} & \xrightarrow{\quad\nabla\sigma\quad} & T^*\boldsymbol{B} \otimes \mathsf{V}\boldsymbol{S} \end{array} \quad .$$

5.2 F–smooth connections in the linear case

Next, let us further suppose that $q : \boldsymbol{G} \to \boldsymbol{F}$ be a *vector bundle* and that the system $(\boldsymbol{S}, \zeta, \epsilon)$ be injective.

Then, we show a natural bijection

$$\mathsf{K} \mapsto \mathcal{D}_{\mathsf{K}}$$

between F–smooth connections K of the F–smooth fibred space $\zeta : \boldsymbol{S} \to \boldsymbol{B}$ and certain smooth differential operators $\mathcal{D}_{\mathsf{K}}$ between finite dimensional smooth manifolds (see Definition 3.2.2)

$$\mathcal{D}_{\mathsf{K}} : \text{tub}_{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{G}) \to \text{tub}(\boldsymbol{F}, T^*\boldsymbol{B} \otimes \boldsymbol{G})\,.$$

We stress that the above smooth differential operators $\mathcal{D}_{\mathsf{K}}$, play a role analogous to the matrix of symbols (K^i_λ) of a standard smooth connection K of a standard smooth fibred manifold.

Thus, let us consider a smooth vector bundle $\boldsymbol{G} \to \boldsymbol{F}$, an injective F–smooth system $(\boldsymbol{S}, \zeta, \epsilon)$ of fibrewisely smooth sections of the smooth double fibred manifold $\boldsymbol{G} \to \boldsymbol{F} \to \boldsymbol{B}$.

We recall that, in the linear case, the F–smooth fibred space $\zeta : \boldsymbol{S} \to \boldsymbol{B}$ inherits naturally a vector structure (see Proposition 3.2.2) and that there is a natural F–smooth linear fibred isomorphism $\mathsf{V}\boldsymbol{S} \to \boldsymbol{S} \underset{\boldsymbol{B}}{\times} \boldsymbol{S}$ over $\boldsymbol{S}$ (see Corollary 3.2.2).

Note 5.2.1. We can regard the covariant differential of a section $\sigma \in \text{F–sec}(\boldsymbol{B}, \boldsymbol{S})$, with respect to an the F–smooth connection K, as an F–smooth section

$$\nabla\sigma : \boldsymbol{B} \to T^*\boldsymbol{B} \otimes \boldsymbol{S}\,.\square$$

The covariant differential ∇_{K} associated with the F–smooth connection K is a differential operator $\mathcal{D}_{\mathsf{K}}$ of a certain type. Indeed, there is a natural bijection between these objects.

This result extends to the present F–smooth framework an analogous result holding for standard smooth connections of smooth fibred manifolds.

Proposition 5.2.1. The following facts hold.

1) Let us consider an F–smooth connection K.

Then, there exists a unique F–smooth sheaf morphism

$$\mathcal{D} \equiv \mathcal{D}[\mathsf{K}] : \mathrm{tub}_{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{G}) \to \mathrm{tub}(\boldsymbol{F}, T^*\boldsymbol{B} \otimes \boldsymbol{G}) : \phi \mapsto \mathcal{D}\phi \,,$$

such that, for each $\phi \in \mathrm{tub}_{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{G})$ and $u \in T\boldsymbol{B}$,

$$\widehat{(\mathcal{D}\phi)(u)} = \nabla_u \widehat{\phi} \,.$$

The sheaf morphism $\mathcal{D}$ turns out to be a differential operator of horizontal order 1, which, for each $b \in \boldsymbol{B}$, factorises fibrewisely, through a smooth sheaf morphism

$$\check{\mathcal{D}}_b : \mathrm{sec}(\boldsymbol{F}_b, \boldsymbol{G}_b) \to \mathrm{sec}(\boldsymbol{F}_b, T^*_b\boldsymbol{B} \otimes \boldsymbol{G}_b) \,,$$

according to the following commutative diagram

$$\begin{array}{ccc} \mathrm{tub}_{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{G}) & \xrightarrow{\quad \mathcal{D} \quad} & \mathrm{tub}(\boldsymbol{F},\ T^*\boldsymbol{B} \otimes \boldsymbol{G}) \\ \downarrow & & \downarrow \\ \mathrm{sec}_{\boldsymbol{S}}(\boldsymbol{F}_b, \boldsymbol{G}_b) & \xrightarrow{\quad \check{\mathcal{D}}_b \quad} & \mathrm{sec}(\boldsymbol{F}_b,\ T^*_b\boldsymbol{B} \otimes \boldsymbol{G}_b) \end{array} \quad .$$

The coordinate expression of the sheaf morphism $\mathcal{D}$ is of the type

$$(\mathcal{D}\phi)^a_\lambda = \partial_\lambda \phi^a - \check{\mathcal{D}}^a_\lambda(\phi) \,,$$

where $\check{\mathcal{D}}^a_\lambda$ are smooth sheaf morphisms

$$\check{\mathcal{D}}^a_\lambda : \mathrm{tub}_{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{G}) \to \mathbb{R} \,,$$

which, for each $b \in \boldsymbol{B}$, factorise fibrewisely, through smooth sheaf morphisms

$$\check{\mathcal{D}}_b{}^a_\lambda : \mathrm{sec}_{\boldsymbol{S}}(\boldsymbol{F}_b, \boldsymbol{G}_b) \to \mathbb{R} \,,$$

according to the following commutative diagram

$$\begin{array}{ccc}
\mathrm{tub}_{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{G}) & \xrightarrow{\check{\mathcal{D}}^a_\lambda} & \mathbb{R} \\
\downarrow & & \downarrow \mathrm{id}_{\mathbb{R}} \\
\mathrm{sec}_{\boldsymbol{S}}(\boldsymbol{F}_b, \boldsymbol{G}_b) & \xrightarrow{\check{\mathcal{D}}_b{}^a_\lambda} & \mathbb{R}
\end{array} \quad .$$

2) Conversely, let us consider an F–smooth sheaf morphism

$$\mathcal{D} : \mathrm{tub}_{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{G}) \to \mathrm{tub}(\boldsymbol{F}, T^*\boldsymbol{B} \otimes \boldsymbol{G}) \,,$$

whose local coordinate expression is of the type

$$(\mathcal{D}\phi)^a_\lambda = \partial_\lambda \phi^a - \check{\mathcal{D}}^a_\lambda(\phi) \,,$$

as in the above item 1).

Then, there exists a unique F–smooth connection K of $\boldsymbol{S}$, such that, for each $\sigma \in \mathrm{sec}(\boldsymbol{B}, \boldsymbol{S})$, the associated sheaf morphism

$$\mathcal{D} \,\equiv\, \mathcal{D}[\mathsf{K}] : \mathrm{tub}_{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{G}) \to \mathrm{tub}(\boldsymbol{F}, T^*\boldsymbol{B} \otimes \boldsymbol{G})$$

be given by

$$\nabla[\mathsf{K}]\, \sigma = \widehat{\mathcal{D}\, \breve{\sigma}} \ .$$

Indeed, we obtain, for each $\sigma \in \mathrm{sec}(\boldsymbol{B}, \boldsymbol{S})$,

$$\mathsf{K} \circ \sigma = d\sigma - \widehat{\mathcal{D}\, \breve{\sigma}} \ . \square$$

Definition 5.2.1. Let us suppose that the fibred manifold $q : \boldsymbol{G} \to \boldsymbol{F}$ be a vector bundle. Then, the F–smooth connection K is said to be *linear* if it is a linear fibred morphism over $\boldsymbol{1}_{\boldsymbol{B}} : \boldsymbol{B} \to T^*\boldsymbol{B} \otimes T\boldsymbol{B}$, according to the following commutative diagram

$$\begin{array}{ccc}
\boldsymbol{S} & \xrightarrow{\mathsf{K}} & T^*\boldsymbol{B} \otimes \mathsf{T}\boldsymbol{S} \\
\zeta \downarrow & & \downarrow \mathrm{id}_{T^*\boldsymbol{B}} \otimes \mathsf{T}\tau_{\boldsymbol{S}} \\
\boldsymbol{B} & \xrightarrow{\boldsymbol{1}_{\boldsymbol{B}}} & T^*\boldsymbol{B} \otimes T\boldsymbol{B}
\end{array} \quad . \square$$

Chapter 6

Applications

We deal with three applications of the above general geometric theory, which are taken in the framework of Covariant Quantum Mechanics (see, for instance, [9, 10, 12, 13] and literature therein).

The 1st application deals with the *upper quantum connection* $Ч^{\uparrow}$ of the upper quantum bundle $\pi^{\uparrow} : \boldsymbol{Q}^{\uparrow} \to J_1\boldsymbol{E}$, which turns out to be the *universal connection of a system of observed quantum connections* $Ч[o]$ of the quantum bundle $\pi : \boldsymbol{Q} \to \boldsymbol{E}$.

The 2nd application deals with the *F–smooth sectional quantum bundle* $\widehat{t} : \widehat{\boldsymbol{Q}} \to \boldsymbol{T}$, whose fibres are equipped with a Hilbert structure.

The 3rd example deals with the Schrödinger operator regarded as an *F–smooth linear connection* of the above F–smooth sectional quantum bundle.

Standard Quantum Mechanics is deeply involved with Mathematical Analysis of Hilbert spaces. However, in Covariant Quantum Mechanics, we have a Hilbert space for each time and there is no covariant isomorphism between these Hilbert spaces. So, besides the standard techniques of Functional Analysis, we would need of the hard techniques of infinite dimensional Differential Geometry. Actually, the simple geometric techniques based on F–smooth spaces, that we propose here (in the above 2nd and 3rd applications), can be regarded as useful tools in order to achieve quickly some concepts and results.

6.1 Introduction to Covariant Quantum Mechanics

We start with a concise introduction to a few basic topics of our approach to Covariant Quantum Mechanics, in view of the applications discussed in the forthcoming Sections (for further details, see, for instance, [9, 10, 12, 13] and literature therein).

0) In both Classical Mechanics and Quantum Mechanics, we explicitly mention, at any step, the *"scale dimensions"* of physical objects by following a formal mathematical language discussed in the paper [13].

Accordingly, we shall refer to the following "positive spaces":

- the space of *time intervals* $\mathbb{T}$,
- the space of *lengths* $\mathbb{L}$,
- the space of *masses* $\mathbb{M}$.

In particular, each charged particle has *mass* and *charge* with the following scale dimensions

$$m \in \mathbb{M} \qquad \text{and} \qquad q \in \mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2} \otimes \mathbb{R}\,.$$

Moreover, we shall be dealing with the Planck constant $\hbar \in \mathbb{T}^{-1} \otimes \mathbb{L}^2 \otimes \mathbb{M}$.

1) *Spacetime* is represented by an oriented 4–dimensional manifold $\boldsymbol{E}$ equipped with a *time fibring* $t : \boldsymbol{E} \to \boldsymbol{T}$, where $\boldsymbol{T}$ is a 1–dimensional affine space associated with the vector space $\mathbb{T} \otimes \mathbb{R}$.

The time fibring yields the scaled *time form* $dt : \boldsymbol{E} \to \mathbb{T} \otimes T^*\boldsymbol{E}$. We denote the fibred spacetime charts adapted to the fibring by $(x^\lambda) \equiv (x^0, x^i)$ and the associated *time unit* and its *dual* by $u_0 \in \mathbb{T}$ and $u^0 \in \mathbb{T}^*$. The induced bases of vector fields and forms are denoted by $(\partial_\lambda) \equiv (\partial_0, \partial_i)$ and $(d^\lambda) \equiv (d^0, d^i)$. We have the natural *vertical subbundle* $V\boldsymbol{E} \subset T\boldsymbol{E}$ and *horizontal subbundle* $H^*\boldsymbol{E} \subset T^*\boldsymbol{E}$. Vertical restriction of forms will be denoted by the check symbol $\check{\ }$.

A *motion* is defined to be a section $s : \boldsymbol{T} \to \boldsymbol{E}$. The *velocity* of s is the section $ds : \boldsymbol{T} \to \mathbb{T}^* \otimes T\boldsymbol{E}$, with coordinate expression $ds = u^0 \otimes (\partial_0 + \partial_0 s^i\, \partial_i)$.

We consider, as classical *phase space* the 1st jet space of motions $J_1\boldsymbol{E}$. The phase space turns out to be an affine bundle $J_1\boldsymbol{E} \to \boldsymbol{E}$ associated with the vector bundle $\mathbb{T}^* \otimes V\boldsymbol{E}$. We denote the fibred charts of phase space by (x^λ, x^i_0). We have the natural *contact map* and *complementary contact map*

$$д : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E} \qquad \text{and} \qquad \theta : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes V\boldsymbol{E}\,,$$

with coordinate expressions $д = u^0 \otimes (\partial_0 + x^i_0\, \partial_i)$ and $\theta = (d^i - x^i_0\, d^0) \otimes \partial_i$.

An *observer* is defined to be a section $o : \boldsymbol{E} \to J_1\boldsymbol{E}$. A spacetime chart (x^λ) is said to be adapted to an observer o if $o^i_0 := x^i_0 \circ o = 0$. For each observers o and $\acute{o}$, we can write $\acute{o} = o + v$, where $v : \boldsymbol{E} \to \mathbb{T}^* \otimes V\boldsymbol{E}$ is the velocity of $\acute{o}$ with respect to o. An observer o yields the observed *contact map* and *complementary contact map*

$$д[o] := \theta \circ o : \boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E} \qquad \text{and} \qquad \theta[o] : \boldsymbol{E} \to T^*\boldsymbol{E} \otimes V\boldsymbol{E} ,$$

with coordinate expressions $д[o] = u^0 \otimes (\partial_0 + o^i_0\, \partial_i)$ and $\theta[o] = (d^i - o^i_0\, d^0) \otimes \partial_i$.

2) We postulate a given *galileian spacetime metric*, i.e. a scaled *spacelike riemannian metric*

$$g : \boldsymbol{E} \to \mathbb{L}^2 \otimes (V\boldsymbol{E} \otimes V\boldsymbol{E}) ,$$

with coordinate expression $g = g_{ij}\, \check{d}^i \otimes \check{d}^j$, where $g_{ij} \in \text{map}(\boldsymbol{E}, \mathbb{L}^2 \otimes \mathbb{R})$.

With reference to a particle of mass $m \in \mathbb{M}$, and to the Planck constant $\hbar$, we define the convenient *rescaled metric*

$$G := \tfrac{m}{\hbar}\, g : \boldsymbol{E} \to \mathbb{T} \otimes (V\boldsymbol{E} \otimes V\boldsymbol{E}) ,$$

with coordinate expression $G = G^0_{ij}\, u_0 \otimes \check{d}^i \otimes \check{d}^j$, where $G^0_{ij} \in \text{map}(\boldsymbol{E}, \mathbb{R})$.

We denote the *rescaled contravariant metric* by

$$\bar{G} : \boldsymbol{E} \to \mathbb{T}^* \otimes (V\boldsymbol{E} \otimes V\boldsymbol{E})$$

and the *musical metric isomorphisms* by

$$g^\flat : V\boldsymbol{E} \to \mathbb{L}^2 \otimes V^*\boldsymbol{E} \qquad \text{and} \qquad g^\sharp : V^*\boldsymbol{E} \to \mathbb{L}^{-2} \otimes V\boldsymbol{E} ,$$
$$G^\flat : V\boldsymbol{E} \to \mathbb{T} \otimes V^*\boldsymbol{E} \qquad \text{and} \qquad G^\sharp : V^*\boldsymbol{E} \to \mathbb{T}^* \otimes V\boldsymbol{E} .$$

The metric g and the time fibring t yield the scaled *spacelike volume form*, *spacetime volume form* and the dual *spacelike volume vector*, *spacetime volume vector*

$$\eta : \boldsymbol{E} \to \mathbb{L}^3 \otimes \Lambda^3 V^*\boldsymbol{E} \qquad \text{and} \qquad \upsilon : \boldsymbol{E} \to (\mathbb{T} \otimes \mathbb{L}^3) \otimes \Lambda^4 T^*\boldsymbol{E} ,$$
$$\bar{\eta} : \boldsymbol{E} \to \mathbb{L}^{-3} \otimes \Lambda^3 V\boldsymbol{E} \qquad \text{and} \qquad \bar{\upsilon} : \boldsymbol{E} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^{-3}) \otimes \Lambda^4 T\boldsymbol{E} ,$$

with coordinate expressions

$$\eta = \sqrt{|g|}\, \check{d}^1 \wedge \check{d}^2 \wedge \check{d}^3 \qquad \text{and} \qquad \upsilon = \sqrt{|g|}\, u_0 \otimes d^0 \wedge d^1 \wedge d^2 \wedge d^3 ,$$
$$\bar{\eta} = 1/\sqrt{|g|}\, \partial_1 \wedge \partial_2 \wedge \partial_3 \qquad \text{and} \qquad \bar{\upsilon} = 1/\sqrt{|g|}\, u^0 \otimes d^0 \wedge \partial_1 \wedge \partial_2 \wedge \partial_3 .$$

The galileian metric yields the *spacelike divergence operator* acting on projectable vector fields and the *spacelike laplacian operator* acting on functions

$$\text{div}_\eta : \text{pro}(\boldsymbol{E}, T\boldsymbol{E}) \to \text{map}(\boldsymbol{E}, \mathbb{R})$$

and

$$\Delta[G] : \mathrm{map}(\boldsymbol{E}, \mathbb{R}) \to \mathrm{map}(\boldsymbol{E}, \mathbb{T}^* \otimes \mathbb{R}) \,,$$

whose coordinate expressions are

$$\mathrm{div}_\eta X = X^0 \, \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}} + \frac{\partial_i (X^i \sqrt{|g|})}{\sqrt{|g|}}$$

and

$$\Delta_0[G]\, f = G_0^{ij}\, \partial_{ij} f + \frac{\partial_i (G_0^{ih} \sqrt{|g|})}{\sqrt{|g|}}\, \partial_h f \,.$$

3) We define a *galileian spacetime connection* to be a spacetime connection

$$K : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E} \,,$$

which fulfills the conditions

$$\nabla dt = 0 \,, \qquad \nabla g = 0 \,, \qquad R_{i\lambda j\mu} = R_{j\mu i\lambda} \,,$$

where R is the curvature tensor of K .

Due to the spacelike feature of g , a galileian spacetime connection K is not fully characterised by g .

However, the galileian metric g and an observer o, yield, in a natural way, a certain distinguished galileian connection

$$\mathsf{K}[o] : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E} \,,$$

whose coordinate expression, in a spacetime chart adapted to o , is

$$\begin{aligned} \mathsf{K}_0{}^i{}_0 &= 0 \,, \\ \mathsf{K}_0{}^i{}_h = \mathsf{K}_h{}^i{}_0 &= -\tfrac{1}{2}\, G_0^{ij}\, \partial_0 G^0_{hj} \,, \\ \mathsf{K}_k{}^i{}_h = \mathsf{K}_h{}^i{}_k &= -\tfrac{1}{2}\, G_0^{ij}\, (\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk}) \,. \end{aligned}$$

Moreover, with reference to an observer o, every galileian spacetime connection K yields the *observed spacetime 2–form*

$$\Phi[o] := 2 \,\mathrm{Ant}\, \big(\theta[o] \overset{2}{\lrcorner} G^\flat(\nabla д[o])\big) : \boldsymbol{E} \to \Lambda^2 T^*\boldsymbol{E} \,,$$

with coordinate expression, in a spacetime chart adapted to o ,

$$\Phi[o] = -2\, G^0_{jh}\, (K_0{}^h{}_0\, d^0 \wedge d^j + K_i{}^h{}_0\, d^i \wedge d^j) \,.$$

Actually, the condition $R_{i\lambda j\mu} = R_{j\mu i\lambda}$ fulfilled by K turns out to be equivalent to the condition

$$d\Phi[o] = 0\,.$$

Hence, we obtain locally a gauge dependent and observer dependent potential

$$A[\mathfrak{b}, o] : \boldsymbol{E} \to T^*\boldsymbol{E}\,, \qquad \text{according to} \qquad \Phi[o] = 2\, dA[\mathfrak{b}, o]\,,$$

whose gauge is labelled by the symbol $\mathfrak{b}$.

Then, we set

$$\widehat{\Phi}\,[o] = G^{ij}_0\, \Phi_{\lambda j}\, u^0 \otimes d^\lambda \otimes \partial_i$$

and, for every galileian connection K, obtain the *observed splitting*

$$K = \mathsf{K}[o] - \tfrac{1}{2}\,(dt \otimes \widehat{\Phi}\,[o] + \widehat{\Phi}\,[o] \otimes dt)\,,$$

with coordinate expression

$$\begin{aligned} K_0{}^i{}_0 &= -G^{ij}_0\, \Phi_{0j}\,, \\ K_0{}^i{}_h = K_h{}^i{}_0 &= -\tfrac{1}{2}\, G^{ij}_0\, (\partial_0 G^0_{hj} + \Phi_{hj})\,, \\ K_k{}^i{}_h = K_h{}^i{}_k &= -\tfrac{1}{2}\, G^{ij}_0\, (\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk})\,. \end{aligned}$$

Indeed, given an observer o, the above observed splitting yields a bijection

$$K \mapsto \Phi[o]$$

between galileian spacetime connections and closed spacetime 2–forms. So, with reference to an observer o, a galileian spacetime connection K turns out to be generated by the 10 spacetime functions (g_{ij}, A_λ).

Thus, we postulate a given *"gravitational" spacetime connection*

$$K^\natural : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E}\,,$$

which represents the *gravitational field.*

4) We postulate a given scaled *electromagnetic field*

$$F : \boldsymbol{E} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 T^*\boldsymbol{E}\,,$$

which fulfills the condition (*1st Maxwell equation*)

$$dF = 0\,.$$

Then, with reference to a particle of mass m and charge q, we set

$$\widehat{F} := G^{ih}_0 \, F_{\lambda h} \, u^0 \otimes d^\lambda \otimes \partial_i$$

and define, by a minimal coupling, the *joined spacetime connection*

$$K \;\equiv\; K^\natural + K^\mathfrak{e} := K^\natural - \tfrac{1}{2} \tfrac{q}{\hbar} \, (dt \otimes \widehat{F} + \widehat{F} \otimes dt) : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E} \,.$$

Indeed, this joined connection turns out to be a galileian spacetime connection.

Then, the joined spacetime connection K yields the associated *joined spacetime 2–form* and *joined observed potential*

$$\Phi[o] = \Phi^\natural[o] + \tfrac{q}{\hbar} \, F : \boldsymbol{E} \to \Lambda^2 T^* \boldsymbol{E}$$

and

$$A[\mathfrak{b}, o] = A^\natural[\mathfrak{b}, o] + \tfrac{q}{\hbar} \, A^\mathfrak{e}[\mathfrak{b}] : \boldsymbol{E} \to T^*\boldsymbol{E} \,.$$

Afterwards, in several respects, we can forget about the gravitational and electromagnetic fields separately and deal with the joined spacetime connection K, which encodes both of them.

The *Newton law* of motion of a charged particle effected by the gravitational and electromagnetic fields can be written, in a compact way, by means of the joined spacetime connection K, as

$$\nabla_{ds} ds = 0 \,.$$

5) The joined spacetime connection K yields in a natural way an affine *joined phase connection* of the phase space

$$\Gamma : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TJ_1\boldsymbol{E} \,,$$

whose coordinate expression is

$$\Gamma = d^\lambda \otimes (\partial_\lambda + \Gamma_\lambda{}^i_0 \, \partial^0_i) \,, \qquad \text{with} \qquad \Gamma_\lambda{}^i_0 = \Gamma_\lambda{}^{i\,0}_{0\,0} + \Gamma_\lambda{}^{i\,0}_{0\,j} \, x^j_0 \,, \quad \Gamma_\lambda{}^{i\,0}_{0\,\mu} = K_\lambda{}^i{}_\mu \,.$$

Further, the above joined phase connection Γ yields in a natural way the joined *dynamical phase connection*, *dynamical phase 2–form* and *dynamical phase 2–vector*

$$\gamma := \text{д} \lrcorner \Gamma : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E} \,,$$
$$\Omega := G \lrcorner \big(\nu[\Gamma] \wedge \theta\big) : J_1\boldsymbol{E} \to \Lambda^2 T^* J_1\boldsymbol{E} \,,$$
$$\Lambda := \bar{G} \lrcorner (\check{\Gamma} \wedge \nu) : J_1\boldsymbol{E} \to \Lambda^2 T J_1\boldsymbol{E} \,,$$

where

- $\nu[\Gamma] : TJ_1\boldsymbol{E} \to \mathbb{T}^* \otimes V\boldsymbol{E}$ is the vertical projection associated with Γ,
- $\check{\Gamma} : J_1\boldsymbol{E} \to V^*\boldsymbol{E} \otimes TJ_1\boldsymbol{E}$ is the vertical restriction of Γ,
- $\nu : J_1\boldsymbol{E} \to (\mathbb{T}\otimes V^*\boldsymbol{E})\otimes(\mathbb{T}^*\otimes V\boldsymbol{E})$ is the natural tensor with coordinate expression $\nu = u_0 \otimes (\check{d}^i \otimes \partial_i^0)$.

We have the coordinate expressions

$$\begin{aligned}\gamma &= u^0 \otimes (\partial_0 + x_0^i\,\partial_i + \gamma_0{}_0^i\,\partial_i^0)\,,\\ \Omega &= G_{ij}^0\,(d_0^i - \Gamma_\lambda{}_0^i\,d^\lambda) \wedge (d^j - x_0^j\,d^0)\,,\\ \Lambda &= G_0^{ij}\,(\partial_i + \Gamma_i{}_0^h\,\partial_h^0) \wedge \partial_j^0\,,\end{aligned}$$

where $\gamma_0{}_0^i = K_0{}^i{}_0 + 2\,K_0{}^i{}_k\,x_0^k + K_h{}^i{}_k\,x_0^h\,x_0^k$.

a) Indeed, the *Newton law* of motion can be equivalently written, by means the dynamical connection γ, as

$$\gamma \circ j_1 s = dj_1 s\,.$$

b) The pair (dt, Ω) equips the phase space with a *cosymplectic structure*, i.e., we have

$$d\Omega = 0\,, \qquad dt \wedge \Omega \wedge \Omega \wedge \Omega \not\equiv 0\,.$$

The cosymplectic 2–form Ω admits locally a *"horizontal potential"*

$$A^\uparrow[\mathfrak{b}] : J_1\boldsymbol{E} \to T^*\boldsymbol{E}\,, \qquad \text{according to} \qquad \Omega = dA^\uparrow[\mathfrak{b}]\,,$$

whose gauge is labelled with the symbol $\mathfrak{b}$, with coordinate expression

$$A^\uparrow[\mathfrak{b}] = -(\tfrac{1}{2}\,G_{ij}^0\,x_0^i\,x_0^j - A_0)\,d^0 + (G_{ij}^0\,x_0^j + A_j)\,d^i\,.$$

We obtain $A[\mathfrak{b}, o] = o^* A^\uparrow[\mathfrak{b}]$.

The cosymplectic phase 2-form yields the gauge dependent, observer independent *classical lagrangian*, the gauge dependent, observer dependent *classical hamiltonian* and the gauge dependent, observer dependent *classical momentum*

$$\begin{aligned}\mathcal{L}[\mathfrak{b}] &:= д \lrcorner A^\uparrow[\mathfrak{b}] && \in \sec(J_1\boldsymbol{E}, H^*\boldsymbol{E})\,,\\ \mathcal{H}[\mathfrak{b}, o] &:= -\,д[o] \lrcorner A^\uparrow[\mathfrak{b}] && \in \sec(J_1\boldsymbol{E}, H^*\boldsymbol{E})\,,\\ \mathcal{P}[\mathfrak{b}, o] &:= \theta[o] \lrcorner A^\uparrow[\mathfrak{b}] && \in \sec(J_1\boldsymbol{E}, T^*\boldsymbol{E})\,,\end{aligned}$$

with coordinate expressions

$$\begin{aligned} \mathcal{L}[\mathtt{b}] &= (\tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 + A_j\, x^j_0 + A_0)\, d^0 \,, \\ \mathcal{H}[\mathtt{b}, o] &= (\tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 - A_0)\, d^0 \,, \\ \mathcal{P}[\mathtt{b}, o] &= (G^0_{ij}\, x^j_0 + A_i)\, d^i \,. \end{aligned}$$

Further, we point out the gauge dependent and observer independent *timelike 1–form*

$$\alpha[\mathtt{b}] \equiv \alpha_0\, d^0 = (A_0 - \tfrac{1}{2}\, A_i\, A^i_0)\, d^0 : \boldsymbol{E} \to H^*\boldsymbol{E} \,, \qquad \text{where} \qquad A^i_0 := G^{ij}_0\, A_j \,.$$

c) The pair (γ, Λ) equips the phase space with a *coPoisson structure*, i.e., we have

$$[\gamma, \Lambda] = 0 \,, \qquad [\Lambda, \Lambda] = 0 \,, \qquad \gamma \wedge \Lambda \wedge \Lambda \wedge \Lambda \not\equiv 0 \,,$$

where $[\,,]$ denotes the Schouten bracket.

6) We postulate as *quantum bundle* a complex 1–dimensional bundle

$$\pi : \boldsymbol{Q} \to \boldsymbol{E}$$

equipped with scaled *hermitian metric*

$$\mathtt{h} : \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to \mathbb{L}^{-3} \otimes \mathbb{C} \,.$$

We consider also the volume valued hermitian metric

$$\mathtt{h}_\eta := \mathtt{h} \otimes \eta : \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to \Lambda^3 V^*\boldsymbol{E} \otimes \mathbb{C} \,.$$

We shall refer to a normalised scaled *quantum basis* and to the associated complex *quantum chart*

$$\mathtt{b} := \mathtt{b}_1 : \boldsymbol{E} \to \mathbb{L}^{3/2} \otimes \boldsymbol{Q} \qquad \text{and} \qquad z : \boldsymbol{Q} \to \mathbb{L}^{-3/2} \otimes \mathbb{C} \,,$$

which fulfill the conditions

$$\mathtt{h}(\mathtt{b}, \mathtt{b}) = 1 \qquad \text{and} \qquad z(\mathtt{b}) = 1 \,.$$

Accordingly, for each *quantum section* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, we have the coordinate expression

$$\Psi = \psi\, \mathtt{b} \,, \qquad \text{where} \qquad \psi := z \circ \Psi \in \operatorname{map}(\boldsymbol{E}, \mathbb{C}) \,.$$

By pullback, we define the *upper quantum bundle*

$$\pi^{\uparrow} : \boldsymbol{Q}^{\uparrow} := J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \,,$$

obtained by enlarging the base space $\boldsymbol{E}$ of the quantum bundle $\pi : \boldsymbol{Q} \to \boldsymbol{E}$ to the classical phase space $J_1\boldsymbol{E}$. In a sense, we might say that the upper quantum bundle $\boldsymbol{Q}^{\uparrow}$ "incorporates" in its base space $J_1\boldsymbol{E}$ all possible observers o.

We denote the natural *Liouville vector fields* of the quantum bundle and of the upper quantum bundle, respectively, by

$$\mathbb{I} : \boldsymbol{Q} \to V_{\boldsymbol{E}}\boldsymbol{Q} \qquad \text{and} \qquad \mathbb{I}^{\uparrow} : \boldsymbol{Q}^{\uparrow} \to V_{J_1\boldsymbol{E}}\boldsymbol{Q}^{\uparrow} \,.$$

6.2 System of observed quantum connections

Now, we discuss the *galileian upper connection* $\text{Ч}^{\uparrow}$ of the upper quantum bundle and the associated *system* $\{\text{Ч}[o]\}$ of *observed quantum connections* as an application of the general theory of *smooth systems of connections* discussed in Section §4.1.1 (for further details, see, for instance, [9, 12] and literature therein).

1) We define a *galileian upper quantum connection* to be a connection of the upper quantum bundle

$$\text{Ч}^{\uparrow} : \boldsymbol{Q}^{\uparrow} \to T^*J_1\boldsymbol{E} \otimes T\boldsymbol{Q}^{\uparrow} \,,$$

which fulfills the following conditions:

a) it is *hermitian*, i.e. $\nabla^{\uparrow} h = 0$,

b) it is *reducible* (see Definition 4.1.2),

c) its *curvature tensor* fulfills the equality

$$R[\text{Ч}^{\uparrow}] = -2\,\mathfrak{i}\,\Omega \otimes \mathbb{I}^{\uparrow} \,,$$

where $\Omega : J_1\boldsymbol{E} \to \Lambda^2 T^* J_1\boldsymbol{E}$ is the classical cosymplectic phase 2–form.

We stress that the cosymplectic phase 2–form Ω is normalised through the rescaled metric G, hence it encodes the Planck constant $\hbar$.

Indeed, the Bianchi identity for an upper quantum connection $\text{Ч}^{\uparrow}$ is assured by the closure of the cosymplectic phase 2–form.

Chosen a quantum basis $\mathfrak{b}$, an upper quantum connection $\text{Ч}^{\uparrow}$ can be locally written as

$$\text{Ч}^{\uparrow} = \chi^{\uparrow}[\mathfrak{b}] + \mathfrak{i}\, A^{\uparrow}[\mathfrak{b}] \otimes \mathbb{I}^{\uparrow} \,,$$

where $\chi^{\uparrow}[\mathfrak{b}] : \boldsymbol{Q}^{\uparrow} \to T^* J_1 \boldsymbol{E} \otimes T\boldsymbol{Q}^{\uparrow}$ is the flat connection of the upper quantum bundle naturally induced by the quantum basis $\mathfrak{b}$ and $A^{\uparrow}[\mathfrak{b}]$ is the horizontal potential of the classical cosymplectic 2–form Ω associated with the classical gauge $\mathfrak{b}$.

Actually, with reference to a given upper quantum connection $\mathrm{Ч}^{\uparrow}$, the above formula allows us to identify the classical gauges $\mathfrak{b}$ of the classical potentials $A^{\uparrow}[\mathfrak{b}]$ of Ω and the quantum bases $\mathfrak{b}$, as we have already anticipated for practical convenience of notation.

Moreover, chosen a quantum basis $\mathfrak{b}$ and an observer o, an upper quantum connection $\mathrm{Ч}^{\uparrow}$ can be locally written as

$$\mathrm{Ч}^{\uparrow} = \chi^{\uparrow}[\mathfrak{b}] + \mathfrak{i}\,\big(-\mathcal{H}[\mathfrak{b}, o] + \mathcal{P}[\mathfrak{b}, o]\big) \otimes \mathbb{I}^{\uparrow} ,$$

where $\mathcal{H}[\mathfrak{b}, o]$ is the observed classical hamiltonian and $\mathcal{P}[\mathfrak{b}, o]$ is the observed classical momentum.

Thus, in a quantum chart adapted to $\mathfrak{b}$ and o, we obtain the coordinate expression

$$\mathrm{Ч}^{\uparrow} = d^{\lambda} \otimes \partial_{\lambda} + d^i_0 \otimes \partial^0_i + \mathfrak{i}\,\big(-(\tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 - A_0)\, d^0 + (G^0_{ij}\, x^j_0 + A_i)\, d^i\big) \otimes \mathbb{I}^{\uparrow} .$$

We denote by

$$\nabla^{\uparrow} : \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(J_1\boldsymbol{E}, T^*\boldsymbol{E} \otimes \boldsymbol{Q}) : \Psi \mapsto \nabla^{\uparrow}\Psi$$

the *covariant differential* associated with the upper quantum connection $\mathrm{Ч}^{\uparrow}$, whose coordinate expression is

$$\nabla^{\uparrow}\Psi = (\partial_{\lambda}\psi - \mathfrak{i}\, A^{\uparrow}{}_{\lambda}\, \psi)\, d^{\lambda} \otimes \mathfrak{b} .$$

2) Given an upper quantum connection $\mathrm{Ч}^{\uparrow}$, we obtain the system of gauge independent *observed quantum connections* parametrised by the family of observers o

$$\mathrm{Ч}[o] := o^* \mathrm{Ч}^{\uparrow} : \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to T\boldsymbol{Q} .$$

Every connection of this system turns out to be hermitian and fulfills the condition

$$R\big[\mathrm{Ч}[o]\big] = -\mathfrak{i}\, \Phi[o] \otimes \mathbb{I} .$$

Moreover, with further reference to a quantum basis $\mathfrak{b}$, we have the splitting

$$\mathrm{Ч}[o] = \chi[\mathfrak{b}] + \mathfrak{i}\, A[\mathfrak{b}, o]\, \mathbb{I} , \qquad \text{with} \qquad A[\mathfrak{b}, o] = o^* A^{\uparrow}[\mathfrak{b}] ,$$

where $\chi[\mathfrak{b}] : \boldsymbol{Q} \to T^*\boldsymbol{E} \otimes T\boldsymbol{Q}$ is the flat connection of the quantum bundle naturally induced by the quantum basis $\mathfrak{b}$ and $A[\mathfrak{b}, o]$ is the potential of the classical observed spacetime 2–form $\Phi[o]$ associated with the classical gauge $\mathfrak{b}$.

In other words, in adapted coordinates, we have the expression

$$Ч[o] = d^\lambda \otimes (\partial_\lambda + \mathfrak{i}\, A_\lambda \, \mathbb{I}) .$$

Indeed, for each observers o and $\acute{o} = o + v$, we have the transition rule

$$Ч[\acute{o}] = Ч[o] + \mathfrak{i} \big(\theta[o] \lrcorner G^\flat(v) - \tfrac{1}{2} G(v, v)\big) \otimes \mathbb{I} ,$$

with coordinate expression, in a chart adapted to o,

$$Ч[\acute{o}] = Ч[o] + \mathfrak{i} \, (G^0_{ij} \, v^i_0 \, d^j - \tfrac{1}{2} \, G^0_{ij} \, v^i_0 \, v^j_0 \, d^0) \otimes \mathbb{I} .$$

Thus, the galileian upper quantum connection $Ч^\uparrow$ turns out to be the *universal connection* of the system of observed connections $Ч[o]$ (see Theorem 4.1.1).

We denote by

$$\nabla[o] : \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, T^*\boldsymbol{E} \otimes \boldsymbol{Q}) : \Psi \mapsto \nabla[o]\Psi$$

the *covariant differential* associated with the observed quantum connection $Ч[o]$, whose coordinate expression, in an adapted chart, is

$$\nabla[o]\Psi = (\partial_\lambda \psi - \mathfrak{i} \, A_\lambda \, \psi) \, d^\lambda \otimes \mathfrak{b} .$$

Further, the observed quantum connection $Ч[o]$ yields the gauge independent *spacelike observed quantum laplacian* acting on quantum sections

$$\Delta[o] : \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \mathbb{T}^* \otimes \boldsymbol{Q}) : \Psi \mapsto (\Delta_0 \psi) \, u^0 \otimes \mathfrak{b} ,$$

whose coordinate expression is

$$\Delta_0 \psi = G^{ij}_0 \, \partial_{ij} \psi + \frac{\partial_i (G^{ih}_0 \sqrt{|g|})}{\sqrt{|g|}} \, \partial_h \psi - 2 \, \mathfrak{i} \, A^i_0 \, \partial_i \psi - \mathfrak{i} \, \frac{\partial_i (A^i_0 \sqrt{|g|})}{\sqrt{|g|}} \, \psi - A_i \, A^i_0 \, \psi .$$

2) One can prove that there exists a global galileian upper quantum connection $Ч^\uparrow$ if and only if the Čech cohomology class $[\Omega]$ is integer (see [26]).

Accordingly, we suppose that the cohomology class of Ω be integer and postulate an *upper quantum connection* $Ч^\uparrow$ as our fundamental quantum object.

6.3 Quantum dynamics

We can derive, in a covariant way, the further main objects of Covariant Quantum Mechanics from the above upper quantum connection $Ч^{\uparrow}$, by means of a heuristic *"criterion of projectability"*. This criterion is aimed at finding distinguished objects which, in principle leave on bundles over the classical phase space $J_1\boldsymbol{E}$, but eventually are projectable on bundles over the classical spacetime $\boldsymbol{E}$, in order to get rid of the family of observers encoded in $J_1\boldsymbol{E}$. In this way, we implement our covariance requirement, by obtaining observer equivariant objects. Actually, all objects obtained via the above covariant procedure turns out to be uniquely defined, up to an unessential multiplicative constant (for further details, see, for instance, [9, 10, 12] and literature therein).

1) The *quantum momentum* can be derived from the upper quantum connection in the following way.

For each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, the following map factorises through the spacetime $\boldsymbol{E}$ (thus cancelling the dependence on classical observers), so yielding, in a covariant way, the gauge independent and observer independent scaled section defined on spacetime

$$\mathrm{Q}[\Psi] := д \otimes \Psi - \mathfrak{i}\, G^{\sharp}(\nabla^{\uparrow}\Psi) \in \sec\big(\boldsymbol{E},\, \mathbb{T}^* \otimes (T\boldsymbol{E} \otimes \boldsymbol{Q})\big)\,.$$

With reference to any observer o, we have the observed splitting (which turns out to be observer equivariant)

$$\mathrm{Q}[\Psi] = д[o] \otimes \Psi - \mathfrak{i}\, G^{\sharp}(\nabla[o]\Psi)\,,$$

with coordinate expression

$$\mathrm{Q}[\Psi] = \big(\psi\, \partial_0 - \mathfrak{i}\, G^{ij}_0\, (\partial_j \psi - \mathfrak{i}\, A_j\, \psi)\, \partial_i\big) \otimes u^0 \otimes \mathfrak{b}\,.$$

2) The *quantum probability current* can be derived from the upper quantum connection in the following way.

For each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, the following map factorises through the spacetime $\boldsymbol{E}$ (thus cancelling the dependence on classical observers), so yielding, in a covariant way, the gauge independent and observer independent scaled section defined on spacetime

$$\mathrm{J}[\Psi] := д \otimes \|\Psi\|^2 - \mathrm{re}\, \mathsf{h}\big(\Psi,\, \mathfrak{i}\, G^{\sharp}(\nabla^{\uparrow}\Psi)\big) \in \sec\big(\boldsymbol{E},\, \mathbb{L}^{-3} \otimes (\mathbb{T}^* \otimes T\boldsymbol{E})\big)\,.$$

With reference to an observer o, we have the observed splitting (which turns out to be observer equivariant)

$$\mathsf{J}[\Psi] = \|\Psi\|^2 \, д[o] - \mathrm{re}\, \mathsf{h}\big(\Psi, \mathfrak{i}\, G^\sharp(\nabla[o])\Psi\big)\,,$$

with coordinate expression

$$\mathsf{J}[\Psi] = \big(|\psi|^2\, \partial_0 + (\mathfrak{i}\, \tfrac{1}{2}\, G^{ij}_0\, (\psi\, \partial_j \bar{\psi} - \bar{\psi}\, \partial_j \psi) - A^i_0\, |\psi|^2)\, \partial_i\big) \otimes u^0\,.$$

3) The *quantum lagrangian* can be derived from the upper quantum connection in the following way.

For each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, the following map factorises through the spacetime $\boldsymbol{E}$ (thus cancelling the dependence on classical observers), so yielding, in a covariant way, the gauge independent and observer independent scaled section defined on spacetime

$$\mathsf{L}[\Psi] := -dt \wedge \Big(\mathrm{im}\, \mathsf{h}_\eta(\Psi,\, д \lrcorner \nabla^\uparrow \Psi) + \tfrac{1}{2}\, (\bar{G} \otimes \mathsf{h}_\eta)(\check{\nabla}^\uparrow \Psi,\, \check{\nabla}^\uparrow \Psi)\Big) : \boldsymbol{E} \to \Lambda^4 T^* \boldsymbol{E}\,.$$

With reference to an observer o, we have the observed expression (which turns out to be observer equivariant)

$$\mathsf{L}[\Psi] = -dt \wedge \big(\mathrm{im}\, \mathsf{h}_\eta(\Psi,\, \nabla[o]_{д[o]}\Psi) + \tfrac{1}{2}\, (\bar{G} \otimes \mathsf{h}_\eta)\, (\nabla[o]\Psi,\, \nabla[o]\Psi)\big)\,,$$

with coordinate expression

$$\begin{aligned} \mathsf{L}[\Psi] = \tfrac{1}{2}\, \big(&- G^{ij}_0\, \partial_i \bar{\psi}\, \partial_j \psi + \mathfrak{i}\, (\bar{\psi}\, \partial_0 \psi - \psi\, \partial_0 \bar{\psi}) \\ &- \mathfrak{i}\, A^j_0\, (\bar{\psi}\, \partial_j \psi - \psi\, \partial_j \bar{\psi}) + 2\, \alpha_0\, \bar{\psi}\, \psi\big)\, \upsilon^0\,. \end{aligned}$$

4) The *Schrödinger operator* can be derived from the upper quantum connection in the following way.

For each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, the following map factorises through the spacetime $\boldsymbol{E}$ (thus cancelling the dependence on classical observers), so yielding, in a covariant way, the gauge independent and observer independent scaled section defined on spacetime

$$\mathsf{S}[\Psi] := \tfrac{1}{2}\, \big(д \lrcorner \nabla^\uparrow \Psi + \delta^\uparrow(\mathrm{Q}[\Psi])\big) \in \sec(\boldsymbol{E},\, \mathbb{T}^* \otimes \boldsymbol{Q})\,.$$

where $\delta^\uparrow$ is the codifferential of vector valued forms associated with the upper quantum connection $Ч^\uparrow$.

With reference to an observer o, we have the observed expression (which turns out to be observer equivariant)

$$\mathsf{S}[\Psi] = \nabla[o]_{д[o]}\Psi + \tfrac{1}{2}\, \mathrm{div}_\eta\, д[o]\, \Psi - \mathfrak{i}\, \tfrac{1}{2}\, \Delta[o]\, \Psi\,.$$

Moreover, we have the coordinate expression

$$\mathsf{S}[\Psi] = \Big(\partial_0\psi - \tfrac{1}{2}\,\mathrm{i}\,G^{ij}_0\,\partial_{ij}\psi - (A^j_0 + \tfrac{1}{2}\,\mathrm{i}\,\frac{\partial_i(G^{ij}_0\sqrt{|g|})}{\sqrt{|g|}})\,\partial_j\psi + \tfrac{1}{2}\,(\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i(A^i_0\sqrt{|g|})}{\sqrt{|g|}} - 2\,\mathrm{i}\,\alpha_0)\,\psi\Big)\,u^0\otimes\mathsf{b}\,.$$

Further, we can prove that the above Schrödinger operator turns out to be the Euler–Lagrange operator associated with the quantum lagrangian.

6.4 F–smooth sectional quantum fibred set

Further, we introduce the "*F–smooth sectional quantum fibred set*"

$$\widehat{t} : \widehat{\boldsymbol{Q}} \to \boldsymbol{T}$$

of the double fibred manifold

$$\boldsymbol{Q} \xrightarrow{\pi} \boldsymbol{E} \xrightarrow{t} \boldsymbol{T} \quad ,$$

as an application of the general theory of *F–smooth systems of smooth sections* discussed in Section §3.2.1 (for further details, see, for instance, [9, 12] and literature therein).

Definition 6.4.1. For each $\mathrm{t}\in\boldsymbol{T}$, we define the *sectional quantum space at* t, to be the subset

$$\widehat{\boldsymbol{Q}}_{\mathrm{t}} := \mathrm{cpt}(\boldsymbol{E}_{\mathrm{t}}, \boldsymbol{Q}_{\mathrm{t}}) \subset \mathrm{Sec}(\boldsymbol{E}_{\mathrm{t}}, \boldsymbol{Q}_{\mathrm{t}})$$

consisting of all *global (fibrewise) smooth sections* $\Psi_{\mathrm{t}} : \boldsymbol{E}_{\mathrm{t}} \to \boldsymbol{Q}_{\mathrm{t}}$ *with compact support.*

For each $\mathrm{t}\in\boldsymbol{T}$, the set $\widehat{\boldsymbol{Q}}_{\mathrm{t}}$ turns out to be a complex vector space, by setting, for each $\widehat{\Psi}_{\mathrm{t}}, \widehat{\acute{\Psi}}_{\mathrm{t}} \in \widehat{\boldsymbol{Q}}_{\mathrm{t}}$ and $k\in\mathbb{C}$,

$$\widehat{\Psi}_{\mathrm{t}} + \widehat{\acute{\Psi}}_{\mathrm{t}} := \widehat{\Psi_{\mathrm{t}} + \acute{\Psi}_{\mathrm{t}}} \qquad \text{and} \qquad k\,\widehat{\Psi}_{\mathrm{t}} := \widehat{k\,\Psi_{\mathrm{t}}}\;.$$

Then, we define the *sectional quantum space* to be the disjoint union

$$\widehat{\boldsymbol{Q}} := \bigsqcup_{\mathrm{t}\in\boldsymbol{T}} \widehat{\boldsymbol{Q}}_{\mathrm{t}}\,.$$

By definition, the sectional quantum space $\widehat{\boldsymbol{Q}}$ is naturally equipped with the natural surjective fibring

$$\widehat{t} : \widehat{\boldsymbol{Q}} \to \boldsymbol{T} : \widehat{\Psi}_{\mathrm{t}} \to \mathrm{t}\,,$$

which makes $\widehat{\boldsymbol{Q}}$ a *fibred set over* $\boldsymbol{T}$. □

Proposition 6.4.1. We have the natural evaluation map

$$\epsilon : \widehat{\boldsymbol{Q}} \underset{\boldsymbol{T}}{\times} \boldsymbol{E} \to \boldsymbol{Q} : (\widehat{\Psi}\,, \mathrm{e}) \mapsto \Psi(\mathrm{e})\,.$$

For each $\mathrm{t} \in \boldsymbol{T}$, the pair $(\widehat{\boldsymbol{Q}}_{\mathrm{t}}, \epsilon_{\mathrm{t}})$ turns out to be an F–smooth system of smooth maps (see Definition 3.2.2).

Then, according to Theorem 3.2.1, for each $\mathrm{t} \in \boldsymbol{T}$, the set $\widehat{\boldsymbol{Q}}_{\mathrm{t}}$ turns out to be an *F–smooth space* (see Definition 1.2.1).

The basic F–smooth curves of $\widehat{\boldsymbol{Q}}_{\mathrm{t}}$ are all curves $c : \boldsymbol{I}_c \to \widehat{\boldsymbol{Q}}_{\mathrm{t}}$, such that the induced curves

$$c_{\boldsymbol{E}_{\mathrm{t}}} : \boldsymbol{I}_c \times \boldsymbol{E}_{\mathrm{t}} \to \boldsymbol{Q}_{\mathrm{t}} : (\lambda, \mathrm{e}) \mapsto \big(c(\lambda)\big)(\mathrm{e})$$

be smooth.

Then, according to Theorem 3.2.1, the set $\widehat{\boldsymbol{Q}}$ turns out to be an F–smooth space.

The basic F–smooth curves of $\widehat{\boldsymbol{Q}}$ are all curves $\widehat{c} : \boldsymbol{I}_{\widehat{c}} \to \widehat{\boldsymbol{Q}}$, such that the induced curves

$$c := \widehat{t} \circ \widehat{c} : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{T} \qquad \text{and} \qquad \widehat{c}^{\,*}(\epsilon) : c^*(\boldsymbol{E}) \to c^*(\boldsymbol{Q})$$

be smooth. Moreover, the maps

$$\widehat{t} : \widehat{\boldsymbol{Q}} \to \boldsymbol{T} \qquad \text{and} \qquad \epsilon : \widehat{\boldsymbol{Q}} \underset{\boldsymbol{T}}{\times} \boldsymbol{E} \to \boldsymbol{Q}$$

turn out to be F–smooth (see Definition 1.2.4).

Thus, $\widehat{t} : \widehat{\boldsymbol{Q}} \to \boldsymbol{T}$ turns out to be an *F–smooth fibred space.* □

Definition 6.4.2. A *tubelike quantum section* is defined to be a *local* quantum section

$$\Psi : \boldsymbol{E} \to \boldsymbol{Q}\,,$$

whose domain is of the type

$$\boldsymbol{D}[\Psi] := t^{-1}\big(\underline{\boldsymbol{D}}[\Psi]\big) \subset \boldsymbol{E}\,, \qquad \text{where} \qquad \underline{\boldsymbol{D}}[\Psi] \subset \boldsymbol{T} \quad \text{is an open subset.}$$

A tubelike quantum section $\Psi : \boldsymbol{E} \to \boldsymbol{Q}$ is said to be

1) *almost regular* if, for each $\mathrm{t} \in \underline{\boldsymbol{D}}[\Psi] \subset \boldsymbol{T}$, its restriction $\Psi_{\mathrm{t}} : \boldsymbol{E}_{\mathrm{t}} \to \boldsymbol{Q}_{\mathrm{t}}$ is *smooth* (no further "transversal" smoothness assumption is required) and with *compact support*,

2)*regular* if it is *almost regular* and *fully smooth* (i.e., also horizontally smooth).

We denote the subsheaf of almost regular quantum sections and the subsheaf of regular quantum sections, respectively, by

$$\underline{\mathrm{reg}}(\boldsymbol{E}, \boldsymbol{Q}) \subset \mathrm{sec}(\boldsymbol{E}, \boldsymbol{Q}) \qquad \text{and} \qquad \mathrm{reg}(\boldsymbol{E}, \boldsymbol{Q}) \subset \underline{\mathrm{reg}}(\boldsymbol{E}, \boldsymbol{Q}) \subset \mathrm{sec}(\boldsymbol{E}, \boldsymbol{Q})\,.$$

We denote the sheaves of local sections $\widehat{\Psi} : \boldsymbol{T} \to \widehat{\boldsymbol{Q}}$, *without any smoothness requirement*, and the subsheaf of local *F–smooth* sections $\widehat{\Psi} : \boldsymbol{T} \to \widehat{\boldsymbol{Q}}$, respectively, by

$$\underline{\mathrm{sec}}(\boldsymbol{T}, \widehat{\boldsymbol{Q}}) \qquad \text{and} \qquad \mathrm{sec}(\boldsymbol{T}, \widehat{\boldsymbol{Q}}) \subset \underline{\mathrm{sec}}(\boldsymbol{T}, \widehat{\boldsymbol{Q}})\,.\square$$

Theorem 6.4.1. *By definition of sectional quantum space, we have natural mutually inverse sheaf isomorphism*

$$\underline{\mathrm{sec}}(\boldsymbol{T}, \widehat{\boldsymbol{Q}}) \to \underline{\mathrm{reg}}(\boldsymbol{E}, \boldsymbol{Q}) : \widehat{\Psi} \mapsto \Psi$$

and

$$\underline{\mathrm{reg}}(\boldsymbol{E}, \boldsymbol{Q}) \to \underline{\mathrm{sec}}(\boldsymbol{T}, \widehat{\boldsymbol{Q}}) : \Psi \mapsto \widehat{\Psi}\,.$$

Then, in virtue of Theorem 3.2.2, we have the following equivalence

$$\Psi : \boldsymbol{E} \to \boldsymbol{Q} \quad \textit{is smooth} \qquad \Leftrightarrow \qquad \widehat{\Psi} : \boldsymbol{T} \to \widehat{\boldsymbol{Q}} \quad \textit{is F–smooth.}$$

Hence, the above sheaf isomorphism restricts to mutually inverse sheaf isomorphisms

$$\mathrm{sec}(\boldsymbol{T}, \widehat{\boldsymbol{Q}}) \to \mathrm{reg}(\boldsymbol{E}, \boldsymbol{Q}) : \widehat{\Psi} \mapsto \Psi$$

and

$$\mathrm{reg}(\boldsymbol{E}, \boldsymbol{Q}) \to \mathrm{sec}(\boldsymbol{T}, \widehat{\boldsymbol{Q}}) : \Psi \mapsto \widehat{\Psi}\,.\square$$

6.5 F–smooth sectional quantum bundle

From now on, for the sake of simplicity, we assume that the spacetime fibring be a *trivial bundle* with *contractible type fibre*, according to the equality

$$\boldsymbol{E} \simeq \boldsymbol{T} \times \mathbb{E}\,.$$

As a consequence, the quantum bundle $\pi : \boldsymbol{Q} \to \boldsymbol{E}$ turns out to be trivialisable and the fibred set $\hat{t} : \widehat{\boldsymbol{Q}} \to \boldsymbol{T}$ turns out to be an F–smooth bundle.

We stress that here we have supposed spacetime to be trivialisable, but we have not chosen any distinguished such trivialisation. As a consequence, the sectional quantum F–smooth bundle $\hat{t} : \widehat{\boldsymbol{Q}} \to \boldsymbol{T}$ has no *distinguished* local splittings into time and a functional type fibre. Indeed, such an F–smooth trivialisation depends on the choice of the bundle trivialisation of spacetime and of a local quantum basis.

For each $\mathsf{t} \in \boldsymbol{T}$, let $\mathrm{cpt}(\boldsymbol{E}_t, \mathbb{L}^{3/2} \otimes \mathbb{C})$ denote the space of global smooth scaled functions with compact support.

Lemma 6.5.1. For each $\mathsf{t} \in \boldsymbol{T}$, a global smooth quantum basis $\mathsf{b}_\mathsf{t} : \boldsymbol{E}_\mathsf{t} \to \mathbb{L}^{-3/2} \otimes \boldsymbol{Q}_\mathsf{t}$ yields a bijection (see Definition 6.4.1)

$$\widehat{\boldsymbol{Q}}_\mathsf{t} \to \mathrm{cpt}(\boldsymbol{E}_t, \mathbb{L}^{3/2} \otimes \mathbb{C}) : \widehat{\Psi}_\mathsf{t} \mapsto \psi_\mathsf{t} \,. \square$$

Each trivialisation of spacetime yields a trivialisation of the sectional quantum space $\widehat{t} : \widehat{\boldsymbol{Q}} \to \boldsymbol{T}$.

Proposition 6.5.1. The sectional quantum fibred set

$$\widehat{t} : \widehat{\boldsymbol{Q}} \to \boldsymbol{T}$$

turns out to be an *F–smooth bundle*, which is globally trivialisable by a global F–smooth bundle isomorphism

$$\widehat{\varPhi} : \widehat{\boldsymbol{Q}} \to \boldsymbol{E} \times \widehat{\mathbb{Q}} \,,$$

where, the type fibre

$$\widehat{\mathbb{Q}} := \mathrm{cpt}(\mathbb{E}, \mathbb{L}^{3/2} \otimes \mathbb{C})$$

is the space of all smooth scaled functions $\mathbb{E} \to \mathbb{L}^{3/2} \otimes \mathbb{C}$ with compact support. □

6.6 The pre–Hilbert sectional quantum bundle

The hermitian quantum metric h_η equips, for each $\mathsf{t} \in \boldsymbol{T}$, the fibres of the sectional quantum bundle $\widehat{t} : \widehat{\boldsymbol{Q}} \to \boldsymbol{T}$, via integration on the fibres $\boldsymbol{E}_\mathsf{t}$ of spacetime, with a *pre–Hilbert fibred metric*

$$\langle \, | \, \rangle : \widehat{\boldsymbol{Q}} \underset{\boldsymbol{T}}{\times} \widehat{\boldsymbol{Q}} \to \mathbb{C} \,,$$

Then, the F–smooth sectional quantum bundle can be made into a true *Hilbert bundle* by a completion procedure.

Indeed, the formulation of standard Quantum Mechanics in terms of a Hilbert space of quantum states is strictly related to the hypothesis of a flat spacetime and to the choice of an inertial observer. Actually, there is no distinguished, observer independent, isomorphism between the Hilbert spaces of quantum states at different times.

Proposition 6.6.1. For each $\mathrm{t} \in \boldsymbol{T}$, the η–hermitian quantum metric $\mathfrak{h}_\eta$ equips the complex vector fibres $\widehat{\boldsymbol{Q}}_\mathrm{t}$ of the sectional quantum bundle $\widehat{t} : \widehat{\boldsymbol{Q}} \to \boldsymbol{T}$ with the scalar product

$$\langle\,|\,\rangle_\mathrm{t} : \widehat{\boldsymbol{Q}}_\mathrm{t} \times \widehat{\boldsymbol{Q}}_\mathrm{t} \to \mathbb{C} : (\widehat{\Psi}_\mathrm{t}, \acute{\widehat{\Psi}}_\mathrm{t}) \mapsto \langle \widehat{\Psi}_\mathrm{t} \mid \acute{\widehat{\Psi}}_\mathrm{t} \rangle_\mathrm{t} := \int_{\boldsymbol{E}_\mathrm{t}} \mathfrak{h}_\mathrm{t}(\Psi_\mathrm{t}, \acute{\Psi}_\mathrm{t})\, \eta_\mathrm{t}\,,$$

with coordinate expression

$$\langle \widehat{\Psi}_\mathrm{t} \mid \acute{\widehat{\Psi}}_\mathrm{t} \rangle_\mathrm{t} = \int_{\boldsymbol{E}_\mathrm{t}} \bar{\psi}_\mathrm{t}\, \acute{\psi}_\mathrm{t} \sqrt{|g|}\; \check{d}^1 \wedge \check{d}^2 \wedge \check{d}^3\,.$$

Clearly, the above scale product $\langle\,|\,\rangle_\mathrm{t}$ makes the complex vector space $\widehat{\boldsymbol{Q}}_\mathrm{t}$ a *pre–Hilbert vector space.*

Indeed, the scalar product $\langle\,|\,\rangle_\mathrm{t} : \widehat{\boldsymbol{Q}}_\mathrm{t} \times \widehat{\boldsymbol{Q}}_\mathrm{t} \to \mathbb{C}$ turns out to be an F–smooth map. □

6.7 Schrödinger connection

Eventually, we show that the *Schrödinger operator* can be regarded in a natural way as the *covariant differential operator* with respect to an *F–smooth connection* of the F–smooth fibred space $\widehat{t} : \widehat{\boldsymbol{Q}} \to \boldsymbol{T}$, as an application of the general theory of *F–smooth connections in the linear case* discussed in Section §5.2.

Indeed, it is remarkable that the spacelike component of the Schrödinger operators behave as the symbol of this "infinite dimensional connection".

Lemma 6.7.1. Each spacelike differential operator

$$\mathrm{O} : \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \boldsymbol{Q})$$

factorises through a spacelike differential operator (denoted by the same symbol)

$$\mathsf{O} : \mathrm{reg}(\boldsymbol{E}, \boldsymbol{Q}) \to \mathrm{reg}(\boldsymbol{E}, \boldsymbol{Q}) \,,$$

according to the following commutative diagram

$$\begin{array}{ccc} \mathrm{sec}(\boldsymbol{E}, \boldsymbol{Q}) & \xrightarrow{\ \mathsf{O}\ } & \mathrm{sec}(\boldsymbol{E}, \boldsymbol{Q}) \\ \cup \big\uparrow & & \big\uparrow \cup \\ \mathrm{reg}(\boldsymbol{E}, \boldsymbol{Q}) & \xrightarrow{\ \mathsf{O}\ } & \mathrm{reg}(\boldsymbol{E}, \boldsymbol{Q}) \end{array} \quad . \square$$

Lemma 6.7.2. In virtue of the above Lemma 6.7.1, each spacelike quantum differential operator

$$\mathsf{O} : \mathrm{sec}(\boldsymbol{E}, \boldsymbol{Q}) \to \mathrm{sec}(\boldsymbol{E}, \boldsymbol{Q}) \,,$$

yields the differential operator

$$\widehat{\mathsf{O}} : \mathrm{sec}(\boldsymbol{T}, \widehat{\boldsymbol{Q}}) \to \mathrm{sec}(\boldsymbol{T}, \widehat{\boldsymbol{Q}}) : \widehat{\Psi} \mapsto \widehat{\mathsf{O}}(\widehat{\Psi}) := \widehat{\mathsf{O}(\Psi)} \,,$$

hence the F–smooth fibred morphism over $\boldsymbol{T}$ (denoted by the same symbol)

$$\widehat{\mathsf{O}} : \widehat{\boldsymbol{Q}} \to \widehat{\boldsymbol{Q}} \,,$$

uniquely defined by the equality

$$\widehat{\mathsf{O}}\big(\widehat{\Psi}_{\mathrm{t}}\big) = \widehat{\mathsf{O}_{\mathrm{t}}(\Psi_{\mathrm{t}})} \,, \qquad \text{where} \qquad \Psi_t \in \mathrm{reg}(\boldsymbol{E}_{\mathrm{t}}, \boldsymbol{Q}_{\mathrm{t}}) \,,$$

for each $\widehat{\Psi}_{\mathrm{t}} \in \widehat{\boldsymbol{Q}}_{\mathrm{t}}$, with $\mathrm{t} \in \boldsymbol{T}$.

Proof. The proof follows from the bijection $\mathrm{reg}(\boldsymbol{E}, \boldsymbol{Q}) \to \mathrm{sec}(\boldsymbol{T}, \widehat{\boldsymbol{Q}})$. QED

Lemma 6.7.3. The Schrödinger operator

$$\mathsf{S} : \mathrm{sec}(\boldsymbol{E}, \boldsymbol{Q}) \to \mathrm{sec}(\boldsymbol{E}, \mathbb{T}^* \otimes \boldsymbol{Q})$$

can be regarded as an F–smooth operator

$$\widehat{\mathsf{S}} : \mathrm{sec}(\boldsymbol{T}, \widehat{\boldsymbol{Q}}) \to \mathrm{sec}(\boldsymbol{T}, \, \mathbb{T}^* \otimes \widehat{\boldsymbol{Q}}) \,. \square$$

Proposition 6.7.1. The Schrödinger operator

$$\widehat{\mathsf{S}} : \mathrm{sec}(\boldsymbol{T}, \widehat{\boldsymbol{Q}}) \to \mathrm{sec}(\boldsymbol{T}, \, \mathbb{T}^* \otimes \widehat{\boldsymbol{Q}})$$

can be uniquely regarded as the covariant differential

$$\nabla[\text{Ш}] : \sec(\boldsymbol{T}, \widehat{\boldsymbol{Q}}) \to \sec(\boldsymbol{T}, \mathbb{T}^* \otimes \widehat{\boldsymbol{Q}})$$

associated with an F–smooth linear connection

$$\text{Ш} : \widehat{\boldsymbol{Q}} \to \mathbb{T}^* \otimes T\widehat{\boldsymbol{Q}} \,.$$

Thus, in the present infinite dimensional framework, the spacelike differential operator acting of spacetime complex functions

$$\psi \mapsto -\tfrac{1}{2}\,\mathfrak{i}\,G_0^{ij}\,\partial_{ij}\psi - (A_0^j + \tfrac{1}{2}\,\mathfrak{i}\,\frac{\partial_i(G_0^{ij}\,\sqrt{|g|})}{\sqrt{|g|}})\,\partial_j\psi$$
$$+\tfrac{1}{2}\,\big(\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i(A_0^i\,\sqrt{|g|})}{\sqrt{|g|}} - \mathfrak{i}\;\alpha_0\big)\,\psi$$

is analogue to the components of standard connections in a finite dimensional framework. □

PROOF. It follows from Proposition 5.2.1. QED

List of Symbols

Introduction

$\mathrm{Map}(\boldsymbol{M},\boldsymbol{N})$ set of global smooth maps $f:\boldsymbol{M}\to\boldsymbol{N}$ §Introduction
$\mathrm{Sec}(\boldsymbol{B},\boldsymbol{F})$ set of global smooth sections $s:\boldsymbol{B}\to\boldsymbol{F}$ §Introduction
$\mathrm{sec}(\boldsymbol{B},\boldsymbol{F})$ sheaf of local smooth sections $s:\boldsymbol{B}\to\boldsymbol{F}$ §Introduction

Smooth manifolds and F–smooth spaces

$(x^i):\boldsymbol{M}\to\mathbb{R}^m$ smooth chart of a smooth manifold Def 1.1.1
$(\boldsymbol{S},\mathcal{C})$ F–smooth space Def 1.2.1
$c:\boldsymbol{I}_c\to\boldsymbol{S}$ basic curve Def 1.2.1

Systems of maps

$(\boldsymbol{S},\epsilon)$ smooth system of smooth maps Def 2.1.1
$\epsilon:\boldsymbol{S}\times\boldsymbol{M}\to\boldsymbol{N}$ evaluation map Def 2.1.1
$\epsilon_{\boldsymbol{S}}:\boldsymbol{S}\to\mathrm{Map}(\boldsymbol{M},\boldsymbol{N}):s\mapsto\breve{s}$ induced map Def 2.1.1
$(\epsilon_{\boldsymbol{S}})^{-1}:\mathrm{Map}_{\boldsymbol{S}}(\boldsymbol{M},\boldsymbol{N})\to\boldsymbol{S}:f\mapsto\widehat{f}$ inverse map Def 2.1.1
$\breve{s}:\boldsymbol{M}\to\boldsymbol{N}:m\mapsto\epsilon(s,m)$ selected map Def 2.1.1
$\mathrm{Map}_{\boldsymbol{S}}(\boldsymbol{M},\boldsymbol{N}):=\epsilon_{\boldsymbol{S}}(\boldsymbol{S})\subset\mathrm{Map}(\boldsymbol{M},\boldsymbol{N})$ subset of selected maps Def 2.1.1
$T\epsilon:T\boldsymbol{S}\times T\boldsymbol{M}\to T\boldsymbol{N}$ tangent prolongation Pro 2.1.1
$T_1\epsilon:T\boldsymbol{S}\times\boldsymbol{M}\to T\boldsymbol{N}$ tangent prolongation Pro 2.1.1
$T_2\epsilon:\boldsymbol{S}\times T\boldsymbol{M}\to T\boldsymbol{N}$ tangent prolongation Pro 2.1.1
$(\boldsymbol{S},\epsilon)$ F–smooth system of smooth maps Def 2.2.1
$\epsilon:\boldsymbol{S}\times\boldsymbol{M}\to\boldsymbol{N}$ evaluation map Def 2.2.1
$\epsilon_{\boldsymbol{S}}:\boldsymbol{S}\to\mathrm{Map}(\boldsymbol{M},\boldsymbol{N}):s\mapsto\breve{s}$ induced map Def 2.2.1
$(\epsilon_{\boldsymbol{S}})^{-1}:\mathrm{Map}_{\boldsymbol{S}}(\boldsymbol{M},\boldsymbol{N})\to\boldsymbol{S}:f\mapsto\widehat{f}$ inverse map Def 2.2.1
$\breve{s}:\boldsymbol{M}\to\boldsymbol{N}:m\mapsto\epsilon(s,m)$ selected map Def 2.2.1
$\mathrm{Map}_{\boldsymbol{S}}(\boldsymbol{M},\boldsymbol{N}):=\epsilon_{\boldsymbol{S}}(\boldsymbol{S})\subset\mathrm{Map}(\boldsymbol{M},\boldsymbol{N})$ subset of selected maps Def 2.2.1
$c^*(\epsilon):\boldsymbol{I}_c\times\boldsymbol{M}\to\boldsymbol{N}:(\lambda,m)\mapsto\epsilon\big(c(\lambda),m\big)$ pullback of a curve The 2.2.1
$\mathsf{X}_s:=\big[(\widehat{c}_s,\lambda)\big]_\sim$ tangent vector Def 2.2.3
$\Xi_s:=\big(T_1(\widehat{c}^*(\epsilon))\big)_{|(\lambda,1)}:\boldsymbol{M}\to T\boldsymbol{N}$ representation of tangent vector The 2.2.2
$\overline{\Xi}_s:T\boldsymbol{M}\to T\boldsymbol{N}$ representation of tangent vector Cor 2.2.2
$\tau_{\boldsymbol{S}}:\mathsf{T}\boldsymbol{S}\to\boldsymbol{S}:\Xi_s\mapsto s$ natural projection Lem 2.2.1
$\mathsf{T}_1\epsilon:\mathsf{T}\boldsymbol{S}\times\boldsymbol{M}\to T\boldsymbol{N}:(\Xi_s,m)\mapsto\Xi_s(m)$ natural evaluation map Lem 2.2.1

Systems of sections

$\mathrm{tub}(\boldsymbol{F},\boldsymbol{G})\subset\mathrm{sec}(\boldsymbol{F},\boldsymbol{G})$ subsheaf of tubelike sections Def 3.1.2

$(\boldsymbol{S},\zeta,\epsilon)$ smooth system of smooth sections Def 3.1.3
$\zeta:\boldsymbol{S}\to\boldsymbol{B}$ smooth projection Def 3.1.3
$\epsilon:\boldsymbol{S}\underset{\boldsymbol{B}}{\times}\boldsymbol{F}\to\boldsymbol{G}$ smooth evaluation map Def 3.1.3
$\epsilon_{\boldsymbol{S}}:\mathrm{sec}(\boldsymbol{B},\boldsymbol{S})\to\mathrm{tub}(\boldsymbol{F},\boldsymbol{G}):\sigma\mapsto\breve{\sigma}$ sheaf morphism Def 3.1.3
$(\epsilon_{\boldsymbol{S}})^{-1}:\mathrm{tub}_{\boldsymbol{S}}(\boldsymbol{F},\boldsymbol{G})\to\mathrm{sec}(\boldsymbol{B},\boldsymbol{S}):\phi\mapsto\widehat{\phi}$ inverse map Def 3.1.3
$\breve{\sigma}:\boldsymbol{F}\to\boldsymbol{G}:f_b\mapsto\epsilon(\sigma(b),f_b)$ selected section Def 3.1.3
$\mathrm{tub}_{\boldsymbol{S}}(\boldsymbol{F},\boldsymbol{G}):=\epsilon_{\boldsymbol{S}}\big(\mathrm{sec}(\boldsymbol{B},\boldsymbol{S})\big)\subset\mathrm{tub}(\boldsymbol{F},\boldsymbol{G})$ subset of selected sections Def 3.1.3
$\boldsymbol{F}^{\uparrow}:=\boldsymbol{S}\underset{\boldsymbol{B}}{\times}\boldsymbol{F}$ lifted fibred manifold Def 3.1.4
$p^{\uparrow}:\boldsymbol{F}^{\uparrow}\to\boldsymbol{S}:(s_b,f_b)\to s_b$ natural projection Def 3.1.4
$\epsilon:\boldsymbol{F}^{\uparrow}\to\boldsymbol{G}$ evaluation map Def 3.1.4
$\underline{\mathrm{tub}}(\boldsymbol{F},\boldsymbol{G})\subset\big\{s:\boldsymbol{F}\to\boldsymbol{G}\big\}$ sheaf of fibrewisely smooth tubelike sections Def 3.2.1
$\mathrm{tub}(\boldsymbol{F},\boldsymbol{G})\subset\underline{\mathrm{tub}}(\boldsymbol{F},\boldsymbol{G})$ subsheaf of smooth tubelike sections Def 3.2.1
$(\boldsymbol{S},\zeta,\epsilon)$ F–smooth system of fibrewisely smooth sections Def 3.2.2
$\zeta:\boldsymbol{S}\to\boldsymbol{B}$ smooth projection Def 3.2.2
$\epsilon:\boldsymbol{S}\underset{\boldsymbol{B}}{\times}\boldsymbol{F}\to\boldsymbol{G}$ smooth evaluation map Def 3.2.2
$\epsilon_{\boldsymbol{S}}:\underline{\mathrm{sec}}(\boldsymbol{B},\boldsymbol{S})\to\underline{\mathrm{tub}}(\boldsymbol{F},\boldsymbol{G}):\sigma\mapsto\breve{\sigma}$ sheaf morphism Def 3.2.2
$(\epsilon_{\boldsymbol{S}})^{-1}:\underline{\mathrm{tub}}_{\boldsymbol{S}}(\boldsymbol{F},\boldsymbol{G})\to\underline{\mathrm{sec}}(\boldsymbol{B},\boldsymbol{S}):\phi\mapsto\widehat{\phi}$ inverse map Def 3.2.2
$\breve{\sigma}:\boldsymbol{F}_b\to\boldsymbol{G}_b:f_b\mapsto\epsilon\big(\sigma(b),f_b\big)$ selected section Def 3.2.2
$\mathrm{tub}_{\boldsymbol{S}}(\boldsymbol{F},\boldsymbol{G}):=\epsilon_{\boldsymbol{S}}\big(\mathrm{sec}(\boldsymbol{B},\boldsymbol{S})\big)\subset\mathrm{tub}(\boldsymbol{F},\boldsymbol{G})$ subset of selected sections Def 3.2.2
$c^*(\boldsymbol{F}):=\{(\lambda,f)\in\boldsymbol{I}_c\times\boldsymbol{F}\mid c(\lambda)=p(f)\}\subset\boldsymbol{I}_c\times\boldsymbol{F}$ pullback space Lem 3.2.1
$c^*(p):c^*(\boldsymbol{F})\to\boldsymbol{I}_c:(\lambda,f)\mapsto\lambda$ pullback map Lem 3.2.1
$c^*_{\boldsymbol{F}}:c^*(\boldsymbol{F})\to\boldsymbol{F}:(\lambda,f)\mapsto f$ pullback map Lem 3.2.1
$\widehat{c}^*(\epsilon):c^*(\boldsymbol{F})\to\boldsymbol{G}:(\lambda,f)\mapsto\epsilon\big(\widehat{c}(\lambda),f\big)$ pullback map Lem 3.2.3
$c^*(\breve{\sigma}):c^*(\boldsymbol{F})\to\boldsymbol{G}$ pullback section Lem 3.2.5
$T\big(\widehat{c}^*(\epsilon)\big):T\big(c^*(\boldsymbol{F})\big)\to T\boldsymbol{G}$ tangent map Lem 3.2.7
$\mathsf{X}_s:=\big[(\widehat{c},\lambda)\big]_{\sim}$ tangent vector Def 3.2.4
$\Xi_u:(T\boldsymbol{F})_u\to(T\boldsymbol{G})_u$ representative of a tangent vector The 3.2.3
$\tau_{\boldsymbol{S}}:\mathsf{T}\boldsymbol{S}\to\boldsymbol{S}:\mathsf{X}_s\mapsto s$ projection Pro 3.2.4
$\mathsf{T}\zeta:\mathsf{T}\boldsymbol{S}\to T\boldsymbol{B}$ projection Pro 3.2.4
$\mathsf{V}\boldsymbol{S}:=(\mathsf{T}\zeta)^{-1}(0)\subset\mathsf{T}\boldsymbol{S}$ vertical subspace Pro 3.2.5
$\mathsf{T}\sigma:T\boldsymbol{B}\to\mathsf{T}\boldsymbol{S}$ tangent prolongation of a section Def 3.2.5
$\mathcal{D}:\mathrm{tub}(\boldsymbol{F},\boldsymbol{G})\to\mathrm{tub}(\boldsymbol{F},\acute{\boldsymbol{G}})$ differential operator Def 3.2.6
$\mathcal{D}:\underline{\mathrm{tub}}(\boldsymbol{F},\boldsymbol{G})\to\underline{\mathrm{tub}}(\boldsymbol{F},\acute{\boldsymbol{G}})$ compatible differential operator Def 3.2.6
$\widehat{\mathcal{D}}:\mathrm{F\text{–}sec}(\boldsymbol{B},\boldsymbol{S})\to\mathrm{F\text{–}sec}(\boldsymbol{B},\acute{\boldsymbol{S}}):\widehat{\sigma}\mapsto\widehat{\mathcal{D}(\sigma)}$ F–smooth differential operator Pro 3.2.7

Systems of connections

$\mathrm{cns\,tub}(\boldsymbol{F},T^*\boldsymbol{B}\otimes T\boldsymbol{F})\subset\mathrm{tub}(\boldsymbol{F},T^*\boldsymbol{B}\otimes T\boldsymbol{F})$ subsheaf of smooth tubelike connections $4.1.1
$(\boldsymbol{C},\zeta,\epsilon)$ smooth system of smooth connections Der 4.1.1
$\zeta:\boldsymbol{C}\to\boldsymbol{B}$ projection Def 4.1.1
$\epsilon:\boldsymbol{C}\underset{\boldsymbol{B}}{\times}\boldsymbol{F}\to T^*\boldsymbol{B}\otimes T\boldsymbol{F}$ evaluation map Def 4.1.1
$\epsilon_{\boldsymbol{C}}:\mathrm{sec}(\boldsymbol{B},\boldsymbol{C})\to\mathrm{cns\,tub}(\boldsymbol{F},T^*\boldsymbol{B}\otimes T\boldsymbol{F}):\gamma\mapsto\breve{\gamma}$ sheaf morphism Def 4.1.1
$(\epsilon_{\boldsymbol{C}})^{-1}:\mathrm{cns\,tub}_{\boldsymbol{C}}(\boldsymbol{F},T^*\boldsymbol{B}\otimes T\boldsymbol{F})\to\mathrm{sec}(\boldsymbol{B},\boldsymbol{C}):c\mapsto\widehat{c}$ inverse sheaf morphism Def 4.1.1
$c^{\uparrow}:\boldsymbol{F}^{\uparrow}\underset{\boldsymbol{C}}{\times}T\boldsymbol{C}\to T\boldsymbol{F}^{\uparrow}$ upper connection Def 4.1.2

$\operatorname{cns}\underline{\operatorname{tub}}(\boldsymbol{F}, T^*\boldsymbol{B} \otimes T\boldsymbol{F}) \subset \{c : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F}\}$ subsheaf of fibrewisely smooth connections Def 4.2.1

$\operatorname{cns}\operatorname{tub}(\boldsymbol{F}, T^*\boldsymbol{B} \otimes T\boldsymbol{F}) \subset \operatorname{cns}\underline{\operatorname{tub}}(\boldsymbol{F}, T^*\boldsymbol{B} \otimes T\boldsymbol{F})$ subsheaf of smooth connections Def 4.2.1

$(\boldsymbol{C}, \zeta, \epsilon)$ system of fibrewisely smooth connections Def 4.2.2

$\zeta : \boldsymbol{C} \to \boldsymbol{B}$ surjective map Def 4.2.2

$\epsilon : \boldsymbol{C} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F}$ evaluation map Def 4.2.2

$\underline{\operatorname{sec}}(\boldsymbol{B}, \boldsymbol{C}) \subset \{\gamma : \boldsymbol{B} \to \boldsymbol{C}\}$ subsheaf of sections Def 4.2.2

F–smooth connections

$\mathsf{K} : \boldsymbol{S} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} \to \mathsf{T}\boldsymbol{S}$ F–smooth connections Def 5.1.1

$\mathcal{D} \equiv \mathcal{D}[\mathsf{K}] : \operatorname{tub}_{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{G}) \to \operatorname{tub}(\boldsymbol{F}, T^*\boldsymbol{B} \otimes \boldsymbol{G}) : \phi \mapsto \mathcal{D}\phi$ differential operator Pro 5.2.1

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