Radiative and non-perturbative corrections to the electron mass and the anomalous magnetic moment in the presence of an external magnetic field of arbitrary strength

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Using the Ritus eingenfunction method we compute corrections to the electron mass m_0 in the presence of a moderate magnetic field $eB \sim m_0^2$. From this we obtain an expression for the anomalous magnetic moment near the critic field. For this we solved numerically the Schwinger-Dyson equations in the rainbow approximation including all Landau levels without make any assumption respect to the field strength.

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I. INTRODUCTION

Radiative corrections to the electron self-energy due to its interaction with an external magnetic field have been widely considered in the literature [1, 2, 3, 4, 5, 6, 7, 8]. The exact form of how the ground state energy of an electron may be shifted from E_0 in the presence of an magnetic field was a source of misleading conclusions [3, 9, 10]. Assuming that this effects can be calculated by adding to the Dirac Hamiltonian a term $\Delta\mu\sigma \cdot \mathbf{B}$, where $\Delta\mu$ is the Schwinger anomalous magnetic moment, it was founded that the ground-state energy of an electron is [3, 11, 12]

$$E_0 = m_0 c^2 |1 - \frac{\alpha}{4\pi} \frac{e\hbar B}{m_0^2 c^3}|, \tag{1}$$

where m_0 is the electron mass without magnetic fields. For $B \sim 10^{16}$Gauss this formula implies that the ground-state energy of an electron is close to zero, a fact that could have dramatical astrophysical and cosmological consequences [9]. However, it was soon realized that this result can only be true for magnetic field intensities that are weak compared with $B_c = m_0^2/e \sim 10^{13}$Gauss, because Eq (1) only represents the linear term in the power expansion in eB [3], which is not the dominant one for strong magnetic fields. In recent publications [8, 13] dynamical effects of a strong magnetic field $B >> m_0^2/e$ were calculated. These approaches, show an enhancement of the dynamical contribution to the dynamical mass. They claim that this is a new effect that could have important consequences. We confirmed this result with an independent calculation and show that this effect is well described by previous calculations which are based on the constant mass approximation [6]. Furthermore, we carry out an analogous calculation for a weak magnetic field $eB \lesssim m_0^2$, from this we obtain an expression for the anomalous magnetic moment near the critic field B_c. Recently this issue has been revived from a non-perturbative point of view for strong fields [14]. In our approach we have included higher Landau levels to allow magnetic field intensities of the same order or less than the electron mass using similar techniques to those used in the context of the magnetic catalysis [15, 16, 17, 18, 19, 20, 21], particularly we follow the last reference.

II. THE MASS FUNCTION

It has been shown [22] that the mass operator in the presence of an electromagnetic field can be written as a combination of the structures

$$\gamma^\mu \Pi_\mu \, , \, \sigma^{\mu\nu} F_{\mu\nu} \, , \, (F_{\mu\nu}\Pi^\nu)^2 \, , \, \gamma_5 F_{\mu\nu} \tilde{F}^{\mu\nu} \tag{2}$$

which commute with the operator $(\gamma \cdot \Pi)^2$, where $\Pi_\mu = i\partial_\mu - eA_\mu^{\text{ext}}$, $F_{\mu\nu} = \partial_\mu A_\nu^{\text{ext}} - \partial_\nu A_\mu^{\text{ext}}$, $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\tau} F_{\lambda\tau}$, $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$ and A^{ext} is the external vector potential. We take $A_\mu^{\text{ext}} = B(0, -y/2, x/2, 0)$, such that the magnetic field is $\mathbf{B} = B\hat{\mathbf{z}}$.

The equation relating the two-point fermion Green's function $G(x,y)$ and the mass operator $M(x,y)$ in coordinate space reads

$$\gamma \cdot \Pi(x) G(x,y) - \int d^4x' M(x,x') G(x',y) = \delta^4(x-y), \tag{3}$$

where the mass function in the rainbow approximation in coordinate space is [23]

$$\begin{aligned} M(x,x') =& m_0 \delta^4(x-x') 1_{1\times 1} \\ & - ie^2 \gamma^\mu G(x,x') \gamma^\nu D_{\mu\nu}^{(0)}(x-x') \, , \end{aligned} \tag{4}$$

where $1_{4\times 4}$ is the 4×4 spinorial identity matrix. In this work, we are interested in exploring the behavior of the mass function for arbitrary values of the magnetic field strength, thus we work near the Landau gauge [24] ($\xi \sim 0$) to compare with the well known results in the weak and strong magnetic field limit.

Schwinger [23] was the first to obtain an exact analytical expression for the fermion Green's function in the presence of a constant electromagnetic field of arbitrary strength. However, in the presence of a constant external field, the fermion asymptotic states are no longer free particle states, but instead are described by eigenfunctions of the operator $(\gamma^\mu \Pi_\mu)^2$. Thus, the alternative representation of $G(x,y)$ proposed by Ritus [22] is more convenient for our purposes, since there the mass operator is diagonal. A matrix constructed out of these eigenfunctions is used to *rotate* the SDE (4) to momentum space, yielding [20]

$$\mathrm{tr}[\Pi(n_p)]\left(\mathcal{M}[p_\parallel,n_p]-m_0\right)=-2ie^2\sum_{\sigma_k,\sigma_p=\pm1}\sum_{n_k,s_k=0}^{\infty}\frac{s_k!(n_p-\frac{(\sigma_p+1)}{2})!}{s_p!(n_k-\frac{(\sigma_k+1)}{2})!}$$
$$\times\int\frac{d^4q}{(2\pi)^4}\frac{e^{-q_\perp^2/2\gamma}}{q^2+i\epsilon}\frac{\mathcal{M}[(p-q)_\parallel,n_k]}{(p-q)_\parallel^2-2eBn_k-\mathcal{M}^2[(p-q)_\parallel,n_k]}$$
$$\times\left(\frac{q_\perp^2}{4\gamma}\right)^{l_k-l_p-\frac{(\sigma_k-\sigma_p)}{2}}\left[2+\frac{(1-\xi)}{q^2}\left(q_\perp^2(1-\delta_{\sigma_p\sigma_k})-q_\parallel^2\delta_{\sigma_p\sigma_k}\right)\right]\left[L_{n_p-\frac{(\sigma_p+1)}{2}}^{n_k-n_p-\frac{(\sigma_k-\sigma_p)}{2}}(q_\perp^2/4\gamma)\right]^2\left[L_{s_k}^{s_p-s_k}(q_\perp^2/4\gamma)\right]^2 \tag{5}$$

where $q_\perp=(0,q_1,q_2,0)$, $q_\parallel=(q_0,0,0,q_3)$, $\gamma=eB/2$, $p^2=E_p^2-p_z^2-2eBn$ and L_n^m are Laguerre functions and

$$\Pi(n_p)=\begin{cases}1, & \text{if } n_p\neq0\\ \Delta(\sigma=1), & \text{if } n_p=0,\end{cases},$$

with $\Delta(\sigma)=\frac{1}{2}(1+i\sigma\gamma^1\gamma^2)$. This factor comes from the normalization [17]

$$\int d^4x\bar{\Psi}_p(x)\Psi_{p'}(x)=\delta_{n_pn_{p'}}\delta_{s_ps_{p'}}\delta^2(p_\parallel-p'_\parallel)\Pi(n_p), \tag{6}$$

and bring out a factor of two in the left-hand side of Eq. (5) for $n_p=0$. In vacuum, working near the Landau gauge, we know that the wave function renormalization equals one. We assume that this is the case also in the presence of a magnetic field of arbitrary strength. Accordingly, to get Eq. (5) we have worked with the ansatz $\Sigma(\bar{p})\sim\mathcal{M}(\bar{p})1_{4\times4}$, where $\Sigma(\bar{p})$ is defined by

$$\int d^4xd^4x'\bar{\Psi}_p(x)M(x,x')\Psi_{p'}(x')$$
$$=\delta_{n_pn_{p'}}\delta_{s_ps_{p'}}\delta^2(p_\parallel-p'_\parallel)\Pi(n_p)\Sigma(\bar{p}). \tag{7}$$

This assumption is good when considering a small external momentum and $n_p=0$ [14, 16, 20, 21]. Since the energy only depends on the principal quantum number n_k, we expect that $\mathcal{M}((p-q)_\parallel,n_k)$ should be independent of s_k. We also assume that $\mathcal{M}((p-q)_\parallel,n_k)$ is a slowly varying function of n_k and thus make the approximation

$$\mathcal{M}((p-q)_\parallel,n_k)\sim\mathcal{M}((p-q)_\parallel,n_k=0). \tag{8}$$

Hereafter, we employ the more convenient notation $\mathcal{M}(k_\parallel,n_k=0)\equiv\mathcal{M}(k_\parallel)$ for generic arguments of the mass function. With these considerations, the sum over s_k can be computed by using the identity

$$L_s^l=(-1)^lx^{-l}\frac{(s+l)!}{s!}L_{s+l}^{-l}, \tag{9}$$

and the Eq 5.11.5.1 in Ref. [25], namely

$$\sum_{k=0}^{\infty}A^kL_k^{m-k}(x)L_n^{k-n}(y)=A^m\left(\frac{1+A}{A}\right)^{m-n}e^{-xA}$$
$$\times L_{m-n}^0(x+y+xA+\frac{y}{A}), \tag{10}$$

yielding

$$\mathrm{tr}[\Pi(n_p)]\left(\mathcal{M}(p)-m_0\right)=-2ie^2\sum_{\sigma_p,\sigma_k=\pm1}\sum_{k=0}^{\infty}\int\frac{d^4q}{(2\pi)^4}$$
$$\times\frac{e^{-\frac{q_\perp^2}{4\gamma}}}{q^2+i\varepsilon}\frac{\mathcal{M}((p-q)_\parallel)}{\left\{(q-p)_\parallel^2-2eB(k+(\sigma_k+1)/2)\right\}-\mathcal{M}^2}$$
$$\times\left\{2+(1-\xi)(1-\delta_{\sigma_p\sigma_k})\frac{q_\perp^2}{q^2}-(1-\xi)\delta_{\sigma_p\sigma_k}\frac{q_\parallel^2}{q^2}\right\}$$
$$\times(-1)^{-m}(-1)^kL_m^{k-m}(\frac{q_\perp^2}{4\gamma})L_k^{-(k-m)}(\frac{q_\perp^2}{4\gamma}), \tag{11}$$

where $k=n_k-\frac{\sigma_k+1}{2}$ and $m=n_p-\frac{\sigma_p+1}{2}$. It is worth mentioning that after summing over s_k, the resulting equation is the same as Eq. (50) in Ref. [16] when considering the case $n_k=0$, which corresponds to the strong field limit. Under a Wick rotation we have

$$\frac{1}{(p-q)_\parallel^2-2eB(k+\frac{\sigma_k+1}{2})-\mathcal{M}^2}\longrightarrow$$
$$\frac{-1}{(p-q)_\parallel^2+2eB(k+\frac{\sigma_k+1}{2})+\mathcal{M}^2}$$
$$=\frac{-1}{2eB}\int_0^1dx\,x^{\left((p-q)_\parallel^2+2eB(k+\frac{\sigma_k+1}{2})+\mathcal{M}^2-2eB\right)/2eB}. \tag{12}$$

With this result the sum over k can be performed also by resorting to Eq. (10) yielding, after carrying out the sums over σ_k and σ_p,

$$\mathrm{tr}[\Pi(n_p)]\,(\mathcal{M}(p)-m_0) = +2e^2\int\frac{d^4Q}{(2\pi)^4}\frac{\mathcal{M}(\frac{p}{4\gamma}-Q)_{\parallel}}{Q^2}$$
$$\int_0^1 dx e^{-Q_\perp^2[1-x]}x^{\left[(2\sqrt{\gamma}Q-p)_\parallel^2+\mathcal{M}^2\right]\frac{1}{4\gamma}}$$
$$\times\left[\left\{2-(1-\xi)\frac{Q_\parallel^2}{Q^2}\right\}\left(x^{n_p}L^0_{n_p}+x^{n_p}L^0_{n_p-1}\right)\right.$$
$$\left.+\left\{2-(1-\xi)\frac{Q_\perp^2}{Q^2}\right\}\left(x^{(n_p+1)}L^0_{n_p}+x^{(n_p-1)}L^0_{n_p-1}\right)\right]. \tag{13}$$

where $Q=\frac{q}{2\sqrt{\gamma}}$ and the argument of the Laguerre functions is $4Q_\perp^2\sin\mathrm{Ln}(\frac{1}{x})$. For consistency with the assumption that the mass function is a slowly varying function of the principal quantum number Eq. (8), we take $n_p=0$. In this case, Eq. (13) gets simplified since $L^0_0=1$ and $L^0_{-1}=0$, so we get

$$\mathcal{M}(p) = m_0+e^2\int\frac{d^4Q}{(2\pi)^4}\frac{\mathcal{M}(\frac{p}{4\gamma}-Q)_\parallel}{Q^2}\int_0^1 dx e^{-Q_\perp^2(1-x)}\;x^{\left[(Q-p)_\parallel^2+\frac{\mathcal{M}^2}{2eB}-1\right]}\left[\left\{2-(1-\xi)\frac{Q_\parallel^2}{Q^2}\right\}+\left\{2-(1-\xi)\frac{Q_\perp^2}{Q^2}\right\}x\right]. \tag{14}$$

This result corresponds with the Eq.(8) in the reference [21]. Following [21] we get for arbitrary gauge fixing parameter the Eq. (15),

$$\mathcal{M}(y) = m_0+\frac{e^2}{4}\int_0^{\frac{\Lambda^2}{4\gamma}}\frac{dz}{(2\pi)^2}\mathcal{M}[z]\int_0^1 dx x^\lambda\Bigg\{$$
$$\times\left[\left\{2(1+x)-(1-\xi)\left(x+(1-x)^2z-y(1-x)^2+\frac{1}{2}yz(1-x)^3\right)\right\}\right.$$
$$\left.\times\, e^{y(1-x)}\Gamma[0,y(1-x)]+(1-\xi)(1-x)\left(\frac{z}{2y}-1+\frac{1}{2}z(1-x)\right)\right]\theta\,(y-z)+$$
$$\left[\left\{2(1+x)-(1-\xi)\left(x+\frac{1}{2}z^2(1-x)^3\right)\right\}\right.$$
$$\left.\times\, e^{z(1-x)}\Gamma[0,z(1-x)]+(1-\xi)(1-x)\left(-\frac{1}{2}+\frac{1}{2}z(1-x)\right)\right]\theta\,(z-y)\Bigg\}, \tag{15}$$

where $z=Q^2$, $y=p^2/4\gamma$ and $\lambda=z+\frac{\mathcal{M}^2[\vec{Q}_\parallel]-i\epsilon}{2eB}-1$ and $\Gamma(x,y)$ is the incomplete gamma function. The set of approximations leading to Eq. (15) are the same that in [21] which make no reference to the strength of the magnetic field, therefore this equation is valid for arbitrary magnetic field intensities. If we take, $\xi=1$, in Eq. (15) it reproduces Eq. (11) in [21] with $m_0=0$. For strong magnetic fields it will be useful to compare the solutions to this equation with the solutions tho the corresponding equation for the lowest Landau level (LLL)

$$\mathcal{M}(p_\parallel) = \frac{e^2}{2(2\pi)^2}\Bigg\{\int_0^{p^2}dq_\parallel^2\frac{\mathcal{M}_A(q_\parallel)}{q_\parallel^2+\mathcal{M}^2(q_\parallel)}e^{\frac{p_\parallel^2}{4\gamma}}\Gamma[0,\frac{p_\parallel^2}{4\gamma}]$$
$$+\int_{p^2}^\infty dq_\parallel^2\frac{\mathcal{M}(q_\parallel)}{q_\parallel^2+\mathcal{M}^2(q_\parallel)}e^{\frac{q_\parallel^2}{4\gamma}}\Gamma[0,\frac{q_\parallel^2}{4\gamma}]\Bigg\} \tag{16}$$

to calculate this expression we have used the same assumptions of the kernel softness respect to the momentum variables that in Eq. (15). We can find numerical solutions to the integral equation Eq. (15) for several magnetic field intensities making an appropriate choice of the cutoff in every range. For weak magnetic fields, $eB < m_0^2$, we take as cutoff the electron mass given that it corresponds to the highest momentum scale and make $\xi = 0$. For weak magnetic fields the dimensional reduction is missing and thus it is important to take into account the ultraviolet divergences since we are in a similar regime to the vacuum. This fact makes the numerical work harder and conceptually some care should be taken. There, contrary to the strong magnetic field case, the difference between the mass function and the bare electron mass m_0 is meaningless because it does not tend to zero when the magnetic field is turn off, leading to to a fictitious dynamical mass. We can make an rough estimate of this contribution by consider the QED one loop contribution to the self-energy in the perturbative case. In the Landau gauge we get

$$\delta m = m_{\text{phys}} - m_0 \sim \frac{3\alpha m_0}{4\pi} \ln \frac{\Lambda^2 + m_0^2}{m_0^2}|_{\Lambda=m_0} \sim 10^{-3} m_0, \tag{17}$$

which is at least two magnitude orders higher than the magnetic field contribution. To remove this additive constant is sufficient consider only differences between the mass function. Numerical results are shown in the fig. 1.

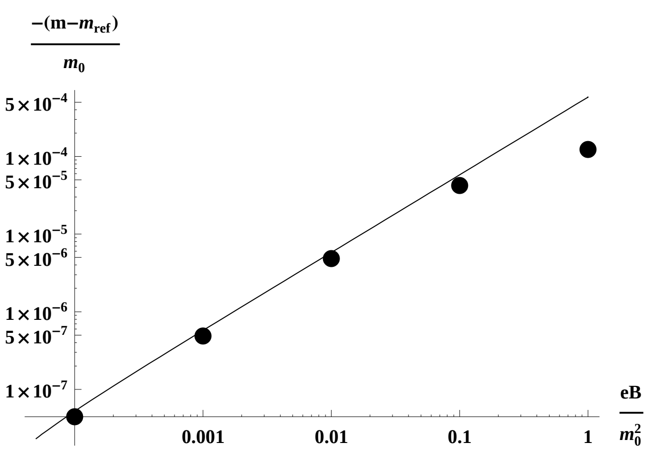

FIG. 1: Dynamical electron mass as a function of the magnetic field in units of m_0^2. The continuous line corresponds to the linear contribution Eq. (1). Dots correspond to the numerical solution including ALL Eq. (15). Here $m_{\text{ref}} = m(eB = 10^{-5} m_0^2) \sim 2.4 \times 10^{-3} m_0$

It is noticeable that the relation, Eq. (1), remains as the leading term even for field intensities close to $B_c = m_0^2/e$. This is an important result because, despite that the old formula Eq. (1) is very well established, to the best of our knowledge no validation of this result including higher Landau levels is known. Besides fig. 1 shows that the assumption $eB << m^2$ is no too restrictive. It is very well known that the anomalous magnetic moment of the electron $\Delta\mu = \frac{\alpha e_0 \hbar}{4\pi m_0 c}$ can be obtained from the real part of the mass operator which depends of the magnetic field [5, 14, 23, 26, 27, 28, 29]. In our approach we can read it from fig. 1 by use the formula $\Delta\mu = |\delta m(B)/B|$, where $\delta m(B)$ is the magnetic correction to the electron mass. We can see in fig. 2 that the anomalous magnetic moment is independent with the magnetic field up to magnetic fields intensities near $0.1 B_c$ in agreement with previous calculations which only include lowest Landau levels [5]. For magnetic fields near B_c the anomalous magnetic field is highly suppressed and the same concept of magnetic moment in this range should be revisited. In the fig. 2 there is an small gap between the analytical value and our numerical solution, this should only be a numerical error. For the strong regime we take

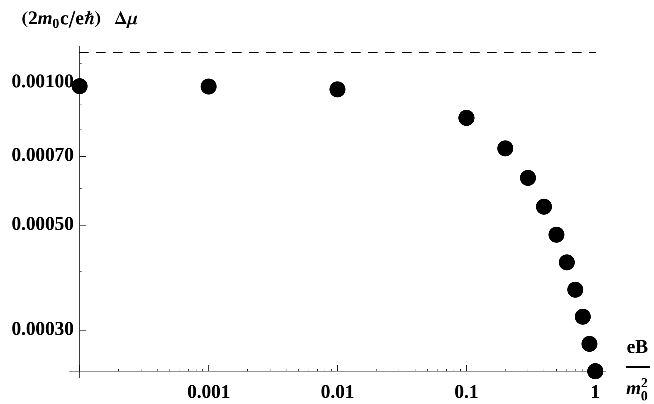

FIG. 2: Anomalous magnetic moment $\Delta\mu$ as a function of the magnetic field in units of m_0^2. Dots correspond to the anomalous magnetic moment $\Delta\mu$ taking into account all Landau levels. Horizontal dashed line corresponds to Schwinger classical result $\frac{2m_0 c \Delta\mu}{e_0 \hbar} = \frac{\alpha}{2\pi}$

$\Lambda = \sqrt{eB}$ and calculate the solutions to Eq. (15) for the interval, $m_0^2 < eB < 10^4 m_0^2$, which corresponds to magnetic fields in the phenomenological range . In this

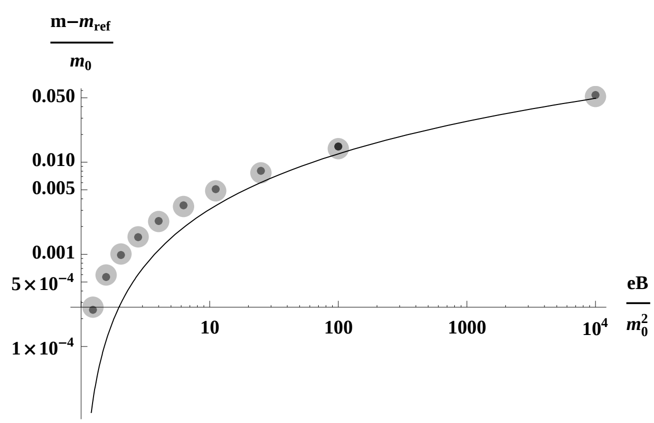

FIG. 3: Dynamical electron mass as a function of the magnetic field in units of m_0^2. The continuous line corresponds to the analytical solution of Gusynin and Smilga. Gray and Black dots correspond to the numerical solution including ALL and the LLL respectively. Here $m_{\text{ref}} = m(B = m_0^2/e)$

regime we make the calculation including higher Landau levels and compare with the LLL solution. The solutions for $eB/m_0^2 = 1$ for ALL and LLL are $m_{\rm ref} = 2.2\times10^{-3}m_0$ and $m_{\rm ref} = 1.4 \times 10^{-3}m_0$ respectively, shown that high Landau levels represent an important contributions at this field intensities. However if, as in the weak magnetic case, we only consider mass differences the results are indistinguishable as is shown in fig. 3. This lead us to conclude that for this regime of coupling, the main effect of higher Landau levels is to include vacuum contributions without changing significantly the dependence of the mass with the magnetic field. For higher magnetic

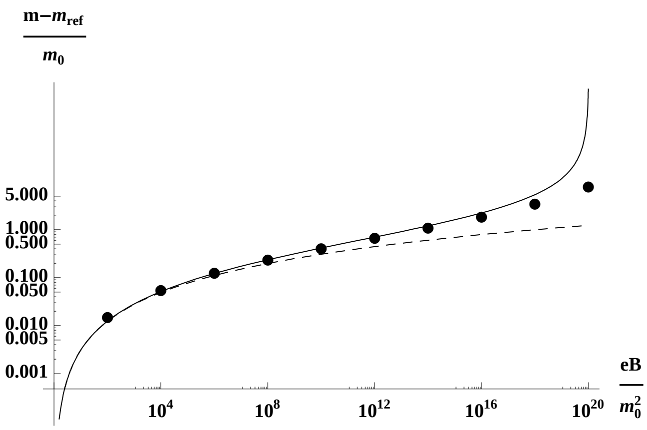

FIG. 4: Dynamical electron mass as a function of the magnetic field in units of m_0^2. The continuous line corresponds to the non-perturbative analytical solution of Gusynin and Smilga. Black dots correspond to the numerical solution to Eq. (16) and the dashed line corresponds to perturbative solution in the Ref. [3]. Here $m_{\rm ref} = m(B = m_0^2/e)$

fields we recover the agreement between the lowest Landau level solution and the complete solution. Given that there are not important contributions by including higher Landau levels for strong magnetic field, we calculate the enhancement of the electron mass and compare the solutions with the previous solutions founded in the literature [6]. From the fig. 4 we can conclude that the dynamical effects are well described by the perturbative solutions which does not take into account the mass dependence with the momenta. We note the excellent agreement between the analytical solution in reference [6] and our numerical solution. This result should not be surprising, since despite that in reference [6] they begin calculating perturbative corrections, at the end they get a self-consistent equation for the dynamical mass and hence non-perturbative effects are expected in their approach. In summary: we have shown that the Eq. (15) can reproduce perturbative as non-perturbative results for different ranges in the magnetic field intensity. This equation is easily solved with lower computational power, allowing to carry out calculations for magnetic fields relevant for the typical astrophysical conditions. To avoid misleading conclusions when dealing numerically with higher Landau levels, we have shown the importance of distinguishing vacuum contributions from magnetic field contributions. Numerically this can be easily implemented by only consider differences between the mass function for different magnetic field intensities. Furthermore, in this work we have shown that the magnetic independence of the anomalous magnetic moment is consistent up to magnetic field intensities near 10^{12}Gauss. At same time, our technique allows to make a precise description of the way in which this assumption is broken in the magnetic field interval $10^{12} - 10^{13}$Gauss. A natural extension of this work is to consider magnetic field intensities in the interval $10^0 - 10^8$Gauss. For this an improvement of our numerical techniques is needed. Due to highly precise measurements of the anomalous magnetic moment non-perturbative effects are strongly bounded. If we extrapolate our calculations to magnetic field intensities of 10^5Gauss, where some experiments currently take place, it is not completely clear that this contributions become less than one part in 10^{12} which represents the nowadays precision in the anomalous magnetic moment. This deserves a future investigation and can be an excellent test for non-perturbative techniques.

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