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Linear Parameter Estimation for Multi-Degree-of-Freedom Nonlinear Systems Using Nonlinear Output Frequency Response Functions

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Abstract: The Volterra series approach has been widely used for the analysis of nonlinear systems. Based on the Volterra series, a novel concept named Nonlinear Output Frequency Response Functions (NOFRFs) was proposed by the authors. This concept can be considered as an alternative extension of the classical frequency response function for linear systems to the nonlinear case. In this study, based on the NOFRFs, a novel algorithm is developed to estimate the linear stiffness and damping parameters of multi-degree-of-freedom (MDOF) nonlinear systems. The validity of this NOFRF based parameter estimation algorithm is demonstrated by numerical studies.

Nomenclature

$x(t)$, $u(t)$	the output and input of the nonlinear system
$X(j\omega)$, $U(j\omega)$	the spectrum of the system output and input
$h_n(\tau_1,\ldots,\tau_n)$	the n^{th} order Volterra kernel
$H_n(j\omega_1,\ldots,j\omega_n)$	the n^{th} order GFRF
$G_n(j\omega)$	the n^{th} order NOFRF
Ω_n	the frequency components of the n^{th} order output of the system subjected to harmonic inputs
Ω	the frequency components of the output of the system
$G_n^H(j\omega)$	the n^{th} order NOFRF of the system subjected to harmonic inputs
M, K, C	the system mass, damping and stiffness matrices
m_i, k_i, c_i	the i^{th} mass, damping and stiffness parameter
$S_{LS}(\Delta)$, $S_{LD}(\dot{\Delta})$	the restoring forces of the nonlinear spring and damper
r_i, w_i $(i=1,\ldots,P)$	the nonlinearity related parameters
$NonF$	the nonlinear force

$x_i(t)$, $X_i(j\omega)$	the displacement and the output frequency response of the i^{th} mass
$h_{(i,j)}(\tau_1,\ldots,\tau_j)$	the j^{th} order Volterra kernel associated to the i^{th} mass
$G_{(i,l)}(j\omega)$	the l^{th} order NOFRF associated to the i^{th} mass
$\lambda_n^{i,i+1}(j\omega)$	the ratio between the n^{th} NOFRFs of the i^{th} and $(i+1)^{\text{th}}$ masses
W	the vector of the unknown parameters to be estimated
$\Gamma_{(L-1,Z)}(j\omega)$	the term introduced by the nonlinear force *NonF* for the Z^{th} order NOFRF.

1 Introduction

Various methods have been developed to estimate the stiffness and damping parameters for linear structures or machines. Most of these are based on modal analysis techniques, which were essentially derived from the Frequency Response Functions (FRFs) [1]-[5]. To tackle the problem with finite element model updating, Arruda and Santos [1] estimated the mechanical parameters via curve fitting for measured frequency response functions using a non-linear least-squares method. Sunder and Ting [2] used the system parameter estimation method to detect the occurrence and location of damage on steel jacket offshore platforms. Also based on the FRFs, Hwang put forward an identification method for stiffness and damping parameters of connections using test data for a structure attached to another structure via connections [3]. Woodgate studied the problem of identifying a positive semi-definite symmetric stiffness matrix for a stable elastic structure from measurements of its displacement in response to some set of static loads [4]. Most recently, Živanović, Pavic and Reynolds [5] described a lively full-scale footbridge from its numerical modelling and dynamic testing. Their work is a successful application of the FRFs to system parameter estimation in practice.

However, there are certain types of qualitative behaviour, which cannot be produced by linear models [6], encountered in engineering, for example, the generation of harmonics and inter-modulation behaviours. In cases where these effects are dominant or significant nonlinear behaviours exist, nonlinear models are required to describe the system, and the linear FRFs are no longer suitable to investigate the system dynamics.

The Volterra series approach [7] is a powerful tool for the analysis of nonlinear systems, which extends the familiar concept of the convolution integral for linear systems to a series of multi-dimensional convolution integrals. The Fourier transforms of the Volterra kernels are known as the kernel transforms, Higher-order Frequency Response Functions (HFRFs) [8], or Generalised Frequency Response Functions (GFRFs), and these provide a convenient tool for analyzing nonlinear systems in the frequency domain. If a differential equation or discrete-time model is available for a system, the GFRFs can be determined using the algorithms in [9]~[11]. The GFRFs can be regarded as the extension of the classical frequency response function (FRF) of linear systems to the nonlinear case. So far only a few researchers have addressed the problem of nonlinear system parameter estimation for nonlinear systems using the GFRFs. Lee proposed a straightforward method to estimate the nonlinear system parameters using the GFRFs [12]. Khan and Vyas [13] employed the relationships between higher order GFRFs and first order GFRF to estimate the non-linear parameters. Later, Chatterjee and Vyas [14] further developed this method by using a method of recursive iteration.

In engineering practice, for many mechanical and structural systems, more than one coordinate is needed to sufficiently describe the system dynamics. The result is a MDOF model. In addition, there are considerable mechanical and structural systems that behave nonlinearly just because one or a few components within the system are nonlinear. One well known example is beam structures [15] with breathing cracks, the global nonlinear behaviours of which are caused only by the cracked elements. Such nonlinear MDOF systems can be regarded as locally nonlinear MDOF systems. An important fact is that, for such nonlinear systems, the linear stiffness and damping are still the decisive characteristics which mainly determine the system behaviour. Therefore, a knowledge of the linear stiffness and damping are still of great significance for understanding the whole system dynamical properties.

In this paper, a novel method is proposed to estimate the linear stiffness and damping parameters for locally nonlinear MDOF systems. The method is based on the concept of Nonlinear Output Frequency Response Functions [16], which was recently proposed by the authors and is an alternative extension of the FRF to the nonlinear case. NOFRFs are one dimensional functions of frequency. This allows the analysis of nonlinear systems to be implemented in a manner similar to the analysis of linear systems and provides great insight into the mechanisms which dominate many important nonlinear behaviours.

The paper is organized as follows. Section 2 provides a brief introduction to the new concept of NOFRFs. Some important properties of the NOFRFs for locally nonlinear MDOF systems, which were first revealed in the authors' recent studies [17], are given in Section 3. The NOFRF based algorithm for the linear stiffness and damping parameter estimation is presented in Section 4. In Section 5, a numerical study is used to verify the effectiveness of the presented algorithm. Finally conclusions are given in Section 6.

2. Nonlinear Output Frequency Response Functions

2.1 Nonlinear Output Frequency Response Functions under General Inputs

The definition of NOFRFs is based on the Volterra series theory of nonlinear systems. The Volterra series extends the well-known convolution integral description for linear systems to a series of multi-dimensional convolution integrals, which can be used to represent a wide class of nonlinear systems [8].

Consider the class of nonlinear systems which are stable at zero equilibrium and which can be described in the neighbourhood of the equilibrium by the Volterra series

$$x(t) = \sum_{n=1}^{N} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, ..., \tau_n) \prod_{i=1}^{n} u(t-\tau_i) d\tau_i \qquad (1)$$

where $x(t)$ and $u(t)$ are the output and input of the system, $h_n(\tau_1,...,\tau_n)$ is the n^{th} order Volterra kernel, and N denotes the maximum order of the system nonlinearity. Lang and Billings [8] derived an expression for the output frequency response of this class of nonlinear systems to a general input. The result is

$$\begin{cases} X(j\omega) = \sum_{n=1}^{N} X_n(j\omega) \quad \text{for } \forall \omega \\ X_n(j\omega) = \dfrac{1/\sqrt{n}}{(2\pi)^{n-1}} \int_{\omega_1+,...,+\omega_n=\omega} H_n(j\omega_1,...,j\omega_n) \prod_{i=1}^{n} U(j\omega_i) d\sigma_{n\omega} \end{cases} \qquad (2)$$

This expression reveals how nonlinear mechanisms operate on the input spectra to produce the system output frequency response. In (2), $X(j\omega)$ and $U(j\omega)$ are the spectrum of the system output and input respectively, $X_n(j\omega)$ represents the n^{th} order output frequency response of the system,

$$H_n(j\omega_1,...,j\omega_n) = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} h_n(\tau_1,...,\tau_n) e^{-(\omega_1\tau_1+,...,+\omega_n\tau_n)j} d\tau_1...d\tau_n \qquad (3)$$

is the n^{th} order Generalised Frequency Response Function (GFRF) [8], and

$$\int_{\omega_1+,...,+\omega_n=\omega} H_n(j\omega_1,...,j\omega_n) \prod_{i=1}^{n} U(j\omega_i) d\sigma_{n\omega}$$

denotes the integration of $H_n(j\omega_1,...,j\omega_n)\prod_{i=1}^{n} U(j\omega_i)$ over the n-dimensional hyper-plane $\omega_1 + \cdots + \omega_n = \omega$. Equation (2) is a natural extension of the well-known linear relationship $X(j\omega) = H(j\omega)U(j\omega)$, where $H(j\omega)$ is the frequency response function, to the nonlinear case.

For linear systems, the possible output frequencies are the same as the frequencies in the input. For nonlinear systems described by equation (1), however, the relationship between the input and output frequencies is more complicated. Given the frequency range of an input, the output frequencies of system (1) can be determined using the explicit expression derived by Lang and Billings in [8].

Based on the above results for the output frequency response of nonlinear systems, a new concept known as Nonlinear Output Frequency Response Function (NOFRF) was recently introduced by Lang and Billings [16]. The NOFRF is defined as

$$G_n(j\omega) = \frac{\int_{\omega_1+,\ldots,+\omega_n=\omega} H_n(j\omega_1,\ldots,j\omega_n)\prod_{i=1}^{n} U(j\omega_i)d\sigma_{n\omega}}{\int_{\omega_1+,\ldots,+\omega_n=\omega} \prod_{i=1}^{n} U(j\omega_i)d\sigma_{n\omega}} \tag{4}$$

under the condition that

$$U_n(j\omega) = \int_{\omega_1+,\ldots,+\omega_n=\omega} \prod_{i=1}^{n} U(j\omega_i)d\sigma_{n\omega} \neq 0 \tag{5}$$

Notice that $G_n(j\omega)$ is valid over the frequency range of $U_n(j\omega)$, which can be determined using the algorithm in [8].

By introducing the NOFRFs $G_n(j\omega)$, $n = 1,\cdots N$, equation (2) can be written as

$$X(j\omega) = \sum_{n=1}^{N} X_n(j\omega) \; = \sum_{n=1}^{N} G_n(j\omega)\; U_n(j\omega) \tag{6}$$

which is similar to the description of the output frequency response for linear systems. The NOFRFs reflect a combined contribution of the system and the input to the system output frequency response behaviour. It can be seen from equation (4) that $G_n(j\omega)$ depends not only on H_n (n=1,…,N) but also on the input $U(j\omega)$. For any structure, the dynamical properties are determined by the GFRFs H_n (n= 1,…,N). However, from equation (3) it can be seen that the GFRF is multidimensional [18][19], which makes the GFRFs difficult to measure, display and interpret in practice. Feijoo, Worden and Stanway [20][21] demonstrated that the Volterra series can be described by a series of associated linear equations (ALEs) whose corresponding associated frequency response functions (AFRFs) are easier to analyze and interpret than the GFRFs. According to equation (4), the NOFRF $G_n(j\omega)$ is a weighted sum of $H_n(j\omega_1,\ldots,j\omega_n)$ over $\omega_1 + \cdots + \omega_n = \omega$ with the weights depending on the test input. Therefore $G_n(j\omega)$ can be used as an alternative representation of the dynamical properties described by H_n. The most important property of the NOFRF $G_n(j\omega)$ is that it is one dimensional, and thus allows the analysis of nonlinear systems to be implemented in a convenient manner similar to the analysis of linear systems. Moreover, there is an effective algorithm [16] available which allows the estimation of the NOFRFs to be implemented directly using system input output data.

2.2 Nonlinear Output Frequency Response Functions under Harmonic Inputs

Harmonic inputs are pure sinusoidal signals which have been widely used for the dynamic testing of many engineering structures. Therefore, it is necessary to extend the NOFRF concept to the harmonic input case.

When system (1) is subject to a harmonic input

$$u(t) = A\cos(\omega_F t + \beta) \tag{7}$$

Lang and Billings [8] showed that equation (2) can be expressed as

$$X(j\omega) = \sum_{n=1}^{N} X_n(j\omega) = \sum_{n=1}^{N} \left(\frac{1}{2^n} \sum_{\omega_{k_1}+\cdots+\omega_{kn}=\omega} H_n(j\omega_{k_1},\cdots,j\omega_{k_n}) A(j\omega_{k_1})\cdots A(j\omega_{k_n}) \right) \tag{8}$$

where

$$A(j\omega_{k_i}) = \begin{cases} |A| e^{j\mathrm{sign}(k)\beta} & if \quad \omega_{k_i} \in \{k\omega_F, k = \pm 1\}, i = 1,\cdots,n \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

Defining the frequency components of the n^{th} order output of the system as Ω_n, then according to equation (8), the frequency components in the system output can be expressed as

$$\Omega = \bigcup_{n=1}^{N} \Omega_n \tag{10}$$

and Ω_n is determined by the set of frequencies

$$\{\omega = \omega_{k_1} + \cdots + \omega_{k_n} \mid \omega_{k_i} = \pm\omega_F, i = 1,\cdots,n\} \tag{11}$$

From equation (11), it is known that if all $\omega_{k_1},\cdots,\omega_{k_n}$ are taken as $-\omega_F$, then $\omega = -n\omega_F$. If k of these are taken as ω_F, then $\omega = (-n+2k)\omega_F$. The maximal k is n. Therefore the possible frequency components of $X_n(j\omega)$ are

$$\Omega_n = \{(-n+2k)\omega_F, k = 0,1,\cdots,n\} \tag{12}$$

Moreover, it is easy to deduce that

$$\Omega = \bigcup_{n=1}^{N} \Omega_n = \{k\omega_F, k = -N,\cdots,-1,0,1,\cdots,N\} \tag{13}$$

Equation (13) explains why some superharmonic components will be generated when a nonlinear system is subjected to a harmonic excitation. In the following, only those components with positive frequencies will be considered.

The NOFRFs defined in equation (4) can be extended to the case of harmonic inputs as

$$G_n^H(j\omega) = \frac{\frac{1}{2^n} \sum_{\omega_{k_1}+\cdots+\omega_{kn}=\omega} H_n(j\omega_{k_1},\cdots,j\omega_{k_n}) A(j\omega_{k_1})\cdots A(j\omega_{k_n})}{\frac{1}{2^n} \sum_{\omega_{k_1}+\cdots+\omega_{kn}=\omega} A(j\omega_{k_1})\cdots A(j\omega_{k_n})} \qquad n = 1,\ldots, N \tag{14}$$

under the condition that

$$A_n(j\omega) = \frac{1}{2^n} \sum_{\omega_{k_1}+\cdots+\omega_{kn}=\omega} A(j\omega_{k_1})\cdots A(j\omega_{k_n}) \neq 0 \tag{15}$$

Obviously, $G_n^H(j\omega)$ is only valid over Ω_n defined by equation (12). Consequently, the output spectrum $X(j\omega)$ of nonlinear systems under a harmonic input can be expressed as

$$X(j\omega) = \sum_{n=1}^{N} X_n(j\omega) = \sum_{n=1}^{N} G_n^H(j\omega)\, A_n(j\omega) \tag{16}$$

When k of the n frequencies of $\omega_{k_1},\cdots,\omega_{k_n}$ are taken as ω_F and the remainders are as $-\omega_F$, substituting equation (9) into equation (15) yields,

$$A_n(j(-n+2k)\omega_F) = \frac{1}{2^n} C_n^k\, |A|^n\, e^{j(-n+2k)\beta} \tag{17}$$

Thus $G_n^H(j\omega)$ becomes

$$G_n^H(j(-n+2k)\omega_F) = \frac{\frac{1}{2^n} H_n(\overbrace{j\omega_F,\cdots,j\omega_F}^{k},\overbrace{-j\omega_F,\cdots,-j\omega_F}^{n-k}) C_n^k\, |A|^n\, e^{j(-n+2k)\beta}}{\frac{1}{2^n} C_n^k\, |A|^n\, e^{j(-n+2k)\beta}}$$

$$= H_n(\overbrace{j\omega_F,\cdots,j\omega_F}^{k},\overbrace{-j\omega_F,\cdots,-j\omega_F}^{n-k}) \tag{18}$$

where $H_n(j\omega_1,...,j\omega_n)$ is assumed to be a symmetric function. Therefore, in this case, $G_n^H(j\omega)$ over the n^{th} order output frequency range $\Omega_n = \{(-n+2k)\omega_F, k=0,1,\cdots,n\}$ is equal to the GFRF $H_n(j\omega_1,...,j\omega_n)$ evaluated at $\omega_1 = \cdots = \omega_k = \omega_F$, $\omega_{k+1} = \cdots = \omega_n = -\omega_F$, $k = 0,\cdots,n$.

3. The NOFRFs of Locally Nonlinear MDOF Systems

The locally nonlinear MDOF systems to be investigated are shown in Figure 1.

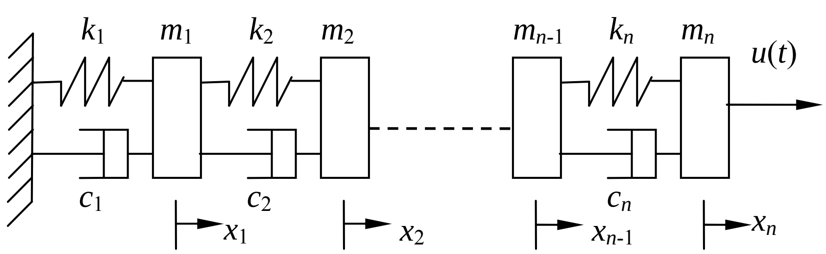

Figure 1, a multi-degree freedom oscillator

If all springs and dampers of the systems have linear properties, then the governing motion equation of the MDOF oscillator can be written as

$$M\ddot{x} + C\dot{x} + Kx = U(t) \tag{19}$$

where M is the system mass matrix,

$$M=\begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_n \end{bmatrix}$$

C and K are the system damping and stiffness matrices respectively,

$$C=\begin{bmatrix} c_1+c_2 & -c_2 & 0 & \cdots & 0 \\ -c_2 & c_2+c_3 & -c_3 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -c_{n-1} & c_{n-1}+c_n & -c_n \\ 0 & \cdots & 0 & -c_n & c_n \end{bmatrix} K=\begin{bmatrix} k_1+k_2 & -k_2 & 0 & \cdots & 0 \\ -k_2 & k_2+k_3 & -k_3 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -k_{n-1} & k_{n-1}+k_n & -k_n \\ 0 & \cdots & 0 & -k_n & k_n \end{bmatrix}$$

x is the displacement, $x=(x_1,\cdots,x_n)'$, and $u(t)$ is the external force acting on the right end of the oscillator, $U(t)=\left(0,\cdots,u(t)\right)'$.

Equation (19) forms the basis of the modal analysis method, which is a well-established approach for determining dynamic characteristics of engineering structures. In the linear case, the displacements $x_i(t)$ ($i=1,\cdots,n$) can be expressed as

$$x_i(t)=\int_{-\infty}^{+\infty} h_{(i)}(t-\tau)u(\tau)d\tau \tag{20}$$

where $h_{(i)}(t)$ ($i=1,\cdots,n$) are the impulse response functions that are determined by equation (19), the Fourier transforms of which are the well-known FRFs of the system.

Assume the characteristics of the L^{th} spring and damper are nonlinear, and the restoring forces $S_{LS}(\Delta)$ and $S_{LD}(\dot{\Delta})$ of the spring and damper are the polynomial functions of the deformation Δ and its derivative $\dot{\Delta}$ respectively, *e.g.*,

$$S_{LS}(\Delta)=\sum_{i=1}^{P} r_i\Delta^i\ ,\ \ S_{LD}(\Delta)=\sum_{i=1}^{P} w_i\dot{\Delta}^i \tag{21}$$

where P is the degree of the polynomials. Without loss of generality, further assume $L\neq 1,n$. Then the motion of the oscillator in Figure 1 can be described by equations (22)~(26) as follows.

For the masses that are not connected to the L^{th} spring and damper, the governing motion equations are

$$m_1\ddot{x}_1+(c_1+c_2)\dot{x}_1-c_2\dot{x}_2+(k_1+k_2)x_1-k_2x_2=0 \tag{22}$$

$$m_i\ddot{x}_i+(c_i+c_{i+1})\dot{x}_i-c_i\dot{x}_{i-1}-c_{i+1}\dot{x}_{i+1}+(k_i+k_{i+1})x_i-k_ix_{i-1}-k_{i+1}x_{i+1}=0$$

$$(i\neq L-1,L) \tag{23}$$

$$m_n\ddot{x}_n+c_n\dot{x}_n-c_n\dot{x}_{n-1}+k_nx_n-k_nx_{n-1}=0 \tag{24}$$

Denote $k_L=r_1$ and $c_L=w_1$, then for the mass that is connected to the left of the L^{th} spring and damper, the governing motion equation is

$$m_{L-1}\ddot{x}_{L-1} + (k_{L-1} + k_L)x_{L-1} - k_{L-1}x_{L-2} - k_L x_L + (c_{L-1} + c_L)\dot{x}_{L-1}$$
$$- c_{L-1}\dot{x}_{L-2} - c_L\dot{x}_L + \sum_{i=2}^{P} r_i(x_{L-1} - x_L)^i + \sum_{i=2}^{P} w_i(\dot{x}_{L-1} - \dot{x}_L)^i = 0 \quad (25)$$

For the mass that is connected to the right of the L^{th} spring and damper, the governing motion equation is

$$m_L\ddot{x}_L + (k_L + k_{L+1})x_L - k_L x_{L-1} - k_{L+1}x_{L+1} + (c_L + c_{L+1})\dot{x}_L$$
$$- c_L\dot{x}_{L-1} - c_{L+1}\dot{x}_{L+1} - \sum_{i=2}^{P} r_i(x_{L-1} - x_L)^i - \sum_{i=2}^{P} w_i(\dot{x}_{L-1} - \dot{x}_L)^i = 0 \quad (26)$$

Denote

$$NonF = \sum_{i=2}^{P} w_i(\dot{x}_{L-1} - \dot{x}_L)^i + \sum_{i=2}^{P} r_i(x_{L-1} - x_L)^i \quad (27)$$

$$NF = \left(\overbrace{0 \cdots 0}^{L-2} \quad NonF \quad -NonF \quad \overbrace{0 \cdots 0}^{n-L} \right)' \quad (28)$$

Then, the governing motion equation of the locally nonlinear oscillator can be written as

$$M\ddot{x} + C\dot{x} + Kx = -NF + U(t) \quad (29)$$

The systems described by (27)~(29) are typical locally nonlinear MDOF systems. The L^{th} nonlinear spring and damper components can lead the whole system to behave nonlinearly. Based on the Volterra series theory of nonlinear systems, the relationships between the displacements $x_i(t)$ ($i = 1, \cdots, n$) and the input force $u(t)$ of the MDOF systems are

$$x_i(t) = \sum_{j=1}^{N} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_{(i,j)}(\tau_1, ..., \tau_j) \prod_{l=1}^{j} u(t - \tau_l) d\tau_l \qquad (i = 1, \cdots, n) \quad (30)$$

where $h_{(i,j)}(\tau_1, ..., \tau_j)$ is the j^{th} order Volterra kernel associated to the i^{th} mass. In the frequency domain, the relationship (30) can be expressed as

$$X_i(j\omega) = \sum_{l=1}^{N} X_{(i,l)}(j\omega) = \sum_{l=1}^{N} G_{(i,l)}(j\omega) U_l(j\omega) \qquad (i = 1, \cdots, n) \quad (31)$$

where $G_{(i,l)}(j\omega)$ is the l^{th} order NOFRF associated to the i^{th} mass.

In a recent study by the authors [17], a series of relationships between the NOFRF $G_{(i,l)}(j\omega)$, ($i = 1, \cdots, n$, $l = 1, \cdots, N$) of locally nonlinear MDOF systems were derived, the results reveal, for the first time, very important characteristics of this general class of nonlinear systems. These relationships in [17] give a comprehensive description how the linear system parameters govern the propagation of the nonlinear effect caused by the nonlinear component in the system and how the nonlinear component affects the vibration propagation in the system. It has been rigorously proved in [17] that, for any two consecutive masses, the NOFRFs satisfy the following relationships

$$\lambda_2^{i,i+1}(j\omega)=\frac{G_{(i,2)}(j\omega)}{G_{(i+1,2)}(j\omega)}=\cdots=\lambda_N^{i,i+1}(j\omega)=\frac{G_{(i,N)}(j\omega)}{G_{(i+1,N)}(j\omega)} \qquad (1\le i\le n-1) \quad (32)$$

$$\lambda_1^{i,i+1}(j\omega)=\frac{G_{(i,1)}(j\omega)}{G_{(i+1,1)}(j\omega)}=\frac{G_{(i,Z)}(j\omega)}{G_{(i+1,Z)}(j\omega)}=\lambda_Z^{i,i+1}(j\omega) \qquad (1\le i\le L-2\,,2\le Z\le N\,) \quad (33)$$

and

$$\lambda_1^{i,i+1}(j\omega)=\frac{G_{(i,1)}(j\omega)}{G_{(i+1,1)}(j\omega)}\neq\frac{G_{(i,Z)}(j\omega)}{G_{(i+1,Z)}(j\omega)}=\lambda_Z^{i,i+1}(j\omega) \quad (L-1\le i\le n-1,2\le Z\le N\,) \quad (34)$$

where

$$\lambda_Z^{i,i+1}(j\omega)=\frac{jc_{i+1}\omega+k_{i+1}}{\left[-m_i\omega^2+j\left(\left(1-\lambda_Z^{i-1,i}(j\omega)\right)c_i+c_{i+1}\right)\omega+\left(1-\lambda_Z^{i-1,i}(j\omega)\right)k_i+k_{i+1}\right]}\left(1+\frac{\Lambda_Z^{i,i+1}(j\omega)}{jc_{i+1}\omega+k_{i+1}}\right)$$

$$(1\le i\le n-1,1\le Z\le N\,) \quad (35)$$

with $\lambda_Z^{0,1}(\omega)=0$,

$$\Lambda_Z^{i,i+1}(j\omega)=\begin{cases}0 & \text{for } Z=1,\text{or } i\neq L-1,L\\ -\dfrac{\Gamma_{(L-1,Z)}(j\omega)}{G_{(L,Z)}(j\omega)} & \text{for } Z\ge 2 \text{ and } i=L-1,\\ \dfrac{\Gamma_{(L-1,Z)}(j\omega)}{G_{(L+1,Z)}(j\omega)} & \text{for } Z\ge 2 \text{ and } i=L\end{cases} \quad (36)$$

and $\Gamma_{(L-1,Z)}(j\omega)$ is a term introduced by the nonlinear force *NonF* for the Z^{th} order NOFRF. As the form of $\Gamma_{(L-1,Z)}(j\omega)$ will not play crucial role in this study, its explicit expression will not be given.

4. The Linear Stiffness and Damping Estimation Method

Based on the results in Section 3, a novel method is developed in this section to estimate the linear stiffness and damping parameters for locally nonlinear MDOF systems. This method requires that the masses m_i, ($i=1,\cdots,n$) of the systems are known a priori. This is a reasonable assumption since the mass distribution of a mechanical structure or machine can usually be predetermined during the design stage and will not change significantly with the change of the structure's or machine's conditions even after years' operation. In addition, the mass distribution can be easily obtained using the FE method or other methods.

Consider *Z*= 1 in (35), it is known that

$$\begin{aligned}&j\omega\left(1-\lambda_1^{i-1,i}(j\omega)\right)\lambda_1^{i,i+1}(j\omega)c_i+j\omega\left(\lambda_1^{i,i+1}(j\omega)-1\right)c_{i+1}\\&+\left(1-\lambda_1^{i-1,i}(j\omega)\right)\lambda_1^{i,i+1}(j\omega)k_i+\left(\lambda_1^{i,i+1}(j\omega)-1\right)k_{i+1}=m_i\lambda_1^{i,i+1}(j\omega)\omega^2\end{aligned} \quad (1\le i\le n-1) \quad (37)$$

Denote

$$\Pi_1^{i-1,i,i+1}(j\omega) = \left(1-\lambda_1^{i-1,i}(j\omega)\right)\lambda_1^{i,i+1}(j\omega),\ \ \Pi_1^{i,i+1}(j\omega) = \lambda_1^{i,i+1}(j\omega)-1$$

$$\Phi_1^{i-1,i,i+1}(j\omega) = \begin{pmatrix} -\mathrm{Im}\left(\Pi_1^{i-1,i,i+1}(j\omega)\right)\omega & \mathrm{Re}\left(\Pi_1^{i-1,i,i+1}(j\omega)\right) & -\mathrm{Im}\left(\Pi_1^{i,i+1}(j\omega)\right)\omega & \mathrm{Re}\left(\Pi_1^{i,i+1}(j\omega)\right) \\ \mathrm{Re}\left(\Pi_1^{i-1,i,i+1}(j\omega)\right)\omega & \mathrm{Im}\left(\Pi_1^{i-1,i,i+1}(j\omega)\right) & \mathrm{Re}\left(\Pi_1^{i,i+1}(j\omega)\right)\omega & \mathrm{Im}\left(\Pi_1^{i,i+1}(j\omega)\right) \end{pmatrix}$$

and

$$W^{i,i+1} = \begin{bmatrix} c_i & k_i & c_{i+1} & k_{i+1} \end{bmatrix}^T$$

Equation (37) can then be written as

$$\Phi_1^{i-1,i,i+1}(j\omega)W^{i,i+1} = \begin{bmatrix} m_i\,\mathrm{Re}\left(\lambda_1^{i,i+1}(j\omega)\right)\omega^2 \\ m_i\,\mathrm{Im}\left(\lambda_1^{i,i+1}(j\omega)\right)\omega^2 \end{bmatrix} \quad (1 \le i \le n-1) \quad (38)$$

where $W^{i,i+1}$ is the parameter vector to be estimated.

Consider $Z = 2$ in (35), an equation similar to (38) can be obtained for $i \ne L-1, L$ as

$$\Phi_2^{i-1,i,i+1}(j\omega)W^{i,i+1} = \begin{bmatrix} m_i\,\mathrm{Re}\left(\lambda_2^{i,i+1}(j\omega)\right)\omega^2 \\ m_i\,\mathrm{Im}\left(\lambda_2^{i,i+1}(j\omega)\right)\omega^2 \end{bmatrix} \quad (1 \le i \le n-1, \text{and } i \ne L-1, L) \quad (39)$$

For i=L-1 and i=L, the extra terms introduced by the nonlinear force *NonF* should be taken into account. When i=L-1, the result is

$$j\omega\left(1-\lambda_2^{L-2,L-1}(j\omega)\right)\lambda_2^{L-1,L}(j\omega)c_{L-1} + j\omega\left(\lambda_2^{L-1,L}(j\omega)-1\right)c_L + \left(1-\lambda_2^{L-2,L-1}(j\omega)\right)\lambda_2^{L-1,L}(j\omega)k_{L-1} + \left(\lambda_2^{L-1,L}(j\omega)-1\right)k_L - \Lambda_2^{L-1,L}(j\omega) = m_{L-1}\lambda_2^{L-1,L}(j\omega)\omega^2 \quad (40)$$

which can further be written as

$$\begin{pmatrix} \Phi_2^{L-2,L-1,L}(j\omega) & \begin{matrix} -1 & 0 \\ 0 & -1 \end{matrix} \end{pmatrix} \begin{bmatrix} W^{L-1,L} \\ \mathrm{Re}(\Lambda_2^{L-1,L}(j\omega)) \\ \mathrm{Im}(\Lambda_2^{L-1,L}(j\omega)) \end{bmatrix} = \begin{bmatrix} m_{L-1}\,\mathrm{Re}\left(\lambda_2^{L-1,L}(j\omega)\right)\omega^2 \\ m_{L-1}\,\mathrm{Im}\left(\lambda_2^{L-1,L}(j\omega)\right)\omega^2 \end{bmatrix} \quad (41)$$

For i=L, the result is

$$\begin{pmatrix} \Phi_2^{L-1,L,L+1}(j\omega) & \begin{matrix} -1 & 0 \\ 0 & -1 \end{matrix} \end{pmatrix} \begin{bmatrix} W^{L,L+1} \\ \mathrm{Re}(\Lambda_2^{L,L+1}(j\omega)) \\ \mathrm{Im}(\Lambda_2^{L,L+1}(j\omega)) \end{bmatrix} = \begin{bmatrix} m_L\,\mathrm{Re}\left(\lambda_2^{L,L+1}(j\omega)\right)\omega^2 \\ m_L\,\mathrm{Im}\left(\lambda_2^{L,L+1}(j\omega)\right)\omega^2 \end{bmatrix} \quad (42)$$

According to equation (36), it can be known

$$\frac{\Lambda_2^{L,L+1}(j\omega)}{\Lambda_2^{L-1,L}(j\omega)} = -\frac{G_{(L,2)}(j\omega)}{G_{(L+1,2)}(j\omega)} = -\lambda_2^{L,L+1}(j\omega) \quad (43)$$

Substituting equation (43) into equation (42) yields

$$\begin{pmatrix} \Phi_2^{L-1,L,L+1}(j\omega) & \begin{matrix} \mathrm{Re}\left(\lambda_2^{L,L+1}(j\omega)\right) & -\mathrm{Im}\left(\lambda_2^{L,L+1}(j\omega)\right) \\ \mathrm{Im}\left(\lambda_2^{L,L+1}(j\omega)\right) & \mathrm{Re}\left(\lambda_2^{L,L+1}(j\omega)\right) \end{matrix} \end{pmatrix} \begin{bmatrix} W^{L,L+1} \\ \mathrm{Re}(\Lambda_2^{L-1,L}(j\omega)) \\ \mathrm{Im}(\Lambda_2^{L-1,L}(j\omega)) \end{bmatrix} = \begin{bmatrix} m_L\,\mathrm{Re}\left(\lambda_2^{L,L+1}(j\omega)\right)\omega^2 \\ m_L\,\mathrm{Im}\left(\lambda_2^{L,L+1}(j\omega)\right)\omega^2 \end{bmatrix} \quad (44)$$

When a sinusoidal input of frequency ω_F is used to excite the system, considering $\lambda_1^{i,i+1}(j\omega_F)$ and $\lambda_2^{i,i+1}(j2\omega_F)$, equations (38), (39), (41) and (44) can be written as

$$\Phi_1^{i-1,i,i+1}(j\omega_F)W^{i,i+1} = \begin{bmatrix} m_i \operatorname{Re}\left(\lambda_1^{i,i+1}(j\omega_F)\right)\omega_F^2 \\ m_i \operatorname{Im}\left(\lambda_1^{i,i+1}(j\omega_F)\right)\omega_F^2 \end{bmatrix} \quad (1 \le i \le n-1) \quad (45)$$

$$\Phi_2^{i-1,i,i+1}(j2\omega_F)W^{i,i+1} = \begin{bmatrix} 4m_i \operatorname{Re}\left(\lambda_2^{i,i+1}(j2\omega_F)\right)\omega_F^2 \\ 4m_i \operatorname{Im}\left(\lambda_2^{i,i+1}(j2\omega_F)\right)\omega_F^2 \end{bmatrix} \quad (1 \le i \le n-1, \text{and } i \ne L-1, L) \quad (46)$$

$$\begin{pmatrix} \Phi_2^{L-2,L-1,L}(j2\omega_F) & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{pmatrix} \begin{bmatrix} W^{L-1,L} \\ \operatorname{Re}(\Lambda_2^{L-1,L}(j2\omega_F)) \\ \operatorname{Im}(\Lambda_2^{L-1,L}(j2\omega_F)) \end{bmatrix} = \begin{bmatrix} 4m_{L-1} \operatorname{Re}\left(\lambda_2^{L-1,L}(j2\omega_F)\right)\omega_F^2 \\ 4m_{L-1} \operatorname{Im}\left(\lambda_2^{L-1,L}(j2\omega_F)\right)\omega_F^2 \end{bmatrix} \quad (47)$$

$$\begin{pmatrix} \Phi_2^{L-1,L,L+1}(j2\omega_F) & \begin{matrix} \operatorname{Re}\left(\lambda_2^{L,L+1}(j2\omega_F)\right) & -\operatorname{Im}\left(\lambda_2^{L,L+1}(j2\omega_F)\right) \\ \operatorname{Im}\left(\lambda_2^{L,L+1}(j2\omega_F)\right) & \operatorname{Re}\left(\lambda_2^{L,L+1}(j2\omega_F)\right) \end{matrix} \end{pmatrix} \begin{bmatrix} W^{L,L+1} \\ \operatorname{Re}(\Lambda_2^{L-1,L}(j2\omega_F)) \\ \operatorname{Im}(\Lambda_2^{L-1,L}(j2\omega_F)) \end{bmatrix}$$
$$= \begin{bmatrix} 4m_L \operatorname{Re}\left(\lambda_2^{L,L+1}(j2\omega_F)\right)\omega_F^2 \\ 4m_L \operatorname{Im}\left(\lambda_2^{L,L+1}(j2\omega_F)\right)\omega_F^2 \end{bmatrix} \quad (48)$$

Consider $\operatorname{Re}(\Lambda_2^{L-1,L}(j2\omega_F))$ and $\operatorname{Im}(\Lambda_2^{L-1,L}(j2\omega_F))$ as unknown parameters to be estimated, then there are totally 2*n*+2 parameters to be estimated in equations (45)~(48). There are clearly 4(*n*-1) equations in total in (45)~(48), and, obviously, when $n \ge 3$, the number of equations is sufficient to estimate the unknown parameters.

Denote

$$W = \left[c_1 \;\; k_1 \cdots c_{L-1} \;\; k_{L-1} \;\; \operatorname{Re}(\Lambda_2^{L-1,L}(j2\omega_F)) \;\; \operatorname{Im}(\Lambda_2^{L-1,L}(j2\omega_F)) \;\; c_L \;\; k_L \cdots c_n \;\; k_n\right]^T$$

and

$$\mathrm{B}_Z(j\omega) = \omega^2\left[m_1 \operatorname{Re}\left(\lambda_Z^{1,2}(j\omega)\right) \;\; m_1 \operatorname{Im}\left(\lambda_Z^{1,2}(j\omega)\right) \;\; \cdots \;\; m_{n-1} \operatorname{Im}\left(\lambda_Z^{n-1,n}(j\omega)\right) \;\; m_{n-1} \operatorname{Im}\left(\lambda_Z^{n-1,n}(j\omega)\right)\right]^T$$

then equations (45)~(48) can be assembled in the following form

$$\begin{pmatrix} \Phi_1(j\omega_F) \\ \Phi_2(j2\omega_F) \end{pmatrix} W = \begin{bmatrix} B_1(j\omega_F) \\ B_2(j2\omega_F) \end{bmatrix} \quad (49)$$

where both $\Phi_1(j\omega_F)$ and $\Phi_2(j2\omega_F)$ are a $2(n-1)\times(2n+2)$ matrix that can be constructed using a similar procedure given in Appendix 1.

From equation (49), a Least Square based approach can be used to estimate the parameters in *W* as below

$$W = \left[\begin{pmatrix} \Phi_1(j\omega_F) \\ \Phi_2(j2\omega_F) \end{pmatrix}^T \begin{pmatrix} \Phi_1(j\omega_F) \\ \Phi_2(j2\omega_F) \end{pmatrix}\right]^{-1} \begin{pmatrix} \Phi_1(j\omega_F) \\ \Phi_2(j2\omega_F) \end{pmatrix}^T \begin{bmatrix} B_1(j\omega_F) \\ B_2(j2\omega_F) \end{bmatrix} \quad (50)$$

Equation (50) provides a simple way to estimate the linear stiffness and damping coefficients for a locally nonlinear MDOF system from the system NOFRFs. This algorithm requires the information of the 1st and 2nd order NOFRFs under sinusoidal inputs, which can readily be evaluated using an effective algorithm developed in [16] from the system input-output data.

The linear parameter estimation algorithm can be summarized as the following procedures:

Step 1: Estimate $G_{(i,1)}(j\omega_F)$ and $G_{(i,2)}(j2\omega_F)(1 \le i \le n)$ under sinusoidal inputs using the algorithm developed in [16].

Step 2: Calculate $\lambda_1^{i,i+1}(j\omega_F)$ and $\lambda_2^{i,i+1}(j2\omega_F)$ $(1 \le i \le n-1)$.

Step 3: Calculate $\Pi_Z^{i-1,i,i+1}(j\omega)$, $\Pi_Z^{i,i+1}(j\omega)$, $(1 \le i \le n-1, Z = 1, 2)$.

Step 4: Construct matrixes $\Phi_1(j\omega_F)$ and $\Phi_2(j2\omega_F)$ using the method in Appendix 1.

Step 5: Estimate the linear parameters using equation (50).

It is worth pointing out that although the algorithm above can only be used to determine the system linear parameters, the results themself are still of great significance in engineering system analysis such as, for example, in the modal analysis of MDOF systems. In addition the algorithm also provides a premise for the identification of all system characteristic parameters. It can be observed that the term $\Gamma_{(L-1,Z)}(j\omega)$ in (36) is related to the system nonlinear parameters. This provides a basis for the system nonlinear parameter estimation. Generally, $\Gamma_{(L-1,N)}(jN\omega_F)$ can be expressed as

$$\Gamma_{(L-1,N)}(jN\omega_F) = Q_{(2)}(r_2, w_2, \omega_F) F_{(N,2)}^{L-1,L}(j\omega_F) + \cdots + Q_{(N)}(r_N, w_N, \omega_F) F_{(N,N)}^{L-1,L}(j\omega_F) \quad (51)$$

where $Q_{(i)}(r_i, w_i, \omega_F)$ $(2 \le i \le N)$ are the functions which only depend on the nonlinear parameters r_i and w_i and driving frequency ω_F, $F_{(N,i)}^{L-1,L}(j\omega_F)$ $(2 \le i \le N-1)$ depend on both the linear parameters and the nonlinear parameters r_l and w_l, $(2 \le l \le N-i+1)$, and $F_{(N,N)}^{L-1,L}(j\omega_F)$ only depends on the system linear parameters. According to equation (51), an effective method for the estimation of both the system linear and nonlinear parameters with the linear parameter estimation using the above algorithm as the first step can be developed. Because of space limitations, this work will be reported in details in a subsequent paper.

5 Numerical Study

To verify the effectiveness of the proposed parameter estimation method, a damped 6-DOF oscillator is used, in which the fourth spring is nonlinear. As widely used in modal analysis, the damping is assumed to be proportional damping, e.g., $C=\mu K$ .The values of the system parameters are

$$m_1=\cdots=m_6=1,\ r_1=k_1=k_2=k_3=3.6\times10^4,\ k_4=k_5=k_6=0.8k_1,$$
$$\mu=0.01,\ r_2=0.8\times r_1^2,\ r_3=0.4\times r_1^3,\ w_1=\mu r_1,\ w_2=0$$

and the input is a harmonic force, $u(t)=A\sin(2\pi\times20t)$.

If only the NOFRFs up to the 4th order are considered, according to equations (16) and (17), the frequency components of the outputs of the 6 masses can be written as

$$\begin{aligned}
X_i(j\omega_F)&=G_{(i,1)}^H(j\omega_F)U_1(j\omega_F)+G_{(i,3)}^H(j\omega_F)U_3(j\omega_F)\\
X_i(j2\omega_F)&=G_{(i,2)}^H(j2\omega_F)U_2(j2\omega_F)+G_{(i,4)}^H(j2\omega_F)U_4(j2\omega_F)\\
X_i(j3\omega_F)&=G_{(i,3)}^H(j3\omega_F)U_3(j3\omega_F)\\
X_i(j4\omega_F)&=G_{(i,4)}^H(j4\omega_F)U_4(j4\omega_F) \qquad (i=1,\cdots,6)
\end{aligned}\tag{52}$$

From equation (52), it can be seen that two different inputs with the same waveform but different strengths are sufficient to estimate the NOFRFs up to 4th order. This is the basic principle of the algorithm proposed in [16] for the evaluation of the NOFRFs. Therefore, in this numerical study, two different inputs are used, *A*=0.8 and *A*=1.0 respectively. The simulation studies were conducted using a fourth-order *Runge–Kutta* method to obtain the forced responses of the system. The evaluated results of $G_1^H(j\omega_F)$, $G_3^H(j\omega_F)$, $G_2^H(j2\omega_F)$ and $G_4^H(j2\omega_F)$ of the six different masses are given in Table 1.

From the evaluated NOFRFs given in Table 1, $\lambda_1^{i,i+1}(j\omega_F)$, $\lambda_3^{i,i+1}(j\omega_F)$, $\lambda_2^{i,i+1}(j2\omega_F)$ and $\lambda_4^{i,i+1}(j2\omega_F)$ (*i*=1,2,3,4,5) can be evaluated using equations (32), (33) and (34). Moreover, the theoretical values of $\lambda_1^{i,i+1}(j\omega_F)$, $\lambda_3^{i,i+1}(j\omega_F)$, $\lambda_2^{i,i+1}(j2\omega_F)$ and $\lambda_4^{i,i+1}(j2\omega_F)$ (*i*=1,2,3,4,5) can also be calculated using the method in [17]. Both the evaluated and theoretical values of $\lambda_1^{i,i+1}(j\omega_F)$, $\lambda_3^{i,i+1}(j\omega_F)$, $\lambda_2^{i,i+1}(j2\omega_F)$ and $\lambda_4^{i,i+1}(j2\omega_F)$ (*i*=1,2,3,4,5) are given in Tables 2 and Table 3. The results in Table 2 and Table 3 clearly show that properties (32)~(34) are tenable. Obviously, $\lambda_1^{i,i+1}(j\omega_F)\neq\lambda_3^{i,i+1}(j\omega_F)$ for *i*=3, 4, 5.

Table 1, the evaluated results of $G_1^H(j\omega_F)$, $G_3^H(j\omega_F)$, $G_2^H(j2\omega_F)$ and $G_4^H(j2\omega_F)$

	$G_1^H(j\omega_F)$ ($\times10^{-6}$)	$G_3^H(j\omega_F)$ ($\times10^{-7}$)	$G_2^H(j2\omega_F)$ ($\times10^{-8}$)	$G_4^H(j2\omega_F)$ ($\times10^{-8}$)
Mass1	-3.5291+6.0326i	1.2996 -2.3216i	2.9244-12.3188i	-2.9300 -1.4749i
Mass2	-7.7472+10.2849i	2.8744-3.9706i	12.5722-19.9209i	-4.2684-4.3621i
Mass3	-12.8459+11.1323i	4.8088-4.3299i	31.2120-15.1678i	-1.9539-8.7759i
Mass4	-19.4063+6.3260i	-1.7437+2.4212i	-31.6321+11.0973i	0.9027+ 8.6380i
Mass5	-23.5870-4.9427i	0.5327+ 3.1966i	-5.1398+17.9423i	4.2148+2.3702i

Mass6	-21.4328-21.4620i	1.8419+ 3.4348i	9.3753+15.5360i	4.4776-1.4327i

Table 2, the evaluated and theoretical values of $\lambda_1^{i,i+1}(j\omega_F)$ and $\lambda_3^{i,i+1}(j\omega_F)$

	$\lambda_1^{i,i+1}(j\omega_F)$		$\lambda_3^{i,i+1}(j\omega_F)$	
	Evaluated	Theoretical	Evaluated	Theoretical
i=1	0.5391 - 0.0630i	0.5391 - 0.0630i	0.5391-0.0630i	0.5391-0.0630i
i=2	0.7407 - 0.1588i	0.7407 - 0.1588i	0.7407-0.1588i	0.7407-0.1588i
i=3	0.7674 - 0.3235i	0.7654 - 0.3206i	-2.1194-0.4597i	-2.1194-0.4597i
i=4	0.7343 - 0.4221i	0.7353 - 0.4228i	0.6485+ 0.6536i	0.6485+0.6536i
i=5	0.6648 - 0.4351i	0.6646 - 0.4359i	0.7874+0.2672i	0.7874+0.2672i

Table 3, the evaluated and theoretical values of $\lambda_2^{i,i+1}(j2\omega_F)$ and $\lambda_4^{i,i+1}(j2\omega_F)$

	$\lambda_2^{i,i+1}(j2\omega_F)$		$\lambda_4^{i,i+1}(j2\omega_F)$	
	Evaluated	Theoretical	Evaluated	Theoretical
i=1	0.5085 - 0.1741i	0.5085 - 0.1741i	0.5085-0.1741i	0.5085 - 0.1741i
i=2	0.5768 - 0.3580i	0.5768 - 0.3580i	0.5768-0.3580i	0.5768 - 0.3580i
i=3	-1.0284 + 0.1187i	-1.0284 + 0.1187i	-1.0284+0.1187i	-1.0284 + 0.1187i
i=4	1.0383 + 1.4656i	1.0383 + 1.4655i	1.0383+1.4656i	1.0383 + 1.4655i
i=5	0.7002 + 0.7534i	0.7002 + 0.7534i	0.7002+0.7534i	0.7002 + 0.7534i

Table 4, the estimated and real values of stiffness and damping

	Evaluated($\times 10^4$)	Real($\times 10^4$)
k_1	3.6001	3.6000
k_2	3.6001	3.6000
k_3	3.6001	3.6000
k_4	2.9208	2.8800
k_5	2.8800	2.8800
k_6	2.8800	2.8800

	Evaluated($\times 10^2$)	Real($\times 10^2$)
c_1	3.6000	3.6000
c_2	3.6000	3.6000
c_3	3.6000	3.6000
c_4	2.8573	2.8800
c_5	2.8799	2.8800
c_6	2.8799	2.8800

Using the evaluated results of $\lambda_1^{i,i+1}(j\omega_F)$ and $\lambda_2^{i,i+1}(j2\omega_F)$ (i=1,2,3,4,5), the linear stiffness and damping can be estimated by the method proposed in the previous section, and the results are given in Table 4. It can be seen that the estimated results match the theoretical results very well except a slightly difference for k_4 and c_4. This difference mainly comes from the facts that the truncated Volterra series have to be applied in practice and the truncation can raise error only to the estimations of k_4 and c_4.

6 Conclusions and Remarks

A new method for the estimation of the linear stiffness and damping parameters of locally nonlinear MDOF systems has been developed. This method is based on the concept of Nonlinear Output Frequency Response Functions (NOFRFs) which were derived from the Volterra series approach of nonlinear systems. This method assumes that the system masses and the position of the system nonlinear component are known priori. The masses can be readily obtained at the system design stage. The position of the nonlinear component can be determined using directly the relationships (32)~(34) [17], which can also be applied to detect the crack position in beams or structures. From these results, all linear stiffness and damping parameters of a nonlinear MDOF system can be estimated using the proposed method directly using the system input-output test data, which are of great engineering significance such as, e.g., in system modal analysis. In addition, although the method is demonstrated on the particular case where the linear oscillators are coupled in series and fixed at one end while excited in the other end, this method can be directly applied (without any modification) to the cases where the excitation force is acting at any position of the system, not only limited to the end.

It is worthy noting here that the method is only valid when the system response can be described using the Volterra series, and basically, the validity of the Volterra series is dependent on the amplitude of the external force. Regarding the problem of determining the domain of validity of the Volterra series, many researchers have made many efforts, including the authors ourselves. The existing literature includes [22]~[26]. Our recent research studies have shown that, for the nonlinear damping system, the validity domain is very wide; on the other hand, the validity domain of the nonlinear stiffness system is relatively narrow, for example, at the region where the jump phenomenon [27] occurs, the Volterra series representation is not valid. However, from the system identification view point, because the amplitude of the external force is controllable, we can try to conduct the identification experiments such that the system always works in a region where the Volterra series theory works. In addition, when conducting the parameter estimation, theoretically, the choice of the excitation frequency can be arbitrary. However, according to our experiences, in practical applications, because the method will make use of the nonlinear effect, therefore, it might be better to choose an excitation frequency that can make the nonlinear effect significant, for example, one half of a resonant frequency where the second harmonic component often reaches a maximum. More information about the resonances in nonlinear systems can be found in [28].

It is worth pointing out that the present study also provides a necessary basis for the identification of all characteristic parameters for the considered MDOF systems. With the algorithm proposed in this paper as the first step, an effective method can be developed to determine both the linear and nonlinear parameters of the MDOF systems. Because of space limitations, this method will be reported in details in a subsequent paper.

Appendix 1: Construction of $\Phi_1(j\omega_F)$ and $\Phi_2(j2\omega_F)$

The construction of $\Phi_1(j\omega_F)$ using this procedure is given as below.

For $1 \le i \le L-2$, i.e., the first 2(L-2) rows of $\Phi_1(j\omega_F)$,

$$\begin{cases} \Phi_1(j\omega_F)(2i-1:2i,\ 1:2i-2) = 0 \\ \Phi_1(j\omega_F)(2i-1:2i,\ 2i-1:2i+2) = \Phi_1^{i-1,i,i+1}(j\omega_F) \\ \Phi_1(j\omega_F)(2i-1:2i,\ 2i+3:2n+2) = 0 \end{cases} \quad (1 \le i \le L-2) \quad \text{(A-1)}$$

For i= L-1,

$$\begin{cases} \Phi_1(j\omega_F)(2(L-1)-1:2(L-1),\ 1:2(L-1)-2) = 0 \\ \Phi_1(j\omega_F)(2(L-1)-1:2(L-1),\ 2(L-1)-1:2(L-1)+4) \\ \qquad = \begin{pmatrix} \Phi_{1,Left}^{L-1}(j\omega_F) & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \Phi_{1,Right}^{L-1}(j\omega_F) \end{pmatrix} \\ \Phi_1(j\omega_F)(2(L-1)-1:2(L-1),\ 2(L-1)+5:2n+2) = 0 \end{cases} \quad \text{(A-2)}$$

where

$$\Phi_{Z,Left}^{L-1}(j\omega) = \begin{pmatrix} -\mathrm{Im}\left(\Pi_Z^{L-2,L-1,L}(j\omega)\right)\omega & \mathrm{Re}\left(\Pi_Z^{L-2,L-1,L}(j\omega)\right) \\ \mathrm{Re}\left(\Pi_Z^{L-2,L-1,L}(j\omega)\right)\omega & \mathrm{Im}\left(\Pi_Z^{L-2,L-1,L}(j\omega)\right) \end{pmatrix} \quad \text{(A-3)}$$

$$\Phi_{Z,Right}^{L-1}(j\omega) = \begin{pmatrix} -\mathrm{Im}\left(\Pi_Z^{L-1,L}(j\omega)\right)\omega & \mathrm{Re}\left(\Pi_Z^{L-1,L}(j\omega)\right) \\ \mathrm{Re}\left(\Pi_Z^{L-1,L}(j\omega)\right)\omega & \mathrm{Im}\left(\Pi_Z^{L-1,L}(j\omega)\right) \end{pmatrix} \quad \text{(A-4)}$$

For i=L,

$$\begin{cases} \Phi_1(j\omega_F)(2L-1:2L,\ 1:2L-2) = 0 \\ \Phi_1(j\omega_F)(2L-1:2L,\ 2L-1:2L+4) = \begin{pmatrix} \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \Phi_1^{L-1,L,L+1}(j\omega_F) \end{pmatrix} \\ \Phi_1(j\omega_F)(2(L-1)-1:2(L-1),\ 2(L-1)+5:2n+2) = 0 \end{cases} \quad \text{(A-5)}$$

For $L < i \le n-1$

$$\begin{cases} \Phi_1(j\omega_F)(2i-1:2i,\ 1:2i) = 0 \\ \Phi_1(j\omega_F)(2i-1:2i,\ 2i+1:2i+4) = \Phi_1^{i-1,i,i+1}(j\omega_F) \\ \Phi_1(j\omega_F)(2i-1:2i,\ 2i+5:2n+2) = 0 \end{cases} \quad (L < i \le n-1) \quad \text{(A-6)}$$

The construction of $\Phi_2(j2\omega_F)$ is similar except for the cases where i=L-1 and i=L.

For i=L-1

$$\begin{cases} \Phi_2(j2\omega_F)(2(L-1)-1:2(L-1),\ 1:2(L-1)-2)=0 \\ \Phi_2(j2\omega_F)(2(L-1)-1:2(L-1),\ 2(L-1)-1:2(L-1)+4)= \\ \qquad \left(\Phi_{2,Left}^{L-1}(j2\omega_F)\begin{matrix}-1 & 0 \\ 0 & -1\end{matrix}\Phi_{2,Right}^{L-1}(j2\omega_F)\right) \\ \Phi_2(j2\omega_F)(2(L-1)-1:2(L-1),\ 2(L-1)+5:2n+2)=0 \end{cases} \quad \text{(A-7)}$$

and for $i=L$

$$\begin{cases} \Phi_2(j2\omega_F)(2L-1:2L,\ 1:2L-2)=0 \\ \Phi_2(j2\omega_F)(2L-1:2L,\ 2L-1:2L+4)= \\ \left(\begin{matrix}\mathrm{Re}\left(\lambda_2^{L,L+1}(j2\omega_F)\right) & -\mathrm{Im}\left(\lambda_2^{L,L+1}(j2\omega_F)\right) \\ \mathrm{Im}\left(\lambda_2^{L,L+1}(j2\omega_F)\right) & \mathrm{Re}\left(\lambda_2^{L,L+1}(j2\omega_F)\right)\end{matrix}\ \Phi_2^{L-1,L,L+1}(j2\omega_F)\right) \\ \Phi_2(j2\omega_F)(2(L-1)-1:2(L-1),\ 2(L-1)+5:2n+2)=0 \end{cases} \quad \text{(A-8)}$$

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