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Linear flows on compact, semisimple Lie groups: stability and periodic orbits

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Abstract. Our first purpose is to study the stability of linear flows on real, connected, compact, semisimple Lie groups. Our second purpose is to study periodic orbits of linear and invariant flows. As an application, we present periodic orbits of linear or invariant flows on $SO(3)$ and $SU(2)$ and we study periodic orbits of linear or invariant flows on $SO(4)$.

Keywords: stability, periodic orbits, linear flow, compact semisimple Lie group.

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1 Introduction

Let G be a real, connected Lie group. A vector field $\mathcal{X}$ on G is called *linear* if its flow, which is denoted by φ_t, is a family of automorphisms of G. In this work, we assume that G is a semisimple Lie group. Our wish is to study some aspects of stability of a linear flow φ_t and periodic orbits of a linear or invariant flows.

Our first task is to study the stability in a fixed point of a linear flow φ_t. In a natural way, we follow the ideas presented in the classical literature of dynamical systems on a Euclidian space (see for instance [3], [6] and [7]). In [5], Da Silva, Santana, and Stelmastchuk show that a necessary and sufficient condition to the asymptotically and exponential stability of φ_t at identity e is that $\mathcal{X}$ is hyperbolic. However, if a linear vector field $\mathcal{X}$ on G is hyperbolic, then G is a nilpotent Lie group. Then, it is obstructed the use of hyperbolic property in the study of the stability of a linear flow on a semisimple Lie group. Thus, we choose to restrict our study to compact, semisimple Lie groups because their algebraic structure allows us to develop some results about stability.

Let G be a real, connected, compact, semisimple Lie group. Consider a linear vector field $\mathcal{X}$ on G and its linear flow φ_t. The first part of our work is about stability. We show that any fixed point of the linear flow φ_t is stable (see Theorem 3.10). Furthermore, we proof that any periodic orbit of the linear flow φ_t is stable (see Theorem 3.13). Also, we proof that the derivation $\mathcal{D} = -ad(\mathcal{X})$ associated to $\mathcal{X}$ has only semisimple eigenvalues since the identity e is stable. The last fact is the key to study periodic orbits of linear flow φ_t.

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The second part of our work is about periodic orbits of the invariant and linear flows on compact, semisimple Lie groups. An important fact is that, in semisimple Lie groups, for any linear vector field $\mathcal{X}$ there is a right invariant vector field X associated to it. Thus, our first step is to show that if the orbits of the invariant flow $\exp(tX)$ are periodic, then the orbits of the linear flow φ_t are also periodic. Done this, we show that the orbits of the invariant flow $\exp(tX)$ are periodic if and only if the derivation $\mathcal{D}$ of $\mathcal{X}$ has as eigenvalues 0 or $\pm\alpha_i \mathrm{i}$ with $\alpha_i \neq 0$, $i = 1, \ldots, r$, and α_i/α_j is a rational number for $i, j = 1, \ldots, r$ (see Theorem 4.2). As a direct consequence, every orbit that is not a fixed point of an invariant flow $\exp(tX)$ on a 3-dimensional, compact, semisimple Lie group is periodic.

To end, we present the periodic orbits of a linear or invariant flows on $SO(3)$ and $SU(2)$, and we study the periodic orbits on $SO(4)$ (see Theorem 5.4).

This paper is organized as follows. Section 2 briefly reviews the notions of the linear vector fields. Section 3 works with stability on compact, semisimple Lie groups. Section 4 develops results in periodic orbits of a linear or invariant flows. Finally, section 5 applies the previous results on compact, semisimple Lie groups $SO(3)$, $SU(2)$ and $SO(4)$.

2 Linear vector fields

Let G be a connected Lie group and $\mathfrak{g}$ be its Lie algebra. We call a vector field $\mathcal{X}$ linear if its flow $(\varphi_t)_{t\in\mathbb{R}}$ is a family of automorphisms of the Lie group G. It is known that for any linear vector field $\mathcal{X}$ we can define a derivation $\mathcal{D}$ by

$$\mathcal{D}(Y) = -[\mathcal{X}, Y], \quad Y \in \mathfrak{g}.$$

Thus, the dynamical system

$$\dot{g} = \mathcal{X}(g), \quad g \in G, \tag{2.1}$$

is associated to the derivation $\mathcal{D}$. In fact, the linearization of system above at the identity is

$$\dot{X} = \mathcal{D}(X), \quad X \in \mathfrak{g}.$$

For the Euclidian case, if $A \in \mathbb{R}^{n\times n}$ and $b, x \in \mathbb{R}^n$, then $\mathcal{D}(b)(x) = [Ax, b] = -Ab$. Thus, we can view the dynamical system (2.1) as a generalization of dynamical system on $\mathbb{R}^n$ given by

$$\dot{x} = Ax.$$

Da Silva, in [4], writes

$$\mathfrak{g}^+ = \bigoplus_{\alpha;\mathrm{Re}(\alpha)>0} \mathfrak{g}_\alpha, \quad \mathfrak{g}^0 = \bigoplus_{\alpha;\mathrm{Re}(\alpha)=0} \mathfrak{g}_\alpha, \quad \text{and} \quad \mathfrak{g}^- = \bigoplus_{\alpha;\mathrm{Re}(\alpha)<0} \mathfrak{g}_\alpha,$$

where α are eigenvalues of the derivation $\mathcal{D}$ such that

$$\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^- \quad \text{and} \quad [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$$

with $\alpha + \beta = 0$ if the sum is not an eigenvalue. Let us denote by G^+, G^0 and G^- the Lie subgroups of the Lie algebras $\mathfrak{g}^+$, $\mathfrak{g}^0$ and $\mathfrak{g}^-$, respectively. It is simple to show that G^+, G^0 and G^+ are φ_t-invariant. The Lie subgroups G^+, G^0 and G^- are called unstable, central and stable groups associated to φ_t, respectively.

For the convenience of the reader we resume some facts about a linear vector field $\mathcal{X}$ and its flow φ_t. The proof of these facts can be found in [2].

Proposition 2.1. *Let $\mathcal{X}$ be a linear vector field, φ_t be its flow, and $\mathcal{D}$ be the derivation associated to $\mathcal{X}$. The following assertions are true:*

(i) φ_t *is an automorphism of Lie groups for each* t*;*

(ii) $\mathcal{X}$ *is linear iff* $\mathcal{X}(gh) = R_{h*}\mathcal{X}(g) + L_{g*}\mathcal{X}(h)$ *for* $g, h \in G$*;*

(iii) $(d\varphi_t)_e = e^{t\mathcal{D}}$ *for all* $t \in \mathbb{R}$.

3 Stability of the linear flow

Let G be a real, connected, semisimple Lie group and $\mathcal{X}$ be a linear vector field on G. In this section, our wish is to study the stability of the linear flow φ_t that is the solution of the differential equation on G given by

$$\dot{g} = \mathcal{X}(g), \quad g \in G. \tag{3.1}$$

Being G semisimple, there is a right invariant vector field X such that $\mathcal{X} = X + I_*X$, where I_*X is the left invariant vector field associated to X and I_* is the differential of inverse map $\mathrm{i}(g) = g^{-1}$ (more details is founded in [9]). It follows that the linear flow can be written as

$$\varphi_t(g) = \exp(tX) \cdot g \cdot \exp(-tX), \quad \forall g \in G.$$

According to the above expression, we have that the identity e is a fixed point for the linear flow φ_t. However, it may exist other fixed points.

Proposition 3.1. *If a point g belongs to center of the Lie group G, then g is a fixed point of the linear flow φ_t.*

Proof. Let g be a point in the center of the Lie group G. Then, for all $t \in \mathbb{R}$,

$$\varphi_t(g) = \exp(tX) \cdot g \cdot \exp(-tX) = \exp(tX) \cdot \exp(-tX) \cdot g = g,$$

which is the desired conclusion. □

Our next step is to present the hyperbolic concept to the linear vector fields. We remember that the stability in Euclidian space is obtained if a dynamical system is hyperbolic (see for instance [7]). As one can see in [5], it is also true if $\mathcal{X}$ is hyperbolic on a Lie group G.

Definition 3.2. Let $\mathcal{X}$ be a linear vector field on a Lie group G. We call $\mathcal{X}$ **hyperbolic** if its associated derivation $\mathcal{D}$ is hyperbolic, that is, $\mathcal{D}$ has no eigenvalues with zero real part.

Let $\mathcal{X}$ be a hyperbolic linear vector field on a semisimple Lie group G. Then $\mathcal{D}$ has no eigenvalues with zero real part. Denoting by $\mathfrak{g}_\alpha$ the generalized eigenspace associated with an eigenvalue α of $\mathcal{D}$ we get

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta},$$

where $\alpha + \beta$ is an eigenvalue of $\mathcal{D}$ and zero otherwise (see for instance Proposition 3.1 in [9]). Since $\dim G < \infty$, it implies that the Lie algebra $\mathfrak{g}$ is nilpotent. In consequence, G is nilpotent.

Proposition 3.3. *There is not hyperbolic linear vector field on semisimple Lie groups.*

We now begin studying the stability of linear flows on semisimple Lie groups. Firstly, we remember some concepts of stability.

Definition 3.4. Let $g \in G$ be a fixed point of the linear vector field $\mathcal{X}$. We call g

1) **stable** if for all g-neighborhood U there is a g-neighborhood V such that $\varphi_t(V) \subset U$ for all $t \geq 0$;

2) **asymptotically stable** if it is stable and there exists a g-neighborhood W such that $\lim_{t\to\infty} \varphi_t(x) = g$ whenever $x \in W$;

3) **exponentially stable** if there exist c, μ and a g-neighborhood W such that for all $x \in W$ it holds that
$$\varrho(\varphi_t(x), g) \leq c e^{-\mu t} \varrho(x, g), \qquad \text{for all} \quad t \geq 0;$$

4) **unstable** if it is not stable.

Since property 3) is local, it does not depend of the metric on G. Because of this reason, we will assume from now on that ϱ is a left invariant Riemannian metric.

In order to study the stability, let us work with the Lyapunov exponents. We follow [5] in assuming that the Lyapunov exponent can be written as
$$\lambda(e, v) = \limsup_{t\to\infty} \frac{1}{t} \log(\|\mathrm{e}^{t\mathcal{D}}(\mathrm{v})\|),$$
where v is in $\mathfrak{g}$ and the norm $\|\cdot\|$ is given by the left invariant metric.

We will use $\lambda_1, \ldots, \lambda_k$ to denote k distinct values of the real parts eigenvalues of the derivation $\mathcal{D}$. Then, the Lie algebra $\mathfrak{g}$ can be written as
$$\mathfrak{g} = \bigoplus_{i=1}^{k} \mathfrak{g}_{\lambda_i} \quad \text{where} \quad \mathfrak{g}_{\lambda_i} := \bigoplus_{\alpha;\mathrm{Re}(\alpha)=\lambda_i} \mathfrak{g}_\alpha.$$
Furthermore, from Theorem 4.2 in [5] we see that
$$\lambda(e, v) = \lambda \quad \Leftrightarrow \quad v \in \mathfrak{g}_\lambda := \bigoplus_{\alpha;\mathrm{Re}(\alpha)=\lambda} \mathfrak{g}_\alpha. \tag{3.2}$$

Using the Lyapunov exponent we show a first result about stability of linear flow φ_t.

Theorem 3.5. *For any linear vector field $\mathcal{X}$ on a semisimple Lie group G, any fixed point is neither asymptotically nor exponentially stable to the linear flow φ_t.*

Proof. We first observe that the Lyapunov exponents satisfy the following: $\lambda(g, v) = \lambda(e, v)$ for each $v \in \mathfrak{g}$. We need only consider the assertion at identity e. Suppose, contrary to our claim, that the identity e is either asymptotically or exponentially stable. By Theorem 4.5 in [5], it follows that all Lyapunov exponents of $\mathcal{D}$ are negatives. From (3.2) it follows that any eigenvalue of $\mathcal{D}$ has the real part negative. It means that $\mathcal{X}$ is hyperbolic, and this contradicts Proposition 3.3. □

Despite any fixed point is neither asymptotically nor exponentially stable, they are stable if G is compact and semisimple as we will show. For this purpose, we begin by introducing an appropriate metric on G.

Let G be a compact, semisimple Lie group. It implies that the Cartan–Killing form is negative defined. Thus we adopt the metric $\langle\cdot,\cdot\rangle$ given by negative of the Cartan–Killing form on $\mathfrak{g}$. Since $\langle\cdot,\cdot\rangle$ satisfies
$$\langle \mathrm{Ad}(g)X, \mathrm{Ad}(g)Y \rangle = \langle X, Y \rangle\,, \quad \forall\, g \in G \text{ and } X, Y \in \mathfrak{g},$$

it follows that $\langle \cdot, \cdot \rangle$ is an invariant Riemannian metric on G (see [1] for more details). From now on we make the assumption: every compact, semisimple Lie group is equipped with the Riemannian metric given by Cartan–Killing form.

Adopting these invariant metrics and using the Lyapunov exponents we obtain an algebraic characterization of linear vector fields on compact, semisimple Lie groups.

Proposition 3.6. *Let $\mathcal{X}$ be a linear vector field on a compact, semisimple Lie group G. Then G is the central group of linear flow φ_t.*

Proof. We begin by writing $\mathcal{X} = X + I_*(X)$ with $X \in \mathfrak{g}$. It is clear that $\mathcal{D} = -\operatorname{Ad}(X)$. Then, for any $v \in \mathfrak{g}$ we have

$$\|e^{t\mathcal{D}}v\| = \|e^{t\,(-\operatorname{Ad}(X))}v\| = \|\operatorname{Ad}(\exp(-tX))v\| = \|v\|,$$

where we used the Ad-invariance of metric at last equality. Thus, Lyapunov exponents can be written as

$$\lambda(e, v) = \limsup_{t\to\infty} \frac{1}{t}\log(\|\mathrm{e}^{t\mathcal{D}}\mathrm{v}\|) = \limsup_{\mathrm{t}\to\infty} \frac{1}{\mathrm{t}}\log(\|\mathrm{v}\|) = 0.$$

Therefore, $\lambda_1 = \ldots = \lambda_k = 0$. Using the relation (3.2) we conclude that $\mathfrak{g} = \mathfrak{g}_0$. Since G is connected, $G = G_0$. It means that G is the central group associated to the linear flow φ_t. $\square$

Despite Proposition above is presented in [4], we proved it because our proof is done by dynamical concepts instead of algebraic concepts.

Our next step is to show that the linear flow φ_t satisfies some metric properties. Let (M, g) be a Riemannian manifold, a Riemannian distance is ρ associated to g is defined by

$$\rho(x, y) = \inf_{\sigma} \left\{ \int_0^1 g(\dot{\sigma}(s), \dot{\sigma}(s))^{1/2} ds \right\},$$

where the infimum is taken over all smooth curves σ such that $\sigma(0) = x$ and $\sigma(1) = y$.

Proposition 3.7. *Let $\mathcal{X}$ be a linear vector field on a compact, semisimple Lie group G. Then φ_t is an isometry for all t.*

Proof. We begin writing $\mathcal{X} = X + I_*X$ where X is a right invariant vector field. Thus for any $g, h \in G$ and $t \in \mathbb{R}$ we see that

$$\rho(\varphi_t(g), \varphi_t(h)) = \rho\left(L_{\exp(tX)} \circ R_{\exp(-tX)}(g), L_{\exp(tX)} \circ R_{\exp(-tX)}(h)\right),$$

where L and R stands for the left and right translation. Since left and right translations are isometries on G to the invariant distance given by Cartan–Killing form, it follows that

$$\rho(\varphi_t(g), \varphi_t(h)) = \rho(g, h),$$

which shows that φ_t is an isometry for any $t \in \mathbb{R}$. $\square$

Before our next result, we need to introduce some notations. For $r > 0$ we will denote an sphere of radius r with center g by $\mathbb{S}_r(g) = \{x \in G : \rho(x, g) = r\}$ and an open ball of radius r with center g by $B_r(g) = \{x \in G; \rho(x, g) < r\}$.

Proposition 3.8. *If G is a compact, semisimple Lie group, then for each $g \in G$ the linear flow $\varphi_t(g)$ is in a sphere.*

Proof. We first choose an arbitrary point $g \in G$ and write $r = \rho(g, e)$. Then

$$\rho(\varphi_t(g), e) = \rho(\varphi_t(g), \varphi_t(e)) = \rho(g, e) = r, \quad \forall\ t.$$

It means that $\varphi_t(g) \in \mathbb{S}_r$ for all t, and the proof is complete. $\square$

A direct consequence of the proposition above is about ω-limit and α-limit sets.

Corollary 3.9. *If G is a compact, semisimple Lie group, then ω-limit and α-limit sets of g are in spheres.*

We can now to prove our main result of our section.

Theorem 3.10. *Let G be a compact, semisimple Lie Group. Then any fixed point of linear flow φ_t is an stable point.*

Proof. We begin by fixing an arbitrary fixed point g of G. We also remember that a Riemannian distance induces the topology of Riemannian manifold. So it is sufficient to consider as neighborhoods of g open balls $B_r(g)$ where $r > 0$ is arbitrary. Choose $r_0 > 0$ such that $r_0 \leq r$ and consider the ball $B_{r_0}(g)$. Taking any $y \in B_{r_0}(g)$ we see that

$$\rho(\varphi_t(y), g) = \rho(y, g) < r_0 \leq r,$$

where we used Proposition 3.7 at first equality. It shows that $\varphi_t(B_{r_0}(g)) \subset B_r(g)$. Consequently, by definition, g is a stable point to the linear flow φ_t. $\square$

Hereafter we give a characterization of derivations on compact, semisimple Lie groups. Before we need to introduce some concepts. Following [3], if for an eigenvalue μ all complex Jordan blocks are one-dimensional, i.e., a complete set of eigenvectors exists, it is called semisimple. Equivalently, the corresponding real Jordan blocks are one-dimensional if μ is real and two-dimensional if $\mu, \bar{\mu} \in \mathbb{C} \setminus \mathbb{R}$.

Theorem 3.11. *On a compact, semisimple Lie group G, every derivation has only semisimple eigenvalues.*

Proof. Let $\mathcal{D}$ be a derivation on G. From Theorem 3.10 we see that e is a stable point of the linear flow φ_t associated to $\mathcal{D}$. Since $(d\varphi_t)_e = e^{t\mathcal{D}}$, it follows that the linearization of $\dot{g} = \mathcal{X}(g)$ is $X = \mathcal{D}(X)$. Being exp a local diffeomorphism and e stable, it follows that 0 is stable. From Proposition 3.6 we know that eigenvalues of $\mathcal{D}$ has real part null. Then Theorem 1.4.10 in [3] assures that every eigenvalue of $\mathcal{D}$ is semisimple, which gives the proof. $\square$

Theorem above is fundamental to study periodic orbits of linear flows.

To end this section, we study the stability of periodic orbits to the linear flows. A periodic orbit Γ of a linear flow φ_t is stable if for each open set V that contains Γ, there is an open set $W \subset V$ such that every solution, starting at a point in W at $t = 0$, stays in V for all $t \geq 0$.

Before presenting our next result, we need to introduce the following notation. Take $g \in G$ and consider the orbit $\varphi_t(g)$. Write for any $r > 0$, $Tube_r(\varphi_t(g)) = \{h \in G : \rho(h, \varphi_t(g)) < r$ for some $t\}$.

Proposition 3.12. *Let $\mathcal{X}$ be a linear vector field on a compact, semisimple Lie group G. If $h \in Tube_r(\varphi_t(g))$, then $\varphi_s(h) \in Tube_r(\varphi_t(g))$ for any $s \in \mathbb{R}$.*

Proof. Suppose that $h \in Tube_r(\varphi_t(g))$. Then for some t we have $\rho(h, \varphi_t(g)) < r$. From Proposition 3.7 it follows for any $s \in \mathbb{R}$ that

$$\rho(\varphi_s(h), \varphi_{t+s}(g)) = \rho(h, \varphi_t(g)) < r,$$

which implies that $\varphi_s(h) \in Tube_r(\varphi_t(g))$. □

Theorem 3.13. *Let $\mathcal{X}$ be a linear vector field on a compact, semisimple Lie group G. Then every periodic orbit is stable.*

Proof. Let $g \in G$ such that $\varphi_t(g)$ is a periodic orbit of linear flow φ_t. We consider a open set V such that $\varphi_t(g) \subset V$. Take $r_0 = \inf\{r : B_r(\varphi_t(g)) \subset V, \ \forall\ t \geq 0\}$. Thus it is sufficient to take $U = Tube_{r_0}(\varphi_t(g))$ and to apply the proposition above. □

4 Periodic orbits

In this section, we study periodic orbits of a linear flow in a compact, semisimple Lie group G. The key of our study is Theorem 3.11 because it describes all eigenvalues of any derivation on G.

We begin by recalling that a linear vector field $\mathcal{X}$ can be written as $\mathcal{X} = X + I_* X$, where X is a right invariant vector field, $I_* X$ is the left invariant vector field associated to X, and I_* is the differential of inverse map $\imath(g) = g^{-1}$. In this way, we can rewrite the differential equation (3.1) as

$$\dot{g} = X(g) + (I_* X)(g).$$

It implies that there exists a relation between flows of the linear dynamical system $\dot{g} = \mathcal{X}(g)$ and of the invariant one $\dot{g} = X(g)$. In fact, direct accounts shows that, for all $g \in G$, $\varphi_t(g)$ is a solution of (3.1) if, and only if, $\varphi_t(g) \cdot \exp(tX)$ is a solution of $\dot{g} = X(g)$. It suggests us that there exists a relation between periodic orbits of the linear flow φ_t and its associated invariant flow $\exp(tX)$. Therefore, our next step is to investigate this fact.

Proposition 4.1. *Let $\mathcal{X}$ be a linear vector field on a compact, semisimple Lie group G. The following sentences are equivalent:*

(i) for every $g \in G$ the invariant flow $\exp(tX)g$ is periodic;

(ii) the identity e is a periodic point of invariant flow $\exp(tX)$;

(iii) for each g the point $Ad(g)$ is periodic with respect to the flow $e^{t\mathcal{D}}$.

Furthermore, any assertion above implies that any point $g \in G$ is a periodic point of linear flow φ_t.

Proof. (i) ⇔ (ii) If for every $g \in G$ the orbit $\exp(tX)g$ is periodic, then e is a periodic point of the curve $\exp(tX)$. On contrary, suppose that e is a periodic point of the flow $\exp(tX)$, that is, there is a $s > 0$ such that $\exp((t+s)X) = \exp(tX)$. Then for any $g \in G$

$$\exp((t+s)X)g = (\exp((t+s)X) \cdot e) \cdot g = \exp(tX)g.$$

(i) ⇔ (iii) Since G is a semisimple Lie group, it follows

$$\mathrm{Ad}(\exp(tX) \cdot g) = e^{t\,\mathrm{Ad}(X)}\,\mathrm{Ad}(g) = e^{t\mathcal{D}}\,\mathrm{Ad}(g).$$

We thus get the equivalence.

Suppose now that e is a periodic point of the flow $\exp(tX)$, then there is a $s > 0$ such that $\exp(tX) = \exp((t+s)X)$. Thus

$$\varphi_{t+s}(g) = \exp((t+s)X) \cdot g \cdot \exp(-(t+s)X) = \exp(tX) \cdot g \cdot \exp(-tX) = \varphi_t(g),$$

which shows that g is a periodic point of φ_t. $\square$

The interest of the proposition above is that periodic orbits of linear or invariant flows are equivalents on compact, semisimple Lie groups.

We are now in position to show our main result.

Theorem 4.2. *Let G be a compact, semisimple Lie group. Assume that $\mathcal{X}$ is a linear vector field on G, that $\mathcal{D}$ and X are its associated derivation and invariant vector field, respectively. The following sentences are equivalent:*

(i) there exists a periodic orbit for the right invariant flow $\exp(tX)$*;*

(ii) the eigenvalues of the derivation $\mathcal{D} = -\operatorname{Ad}(X)$ are the form 0 or $\pm\alpha_1 \mathrm{i}, \dots, \pm\alpha_r \mathrm{i}$ where $\alpha_i \neq 0$, $i = 1, \dots, r$, and α_i/α_j is a rational for $i, j = 1, \dots, r$.

Furthermore, the sentences above implies that there exists a periodic orbit for the linear flow φ_t.

Proof. We first observe that (i) assures that φ_t has a periodic orbit by Proposition 4.1. We are going to show that (i) is equivalent to (ii). For this, it is sufficient to consider e as a periodic point to the flow $\exp(tX)$ with period $T > 0$. Then for all $t \in \mathbb{R}$

$$\exp((t+T)X) = \exp(tX) \Leftrightarrow \exp(TX) = e \Leftrightarrow e^{T\mathcal{D}} = Id.$$

Take the Jordan form J of $\mathcal{D}$. A simple account shows that $e^{TJ} = Id$. Since any eigenvalues of $\mathcal{D}$ is semisimple, its real Jordan Block has dimension 1 or 2 if it is real or complex, respectively. If 0 is eigenvalue of $\mathcal{D}$, then its real Jordan block is written as $J_0 = [0]$. Therefore e^{tJ_0} is constant. It implies that in direction of 0 the e^{tJ} is constant. Consequently, solutions associated to 0 are trivially periodic. Suppose that there are non-null eigenvalues. From Proposition 3.6 these eigenvalues are of the form $\pm\alpha_i \mathrm{i}$, $i = 1, \dots, r$. By Theorem 3.11, its real Jordan blocks are

$$\begin{pmatrix} \cos(t\alpha_i) & -\sin(t\alpha_i) \\ \sin(t\alpha_i) & \cos(t\alpha_i) \end{pmatrix}, \quad i = 1, \dots r.$$

As $e^{TJ} = Id$ we have $\alpha_i \cdot T = p_i \cdot 2\pi$ for some $p_i \in \mathbb{Z}$, $i = 1, \dots, r$. It entails for any $i, j = 1, \dots, r$ that $\alpha_i/\alpha_j = p_i/p_j$ is a rational number

Reciprocally, suppose that the eigenvalues of $\mathcal{D}$ are 0 or $\pm\alpha_1 \mathrm{i}, \dots, \pm\alpha_r \mathrm{i}$ where $\alpha_i \neq 0$, $i = 1, \dots, r$ and α_i/α_j is rational for $i, j = 1, \dots n$. For the eigenvalue 0 we have that the solution is constant. We thus consider the eigenvalues $\pm\alpha_i \mathrm{i}$ with $\alpha_i \neq 0$. Being $\pm\alpha_i \mathrm{i}$ semisimple, every real Jordan block associated to it has dimension two and the solution applied at this block gives the following matrix

$$\begin{pmatrix} \cos(t\alpha_i) & -\sin(t\alpha_i) \\ \sin(t\alpha_i) & \cos(t\alpha_i) \end{pmatrix}.$$

By assumption, there exists $p_{ij}, q_{ij} \in \mathbb{Z}$ with $q_{ij} > 0$ such that $\alpha_i/\alpha_j = p_{ij}/q_{ij}$ for $i, j = 1, \dots r$. In particular, we can written $\alpha_i = (p_{i1}/q_{i1})\alpha_1$ for $i = 2, \dots, r$. Assuming that $\alpha_1 > 0$ it is sufficient to take $T = q_{21}q_{31} \dots q_{r1}(2\pi/\alpha_1)$ to see that J satisfies $e^{TJ} = Id$. In other words, Id is a periodic point of e^{TJ} with period $T > 0$, which is equivalent Id to be periodic point of $e^{T\mathcal{D}}$. Consequently, by Proposition 4.1, the right invariant flow $\exp(tX)$ is periodic with period of $T > 0$. $\square$

Remark 4.3. The theorem above fails if α_i/α_j is irrational for some i and j in $\{1,\dots r\}$ because the flow e^{tJ} in the proof of theorem above is a flow of a harmonic oscillator. The bidimensional case is treated in Section 6.2 of [6].

Corollary 4.4. *Let G be a compact, semisimple Lie group with dimension 3. If X is a right invariant flow, then for every $g \in G$ the orbit $\exp(tX)\cdot g$ of the invariant flow is periodic. In consequence, the orbit $\varphi_t(g)$ of the linear flow is periodic for all $g \in G$.*

Proof. It is sufficient to observe that the derivation $\mathcal{D} = -\operatorname{ad}(X)$ has only eigenvalues 0, $\alpha\,\mathrm{i}$, and $-\alpha\,\mathrm{i}$ with $\alpha \in \mathbb{R}^*$. $\square$

5 Applications

In this section, our wish is to study the periodic orbits on compact, semisimple Lie groups of lower dimension. In fact, we are interested to describe the periodic orbits of linear flows on $SO(3)$ and $SU(2)$ and to study the periodic orbits of linear flows on $SO(4)$.

5.1 Linear flows on $SO(3)$ and $SU(2)$

Our first case is the orthogonal group

$$SO(3) = \{g \in \mathbb{R}^{3\times 3} : gg^T = 1,\ \det g = 1\}.$$

It is well known that its Lie algebra is

$$\mathfrak{so}(3) = \left\{ \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} : x, y, z \in \mathbb{R} \right\}.$$

Let $\mathcal{X}$ be a linear vector field on $SO(3)$. Then there exists a right invariant vector field X such that $\mathcal{X} = X + I_* X$. A direct calculus shows that eigenvalues of $\mathcal{D} = -\operatorname{ad}(X)$ are

$$\left\{0, -\sqrt{-x^2-y^2-z^2}, \sqrt{-x^2-y^2-z^2}\right\}.$$

Write $\lambda_1 = -\sqrt{-x^2-y^2-z^2}$ and $\lambda_2 = \sqrt{-x^2-y^2-z^2}$. Using functional calculus we obtain

$$\exp(tX) = \frac{\cosh(t\lambda_1)-1}{\lambda_1^2}X^2 + \frac{\sinh(t\lambda_1)}{\lambda_1}X + Id.$$

Therefore it is possible to give the solution of linear flow φ_t on $SO(3)$.

Proposition 5.1. *Let $\mathcal{X}$ be a linear vector field on $SO(3)$. Then the solution of linear flow $\varphi_t(g)$ associated to $\mathcal{X}$ is*

$$\left(\frac{\cosh(t\lambda_1)-1}{\lambda_1^2}X^2 + \frac{\sinh(t\lambda_1)}{\lambda_1}X + Id\right)\cdot g\cdot\left(\frac{\cosh(t\lambda_2)-1}{\lambda_2^2}X^2 + \frac{\sinh(t\lambda_2)}{\lambda_2}X + Id\right),$$

where X is the right invariant vector field associated to $\mathcal{X}$ and

$$\lambda_1 = -\sqrt{-x^2-y^2-z^2} \quad \text{and} \quad \lambda_2 = \sqrt{-x^2-y^2-z^2}.$$

Corollary 4.4 now assures the characterization of periodic orbits of the linear flow φ_t.

Proposition 5.2. *Under assumptions above,*

(i) every orbit of the invariant flow $\exp(tX)$ *is periodic;*

(ii) every orbit of the linear flow φ_t *is periodic.*

Our other case is the unitary group $SU(2)$, which is a matrix group given by

$$\mathrm{SU}(2) = \left\{ g \in \mathbb{C}^{2\times 2} : gg^T = 1, \det g = 1 \right\}.$$

The Lie algebra associated to $SU(2)$ is described as

$$\mathfrak{su}(2) = \left\{ \begin{bmatrix} \frac{i}{2}x & \frac{1}{2}(iz+y) \\ \frac{1}{2}(iz-y) & -\frac{1}{2}x \end{bmatrix} : x, y, z \in \mathbb{R} \right\}.$$

Let $\mathcal{X}$ be a linear vector field on $SU(2)$ and X the right invariant vector field associated to it. In analogous way to the case of $SO(3)$, it is easily to see that eigenvalues of a derivation $\mathcal{D} = -\operatorname{ad}(X)$ are

$$\left\{ 0, -\sqrt{-x^2-y^2-z^2}, \sqrt{-x^2-y^2-z^2} \right\}.$$

In consequence,

Proposition 5.3. *Under assumptions above,*

(i) every orbit of some invariant flow $\exp(tX)$ *is periodic;*

(ii) every orbit of some linear flow φ_t *is periodic.*

5.2 Periodic orbits on $SO(4)$

In this subsection, our wish is to give a condition for the orbits of invariant or linear flow on $SO(4)$ be or not be periodic. Let $\mathfrak{so}(4)$ be the Lie algebra of $SO(4)$ given by

$$\left\{ \begin{bmatrix} 0 & -x & -y & -z \\ x & 0 & -u & -v \\ y & u & 0 & -w \\ z & v & w & 0 \end{bmatrix} : x, y, z, u, v, w \in \mathbb{R} \right\}.$$

Consider the basis β for $\mathfrak{so}(4)$ that consists of 4×4 matrices $e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}$ that have 1 in the (i,j) entry, -1 in the (j,i) entry, and 0 elsewhere ($1 \le i < j \le 4$). A computation of Lie brackets gives

$$\begin{array}{lllll} [e_{12}, e_{13}] = e_{23}, & [e_{12}, e_{14}] = e_{24}, & [e_{12}, e_{23}] = -e_{13}, & [e_{12}, e_{24}] = -e_{14}, & [e_{12}, e_{34}] = 0, \\ [e_{13}, e_{14}] = e_{34}, & [e_{13}, e_{23}] = e_{12}, & [e_{13}, e_{24}] = 0, & [e_{13}, e_{34}] = -e_{14}, & [e_{14}, e_{23}] = 0, \\ [e_{14}, e_{24}] = e_{12}, & [e_{14}, e_{34}] = e_{13}, & [e_{23}, e_{24}] = e_{34} & [e_{23}, e_{34}] = -e_{24}, & [e_{24}, e_{34}] = e_{23}. \end{array}$$

Let $\mathcal{X}$ be a linear vector field on $SO(4)$. Let us denote by $\mathcal{D} = -\operatorname{Ad}(X)$ the associated derivation to $\mathcal{X}$ where X is an right invariant vector field on $SO(4)$. Our next step is to describe the derivation $\mathcal{D}$. To do this, write

$$X = ae_{12} + be_{13} + ce_{14} + de_{23} + ee_{24} + fe_{34}, \quad a, b, c, d, e, f \in \mathbb{R}.$$

By Lie brackets above, we compute

$$\mathcal{D} = -\operatorname{ad}(X) = \begin{pmatrix} 0 & -d & -e & b & c & 0 \\ d & 0 & -f & -a & 0 & c \\ e & f & 0 & 0 & -a & -b \\ -b & a & 0 & 0 & -f & e \\ -c & 0 & a & f & 0 & -d \\ 0 & -c & b & -e & d & 0 \end{pmatrix}.$$

Some calculus show that the eigenvalues of $\mathcal{D} = -\operatorname{ad}(X)$ are

$$\left\{0, 0, \pm\sqrt{-(a+f)^2 - (b-e)^2 - (c+d)^2}, \pm\sqrt{-(a+f)^2 - (b+e)^2 - (c-d)^2}\right\}.$$

We observe that the eigenvalues are according to Theorem 3.11. We now are in a position to give a condition that characterizes periodic orbits of an invariant or linear flow.

Theorem 5.4. *Let $\mathcal{X}$ be a linear vector field on $SO(4)$. Consider the derivation $\mathcal{D} = -\operatorname{ad}(X)$ of $\mathcal{X}$, where X is a right invariant vector field such that*

$$X = ae_{12} + be_{13} + ce_{14} + de_{23} + ee_{24} + fe_{34}, \quad a, b, c, d, e, f \in \mathbb{R}.$$

A necessary and sufficient condition to every orbit that is not a fixed point of the invariant flow $\exp(tX)$ *be periodic is that*

$$\sqrt{\frac{(a+f)^2 + (b-e)^2 + (c+d)^2}{(a+f)^2 + (b+e)^2 + (c-d)^2}} \tag{5.1}$$

is a rational number. The last condition is satisfies if $be = cd$.

Proof. It is a direct application of Theorem 4.2. □

Corollary 5.5. *Under conditions of Theorem above, if (5.1) is a rational number, then every orbit of the linear flow φ_t associated to derivation $\mathcal{D}$ that is not a fixed point is periodic.*

As a direct application of the theorem above, each right invariant vector field of the basis $\beta = \{e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}\}$ yields periodic orbits for the linear or invariant flows.

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